PROXIMITY EFFECT OF THE SUPERCONDUCTIVITY IN METALLIC SUPERLATTICES AND A SELF-SIMILAR MULTILAMELLAR SYSTEM

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Proximity Effect of the Superconductivity in Metallic Superlattices and a Self-Similar Multilamellar System

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Abstract

In the thesis, the various theories of inhomogeneous superconductivity based on Gor'kov's equations(Chapter 2) are reviewed including de Gennes-Werthamer theory and Eilenberger equations as well as their applications to study the proximity effect of a bilayer system. McMillan's tunnelling model is also introduced. The characteristics of the thickness dependence of the transition temperature T_c in the thin film limit from the above theories are discussed and will be compared with the results from our calculation.

The superconductivity of a superlattice consisting of alternating superconducting and normal layers has been investigated(Chapter 3) by means of the Bogoliubov equation. A relation, which determines the transition temperature T_c, is obtained from the self-consistency equation of the order parameter. The analytical expression of the dependence of the transition temperature T_c on the thickness of a superconducting layer in the thin film limit has been obtained through an analysis of the equation and the comparison with the results from the other theories discussed in Chapter 2 has been done. A periodic energy-momentum relation reflecting the periodicity of the structure has been obtained which gives rise to the basis for any further calculation for the presence of an external magnetic field or a finite order parameter.

In Chapter 4, we applied the Werthamer theory to a quasi-periodic geometry with a fractional dimension $\mathcal{D}$. We build the recurrence relation of the coefficients in the linear fractional transformation in terms of the self-similarity of the geometry, which provides a systematic method to deal with the complicated boundary conditions. The dependence of the transition temperature on both the thickness of the superconducting layer and fractional dimension $\mathcal{D}$ has been obtained by the scaling argument in some limiting cases. The scaling argument is also extended to the case of the presence of a perpendicular magnetic field. The numerical calculation for transition temperature as a function of the thickness is completed and shown in the corresponding figures. We also calculate certain values of T_c corresponding to the given parameters, such as dimension $\mathcal{D}$, thickness d_s and coherence length $\xi_{||}(0)$, used in the experiment and compare the theoretical results from the calculation with the experimental values. The result of the comparison is satisfactory and the discrepancies ranging from $2\% \rightarrow 10\%$, consistent with the Werthamer theory.

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Chapter 1

Introduction

Since the early 1960's, growing attention has been paid to the proximity effect in superconducting systems. Most of the early studies of the proximity effect were concerned with the transition temperature for superconducting bilayer structure and were based on the Ginzberg-Landau(GL) theory of inhomogeneous superconductivity or on the linearized Gor'kov equations[1]. In particular de Gennes[2] [3] derived an expression for the kernel in the linearized Gor'kov equation based on the solution of the diffusion equation together with the appropriate boundary conditions. Using this method he was able to obtain certain analytical results in the limiting case of thick and thin films. Werthamer[4] has also derived an expression for the kernel, although in a somewhat more straightforward manner, and obtained a result equivalent to that of One-Frequency approach found in de Gennes[2] work for thick films. We will discuss the derivation and the application of the de Gennes-Werthamer theory of the proximity effect in some detail in the thesis. A substantial number of theoretical studies have examined the generalisation of the basic de Gennes-Werthamer theory.

An extension of the application of the de Gennes-Werthamer theory to the more complicated geometry is straightforward provided the boundary conditions can be properly handled. Of particular interest are systems comprising alternating layers of superconducting and normal metal (SNS ... SNS) or more generally systems comprising alternating layers of superconducting materials with different material characteristics (SS'S ... SS'S) and many length scales. Much of the interest

has concerned superlattice structures which is a relatively simple system involving only two length scales. Experimental studies on such superlattices have concerned largely the dependence of the superconducting transition temperature on the relative thickness of the layers and the modulation length of the superlattice,[5] [6] [7] [8] and the effects of applied magnetic field.[9] [10] [11] [12] [13] Such studies reveal a rich variety of phenomena associated with the complex interaction of competing mechanisms and, in the case of the critical field measurements, the often subtle interplay between the magnetic length scale and the inhomogeneity in the superconducting coherence length associated with the superlattice structure[14]. Since the de Gennes-Werthamer theory is concerned with the calculation of the kernel in the linearized Gor'kov equations, it is applicable only close to the transition temperature, where the superconducting order parameter vanishes.

If one wishes to explore the superconducting properties of such systems away from the transition temperature one would be involved in the complication of the Gor'kov equations due to the non-linear feature. A much simplified version of the Gor'kov equations, the Eilenberger equations[15] [16] provides the basis of studying such properties of the system. We will outline the basis of the Eilenberger formalism. Close to the transition temperature, it can be shown that the Eilenberger equations reduce to the de Gennes-Werthamer equations in the appropriate limits and recent quantitative calculations of the upper critical field have been presented based on the Eilenberger theory[15] [16] [17]. However, in general, the calculations are extremely difficult to handle. Hence, alternative approximation schemes must be employed.

Another approach is based on the Bogoliubov equations, which with a suitably simple assumption of the form of the pair potential can be solved and various properties of the system thereby obtained[18] [19] [20] [21]. The relation between the transition temperature and the reduced thickness is easily obtained, however

any further calculation involving a finite superconducting order parameter Δ or the presence of the critical magnetic field will be involved in the complication of the periodic structure of the energy-momentum spectrum. Another phenomenological approach is the tunnelling model of McMillan[18], where a potential barrier is assumed to exist at the interface; tunnelling through the barrier then is treated by the transfer Hamiltonian method[22] [23]. While such theories have been extremely successful in providing a quantitative description of many of the observed phenomena there are in the literature a number of data which are not consistent with the predictions of the Werthamer theory of the proximity effect[5] [6] [8]. That such theories fail in the case of thin layers and for clean systems is not surprising given the approximations involved in the derivation of the Werthamer formula[24] [25] [26].

Other geometries which have been studied include one dimensional quasiperiodic structures[27] and self similar or fractal geometries[28]. The interest in such novel geometries stems in part from the hope that despite the relatively complex nature of the geometries in question, it is nevertheless possible to understand many qualitative aspects of their behaviour in terms of fairly general arguments. In the case of the self similar geometry the fact that the structure repeats itself over many length scales suggests that one should be able to describe certain aspects of their behaviour in terms of certain scaling arguments. This has indeed proved to be the case in a number of recent studies on the closely related problems involving superconducting and normal fractal networks[29] [30].

In this thesis, we calculate dependence of the transition temperature T_c on the thickness of the superconducting layer for both the periodic and quasiperiodic geometries of superlattice. The outline of the thesis is as follows: In Chapter 2, we review briefly the main microscopic theories starting from the effective BCS

Hamiltonian and discuss the de Gennes-Werthamer theory, the Eilenberger equations and the McMillan model. We change the notations in this chapter from time to time in order to follow the notations in those original papers cited in the chapter so that one can easily compare the results in the thesis with those in the papers. In chapter 3, we apply the Bogoliubov equation to a periodic geometry with a simple assumption of the form of the pair potential, and then calculate the pair potential self-consistently which allows us to obtain the relationship between the transition temperature and the thickness of the single layer embedded in the superlattice. An energy-momentum spectrum has been obtained with the application of Bloch's theorem. We compare and contrast the results obtained through this technique with those obtained by the other approximation schemes in various limiting situations. In Chapter 4, we calculate the transition temperature of the quasiperiodic geometry with fractional dimension by the generalized Werthamer theory proposed by Takahashi and Tachiki[31]. A comparison of the theoretical predictions with the measured values[28] is presented and the agreement shown to be quite satisfactory. In two limiting cases that the length scale of the whole fractal structure $L >> \xi$ and $L << \xi$ with ξ being the coherence length, the analytical results of the scaling law which reflects the thickness dependence of the transition temperature are obtained. In Chapter 5, we summarize the results of the work presented and offer some conclusions and discussion concerning them.

Chapter 2

The Inhomogeneous Microscopic Theory of Superconductivity

In this chapter, we introduce the effective BCS Hamiltonian that arises as a result of the electron-phonon interaction[32] together with the Gorkov's equations for the Green's functions.[1] In order to apply Gor'kov equations to the inhomogeneous superconductor, some further theories with appropriate approximations have to be introduced. Near the transition temperature T_c, one can linearize the Gor'kov equation since the order parameter is infinitesimally small. For a bilayer system composed of the alternating superconducting and normal metal layers, de Gennes assumed that the kernel in the linearized Gor'kov equation satisfies the diffusion equation and introduced the boundary conditions at free surface and interfaces so that the transition temperature can be completely solved in various limiting cases. Werthamer derived the same result as that found in de Gennes' theory in more straightforward way rather than deal with the diffusion equation. Away from the transition temperature, a calculation scheme suitable for a finite order parameter is needed. Eilenberger simplified the Gor'kov equation through reducing the unknown Green's functions from four to two and the theory has been widely used to study the effect of the presence of a external magnetic field. Another approach is the tunnelling model Hamiltonian proposed by McMillan which introduces a transfer Hamiltonian and treats the interface as an energy barrier. The various theories mentioned above will be discussed briefly below.

§2-1 Gor'kov Equation

§2-1-1 The Electron-Phonon Interaction Hamiltonian

In the second quantization representation, the model Hamiltonian of the elec-

tron-phonon system may be written as[32]

$$H = H_0 + H_1 \ , \tag{2-1-1}$$

where H_0 is the Hamiltonian for free electrons and free phonons,

$$H_0 = \sum_{\vec{q}} \hbar\omega_{\vec{q}} a^{\dagger}_{\vec{q}} a_{\vec{q}} + \sum_{\vec{k},\sigma} \epsilon_{\vec{k}} c^{\dagger}_{\vec{k}\sigma} c_{\vec{k}\sigma} \ , \tag{2-1-2}$$

and H_1 is the interaction Hamiltonian of the electrons and the phonons,

$$H_1 = \sum_{\vec{k},\vec{q},\sigma} (D_{\vec{q}} a_{\vec{q}} c^{\dagger}_{\vec{k}+\vec{q},\sigma} c_{\vec{k},\sigma} + D^{*}_{\vec{q}} a^{\dagger}_{\vec{q}} c^{\dagger}_{\vec{k}-\vec{q},\sigma} c_{\vec{k},\sigma}) \ . \tag{2-1-3}$$

The indices $\vec{q}$ and $\vec{k}$ denote the wave vectors of the phonons and the electrons respectively, $\epsilon_{\vec{k}} = \frac{k^2}{2m} - \mu$ is the energy of a single electron in the state $\vec{k}$ relative to the chemical potential μ, $\hbar\omega_{\vec{q}}$ denotes the energy of a phonon in the state $\vec{q}$, σ is the spin index of an electron; $D_{\vec{q}}(D^{*}_{\vec{q}})$ is the electron-phonon coupling constant, which depends on the interaction potential, and $a^{\dagger}_{\vec{q}}$ $(a_{\vec{q}})$ and $c^{\dagger}_{\vec{k},\sigma}$ $(c_{\vec{k},\sigma})$ are the creation (annihilation) operators of the phonons and electrons, which satisfy the algebra

$$[a^{\dagger}_{\vec{q}}, a_{\vec{q}'}] = a^{\dagger}_{\vec{q}} a_{\vec{q}'} - a_{\vec{q}'} a^{\dagger}_{\vec{q}} = \delta_{\vec{q}\,\vec{q}'} \ , \tag{2-1-4}$$

$$\{c^{\dagger}_{\vec{k},\sigma}, c_{\vec{k}',\sigma'}\} = c^{\dagger}_{\vec{k},\sigma} c_{\vec{k}',\sigma'} + c_{\vec{k}',\sigma'} c^{\dagger}_{\vec{k},\sigma} = \delta_{\vec{k}\,\vec{k}'}\, \delta_{\sigma\sigma'} \ , \tag{2-1-5}$$

respectively.

Starting from the Hamiltonian (2-1-1), Fröhlich[32] derived the effective Hamiltonian by means of the canonical transformation

$$H_s = e^{-S} H e^{S} \ , \tag{2-1-6}$$

where the operator S is chosen such that the transformed Hamiltonian H_s has the same eigenvalue spectrum as that of H.

Equation (2-1-6) may be expanded as

$$\begin{aligned} H_s &= H + [H,S] + \frac{1}{2}[[H,S],S] + \cdots \\ &= H_0 + (H_1 + [H_0,S]) + \frac{1}{2}[(H_1 + [H_0,S]),S] \\ &\quad + \frac{1}{2}[H_1,S] + \frac{1}{2}[H_1 + [H_1,S],S] + \cdots . \end{aligned} \tag{2-1-7}$$

We can eliminate the interaction to lowest order if we choose S to satisfy the equation

$$H_1 + [H_0,S] = 0\,, \tag{2-1-8}$$

so that there is no first order electron-phonon interaction and we obtain the following expression for H_S

$$\begin{aligned} H_s &= H_0 + \frac{1}{2}[H_1,S] + \frac{1}{2}[H_1 + [H_1,S],S] + \cdots \\ &\approx H_0 + \frac{1}{2}[H_1,S]\,. \end{aligned} \tag{2-1-9}$$

If the operator S is assumed to be of the form

$$S = \sum_{\vec{k},\vec{q},\sigma} (A_{\vec{q}} a_{\vec{q}} c^{\dagger}_{\vec{k}+\vec{q},\sigma} c_{\vec{k},\sigma} + B_{\vec{q}} a^{\dagger}_{\vec{q}} c^{\dagger}_{\vec{k}-\vec{q},\sigma} c_{\vec{k},\sigma})\,, \tag{2-1-10}$$

then using equation (2-1-8), we can determine the coefficients $A_{\vec{q}}$ and $B_{\vec{q}}$

$$\begin{cases} A_{\vec{q}} = D_{\vec{q}} (\epsilon_{\vec{k}} + h\omega_{\vec{q}} - \epsilon_{\vec{k}+\vec{q}})^{-1}\,, \\ B_{\vec{q}} = D^{*}_{\vec{q}} (\epsilon_{\vec{k}} - h\omega_{\vec{q}} - \epsilon_{\vec{k}-\vec{q}})^{-1}\,, \end{cases} \tag{2-1-11}$$

thus S is expressed as

$$S = \sum_{\vec{k},\vec{q},\sigma} (D_{\vec{q}} \frac{a_{\vec{q}} c^{\dagger}_{\vec{k}+\vec{q},\sigma} c_{\vec{k},\sigma}}{\epsilon_{\vec{k}} - \epsilon_{\vec{k}+\vec{q}} + h\omega_{\vec{q}}} + D^{*}_{\vec{q}} \frac{a^{\dagger}_{\vec{q}} c^{\dagger}_{\vec{k}-\vec{q},\sigma} c_{\vec{k},\sigma}}{\epsilon_{\vec{k}} - \epsilon_{\vec{k}-\vec{q}} - h\omega_{\vec{q}}})\,. \tag{2-1-12}$$

To obtain the effective Hamiltonian which describes the scattering process of the electrons through exchange of one phonon, we can take the expectation value of the term H_s with respect to the phonon vacuum state $|0>$, which is defined as

$$a_{\vec{q}}\,|0> = 0, \tag{2-1-13}$$

to give

$$
\begin{aligned}
H_{eff} &= <0|H_s|0> \\
&= \frac{1}{2}\sum_{n_{\vec{q}}}\sum_{\vec{q}}\{<0|H_1|n_{\vec{q}}><n_{\vec{q}}|S|0> - <0|S|n_{\vec{q}}><n_{\vec{q}}|H_1|0>\} \\
&= \frac{1}{2}\sum_{\vec{q}}\{<0|H_1|1_{\vec{q}}><1_{\vec{q}}|S|0> - <0|S|1_{\vec{q}}><1_{\vec{q}}|H_1|0>\} .
\end{aligned}
\tag{2-1-14}
$$

where we have used the completeness of the phonon eigenstates

$$
\sum_{n_{\vec{q}},\vec{q}} |n_{\vec{q}}><n_{\vec{q}}| = 1 , \tag{2-1-15}
$$

and the property of the operator S

$$
<n_{\vec{q}}|S|0> = 0 \qquad \text{for} \qquad n_{\vec{q}} \neq 1_{\vec{q}} . \tag{2-1-16}
$$

Substituting (2-1-12) and (2-1-3) into (2-1-14) and completing the algebraic calculation, we obtain the effective interaction Hamiltonian

$$
H_{eff} = \frac{1}{2}\sum_{\vec{k},\vec{k}',\vec{q}}\sum_{\sigma,\sigma'} |D_{\vec{q}}|^2 \frac{2\hbar\omega_{\vec{q}}}{(\epsilon_{\vec{k}} - \epsilon_{\vec{k}+\vec{q}})^2 - (\hbar\omega_{\vec{q}})^2} c^{\dagger}_{\vec{k}+\vec{q},\sigma} c^{\dagger}_{\vec{k}'-\vec{q},\sigma'} c_{\vec{k}',\sigma'} c_{\vec{k},\sigma} . \tag{2-1-17}
$$

which describes the interaction between two electrons through the one-phonon-exchange process The other terms describing multiphonon processes can be omitted due to the Migdal's theorem.[33]

§2-1-2 The Effective BCS Hamiltonian

The effective electron-phonon interaction may be written as

$$
H_{eff} = \frac{1}{2}\sum_{\vec{k},\vec{k}',\vec{q}}\sum_{\sigma,\sigma'} V_{\vec{k},\vec{q}} c^{\dagger}_{\vec{k}+\vec{q},\sigma} c^{\dagger}_{\vec{k}'-\vec{q},\sigma'} c_{\vec{k}',\sigma'} c_{\vec{k},\sigma} , \tag{2-1-18}
$$

where

$$
V_{\vec{k},\vec{q}} = |D_{\vec{q}}|^2 \frac{2\hbar\omega_{\vec{q}}}{(\epsilon_{\vec{k}} - \epsilon_{\vec{k}+\vec{q}})^2 - (\hbar\omega_{\vec{q}})^2} . \tag{2-1-19}
$$

When both of the electrons scattered by exchanging a phonon are very close to the Fermi surface determined by chemical potential μ, the difference of the energies satisfies the condition $|\epsilon_{\vec{k}} - \epsilon_{\vec{k}+\vec{q}}| < \hbar\omega_{\vec{q}} \approx \hbar\omega_D$ (ω_D is the Debye frequency) so that the coupling constant $V_{\vec{k},\vec{q}} < 0$, which yields an attractive interaction. If the difference of the energies is larger than $\hbar\omega_D$, the coupling constant $V_{\vec{k},\vec{q}}$ is positive so that H_{eff} is a repulsive interaction which decreases rapidly with increasing energy difference.

In the superconducting state, only the electrons occupying the states with energy in the range of $\mu \pm \hbar\omega_D$ can be scattered to the new states through the phonon-exchange process. Those occupying states far below the Fermi surface can be treated as free electrons which prohibit other electrons from occupying the same states by the Pauli exclusion principle. Thus, in the application of (2-1-19) to superconductivity, Cooper[34] took the appropriate assumption that for the electrons in the states $\epsilon_{\vec{k}} < \hbar\omega_D$ the coupling induced by phonons is a constant $V > 0$ and for those in the states $\epsilon_{\vec{k}} > \hbar\omega_D$ the coupling vanishes. So the effective Hamiltonian becomes, in the BCS approximation,

$$H_{eff} = -\frac{1}{2}V \sum_{\vec{k},\vec{k}',\vec{q}} \sum_{\sigma,\sigma'} c^{\dagger}_{\vec{k}+\vec{q},\sigma} c^{\dagger}_{\vec{k}'-\vec{q},\sigma'} c_{\vec{k}',\sigma'} c_{\vec{k},\sigma} \ . \qquad (2\text{-}1\text{-}20)$$

The approximation obtained here is valid only for a "weak coupling" superconductor, since the form of the H_{eff} implies two assumptions:

(i)In assuming a coupling constant which is independent of the energy variables $\epsilon_{\vec{k}}$ and $\hbar\omega_{\vec{q}}$, we have neglected the effect of retardation.

(ii) All the processes of absorption and emission of phonons by creating and annihilating a pair of quasi-particles have been neglected so that the quasi-particles are treated as those with infinite lifetime.

To consider both of the influences mentioned above, a strong coupling theory should be built and we will discuss it in the last section of this chapter briefly.

In applying the theory to the inhomogeneous case, it is convenient to transform the H_{eff} into the coordinate representation by means of the transformation,

$$\begin{cases} \hat{\psi}_\sigma(\vec{x}) = \sum_{\vec{k}} c_{\vec{k},\sigma} e^{i\vec{k}\cdot\vec{r}} , \\ \hat{\psi}^\dagger_\sigma(\vec{x}) = \sum_{\vec{k}} c^\dagger_{\vec{k},\sigma} e^{-i\vec{k}\cdot\vec{r}} , \end{cases} \tag{2-1-21}$$

which may be inverted to yield

$$\begin{cases} c_{\vec{k},\sigma} = \int e^{-i\vec{k}\cdot\vec{r}} \hat{\psi}_\sigma(\vec{x})\, d^3x , \\ c^\dagger_{\vec{k},\sigma} = \int e^{i\vec{k}\cdot\vec{r}} \hat{\psi}_\sigma(\vec{x})\, d^3x , \end{cases} \tag{2-1-22}$$

where we have chosen the volume as unity. The operators $\hat{\psi}_\sigma(\vec{x})$ and $\hat{\psi}^\dagger_\sigma(\vec{x})$ can be shown to satisfy the anti-commutating algebra

$$\{\hat{\psi}_\sigma(\vec{x}) , \hat{\psi}^\dagger_{\sigma'}(\vec{x}')\} = \delta(\vec{x} - \vec{x}')\delta_{\sigma\sigma'}. \tag{2-1-23}$$

The effective Hamiltonian in the BCS approximation may be written in terms of the operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ as

$$\begin{aligned} H_{eff} \to \hat{V} &= -\frac{1}{2} V \sum_{\alpha,\beta} \int \hat{\psi}^\dagger_\alpha(\vec{x})\, \hat{\psi}^\dagger_\beta(\vec{x})\, \hat{\psi}_\beta(\vec{x})\, \hat{\psi}_\alpha(\vec{x})\, d^3x \\ &= -V \int \hat{\psi}^\dagger_\uparrow(\vec{x})\, \hat{\psi}^\dagger_\downarrow(\vec{x})\, \hat{\psi}_\downarrow(\vec{x})\, \hat{\psi}_\uparrow(\vec{x})\, d^3x , \end{aligned} \tag{2-1-24}$$

where we have further simplified the effective Hamiltonian by considering the Pauli exclusion principle, which prohibits two electrons from staying at the same state, i.e.

$$\hat{\psi}^\dagger_\uparrow(\vec{x})\, \hat{\psi}^\dagger_\uparrow(\vec{x}) = \hat{\psi}^\dagger_\downarrow(\vec{x})\, \hat{\psi}^\dagger_\downarrow(\vec{x}) = \hat{\psi}_\uparrow(\vec{x})\, \hat{\psi}_\uparrow(\vec{x}) = \hat{\psi}_\downarrow(\vec{x})\, \hat{\psi}_\downarrow(\vec{x}) = 0$$

For the Hamiltonian (2-1-2) of the free phonons and electrons, we omit the the free phonon term since in superconductivity, the physical properties of a superconducting state mainly depend on the behavior of the electrons, and consider

the phonons only in a role which induces a new attractive interaction H_{eff}. The free part of the Hamiltonian, $\hat{H}_0$ may then be written as

$$H_0 \approx \hat{K}_0 = \int d^3x \ \hat{\psi}^\dagger_\sigma(\vec{x})\epsilon(i\nabla)\hat{\psi}_\sigma(\vec{x}), \tag{2-1-25}$$

with

$$\epsilon(i\nabla) = \frac{-h^2\nabla^2}{2m} - \mu \ .$$

Combining Eq. (2-1-24) and (2-1-25), we finally obtain the BCS Hamiltonian $\hat{K}$

$$\hat{K} = \hat{K}_0 + \hat{V} \ , \tag{2-1-26}$$

which is a well defined theory including the phonon-induced attractive interaction among the electrons. Such a new mechanism enables people to calculate various parameters in superconductivity and explain the experimental phenomena.

§2-1-3 Gor'kov Equation

The theory obtained in last subsection can be easily generalized to include the influences of both external magnetic field and the existence of non-magnetic impurities. Following the method of A. L. Fetter and J. D. Walecka[35], we obtain

$$\begin{cases} \hat{K} = \hat{K}_0 + \hat{V} \ , \\ \hat{K}_0 = \int d^3x \ \hat{\psi}^\dagger_\alpha(\vec{x}) \{ \frac{1}{2m}[-ih\nabla + \frac{e}{c}\vec{A}(\vec{x})]^2 - \mu + U(\vec{x}) \} \hat{\psi}_\alpha(\vec{x}) \ , \\ \hat{V} = -\frac{1}{2}V \sum_{\alpha,\beta} \int \hat{\psi}^\dagger_\alpha(\vec{x}) \ \hat{\psi}^\dagger_\beta(\vec{x}) \ \hat{\psi}_\beta(\vec{x}) \ \hat{\psi}_\alpha(\vec{x}) \ d^3x \ , \end{cases} \tag{2-1-27}$$

where $\vec{A}(\vec{x})$ is the vector potential and $U(\vec{x})$ is the impurity potential which is of the form

$$U(\vec{x}) = \sum_a u(\vec{x} - \vec{x}_a) \ , \tag{2-1-28}$$

with $\vec{x}_a$ being the position of a^{th} impurity.

Generally, one can obtain a non-linear differential equation describing the spatial and temporal evolution of the operator fields $\hat{\psi}_\sigma(\vec{x})$ by means of the Hamiltonian (2-1-27). However, it is too complicated to be solved completely so a mean field approximation has to be taken. In this approximation, one can decompose the product of four field operators in the interaction term $\hat{V}$ into the following

$$\begin{aligned}\hat{V} &\approx \hat{V}_{eff} \\ &= -V \int d^3x \{ < \hat{\psi}_\downarrow^\dagger(\vec{x})\hat{\psi}_\uparrow^\dagger(\vec{x}) > \hat{\psi}_\uparrow(\vec{x})\hat{\psi}_\downarrow(\vec{x}) + \hat{\psi}_\downarrow^\dagger(\vec{x})\hat{\psi}_\uparrow^\dagger(\vec{x}) < \hat{\psi}_\uparrow(\vec{x})\hat{\psi}_\downarrow(\vec{x}) > \}.\end{aligned} \tag{2-1-29}$$

The reason for making such a decomposition is that an essential characteristic of superconductivity is the formation of a Cooper pairs by two electrons with opposite spins, which results in the appearance of an order parameter, since $< \hat{\psi}\hat{\psi} > \neq 0$. The other terms coming from the Hartree-Fock decomposition such as $< \hat{\psi}_\alpha^\dagger(\vec{x})\hat{\psi}_\alpha(\vec{x}) >$ and $< \hat{\psi}_\alpha^\dagger(\vec{x})\hat{\psi}_\beta(\vec{x}) >$ have been omitted since only the difference between superconducting state and normal states are of interest; nevertheless, those Hartree-Fock terms are assumed to be the same in both of the states and they have no influence on the comparison between these two states.

With the mean field approximation, the effective Hamiltonian now becomes

$$\hat{K} \approx \hat{K}_{eff} = \hat{K}_0 + \hat{V}_{eff} \ . \tag{2-1-30}$$

and the pair amplitude in the decomposition is defined as

$$< \hat{\psi}_\downarrow^\dagger(\vec{x})\hat{\psi}_\uparrow^\dagger(\vec{x}) > = \frac{T_r[e^{-\beta K_{eff}}\hat{\psi}_\downarrow^\dagger(\vec{x})\hat{\psi}_\uparrow^\dagger(\vec{x})]}{T_r[e^{-\beta K_{eff}}]} \ . \tag{2-1-31}$$

with

$$\beta^{-1} = k_B T$$

Eq. (2-1-31) provides us with a self-consistent definition of the pair amplitude which depends on the effective Hamiltonian including the pair amplitude itself.

To calculate the pair amplitude, one needs to introduce the generalized Heisenberg field operators defined as

$$\begin{cases} \hat{\psi}_{K\uparrow}(\vec{x},\tau) = e^{\hat{K}_{eff}\frac{\tau}{\hbar}}\ \hat{\psi}_{\uparrow}(\vec{x})\ e^{-\hat{K}_{eff}\frac{\tau}{\hbar}}\ , \\ \hat{\psi}^{\dagger}_{K\uparrow}(\vec{x},\tau) = e^{\hat{K}_{eff}\frac{\tau}{\hbar}}\ \hat{\psi}^{\dagger}_{\uparrow}(\vec{x})\ e^{-\hat{K}_{eff}\frac{\tau}{\hbar}}\ , \\ \hat{\psi}_{K\downarrow}(\vec{x},\tau) = e^{\hat{K}_{eff}\frac{\tau}{\hbar}}\ \hat{\psi}_{\downarrow}(\vec{x})\ e^{-\hat{K}_{eff}\frac{\tau}{\hbar}} \\ \hat{\psi}^{\dagger}_{K\downarrow}(\vec{x},\tau) = e^{\hat{K}_{eff}\frac{\tau}{\hbar}}\ \hat{\psi}^{\dagger}_{\downarrow}(\vec{x})\ e^{-\hat{K}_{eff}\frac{\tau}{\hbar}}\ , \end{cases} \tag{2-1-32}$$

where we may regard τ as an the imaginary time.

With this definition, one can establish the equation of motion for the field operators from the effective Hamiltonian $\hat{K}_{eff}$ using the Heisenberg equation

$$\hbar\frac{\partial}{\partial\tau}\hat{O}_K = [\hat{O}_K\ ,\hat{K}_{eff}]\ , \tag{2-1-33}$$

where $\hat{O}_K$ is a arbitrary operator defined by $\hat{O}_K = e^{\hat{K}_{eff}\frac{\tau}{\hbar}}\hat{O}e^{-\hat{K}_{eff}\frac{\tau}{\hbar}}$. By means of the anti-commutating algebra (2-1-23) we obtain the equation of motion for the field operators as follows

$$\begin{cases} \hbar\frac{\partial}{\partial\tau}\hat{\psi}_{K\downarrow} = -[\frac{1}{2m}(-i\hbar\nabla+\frac{e}{c}\vec{A}(\vec{x}))^2-\mu+U(\vec{x})]\hat{\psi}_{K\downarrow} - V<\hat{\psi}_{\downarrow}\hat{\psi}_{\downarrow}>\hat{\psi}^{\dagger}_{K\downarrow}\ , \\ \hbar\frac{\partial}{\partial\tau}\hat{\psi}^{\dagger}_{K\downarrow} = [\frac{1}{2m}(i\hbar\nabla+\frac{e}{c}\vec{A}(\vec{x}))^2-\mu+U(\vec{x})]\hat{\psi}^{\dagger}_{K\downarrow} - V<\hat{\psi}^{\dagger}_{\downarrow}\hat{\psi}^{\dagger}_{\downarrow}>\hat{\psi}_{K\downarrow}\ . \end{cases} \tag{2-1-34}$$

Note that the pair amplitudes may be expressed in terms of the Heisenberg field operators transformation, i.e.

$$\begin{cases} <\hat{\psi}^{\dagger}_{\uparrow}\hat{\psi}^{\dagger}_{\downarrow}> = <\hat{\psi}^{\dagger}_{K\downarrow}(\tau)\hat{\psi}^{\dagger}_{K\downarrow}(\tau)>\ , \\ <\hat{\psi}_{\downarrow}\hat{\psi}_{\uparrow}> = <\hat{\psi}_{K\downarrow}(\tau)\hat{\psi}_{K\downarrow}(\tau)>\ . \end{cases} \tag{2-1-35}$$

Since we are only interested in finding the solution of the pair amplitude, which gives the order parameter in a superconductor, rather than the detailed representation of the quantized wave functions $\hat{\psi}$ and $\hat{\psi}^{\dagger}$ themselves, we introduce the Matsubara function given by

$$\mathcal{G}(\vec{x}\tau,\vec{x}'\tau') = -<T_{\tau}[\hat{\psi}_{K\downarrow}(\vec{x}\tau)\hat{\psi}^{\dagger}_{K\downarrow}(\vec{x}'\tau')]>\ , \tag{2-1-36}$$

together with anomalous Matsubara functions, which are closely related to the pair amplitude, given by

$$\begin{cases} \mathcal{F}(\vec{x}\tau, \vec{x}'\tau') = - < T_\tau[\hat{\psi}_{K\uparrow}(\vec{x}, \tau)\hat{\psi}_{K\downarrow}(\vec{x}', \tau')] > , \\ \mathcal{F}^\dagger(\vec{x}\tau, \vec{x}'\tau') = - < T_\tau[\hat{\psi}^\dagger_{K\downarrow}(\vec{x}, \tau)\hat{\psi}^\dagger_{K\uparrow}(\vec{x}', \tau')] > . \end{cases} \tag{2-1-37}$$

The self-consistent expression of the order parameter is defined as

$$\Delta(\vec{x}) = V\mathcal{F}(\vec{x}\tau^{0^+}, \vec{x}\tau) = V < \hat{\psi}_\downarrow(\vec{x})\hat{\psi}_\uparrow(\vec{x}) > . \tag{2-1-38}$$

The time-order operator T_τ which appears in the definition of those Matsubara functions with respect to the imaginary time τ is defined as

$$\begin{aligned} & < T_\tau(\hat{A}(\tau)\hat{B}(\tau')) > \\ & = \theta(\tau - \tau') < \hat{A}(\tau)\hat{B}(\tau') > \\ & - \theta(\tau' - \tau) < \hat{B}(\tau)\hat{A}(\tau') >, \end{aligned}$$

where $\hat{A}$ and $\hat{B}$ are any fermion operators and $\theta(\tau)$ is the step function

$$\theta(\tau) = \begin{cases} 1, & \tau > 0, \\ 0, & \tau < 0. \end{cases}$$

If we use the Eq. (2-1-34) and take the derivative with respect to imaginary time τ of these defined Matsubara functions, we can have the Gor'kov equations

$$[-\hbar\frac{\partial}{\partial\tau} - \frac{1}{2m}(-i\hbar\nabla + \frac{e}{c}\vec{A}(\vec{x}))^2 + \mu - U(\vec{x})]\mathcal{G}(\vec{x}\tau, \vec{x}'\tau') + \Delta(\vec{x})\mathcal{F}^\dagger(\vec{x}\tau, \vec{x}'\tau') = \hbar\delta(\vec{x} - \vec{x}') , \tag{2-1-39}$$

$$[-\hbar\frac{\partial}{\partial\tau} - \frac{1}{2m}(-i\hbar\nabla + \frac{e}{c}\vec{A}(\vec{x}))^2 + \mu - U(\vec{x})]\mathcal{F}(\vec{x}\tau, \vec{x}'\tau') - \Delta(\vec{x})\mathcal{G}(\vec{x}\tau, \vec{x}'\tau') = 0 , \tag{2-1-40}$$

$$[\hbar\frac{\partial}{\partial\tau} - \frac{1}{2m}(i\hbar\nabla + \frac{e}{c}\vec{A}(\vec{x}))^2 + \mu - U(\vec{x})]\mathcal{F}^\dagger(\vec{x}\tau, \vec{x}'\tau') - \Delta^\dagger(\vec{x})\mathcal{G}(\vec{x}\tau, \vec{x}'\tau') = 0 , \tag{2-1-41}$$

In the case that the Hamiltonian is independent of imaginary time τ, the Matsubara functions only depend on the difference of the imaginary time $(\tau - \tau')$. The Fourier transformation of these functions with respect to τ yields

$$\begin{cases} \mathcal{G}(\vec{x}\tau, \vec{x}'\tau') = (\beta h)^{-1} \sum_n e^{-i\omega_n(\tau-\tau')} \mathcal{G}(\vec{x}, \vec{x}', \omega_n) , \\ \mathcal{F}^\dagger(\vec{x}\tau, \vec{x}'\tau') = (\beta h)^{-1} \sum_n e^{-i\omega_n(\tau-\tau')} \mathcal{F}^\dagger(\vec{x}, \vec{x}', \omega_n) , \end{cases} \tag{2-1-42}$$

where $\omega_n = (2n+1)\pi/\beta h$ and $n = 0, \pm 1, \pm 2, \ldots$ and we have used the periodicity of the Matsubara function[35] given by

$$\mathcal{G}(\tau < 0) = \mp \mathcal{G}(\tau + \beta > 0) ,$$

where the sign on the right side depends on whether the field operators constructing the Matsubara function are fermions or bosons.[35]

The equations of motion for the Fourier components can be written as

$$\begin{cases} [ih\omega_n - \frac{1}{2m}(-ih\nabla + \frac{e}{c}\vec{A}(\vec{x}))^2 + \mu - U(\vec{x})]\mathcal{G}(\vec{x}, \vec{x}'\omega_n) + \Delta(\vec{x})\mathcal{F}^\dagger(\vec{x}, \vec{x}', \omega_n) \\ \qquad = h\delta(\vec{x} - \vec{x}') , \\ [-ih\omega_n - \frac{1}{2m}(ih\nabla + \frac{e}{c}\vec{A}(\vec{x}))^2 + \mu - U(\vec{x})]\mathcal{F}^\dagger(\vec{x}, \vec{x}', \omega_n) - \Delta^\dagger(\vec{x})\mathcal{G}(\vec{x}, \vec{x}', \omega_n) \\ \qquad = 0 . \end{cases} \tag{2-1-43}$$

The self-consistent condition becomes

$$\begin{aligned} \Delta(\vec{x}) &= V < \hat{\psi}_\downarrow(\vec{x})\hat{\psi}_\uparrow(\vec{x}) > \\ &= V\mathcal{F}(\vec{x}\tau^{0^+}, \vec{x}\tau) \\ &= V(\beta h)^{-1} \sum_n e^{-i\omega_n 0^+} \mathcal{F}(\vec{x}, \vec{x}', \omega_n) . \end{aligned} \tag{2-1-44}$$

The Gor'kov equations obtained above provide the basis for the self-consistent calculation of the order parameter. One can assume a simple form of Δ in Eq.(2-1-43) and obtain the solution of $\mathcal{F}$. The Eq.(2-1-44) then gives a new form of Δ,

so that by substituting the obtained Δ into Eq.(2-1-43) again and repeating the same procedure, one can finally obtain the self-consistent solution for Δ.

The theory obtained above covers most of the early theories such as Ginzberg-Landau theory and BCS theory. Gor'kov[36] succeeded in deriving the G-L equation which is valid near the critical temperature T_c from Gor'kov equations. The derivation determines the phenomenological constants appearing in the G-L theory in terms of the microscopic constants and the appropriate range for which the G-L theory is applicable so that the G-L theory has the firm basis of microscopic theory and can be generalized to much more complicated systems such as magnetic superconductors etc.

If one applies the Gor'kov equations to the bulk superconductor which is spatially homogeneous without the external magnetic field , one can rapidly find the results as obtained from the BCS[34] theory. As an example of the application of the Gor'kov equations, we will show the derivation briefly below.

In a spatially homogenous superconductor, one can take

$$\vec{A}(\vec{x}) = 0 \ ,$$
$$U(\vec{x}) = 0 \ ,$$
$$\Delta(\vec{x}) = \Delta \ ,$$

so that the Eq.(2-1-43) is of the form

$$[i\hbar\omega_n - \frac{1}{2m}(-i\hbar\nabla)^2 + \mu]\mathcal{G}(\vec{x},\vec{x}'\omega_n) + \Delta\mathcal{F}^\dagger(\vec{x},\vec{x}',\omega_n) = \hbar\delta(\vec{x}-\vec{x}') \ , \qquad (2\text{-}1\text{-}45)$$

$$[-i\hbar\omega_n - \frac{1}{2m}(i\hbar\nabla)^2 + \mu]\mathcal{F}^\dagger(\vec{x},\vec{x}',\omega_n) - \Delta^\dagger\mathcal{G}(\vec{x},\vec{x}',\omega_n) = 0 \ . \qquad (2\text{-}1\text{-}46)$$

The translational invariance of the above Eqs. implies that

$$\begin{aligned} \mathcal{G}(\vec{x},\vec{x}',\omega_n) &= \mathcal{G}(\vec{x}-\vec{x}',\omega_n) \ , \\ \mathcal{F}^\dagger(\vec{x},\vec{x}',\omega_n) &= \mathcal{F}^\dagger(\vec{x}-\vec{x}',\omega_n) \ , \end{aligned} \qquad (2\text{-}1\text{-}47)$$

and hence we may have the Fourier transformation of the Matsubara functions with respect to spatial coordinates

$$\mathcal{G}(\vec{x}-\vec{x}',\omega_n) = (2\pi)^{-3}\int d^3k\; e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\mathcal{G}(\vec{k},\omega_n)\;, \tag{2-1-48}$$

$$\mathcal{F}^\dagger(\vec{x}-\vec{x}',\omega_n) = (2\pi)^{-3}\int d^3k\; e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\mathcal{F}^\dagger(\vec{k},\omega_n)\;, \tag{2-1-49}$$

Substituting the Eq.(2-1-48) and (2-1-49) into (2-1-45) and (2-1-46), we have the equations

$$[ih\omega_n - \epsilon_{\vec{k}}]\mathcal{G}(\vec{k},\omega_n) + \Delta\mathcal{F}^\dagger(\vec{k},\omega_n) = h\;, \tag{2-1-50}$$

$$[-ih\omega_n - \epsilon_{\vec{k}}]\mathcal{F}^\dagger(\vec{k},\omega_n) - \Delta^*\mathcal{G}(\vec{k},\omega_n) = 0\;. \tag{2-1-51}$$

where the order parameter Δ is of the form

$$\Delta^* = V(\beta h)^{-1}\sum_n e^{-i\omega_n 0^+}\mathcal{F}^\dagger(\vec{x}=0,\omega_n) \tag{2-1-52}$$

The solutions of the Eqs.(2-1-50) and (2-1-51) are easy to obtain

$$\mathcal{G}(\vec{k},\omega_n) = \frac{-h(ih\omega_n + \epsilon_{\vec{k}})}{h^2\omega_n^2 + \epsilon_{\vec{k}}^2 + |\Delta|^2}\;, \tag{2-1-53}$$

$$\mathcal{F}^\dagger(\vec{k},\omega_n) = \frac{h\Delta^*}{h^2\omega_n^2 + \epsilon_{\vec{k}}^2 + |\Delta|^2}\;, \tag{2-1-54}$$

so the self-consistent equation for the order parameter is written as

$$\Delta = V(h\beta)^{-1}\sum_n\int \frac{d^3k}{(2\pi)^3}\frac{h\Delta}{(h\omega_n)^2 + E_{\vec{k}}^2}\;, \tag{2-1-55}$$

with

$$E_{\vec{k}} = (\epsilon_{\vec{k}}^2 + \Delta^2)^{\frac{1}{2}}\;.$$

Eliminating the common factor Δ in (2-1-55) and completing the summation over ω_n, we have

$$1 = V(2\pi)^{-3}\int d^3k\,(2E_{\vec{k}})^{-1}\;\tanh(\beta E_{\vec{k}}/2)\;. \tag{2-1-56}$$

This equation can determine the transition temperature and order parameter $\Delta(T)$. When $T = T_c$, the system goes to normal state so that $\Delta(T_c) = 0$. Eq. (2-1-56) is of the form

$$1 = VN(0) \int_0^{\hbar\omega_D} \frac{d\epsilon}{\epsilon} \tanh(\beta_c \epsilon/2) , \qquad (2\text{-}1\text{-}57)$$

This equation allows us to write down the expression for transition temperature T_c in the bulk superconductor as

$$T_c = \frac{2e^\gamma}{\pi} \exp(-1/N(0)V) \qquad (2\text{-}1\text{-}58)$$

where γ is Euler's constant. This is the same result as that of BCS theory obtained by means of variation method.[34]

Another interesting limiting case is the value of the order parameter at $T = 0$. In this case, Eq.(2-1-56) becomes

$$1 = VN(0) \int_0^{\hbar\omega_D} \frac{d\epsilon}{(\epsilon^2 + \Delta^2)^{\frac{1}{2}}} = VN(0) \ln(\frac{2\omega_D}{\Delta}) , \qquad (2\text{-}1\text{-}59)$$

so that

$$\Delta(0) = 2\omega_D \exp(-1/N(0)V) . \qquad (2\text{-}1\text{-}60)$$

The more general relation between the order parameter and temperature needs numerical calculation and we do not discuss it here.

We have discussed the theoretical description of a superconductor based on the Gor'kov equations which is quite general and includes the G-L theory and BCS theory as the special cases. In addition we can apply this theory to a more complicated case with the external magnetic field and impurity potential or, as we will show, to consider inhomogeneous structures. Nevertheless, the non-linear feature in the Gor'kov equations makes it very difficult to solve the equations directly so that further approximation methods have to be considered corresponding to different cases. We will discuss these methods briefly below.

§2-2 de Gennes-Werthamer Theory

§2-2-1 Linearized Gap Equation

To calculate the transition temperature T_c, the integral solutions for both the functions $\mathcal{G}$ and $\mathcal{F}^\dagger$ from equations (2-1-43) can be worked out as

$$\mathcal{G}(\vec{x},\vec{x}',\omega) = \mathcal{G}^n(\vec{x},\vec{x}',\omega) - \int d^3x_1\, d^3x_2 \mathcal{G}^n(\vec{x},\vec{x}_1,\omega)\Delta(\vec{x}_1)\mathcal{G}^n(\vec{x}_2,\vec{x}_1,-\omega)\Delta^\dagger(\vec{x}_2)\mathcal{G}^n(\vec{x}_2,\vec{x}',\omega)\,, \tag{2-2-1}$$

$$\mathcal{F}^\dagger(\vec{x},\vec{x}',\omega) = \int d^3x_1\, \mathcal{G}^n(\vec{x}_1,\vec{x},-\omega)\Delta^\dagger(\vec{x}_1)\mathcal{G}^n(\vec{x}_1,\vec{x}',\omega) - \int d^3x_1\, d^3x_2 \mathcal{G}^n(\vec{x}_1,\vec{x},-\omega)\Delta^\dagger(\vec{x}_1)\mathcal{G}^n(\vec{x}_1,\vec{x}_2,-\omega)\Delta(\vec{x}_2)\mathcal{F}^\dagger(\vec{x}_2,\vec{x}',\omega)\,, \tag{2-2-2}$$

where we have introduced the Green's function which describes the behavior of a single electron in the normal state, $\mathcal{G}^n(\vec{x},\vec{x}',\omega_n)$, satisfying

$$[ih\omega_n - \frac{1}{2m}(-ih\nabla + \frac{e}{c}\vec{A}(\vec{x}))^2 + \mu - U(\vec{x})]\mathcal{G}^n(\vec{x},\vec{x}',\omega_n) = h\delta(\vec{x}-\vec{x}')\,, \tag{2-2-3}$$

where $\omega = \omega_n = (2n+1)\pi/\beta h$ and $n = 0, \pm1, \pm2, \ldots$ as before. One can notice that, near the critical transition temperature, the order parameter which describes a superconducting phase is supposed to be small so that a linearization approximation for the order parameter can be considered, hence, the self-consistent equation for the order parameter, $\Delta(\vec{x})$, is given by

$$\Delta^\dagger(\vec{x}) = V(\vec{x})(\beta h)^{-1}\sum_\omega \int d^3y\, Q_\omega(\vec{x},\vec{y})\Delta^\dagger(\vec{y})\,, \tag{2-2-4}$$

where the coupling constant has been written as a function of $\vec{x}$ since in the application to the inhomogeneous superconductor, it takes different values for different regimes and the kernel is

$$Q_\omega(\vec{x},\vec{y}) = \overline{\mathcal{G}^n(\vec{y},\vec{x},-\omega)\mathcal{G}^n(\vec{y},\vec{x},\omega)}\,, \tag{2-2-5}$$

where the bar means the average over the randomly distributed impurity configurations and the order parameter has been assumed real without the magnetic field. If one can have the solution for the kernel by any means, one can calculate the transition temperature T_c from the Eq. (2-2-4).

§2-2-2 de Gennes Theory in a Bilayer Structure

Perhaps the simplest geometry one can consider is a bilayer structure with one layer of superconductor labeled by A and another of normal metal labeled by B. Related to the correlation function de Gennes [2] showed that the kernel $Q_\omega(\vec{x}, \vec{y})$ satisfies the diffusion equation. In one dimensional case, it is of the form

$$[2|\omega| - D(x')\frac{d^2}{dx'^2}]Q_\omega(x, x') = 2\pi N(x)\delta(x - x') , \qquad (2\text{-}2\text{-}6)$$

where $D(x)$ is the diffusion constant and $N(x)$ is the density of states near the Fermi surface, both of the two constants take certain values according as x is in the superconductor region or normal metal region. From the Eq. (2-2-6) we have the general solution

$$\begin{cases} Q_\omega(x, x') = \dfrac{N_A\pi}{2|\omega|\xi_A}[e^{-|x-x'|/\xi_A} + \lambda e^{-(x+x')/\xi_A}] & \text{for} \begin{cases} x > 0 \\ x' > 0 \end{cases} , \\ Q_\omega(x, x') = \dfrac{N_B\pi}{2|\omega|\xi_B}\mu e^{-x/\xi_A + x'/\xi_B} & \text{for} \begin{cases} x > 0 \\ x' < 0 \end{cases} , \end{cases} \qquad (2\text{-}2\text{-}7)$$

where $\xi_A = (\frac{D_A}{2|\omega|})^{\frac{1}{2}}$, $\xi_B = (\frac{D_B}{2|\omega|})^{\frac{1}{2}}$ and A and B are the labels defined above denoting the different layers. The coefficients μ and λ should be determined by the proper boundary conditions.

The first boundary condition given by de Gennes[2] is

$$\frac{d\Delta(\vec{x})}{dx} = 0 \qquad \text{at free surface} , \qquad (2\text{-}2\text{-}8)$$

This condition means that there is no electron flow out of the free surface, which yields

$$\frac{\partial}{\partial x} Q_\omega(\vec{x}, \vec{x}') = 0 \qquad \text{at free surface .} \tag{2-2-9}$$

The second condition comes from the Eq. (2-2-6) directly if one completes the integral with respect to x' in the Eq., the boundary condition is

$$\left[D(x') \frac{d}{dx'} Q_\omega(x, x') \right]_{x'=0^-}^{x'=0^+} = 0 \,, \tag{2-2-10}$$

which yields

$$\frac{D}{V} \frac{d\Delta}{dx} \qquad \text{continuous at interface .} \tag{2-2-11}$$

The last boundary condition given by de Gennes[2], having considered the dirty limit $l \ll \xi$ with l being the mean-free path, is that

$$N_B Q_\omega(x, 0^+) = N_A Q_\omega(x, 0^-) \tag{2-2-12}$$

which yields

$$\frac{\Delta}{NV} \qquad \text{continuous at interface .} \tag{2-2-13}$$

From those boundary conditions, it is straightforward to solve the coefficients μ and λ

$$\begin{cases} \mu = \dfrac{2\xi_B N_A}{[\xi_B N_B + \xi_A N_A]} \,, \\ \lambda = \dfrac{N_A \xi_A - N_B \xi_B}{N_A \xi_A + N_B \xi_B} \,. \end{cases} \tag{2-2-14}$$

If we assume that the thickness a, b of the slabs are much smaller than the respective coherence lengths $\xi_N(T), \xi_S(T)$ so that the kernel Q_ω in each region can be treated as constant, we can have the quite simple equations for those order parameters given as

$$\begin{cases} \Delta_n = \displaystyle\sum_\omega V_n T \frac{\pi}{|\omega|} \frac{1}{N_n b + N_s a} (N_n^2 b \Delta_n + N_n N_s a \Delta_s) \,, \\ \Delta_s = \displaystyle\sum_\omega V_s T \frac{\pi}{|\omega|} \frac{1}{N_n b + N_s a} (N_s^2 b \Delta_s + N_n N_s a \Delta_n) \,, \end{cases} \tag{2-2-15}$$

where a and b are the thicknesses of the superconducting and normal layer respectively and $a, b << \xi$. The thickness dependence of the transition temperature T_c may be obtained

$$T_c(a,b) = 1.14\omega_D e^{-1/\rho} , \tag{2-2-16}$$

with

$$\rho = \frac{N_s^2 V_s a + N_n^2 V_n b}{N_n b + N_s a} , \tag{2-2-17}$$

being the "effective NV" in the BCS formula for T_c. When the superlattice is composed of superconducting and pure normal metals($V_n = 0$), we have

$$\frac{T_c(a,b)}{T_{ca}} = (\frac{T_{ca}}{1.14\omega_D})^{b/a} . \tag{2-2-18}$$

The other interesting limiting case is that the thicknesses of the layers are much larger than the coherence lengths. In this case, only the lowest frequency is important since $\xi_\omega = (D/2|\omega|)^{\frac{1}{2}}$ in the kernel $Q_\omega(x,x')$ has the maximum $\xi_{\omega_0} = (D/2\pi T)^{\frac{1}{2}}$. The details of the one-frequency approximation, which is valid for this limit, can be found in de Gennes' work[2] and we only present here the resultant equation determining T_c

$$q\tan(qa) = \eta K \tanh(Kb)$$

where $\eta = D_n N_n / D_s N_s$ and q and K are the wave vectors of the electrons in the superconductor and normal metal respectively.

§2-2-3 Werthamer's Kernel and the Transition Temperature

Instead of relating the kernel in the linearized self-consistent Eq. (2-2-4) to the diffusion equation, Werthamer[4] dealt with the kernel in a more straightforward

way and obtained the explicit expression as follows

$$\begin{cases} \sum_{\omega} Q_{\omega}(\vec{x},\vec{y}) = N(0)[\ln(\frac{1.14\theta_D}{T_c})\delta(\vec{x}-\vec{y}) - X(\vec{x}-\vec{y})] \,, \\ X(\vec{x}-\vec{y}) = (2\pi)^{-3}\int e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\chi(\xi^2 k^2)\, d^3k \,, \\ \chi(z) = \psi(\frac{1}{2}+\frac{1}{2}z) - \psi(\frac{1}{2}) \,, \end{cases} \tag{2-2-19}$$

where $\psi(z)$ is a digamma function and $N(0)$ is the density of states for one spin projection at Fermi-surface. The kernel obtained above is quite similar to that worked out by de Gennes[2] through the one-frequency approximation. Indeed, we will see below that both of the kernels yield exactly the same implicit equations which determine the transition temperature T_c of a bilayer system.

The substitution of equation (2-2-19) into the linearized self-consistent gap equation

$$\Delta(\vec{x}) = V(\vec{x})\int \sum_{\omega} Q_{\omega}(\vec{x},\vec{y})\ \Delta(\vec{y})\, d^3y \,, \tag{2-2-20}$$

casts the integral equation into the differential form

$$\chi[-\xi^2\nabla^2]\Delta(\vec{x}) = \ln(\frac{T_c(\vec{x})}{T_c})\Delta(\vec{x}) \,, \tag{2-2-21}$$

where

$$\xi^2 = \frac{hv_F l}{6\pi k_B T_c} = \frac{hD}{2\pi k_B T_c} \,, \tag{2-2-22}$$

$$[N(0)V(\vec{x})]^{-1} = \ln(\frac{1.14\theta_D}{T_c(\vec{x})}) \,. \tag{2-2-23}$$

The boundary conditions, based on the conservation of the current when the electrons cross the interface, are

$$\begin{cases} \dfrac{d\Delta(x)}{dx} = 0 & \text{at free surfaces} \,, \\ \dfrac{1}{\Delta(x)}\dfrac{d\Delta(x)}{dx} & \text{continuous at interface} \,. \end{cases} \tag{2-2-24}$$

It worth noting that the second boundary condition adopted here without the diffusion coefficient D and the density of states at the Fermi surface N is slightly

different from that of de Gennes. An implicit assumption has been made that the densities of the states at Fermi surface are the same in both of the regions and the diffusion constants only influence the coherence lengths in each region. The detailed discussion on the effects of introducing the diffusion constants and the densities of the states at interfaces can be found in the works by S. Takahashi and M. Tachiki[14].

If one considers the solutions of the order parameter in both regions to be of the following form

$$\begin{aligned} \Delta(x) &= e^{\pm ik_s x}\,, \qquad 0 < x \leq D_s\,, \\ &= e^{\pm k_n x}\,, \qquad -D_n \geq x < 0\,, \end{aligned} \tag{2-2-25}$$

one can easily find the following equations with the boundary conditions

$$\begin{cases} \chi(\xi^2 k_s^2) = \ln(\dfrac{T_{cs}}{T_c})\,, \\ \chi(-\xi^2 k_n^2) = \ln(\dfrac{T_{cn}}{T_c})\,, \\ k_s \tan(k_s a) = k_n \tanh(k_n b)\,. \end{cases} \tag{2-2-26}$$

which are sufficient to determine T_c. As mentioned above, the last equation of the above, if one included diffusion constants and densities of the states at the the interface, would be the same as that obtained in previous subsection by de Gennes theory under the one-frequency approximation. It is in that sense that we consider the Werthamer theory valid for thick films.

In this thick films case, one can consider the limit $\xi \ll k_{s,n}^{-1} \approx d_{s,n}$ so that the function $\chi(z)$ can be replaced by

$$\chi(z) \to \begin{cases} \ln[1 + (\frac{\pi^2}{4} z)], & \text{for } z \geq 0\,, \\ \frac{\pi^2}{4} \ln(1+z), & \text{for } z \leq 0\,. \end{cases} \tag{2-2-27}$$

Choosing the transition temperature $T_{CN} = 0$ for the normal metal, the transition temperature for the bilayer system is obtained

$$\frac{T_c(a,b)}{T_{cs}} \approx \frac{1}{1 + \frac{\pi^2}{4}\frac{b}{a}}\,. \tag{2-2-28}$$

The results obtained from de Gennes-Werthamer theory will be discussed further in next chapter with the comparison to our calculation.

§2-3 Eilenberger's Equation and Its Dirty Limit Version

§2-3-1 Eilenberger's Equation

Since de Gennes-Werthamer theory is based on linearized self-consistent equation (2-2-4), it is valid only when the temperature is very close to the transition temperature T_c. To study the behavior of the inhomogeneous system with the external magnetic field, one has to consider the finite order parameter so that a more general theory, but simpler than solving the Gor'kov equations is needed. Such a theory was established by Eilenberger[15] [16]. We will only outline the results of Eilenberger's theory below. A somewhat more detailed discussion can be found in Appendix A or from Eilenberger's works[15] [16].

Eilenberger introduced the gauge-invariant Green functions

$$\begin{cases} G_\alpha(t,\vec{x},\vec{x}') = -i < T(\hat{\psi}_\alpha(\vec{x},t)\hat{\psi}^\dagger_\alpha(\vec{x}',0)) > e^{-iI(\vec{x},\vec{x}')} , \\ G^\dagger_\alpha(t,\vec{x},\vec{x}') = i < T(\hat{\psi}^\dagger_\alpha(\vec{x},t)\hat{\psi}_\alpha(\vec{x}',0)) > e^{iI(\vec{x},\vec{x}')} , \end{cases} \tag{2-3-1}$$

and the anomalous Green's functions

$$\begin{cases} F_(t,\vec{x},\vec{x}') = -i < T(\hat{\psi}_\downarrow(\vec{x},t)\hat{\psi}_\downarrow(\vec{x}',0)) > e^{iI(\vec{x},\vec{x}')} , \\ F^\dagger(t,\vec{x},\vec{x}') = i < T(\hat{\psi}^\dagger_\downarrow(\vec{x},t)\hat{\psi}^\dagger_\downarrow(\vec{x}',0)) > e^{-iI(\vec{x},\vec{x}')} , \end{cases} \tag{2-3-2}$$

where

$$< \cdots >= T_r(e^{-\beta K_{eff}} \cdots); \qquad \hat{\psi}(\vec{x},t) = e^{itK_{eff}}\hat{\psi}(\vec{x})e^{-itK_{eff}} , \tag{2-3-3}$$

and

$$I(\vec{x},\vec{x}') = \frac{e}{c}\int_{\vec{x}'}^{\vec{x}} \vec{A}(\vec{x})\cdot d\vec{x} . \tag{2-3-4}$$

To coincide the notations used in previous section with those in Eilenberger's works, we have introduced the transformation from Schrödinger field to Heisenberg field (2-3-3) in which the time t is real rather than imaginary as before and chosen $h = 1$. The operator $\hat{K}_{eff}$ is defined by (2-1-30) and we rewrite it below

$$\hat{K}_{eff} = \int d^3x \hat{\psi}^{\dagger}_{\alpha}(\vec{x}) \hat{K}_0(\vec{x}) \hat{\psi}_{\alpha}(\vec{x}) - \Delta(\vec{x}) \hat{\psi}^{\dagger}_{\uparrow}(\vec{x}) \hat{\psi}^{\dagger}_{\downarrow}(\vec{x}) - \Delta^{\dagger}(\vec{x}) \hat{\psi}_{\downarrow}(\vec{x}) \hat{\psi}_{\uparrow}(\vec{x}) \ , \quad (2\text{-}3\text{-}5)$$

with

$$\hat{K}_0 = \frac{1}{2m}(-ih\nabla - \frac{e}{c}\vec{A}(\vec{x}))^2 - \mu + U(\vec{x}) \ . \quad (2\text{-}3\text{-}6)$$

The phase factor I introduced in the definition of the Green's functions above removes the influence of gauge transformation on the Green's functions $G_{\alpha}(t, \vec{x}, \vec{x}')$ and $G^{\dagger}_{\alpha}(t, \vec{x}, \vec{x}')$ but the $F^{\dagger}(t, \vec{x}, \vec{x}')$ and $F(t, \vec{x}, \vec{x}')$ are remained gauge dependent.

The Fourier transformations of the Green functions $G(t, \vec{x}, \vec{x}')$ are defined as

$$\begin{cases} G_{\omega}(\vec{x}, \vec{x}') = \dfrac{1}{2} \displaystyle\int_{i\beta}^{-i\beta} G(t, \vec{x}, \vec{x}') \, e^{-\omega t} \, dt \ , \\ G^{\dagger}_{\omega}(\vec{x}, \vec{x}') = \dfrac{1}{2} \displaystyle\int_{i\beta}^{-i\beta} G^{\dagger}(t, \vec{x}, \vec{x}') \, e^{-\omega t} \, dt \ , \\ F_{\omega}(\vec{x}, \vec{x}') = \dfrac{1}{2} \displaystyle\int_{i\beta}^{-i\beta} F(t, \vec{x}, \vec{x}') \, e^{-\omega t} \, dt \ , \\ F^{\dagger}_{\omega}(\vec{x}, \vec{x}') = \dfrac{1}{2} \displaystyle\int_{i\beta}^{-i\beta} F^{\dagger}(t, \vec{x}, \vec{x}') \, e^{-\omega t} \, dt \ , \end{cases} \quad (2\text{-}3\text{-}7)$$

For the Green functions including the phase factors, the Gor'kov equations

can be easily written as

$$\begin{cases} [i\omega+\mu-\frac{1}{2m}(-i\nabla-\frac{e}{c}\vec{A}(\vec{x})+\vec{K})^2-U(\vec{x})]G_\omega(\vec{x},\vec{x}')+\Delta(\vec{x})F^\dagger_\omega(\vec{x},\vec{x}') \\ \qquad\qquad =\delta(\vec{x}-\vec{x}')\,, \\ [-i\omega+\mu-\frac{1}{2m}(-i\nabla+\frac{e}{c}\vec{A}(\vec{x})-\vec{K})^2-U(\vec{x})]G^\dagger_\omega(\vec{x},\vec{x}')-\Delta^\dagger(\vec{x})F_\omega(\vec{x},\vec{x}') \\ \qquad\qquad =\delta(\vec{x}-\vec{x}')\,, \\ [i\omega+\mu-\frac{1}{2m}(-i\nabla-\frac{e}{c}\vec{A}(\vec{x})-\vec{K})^2-U(\vec{x})]F_\omega(\vec{x},\vec{x}')+\Delta(\vec{x})G^\dagger_\omega(\vec{x},\vec{x}') \\ \qquad\qquad =0\,, \\ [-i\omega+\mu-\frac{1}{2m}(-i\nabla+\frac{e}{c}\vec{A}(\vec{x})+\vec{K})^2-U(\vec{x})]F^\dagger_\omega(\vec{x},\vec{x}')-\Delta^\dagger(\vec{x})G_\omega(\vec{x},\vec{x}') \\ \qquad\qquad =0\,, \end{cases} \tag{2-3-8}$$

where

$$\vec{K}=e^{-il(\vec{r},\vec{r}')}(-i\nabla)e^{il(\vec{r},\vec{r}')}\,, \tag{2-3-9}$$

In order to solve the above Eqs. (2-3-8), Eilenberger considers a more general expression for the Green's function $G_\omega(\vec{x}_0;\vec{x},\vec{x}')$ which satisfies

$$\begin{pmatrix} i\omega+\mu-\frac{1}{2m}(-i\nabla-i\hat{\partial}_{\vec{r}_0})^2-U(\vec{x}) & \Delta(\vec{x}_0) \\ -\Delta^\dagger(\vec{x}_0) & -i\omega+\mu-\frac{1}{2m}(-i\nabla-i\hat{\partial}_{\vec{r}_0})^2-U(\vec{x}) \end{pmatrix}\hat{G}_\omega(\vec{x}_0,\vec{x},\vec{x}') = \hat{I}\delta(\vec{x}-\vec{x}')\,, \tag{2-3-10}$$

where $\hat{\partial}_{\vec{r}_0}$ is the Eilenberger's gauge-invariant differentiation

$$\hat{\partial}_{\vec{r}_0}=\begin{cases} -i\nabla-2\frac{e}{c}\vec{A}(\vec{r}_0) & \text{when working on } \Delta(\vec{x}_0)\,, \\ -i\nabla+2\frac{e}{c}\vec{A}(\vec{r}_0) & \text{when working on } \Delta^\dagger(\vec{x}_0)\,, \\ -i\nabla & \text{when working on some function of } |\Delta(\vec{x}_0)|^2\,. \end{cases} \tag{2-3-11}$$

and $\hat{G}_\omega(\vec{x}_0,\vec{x},\vec{x}')$ is the matrix form of the Green's functions

$$\hat{G}_\omega(\vec{x}_0;\vec{x},\vec{x}')=\begin{pmatrix} G_\omega(\vec{x}_0;\vec{x},\vec{x}') & F_\omega(\vec{x}_0;\vec{x},\vec{x}') \\ F^\dagger_\omega(\vec{x}_0;\vec{x},\vec{x}') & G^\dagger_\omega(\vec{x}_0;\vec{x},\vec{x}') \end{pmatrix}\,, \tag{2-3-12}$$

Obviously we have

$$G_\omega(\vec{x},\vec{x}')=\lim_{\vec{x}_0\to\vec{x}}G_\omega(\vec{x}_0;\vec{x},\vec{x}') \tag{2-3-13}$$

Eilenberger then assumes that the average of the Green's function over the randomly distributed impurities restores the spatial translational invariance so that the averaged Green's function $\overline{\hat{G}_\omega(\vec{r}_0;\vec{r},\vec{r}')}$ can be expressed as

$$\begin{aligned}\overline{\hat{G}_\omega(\vec{r}_0;\vec{r},\vec{r}')} &= \overline{\hat{G}_\omega(\vec{r}_0;\vec{r}-\vec{r}')} \\ &= \int \frac{d^3k}{(2\pi)^3}\overline{\hat{G}_\omega(\vec{r}_0;\vec{k})}e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\end{aligned} \tag{2-3-14}$$

Further, Eilenberger changes the variables from $\vec{k}$ to $(\vec{k}_F^0,\zeta)$ with ζ being the energy variable $\zeta=\zeta(|\vec{k}-\vec{k}_F|)$ so that

$$\overline{\hat{G}_\omega(\vec{r};\vec{k})} = \overline{\hat{G}_\omega(\vec{r};\vec{k}_F,\zeta)} \tag{2-3-15}$$

and defines the simplified Green functions as

$$\begin{cases} g(\omega,\vec{k}_F,\vec{x}) = i\oint\frac{d\zeta}{2\pi}\overline{G}(\omega,\zeta,\vec{k}_F,\vec{x})\,, \\ f(\omega,\vec{k}_F,\vec{x}) = -\oint\frac{d\zeta}{2\pi}\overline{F}(\omega,\zeta,\vec{k}_F,\vec{x})\,, \\ f^\dagger(\omega,\vec{k}_F,\vec{x}) = \oint\frac{d\zeta}{2\pi}\overline{F}^\dagger(\omega,\zeta,\vec{k}_F,\vec{x})\,. \end{cases} \tag{2-3-16}$$

where, on the right side, the argument ζ denotes complex energy variable which should be integrated over along the contour circling the poles of the Green functions, parameter $\vec{x}$ is the replacement for $\vec{x}_0$.

Starting from the Gor'kov equations (2-3-8) and following a tedious derivation, Eilenberger obtained equations for those simplified Green functions. We only present the final results here and leave the detailed discussion in Appendix A.

$$\begin{aligned}&\{2\omega+\vec{v}_F\cdot(\nabla-i\frac{2e}{c}\vec{A}(\vec{x})\}f(\omega,\vec{k}_F,\vec{x}) \\ &= 2\Delta(\vec{x})g(\omega,\vec{k}_F,\vec{x}) + \int_{S_F}\frac{d^2\vec{q}_F^0}{4\pi}W(\vec{k}_F-\vec{q}_F) \\ &\quad\cdot\{g(\omega,\vec{k}_F,\vec{x})f(\omega,\vec{q}_F,\vec{x}) - f(\omega,\vec{k}_F,\vec{x})g(\omega,\vec{q}_F,\vec{x})\}\,,\end{aligned} \tag{2-3-17}$$

$$\{2\omega - \vec{v}_F \cdot (\nabla + i\frac{2e}{c}\vec{A}(\vec{x})\}f^\dagger(\omega, \vec{k}_F, \vec{x})$$
$$= 2\Delta^\dagger(\vec{x})g(\omega, \vec{k}_F, \vec{x}) + \int_{S_F} \frac{d^2\vec{q}_F^0}{4\pi} W(\vec{k}_F - \vec{q}_F) \tag{2-3-18}$$
$$\cdot \{g(\omega, \vec{k}_F, \vec{x})f^\dagger(\omega, \vec{q}_F, \vec{x}) - f^\dagger(\omega, \vec{k}_F, \vec{x})g(\omega, \vec{q}_F, \vec{x})\} ,$$

and

$$g(\omega, \vec{k}_F, \vec{x}) = (1 - f(\omega, \vec{k}_F, \vec{x})f^\dagger(\omega, \vec{k}_F, \vec{x}))^{\frac{1}{2}} . \tag{2-3-19}$$

The system of equations is completed by the self-consistency condition

$$\Delta(\vec{x}) = V \prec \hat{\psi}_\downarrow(\vec{x})\hat{\psi}_\uparrow(\vec{x}) > , \tag{2-3-20}$$

where V is the coupling constant and may be replaced by the standard cutoff procedure

$$\frac{1}{VN(0)} \rightarrow \ln(\frac{T}{T_c}) + \frac{\pi}{\beta}\sum_\omega \frac{1}{|\omega|} . \tag{2-3-21}$$

The explicit expression of $\Delta(\vec{x})$ in terms of the simplified Green's functions is given by

$$\Delta(\vec{x})\ln(\frac{T}{T_c}) + \frac{2\pi}{\beta}\sum_\omega \frac{\Delta(\vec{x})}{|\omega|} - \int_{S_F} \frac{d^2\vec{k}_F^0}{4\pi} f(\omega, \vec{k}_F, \vec{x}) = 0 . \tag{2-3-22}$$

where S_F denotes the Fermi surface. The same procedure gives rise to the equation for the current density $\vec{j}(\vec{x})$

$$\frac{1}{4\pi}\nabla \times (\vec{B}(\vec{x}) - \vec{B}_c(\vec{x})) + \frac{2ie}{c}\frac{2\pi N(0)}{\beta}\sum_\omega \int_{S_F} \frac{d^2\vec{k}_F^0}{4\pi}\vec{v}_F g(\omega, \vec{k}_F, \vec{x}) = 0 . \tag{2-3-23}$$

where B_c is the magnetic field induced by the motion of the charged particles in a superconductor. Compared with the original Gor'kov's equation, the Eilenberger's equations are much easier to handle since the number of the variables is reduced to two from four. In the procedure of obtaining the equations, no assumption of small order parameter has been made so that Eilenberger's equations are suitable for discussing the effect of the magnetic field.

§2-3-2 Eilenberger's Equation in Dirty Limit

Usadel[37] studied Eilenberger's equations for a dirty superconductor in which the mean-free path is very short so that the motion of the electrons is nearly isotropic with respect to Fermi velocity $\vec{v}_F = \vec{k}_F/m$. One thus can expand the simplified Green function $f(\omega, \vec{x}, \vec{k})$ as

$$f(\omega, \vec{x}, \vec{v}) = F(\omega, \vec{x}) + \frac{\vec{v}}{v} \cdot \vec{F}(\omega, \vec{x}) , \tag{2-3-24}$$

and assume

$$|F| >> |\vec{v} \cdot \vec{F}|$$

From the Eq.(2-3-19), one has

$$g(\omega, \vec{x}, \vec{v}) = G(\omega, \vec{x}) + \frac{\vec{v}}{v} \cdot \vec{G}(\omega, \vec{x}) , \tag{2-3-25}$$

where $G(\omega, \vec{x})$ is defined by

$$G(\omega, \vec{x}) = \left[1 - |F(\omega, \vec{x})|^2\right]^{1/2} \tag{2-3-26}$$

and $\vec{G}(\omega, \vec{x})$, by

$$\vec{G}(\omega, \vec{x}) = \frac{1}{2} \frac{F(\omega, \vec{x})\vec{F}^*(\omega, \vec{x}) - F^*(\omega, \vec{x})\vec{F}(\omega, \vec{x})}{G(\omega, \vec{x})} \tag{2-3-27}$$

Substituting Eqs.(2-3-24) and (2-3-25) into the Eilenberger's equations given in last subsection and completing some basic calculation[37], one has Eilenberger's equations in the dirty limit version

$$\begin{cases} 2\omega F(\omega, \vec{x}) - D\hat{\partial}[G(\omega, \vec{x})\hat{\partial}F(\omega, \vec{x}) + \frac{1}{2}\frac{F(\omega, \vec{x})}{G(\omega, \vec{x})}\nabla|F(\omega, \vec{x})|^2] = 2\Delta(\vec{x})G(\omega, \vec{x}) , \\ \Delta(\vec{x}) \ln(\frac{T}{T_c}) + 2\pi T \sum_{\omega>0} [\frac{\Delta(\vec{x})}{\omega} - F(\omega, \vec{x})] = 0 , \\ \vec{j}(\vec{x}) = 2ieN(0)\pi TD \sum_{\omega>0} [F^\dagger(\omega, \vec{x})\hat{\partial}F(\omega, \vec{x}) - F(\omega, \vec{x})(\hat{\partial}F(\omega, \vec{x}))^\dagger] \end{cases} \tag{2-3-28}$$

where $\hat{\partial} = \nabla + 2ie\vec{A}(\vec{x})$ and $D = \frac{1}{3}v_F l$ is the diffusion constant which can be used in studying the effect of the magnetic field in inhomogeneous dirty superconductor.

Biagi[17] and co-workers applied the equation to the SN multilayers case with the boundary conditions

$$\begin{cases} F_S = F_N \ , \\ \dfrac{\partial F_S}{\partial z} = \eta \dfrac{\partial F_N}{\partial z} \end{cases} \qquad \text{at SN interfaces} \ . \tag{2-3-29}$$

where $\eta = \sigma_n/\sigma_s$ with σ standing for conductivity of normal metal. For the perpendicular upper critical field, the result may be summarized[17] to

$$\begin{cases} \ln(t_S) = \hat{\psi}(\dfrac{1}{2}) - \hat{\psi}(\dfrac{1}{2} + \dfrac{y(t_S)}{2}) \ , \\ \ln(t_N) = \hat{\psi}(\dfrac{1}{2}) - \hat{\psi}(\dfrac{1}{2} + \dfrac{y(t_N)}{2}) \ , \\ k_S^2 = 2\pi T y(t_S)/\hbar D_S \ , \\ k_N^2 = -2\pi T y(t_N)/\hbar D_N \ , \\ q_S \tan(q_S d_S/2) = \eta q_N \tanh(q_N d_N/2) \ , \\ q_S^2 = k_S^2 - \dfrac{2\pi H_{c2}}{\Phi_0} \ , \\ q_N^2 = k_N^2 + \dfrac{2\pi H_{c2}}{\Phi_0} \ , \end{cases} \tag{2-3-30}$$

where $t_i = \frac{T}{T_{ci}}, (i = S, N)$ is the reduced temperature and $y(t)$ is the function of the reduced temperature determined by the eigenvalue k_S and k_N. Given the needed values of t_S, t_N, D_S, D_N and η, one obtains the relation between the magnetic field and temperature

$$\begin{aligned} &\left[2\pi T y(t_S)/\hbar D_S\right]^{1/2} \tan\left(\left[2\pi T y(t_S)/\hbar D_S\right]^{1/2} d_S/2\right) \\ &\qquad = \eta\left[2\pi T y(t_N)/\hbar D_N\right]^{1/2} \tanh\left(\left[2\pi T y(t_N)/\hbar D_N\right]^{1/2} d_N/2\right) \end{aligned} \tag{2-3-31}$$

where d_S and d_N are the thicknesses of the superconducting and normal metal layers respectively. At the critical transition temperature of the superlattice, the

upper critical field vanishes, the Eq. (2-3-31) reduces to the result of de Gennes, Werthamer and co-workers for a dirty SN proximity system.

$$k_S \tan(k_S d_S/2) = \eta k_N \tanh(k_N d_N/2)\,. \tag{2-3-32}$$

§2-4 Tunnelling Model in Superconductivity

In this section, we will mainly discuss the tunneling model Hamiltonian and the resultant expression for the transition temperature obtained by McMillan.

Rather than starting from the Gor'kov equation, McMillan studied the proximity effect of a sandwich by means of the tunnelling model Hamiltonian given as

$$H = H_S + H_N + H_T\,, \tag{2-4-1}$$

where H_S and H_N denote the contribution to the Hamiltonian from the electronic states in the superconducting and normal layers respectively and H_T describes the process whereby the electrons tunnel from one layer to the other. In terms of the Nambu doublets,

$$\begin{aligned} H_S = &\sum_{\vec{k}} \epsilon^0_{\vec{k}} \hat{\Psi}^\dagger_{\vec{k}} \hat{\tau}_3 \hat{\Psi}_{\vec{k}} + \sum_{\vec{q}} \omega_{\vec{q}} a^\dagger_{\vec{q}} a_{\vec{q}} + \sum_{\vec{k}_1,\vec{k}_2} g^S_{\vec{k}_1-\vec{k}_2} (a_{\vec{k}_1-\vec{k}_2} + a^\dagger_{\vec{k}_2-\vec{k}_1}) \hat{\Psi}^\dagger_{\vec{k}_1} \hat{\tau}_3 \hat{\Psi}_{\vec{k}_2} \\ &+ \sum_{\vec{k}_1,\vec{k}_2,\vec{q}} V_{\vec{q}} (\hat{\Psi}^\dagger_{\vec{k}_1-\vec{q}} \hat{\tau}_3 \hat{\Psi}_{\vec{k}_1}) (\hat{\Psi}^\dagger_{\vec{k}_2+\vec{q}} \hat{\tau}_3 \hat{\Psi}_{\vec{k}_2})\,, \end{aligned} \tag{2-4-2}$$

with the Nambu doublet being defined as

$$\hat{\Psi}_{\vec{k}} = \begin{pmatrix} c_{\vec{k}\uparrow} \\ c^\dagger_{-\vec{k}\downarrow} \end{pmatrix}\,, \qquad \hat{\Psi}^\dagger_{\vec{k}} = \begin{pmatrix} c^\dagger_{\vec{k}\uparrow} & c_{-\vec{k}\downarrow} \end{pmatrix}\,, \tag{2-4-3}$$

where $c^\dagger_{\vec{k}\alpha}$ and $c_{\vec{k},\alpha}$ are the creation and annihilation operators for electrons in single-particle eigenstate $\Phi_{\vec{k}}$ of the superconductor, the spin and the polarization

indices are suppressed and $a^\dagger(a)$ creates(annihilates) bare phonons with energy ω^0, $V_{\vec{q}}$ is the bare Coulomb interaction, $g^S_{\vec{k}_1-\vec{k}_2}$ is the electron-phonon coupling in superconductor, $\epsilon^0_{\vec{k}} = \frac{k^2}{2m} - \mu$ is the energy spectrum of the free electron in superconductor.

Similarly, H_N is of the form

$$H_N = \sum_{\vec{p}} \epsilon^0_{\vec{p}} \hat{\Psi}^\dagger_{\vec{p}} \hat{\tau}_3 \hat{\Psi}_{\vec{p}} + \sum_{\vec{q}} \omega_{\vec{q}} a^\dagger_{\vec{q}} a_{\vec{q}} + \sum_{\vec{p}_1,\vec{p}_2} g^N_{\vec{p}_1-\vec{p}_2} (a_{\vec{p}_1-\vec{p}_2} + a^\dagger_{\vec{p}_2-\vec{p}_1}) \hat{\Psi}^{'}_{\vec{p}_1} \hat{\tau}_3 \hat{\Psi}_{\vec{p}_2}$$
$$+ \sum_{\vec{p}_1,\vec{p}_2,\vec{q}} V_{\vec{q}} (\hat{\Psi}^\dagger_{\vec{p}_1-\vec{q}} \hat{\tau}_3 \hat{\Psi}_{\vec{p}_1}) (\hat{\Psi}^\dagger_{\vec{p}_2+\vec{q}} \hat{\tau}_3 \hat{\Psi}_{\vec{p}_2}) . \tag{2-4-4}$$

In fact, one can obtain the H_N from H_S by replacing $\vec{k}$ with $\vec{p}$ and the index S with N.

The tunneling Hamiltonian H_T is of the form

$$H_T = \sum_{\vec{p},\vec{k}} T_{\vec{p},\vec{k}} (c^\dagger_{\vec{k},\downarrow} c_{\vec{p},\downarrow} + c^\dagger_{\vec{p}\downarrow} c_{\vec{k},\downarrow}) + h.c. . \tag{2-4-5}$$

where the term $c^\dagger_{\vec{k}\downarrow} c_{\vec{p},\downarrow}$ indicates a process of creating an electron in a state $\Phi_{\vec{k}}$ in superconductor and annihilating one in a state $\Phi_{\vec{p}}$ in normal metal. The probability amplitude for realization of such a process is denoted by $T_{\vec{p},\vec{k}}$. A further simplification will be made by treating the tunneling matrix elements $T_{\vec{p},\vec{k}}$ of equal magnitude T between every state $\Phi_{\vec{k}}$ and every state $\Phi_{\vec{p}}$.

Using the Hamiltonian H_S and H_N and completing the transformation (2-1-32), one can define the Nambu Green's function in superconductivity as

$$\hat{\mathcal{G}}_S(\vec{k},\tau) = - < T_\tau (\hat{\Psi}_{\vec{k}}(\tau) \hat{\Psi}^\dagger_{\vec{k}}(0) >$$
$$= - \begin{pmatrix} < T_\tau (c_{\vec{k},\downarrow}(\tau) c^\dagger_{\vec{k},\downarrow}(0)) > & < T_\tau (c_{\vec{k},\downarrow}(\tau) c_{-\vec{k},\downarrow}(0)) > \\ < T_\tau (c^\dagger_{-\vec{k},\downarrow}(\tau) c^\dagger_{\vec{k},\downarrow}(0)) > & < T_\tau (c^\dagger_{-\vec{k},\downarrow}(\tau) c_{-\vec{k},\downarrow}(0)) > \end{pmatrix} \tag{2-4-6}$$

where $< \cdots >$ denotes the average over the *Grand Canonical Ensemble*. In a similar way, one can define the Nambu Green's function in the normal metal as

$$\hat{\mathcal{G}}_N(\vec{p},\tau) = - < T_\tau(\hat{\Psi}_{\vec{p}}(\tau)\hat{\Psi}^\dagger_{\vec{p}}(0) > $$
$$= - \begin{pmatrix} < T_\tau(c_{\vec{p},\uparrow}(\tau)c^\dagger_{\vec{p},\uparrow}(0)) > & < T_\tau(c_{\vec{p},\uparrow}(\tau)c_{-\vec{p},\downarrow}(0)) > \\ < T_\tau(c^\dagger_{-\vec{p},\downarrow}(\tau)c^\dagger_{\vec{p},\uparrow}(0)) > & < T_\tau(c^\dagger_{-\vec{p},\downarrow}(\tau)c_{-\vec{p},\downarrow}(0)) > \end{pmatrix} \qquad (2\text{-}4\text{-}7)$$

The Green's functions $\hat{\mathcal{G}}_{S,N}$ defined here are, in fact, the spatial Fourier components of the Green's functions appearing in (2-1-36) and (2-1-37).

If one only considers the electron-phonon interaction in $H_{S,N}$ and lets H_T go to zero, the Fourier components of the Green's function $\hat{\mathcal{G}}_{S,N}$ with respect to imaginary time τ may be expressed, in terms of Dyson equation, as

$$\begin{cases} \hat{\mathcal{G}}_S^{-1}(\vec{k},\epsilon) = G_0^{-1}(\vec{k},\epsilon) - \hat{\Sigma}_S^{ph}(\vec{k},\epsilon) \\ \hat{\mathcal{G}}_N^{-1}(\vec{p},\epsilon) = G_0^{-1}(\vec{p},\epsilon) - \hat{\Sigma}_N^{ph}(\vec{p},\epsilon) \end{cases} \qquad (2\text{-}4\text{-}8)$$

where $\hat{\mathcal{G}}_0^{-1}(\vec{p},\epsilon) = (\epsilon - \epsilon_{\vec{p}}\hat{\tau}_3)^{-1}$ denotes the Green's function for a free electron in normal metal and $\hat{\Sigma}_N^{ph}(\vec{p},\epsilon)$ denotes the matrix self-energy coming from the electron-phonon interaction shown in Fig.1 $D_{S,N}$ denote the Green's functions of the phonon in the respective regions.

If one treats the tunneling Hamiltonian H_T to second-order, in self-consistent perturbation theory, the matrix self-energy is modified diagrammatically as shown in Fig.2 which gives rise to the matrix self-energy of the form

$$\begin{cases} \hat{\Sigma}_N(E) = \hat{\Sigma}_N^{ph} + \bar{T}^2 \sum_{\vec{k}'} G_{S,\vec{k}'}(E)\,, \\ \hat{\Sigma}_S(E) = \hat{\Sigma}_S^{ph} + \bar{T}^2 \sum_{\vec{k}'} G_{N,\vec{k}'}(E)\,, \end{cases} \qquad (2\text{-}4\text{-}9)$$

where $\bar{T}$ denotes the transition matrix element and the matrix self-energy $\hat{\Sigma}_{S,N}^{ph}$ is assumed to be E independent. In general, the 2×2 matrix self-energy $\hat{\Sigma}_{S,N}$ may be expanded as the linear combination of the matrix set $\{\hat{I},\vec{\tau}\}$ with $\vec{\tau}$ being the Pauli matrices and $\hat{I}$ being the unit matrix,

$$\hat{\Sigma}_N(E) = (1 - Z_N(E))E\hat{I} + \chi(E)\hat{\tau}_3 + \Phi_N(E)\hat{\tau}_1 + \zeta(E)\hat{\tau}_2\,. \qquad (2\text{-}4\text{-}10)$$

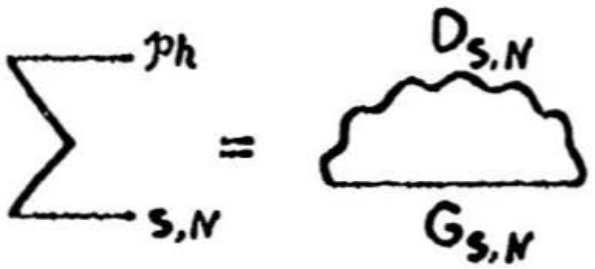

Fig. 1. Self-Energy

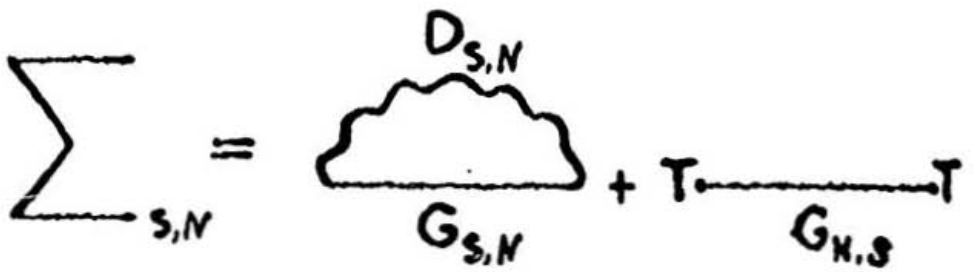

Fig. 2.[18] Self-Energy of the superconductor is the sum of the second-order phonon and tunnelling contributions. The Green's functions are determined self-consistently from the self-energies.

One can simplify the expression by properly choosing the gauge and renormalizing the energy spectrum[38] so that the components of $\hat{\tau}_2$ and $\hat{\tau}_3$ can be eliminated. The self-energy may be expressed as

$$\hat{\Sigma}_N(E) = (1 - Z_N(E))E\hat{I} + \Phi_N(E)\hat{\tau}_1 \,. \tag{2-4-11}$$

The function $Z_N(E)$ and $\Phi(E)$ should be determined self-consistently. The Green's function for a free electron is given by

$$G^{-1}_{0,\epsilon_{\vec{k}'}}(E) = E\hat{I} - \epsilon_{\vec{k}'}\hat{\tau}_3 \,. \tag{2-4-12}$$

Substituting (2-4-11) into the Dyson's equation gives rise to

$$G^{-1}_{N,\epsilon_{\vec{k}'}}(E) = Z_N(E)E\hat{I} - \epsilon_{\vec{k}'}\hat{\tau}_3 - \Phi_N(E)\hat{\tau}_1 \,, \tag{2-4-13}$$

or

$$G_{N,\epsilon_{\vec{k}'}} = \frac{Z_N(E)E\hat{I} + \epsilon_{\vec{k}'}\hat{\tau}_3 + \Phi_N(E)\hat{\tau}_1}{Z^2_N(E)E^2 - \epsilon^2_{\vec{k}'} + \Phi^2_N(E)} \,. \tag{2-4-14}$$

Performing the sum over states, we have

$$\begin{aligned} \sum_{\epsilon_{\vec{k}'}} G_{N,\epsilon_{\vec{k}'}} &= \frac{\Omega_N}{(2\pi)^3}\int d^3k' \frac{Z_N(E)E\hat{I} + \epsilon_{\vec{k}'}\hat{\tau}_3 + \Phi_N(E)\hat{\tau}_1}{Z^2_N(E)E^2 - \epsilon^2_{\epsilon_{\vec{k}'}} + \Phi^2_N(E)} \\ &= -i\pi\Omega_N N(0)\frac{E\hat{I} + \Delta_N(E)\hat{\tau}_1}{[E^2 - \Delta^2_N(E)]^{\frac{1}{2}}} \,, \end{aligned} \tag{2-4-15}$$

where $\Omega_N = Ad_N$ is the volu · for N slab, A and d_N denote the area and the thickness of the normal metal, $N(0)$ is the density of the states at Fermi surface and Δ_N is the renormalized order parameter defined by

$$\Delta_N(E) = \frac{\Phi_N(E)}{Z_N(E)} \,. \tag{2-4-16}$$

Substituting (2-4-15) into (2-4-9) and taking $\Sigma^{ph}_N = \Delta^{ph}_N\hat{\tau}_1$,[38] we obtain the self-energy equations;

$$\begin{cases} Z_N(E) = 1 + i\pi\Gamma_S\dfrac{E}{[E^2 - \Delta^2_S]^{\frac{1}{2}}} \,, \\ \Delta_N(E) = Z^{-1}_N(E)\left[\Delta^{ph}_N - \dfrac{i\pi\Gamma_N\Delta_S(E)}{[E^2 - \Delta^2_S]^{\frac{1}{2}}}\right] \,, \end{cases} \tag{2-4-17}$$

where $\Gamma_N = \bar{T}^2 A d_S N_S(0)$ and $\Gamma_S = \bar{T}^2 A d_N N_N(0)$.

The self-consistency gap equation for the BCS potential[38] is

$$\begin{aligned}\Delta_N^{ph} &= V_N \int_0^{\omega_c} \frac{dE}{\pi} \hat{\mathcal{G}}_N(E)_{12} \\ &= V_N N_N(0) \int_0^{\omega_c} dE\, Re\{\frac{\Delta_N(E)}{[E^2 - \Delta_N^2]^{\frac{1}{2}}}\} ,\end{aligned} \tag{2-4-18}$$

where the V_N is the BCS coupling constant. An factor $\tanh(\frac{1}{2}\beta E)$ can be inserted into the integral of (2-4-18) at finite temperature. For the S slab , there are a set of equations identical to those given above with the subscripts N and S interchanged expressed as

$$\begin{cases} Z_S(E) = 1 + i\pi\Gamma_N \dfrac{E}{[E^2 - \Delta_N^2]^{\frac{1}{2}}} , \\ \Delta_S(E) = Z_S^{-1}(E)[\Delta_S^{ph} - \dfrac{i\pi\Gamma_S \Delta_N(E)}{[E^2 - \Delta_N^2]^{\frac{1}{2}}}] , \end{cases} \tag{2-4-19}$$

$$\begin{aligned}\Delta_S^{ph} &= V_S \int_0^{\omega_c} \frac{dE}{\pi} \hat{\mathcal{G}}_S(E)_{12} \\ &= V_S N_S(0) \int_0^{\omega_c} dE\, Re\{\frac{\Delta_S(E)}{[E^2 - \Delta_S^2]^{\frac{1}{2}}}\} .\end{aligned} \tag{2-4-20}$$

The transition temperature may be obtained by evaluating the gap equation at the limit $\Delta_{S,N} \to 0$. For the SN sandwich, we take $V_N = 0$. The self-consistency equation for Δ_S^{ph} at T_c is

$$\Delta_S^{ph} = V_S N_S(0) Re \int_0^{\omega_c} \frac{\Delta_S(E)}{E} \tanh(\frac{E}{2T_c})\, dE . \tag{2-4-21}$$

The relation between Δ_S^{ph} and Δ_S is obtained from (2-4-19) as

$$\Delta_S(E) = \Delta_S^{ph} \frac{1 - i\pi\Gamma_N/E}{1 - i\Gamma/E} , \tag{2-4-22}$$

where $\Gamma = \Gamma_S + \Gamma_N$. By substituting (2-4-22) into (2-4-21) and completing the contour integration, we have

$$\ln(\frac{T_{Sc}}{T_c}) = (\frac{\Gamma_N}{\Gamma})[\psi(-\frac{1}{2} + \frac{\Gamma}{2\pi T_c}) - \psi(-\frac{1}{2})] , \tag{2-4-23}$$

where ψ is the digamma function and T_{Sc} is the transition temperature of the bulk superconductor which satisfies

$$\begin{aligned}\frac{1}{V_S N_S(0)} &= \int_0^{\omega_c} \tanh(\frac{E}{2T_{Sc}})\,\frac{dE}{E} \\ &= \ln(\frac{\omega_c}{2T_{Sc}}) + \ln(\frac{4e^\gamma}{\pi})\end{aligned} \tag{2-4-24}$$

In the limit of large Γ one can make use of the asymptotic form $\psi(z) \approx \ln(z)$ for large z and find[18]

$$\frac{T_c}{T_{Sc}} \approx (\frac{\pi T_{Sc}}{2\gamma\Gamma})^{\Gamma_S/\Gamma_N} \quad ,\Gamma/T_{Sc} >> 1 \tag{2-4-25}$$

where

$$\begin{aligned}\Gamma &= \Gamma_N(1+\frac{\Gamma_S}{\Gamma_N}) \\ &= d_S\bar{T}^2 A N_S(0)(1+\frac{\Gamma_S}{\Gamma_N})\end{aligned} \tag{2-4-26}$$

One can notice from (2-4-25) that the transition temperature T_c vanishes with the thickness d_S going to zero. This is a different result from that found in de Gennes-Werthamer theory and we will compare this it with the result from our calculation in Chapter 3.

Chapter 3

A Calculation of the Critical Temperature in Metallic Superlattices

In this chapter of the thesis, we[39] apply the Bogoliubov equations to the metallic superlattice with a step-function approximation for the pair amplitude, the height of which is then determined by minimization of the free energy of the system. We shall find, using this method for a calculation of the critical temperature of a superlattice, that our results agree most closely with the bilayer approach of the McMillan model, the thin film limit providing the clearest separation of the various approaches. Further calculation with finite pair amplitude will involve the complication of the periodic structure and the subsequent implications of Bloch's theorem[40][41].

§3-1 Bogoliubov Equation

To apply the effective BCS Hamiltonian to inhomogeneous geometries, we treat it in the mean field approximation as[39]

$$\begin{aligned}\hat{\Psi}_\uparrow^\dagger(\vec{x})\hat{\Psi}_\downarrow^\dagger(\vec{x})\hat{\Psi}_\downarrow(\vec{x})\hat{\Psi}_\uparrow(\vec{x}) \approx & < \hat{\Psi}_\uparrow^\dagger(\vec{x})\hat{\Psi}_\downarrow^\dagger(\vec{x}) > \hat{\Psi}_\downarrow(\vec{x})\hat{\Psi}_\uparrow(\vec{x}) \\ & + \hat{\Psi}_\uparrow^\dagger(\vec{x})\hat{\Psi}_\downarrow^\dagger(\vec{x}) < \hat{\Psi}_\downarrow(\vec{x})\hat{\Psi}_\uparrow(\vec{x}) > \\ & - < \hat{\Psi}_\uparrow^\dagger(\vec{x})\hat{\Psi}_\downarrow^\dagger(\vec{x}) >< \hat{\Psi}_\downarrow(\vec{x})\hat{\Psi}_\uparrow(\vec{x}) > \ .\end{aligned} \tag{3-1-1}$$

In terms of Nambu doublet

$$\hat{\Psi}(\vec{x}) = \begin{pmatrix} \hat{\psi}_\uparrow(\vec{x}) \\ \hat{\psi}_\downarrow^\dagger(\vec{x}) \end{pmatrix} \ . \tag{3-1-2}$$

The Hamiltonian in a superlattice may be written as

$$\hat{H} = \int d^3x\, \hat{\Psi}^\dagger(\vec{x})\epsilon(-i\nabla)\hat{\tau}_3\hat{\Psi}(\vec{x}) - V(\vec{x})\int d^3x\, F(\vec{x})\hat{\Psi}^\dagger(\vec{x})\hat{\tau}_1\hat{\Psi}(\vec{x}) \ , \tag{3-1-3}$$

where the parabolic band model has been assumed so that

$$\epsilon(-i\nabla) = \frac{-\nabla^2}{2m} - \mu \,, \tag{3-1-4}$$

and the pair amplitude $F(\vec{x})$ is defined as

$$F(\vec{x}) = F^{\dagger}(\vec{x}) = - < \hat{\psi}_{\uparrow}(\vec{x})\hat{\psi}_{\downarrow}(\vec{x}) > = - < \hat{\Psi}_{\uparrow}(\vec{x})\hat{\Psi}^{\dagger}_{\downarrow}(\vec{x}) >_{12} \,. \tag{3-1-5}$$

For further simplification, we assume that the chemical potential and the effective mass of the quasi-particle in each region are the same, i.e.

$$\mu = \mu_1 = \mu_2 \qquad m = m_1 = m_2 \,. \tag{3-1-6}$$

Completing the transformation from Schrödinger(S) field to Heisenberg(H) field defined as

$$O_H(\vec{x},t) = e^{it\hat{H}} O_S(\vec{x}) e^{-it\hat{H}}$$

with $O_S(O_H)$ being any field operator, $\hat{\psi}$ or $\hat{\psi}^{\dagger}$, in S(H) picture and using the Heisenberg equation

$$i\frac{\partial}{\partial t} O_H(\vec{x},t) = [O_H(\vec{x},t)\,, \hat{H}],$$

the equation of motion for the field operators can be written as

$$\begin{cases} [i\dfrac{\partial}{\partial t} - \epsilon(-i\nabla)\hat{\tau}_3 + \Delta_1\hat{\tau}_1]\hat{\Psi}(\vec{x}t) = 0 & \text{for } -a \le x \le 0 \,, \\ [i\dfrac{\partial}{\partial t} - \epsilon(-i\nabla)\hat{\tau}_3 + \Delta_2\hat{\tau}_1]\hat{\Psi}(\vec{x}t) = 0 & \text{for } 0 \le x \le b \,, \end{cases} \tag{3-1-7}$$

where Δ_i is the order parameter in the region

$$\Delta_i(\vec{x}) = V_i(\vec{x})F(\vec{x}) \qquad (i = 1,2) \,. \tag{3-1-8}$$

Expanding the field operator $\hat{\Psi}(\vec{x})$ as the sum of the positive and negative energy component and completing the Fourier transformation with respect to the coordinates y and z yields

$$\begin{aligned} \hat{\Psi}(\vec{x},t) = \int_0^{\omega_D} \frac{dE}{2\pi} \int \frac{d^2l}{(2\pi)^2} \{ & \Phi(E,l,x) e^{-i(Et - \vec{l}\cdot\vec{\rho})} a(E,l) \\ & + \Phi(-E,l,x) e^{i(Et - \vec{l}\cdot\vec{\rho})} b^{\dagger}(E,l) \} \,, \end{aligned} \tag{3-1-9}$$

where $\vec{l} = (k_y, k_z), \vec{\rho} = (y, z)$ and $\omega_D (\hbar = 1)$ is the Debye frequency .

The equation of motion (3-1-7) becomes

$$\begin{cases} \{E + \frac{1}{2m}\frac{d^2}{dx^2} + (\mu - \frac{1}{2m}l^2)\hat{\tau}_3 + \Delta_i\hat{\tau}_1\}\Phi(E, l, x) = 0 , \\ \{-E + \frac{1}{2m}\frac{d^2}{dx^2} + (\mu - \frac{1}{2m}l^2)\hat{\tau}_3 + \Delta_i\hat{\tau}_1\}\Phi(-E, l, x) = 0 . \end{cases} \tag{3-1-10}$$

Note that once we obtain the solution $\Phi(E, l, x)$ with positive energy, the negative energy solution $\Phi(-E, l, x)$ may be written down immediately by the replacement $E \rightarrow -E$ in the solution $\Phi(E, l, x)$, so we only concentrate on obtaining the solution $\Phi(E, l, x)$.

The further approximation is made by taking

$$\Phi(E, l, x) = u(E, x)e^{iQx} , \tag{3-1-11}$$

where $u(E, l, x)$ is assumed a smoothly varying function so that the second derivative term of $u(E, x)$ with respect to x can be dropped off and Q is assumed to be

$$2mQ^2 = 2m\mu - l^2 = k_F^2 - l^2 , \tag{3-1-12}$$

where k_F is the Fermi wave vector.

The equations of motion (3-1-10) now become

$$\begin{cases} \{E + \frac{i}{m}Q\hat{\tau}_3\frac{d}{dx} + \Delta_1(\vec{x})\hat{\tau}_1\}u_S(E, x) = 0 \qquad S , \\ \{E + \frac{i}{m}Q\hat{\tau}_3\frac{d}{dx} + \Delta_2(\vec{x})\hat{\tau}_1\}u_N(E, x) = 0 \qquad N . \end{cases} \tag{3-1-13}$$

We now assume that the localized order parameter $\Delta_i(\vec{x})$ is simply the bulk value Δ_1 and Δ_2 in each region, and also introduce the step-function approximation for the pair potential

$$\Delta_i(x) = \begin{cases} \Delta_1 & \text{for } nd - a < x < nd , \\ \Delta_2 & \text{for } nd < x < nd + b , \end{cases} \tag{3-1-14}$$

where n is any integer and $d = a + b$ is the lattice period. The parameters Δ_1 and Δ_2 appearing in (3-1-14) are next determined by the minimization of the free energy of the system, which can be expressed as

$$< \frac{\partial H}{\partial \Delta_1^2} >= 0 =< \frac{\partial H}{\partial \Delta_2^2} > , \tag{3-1-15}$$

and results in the set of coupled equations

$$\begin{cases} \Delta_1 = \dfrac{1}{a} \displaystyle\int_{-a}^{0} dx\, V_1(x) F(x) , \\ \Delta_2 = \dfrac{1}{b} \displaystyle\int_{0}^{b} dx\, V_2(x) F(x) . \end{cases} \tag{3-1-16}$$

To derive the explicit form of these equations,we require the solutions of (3-1-13). These are readily found to be

$$\begin{cases} u_S(E,x) &= A_1 \begin{pmatrix} 1+\gamma_1 \\ \delta_1 \end{pmatrix} e^{ip_1 x} + B_1 \begin{pmatrix} \delta_1 \\ 1+\gamma_1 \end{pmatrix} e^{-ip_1 x} \qquad nd - a < x < nd , \\ u_N(E,x) &= A_2 \begin{pmatrix} 1+\gamma_2 \\ \delta_2 \end{pmatrix} e^{ip_2 x} + B_2 \begin{pmatrix} \delta_2 \\ 1+\gamma_2 \end{pmatrix} e^{-ip_2 x} \qquad nd < x < nd + b , \end{cases} \tag{3-1-17}$$

where, for $i = 1(S), 2(N)$

$$\delta_i = -\frac{\Delta_i}{E} , \qquad \gamma_i = \sqrt{1 - \delta_i^2} , \qquad p_i = \gamma_i m E / Q .$$

§3-2 The Periodic Solution and the Energy-Spectrum

Since we are considering the superlattice structure, the Bloch theorem requires that

$$\begin{cases} u(E,x) = \tilde{u}(E,x) e^{iqx} , \\ \tilde{u}(E,x) = \tilde{u}(E,x+d) , \end{cases} \tag{3-2-1}$$

which yields

$$u(E,x+d) = u(E,x) e^{iqd} . \tag{3-2-2}$$

Then the boundary conditions may be written as

$$\begin{cases} u_S(E, x = 0^-) = u_N(E, x = 0^+), \\ u_S(E, -a) = u_N(E, b)e^{-iqd}. \end{cases} \tag{3-2-3}$$

For the nontrivial solutions the following condition is then necessary:

$$\cos(dq) = \cos(ap_1)\cos(bp_2) - [\frac{(1-\delta_1\delta_2)}{\gamma_1\gamma_2}]\sin(ap_1)\sin(bp_2), \tag{3-2-4}$$

which generally gives rise to gaps in the energy momentum spectrum [40] [41]. The final condition on the normalization of the solutions

$$\int_{-\infty}^{\infty} u^{\dagger}(E', x)u(E, x)\,dx = 2\pi\delta(E - E'), \tag{3-2-5}$$

allows one then to completely determine the coefficients, which is shown in Appendix B.

With the solution at hand, the pair amplitude $F(x)$ may be written down

$$\begin{aligned} F(x) &= - < \hat{\Psi}(\vec{x})\hat{\Psi}^{\dagger}(\vec{x}) >_{12} \\ &= \int_0^{\omega_D} \frac{dE}{2\pi}\frac{dl^2}{(2\pi)^2}[u(E, x)u^{\dagger}(E, x)]_{12}\tanh(\frac{\beta E}{2}). \end{aligned} \tag{3-2-6}$$

The gap equations

$$\begin{aligned} \frac{\Delta_1}{V_1} &= \frac{1}{a}\int_{-a}^{0} dx\, F(x) \\ &= \int_0^{\omega_D} \frac{dE}{2\pi}\frac{dl^2}{(2\pi)^2}\bar{F}(\Delta_1)\tanh(\frac{\beta E}{2}), \end{aligned} \tag{3-2-7}$$

with

$$\begin{aligned} \bar{F}(\Delta_1) &= \frac{1}{a}\int_{-a}^{0} dx\,[u_S(E, x)u_S(E, x)]_{12} \\ &= a\delta_1(1+\gamma_1)(|A|^2 + |B|^2) + \frac{1}{p_1}\sin(ap_1)AB^* e^{-ip_1 a}2(1+\gamma_1). \end{aligned} \tag{3-2-8}$$

The corresponding expressions for region N are

$$\begin{aligned} \frac{\Delta_2}{V_2} &= \frac{1}{b}\int_0^{b} dx\, F(x) \\ &= \int_0^{\omega_D} \frac{dE}{2\pi}\frac{dl^2}{(2\pi)^2}F(\Delta_2)\tanh(\frac{\beta E}{2}), \end{aligned} \tag{3-2-9}$$

with

$$F(\Delta_2) = \frac{1}{b}\int_0^b dx\,[u_N(E,x)u_N(E,x)]_{12}$$
$$= b\delta_2(1+\gamma_2)(|C|^2+|D|^2) + \frac{1}{p_2}\sin(bp_2)CD^* e^{ip_2 b}2(1+\gamma_2)\,. \tag{3-2-10}$$

The full expression of the coefficients is quite complicated. However, to find the critical temperature we let the gap parameters Δ_1 and Δ_2 approach zero, in which case (3-2-4) gives $\cos(dq) = \cos(\frac{dmE}{Q})$.In this limit the gap equations also simplify to

$$\begin{cases} \dfrac{1}{V_2 N(0)} = \displaystyle\int_0^{\omega_D}\frac{dE}{E}\int_0^{k_F}\frac{dQ}{k_F}[1+(\frac{\Delta_1}{\Delta_2}-1)\frac{\sin(ak)\sin(bk)}{bk\sin(dk)}]\tanh(\frac{\beta E}{2})\,, \\ \dfrac{1}{V_1 N(0)} = \displaystyle\int_0^{\omega_D}\frac{dE}{E}\int_0^{k_F}\frac{dQ}{k_F}[1+(\frac{\Delta_2}{\Delta_1}-1)\frac{\sin(ak)\sin(bk)}{ak\sin(dk)}]\tanh(\frac{\beta E}{2})\,, \end{cases} \tag{3-2-11}$$

where $N(0) = \frac{mk_F}{2\pi^2}$ is the density of states at Fermi surface for one spin projection and $k = \frac{mE}{Q}$.

§3-3 Transition Temperature

Completing the calculation (Appendix C) of (3-2-11), we obtain the equation for determining the transition temperature $T_c(a,b)$:

$$\ln(\frac{a}{a_c})\ln(\frac{b}{b_c}) = \ln(\frac{a}{a_c})[I(r\tau)-J(r\tau,\frac{1}{r})] + \ln(\frac{b}{b_c})[I(\tau)-J(\tau,r)]$$
$$-[I(\tau)-J(\tau,r)][I(r\tau)-J(r\tau,\frac{1}{r})] \tag{3-3-1}$$
$$+[J(\tau,r)-K(\tau)][J(r\tau,\frac{1}{r})-K(r\tau)]\,,$$

where

$$\tau = \frac{2}{\pi}\frac{a}{a_c}\frac{T_c(a,b)}{T_{Sc}}\,, \qquad r = \frac{b}{a}\,,$$
$$a_c = \frac{1}{\pi k_F}\frac{T_F}{T_{Sc}}\,, \qquad b_c = \frac{1}{\pi k_F}\frac{T_F}{T_{Nc}}\,. \tag{3-3-2}$$

Here T_{Sc} and T_{Nc} are the respective bulk critical temperatures, and

$$\begin{cases} I(x) = \frac{1}{2}\pi x \ln 2 - x\int_0^\infty \frac{dz}{z^2}[\frac{\sin(2z)}{2z}]\ln(1+e^{-\frac{z}{x}}) , \\ J(x,y) = x\int_0^\infty \frac{dz}{z^3}\sin^2 z\cot[z(1+y)]\ln\cosh(\frac{z}{x}) , \\ K(x) = \frac{1}{2}x\int_0^\infty \frac{dz}{z^3}\sin(2z)\ln\cosh(\frac{z}{x}) ; \end{cases} \tag{3-3-3}$$

a_c and b_c are signatures of the Debye cutoff ω_D. These length scales also arise in the case of a single superconducting film embedded in a normal metal [12], where a_c is the critical thickness at which the transition temperature vanishes. This can be seen by considering the case $V_2 = 0$, whereby (3-3-1) becomes

$$\ln(a/a_c) = I(\tau) - J(\tau, r) . \tag{3-3-4}$$

For infinite b, the function $J(\tau, r)$ vanishes, and an analysis of the other terms yields the solution $T_c(a_c) = 0$. We also see from this that $J(\tau, r)$ then represents coherence effects the other layers introduce to the single film geometry.

In Fig.3 we show some results for the transition temperature determined by (3-3-4) in this simpler case of $V_2 = 0$. Here we plot the reduced critical temperature $T_c(a, b)/T_{c1}$ as a function of a/a_c for various values of b/a_c. We see that for about $b > a_c$, the critical temperature corresponds closely to that of a single film embedded in an infinite normal metal, significant deviation occurring only for around $a < a_c$. Also, we find for finite b that $T_c(a, b) = 0$ only for $a = 0$, indicating an absence of a critical thickness as happens for infinite b.

Fig.4 and Fig.5 show results in the more general case of $V_2 \neq 0$. In Fig.4 the ratio T_{c2}/T_{c1} is chosen as 0.1, and we plot the reduced critical temperature as a function of a/d for various values of d/a_c, where d is the modulation length $a + b$. In Fig.5 we show the dependence of the reduced transition temperature on the modulation length d/a_c for various values of T_{c2}/T_{c1} in the particular case $a = b$.

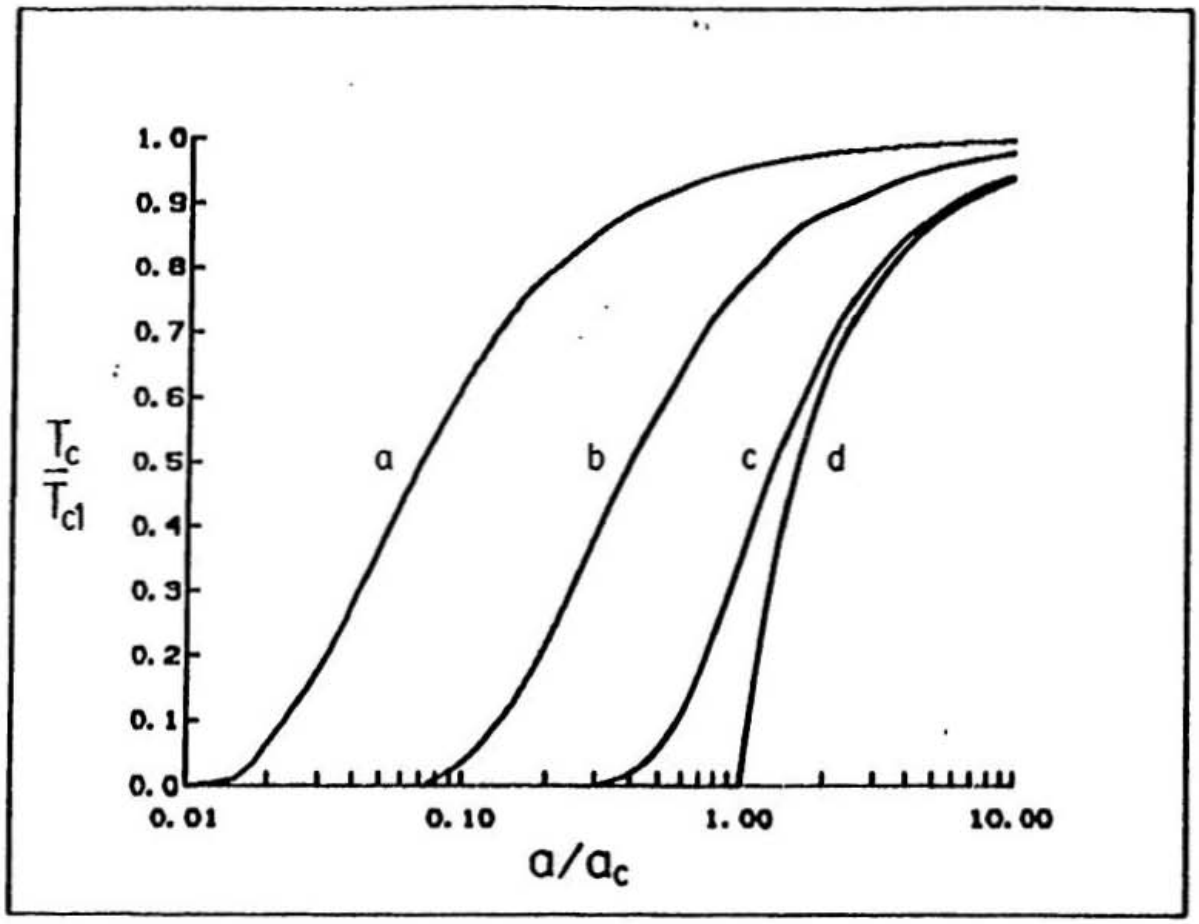

Fig. 3.[26] The dependence of the reduced transition temperature $T_c(a,b)/T_{c1}$ on the thickness a/a_c. (a) $b/a_c = 0.01$, (b) $b/a_c = 0.1$, (c) $b/a_c = 1.0$ (d) infinite b.

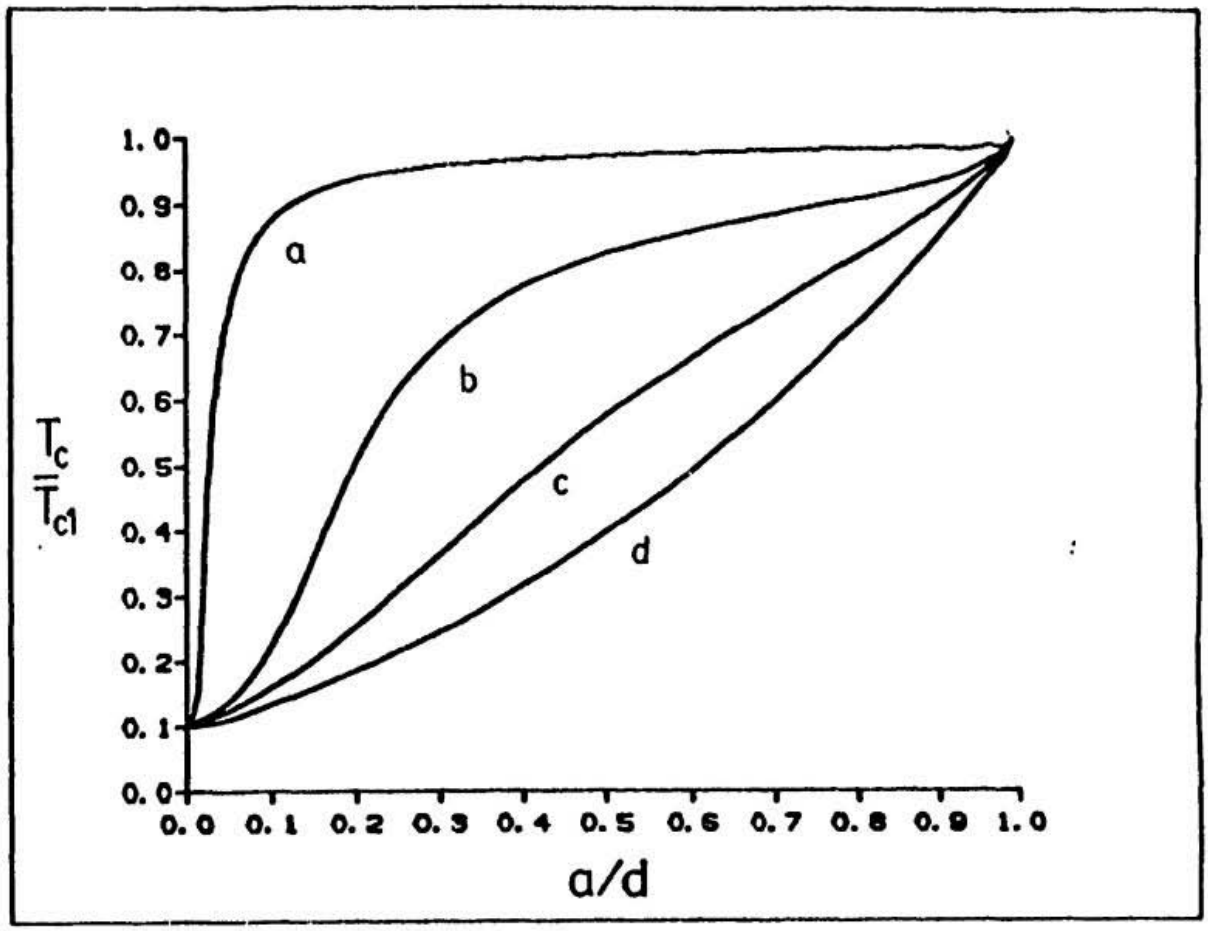

Fig. 4.[26] The dependence of the reduced transition temperature $T_c(a,b)/T_{c1}$ on the thickness a/d for T_{c2}/T_{c1} =0.1. (a)$d/a_c = 0.05$, (b)$d/a_c = 2.0$, (c)$d/a_c = 7.0$, and (d) $d/a_c = 50.0$.

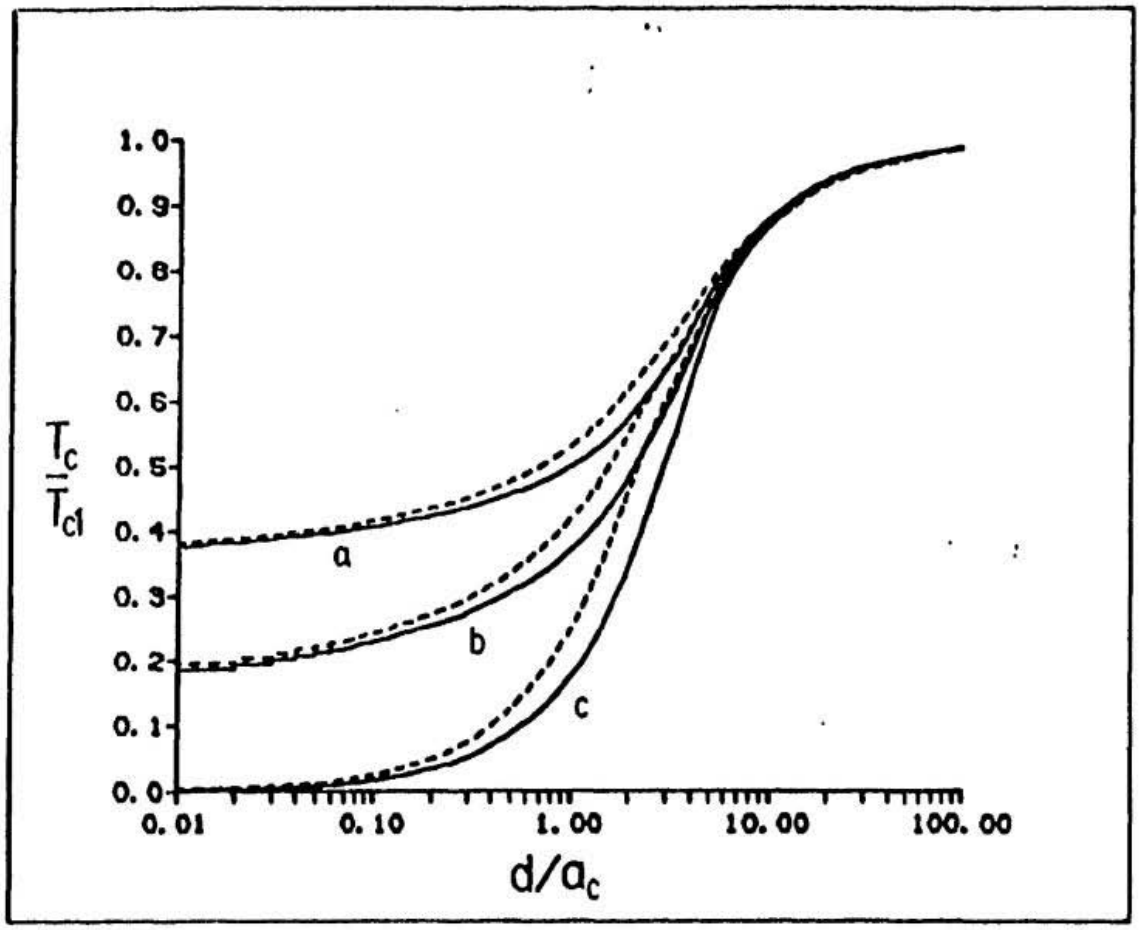

Fig. 5.[26] The dependence of the reduced transition temperature $T_c(a,b)/T_{c1}$ on the thickness d/a_c for $a = b$. (a)$T_{c2}/T_{c1} = 0.1$, (b)$T_{c2}/T_{c1} = 0.01$, and (c)$T_{c2}/T_{c1} = 0$. The corresponding results of the McMillan model are shown by the dotted line.

Note on this graph that for $T_{c2}/T_{c1} = 0$ (curve c) $T_c(a,b)$ vanishes in the limit d tends to zero.

To compare these results with those of a bilayer approximation we have calculated the the corresponding curves from the McMillan model [18]. In this model a free parameter exists which we have chosen so that the value of a_c coincides with ours. In all cases good qualitative agreement is obtained, as Fig.5 indicates, although quantitatively differences up to 30% are present in some sections of the curves. Nevertheless, it is significant that even qualitative agreement is obtained, given the different geometries considered and the fact that we are assuming clean materials with ideal interfaces while the McMillan model assumes dirty materials with a tunneling interface. In a sense it may then be possible to include dirt and reflecting interfaces in this approach by a simple reinterpretation of the parameter a_c, as has been suggested by good agreement of the step-function approximation with more realistic pair potentials in the context of the quasiclassical approximation [43]

A useful contrast between our approach and the various bilayer approximations arises in the Cooper limit of thin films, since it is in this limit that the effects of the coupling between layers is most pronounced. It is also known in this limit that the common approaches of de Gennes and Werthamer[2] [4] are unable to completely account for the observed behaviour of the transition temperature.[8] [44] [24] The differences among the various approaches are most readily seen in the case of $V_2 = 0$. In the thin film limit, Werthamer's result[4] is, as expressed in (2-2-28),

$$\frac{T_c(a,b)}{T_{c1}} \approx \frac{1}{1 + \frac{1}{4}\pi^2 b/a} ,$$

whereas the de Gennes result[2], in terms of (2-2-18), reduces to that of Cooper[45]

$$\frac{T_c(a,b)}{T_{c1}} = \left(\frac{T_{c1}}{1.14\omega_D}\right)^{b/a} .$$

Although quantitatively different, both of these expressions predict that, for a fixed ratio b/a, the critical temperature approaches some finite constant as the thickness a tends to zero. On the other hand, an analysis of (3-3-4) for small thicknesses yields the result

$$\frac{T_c(a,b)}{T_{c1}} \approx (\frac{a}{a_c})^{b/a} \ , \tag{3-3-5}$$

which, again for fixed b/a, approaches zero as a vanishes. This approach to zero is of the same form as that found in the McMillan model[18] expressed in (2-4-25).

A similar difference arises in the case of two different superconductors. Taking the thin film limit of (3-3-1) results in

$$\frac{T_c(a,b)}{T_{c1}} = (\frac{T_{c2}}{T_{c1}})^{b/(a+b)} \ , \tag{3-3-6}$$

valid for a/a_c and b/b_c but not for T_{c2}/T_{c1} small. This is the same result as in the McMillan model, but again differs with the approaches of de Gennes and Werthamer.

Chapter 4

The Transition Temperature of a Self-Similar Multilamellar Superconductor

In this chapter we examine the transition temperature of a self similar layered system similar to that described by Matijašević and Beasley[28] by means of the Werthamer theory of the proximity effect. The outline of the chapter is as follows. In the next section we briefly outline the Werthamer theory of the proximity effect and show how it can be applied to consider the case of layered structures. In the third section we describe the self similar geometry we wish to consider in some detail. By means of the Werthamer theory of the proximity effect we obtain a relation between the logarithmic derivative of the order parameter on the bounding surface of a cell, consisting of several layers, in terms of the logarithmic derivative of the other bounding surface of the cell. This relation together with the self similar nature of the geometry allows us to compute by means of a relatively straightforward recurrence relation an expression for the logarithmic derivative of the order parameter on one side of the sample in terms of the logarithmic derivative on the opposite side, and hence determine the transition temperature of the composite system. The results of numerical calculations are presented for a particular choice of parameters and a comparison of the calculated transition temperature and the measured values[28] is made and the agreement is shown to be satisfactory. In section four we discuss a number of limiting cases from which we can obtain certain scaling relations describing the dependence of the transition temperature on the layer thickness and the size of the sample. Finally, we extend the arguments presented here to consider the transition in the presence of a perpendicular magnetic field. The extension of this analysis to the more difficult, yet more interesting case, of the parallel critical field is currently in progress and will be presented in a future

publication, permitting a comparison with the phenomenological theory presented by Matijašević and Beasley[28], based on the assumption of Josephson coupling between the layers.

§4-1 Formulation

§4-1-1 de Gennes-Werthamer Theory

In this subsection, we summarize the de Gennes-Werthamer theory in order to collect the needed formulae being used in this chapter.

When the phase transition at H_{c2} is of the second order,the superconducting order parameter $\Delta(\vec{x})$ is governed by the linearized integral equation[1]

$$\Delta(\vec{x}) = V(\vec{x}) \int Q(\vec{x},\vec{y})\, \Delta(\vec{y})\, d^3y, \tag{4-1-1}$$

where the kernel $Q(\vec{x},\vec{y})$ is defined as

$$Q(\vec{x},\vec{y}) = \beta^{-1} \sum_{\omega} g_{-\omega}(\vec{y},\vec{x}) g_{\omega}(\vec{y},\vec{x})\ . \tag{4-1-2}$$

The one-electron Green's function $g_\omega(\vec{x},\vec{y})$ in Eq. (4-1-2) is given by

$$\{i\omega - \frac{1}{2m}[-ih\nabla - \frac{e}{c}\vec{A}(\vec{x})]^2 + \sum_j u(\vec{x}-\vec{x}_j) + \mu\} g_\omega(\vec{x},\vec{y}) = \delta(\vec{x}-\vec{y}), \tag{4-1-3}$$

where $u(\vec{x}-\vec{x}_j)$ denotes the scattering potential due to an impurity at position $\vec{x}_j$.

Within the framework of the semi-classical phase integral approximation we can write the kernel Q in the form[46]

$$Q(\vec{x},\vec{y}) \approx e^{2ie\vec{A}(\vec{x})\cdot(\vec{x}-\vec{y})} k(\vec{x},\vec{y}), \tag{4-1-4}$$

where the reduced kernel $k(\vec{x},\vec{y})$ may be evaluated in the single mode approximation as[4]

$$k(\vec{x},\vec{y}) = V N(0)[\ln(\frac{1.14\theta_D}{T})\delta(\vec{x}-\vec{y}) - N(\vec{x}-\vec{y})]\ , \tag{4-1-5}$$

with

$$X(\vec{x}-\vec{y}) = (2\pi)^{-3} \int e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \chi(\xi^2 k^2)\, d^3k\,, \tag{4-1-6}$$

and

$$\chi(z) = \psi(\frac{1}{2} + \frac{1}{2}z) - \psi(\frac{1}{2})\,, \tag{4-1-7}$$

where $\psi(z)$ is the digamma function, $N(0)$ is the density of states for one spin projection at the Fermi surface and ξ denotes the temperature dependent coherence length, which is defined as

$$\xi^2 = \frac{hv_F l}{6\pi k_B T} = \frac{hD}{2\pi k_B T}\,. \tag{4-1-8}$$

§4-1-2 Eigenequation and Boundary Conditions

The above approximations allow us to recast the integral equation Eq. (4-1-1) for the order parameter in differential form,as

$$\chi\big(-\xi^2[\nabla - \frac{2\pi i}{\Phi_0}\vec{A}(\vec{x})]^2\big)\Delta(\vec{x}) = \ln(\frac{T_c}{T})\Delta(\vec{x})\,, \tag{4-1-9}$$

where Φ_0 denotes the flux quantum and T_c denotes the transition temperature of the bulk superconductor, which may be expressed as

$$[N(0)V]^{-1} = \ln(\frac{1.14\theta_D}{T_c})\,, \tag{4-1-10}$$

where θ_D denotes the Debye temperature and constant V denotes the BCS coupling constant. In addition to the differential equation, Eq. (4-1-9), we also require that the order parameter satisfy certain boundary conditions at the surface of the superconductor. The boundary condition for a free surface S is given by

$$\frac{\hat{n}}{\Delta(\vec{x})} \cdot \big[\nabla - \frac{2\pi i}{\Phi_0}\vec{A}\big] \mid_{\vec{x}\in S} = 0\,, \tag{4-1-11}$$

where $\hat{n}$ denotes the unit vector normal to the bonding surface. Eq. (4-1-9) together with the boundary conditions given by Eq. (4-1-11) define an eigenvalue problem which serves to determine the transition temperature of the metal as a function of the applied field H. In order to solve the eigenvalue problem, we define a linear operator $\hat{L}(\nabla)$

$$\hat{L}(\nabla) = -[\nabla - \frac{2\pi i}{\Phi_0}\vec{A}(\vec{x})]^2 \ , \tag{4-1-12}$$

and introduce eigenfunctions $\Delta^n(\vec{x})$ together with the corresponding eigenvalues ϵ^n which satisfy

$$\hat{L}(\nabla)\Delta^n(\vec{x}) = \epsilon^n\Delta^n(\vec{x}) \ , \tag{4-1-13}$$

together with the boundary conditions given by Eq. (4-1-11). The transition temperature T is then determined from the lowest eigenvalue of the set $\{\epsilon^n\}$, which we simply denote as ϵ, from the requirement that

$$\chi(\xi^2\epsilon) = \ln(\frac{T_c}{T}) \ . \tag{4-1-14}$$

By means of the above system of equations one can calculate the dependence of the superconducting transition temperature on the applied field and the geometry of the sample.

The above procedure is easily generalized to an inhomogeneous system comprising several different metals. Specifically let us consider a layered structure consisting of several different superconductors characterized by their bulk transition temperatures, which we denote by $T_{c\lambda}$, together with the corresponding diffusion lengths, which we denote by D_λ. We assume that the planes separating the different regions are aligned normal to the x-axis and we denote the interface separating layer λ from the adjacent layer $\lambda+1$ as x_λ. The order parameter Δ may then be written as

$$\Delta(\vec{x}) = f_\lambda(\vec{x}) \quad \text{for} \quad x_{\lambda-1} < x < x_\lambda \ ,$$

where $f_\lambda(\vec{x})$ satisfies the equation

$$\chi\left(-\xi_\lambda^2[\nabla - \frac{2\pi i}{\Phi_0}\vec{A}(\vec{x})]^2\right) f_\lambda(\vec{x}) = \ln(\frac{T_{c\lambda}}{T}) f_\lambda(\vec{x}) \,. \tag{4-1-15}$$

in layer λ, together with the boundary conditions

$$\frac{D_\lambda}{f_\lambda}\left[\frac{d}{dx} - \frac{2\pi i}{\Phi_0}A_x\right] f_\lambda(x)\,\Big|_{x_\lambda - \epsilon} = \frac{D_{\lambda+1}}{f_{\lambda+1}}\left[\frac{d}{dx} - \frac{2\pi i}{\Phi_0}A_x\right] f_{\lambda+1}(x)\,\Big|_{x_\lambda + \epsilon} \,, \tag{4-1-16}$$

at the interface separating the layers λ and $\lambda + 1$ and

$$\frac{1}{f_1}\left[\frac{d}{dx} - \frac{2\pi i}{\Phi_0}A_x\right] f_1(\vec{x})\,\Big|_{x=x_0} = 0 \,, \tag{4-1-17}$$

$$\frac{1}{f_N}\left[\frac{d}{dx} - \frac{2\pi i}{\Phi_0}A_x\right] f_N(\vec{x})\,\Big|_{x=x_N} = 0 \,, \tag{4-1-18}$$

at the free surface bounding the system. The parameters $T_{c\lambda}$ and ξ_λ are given by the generalisation of Eqs. (4-1-8) and (4-1-10)

$$\xi_\lambda^2 = \frac{hD_\lambda}{2\pi k_B T} \,, \tag{4-1-19}$$

and

$$[N(0)_\lambda V_\lambda]^{-1} = \ln(\frac{1.14\theta_D}{T_{c\lambda}}) \,, \tag{4-1-20}$$

respectively.

In order to determine the transition temperature for such a system we must determine first the eigenvalues, ϵ_λ, defined by the set of differential equations

$$\hat{L}(\nabla) f_\lambda(\vec{x}) = \epsilon_\lambda f_\lambda(\vec{x}) \quad \text{for} \quad x_\lambda < x < x_{\lambda+1} \,, \tag{4-1-21}$$

together with the boundary conditions imposed by Eq. (4-1-16) and (4-1-17). The transition temperature is determined by the requirement that

$$\chi(\xi^2 c_\lambda) = \ln(\frac{T_{c\lambda}}{T}) \,, \tag{4-1-22}$$

In general the above system of equations will yield a set of solutions which we may write $\{\Delta_\lambda^n\}$ together with the corresponding coefficients $\{\epsilon_\lambda^n\}$, each solution corresponding to a distinct transition temperature T_c^n. The observed transition temperature is simply given by the highest member of the set $\{T_c^n\}$

As discussed earlier, despite the complexity of the above equations it is possible to obtain solutions for certain simple geometries[2] [4]. For example solutions have been presented for the case of a bilayer structure in the presence of a homogeneous magnetic field. This solution can also be applied to the case of a superlattice structure consisting of two different superconducting materials and the upper critical field calculated for such a geometry.[14] [31] [25]

§4-2 Application to a Fractal Geometry

§4-2-1 Eigenfunctions

We wish to apply the above analysis to consider a multilamellar superconductor of length L with a self similar structure consisting of alternating layers of two superconducting metals. The geometry of the system is determined by subdividing the system into three cells, a central cell of length $\sigma L'$ and two outer cells of length L', with $L = (2+\sigma)L'$. The two outer cells are then further subdivided into three cells, a central cell of length $\sigma L''$ and two outer cells of width L'', with $L' = (2+\sigma)L''$. This process of subdivision is repeated M times until we obtain a system of N layers with $N = 2^{M+1} - 1$. Alternating layers consist of metals a and b, respectively, such that

$$T_{c\lambda} = \begin{cases} T_{ca} & \text{for } \lambda \text{ odd}, \\ T_{cb} & \text{for } \lambda \text{ even}, \end{cases} \tag{4-2-1}$$

and

$$D_\lambda = \begin{cases} D_a & \text{for } \lambda \text{ odd}, \\ D_b & \text{for } \lambda \text{ even}. \end{cases} \tag{4-2-2}$$

where $\lambda = 1, \ldots, N$. Such a geometry is shown schematically in Fig.6. Denoting by t_λ the thickness of layer λ we obtain for the odd numbered layers the result that

$$t_\lambda = d \qquad \text{for} \quad \lambda = 2n-1 \quad \text{with} \quad 1 \le n \le 2^M \,, \tag{4-2-3}$$

where $d = L/(2+\sigma)^M$. The thickness of the even layers, on the other hand, is not constant but is dependent upon the particular value of λ. For the even numbered layers we obtain the result that

$$t_\lambda = \begin{cases} d\sigma, & \text{for} \quad \lambda = 2(2n-1) \quad \text{with} \quad 1 \le n \le 2^{M-1} \\ d\sigma(2+\sigma), & \text{for} \quad \lambda = 4(2n-1) \quad \text{with} \quad 1 \le n \le 2^{M-2} \\ d\sigma(2+\sigma)^2, & \text{for} \quad \lambda = 8(2n-1) \quad \text{with} \quad 1 \le n \le 2^{M-3} \\ \ldots \text{etc.} \end{cases}$$

The general result may be expressed as

$$t_\lambda = d\sigma(2+\sigma)^{m-1} \qquad \text{for} \quad \lambda = 2^m(2n-1) \quad \text{with} \quad 1 \le n \le 2^{M-m} \,, \tag{4-2-4}$$

where $m = 1, \ldots, M$.

If we consider such a geometry in the limit $M \to \infty$ and $d \to 0$ such that the product $N(M)d = L$ remains finite, then our structure describes a fractal[47] with fractal dimension $\mathcal{D}$ defined as

$$\mathcal{D} = \frac{\ln 2}{\ln(2+\sigma)} \,, \tag{4-2-5}$$

In this section we wish to apply the analysis described in the preceeding section to study the transition temperature of the fractal geometry described above. In order to keep the discussion relatively simple we restrict our considerations to the zero field case, $(H = 0)$. In a subsequent section we will generalize the arguments developed to the case of a perpendicular magnetic field. In the zero field limit Eq. (4-1-21) reduces to the form

$$-\nabla^2 f_\lambda(\vec{x}) = \begin{cases} \epsilon_a f_\lambda(\vec{x}), & \text{for } \lambda \text{ odd} \\ \epsilon_b f_\lambda(\vec{x}), & \text{for } \lambda \text{ even} \end{cases} \,, \tag{4-2-6}$$

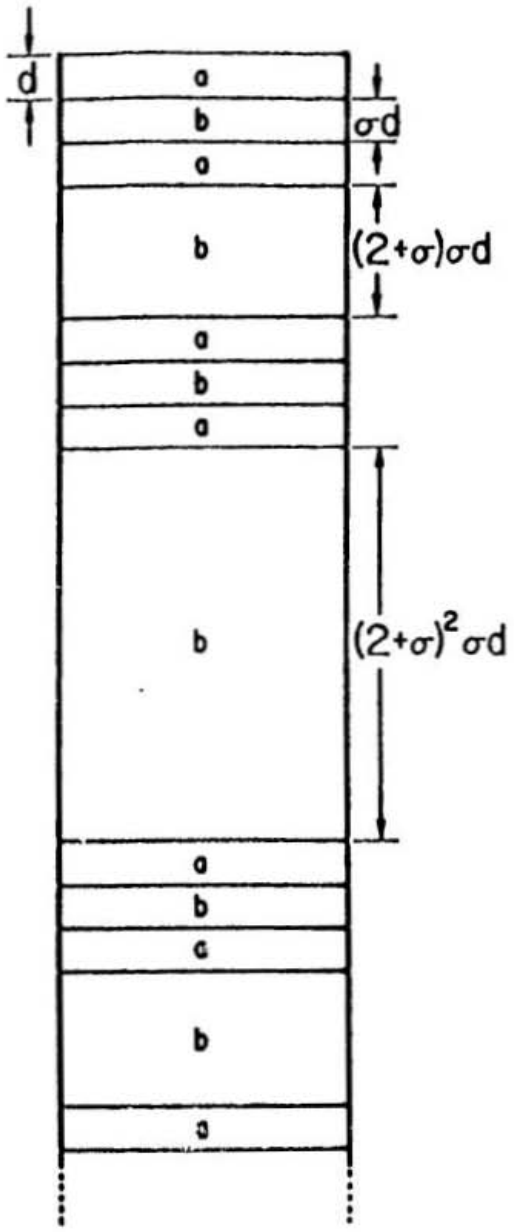

Fig. 6. Geometrical configuration of a fractal structure.

where ϵ_a and ϵ_b are determined from Eq. (4-1-22) by

$$\lambda(\xi^2\epsilon_a) = \ln(\frac{T_{ca}}{T}) , \tag{4-2-7}$$

$$\lambda(\xi^2\epsilon_b) = \ln(\frac{T_{cb}}{T}) , \tag{4-2-8}$$

respectively. If we assume that bulk transition temperatures T_{ca} and T_{cb} are chosen such that

$$T_{ca} > T_{cb} , \tag{4-2-9}$$

then, we would expect the transition temperature, T, of the composite system to be bounded, namely $T_{ca} > T > T_{cb}$. Equation (4-1-22) together with the assumption given by Eq. (4-2-9) yields the result that

$$\begin{aligned} \epsilon_a &> 0 , \\ \epsilon_b &< 0 \end{aligned} \tag{4-2-10}$$

From Eq. (4-2-6) we obtain the following expression for Δ_λ

$$\begin{cases} f_\lambda(x) = \cos(k_a x + \alpha_\lambda) & \text{for } \lambda = 1, 3, 5, \cdots , \\ f_\lambda(x) = \cosh(\kappa_b x + \beta_\lambda) & \text{for } \lambda = 2, 4, 6, \cdots . \end{cases} \tag{4-2-11}$$

where k_a and κ_b are defined as

$$k_a = \sqrt{\epsilon_a} , \tag{4-2-12}$$

$$\kappa_b = \sqrt{-\epsilon_b} . \tag{4-2-13}$$

From this we obtain the following expressions for the logarithmic derivatives

$$\frac{d\ln(f_\lambda(x))}{dx} = -k_a \tan(k_a x + \alpha_\lambda) \quad \text{for } \lambda = 1, 3, 5, \cdots . \tag{4-2-14}$$

$$\frac{d\ln(f_\lambda(x))}{dx} = \kappa_b \tanh(\kappa_b x + \beta_\lambda) \quad \text{for } \lambda = 2, 4, 6, \cdots . \tag{4-2-15}$$

where α_λ and β_λ denote constants to be determined by the boundary conditions, given by Eqs. (4-1-16) and (4-1-17). We can use these equations to relate the

logarithmic derivative on one side of a layer in terms of the logarithmic derivative on the other side of the layer. In particular let us consider the $(N+1)/2$ layers of width d with $\lambda = 2n$ such that $x_{2n-1} < x < x_{2n}$. We denote the logarithmic derivative on the bounding surfaces of the layer as

$$g_n^L(0) = \lim_{\epsilon \to 0} \frac{d \ln \Delta(x)}{dx} \Big|_{x = x_{2n-1}+\epsilon} , \tag{4-2-16}$$

$$g_n^R(0) = \lim_{\epsilon \to 0} \frac{d \ln \Delta(x)}{dx} \Big|_{x = x_{2n}-\epsilon} , \tag{4-2-17}$$

§4-2-2 Recurrence Relation

For the particular case $D_a = D_b$ we can use the above equation together with the boundary equations in Eq. (4-1-16) to obtain the following fractional linear transformation relating the logarithmic derivatives on either side of the layer

$$g_n^R(0) = \frac{g_n^L(0) + \tan(k_a d_a)}{1 - g_n^L(0) \tan(k_a d_a)} . \tag{4-2-18}$$

This we can write in the following form as

$$g_n^R(0) = \frac{\alpha(0) g_n^L(0) + \beta(0)}{\mu(0) g_n^L(0) + \nu(0)} , \tag{4-2-19}$$

where we have defined

$$\begin{pmatrix} \alpha(0) & \beta(0) \\ \mu(0) & \nu(0) \end{pmatrix} = \begin{pmatrix} 1 & \tan(k_a d_a) \\ -\tan(k_a d_a) & 1 \end{pmatrix} . \tag{4-2-20}$$

The expressions given by Eq. (4-2-14) and (4-2-15) for the logarithmic derivative together with the boundary conditions given in Eq. (4-1-16) allow us to generalize the above argument and express the logarithmic derivative on one side of trilayer in terms of the logarithmic derivative on the other side. In particular let us consider the $(N+1)/4$ trilayers comprising two layers of width d, and composed of the metal of type a, separated by a layer of width σd composed of the metal of type b. In

analogy with the terms $g_n^L(0)$ and $g_n^R(0)$, defined in Eq. (4-2-16) and (4-2-17), let us define $g_n^L(1)$ and $g_n^R(1)$ for the trilayer as

$$g_n^L(1) = \lim_{\epsilon \to 0} \frac{d \ln \Delta(x)}{dx} \Big|_{x = x_{4n-3}+\epsilon} , \tag{4-2-21}$$

$$g_n^R(1) = \lim_{\epsilon \to 0} \frac{d \ln \Delta(x)}{dx} \Big|_{x = x_{4n}-\epsilon} , \tag{4-2-22}$$

where $x = x_{4n-3}$ and $x = x_{4n}$ denote the left and right boundaries of the n^{th} trilayer respectively. From Eq. (4-2-14) and (4-2-15) together with the boundary conditions given in Eq. (4-1-16) we obtain the following result

$$g_n^R(1) = \frac{\alpha(1) g_n^L(1) + \beta(1)}{\mu(1) g_n^L(1) + \nu(1)} , \tag{4-2-23}$$

where the coefficients $\alpha(1)$, $\beta(1)$, $\mu(1)$ and $\nu(1)$ are given as

$$\begin{pmatrix} \alpha(1) & \beta(1) \\ \mu(1) & \nu(1) \end{pmatrix} = \begin{pmatrix} \alpha(0) & \beta(0) \\ \mu(0) & \nu(0) \end{pmatrix} \begin{pmatrix} -\tau & \tau^2 \tanh(\kappa_b d_b) \\ \tanh(\kappa_b d_b) & -\tau \end{pmatrix} \begin{pmatrix} \alpha(0) & \beta(0) \\ \mu(0) & \nu(0) \end{pmatrix} , \tag{4-2-24}$$

with $\tau \equiv \kappa_b / k_a$.

The self similar nature of the geometry under consideration allows us to generalize the above procedure to higher order. In particular let us consider the $(N+1)/2^{m+1}$ cells each of width $D^{(m)}$, where $D^{(m)}$ is given by

$$D^{(m)} = d(2+\sigma)^m . \tag{4-2-25}$$

Defining

$$g_n^L(m) = \lim_{\epsilon \to 0} \frac{d \ln \Delta(x)}{dx} \Big|_{x = x_{2^{m+1}(n-1)+1}+\epsilon} , \tag{4-2-26}$$

$$g_n^R(m) = \lim_{\epsilon \to 0} \frac{d \ln \Delta(x)}{dx} \Big|_{x = x_{2^{m+1}n}-\epsilon} , \tag{4-2-27}$$

where $x = x_{2^{m+1}(n-1)+1}$ and $x = x_{2^{m+1}n}$ denote the left and right boundaries of the n^{th} cell, we obtain the following relation

$$g_n^R(m) = \frac{\alpha(m) g_n^L(m) + \beta(m)}{\mu(m) g_n^L(m) + \nu(m)} . \tag{4-2-28}$$

The coefficients appear in the Eq. (4-2-28) may be obtained through the recurrence relation

$$\begin{pmatrix} \alpha(m) & \beta(m) \\ \mu(m) & \nu(m) \end{pmatrix} = \begin{pmatrix} \alpha(m-1) & \beta(m-1) \\ \mu(m-1) & \nu(m-1) \end{pmatrix} \times \begin{pmatrix} -\tau & \tau^2 \tanh(\kappa_b \sigma D^{(m-1)}) \\ \tanh(\kappa_b \sigma D^{(m-1)}) & -\tau \end{pmatrix} \times \begin{pmatrix} \alpha(m-1) & \beta(m-1) \\ \mu(m-1) & \nu(m-1) \end{pmatrix} . \tag{4-2-29}$$

The solution to the recurrence relation given in Eq (4-2-29) may be written as

$$\begin{pmatrix} \alpha(m) & \beta(m) \\ \mu(m) & \nu(m) \end{pmatrix} = -\tau \Phi(m) \begin{pmatrix} 1 & -\tau \Lambda_+(m) \\ -\Lambda_-(m)/\tau & 1 \end{pmatrix} ,$$

where $\Phi(m)$, $\Lambda_+(m)$ and $\Lambda_-(m)$ satisfy the following recurrence relation

$$\Phi(m) = -\tau^2 \Phi(m-1)^2 \left(1 + \tanh(\kappa_b \sigma D^{(m-1)})(\Lambda_-(m-1) + \Lambda_+(m-1)) + \Lambda_-(m-1)\Lambda_+(m-1)\right) , \tag{4-2-30}$$

$$\Lambda_+(m) = \frac{2\Lambda_+(m-1) + \tanh(\kappa_b \sigma D^{(m-1)})(1 + \Lambda_+(m-1)^2)}{1 + \tanh(\kappa_b \sigma D^{(m-1)})(\Lambda_-(m-1) + \Lambda_+(m-1)) + \Lambda_-(m-1)\Lambda_+(m-1)} , \tag{4-2-31}$$

$$\Lambda_-(m) = \frac{2\Lambda_-(m-1) + \tanh(\kappa_b \sigma D^{(m-1)})(1 + \Lambda_-(m-1)^2)}{1 + \tanh(\kappa_b \sigma D^{(m-1)})(\Lambda_-(m-1) + \Lambda_+(m-1)) + \Lambda_-(m-1)\Lambda_+(m-1)} . \tag{4-2-32}$$

The proof, which proceeds by induction, is presented in Appendix D. The particular values $\Lambda_+(0)$ and $\Lambda_-(0)$, may be obtained from Eq. (4-2-20) and are given by

$$\Lambda_+(0) = -\frac{\tan k_a d}{\tau} . \tag{4-2-33}$$

$$\Lambda_-(0) = \tau \tan k_a d . \tag{4-2-34}$$

respectively.

There are two points worth noting with regard to the above relation for the variables $\Lambda_+(m)$ and $\Lambda_-(m)$ namely

i) The recurrence relation involves only $\Lambda_-(m-1)$ and $\Lambda_+(m-1)$

ii) The ratio τ does not appear *explicitly* in the recurrence relation

Both of the above features will play an important role in our subsequent analysis.

Since the boundary conditions on either side of the sample may be expressed as

$$\begin{cases} g_n^L(M) &= 0 \\ g_n^R(M) &= 0 \,, \end{cases}$$

then Eq. (4-2-28) will yield the following result

$$\beta(M) = 0 \Rightarrow \Lambda_+(M) = 0 \,. \qquad (4\text{-}2\text{-}35)$$

Eq. (4-2-35) will only have a solution for particular values of $\tau = \kappa_b/k_a$. The lowest value of τ satisfying Eq. (4-2-35) together with Eqs. (4-2-7) and (4-2-8) yields the transition temperature for the composite system.

In Fig.7 the calculated transition temperature is plotted as a function of $\bar{d} = d/\xi_0$, for several values of M for the particular case $T_{c2} = 0$ and $\sigma = 1$. For this particular choice of parameters we have, from Eqs. (4-1-22) and (4-2-13), that $\kappa_b = \xi^{-1}$ and that the fractal dimension $\mathcal{D} = \ln 2/\ln 3$. Two separate regimes of behaviour may be clearly identified in each of the curves. For $L/\xi_0 \ll 1$ the transition temperature is observed to be relatively independent of the layer thickness with a limiting value that depends on the order of the fractal as defined by the variable M. For $L/\xi_0 \gg 1$ the calculated transition temperature is observed to become independent of M and the various curves merge to form a single curve. In the next section we will discuss this behaviour in some detail and show explicitly

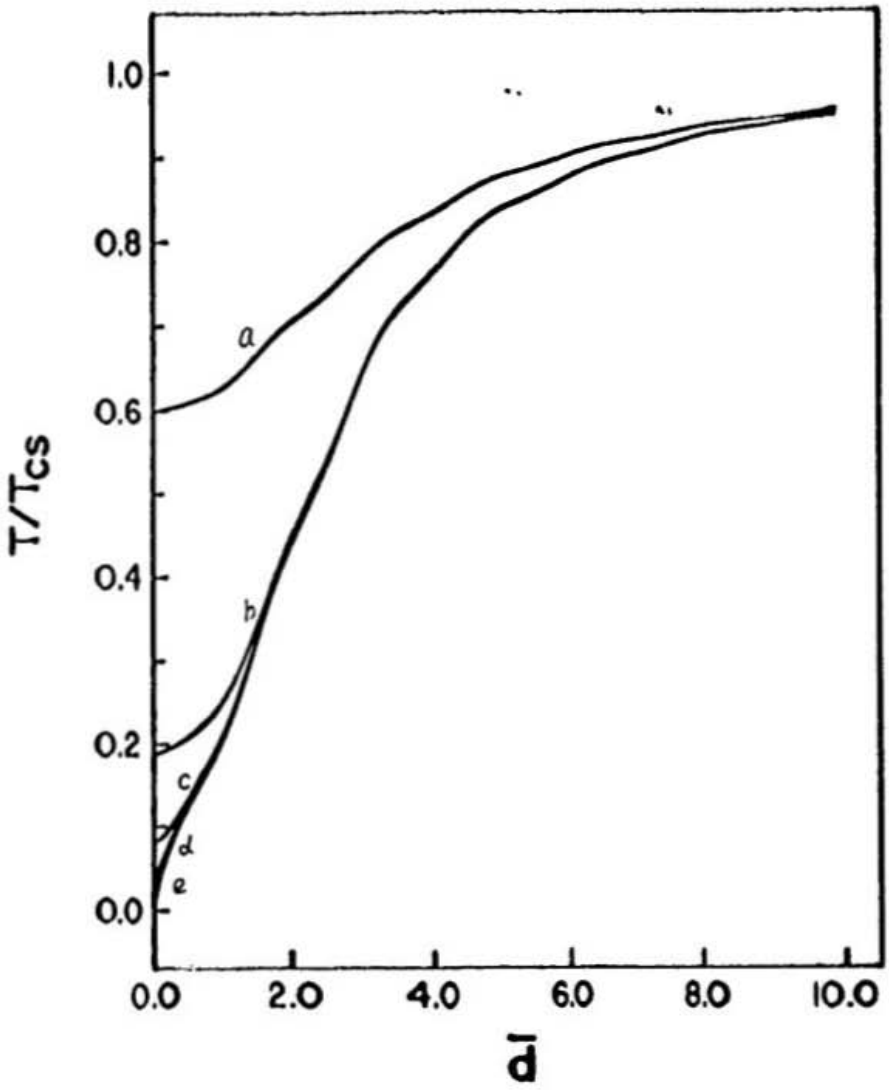

Fig. 7. The dependence of the reduced transition temperature T_c/T_{cS} on the thickness $\bar{d} = d_S/\xi_0$ for $T_{cN} = 0$ and $\mathcal{D} = \ln(2)/\ln(3)$ with order of the fractal M. (a) $M = 1$, (b) $M = 2$, (c) $M = 3$, (d) $M = 4$, and (e) $M = 5$.

how the self similarity of the geometry can be exploited to obtain the various limiting forms for the transition temperature in each of these regimes.

§4-2-3 Comparison with the Measured Values

In order to obtain a comparison with the measured values of the transition temperature [28] of a sample consisting of amorphous $Mo_{1-x}Ge_x$ with two different compositions, $x \approx 20\%$, with bulk $T_c \approx 7.5K$, for the superconducting layers and $x \approx 60\%$, with bulk $T_c \leq 1K$, for the "normal"layers, we have also computed the transition temperature for the parameters presented in Table I. We have considered the dirty limit case $\xi_{//}^2(0) = \frac{\pi^2}{4}\xi_0^2$ with $\xi_0^2 = \frac{\hbar v_F l}{6k_B T_{ca}}$ and have chosen $T_{ca} = 7.5K$ and $T_{cb} = 1K$, which is consistent with the experimental values. For a given value of M we obtain the relationship between τ and $\kappa_b d$ from the solution of Eq. (4-2 35) with $\Lambda_+(M)$, calculated by means of the recurrence relation given by Eq. (4-2 31) and (4-2-32), and from this we obtain the transition temperature by means of (4 2 7) and (4-2-8). The transition temperature is computed for increasing values of M, until the successive values agree to within 4 significant figures. Increasing the value of M beyond this does not affect the final result. In this way we can obtain an accurate estimate of the transition temperature of the system in the limit $M \to \infty$. The resultant temperature are presented in Table I together with the corresponding measured values. The agreement is seen to be satisfactory.

§4-3 Scaling Laws

In the previous section we examined the structure of the proximity equations for the case of a self-similar geometry and presented numerical results for a particular choice of variables. Despite the complexity of the above equation it is nevertheless possible to exploit the self similar nature of the geometry and obtain analytical results in certain limiting cases. We consider two such limiting cases in this section.

Table I. The theoretical and experimental values of the superconducting transition temperatures of the geometries with different fractal dimensions.

Dimension	Layer Spacing (Å)	T_c (K)(exp.)	T_c (K)(theory)	$\xi_{//}(0)$(Å)	d/ξ_0
0.73	50	3.72	3.33	58	0.525
0.83	220	5.61	5.74	58	2.579
0.57	80	2.80	2.56	58	0.642

§4-3-1 Thin Film Limit

In the regime $L \ll \xi$ we can assume that $\kappa_b L \ll 1$ and hence that

$$k_a d \ll 1 ,$$

$$\kappa_b \sigma D^{(m)} \ll 1 \quad \text{for } 1 \leq m \leq M .$$

This suggests that we need only consider the terms linear in $\kappa_b \sigma D^{(m)}$, in the recurrence relation for the coefficients $\Lambda_-(m)$ and $\Lambda_+(m)$. Retaining only the lowest order terms in the recurrence relations Eq. (4-2-31) and (4-2-32), we obtain

$$\Lambda_+(m) = 2\Lambda_+(m-1) + \kappa_b \sigma D^{(m-1)} , \tag{4-3-1}$$

$$\Lambda_-(m) = 2\Lambda_-(m-1) + \kappa_b \sigma D^{(m-1)} , \tag{4-3-2}$$

while from Eq. (4-2-33) and (4-2-33) $\Lambda_-(0) = \kappa_b d$ and $\Lambda_+(0) = -\kappa_b d/\tau^2$. These may be solved to give the following expression for $\Lambda_+(m)$ and $\Lambda_-(m)$

$$\Lambda_+(m) = 2^m \left(\left(\frac{2+\sigma}{2} \right)^m - \left(1 + \frac{1}{\tau^2} \right) \right) \kappa_b d , \tag{4-3-3}$$

$$\Lambda_-(m) = (2+\sigma)^m \kappa_b d . \tag{4-3-4}$$

From the requirement that $\Lambda_+(M) = 0$ we therefore obtain the result that

$$\tau^{-2} = \left(\frac{2+\sigma}{2} \right)^M - 1 , \tag{4-3-5}$$

and hence we obtain the following expression

$$\frac{\epsilon_a}{\epsilon_b} = 1 - \left(\frac{2+\sigma}{2} \right)^M . \tag{4-3-6}$$

An alternate form for the above expression which demonstrates the role of the fractal dimension is obtained by noting that $L = d(2+\sigma)^M$ and hence

$$M = \frac{\ln(L/d)}{\ln(2+\sigma)} , \tag{4-3-7}$$

which allows us to write

$$(\frac{2+\sigma}{2})^M = (\frac{L}{d})^{(1-\mathcal{D})} , \tag{4-3-8}$$

where $\mathcal{D}$ denotes the fractal dimension defined by Eq. (4-2-5). Equation (4-3-6) then reduces to the form

$$\frac{\epsilon_a}{\epsilon_b} = 1 - (\frac{L}{d})^{(1-\mathcal{D})} . \tag{4-3-9}$$

The result contained in Eq. (4-3-9) together with expressions for ϵ_a and ϵ_b, given by Eq. (4-2-7) and (4-2-8) yields three equations for the three unknown quantities ϵ_a, ϵ_b and the transition temperature T. In general the above equations can only be solved numerically. However if we consider the particular case discussed in the previous section, namely $T_{cb} = 0$ then Eq. (4-2-8) may be solved explicitly to give

$$\epsilon_b = -\xi^{-2} ,$$

and hence the Eq. (4-2-7) may be written as

$$\chi(-\frac{\epsilon_a}{\epsilon_b}) = -\ln t , \tag{4-3-10}$$

where we have defined the reduced transition temperature $t = T/T_{ca}$. Thus we obtain the following expression for the reduced transition temperature

$$t = \exp\left(-\chi((\frac{L}{d})^{(1-\mathcal{D})} - 1)\right) . \tag{4-3-11}$$

If we consider the case in which $L/d \gg 1$, we may use the asymptotic limit of the function $\chi(z)$

$$\begin{aligned} \lim_{z\to\infty} \chi(z) &= \lim_{z\to\infty} (\psi(\frac{1}{2} + \frac{1}{2}z) - \psi(\frac{1}{2})) \\ &\approx \ln z + \gamma + \ln 2 + \mathcal{O}(1/z) , \end{aligned} \tag{4-3-12}$$

where γ denotes the Euler constant. From this we obtain the result that

$$t \approx \frac{e^{-\gamma}}{2}(\frac{d}{L})^{1-\mathcal{D}} . \tag{4-3-13}$$

A more general result may be obtained by means of the approximate expression for χ

$$\chi(z) \approx \ln(1+\frac{\pi^2 z}{4}) , \tag{4-3-14}$$

which is valid for all $z > 0$[4], yielding the result that

$$t = \left(1+\frac{\pi^2}{4}\left(\left(\frac{L}{d}\right)^{(1-P)}-1\right)\right)^{-1} . \tag{4-3-15}$$

In the limit $L/d \to \infty$ the result obtained above agrees with that given by Eq. (4-3-13) up to a simple numerical factor. The results obtained agree reasonably well with the numerical results presented in Fig.8. It is worth noting that this result corresponds to the generalisation, to a self similar geometry, of the result obtained by Werthamer for the case of a simple bilayer[4], namely

$$t = \left(1+\frac{\pi^2}{4}\left(\frac{d_N}{d_S}\right)\right)^{-1} , \tag{4-3-16}$$

where, d_N and d_S are the thicknesses of the normal and superconducting layers respectively. Indeed we may recover this result as a special case of the above analysis by setting $M = 1$ and identifying $d_S = d$ and $d_N = \sigma d/2$.

§4-3-2 Decoupling of the Structure

As we demonstrated in the previous section in the limit $L \gg \xi$, the relationship between the transition temperature T and the reduced thickness d/ξ becomes independent of the particular value of M. In order to understand this result we should note that the linear fractional transformation given by Eq. (4-2-28) allows us to relate the logarithmic derivative $g_n^L(m)$ to the logarithmic derivative $g_n^R(m)$ if and only if the coefficients $\alpha(m)$ satisfy the condition

$$\det\begin{vmatrix}\alpha(m) & \beta(m)\\ \mu(m) & \nu(m)\end{vmatrix} \neq 0 . \tag{4-3-17}$$

If this condition is not satisfied then the expression given by Eq. (4-2-28) reduces to the form

$$g_n^R(m) = \frac{\beta(m)}{\mu(m)}\,, \tag{4-3-18}$$

and hence the boundary condition on the right hand side of the cell is independent of the boundary condition on the left hand side of the cell. In essence the two surfaces bounding the cell are independent.

From Eq. (4-2-29) we obtain that

$$\det\begin{vmatrix}\alpha(m) & \beta(m)\\ \mu(m) & \nu(m)\end{vmatrix} = \det\begin{vmatrix}\alpha(m-1) & \beta(m-1)\\ \mu(m-1) & \nu(m-1)\end{vmatrix}^2 \det\begin{vmatrix}-\tau & \tau^2\tanh(\kappa_b\sigma D^{(m-1)})\\ \tanh(\kappa_b\sigma D^{(m-1)}) & -\tau\end{vmatrix}\,.$$

Now for the case that $\kappa_b\sigma D^{(m-1)} \gg 1$ we have the result that

$$\det\begin{vmatrix}-\tau & \tau^2\tanh(\kappa_b\sigma D^{(m-1)})\\ \tanh(\kappa_b\sigma D^{(m-1)}) & -\tau\end{vmatrix} \approx \mathcal{O}\left(e^{-\kappa_b\sigma D^{(m-1)}}\right)\,,$$

and hence that

$$\det\begin{vmatrix}\alpha(m) & \beta(m)\\ \mu(m) & \nu(m)\end{vmatrix} \approx \mathcal{O}\left(e^{-\kappa_b\sigma D^{(m-1)}}\right)\,.$$

Thus from Eq. (4-3-18) the logarithmic derivative $g_n^R(m)$ is given as

$$g_n^R(m) \approx \frac{\beta(m)}{\mu(m)} + \mathcal{O}\left(e^{-\kappa_b\sigma D^{(m-1)}}\right)\,. \tag{4-3-19}$$

The physics behind this last result is relatively straight forward. If we consider the specific example considered in the previous section with $T_{cb} = 0$ then the above result simply reflects the fact that if the "normal" layers separating the multilamellar cells is much greater than the coherence length of the order parameter then the cells on either side of the "normal" layer decouple and the logarithmic derivatives $g_n^L(m)$ and $g_n^R(m)$ are essentially (i.e. to order $\approx \mathcal{O}(e^{\kappa_b\sigma D^{(m)}})$) independent. In order to understand how this gives rise to the fact that the transition temperature

becomes independent of M we note that for all m such that $\kappa_b \sigma D^{(m)} \gg 1$, we can obtain the result that

$$\begin{pmatrix} \alpha(m) & \beta(m) \\ \mu(m) & \nu(m) \end{pmatrix} = -\tau \left(\alpha(m-1) + \mu(m-1) - \beta(m-1)/\tau - \nu(m-1)\tau \right) \begin{pmatrix} \alpha(m-1) & \beta(m-1) \\ \mu(m-1) & \nu(m-1) \end{pmatrix} . \tag{4-3-20}$$

Iterating this expression we obtain, for any $m' > m$, the result that

$$\begin{pmatrix} \alpha(m') & \beta(m') \\ \mu(m') & \nu(m') \end{pmatrix} \propto \begin{pmatrix} \alpha(m) & \beta(m) \\ \mu(m) & \nu(m) \end{pmatrix} .$$

Let us now consider two similar geometries one $L = L_1 = d(2+\sigma)^{M_1}$ and $L = L_2 = d(2+\sigma)^{M_2}$, with $M_1 < M_2$ and $\kappa_b \sigma L_1 > \kappa_b \sigma L_2 \gg 1$. From the above arguments we see that for the case $L = L_2$ we obtain the result that

$$\begin{aligned} g_n^R(M_2) &= \frac{\beta(M_2)}{\mu(M_2)} \\ &= \frac{\beta(M_1)}{\mu(M_1)} . \end{aligned} \tag{4-3-21}$$

Thus we see that the same value of τ which satisfies the boundary condition for the case $L = L_2$, namely that $g_n^R(M_2) = 0$, also satisfies the boundary conditions for the case $L = L_1$, $g_n^R(M_1) = 0$, and hence the resultant transition temperature T calculated will be the same for both cases.

§4-3-3 Scaling Law

While transition temperature becomes independent of the length L when $L \ll \xi$ it is only possible to obtain analytical results for the limiting cases $d \ll \xi$ and $d \gg \xi$. We will limit our discussion to the situation $d \ll \xi$, since this represents the more interesting limit.

The form of the recurrence relation given by Eqs. (4-2-31) and (4-2-32) allows us to write $\Lambda_+(M)$ and $\Lambda_-(M)$ in the following manner

$$\Lambda_+(M) = F^+_{M-m}(\Lambda_+(m), \Lambda_-(m), \kappa_b D^{(m)}) \,, \tag{4-3-22}$$

$$\Lambda_-(M) = F^-_{M-m}(\Lambda_+(m), \Lambda_-(m), \kappa_b D^{(m)}) \,, \tag{4-3-23}$$

for $m < M$. The functions F^+ and F^- are as yet undetermined functions which are *independent of* τ and depend only on M and m through the difference $M - m$. If we consider the limit $M \to \infty$, then setting $\Lambda_+(\infty) = 0$ and choosing a value of m such that $\kappa_b D^{(m)} \ll 1$ we can write

$$F^+_\infty(\Lambda_+(m), \kappa_b D^{(m)}, \kappa_b D^{(m)}) = 0 \,. \tag{4-3-24}$$

$$F^-_\infty(\Lambda_+(m), \kappa_b D^{(m)}, \kappa_b D^{(m)}) = 0 \,. \tag{4-3-25}$$

This suggests that for $\kappa_b D^{(m)} \ll 1$ the solution of the boundary value problem provides a unique and unambiguous relation between $\Lambda_+(m)$ and $\kappa_b D^{(m)}$ which we write as follows.

$$\Lambda_+(m) = \kappa_b D^{(m)} \mathcal{G}(\kappa_b D^{(m)}) \,. \tag{4-3-26}$$

Taking the limit $m \to 0$ and noting the result contained in Eq. (4-3-3), we obtain

$$\tau^{-2} = -\mathcal{G}(\kappa_b d) \,.$$

In order to determine the explicit form of the function $\mathcal{G}$ we substitute in the expression for $\Lambda_+(m)$ from Eq. (4-3-26) which yields the following expression

$$2^m \left(\left(\frac{2+\sigma}{2} \right)^m - (1 + \tau^{-2}) \right) = (2+\sigma) \mathcal{G}(\kappa_b D^{(m)}) \,.$$

Using the above expression for τ^{-2} we may rearrange this to give

$$\frac{\mathcal{G}(\kappa_b D^{(m)}) - 1}{\mathcal{G}(\kappa_b d) - 1} \left(\frac{2+\sigma}{2} \right)^m = 1 \,.$$

Eliminating the explicit m dependence by means of Eq (4-3-8) we obtain

$$\frac{\mathcal{G}(\kappa_b D) - 1}{\mathcal{G}(\kappa_b d) - 1}\left(\frac{D}{d}\right)^{1-p} = 1 \,. \tag{4-3-27}$$

from which we can conclude that

$$\mathcal{G}(\kappa_b D) = 1 - \frac{A}{(\kappa_b D)^{1-p}} \,. \tag{4-3-28}$$

where A denotes some undetermined constant which is *independent* of τ. Setting $m = 0$ we therefore obtain the result that

$$1 + \frac{1}{\tau^2} = \frac{A}{(\kappa_b d)^{1-p}} \,. \tag{4-3-29}$$

For the particular case $T_{cb} = 0$ we have that

$$\kappa_b = \xi^{-1} \,,$$

and hence

$$\begin{aligned} \frac{\epsilon_a}{\epsilon_b} &= \tau^{-2} \\ &= A\left(\frac{\xi}{d}\right)^{1-p} - 1 \,. \end{aligned}$$

This yields the following expression

$$t = \exp\left(-\gamma\left(A\left(\frac{\xi}{d}\right)^{1-p} - 1\right)\right) \,.$$

Since we are considering the limit in which $\xi/d \gg 1$ we may use the asymptotic expansion for the χ function, given by Eq. (4-3-12), to yield the following expression for the transition temperature in terms of the temperature dependent coherence length ξ

$$t \approx A\frac{e^{-\gamma}}{2}\left(\frac{d}{\xi}\right)^{1-p} \,. \tag{4-3-30}$$

Comparing this with (4-3-13), one can see that only the length scale L is replaced by the coherence length ξ. Eq. (4-3-30) together with the expression for $\xi^2 = \hbar D/2\pi k_B T$ allows us to write an explicit form for the transition temperature as

$$t \approx A\left(\frac{d}{\xi_0}\right)^{\frac{2(1-p)}{1+p}} \,, \tag{4-3-31}$$

where ξ_0 is defined by the expression

$$\xi_0^2 = \frac{hD_a}{2\pi k_B T_{ca}} ,$$

and the coefficient $A = (Ac^{-\gamma}/2)^{\frac{2}{1+\gamma}}$

§4-4 The Influence of the Perpendicular Magnetic Field

In this section we wish to extend the analysis of the previous section to evaluate the transition temperature of the multilamellar system, described in Section 2, in the presence of a magnetic field $\vec{B} = -H\vec{e}_z$.

§4-4-1 Eigenfunctions in the Presence of Perpendicular Magnetic Field

We represent the magnetic field $\vec{B}$ by the vector potential $\vec{A}$ as

$$\vec{A}(\vec{x}) = (0, Hz, 0) .$$

Eq. (4-1-21) then reduces to the form

$$(-\frac{\partial^2}{\partial x^2} - (\frac{\partial}{\partial y} - \frac{2\pi i H}{\Phi_0} z)^2 - \frac{\partial^2}{\partial z^2})\Delta(\vec{x}) = \begin{cases} \ln(\frac{T_{ca}}{T_c})\Delta(\vec{x}) & \text{for } \lambda \text{ odd} , \\ \ln(\frac{T_{cb}}{T_c})\Delta(\vec{x}) & \text{for } \lambda \text{ even} . \end{cases} \qquad (4\text{-}4\text{-}1)$$

The order parameter then has the form

$$\Delta(\vec{x}) = e^{-\frac{1}{2}\frac{2\pi H}{\Phi_0} z^2} f_\lambda(x) ,$$

where $f_\lambda(x)$ satisfies the differential equation

$$\frac{d^2 f_\lambda(x)}{dx^2} = \begin{cases} (\epsilon_a - \frac{2\pi H}{\Phi_0}) f_\lambda(x) & \text{for } \lambda \text{ odd} , \\ (\epsilon_b - \frac{2\pi H}{\Phi_0}) f_\lambda(x) & \text{for } \lambda \text{ even} , \end{cases}$$

where ϵ_a and ϵ_b satisfy

$$\chi(\xi^{-2}\epsilon_a) = \ln(\frac{T_{ca}}{T}) , \tag{4-4-2}$$

$$\chi(\xi^{-2}\epsilon_b) = \ln(\frac{T_{cb}}{T}) , \tag{4-4-3}$$

and the functions f_λ and $f_{\lambda+1}$ are required to have a continuous logarithmic derivative at the interface separating the two layers. The function $f_\lambda(x)$ is therefore given by

$$f_\lambda(x) = \begin{cases} \cos(k_a x + \alpha_\lambda) & \text{for} \quad \lambda = 1, 3, 5, \cdots , \\ \cosh(\kappa_b x + \beta_\lambda) & \text{for} \quad \lambda = 2, 4, 6, \cdots , \end{cases}$$

where the variables k_a and κ_b are defined as

$$k_a^2 = \epsilon_a - \frac{2\pi H}{\Phi_0} , \tag{4-4-4}$$

$$\kappa_b^2 = -\epsilon_b + \frac{2\pi H}{\Phi_0} . \tag{4-4-5}$$

If we restrict our attention to the case with $T_{cb} = 0$ then from Eq. (4-4-3) we obtain that

$$\begin{aligned} & \xi^2\epsilon_b = -1 , \\ \Rightarrow \quad & \xi^2\kappa_b^2 = 1 + \frac{2\pi H}{\Phi_0}\xi^2 \\ & \qquad = 1 + \frac{h}{t} , \end{aligned} \tag{4-4-6}$$

where we have defined $h = \frac{H}{H_0}$, where $H_0 = \frac{2\pi}{\Phi_0/\xi_0^2}$ and denotes the zero temperature critical field of the bulk metal of type a. The transition temperature of the composite system may therefore be expressed in terms of the ratio $\tau = \kappa_b/k_a$ from Eq. (4-4-3) as

$$\begin{aligned} \ln t &= \chi(\xi^2\epsilon_a) \\ &= \chi\big(\xi^2(k_a + \frac{2\pi H}{\Phi_0})\big) \\ &= \chi\big((\frac{\kappa_b\xi}{\tau})^2 + \frac{h}{t}\big) \\ &= \chi\big((1 + \frac{h}{t})/\tau^2 + \frac{h}{t}\big) \\ &= \chi\big(\tau^{-2} + \frac{h}{t}(1 + \tau^{-2})\big) . \end{aligned} \tag{4-4-7}$$

It remains only to compute the value of τ for the particular system of interest, by solving the eigenvalue problem posed by Eq. (4-4-1) together with the corresponding boundary conditions. However one readily notes that this aspect of the problem is identical to the problem discussed in the previous section for the zero field case and hence the results obtained may be applied to the problem of the perpendicular critical field. We discuss the two limiting cases $\xi \gg L$ and $\xi \ll L$ separately.

§4-4-2 Limiting Cases

Case 1: $L \ll \xi$

In this limit we have the result from Eq. (4-3-29)

$$\tau^{-2} = 1 - \left(\frac{L}{d}\right)^{(1-\mathcal{D})} ,$$

If we consider the case for which $L \gg d$ then $\tau \to 0$ and we may use the asymptotic form of the $\setminus$ function to obtain the result that

$$\begin{aligned} t + h &= \frac{e^{-\gamma}}{2}\left(\frac{d}{L}\right)^{(1-\mathcal{D})} \\ &= t_0 , \end{aligned} \tag{4-4-8}$$

where t_0 denotes the reduced transition temperature at zero field, given by Eq. (4-3-13).

Case 2: $L \gg \xi$

In this limit we have the result from Eq. (4-3-29)

$$\tau^{-2} \approx \frac{A}{(\kappa_b d)^{1-\mathcal{D}}} .$$

From Eq. (4-4-6) we have that

$$\kappa_b = \xi^{-1}\sqrt{1+\frac{h}{t}}\ . \tag{4-4-9}$$

Substituting this into Eq. (4-4-7) yields the following relationship between the magnetic field, the transition temperature and the temperature dependant coherence length ξ :

$$(\frac{\xi}{d})^{(1-\mathcal{D})}t(1+\frac{h}{t})^{\frac{1+\mathcal{D}}{2}} = 1\ . \tag{4-4-10}$$

Writing $\xi = \xi_0/\sqrt{t}$ we obtain

$$(\frac{\xi_0}{d})^{(1-\mathcal{D})}(t+h)^{\frac{1+\mathcal{D}}{2}} = 1\ , \tag{4-4-11}$$

which yields the following expression for the transition temperature

$$\begin{aligned} t+h &= (\frac{d}{\xi_0})^{\frac{2(1-\mathcal{D})}{1+\mathcal{D}}} \\ &= t_0\ . \end{aligned} \tag{4-4-12}$$

§4-5 Discussion

In summary we have applied the Werthammer theory of the proximity effect to the study the transition temperature of a self similar multilamellar system consisting of alternating layers of superconducting and normal metal. The comparison with the limited amount of experimental data suggest that the description is reasonable.

In the limit $d/\xi \ll 1$ the dependence of the transition temperature on the dimensions of the sample may be expressed in terms of certain scaling laws in certain limiting cases. For $T_{cb} = 0$ we obtain

$$t \approx \left(\frac{d}{L}\right)^{1-\mathcal{D}} \quad \text{for} \quad L \ll \xi ,$$
$$t \approx \left(\frac{d}{\xi_0}\right)^{\frac{2(1-\mathcal{D})}{1+\mathcal{D}}} \quad \text{for} \quad L \gg \xi .$$

Not surprisingly the exponents obtained are simply expressed in terms of the fractal dimension associated with the geometry. We have also extended the analysis to consider the effect of a perpendicular magnetic field on the transition temperature and have shown how, in certain limiting cases, we can again derive certain simple scaling laws to describe the upper critical field. The much more difficult extension of this work to the case of parallel critical field is in progress and should provide an interesting comparison with the results obtained from phenomenological theories[28]. The results of such a calculation would provide an extremely valuable test of the Werthamer theory of the proximity effect, and its application to the superconducting properties of layered structures, as it would it would involve a system with many length scales. We regard the present work and the results obtained as an essential first step in this calculation.

To what extent the scaling laws we obtain in the present calculation are experimentally accessible is not addressed and is perhaps a question best left to experimentalists. Perhaps a more pertinent criticism concerning the scaling laws presented in sections 4 and 5 is the validity of the Werthamer theory of the proximity effect in the appropriate limit, since contrary to the statements made in the literature the theory is *not* exact and one might expect nonlocal effects, for example, to play an important role. Suffice to say however that we have presented the theoretical predictions of a theory which enjoys a wide acceptance and has been extensively and successfully applied to a variety problems of layered media.

Chapter 5

Conclusions

The conclusions presented in previous chapters can be summarized as follows.

In Chapter 3, we applied the Bogoliubov equation to a superlattice composed of alternating superconductor and normal metal layers. The periodic energy spectrum has been obtained relating to the periodic structure for finite superconducting order parameter Δ.

$$\cos(dq) = \cos(ap_1)\cos(bp_2) - [\frac{(1-\delta_1\delta_2)}{\gamma_1\gamma_2}]\sin(ap_1)\sin(bp_2)\,.$$

This relation allows one to completely calculate, in principle, the normalization constant of the wave functions u_S and u_N. However, the calculation will be extremely difficult in the finite order parameter case due to the periodic structure.

The transition temperature is calculated in the limit $\Delta \to 0$ and it is plotted in Fig.3 as a function of the reduced thickness. The analytical result of the transition temperature in the thin films limit can be expressed as

$$\frac{T_c(a,b)}{T_{c1}} \approx (\frac{a}{a_c})^{b/a}\,.$$

It is seen that the transition temperature, for a fixed ratio b/a, goes to zero when the thickness of the superconducting layer a vanishes.

A useful contrast between our approach and the various bilayer approximations arises in this Cooper limit of thin films, since the effect of the coupling between layers is most pronounced in such a limit. As a comparison, we present the results of the other theories below. Werthamer's result[4] is

$$\frac{T_c(a,b)}{T_{c1}} \approx \frac{1}{1+\frac{1}{4}\pi^2 b/a}\,,$$

and de Gennes result[2] is

$$\frac{T_c(a,b)}{T_{c1}} = (\frac{T_{c1}}{1.14\omega_D})^{b/a} .$$

Although quantitatively different, both of these expressions predict that, for a fixed ratio b/a, the critical temperature approaches some finite constant and is independent of the thickness of the superconducting layer a. The reason for the existence of such a difference is known in that the common approaches of de Gennes and Werthamer are unable to completely account for the observed behavior of the transition temperature in this limit, since the One-Frequency approximation taken in their theories is only valid for thick films. On the other hand, our result agrees most closely with that of McMillan's tunnelling model[18], which, in the same limit, is expressed as

$$\frac{T_c}{T_{Sc}} \approx (\frac{\pi T_{Sc}}{2\gamma\Gamma})^{\Gamma_S/\Gamma_N} \quad , \Gamma/T_{Sc} >> 1$$

where the Γ is inversely proportional to the thickness of the superconducting layer a. The curves from the numerical calculations for both our and McMillan's model are shown in Fig.5 and a good qualitative agreement is obtained. Nevertheless, it is significant that even qualitative agreement is obtained since we are assuming clean materials with ideal interfaces while the McMillan model assumes dirty materials with a tunnelling interface. In a sense it may then be possible to include dirt and reflecting interfaces in this approach by reinterpretation of the parameter a_c, as has been suggested by the good agreement of the step-function approximation with more realistic pair potentials in the context of the quasiclassical approximation[43].

In Chapter 4, we applied the Werthamer theory to a geometry with fractional dimension. In each of the superconducting and normal layers, we assume the Werthamer equations

$$\psi(\xi^2\epsilon_a) = \ln(\frac{T_{ca}}{T}) ,$$

$$\psi(\xi^2\epsilon_b) = \ln(\frac{T_{cb}}{T}) .$$

to be valid and then we construct a linear fractional transformation

$$g_n^R(m) = \frac{\alpha(m) g_n^L(m) + \beta(m)}{\mu(m) g_n^L(m) + \nu(m)} .$$

where $g_n^R(m)$ and $g_n^L(m)$ are boundary values on right and left side of a supercell of order m. In order to determine the transition temperature T_c by means of the above equation, the coefficients $\alpha(m), \beta(m), \mu(m)$ and $\nu(m)$ need to be known. Hence, we extract the recurrence relation of these coefficients, through an analysis to the self-similar geometry, given as

$$\begin{pmatrix} \alpha(m) & \beta(m) \\ \mu(m) & \nu(m) \end{pmatrix} = \begin{pmatrix} \alpha(m-1) & \beta(m-1) \\ \mu(m-1) & \nu(m-1) \end{pmatrix} \times \begin{pmatrix} -\tau & \tau^2 \tanh(\kappa_b \sigma D^{(m-1)}) \\ \tanh(\kappa_b \sigma D^{(m-1)}) & -\tau \end{pmatrix} \times \begin{pmatrix} \alpha(m-1) & \beta(m-1) \\ \mu(m-1) & \nu(m-1) \end{pmatrix} ,$$

which provides a basis of the numerical calculation and the detailed analysis in some limiting cases for the transition temperature T_c. To compare with the experimentally measured values[28], we input the same parameters as those in the paper[28] and use the technique of the recurrence relation together with the Werthamer's equation. We obtain the theoretical values of the transition temperature which are shown in Table I. The discrepancies are 10%, 8.6% and 2% corresponding to the reduced thicknesses 0.525, 0.642 and 2.579 respectively. The decreasing of the discrepancy with increasing reduced thickness is consistent with the Werthamer theory for the thick films case as we mentioned above. A numerical calculation has also been done for the transition temperature as a function of the reduced thickness $\bar{d}$ and the result is shown in Fig.7. This part of the calculation reveals that the various curves corresponding to the different order of the fractal m merge to form a single curve at some values of the d and hence, we conclude that only those length scales comparable with the coherence length ξ dominate the behavior of the system.

The analytical results for the dependence of the transition temperature on the dimensions of the sample are obtained in the limiting cases $L << \xi$ and $L >> \xi$ through the scaling argument and expressed as follows

$$t \approx (\frac{d}{L})^{1-\mathcal{D}} \quad \text{for} \quad L \ll \xi ,$$
$$t \approx (\frac{d}{\xi_0})^{\frac{2(1-\mathcal{D})}{1+\mathcal{D}}} \quad \text{for} \quad L \gg \xi .$$

The exponents obtained are simply expressed in terms of the fractal dimension associated with the geometry. Since the fractal dimension $\mathcal{D}$ relates to the manner of increasing the portion of the superconductor in the whole fractal structure while enlarging the whole structure, the limit $\mathcal{D} \rightarrow 1$ is equivalent to working on a bulk superconductor in our geometry. The fact that the reduced transition temperature t for a system of length L has a finite value independent of the reduced thickness remains the feature of the Werthamer's theory, while in a fractal structure with infinite layers, t vanishes as d goes to zero.

We have also extended the analysis to consider the effect of a perpendicular magnetic field on the transition temperature and derived certain simple scaling laws to describe the upper critical field. The scaling laws are expressed as

$$t + h = \frac{e^{-\gamma}}{2}(\frac{d}{L})^{(1-\mathcal{D})} \qquad L << \xi ,$$
$$t + h = (\frac{d}{\xi_0})^{\frac{2(1-\mathcal{D})}{1+\mathcal{D}}} \qquad L >> \xi .$$

In both of the cases, the dependence of the critical magnetic field on the temperature t is linear as found in other theories.

Appendix A:

Eilenberger's Equation

Based on the linearized self-consistent equation (2-2-4) for the order parameter, the de Gennes-Werthamer theory is valid only when the temperature is very close to the critical transition temperature T_c. To study the effect of the external magnetic field on the inhomogeneous system, a more general theory, but simpler than solving the Gor'kov equations, is needed. Such a well known theory was established by Eilenberger[15] [16]. We will discuss the theory briefly by mainly following Eilenberger's works. Eilenberger introduced the gauge-invariant Green functions

$$\begin{cases} G_\alpha(t,\vec{x},\vec{x}') = -i < T(\hat{\psi}_\alpha(\vec{x},t)\hat{\psi}^\dagger_\alpha(\vec{x}',0)) > e^{-iI(\vec{x},\vec{x}')} , \\ G^\dagger_\alpha(t,\vec{x},\vec{x}') = i < T(\hat{\psi}^\dagger_\alpha(\vec{x},t)\hat{\psi}_\alpha(\vec{x}',0)) > e^{iI(\vec{x},\vec{x}')} , \end{cases} \tag{A-1}$$

and the anomalous Green's functions

$$\begin{cases} F(t,\vec{x},\vec{x}') = -i < T(\hat{\psi}_\downarrow(\vec{x},t)\hat{\psi}_\uparrow(\vec{x}',0)) > e^{iI(\vec{x},\vec{x}')} , \\ F^\dagger(t,\vec{x},\vec{x}') = i < T(\hat{\psi}^\dagger_\downarrow(\vec{x},t)\hat{\psi}^\dagger_\uparrow(\vec{x}',0)) > e^{-iI(\vec{x},\vec{x}')} , \end{cases} \tag{A-2}$$

where

$$< \cdots >= T_r(e^{-\beta H}\cdots); \qquad \hat{\psi}(\vec{x},t) = e^{itK_{eff}}\hat{\psi}(\vec{x})e^{-itK_{eff}} , \tag{A-3}$$

and

$$I(\vec{x},\vec{x}') = \frac{e}{c}\int_{\vec{x}'}^{\vec{x}} \vec{A}(\vec{x})\cdot d^3x . \tag{A-3}$$

To make the notations used in previous section coincide with those in Eilenberger's works, we have adopted the definition of the transformation from Schrödinger field to Heisenberg field (A-3) in which the time t is real rather than imaginary and also $h = 1$. The operator $\hat{K}_{eff}$ is defined by (2-1-30).

The phase factors are introduced to reduce the influence of gauge transformation. H is the effective BCS hamiltonian in the form of the second quantization,

$$H = \int d^3x \hat{\psi}^\dagger_\alpha(\vec{x}) H_0(\vec{x}) \hat{\psi}_\alpha(\vec{x}) - \Delta(\vec{x}) \hat{\psi}^\dagger_\uparrow(\vec{x}) \hat{\psi}^\dagger_\downarrow(\vec{x}) - \Delta^\dagger(\vec{x}) \hat{\psi}_\downarrow(\vec{x}) \hat{\psi}_\uparrow(\vec{x}) \,, \quad \text{(A-4)}$$

with

$$H_0 = \frac{1}{2m}(-i\nabla - \frac{e}{c}\vec{A}(\vec{x}))^2 - \mu + U(\vec{x}) \,. \quad \text{(A-5)}$$

Here we have chosen $h = 1$ and $U(\vec{x})$ is the impurity potential. It is obvious that a gauge transformation

$$\frac{e}{c}\vec{A}(\vec{x}) + \nabla\chi(\vec{x}) \,, \quad \text{(A-6)}$$

results in

$$\hat{\psi}(\vec{x}) \rightarrow e^{i\chi(\vec{x})}\hat{\psi}(\vec{x}) \,. \quad \text{(A-7)}$$

Hence, the normal Green's functions defined above will not be affected by the gauge transformation but the anomalous ones will be multiplied by factors $e^{2i\chi(\vec{x})}$ and $e^{-2i\chi(\vec{x})}$ respectively.

The Gor'kov equation may be written as

$$\begin{pmatrix} i\omega + \mu - \frac{1}{2m}(-i\nabla - \frac{e}{c}\vec{A}(\vec{x}) \pm \vec{K})^2 - U(\vec{x}) & \Delta(\vec{x}) \\ -\Delta^\dagger(\vec{x}) & -i\omega + \mu - \frac{1}{2m}(-i\nabla + \frac{e}{c}\vec{A}(\vec{x}) \pm \vec{K})^2 - U(\vec{x}) \end{pmatrix} \hat{G}_\omega(\vec{x}, \vec{x}') = \hat{I}\delta(\vec{x} - \vec{x}') \,, \quad \text{(A-8)}$$

where

$$\vec{K} = e^{-il(\vec{x},\vec{x}')}(-i\nabla)e^{il(\vec{x},\vec{x}')} \,, \quad \text{(A-9)}$$

and $\hat{G}_\omega(\vec{x}, \vec{x}')$ is the Fourier transformation of the Nambu matrix,

$$\hat{G}(t, \vec{x}, \vec{x}') = \begin{pmatrix} G(t, \vec{x}, \vec{x}') & F(t, \vec{x}, \vec{x}') \\ F^\dagger(t, \vec{x}, \vec{x}') & G^\dagger(t, \vec{x}, \vec{x}') \end{pmatrix} \,, \quad \text{(A-10)}$$

$$G_\omega(\vec{x}, \vec{x}') = \frac{1}{2}\int_{i\beta}^{-i\beta} \hat{G}(t, \vec{x}, \vec{x}')e^{-\omega t}\, dt \,. \quad \text{(A-11)}$$

The sign in front of $\vec{K}$ depends on whether it operates on the first or second column of $\hat{G}$. Since the solutions of $\hat{G}$ depend on the order parameter Δ and $\Delta^\dagger$, we can separate the influence of the spatial variation of the order parameter from the Green's function by considering the following equation

$$\begin{pmatrix} i\omega+\mu-\frac{1}{2m}(-i\nabla-i\hat{\partial}_{\vec{r}_0})^2-U(\vec{x}) & \Delta(\vec{r}_0) \\ -\Delta^\dagger(\vec{x}_0) & -i\omega+\mu-\frac{1}{2m}(-i\nabla-i\hat{\partial}_{\vec{r}_0})^2-U(\vec{x}) \end{pmatrix}\hat{G}_\omega(\vec{x}_0,\vec{x},\vec{x}') = \hat{I}\delta(\vec{x}-\vec{x}') , \tag{A-12}$$

where $\hat{\partial}_{\vec{r}_0}$ is Eilenberger's gauge-invariant differentiation

$$\hat{\partial}_{\vec{r}_0} = \begin{cases} -i\nabla - 2\frac{e}{c}\vec{A}(\vec{x}_0) & \text{when working on } \Delta(\vec{x}_0) , \\ -i\nabla + 2\frac{e}{c}\vec{A}(\vec{x}_0) & \text{when working on } \Delta^\dagger(\vec{x}_0) , \\ -i\nabla & \text{when working on some function of } |\Delta(\vec{x}_0)|^2 . \end{cases} \tag{A-13}$$

The Fourier transformation of spatial coordinates yields

$$\begin{pmatrix} i\omega+\mu-\frac{k^2}{2m}, & \Delta(\vec{x}) \\ -\Delta^\dagger(\vec{x}), & -i\omega+\mu-\frac{k^2}{2m} \end{pmatrix}\hat{G}_\omega(\vec{x},\vec{k},\vec{k}')- \\ -\int\frac{d^3q}{(2\pi)^3}U(\vec{k}-\vec{q})\hat{G}_\omega(\vec{x},\vec{q},\vec{k}) - \hat{I}(2\pi)^3\delta(\vec{k}-\vec{k}') \\ = \left(\frac{\vec{k}}{m}\cdot\frac{\hat{\partial}_{\vec{x}}}{i}+\frac{1}{2m}(\frac{\hat{\partial}_{\vec{x}}}{i}\cdot)^2\right)\hat{G}_\omega(\vec{x},\vec{q},\vec{k}) , \tag{A-14}$$

where $U(\vec{q})$ is the Fourier component of the impurity potential. Assuming that the order parameter varies slowly in space enables us to neglect the term $(\frac{\partial_x}{i})^2$. Then the final version of Gor'kov's equations with all the standard approximation become

$$\begin{pmatrix} i\omega+\mu-\frac{k^2}{2m}-\frac{\vec{k}}{m}\cdot\frac{\partial_{\vec{x}}}{i}, & \Delta(\vec{x}) \\ -\Delta^\dagger(\vec{x}), & -i\omega+\mu-\frac{k^2}{2m}-\frac{\vec{k}}{m}\cdot\frac{\partial_{\vec{x}}}{i} \end{pmatrix}\hat{G}_\omega(\vec{x},\vec{k},\vec{k}')- \\ -\int\frac{d^3q}{(2\pi)^3}U(\vec{k}-\vec{q})\hat{G}_\omega(\vec{x},\vec{q},\vec{k}) = \hat{I}(2\pi)^3\delta(\vec{k}-\vec{k}') . \tag{A-15}$$

The standard average procedure[48] over impurity potential results in

$$\bar{\hat{G}}_\omega(\vec{x},\vec{k},\vec{k}') = (2\pi)^3\delta(\vec{k}-\vec{k}')\bar{G}_\omega(\vec{x},\vec{k}) \\ = (2\pi)^3\delta(\vec{k}-\vec{k}')\{G_0(\omega,\vec{x},\vec{k})+G_1(\omega,\vec{x},\vec{k})+G_2(\omega,\vec{x},\vec{k})+\cdots\} , \tag{A-16}$$

where the indices $0, 1, 2, \cdots$ denote the power of $(\hat{\partial}_{\vec{x}}/i)$ contained in the corresponding Green's functions. The first term $\bar{G}_0$ satisfies Dyson's equation or

$$\begin{aligned}\bar{G}_0^{-1}(\omega,\vec{x},\vec{k}) &= G_{00}^{-1}(\omega,\vec{x},\vec{k}) - n\int \frac{d^3q}{(2\pi)^3}|u(\vec{k}-\vec{q})|^2 G_0(\omega,\vec{x},\vec{q}) \\ &= G_{00}^{-1}(\omega,\vec{x},\vec{k}) - \Sigma(\omega,\vec{x},\vec{k}) \ ,\end{aligned} \tag{A-17}$$

where n is the impurity density and $G_{00}^{-1}(\omega,\vec{x},\vec{k})$ is the solution of equation (A-14) in the case $U = 0$. From the Dyson equation, the solution for G_0 may be written as

$$\begin{aligned}\bar{G}_0(\omega,\vec{x},\vec{k}) =& G_{00}(\omega,\vec{x},\vec{k}) + \\ &+ G_{00}(\omega,\vec{x},\vec{k}) n \int \frac{d^3q}{(2\pi)^3}|u(\vec{k}-\vec{q})|^2 \bar{G}_0(\omega,\vec{x},\vec{q}) G_0(\omega,\vec{x},\vec{k}) \ .\end{aligned} \tag{A-18}$$

The well known solution is

$$\bar{G}_0(\omega,\vec{x},\vec{k}) = \frac{\hat{I}\zeta + \eta(\omega,\vec{x}) \begin{pmatrix} -i\omega & -\Delta(\vec{x}) \\ \Delta^\dagger(\vec{x}) & i\omega \end{pmatrix}}{\eta^2(\omega,\vec{x})(|\Delta(\vec{x})|^2 + \omega^2) + \zeta^2} \ , \tag{A-19}$$

with

$$\begin{aligned}\zeta &= \mu - \frac{k^2}{2m} \ , \\ \eta^2(\omega,\vec{x}) &= 1 + \frac{1}{2\tau\sqrt{|\Delta(\vec{x})|^2 + \omega^2}} \ ,\end{aligned} \tag{A-20}$$

where τ depends on the impurity density n, Fermi velocity $v_F = \sqrt{\frac{2\mu}{m}}$ and an integral over $|u(\vec{k}-\vec{q})|^2$. Since we are only interested in the scattering effects of those electrons at Fermi surface on the impurity atoms, the momentum variation of the electrons is limited in a range near the Fermi surface. Thus, another approximation is made that

$$\begin{cases} |u(\vec{k}-\vec{q})|^2 \approx |u(\vec{k}_F - \vec{q}_F)|^2 \ , \\ \dfrac{1}{\tau} = 2\pi N(0) n \displaystyle\int \frac{d^2\vec{q}_F^{\,0}}{4\pi} |u(\vec{k}_F - \vec{q}_F)|^2 \ . \end{cases} \tag{A-21}$$

where $N(0) = \frac{mk_F}{2\pi^2}$ is the density of states at Fermi surface for one spin projection . Iterating equation (A-15) gives the higher order terms of $\hat{G}$:

$$\begin{cases} \bar{G}_1 = \bar{G}_0 \vec{v}_F \cdot \frac{\hat{\partial}_{\vec{x}}}{i} \bar{G}_0 + \text{corr} , \\ \bar{G}_2 = \bar{G}_0 \vec{v}_F \cdot \frac{\hat{\partial}_{\vec{x}}}{i} \bar{G}_0 \vec{v}_F \cdot \frac{\hat{\partial}_{\vec{x}}}{i} \bar{G}_0 + \text{corr} , \\ \cdots \end{cases} \tag{A-22}$$

where *corr.* means the vertices correction. Now we define the operator $\hat{H}_0$ as

$$\hat{H}_0 = \begin{pmatrix} i\omega - \zeta - \vec{v}_F \cdot \frac{\partial_{\vec{x}}}{i} & \Delta(\vec{x}) \\ -\Delta(\vec{x}) & -i\omega - \zeta - \vec{v}_F \cdot \frac{\partial_{\vec{x}}}{i} \end{pmatrix} , \tag{A-23}$$

and we have

$$\hat{H}_0 G_{00}(\omega, \vec{x}, \vec{k}) = \hat{I} . \tag{A-24}$$

We only consider the lowest order term of $\hat{G}$, then the Gor'kov equation under the lowest order approximation of $(\frac{\partial_{\vec{x}}}{i})$ may be written as

$$\left\{ \begin{pmatrix} i\omega - \zeta - \vec{v}_F \cdot \frac{\partial_{\vec{x}}}{i} & \Delta(\vec{x}) \\ -\Delta(\vec{x}) & -i\omega - \zeta - \vec{v}_F \cdot \frac{\partial_{\vec{x}}}{i} \end{pmatrix} - \hat{\Sigma}^{OP}(\omega, \vec{x}, \vec{k}_F) \right\} \bar{G}_0^{OP}(\omega, \zeta, \vec{x}, \vec{k}) = \hat{I} , \tag{A-25}$$

where

$$\begin{aligned} \hat{\Sigma}^{OP}(\omega, \vec{x}, \vec{k}_F) &= n \int \frac{d^3q}{(2\pi)^3} |u(\vec{k} - \vec{q})|^2 \bar{G}_0^{OP}(\omega, \vec{x}, \vec{q}_F) \\ &= 2\pi N(0) n \int \frac{d^2\vec{q}_F^0}{4\pi} |u(\vec{k}_F - \vec{q}_F)|^2 \oint \frac{d\zeta}{2\pi} \bar{G}_0^{OP}(\omega, \zeta, \vec{x}, \vec{q}_F) \\ &= \int \frac{d^2\vec{q}_F^0}{4\pi} W(\vec{k}_F - \vec{q}_F) \oint \frac{d\zeta}{2\pi} \bar{G}_0^{OP}(\omega, \zeta, \vec{x}, \vec{q}_F) , \end{aligned} \tag{A-26}$$

with

$$W(\vec{k}_F - \vec{q}_F) = 2\pi N(0) n |u(\vec{k}_F - \vec{q}_F)|^2 . \tag{A-27}$$

Note that the functions $\bar{G}$ and $\hat{\Sigma}$ have been replaced by operators since $\hat{H}_0$ is a function of $(\frac{\partial_{\vec{x}}}{i})$.

Completing the tedious derivation of the operator algebra[16] we may have the equation of motion for simplified Green's function defined by

$$\begin{cases} g(\omega,\vec{k}_F,\vec{x}) = i\oint \frac{d\zeta}{2\pi}\bar{G}(\omega,\zeta,\vec{k}_F,\vec{x}) , \\ f(\omega,\vec{k}_F,\vec{x}) = -\oint \frac{d\zeta}{2\pi}\bar{F}(\omega,\zeta,\vec{k}_F,\vec{x}) , \\ f^{\dagger}(\omega,\vec{k}_F,\vec{x}) = \oint \frac{d\zeta}{2\pi}\bar{F}^{\dagger}(\omega,\zeta,\vec{k}_F,\vec{x}) . \end{cases} \tag{A-28}$$

The equations are

$$\begin{aligned} &\{2\omega + \vec{v}_F\cdot(\nabla - i\frac{2e}{c}\vec{A}(\vec{x})\}f(\omega,\vec{k}_F,\vec{x}) \\ &= 2\Delta(\vec{x})g(\omega,\vec{k}_F,\vec{x}) + \int_{S_F}\frac{d^2\vec{q}_F^0}{4\pi}W(\vec{k}_F-\vec{q}_F) \\ &\quad\cdot\{g(\omega,\vec{k}_F,\vec{x})f(\omega,\vec{q}_F,\vec{x}) - f(\omega,\vec{k}_F,\vec{x})g(\omega,\vec{q}_F,\vec{x})\} , \end{aligned} \tag{A-29}$$

$$\begin{aligned} &\{2\omega - \vec{v}_F\cdot(\nabla + i\frac{2e}{c}\vec{A}(\vec{x})\}f^{\dagger}(\omega,\vec{k}_F,\vec{x}) \\ &= 2\Delta^{\dagger}(\vec{x})g(\omega,\vec{k}_F,\vec{x}) + \int_{S_F}\frac{d^2\vec{q}_F^0}{4\pi}W(\vec{k}_F-\vec{q}_F) \\ &\quad\cdot\{g(\omega,\vec{k}_F,\vec{x})f^{\dagger}(\omega,\vec{q}_F,\vec{x}) - f^{\dagger}(\omega,\vec{k}_F,\vec{x})g(\omega,\vec{q}_F,\vec{x})\} , \end{aligned} \tag{A-30}$$

and

$$g(\omega,\vec{q}_F,\vec{x}) = (1 - f(\omega,\vec{k}_F,\vec{x})f^{\dagger}(\omega,\vec{q}_F,\vec{x}))^{\frac{1}{2}} . \tag{A-31}$$

The system of equations is completed by the self-consistency condition

$$\Delta(\vec{x}) = \lambda < \hat{\psi}_{\uparrow}(\vec{x})\hat{\psi}_{\downarrow}(\vec{x}) > , \tag{A-32}$$

where λ is the Gor'kov coupling constant and may be replaced by the standard cutoff procedure

$$\frac{1}{\lambda N(0)} \to \ln(\frac{T}{T_c}) + \frac{\pi}{\beta}\sum_{\omega}\frac{1}{|\omega|} . \tag{A-33}$$

The explicit expression of $\Delta(\vec{x})$ in terms of the simplified Green's functions is given by

$$\Delta(\vec{x})\ln(\frac{T}{T_c}) + \frac{2\pi}{\beta}\sum_{\omega}\frac{\Delta(\vec{x})}{|\omega|} - \int_{S_F}\frac{d^2\vec{k}_F^0}{4\pi}f(\omega,\vec{k}_F,\vec{x}) = 0 . \tag{A-34}$$

The same procedure gives rise to the equation for the current density $\vec{j}(\vec{x})$

$$\frac{1}{4\pi}\nabla\times(\vec{B}(\vec{x})-\vec{B}_e(\vec{x}))+\frac{2ic}{c}\frac{2\pi N(0)}{\beta}\sum_{\omega}\int_{S_F}\frac{d^2\vec{k}_F^0}{4\pi}\vec{v}_F g(\omega,\vec{k}_F,\vec{x})=0\,. \quad \text{(A-35)}$$

Compared with the original Gor'kov equation, the Eilenberger's equations are much easier to handle since the number of the variables is reduced to two from four. For small order parameter $\Delta(\vec{x})$, Eilenberger equation reduces to the Boltzmann transport-like equation introduced by Lüders[49] which, in the dirty limit, is very close to de Gennes's diffusion equation coming from the linearized gap equation.

Eilenberger's Equation the in Dirty Limit

Instead of considering the dirty limit in a linearized equation, Usadel [37] considered the dirty limit form of the Eilenberger equations which may be written as

$$\begin{cases} 2\omega F(\omega,\vec{r})-D\hat{\partial}[G(\omega,\vec{r})\hat{\partial}F(\omega,\vec{r})+\dfrac{1}{2}\dfrac{F(\omega,\vec{r})}{G(\omega,\vec{r})}\nabla|F(\omega,\vec{r})|^2]=2\Delta(\vec{r})G(\omega,\vec{r})\,, \\ \Delta(\vec{r})\ln(\dfrac{T}{T_c})+2\pi T\sum_{\omega>0}[\dfrac{\Delta(\vec{r})}{\omega}-F(\omega,\vec{r})]=0\,, \\ \vec{j}(\vec{r})=2icN(0)\pi TD\sum_{\omega>0}[F^{\dagger}(\omega,\vec{r})\hat{\partial}F(\omega,\vec{x})-F(\omega,\vec{r})(\hat{\partial}F(\omega,\vec{r}))^{\dagger}] \end{cases} \quad \text{(A-36)}$$

where $\hat{\partial}=\nabla+2ie\vec{A}(\vec{r})$ and $D=\frac{1}{3}v_F l$. These equations are valid for arbitrary Δ provided that the dirty limit holds,i.e.

$$G>>2\tau\omega, \qquad F>>2\tau\Delta, \qquad |F|>>|\frac{\vec{v}}{v}\cdot\vec{F}|\,. \quad \text{(A-37)}$$

These conditions were used in deriving (A-36) when expanding $f(\omega,\vec{r},\vec{v})$:

$$\begin{cases} f(\omega,\vec{r},\vec{v})=F(\omega,\vec{r})+\dfrac{\vec{v}}{v}\cdot\vec{F}(\omega,\vec{r})\,, \\ g(\omega,\vec{r},\vec{v})=G(\omega,\vec{r})+\dfrac{\vec{v}}{v}\cdot\vec{G}(\omega,\vec{r})\,, \end{cases} \quad \text{(A-38)}$$

where f and g are the Green's functions in Eilenberger's equations.

Application of Eilenberger's Equation to Superlattice

The dirty-limit version of the Eilenberger theory is particularly useful in studying the upper critical field $H_{c2}(T)$ of an inhomogeneous superconducting structure. Biagi [17] and co-workers applied the equation to the SN multilayers case with the boundary conditions:

$$\begin{cases} F_S = F_N \,, \\ \dfrac{\partial F_S}{\partial z} = \eta \dfrac{\partial F_N}{\partial z} \end{cases} \qquad \text{at SN interfaces} \,. \tag{A-39}$$

For the perpendicular upper critical field, the result may be summarized to

$$\begin{cases} \ln(t_i) = \hat{\psi}(\dfrac{1}{2}) - \hat{\psi}(\dfrac{1}{2} + \dfrac{y(t_i)}{2}) \,, \\ k_S^2 = 2\pi T y(t_S)/hD_S \,, \\ k_N^2 = -2\pi T y(t_N)/hD_N \,, \\ q_S \tan(q_S d_S/2) = \eta_{1N} \tanh(q_N d_N/2) \,, \\ q_S^2 = k_S^2 - \dfrac{2\pi H_{c2}}{\Phi_0} \,, \\ q_N^2 = k_N^2 + \dfrac{2\pi H_{c2}}{\Phi_0} \,, \end{cases} \tag{A-40}$$

where $t_i = \frac{T}{T_{ci}}, (i = S, N)$ is the reduced temperature and $y(t)$ is the function of the reduced temperature determined by the eigenvalue k_S and k_N.

This set of equations completely describes the $H_{c2\perp}(T)$. In general, they must be solved numerically. But at the critical temperature of the superlattice, the upper critical field vanishes, the equations reduce to the result of de Gennes, Werthamer and co-workers for a dirty SN proximity system.

$$k_S \tan(k_S d_S/2) = \eta k_N \tanh(k_N d_N/2) \,. \tag{A-41}$$

Appendix B:

Relations among the Coefficients

Define:

$$x_i = \delta_i \qquad y_i = (1 + \gamma_i) .$$

The boundary conditions yield the following equations

$$\begin{cases} Ay_1 + Bx_1 = Cy_2 + Dx_2 , \\ Ax_1 + By_1 = Cx_2 + Dy_2 , \\ Ay_1 e^{-iap_1} + Bx_1 e^{iap_1} = (Cy_2 e^{ibp_2} + Dx_2 e^{-ibp_2}) e^{-idq} , \\ Ax_1 e^{-iap_1} + By_1 e^{iap_1} = (Cx_2 e^{ibp_2} + Dy_2 e^{-ibp_2}) e^{-idq} . \end{cases} \tag{B-1}$$

$(B-1-1)x_2 - (B-1-2)y_2 .\ (B-1-1)y_2 - (B-1-2)x_2 \ldots .$

$$\begin{cases} A(y_1 x_2 - x_1 y_2) + B(x_1 x_2 - y_1 y_2) = D(x_2^2 - y_2^2) , \\ A(y_1 y_2 - x_1 x_2) + B(x_1 y_2 - y_1 x_2) = C(y_2^2 - x_2^2) , \\ A(y_1 x_2 - x_1 y_2) e^{-iap_1} + B(x_1 x_2 - y_1 y_2) e^{iap_1} = D(x_2^2 - y_2^2) e^{-ibp_2} e^{-idq} , \\ A(y_1 y_2 - x_1 x_2) e^{-iap_1} + B(x_1 y_2 - y_1 x_2) e^{iap_1} = C(y_2^2 - x_2^2) e^{ibp_2} e^{-idq} . \end{cases} \tag{B-2}$$

Now we define:

$$u = (y_1 y_2 - x_1 x_2) , \qquad v = -(y_1 x_2 - x_1 y_2) , \qquad z_i = (y_i^2 - x_i^2) = 2\gamma_i y_i ,$$

we have

$$\begin{cases} Av + Bu = Dz_2 , \\ Au + Bv = Cz_2 , \\ Ave^{-iap_1} + Bue^{iap_1} = Dz_2 e^{-i(bp_2 + qd)} , \\ Aue^{-iap_1} + Bve^{iap_1} = Cz_2 e^{i(bp_2 - qd)} . \end{cases} \tag{B-3}$$

Eliminating C and D from (B-3) yields

$$\begin{cases} A(y_1x_2 - x_1y_2)[e^{-iap_1} - e^{-i(bp_2+qd)}] = B(x_1x_2 - y_1y_2)[e^{-i(bp_2+qd)} - e^{iap_1}] , \\ A(y_1y_2 - x_1x_2)[e^{-iap_1} - e^{i(bp_2-qd)}] = B(x_1y_2 - y_1x_2)[e^{i(bp_2-qd)} - e^{iap_1}] . \end{cases} \tag{B-4}$$

so the coefficients A and B can be solved as

$$|A|^2[e^{iap_1} - e^{i(bp_2+qd)}][e^{-iap_1} - e^{i(bp_2-qd)}]$$
$$= |B|^2[e^{-iap_1} - e^{i(bp_2+qd)}][e^{iap_1} - e^{i(bp_2-qd)}] ,$$

or

$$|A|^2[\cos(bp_2) - \cos(dq - ap_1)] = |B|^2[\cos(bp_2) - \cos(dq + ap_1)] , \tag{B-5}$$

and

$$A^*B[e^{iap_1} - e^{i(bp_2+qd)}][e^{iap_1} - e^{i(bp_2-qd)}]v^2$$
$$= B^*A[e^{-iap_1} - e^{i(bp_2+qd)}][e^{-iap_1} - e^{i(bp_2-qd)}]u^2 ,$$

or

$$A^*Be^{iap_1}v^2[\cos(dq) - \cos(bp_2 - ap_1)] = B^*Ae^{-iap_1}u^2[\cos(dq) - \cos(bp_2 + ap_1)] . \tag{B-6}$$

We may also eliminate coefficient B from (B-3) to obtain the relations among A, C and D

$$\begin{cases} A(v^2 - u^2) = z_2(Dv - Cu) , \\ A(v^2 - u^2)e^{-iap_1} = z_2e^{-idq}(Dve^{-ibp_2} - Cue^{ibp_2}) . \end{cases} \tag{B-7}$$

Using the relation

$$v^2 - u^2 = -z_1z_2 ,$$

we find that

$$\begin{cases} C = \dfrac{-Az_1}{2iu\sin(bp_2)}[e^{-ibp_2} - e^{i(dq-ap_1)}] , \\ D = \dfrac{-Az_1}{2iv\sin(bp_2)}[e^{ibp_2} - e^{i(dq-ap_1)}] . \end{cases} \tag{B-8}$$

Energy-Momentum Spectrum

The nontrivial solution condition from (B-4) requires that

$$(y_1x_2 - x_1y_2)[e^{-ia p_1} - e^{-i(bp_2+qd)}](x_1y_2 - y_1x_2)[e^{i(bp_2-qd)} - e^{ia p_1}] -$$
$$- (x_1x_2 - y_1y_2)[e^{-ia p_1} - e^{i(bp_2-qd)}](y_1y_2 - x_1x_2)[e^{-i(bp_2+qd)} - e^{ia p_1}] = 0 . \tag{B-9}$$

Using

$$(y_1y_2 - x_1x_2)^2 - (x_1y_2 - y_1x_2)^2 = 4\gamma_1\gamma_2(1+\gamma_1)(1+\gamma_2) ,$$
$$(y_1y_2 - x_1x_2)^2 + (x_1y_2 - y_1x_2)^2 = 4(1+\gamma_1)(1+\gamma_2)[1-\delta_1\delta_2] ,$$

the energy-momentum spectrum may expressed as

$$\cos(dq) = \cos(bp_2)\cos(ap_1) - \frac{(1-\delta_1\delta_2)}{\gamma_1\gamma_2}\sin(ap_1)\sin(bp_2) , \tag{B-10}$$

Normalization

$$\begin{aligned}
\int_{-\infty}^{\infty} u^*(E',x)u(E,x)\,dx &= 2\pi\delta(E-E')\int_{-a}^{b} u^*(E',x)u(E,x)\,dx \sum_n e^{i(q-q')nd} \\
&= 2\pi\delta(E-E')\int_{-a}^{b} u^*(E',x)u(E,x)\,dx\, N\delta_{qq'} \\
&= 2\pi\delta(E-E')\int_{-a}^{b} u^*(E',x)u(E,x)\,dx\, N\frac{2\pi}{V}\delta(q-q') \\
&= 2\pi\delta(E-E')\frac{1}{\Omega}\int_{-a}^{b} u^*(E',x)u(E,x)\,dx \\
&= |\frac{dq}{dE}| .
\end{aligned} \tag{B-11}$$

where $\Omega = d = a+b$ is the volume for one cell. The normalization may be completed by using (B-10). In the limit of Δ_1 and $\Delta_2 \to 0$, the energy-momentum spectrum is simplified to

$$\cos(dq) = \cos(ap_1 + bp_2) = \cos(\frac{mEd}{Q}) . \tag{B-12}$$

Substituting (B-12) into (B-11) and using the relations of the coefficients, we obtain

$$
\begin{aligned}
|A| &= \frac{1}{2}\sqrt{\frac{m}{Q}} \qquad q > 0\,, \\
|B| &= \frac{1}{2}\sqrt{\frac{m}{Q}} \qquad q < 0\,.
\end{aligned}
\tag{B-13}
$$

Appendix C:

Calculation of $F(\vec{x})$

$$F(\vec{x}) = - < \hat{\Psi}(\vec{x})\hat{\Psi}^{\dagger}(\vec{x}) >_{12} , \tag{C-1}$$

$$\begin{aligned}
< \hat{\Psi}(\vec{x})\hat{\Psi}^{\dagger}(\vec{x}) > &= \int \frac{dE}{2\pi}\frac{dE'}{2\pi}\frac{dl^2}{(2\pi)^2}\frac{dl'^2}{(2\pi)^2} \\
&< \{\Phi(E,l,x)e^{i\vec{l}\cdot\vec{\rho}}a(E,l) + \Phi(-E,l,x)e^{-i\vec{l}\cdot\vec{\rho}}b^{\dagger}(E,l)\} \\
&\{\Phi^*(E',l',x)e^{-i\vec{l}'\cdot\vec{\rho}}a^{\dagger}(E',l') + \Phi^*(-E',l',x)e^{-i\vec{l}'\cdot\vec{\rho}}b(E',l')\} > \\
&= \int \frac{dE}{2\pi}\frac{dE'}{2\pi}\frac{dl^2}{(2\pi)^2}\frac{dl'^2}{(2\pi)^2} \\
&\{\Phi(E,l,x)\Phi^*(E',l',x)e^{i(\vec{l}-\vec{l}')\cdot\vec{\rho}} < a(E,l)a^{\dagger}(E',l') > \\
&+ \Phi(-E,l,x)\Phi^*(-E',l',x)e^{-i(\vec{l}-\vec{l}')\cdot\vec{\rho}} < b^{\dagger}(E,l)b(E',l') >\} \\
&= \int \frac{dE}{2\pi}\frac{dl^2}{(2\pi)^2} \\
&\{\Phi(E,l,x)\Phi^*(E,l,x)\frac{e^{\beta E}}{1+e^{\beta E}} + \Phi(-E,l,x)\Phi^*(-E,l,x)\frac{1}{1+e^{\beta E}} ,
\end{aligned} \tag{C-2}$$

where the thermal averages of the operators , according to Fermi-Dirac statistics, are expressed as

$$\begin{aligned}
< a(E,l)a^{\dagger}(E',l') > &= \frac{e^{\beta E}}{1+e^{\beta E}}2\pi\delta(E-E')(2\pi)^2\delta(\vec{l}'-\vec{l}) , \\
< b^{\dagger}(E,l)b(E',l') > &= \frac{1}{1+e^{\beta E}}2\pi\delta(E-E')(2\pi)^2\delta(\vec{l}'-\vec{l}) .
\end{aligned} \tag{C-3}$$

From the anti-commutator relation of the Nambu doublet

$$\{\hat{\Psi}(\vec{x}), \hat{\Psi}^{\dagger}(\vec{x}')\}_{ij} = \delta_{ij}\delta_{(\vec{x}-\vec{x}')} , \tag{C-4}$$

we may have

$$\{\hat{\Psi}(\vec{x}),\hat{\Psi}^{\dagger}(\vec{x}')\} = \int \frac{dE}{2\pi}\frac{dE'}{2\pi}\frac{dl^2}{(2\pi)^2}\frac{dl'^2}{(2\pi)^2}$$
$$[\Phi(E,l,x)\Phi^*(E',l',x)e^{i(\vec{l}-\vec{l}')\cdot\vec{r}}\{a(E,l),a^{\dagger}(E',l')\}$$
$$+\Phi(-E,l,x)\Phi^*(-E',l',x)e^{-i(\vec{l}-\vec{l}')\cdot\vec{r}}\{b^{\dagger}(E,l),b(E',l')\}]$$
$$\{\hat{\Psi}(\vec{x}),\hat{\Psi}^{\dagger}(\vec{x}')\} = \int \frac{dE}{2\pi}\frac{dl^2}{(2\pi)^2}\{\Phi(E,l,x)\Phi^*(E,l,x)+\Phi(-E,l,x)\Phi^*(-E,l,x)\}\ . \tag{C-5}$$

The vanishing of the 1-2 component of the anticommutator

$$\{\hat{\Psi}(\vec{x}),\hat{\Psi}^{\dagger}(\vec{x}')\}_{12} = 0\ ,$$

yields

$$[\Phi(E,l,x)\Phi^*(E,l,x)]_{12} = -[\Phi(-E,l,x)\Phi^*(-E,l,x)]_{12}\ . \tag{C-6}$$

Thus

$$<\hat{\Psi}(\vec{x})\hat{\Psi}^{\dagger}(\vec{x}')>_{12} = \int \frac{dE}{2\pi}\frac{dl^2}{(2\pi)^2}[\Phi(E,l,x)\Phi^*(E,l,x)]_{12}\tanh(\frac{\beta E}{2})\ . \tag{C-7}$$

The Gap Equation

Now we can calculate the order parameter self-consistently

$$\frac{\Delta_2}{V_2} = \frac{1}{b}\int_0^b dx\, F(\vec{x})$$
$$= \frac{-1}{b}\int_0^b dx\ <\hat{\Psi}(\vec{x})\hat{\Psi}^{\dagger}(\vec{x}')>_{12} \tag{C-8}$$
$$= \frac{-1}{b}\int \frac{dE}{2\pi}\frac{dl^2}{(2\pi)^2}\tanh(\frac{\beta E}{2})\int_0^b dx\,[\Phi(E,l,x)\Phi^*(E,l,x)]_{12}\ .$$

The integral over x can be completed by using the results in Appendix B

$$\int_0^b dx\,[\Phi(E,l,x)\Phi^*(E,l,x)]_{12}$$
$$= \int_0^b dx\,[u_N(E,l,x)u_N^*(E,l,x)]_{12} \tag{C-9}$$
$$= b\delta_2(1+\gamma_2)[|C|^2+|D|^2]+\frac{1}{p_2}\sin(bp_2)\{CD^*e^{ibp_2}2(1+\gamma_2)\}\ .$$

Substituting $|C|^2, |D|^2$, and $CD^* e^{ibp_2}$ into (C-8) and taking the limit $\Delta_1, \Delta_2 \to 0$ results in

$$\int_0^b dx\,[\Phi(E,l,x)\Phi^*(E,l,x)]_{12} = \frac{b}{2}\frac{m}{Q}[\delta_2 - \frac{(\delta_2-\delta_1)\sin(ak)\sin(bk)}{bk\sin(dk)}]\,. \qquad \text{(C-10)}$$

The gap equation, which determines the transition temperature, can be expressed as

$$\frac{1}{V_2 N(0)} = \int_0^{\omega_D} \frac{dE}{E}\tanh(\frac{\beta E}{2})\int_0^{k_F}\frac{dQ}{k_F}[1+(\frac{\Delta_1}{\Delta_2}-1)\frac{\sin(ak)\sin(bk)}{kb\sin(dk)}]\,, \qquad \text{(C-11)}$$

where a factor 1/4 is dropped off since we have four independent solutions corresponding to $\pm Q$ and $\pm q$. It is obvious that when $\Delta_1 = \Delta_2$ the result reduces to the standard BCS formula.

The further calculation will be completed numerically, but it is worth making some simplification

$$\begin{aligned}
\frac{1}{V_2 N(0)} &= \int_0^{\omega_D}\frac{dE}{E}\tanh(\frac{\beta E}{2})\int_0^{k_F}\frac{dQ}{k_F}\{1-\frac{\sin(2bk)}{2bk}\} \\
&+ \int_0^{\omega_D}\frac{dE}{E}\tanh(\frac{\beta E}{2})\int_0^{k_F}\frac{dQ}{k_F}\frac{\sin(bk)}{bk}\{[1-\frac{\Delta_1}{\Delta_2}]\sin(bk)\cot(dk)+\frac{\Delta_1}{\Delta_2}\cos(bk)\} \\
&= \int_0^{\omega_D}\frac{dE}{E}\int_0^{k_F}\frac{dQ}{k_F}\{1-\frac{\sin(2bk)}{2bk}\} \\
&+ \int_0^{\infty}\frac{dE}{E}[\tanh(\frac{\beta E}{2})-1]\int_0^{k_F}\frac{dQ}{k_F}\{1-\frac{\sin(2bk)}{2bk}\} \\
&+ \int_0^{\omega_D}\frac{dE}{E}\tanh(\frac{\beta E}{2})\int_0^{k_F}\frac{dQ}{k_F}\frac{\sin^2(bk)}{bk}[1-\frac{\Delta_1}{\Delta_2}]\cot(dk) \\
&+ \int_0^{\omega_D}\frac{dE}{E}\tanh(\frac{\beta E}{2})\int_0^{k_F}\frac{dQ}{k_F}\frac{\sin(2bk)}{2bk}\frac{\Delta_1}{\Delta_2}\,.
\end{aligned} \qquad \text{(C-12)}$$

Define

$$I(\omega) = \frac{1}{k_F}\int_0^{\omega_D}\frac{dE}{E}\int_0^{k_F} dQ[1-\frac{\sin(2bk)}{2bk}]\,, \qquad \text{(C-13)}$$

we may have

$$\begin{aligned}
&\frac{1}{k_F}\int_0^{\omega_D}\frac{dE}{E}\tanh(\frac{\beta E}{2})\int_0^{k_F} dQ[1-\frac{\sin(2bk)}{2bk}] \\
&= I(\omega_D) - \frac{\beta}{2}\int_0^{\infty} d\omega I(\omega)[\cosh(\frac{\beta\omega}{2})]^{-2}\,.
\end{aligned} \qquad \text{(C-14)}$$

The function $I(\omega)$ may be calculated to give

$$I(\omega) = I(\Lambda\Omega) , \tag{C-15}$$

where

$$\Lambda = \frac{2k_F}{b} , \qquad \Omega = \frac{\omega}{E_F} ,$$

and

$$I(x) = \gamma + \ln(x) + \frac{1}{2}\{\frac{\sin(x)}{x} - \cos(x) - 2C_i(x) + x[\frac{\pi}{2} - S_i(x)]\} , \tag{C-16}$$

where $C_i(x)$ and $S_i(x)$ denoting the cosine and sine integrals, with γ being Euler's constant.One can show that $I(x)$ has the following limits.

$$\lim_{x\to\infty} = \gamma + \ln(x) + 0(\frac{1}{x}) ,$$

so we have

$$\frac{1}{V_2 N(0)} = \gamma + \ln(\Lambda_c \Omega_d) .$$

The equation (C-12) may be written as

$$\begin{aligned}
\ln(\frac{\Lambda_c}{\Lambda}) &= \int_0^\infty \frac{dE}{E}[\tanh(\frac{\beta E}{2}) - 1] \int_0^{k_F} \frac{dQ}{k_F}\{1 - \frac{\sin(2bk)}{2bk}\} \\
&+ \int_0^\infty \frac{dE}{E} \tanh(\frac{\beta E}{2}) \int_0^{k_F} \frac{dQ}{k_F} \frac{\sin^2(bk)}{bk}\{[1 - \frac{\Delta_1}{\Delta_2}]\cot(dk) \\
&+ \int_0^\infty \frac{dE}{E} \tanh(\frac{\beta E}{2}) \int_0^{k_F} \frac{dQ}{k_F} \frac{\sin(2bk)}{2bk} \frac{\Delta_1}{\Delta_2} \\
&= I_1 + I_2 + I_3 ,
\end{aligned} \tag{C-17}$$

where

$$\begin{aligned}
I_1 &= \int_0^\infty \frac{dz}{z}[1 - \frac{\sin(z)}{z}][\frac{\tau_1}{z}\ln\cosh(\frac{z}{\tau_1}) - 1] , \\
I_2 &= \int_0^\infty \frac{dz}{z}[1 - \frac{\delta_1}{\delta_2}]\frac{\tau_1}{z} \ln\cosh(\frac{z}{\tau_1}) \frac{\sin^2 z}{z} \cot[(1+r)z] , \\
I_3 &= \frac{\delta_1}{\delta_2}\int_0^\infty \frac{dz}{z}\frac{\tau_1}{z} \ln\cosh(\frac{z}{\tau_1}) \frac{\sin(2z)}{2z} ,
\end{aligned} \tag{C-18}$$

where

$$z = bk = \frac{bEm}{Q} \qquad r = \frac{a}{b} \qquad \tau_1 = \frac{2mb}{\beta k_F} ,$$

and then

$$\begin{aligned} \ln(\frac{\Lambda_c}{\Lambda}) = & \frac{\pi}{2}\tau_1 \ln 2 + \tau_1 \int_0^\infty \frac{dz}{z^2}[1 - \frac{\sin(2z)}{2z}] \ln[1 + e^{\frac{-2z}{\tau_1}}] \\ & + \int_0^\infty \frac{dz}{z^3}[1 - \frac{\delta_1}{\delta_2}]\tau_1 \ln\cosh(\frac{z}{\tau_1}) \sin^2 z \cot[(1+r)z] \\ & + \frac{\delta_1}{\delta_2}\frac{\tau_1}{2}\int_0^\infty \frac{dz}{z^3} \ln\cosh(\frac{z}{\tau_1}) \sin(2z) . \end{aligned} \tag{C-19}$$

Define

$$\begin{cases} I(x) = \dfrac{\pi}{2}x \ln 2 - x\displaystyle\int_0^\infty \frac{dz}{z^2}[1 - \frac{\sin(2z)}{2z}] \ln[1 + e^{\frac{-2z}{x}}] , \\ J(x,y) = x\displaystyle\int_0^\infty \frac{dz}{z^3} \ln\cosh(\frac{z}{x}) \sin^2 z \cot[(1+y)z] , \\ K(x) = \dfrac{x}{2}\displaystyle\int_0^\infty \frac{dz}{z^3} \ln\cosh(\frac{z}{x}) \sin(2z) , \end{cases} \tag{C-20}$$

we can have

$$\ln(\frac{b}{b_c}) = I(\tau_1) - [1 - \frac{\Delta_1}{\Delta_2}]J(\tau_1, r) - \frac{\Delta_1}{\Delta_2}K(\tau_1) . \tag{C-21}$$

Another equation about Δ_1 can be obtained by replacement

$$b \to a \qquad a \to b \qquad V_2 \to V_1 ,$$

$$\begin{aligned} \ln(\frac{a}{a_c}) = & \frac{\pi}{2}\tau_2 \ln 2 - \tau_2 \int_0^\infty \frac{dz}{z^2}[1 - \frac{\sin(2z)}{2z}] \ln[1 + e^{\frac{-2z}{\tau_2}}] \\ & - \int_0^\infty \frac{dz}{z^3}[1 - \frac{\delta_2}{\delta_1}]\tau_2 \ln\cosh(\frac{z}{\tau_2}) \sin^2 z \cot[(1+1/r)z] \\ & - \frac{\delta_2}{\delta_1}\frac{\tau_2}{2}\int_0^\infty \frac{dz}{z^3} \ln\cosh(\frac{z}{\tau_2}) \sin(2z) , \end{aligned} \tag{C-22}$$

$$\begin{aligned} \ln(\frac{a}{a_c}) &= \ln(\frac{b}{b_c}) + \ln(r) + \ln(\frac{T_{c2}}{T_{c1}}) \\ &= I(r\tau_1) - [1 - \frac{\Delta_2}{\Delta_1}]J(r\tau_1, 1/r) - \frac{\Delta_2}{\Delta_1}K(r\tau_1) . \end{aligned} \tag{C-23}$$

One finds

$$\frac{\Delta_1}{\Delta_2} = \frac{J(r\tau_1, 1/r) - K(r\tau_1)}{\ln(\frac{b}{b_c}) + \ln(r) + \ln(\frac{T_{c2}}{T_{c1}}) - I(r\tau_1) + J(r\tau_1, 1/r)} . \tag{C-24}$$

Substituting this result into the equation (C-21), we have the result of (3-3-1).

Appendix D:

We express the matrix defined by coefficients appearing in Eq. (4-2-28) as

$$T(m) = \begin{pmatrix} \alpha(m) & \beta(m) \\ \mu(m) & \nu(m) \end{pmatrix} , \tag{D-1}$$

and the matrix corresponding to the mapping of the structure from order m to $m+1$ as

$$P(m) = (-\tau)\left[I - \tanh\left(\kappa_b \sigma D^{(m)}\right)\begin{pmatrix} 0 & \tau \\ 1/\tau & 0 \end{pmatrix}\right] . \tag{D-2}$$

Then the recurrence relation given by Eq. (4-2-29)may be written as

$$T(m+1) = T(m) \cdot P(m) \cdot T(m) . \tag{D-3}$$

If we assume $T(m)$ to be of the form

$$T(m) = \tau\Phi(m)\begin{pmatrix} -1 & \tau\Lambda_+(m) \\ \Lambda_-(m) & -1 \end{pmatrix} , \tag{D-4}$$

then from the above expression it is straightforward to show that the matrix $T(m+1)$ is also of this form with $\Phi(m+1)$, $\Lambda_+(m+1)$ and $\Lambda_-(m+1)$ given by

$$\Phi(m+1) = \tau^2\Phi(m)^2\left\{1 + \Lambda_+(m)\Lambda_-(m) + \tanh\left(\kappa_b\sigma D^{(m)}\right)\left[\Lambda_+(m) + \Lambda_-(m)\right]\right\} , \tag{D-5}$$

$$\Phi(m+1)\Lambda_+(m+1) = \tau^2\Phi(m)^2\left\{2\Lambda_+(m) + \tanh\left(\kappa_b\sigma D^{(m)}\right)\left(1 + \Lambda_+(m)^2\right)\right\} , \tag{D-6}$$

$$\Phi(m+1)\Lambda_-(m+1) = \tau^2\Phi(m)^2\left\{2\Lambda_-(m) + \tanh\left(\kappa_b\sigma D^{(m)}\right)\left(1 + \Lambda_-(m)^2\right)\right\} , \tag{D-7}$$

to complete the proof we note that $T(0)$ may be written as

$$T(0) = \tau\Phi(0)\begin{pmatrix} -1 & \tau\Lambda_+(0) \\ \Lambda_-(0) & -1 \end{pmatrix} , \tag{D-8}$$

with $\Phi(0) = -1/\tau$, $\Lambda_+(0) = -\tan(k_a d_a)/\tau$ and $\Lambda_-(0) = \tau\tan(k_a d_a)$.

which allow us to obtain the Eq.(4-2-31) and (4-2-32).

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