SOLUTIONS OF THE STEADY-STATE LANDAU-GINZBURG EQUATION IN EXTERNAL DRIVING FIELDS

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Solutions of the Steady-state Landau-Ginzburg Equation in External Driving Fields

©Jennifer M. Rendell, B. Sc.(Hon.)

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This thesis is dedicated to my parents

who have always encouraged me to do my best.

Abstract

A steady state equation derived from the variation with respect to the order parameter $M(\vec{x})$ of a Landau-Ginzburg free energy density of the form

$$f = f_0 - hM + \frac{A}{2}M^2 + \frac{B}{4}M^4 + \frac{C}{6}M^6 + \nabla M \cdot \mathbf{D} \cdot \nabla M$$

is considered, where $h \neq 0$, $C \geq 0$, $\mathbf{D} \neq 0$ is a second rank tensor. This is a generalization of prior work by Winternitz *et al.* [J.Phys. C **21** 4931-4953 (1988)], who studied the case $h = 0$ and $C = 0$. Applied to a magnetic system, it describes the behaviour of the magnetization of a critical system in the presence of an external magnetic field h and near a structural phase transition. The Landau coefficients A, B, and C are weakly temperature dependent, but are considered constant near the transition temperature T_{tr} (the Curie point in magnetic systems), except for $A \propto (T - T_{tr})$. The gradient term allows for spatial inhomogeneities due to nearest neighbour interactions. Two cases are examined: $C = 0$ $(B > 0)$ and $C > 0$ $(B < 0)$ which correspond to second and first order phase transitions, respectively. The symmetries of the equation are exploited by the symmetry reduction method to find exact solutions in terms of varied symmetry variables. These solutions are in the form of kinks, bumps, singular, periodic, and doubly periodic solutions. The physical interpretation of these results and other calculations (*e.g.* energy, susceptibility) based on these results is discussed.

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1 INTRODUCTION

The study of magnetic phase transitions, and the behaviour of magnetic systems near a structural phase transition is a topic of major interest in condensed matter physics. Landau first pointed out the importance of symmetry in phase transitions and suggested that second order phase transitions could only occur between phases of different symmetry [1], usually involving a group-subgroup transformation.[†] Phase transitions of the second kind have a continuous change of state across the transition, but each phase is still characterized by different symmetries. In other words, a state where the scale of correlations is unbounded is continuously approached. In field theory this is termed approaching a zero mass theory. At the phase transition, the states of the two phases are the same, and there is no latent heat associated with the transition.[‡]

At a second order phase transition the free energy (potential) and its derivative, entropy are continuous, but its second derivative (*e.g.* heat capacity), is discontinuous. This is the Ehrenfest classification, which applies to most phase transitions. Mathematically, then, the phase transition point is a singularity of the free energy, F. A quantity, called the order parameter (M), is defined to describe the change in the structure of the body when it passes through a phase transition.

[†]Ironically, the classic example of a second order phase transition, the critical point of the gas-liquid transition, involves no symmetry change.

[‡]Phase transitions of the first kind have a discontinuous transition between two phases of different symmetry. At the phase transition, bodies in two different states are in equilibrium, and there is a latent heat, hysteresis and a finite change in volume associated with the transition.

The order parameter is defined such that it is zero† in the 'symmetrical' phase (the phase with a higher symmetry, *i.e.* more disordered), and non-zero in the 'unsymmetrical' phase (the phase with lower symmetry, *i.e.* more ordered). The symmetry group of each phase must be different, and one is usually a sub-group of the other. The more symmetrical phase usually corresponds to the higher temperature. Some example order parameters are: a concentration difference of atoms in a lattice, displacement from an original site in a lattice, a macroscopic magnetic moment per unit volume (ferromagnet), and a magnetic moment of the sub-lattice (anti-ferromagnet).

In 1937 Landau combined several theories: the Van der Waals equation of state (the gas-liquid transition), the Weiss theory (ferromagnets) and the Curie-Weiss theory (anti-ferromagnets) into a single phenomenological (or mean field) theory. Each of these theories had assumed an interacting system could be replaced by a system in an external field — if only that field is properly chosen [2]. A non-interacting system in an external field can be exactly solved; the field is then determined by a variational calculation. It is equivalent to selecting, out of all possible configurations, the one that gives the largest contribution to the partition function. In field theory, this is called the classical or tree approximation. This means that all fluctuations are ignored. So, while this type of theory may be

†The order parameter is only zero in the symmetrical phase when there is no external field. The presence of an external field will be discussed later.

quantitatively wrong (because of the fluctuations near the transition point), it can be a guide, and is a good starting point for more complicated theories that take these fluctuations into account [2] since scaling and scaling laws are still obeyed.†

Landau's phenomenological theory of phase transitions [4] is based on the assumption that, near second order phase transition temperatures, (*i.e.* near a structural phase transition or in particular, a magnetic phase transition), the free energy density f may be expanded in a power series of the order parameter, which is continuous and thus can take arbitrarily small values near the transition point;

$$f(H,T,M) = f_0(H,T) + b_1(H,T)M + \frac{1}{2}A(H,T)M^2 + b_3(H,T)M^3 + \frac{1}{4}B(H,T)M^4$$

where f_0 sets the energy scale, (*i.e.* it is the free energy density of the disordered phase). The arguments of the potential, external field (H)‡ and temperature (T) can be arbitrarily specified, while only the particular values of the order parameter (M) that correspond to an equilibrium state (*i.e.* minimize the potential) are allowed. (*N.B.* This expansion does not take account of the singularity at the transition point; it will be shown later it is not necessary.) For magnetic phase transitions, which will be the primary application in this thesis, M represents the magnetization component along a given direction (*i.e.* the projection of the magnetization along the axis in the direction of the spontaneous magnetization),

†Hyperscaling relations do not apply here since the dimension is not four. At that marginal dimension, another relation involving the critical index of the correlation length exists [3].

‡In other systems, Landau coefficients will be functions of the pressure, and not of the external field.

H represents the external magnetic field, and T_{tr} is the Curie temperature.

Two conditions must be met for the potential to be a minumum at particular values of M:

$$\frac{\partial f}{\partial M} = b_1 + AM + 3b_3M^2 + BM^3 = 0$$

$$\frac{\partial^2 f}{\partial M^2} = A + 6b_3M + 3BM^2 > 0$$

in both the symmetric phase ($M = 0$) and the unsymmetric phase ($M \neq 0$). The first implies the coefficient b_1 must be identically zero for all pressures and temperatures. The second implies immediately that in the symmetric phase, $A(H,T) > 0$. For the unsymmetric phase, combining the two conditions gives the inequality:

$$-2A > -3b_3M$$

which is satisfied, regardless of the sign of b_3M, if $A(H,T) < 0$. Thus at the transition point $A_{tr} = 0$. For an absolute minimum (not just a relative minimum) to exist, the coefficient $B_{tr}(H,T)$ must be positive. Then the first condition must be satisfied for $M = 0$, requiring $b_{3\ tr} = 0$. (*N.B.* the subscript tr refers to the transition point.) If $b_3(H,T) \equiv 0$, then there is a line of second order phase transition points in the HT plane. If $b_{3\ tr} = 0$ and $A_{tr} = 0$ only at the transition point, there are isolated (second order) transition points in the HT plane. Only lines of second order phase transitions are considered here; $b_3(H,T) \equiv 0$. Now B_{tr} is positive, thus $B(H,T)$ must be positive in the vicinity of the transition temperature (T_{tr}) and is assumed to be slowly changing, so it is sufficient to use $B(H, T_{tr})$. The coefficient A can be written as $a(H)(T - T_{tr}(H))$, assuming $a(H) > 0$.[†]

For magnetic phase transitions, it is necessary that under time reversal (*i.e.* $M \rightarrow -M$) the potential be invariant, so all odd order terms must be identically zero. (*N.B.* This is another reason for no linear term when discussing magnetic phase transitions in the absence of an external field.)

At the critical point [4] (the point where a line of second order phase transitions becomes a line of first order phase transitions) it is necessary to have $A_{cr}(H,T) = 0$ and $B_{cr}(H,T) = 0$, since a curve of second order transitions requires $B > 0$. Thus for the state of the body to be stable, another order is added to the expansion of the free energy density;

$$f(H,T,M) = f_0 + \frac{1}{2}AM^2 + \frac{1}{4}BM^4 + \frac{1}{6}CM^6$$

where $C_{cr}(H,T) > 0$. By the same reasoning C is positive in the vicinity of the critical temperature T_{cr}, and it can be assumed it is always positive, and approximately constant near T_{cr}. It seems reasonable that $B < 0$ is the line of first order phase transitions ($C > 0$). At a phase transition of the first kind, $f = f_0$ and $\partial f/\partial M = 0$ must be satisfied together. This means that $M = 0$, or $M^2 = -3B/(4C) > 0$, so $B < 0$ for a line of first order phase transitions. Substituting this back into either equation gives the equation of that line, $16AC = 3B^2$. This

[†]This is true for most, but not all, transitions.

in turn gives the new transition temperature for a first order transition, in terms of the second order transition temperature:

$$T_{tr(I)} = T_{tr(II)} + \frac{3B^2}{16aC}$$

Lifshitz and Pitaevskii [4] show that the curve of transitions of the first kind passes continuously into the curve of phase transitions of the second kind at the critical point (*i.e.* dT/dH is continuous), and that the two curves are discontinuous in the second derivative at the critical point (*i.e.* d^2T/dH^2 is discontinuous).

The difference between the two polynomials for each kind of phase transition can be understood intuitively by considering the graph of free energy density vs. the magnetization. The fourth order polynomial, which is quadratic in M^2, has a single well (above T_{tr}) that continuously splits into a double well as the temperature is lowered throught the transition point. The appearance of a pair of degenerate ground states is continuous. The sixth order polynomial, which is cubic in M^2, has a single well (and may have local minima) above the transition point, and as the temperature is lowered, local minima form and drop down to the ground state. The appearance of three degenerate ground states is discontinuous.

Up to this point, the treatment has been purely mean field, but the order parameter can be considered to be slowly varying in space; to include effects due to nearest neighbour interactions. This idea is due to Ginzburg (hence Landau-

Ginzburg theory) and was first applied to superconductors [5].[†] For long wavelength fluctuations, consider derivatives of the lowest order. The terms $M\nabla M$ and ∇M will contribute only to surface effects, and can be neglected, but terms proportional to $(\nabla M)^2$ will contribute to bulk volume effects:

$$f(M(\vec{x}), \nabla M, H, T) = f_0 + \frac{1}{2}AM^2 + \frac{1}{4}BM^4 + \frac{1}{6}CM^6 + \nabla M \cdot \mathbf{D} \cdot \nabla M. \quad (1.1)$$

The free energy is now a functional:

$$F([M]; H, T) = \int f(M, \nabla M, H, T) dV. \quad (1.2)$$

In the most general case, the constant of proportionality $\mathbf{D}$ will be a real second rank tensor, which can be diagonalized. The three diagonal elements D_i, $(i = 1, 2, 3)$ represent the principal axes of the lattice. If D_i is positive, the lattice is ferromagnetic along the x_i axis; if D_i is negative, the lattice is anti-ferromagnetic along the x_i axis. Thus, anisotropy effects due to nearest neighbour interactions can also be incorporated. It is useful to scale x_i with respect to the coefficients D_i, so that the gradient term can be written as a scalar times the gradient of the order parameter.

$$\left.\begin{array}{l} D_i \text{ all positive} \\ D_i \text{ all negative} \end{array}\right\} : D\left[(\partial_1 M)^2 + (\partial_2 M)^2 + (\partial_3 M)^2\right] \equiv D(\nabla M)^2_+ \quad (1.3)$$

$$D_i \text{ mixed signature}: D\left[(\partial_0 M)^2 - (\partial_1 M)^2 - (\partial_2 M)^2\right] \equiv D(\nabla M)^2_- \quad (1.4)$$

[†] An English translation is given in [6], and a review is given by [7].

where D is a real non-zero scalar and $\partial_i \equiv \partial/\partial_{x_i}$. (Here, the x_i's are the new, scaled axes.) The gradient term is now in one of the two above forms; the first being Euclidean space $E(3)$, and the second a Minkowski space $M(2,1)$ with a pseudo-time variable.

The external field may be explicitly included in the free energy density by adding a linear term $(-hM)$ [4], where h is proportional to the external field H. The symmetry of the more symmetric phase is then reduced, since the order parameter is non-zero everywhere. The phase transition point is no longer discrete at H_{tr} and T_{tr}, but smoothed out. In particular, the specific heat no longer has a sharp disconitinuity; it is smeared out. The final free energy density to be used in this thesis is then:

$$f(M(\vec{x}), \nabla M, H, T) = f_0 - hM + \frac{1}{2}AM^2 + \frac{1}{4}BM^4 + \frac{1}{6}CM^6 + D(\nabla M)^2_{\pm}. \quad (1.5)$$

where

$$A = a(H)(T - T_{tr}(h)), h \propto H .$$

For a second order transition

$$C(H, T_{tr}) = 0,\ B(H, T_{tr}) > 0 ,$$

and for a first order transition

$$C(H, T_{tr}) > 0,\ B(H, T_{tr}) < 0 .$$

The Ginzburg criterion [4] gives the temperature range, outside of which the Landau theory is valid. So this theory is useful close, but not too close, to the transition temperature. Inside this fluctuation range, the fluctuations resulting from the singular nature of the free energy near the phase transition are dominant. Near the transition temperature, large scale correlations appear. This is manifested, for example, by critical opalescence at the gas-liquid critical point when regions the size of microns fluctuate coherently [2,8]. In a magnet, divergence of the susceptibility indicates the approach of the transition temperature. The conditions for validity of this theory can be more easily satisfied as the critical point is approached.

There are other theories that model critical phenomena: *e.g.* Ising (2 and 3 dimensions), Heisenberg, and spherical models. Of these, the 2-dimensional Ising model is exactly solvable, and produces remarkable results at phase transition temperatures. A mechanism is needed to compare the results of these various theories. The method used is to look at how physical quantities (*e.g.* heat capacity, susceptibility, correlation length, the correlation function, and the magnetization) change as the transition point is approached. This is written as the power of the reduced temperature ($t = (T - T_{tr})/T_{tr}$) near the transition point. These powers are collectively called the critical indices. They are experimentally verifiable and provide a straightforward way to compare the results of the various models. Unfortunately, because Landau-Ginzburg theory doesn't apply in the fluctuation range, critical exponents calculated using it are not very reliable.

The next step is to find the function M that minimizes the free energy. The stationary points of F with respect to M can be found by solving this equation:

$$\delta F = \int dV[-h + AM + BM^3 + CM^5 - 2D \begin{Bmatrix} \Delta M \\ \square M \end{Bmatrix}]\delta M = 0, \tag{1.6}$$

where

$$\Delta \equiv \partial_1^2 + \partial_2^2 + \partial_3^2 \tag{1.7}$$

$$\square \equiv \partial_0^2 - \partial_1^2 - \partial_2^2 \tag{1.8}$$

which results in a non-linear partial differential equation (PDE):

$$2D \begin{Bmatrix} \Delta M(\vec{x}) \\ \square M(\vec{x}) \end{Bmatrix} = -h + AM + BM^3 + CM^5. \tag{1.9}$$

It is important to note that it will be necessary to check which functions M actually minimize F, and thus are stable solutions.

In the context of quantum field theory Burt [9] studied a class of non-linear field equations (in Minkowski space) of the form:

$$(\partial_\mu \partial^\mu + m^2)\varphi + \alpha\varphi^{2p+1} + \lambda\varphi^{4p+1} = 0 \tag{1.10}$$

where $p \neq 0, -1/2, -1$; and $\mu \in (0, 1, 2, 3)$. He found solitary wave solutions of a class of field equations for systems with polynomial self-interactions which reduce to plane wave solutions when the coupling constants α and λ are set to zero. He postulated a plane wave variable $y = k_\mu x^\mu$, with $k_\mu k^\mu = m^2$ and developed a general solution. For $p = 1/2$, his solution is exactly the form of a case presented later (2SD; see equs. (3.13), (3.14); $\delta/D < 0$). However, he set the first integration constant to zero, and considered only the case when $k_\mu k^\mu = m^2 > 0$, thus he missed the other bounded solutions detailed in the 2SD case, and did not get the singular solutions.

Khan [10] studied magnetic phase transitions as a basis for understanding critical phenomena. He analysed a one dimensional version of equation (1.9) with no external field, and no sixth order term ($h, C = 0$). Because he worked on the one-dimensional case (*i.e.* the equation is equivalent to the ordinary differential equation analysed here), the methods of integration are the same as used in this thesis. Thus, his solutions correspond to the elliptic solutions presented here for the $C = 0$ case. All of his solutions are periodic, but some are discontinuous. He states that the discontinuous solutions are physically unrealistic and perhaps these discontinuities may be eliminated by the inclusion of higher order terms in the free energy expression.

In a more general context of the kinetics of a first order phase transition with the inclusion of dissipation, Gordon [11] considered a time dependent Landau-Ginzburg equation (one spatial dimension) for the evolution of the order parameter φ as given by:

$$2\Gamma D\frac{d^2\varphi}{ds^2} + v\frac{d\varphi}{ds} = \Gamma(a\varphi - b\varphi^3 + c\varphi^5) \tag{1.11}$$

where $s = x - vt$ represents a plane wave that takes the original PDE into a solvable ordinary differential equation (ODE), Γ is the Landau-Khalatnikov damping coefficient, $a, b, c > 0$ are the Landau coefficients, and D is the coefficient of the inhomogeneity term. Assuming kink-like boundary conditions $\lim_{s\to\pm\infty}\varphi'(s) = 0$, $\lim_{s\to\infty}\varphi = \varphi_1$, and $\lim_{s\to-\infty}\varphi = \varphi_2$, where φ_1, φ_2 are the minimum and maximum, respectively, of the free energy, he gets a kink solution. One example profile shows the interface between the ferroelectric phase $\varphi_1 > 0$ and a para-electric phase $\varphi_2 = 0$ where v is the propagation rate of the interface. Comparing his results with experimental values of the Landau coefficients, he found satisfactory agreement between theory and experiment.

Winternitz *et al.* [12] investigated equation (1.9) for second order phase transitions ($C = 0$, $B > 0$) in the absence of an external magnetic field. This work is a three dimensional analogue of the calculations by Khan [10] discussed above. They found a large class of symmetry variables, and corresponding solutions. The solutions presented here will be similar in many ways, but generally more restricted since the presence of the external magnetic field lowers the symmetry of the problem.

The aim of this thesis is to analyse a more general situation where the space of the independent variables is three-dimensional, and external fields play an important role, but no time is involved. Chapter 2 will be a short outline on the symmetry reduction method as it applies to this equation, and the benefits of using such a method. The solutions will be presented, with some discussion, in Chapter 3. A more general discussion of the solutions and the calculations then possible, is presented in Chapter 4.

2 SYMMETRY REDUCTION

This is a brief foray into symmetry reduction as a method for finding solutions to (systems of) partial differential equations. The equation of interest is:

$$\left.\begin{array}{c}\Delta M \\ \\ \Box M\end{array}\right\} = \frac{1}{2D}(-h + AM + BM^3 + CM^5) \equiv V'(M) \tag{2.1}$$

(*N.B.* V' has been used here because the polynomial above is proportional to the derivative of the polynomial in the free energy density equation (1.5).) It is an example of the non-linear Klein-Gordon equation, which has the general form:

$$\Box M = H(M, (\nabla M)^2)$$

where the functional $H(M, (\nabla M)^2)$ can be $\sin M$ (the sine-Gordon equation), $\sin M + \sin 2M$ (double sine-Gordon equation) or, as in this case, a polynomial in M. Symmetry reduction uses the properties of a Lie group of continuous symmetries of an equation to introduce new, independent symmetry variables and reduce the dimensionality of the system. The method is very general and can be used with a system of multi-dimensional PDEs of k^{th} order. It is well documented by Olver [13], and this chapter will only attempt to give as much of an overview of the method as is needed for this particular example. The equation is solved in Euclidean space $E(3)$ and in Minkowski space $M(2,1)$, depending on each particular $\mathbf{D}$ (eqns. (1.3), (1.4)). In this example the symmetry variables used reduce the PDE to an ODE. The basic procedure follows:

a) Calculate and solve the system of determining equations. The calculation is algebraic and tedious when done by hand, but ideally suited to computer operations. Recently, symbolic computer programs (*e.g.* in MACSYMA^{TM}[14] and REDUCE [15]) have been developed to implement the algorithms for finding the system of determining equations. However, solving the system of determining equations is quite straightforward. The result is the Lie algebra **g** which is defined by differential operators g_i called generators. In this case, generators of equation (2.1) in Euclidean space $E(3)$ are:

$$P_i = \partial_i, \; L_i = -\epsilon_{ijk} x_j P_k, \tag{2.2}$$

where $i, j, k \in (1, 2, 3)$, and ϵ_{ijk} is the anti-symmetric Levi-Civita tensor. These are the vector fields of translation and rotation. The generators of equation (2.1) in Minkowski space $M(2,1)$ are:

$$\begin{aligned} P_j &= \partial_j, \quad L_3 = x_2 P_1 - x_1 P_2, \\ P_0 &= \partial_0, \quad K_j = -x_0 P_j - x_j P_0, \end{aligned} \tag{2.3}$$

where $j \in (1, 2)$. The new elements are pseudo-time translation and the Lorentz boost. The Lie algebra, defined by the Lie (commutation) bracket and equations (2.2) and (2.3), is:

$$\begin{gathered} [P_0, P_0] = [P_0, P_i] = [P_i, P_j] = [P_0, L_i] = 0, \\ [P_0, K_i] = -P_i, \; [P_i, L_j] = \epsilon_{ijk} P_k, \; [P_i, K_j] = -\delta_{ij} P_0, \\ [L_i, L_j] = [K_i, K_j] = \epsilon_{ijk} L_k, \; [L_i, K_j] = \epsilon_{ijk} K_k \end{gathered} \tag{2.4}$$

Table 1: The action of the one parameter symmetry group

Generators g_i	$(x_0', x_1', x_2', x_3') = e^{-g_i\rho}(x_0, x_1, x_2, x_3)$
$P_0 = \partial_0$	$(x_0 + \rho, x_1, x_2, x_3)$
$P_1 = \partial_1$	$(x_0, x_1 + \rho, x_2, x_3)$
$P_2 = \partial_2$	$(x_0, x_1, x_2 + \rho, x_3)$
$P_3 = \partial_3$	$(x_0, x_1, x_2, x_3 + \rho)$
$L_1 = x_3P_2 - x_2P_3$	$(x_0, x_1, x_2 \cos\rho - x_3 \sin\rho, x_3 \cos\rho + x_2 \sin\rho)$
$L_2 = x_1P_3 - x_3P_1$	$(x_0, x_1 \cos\rho + x_3 \sin\rho, x_2, x_3 \cos\rho - x_1 \sin\rho)$
$L_3 = x_2P_1 - x_1P_2$	$(x_0, x_1 \cos\rho - x_2 \sin\rho, x_2 \cos\rho + x_1 \sin\rho, x_3)$
$K_1 = -x_0P_1 - x_1P_0$	$(x_0 \cosh\rho + x_1 \sinh\rho, x_1 \cosh\rho + x_0 \sinh\rho, x_2, x_3)$
$K_2 = -x_0P_2 - x_2P_0$	$(x_0 \cosh\rho + x_2 \sinh\rho, x_1, x_2 \cosh\rho + x_0 \sinh\rho, x_3)$
$K_3 = -x_0P_3 - x_1P_3$	$(x_0 \cosh\rho + x_3 \sinh\rho, x_1, x_2, x_3 \cosh\rho + x_0 \sinh\rho)$

where $i, j, k \in (1, 2, 3)$ and δ_{ij} is the Kronecker delta. *N.B.* The commutation relation implies the operators are acting on an arbitrary function of the independant variables. Each generator $g_i \in \mathbf{g}$ has a first integral that is the one-parameter symmetry group G_i or the local group of point transformations that takes one solution into a new solution. The action of this group is:

$$(x_0', x_1', x_2', x_3') = e^{-g\rho}(x_0, x_1, x_2, x_3)$$

where ρ is the parameter of the group. That is, given a solution in terms of x_ν, $\nu \in (0, 1, 2, 3)$, a new solution can be found using x_ν'. The action of the one-parameter group for the two spaces are shown together in Table 1.

b) Find all possible (closed) sub-algebras and choose a representative of each conjugacy class. This is equivalent to finding the optimal system of subalgebras.

Table 2: Symmetry variables $E(3)$

ξ	ODE: $(\nabla\xi)^2_+ W_{\xi\xi} + \Delta\xi W_\xi = V'(W(\xi))$
x_1	$W_{\xi\xi} = V'(W)$
$(x_1^2 + x_2^2)^{1/2}$	$W_{\xi\xi} + \frac{1}{\xi}W_\xi = V'(W)$
$(x_1^2 + x_2^2 + x_3^2)^{1/2}$	$W_{\xi\xi} + \frac{2}{\xi}W_\xi = V'(W)$

Methods for finding the subalgebras are described by [16]. The results used in this thesis are taken from [16,17,18].

c) Integrate each linear system of first order PDEs corresponding to the subalgebra, using the method of characteristics [19]. This results in a function $\xi(x_i)$ (the symmetry variable) which is an invariant of the group. The dependent variable, (M), is then expressed in terms of this known function, and a new unknown function (W):

$$M(\vec{x}) = W(\xi(\vec{x})).$$

Substituting this expression for M in the PDE will generate ODEs of W in ξ. The symmetry variables used in this thesis were taken from results for an implicit, general, and non-linear Klein-Gordon equation. Grundland et al. [20] solved the problem $H(\Box u, (\nabla u)^2, u) = 0$ for $(n+1)$-dimensional Minkowski space $M(n,1)$, where H is an arbitrary function. The results for this example are given in Tables 2 and 3 ($W_\xi \equiv dW/d\xi$ and v is an arbitrary function).

Table 3: Symmetry variables $M(2,1)$

ξ	ODE: $(\nabla\xi)^2_- W_{\xi\xi} + \Box\xi W_\xi = V'(W(\xi))$
x_0	$W_{\xi\xi} = V'(W)$
x_1	$W_{\xi\xi} = -V'(W)$
$x_0 \pm x_1$	$0 = V'(W)$
$x_2 + v(x_0 + x_1)$	$W_{\xi\xi} = -V'(W)$
$(x_1^2 + x_2^2)^{1/2}$	$W_{\xi\xi} + \frac{1}{\xi} W_\xi = -V'(W)$
$(x_0^2 - x_1^2)^{1/2}$	$W_{\xi\xi} + \frac{1}{\xi} W_\xi = V'(W)$
$(x_0^2 - x_1^2 - x_2^2)^{1/2}$	$W_{\xi\xi} + \frac{2}{\xi} W_\xi = V'(W)$

The symmetry variables for Euclidean space are translationally, cylindrically and spherically invariant, respectively. The translationally invariant variable can be thought of as a plane wave $\vec{k} \cdot \vec{x}$ where $\vec{k}$ is the wave vector.

The symmetry variables for Minkowski space are somewhat more interesting. The first and second are translationally invariant with respect to pseudo-time and space, respectively. The third is translationally invariant on a plane perpendicular to the $x_1 = \mp x_0$ line, respectively. The fourth describes a sheet, the projection of which on the x_2, $x_0 = x_1$ plane is an arbitrary function of $x_0 + x_1$. The fifth is cylindrically invariant. The sixth describes a hyperboloid sheet, and the seventh a hyperboloid. The first three variables can be thought of as plane waves; with x_0 representing a light-like vector ($k^2 = 1$), x_1 representing a space-like vector ($k^2 = -1$) and $x_0 \pm x_1$ representing vectors on the light cone. While the translationally invariant variables may be an obvious choice to make, the others, are much less obvious. This is the power of the symmetry reduction method — to calculate, in a methodical manner, symmetry variables that apply to an equation or system of equations. In the cases calculated earlier [12], no external field meant that there was a much richer field of symmetry variables, and of interesting ODEs. This was due to the presence of scaling symmetries (dilations) which are valid when the polynomial non-linearity is homogeneous (i.e. for some function $P(\lambda\phi) = {}_{\wedge}P(\phi)$).

The solutions presented in the following chapter are for the equations of the form:

$$W_{\xi\xi} = \pm V'(W) \tag{2.5}$$

corresponding to the translationally invariant and degenerate symmetry variables. Note this equation also has two discrete symmetries: $\xi \to -\xi$ and $\xi \to i\xi$. (*N.B.* In the following chapters, the $\pm$ (eqn. (2.5)) is absorbed into the coefficient D.) When $C = 0$, the equation is of Painlevé type, but the general equation $C > 0$ (and $h \neq 0$) is not. An ODE has the Painlevé property if its general solution has no essential singularities, other than poles, which depend on the initial conditions [19,21]. If all the critical points are fixed and ξ is not a critical point, the critical points are independent of the initial conditions [20], the solution is unique, and completely determined by the initial points $W_0 = W(\xi_0)$ and $\dot{W}_0 = \dot{W}|_{\xi=\xi_0}$ [19]. (*N.B.* Choosing a boundary condition restricts the symmetry of the solution to a smaller symmetry than that of the equation itself.) The Painlevé property is a necessary but not sufficient condition for solutions without moving essential singularities. An equation with the Painlevé property will most likely be integrable in terms of elementary, transcendental or Painlevé transcendental functions. A method of testing for the Painlevé property is given by [22] and a MACSYMA™ program based on those results given by [23]. Thus, the equation (2.5) will have analytic solutions for $C = 0$, and probably solutions with moving essential singularities for $C > 0$. Thus, the following chapter presents solutions to a less tractable equation. The equations of the form:

$$W_{\xi\xi} + \frac{k}{\xi} W_{\xi} = \pm V'(W) \tag{2.6}$$

(corresponding to the cylindrically, spherically and hyperbolically invariant symmetry variables) asymptotically have the same solutions as (2.5), but also are not of Painlevé type. It is expected the solutions of (2.6) will also have movable algebraic branch points and may or may not have analytic solutions.

3 SOLUTIONS OF THE STEADY-STATE EQUATION

The equation that will be concentrated on is

$$\ddot{W}(\xi) = \frac{1}{2D}(-h + AW + BW^3 + CW^5),\ C \geq 0\,, \tag{3.1}$$

where $\ddot{W} \equiv d^2W/d\xi^2$. Equation (3.1) can immediately be integrated to

$$\dot{W}^2(\xi) = \frac{1}{D}(s_0 - hW + \frac{A}{2}W^2 + \frac{B}{4}W^4 + \frac{C}{6}W^6) \tag{3.2}$$

where s_0 is a real integration constant defined by the boundary surface $\xi(\vec{x}) = \xi_0$. The inital conditions are determined by the boundary surface of the ODE:

$$W_0 := W(\xi_0),\ \dot{W}_0 := \dot{W}(\xi)|_{\xi=\xi_0}\,.$$

Thus s_0 is defined as follows:

$$s_0 = D\dot{W}_0^{\,2} + hW_0 - \frac{A}{2}W_0^2 - \frac{B}{4}W_0^4 - \frac{C}{6}W_0^6\,. \tag{3.3}$$

The analysis of (3.2) is divided into two cases: second order phase transitions $C = 0$ $(B > 0)$ and first order phase transitions $C > 0$ $(B < 0)$. Recall A is positive above the transition temperature, and negative below. Real solutions exist when $\dot{W}^2(\xi) \geq 0$. The former has a complete set of analytic solutions, and the latter whenever a multiple root occurs in the polynomial. For simplicity, a scaled variable $\eta = \kappa_{(0)}^{-1}\xi$ is chosen, such that the coefficient of the highest power (with the exception of the sign) is unity. The new equations to be solved are:

$$W^4\ :\ \dot{W}^2(\eta) = \epsilon(\lambda + \alpha W + \beta W^2 + W^4),\ \ \epsilon = \pm 1 \tag{3.4}$$

where

$$C = 0: \ \kappa_0^{-2} = \frac{\epsilon B}{4D}, \ \ (\lambda, \alpha, \beta) = \frac{4}{B}(s_0, -h, \frac{A}{2})$$

and

$$W^6 \ : \ \dot{W}^2(\eta) = \epsilon(\lambda + \alpha W + \beta W^2 + \gamma W^4 + W^6), \ \ \epsilon = \pm 1 \tag{3.5}$$

where

$$C > 0: \ \kappa^{-2} = \frac{\epsilon C}{6D}, \ \ (\lambda, \alpha, \beta, \gamma) = \frac{6}{C}(s_0, -h, \frac{A}{2}, \frac{B}{4}).$$

Solutions are analysed on the basis of the root structure of the polynomial on the right hand side of the W^4 (W^6) equation. Throughout this chapter η is real, thus the value of ϵ is chosen to keep ϵD positive. For each case, $D > 0$ denotes $\dot{W}^2(\eta)$ bounded below, and $D < 0$ denotes $\dot{W}^2(\eta)$ bounded above. (*N.B.* The free energy density equation (1.5) is proportional to equation (3.2);

$$f - f_0 + s_0 = 2D\dot{W}^2,$$

but for physical reasons discussed later, it is not a problem that some of the graphs are unbounded from below.) The shaded areas shown in Fig. 1 are the regions that can be integrated to find real solutions. The vertical axes are the polynomial function $\dot{W}^2(\eta)$, and the horizontal axis is the dependent function W.

For the remaining graphs, the usual pair of axes is the graph $\zeta_i(\eta)$ vs. the solution $W(\zeta_i)$, and the extra (righthand) axis gives the graph $\dot{W}^2(\eta)$ vs. $W(\zeta_i)$. The two graphs are placed together to emphasize the connection between the real roots of $\dot{W}^2$ and the limits of the solutions W. The roots w_i, r, p_i, q_i, and other

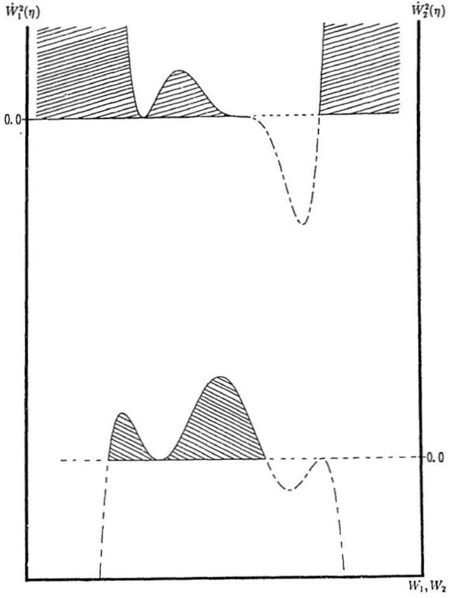

Figure 1: The role of $\dot{W}^2(\eta)$ in determining the regions of real solutions. W may be integrated in the shaded regions.

constants (*e.g.* α_i,k_i) are understood to be local to each particular solution. The arguments ζ_i, and functions $R_i(W)$ have been numbered. Each $\mu_{(i)} = \pm 1$. Each ζ_i is understood to have $\pm$ associated with it. This merely gives the mirror image solution and is equivalent to $W \to -W$. This sign is thus dropped. W is graphed from -1 to 1 because derivations from microscopic Hamiltonian densities require this bound for the series expansion [5,12]. Each multiple root has a constant solution associated with it; these are graphed, but not explicitly included in the solutions. Sections 3.1 and 3.2 detail the exact solutions and their particular graphs. Chapter 4 discusses these solutions.

The following terms will be used to describe the solutions in the remainder of this chapter. Their definitions are given here for clarity. There are two kinds of solitary waves: bumps and kinks. Each is a localized travelling wave. A bump is characterized by the same asymptotic value, and a kink by different asymptotic values. They are related in that a kink is the derivative of a bump. It has not been shown whether these solitary waves are in fact solitons, which have the special property of retaining their shape and velocity upon collision with other solitary waves [24]. Periodic bounded solutions will be called spin waves [20] to emphasize the periodic change in the magnetic lattice of the spin vector orientations. The term 'layers' is used to describe solutions that, except for singularities, are essentially constant between real roots of $\dot{W}^2$. When these constant values are different in various regions, the term 'gap' is used to describe the inbetween region that may be inaccessible (corresponding to negative portions of $\dot{W}^2$), or may have other bounded real solutions. A nucleation center refers to a localized region of magnetic order, *i.e.* a bump.

3.1 Exact solutions of W^4 equation

The solutions of second order phase transitions are found by analysing the root structure of the fourth-order polynomial:

$$
\begin{aligned}
W^4 \ : \ \dot{W}^2(\eta) &= \epsilon(\lambda + \alpha W + \beta W^2 + W^4) \\
&= \epsilon(W - w_1)(W - w_2)(W - w_3)(W - w_4)\,; \qquad (3.6)
\end{aligned}
$$

$$
\epsilon = \pm 1,\ \eta = \kappa_0^{-1}\xi; \qquad \kappa_0^{-2} = \frac{\epsilon B}{4D},\ (\lambda, \alpha, \beta) = \frac{4}{B}(s_0, -h, \frac{A}{2})\ , B > 0\,.
$$

The absence of a cubic term (W^3) gives one condition on the four roots (real or complex):

$$
w_1 + w_2 + w_3 + w_4 = 0\,.
$$

Using this to determine w_4, the coefficients can be written

$$
\begin{aligned}
\lambda &= -w_1 w_2 w_3 (w_1 + w_2 + w_3)\,, \\
\alpha &= (w_1 + w_2)(w_2 + w_3)(w_3 + w_1) \neq 0\,, \\
\beta &= w_1 w_2 + w_2 w_3 + w_3 w_1 - (w_1 + w_2 + w_3)^2\,, \qquad (3.7)
\end{aligned}
$$

where λ is non-zero to maintain the generality of the solutions. Note $\alpha \propto h$ and non-zero, requires each root $\{w_i\}$ to be different from the negative of any other $i \neq j$, $w_i \neq -w_j$. Any complex roots must occur in conjugate pairs since each coefficient is real. This eliminates a number of simpler cases. There are seven cases which can be divided as follows: one constant, three elementary and three elliptic solutions.

3.1.1 Constant solutions

The solutions correspond to the three roots of equation (3.1) ($C = 0$), and are:

$$W(\xi) = \begin{cases} -\frac{1}{2}(1 \pm \sqrt{3}i)c_0^{1/3} + \frac{A}{6B}(1 \mp \sqrt{3}i)c_0^{-1/3} \\ c_0^{1/3} - \frac{A}{3B}c_0^{-1/3} \end{cases} \tag{3.8}$$

where the constant c_0 is given by:

$$c_0 = \frac{1}{6\sqrt{3}B}\sqrt{4\frac{A^3}{B} + 27h^2} + \frac{h}{2B},$$

and the integration constant s_0 can be found by substituting the above solutions into (3.2) with $\dot{W}_0 = 0$. (*N.B.* $W \in \Re$, but this depends on the actual values of the constants.) When $h \to 0$, $W(\xi) = 0, \pm i\sqrt{A/B}$. This solution also corresponds to the symmetry variable on the light cone $\xi = x_0 \pm x_1$. The magnetization is piecewise continuous in this case.

3.1.2 Elementary solutions

Each case in this section has at least one multiple root. The solutions will be bumps, singular and periodic solutions. There will be no kink solutions in the W^4 case because of the external field. All of the integrals used for solutions in this section can be found in [25].

Single and triple real roots (ST)

The roots are:

$$w_1 = w_2 = w_3 = w, \; w_4 = -3w; \; 0 \neq w \in \Re,$$

with the conditions $s_0 = -3w^4B/4 < 0$, $A = -3w^2B < 0$ (*i.e.* below the transition temperature.) and the field is $h = -2w^3B$. The solution is (Figs. 2, 3):

$$W(\zeta_1) = w(1 - \frac{4}{1-\zeta_1^2}), \tag{3.9}$$

where

$$\zeta_1^2 = w^2\frac{B}{D}(\xi - \xi_0)^2.$$

a) When $D > 0$ this solution (eqn. (3.9), Fig. 2) is singular at $(\xi - \xi_0) = \pm\frac{1}{w}\sqrt{\frac{D}{B}}$ and could be interpreted as representing a magnetic 'tri-layer'. The magnetization is constant at $w \propto -h^{1/3}$ when $(\xi - \xi_0)^2 > D/(w^2B)$ and abruptly switches to $-3w$ when $(\xi - \xi_0)^2 < D/(w^2B)$. Increasing the field will increase the difference ($4w$) in the relative magnetizations of the 'tri-layer' structure. Maximum and minimum cutoffs are imposed (as a result of finiteness of spin magnitude) near the singularities as the gradient changes rapidly. What happens between these cutoffs is not revealed through this theory, since the continuum approximation used apparently breaks down in this regime.

b) When $D < 0$ this solution (eqn. (3.9), Fig. 3) is a bump with height $4w$ and limit w as $(\xi - \xi_0) \to \pm\infty$. The minimun magnetization is $-3w$. It represents a nucleation center of magnetic order separate from the constant magnetization w throughout the rest of the sample. Increasing the field h will increase the height of this bump.

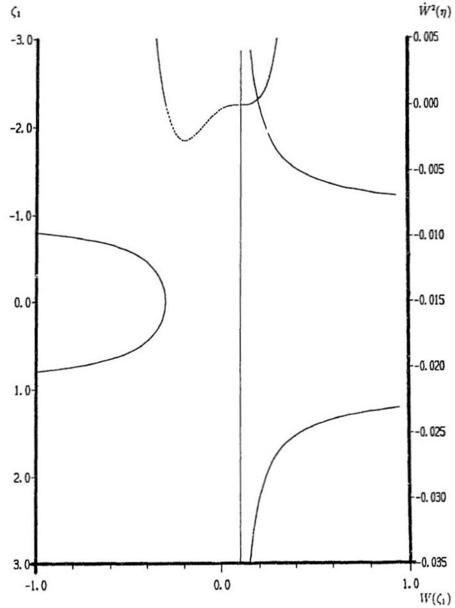

Figure 2: W^4 ST Single and triple real roots, $D > 0$, $w = 0.1$

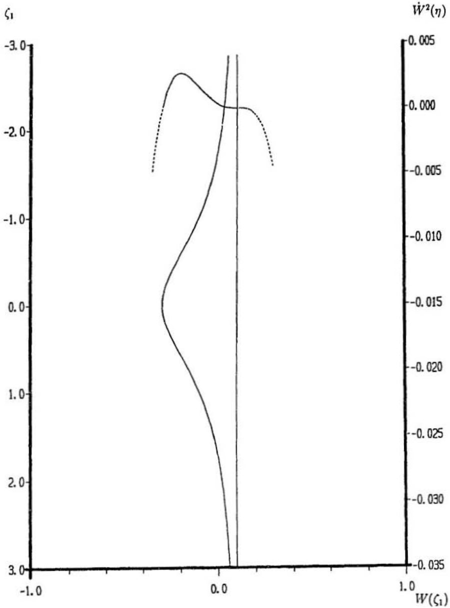

Figure 3: W^4 ST Single and triple real roots, $D < 0$, $w = 0.1$

A complex conjugate pair and a real double root (CD)

The roots are:

$$w_1 = w_2 = -p,\ w_3 = w_4^* = p(-1+ir); 0 \neq p, r \in \Re\,,$$

requiring $s_0 = p^4(1+r^2)B/4 > 0$, $A = p^2(r^2-2)B/2$, and the field $h = r^2p^3B/2$. The only real solution occurs when $D > 0$:

$$W(\zeta_2) = p(1 + \frac{r^2+4}{r\sinh\zeta_2 - 2})\,, \tag{3.10}$$

where

$$\zeta_2 = \frac{p}{2}\sqrt{r^2+4}\sqrt{\frac{B}{D}}(\xi - \xi_0),\ D > 0\,.$$

This solution (Fig. 4) is singular at $\sinh\zeta_2 = -2/r$. The $\lim_{\zeta_2\to\pm\infty} W = p$. So the magnetization ($p$) is constant over most of the sample, but near the singularity the magnetization changes exponentially and flips to the opposite sign, returning to the constant magnetization (p). This, once cutoffs are imposed, appears to be essentially a homogeneous field, with an inhomogeneity near the singularity. Increasing the field will drive the constant field away from the zero field.

Two distinct single and a double real root (2SD)

The roots are:

$$w_1, w_2,\ w_3 = w_4 = -(w_1 + w_2)/2 \equiv w\,;$$

$$w_1, w_2 \in \Re,\ w_1^2 \neq w_2^2.$$

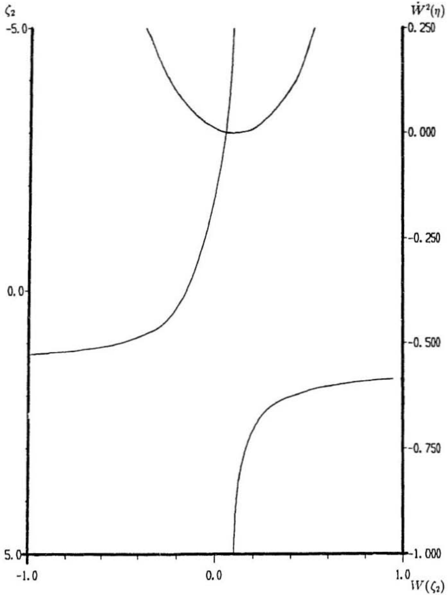

Figure 4: W^1 CD A complex conjugate pair and a real double root, $D > 0$, $p = 0.1$, $r = 1$

The field is $h = -Bw(w_1 - w_2)^2/8$; the other coefficients are $s_0 = w_1 w_2 w^2 B/4$, and $A = (w_1 w_2 - 3w^2)B/2$. Choose $w_2 < w_1$, and introduce the parameter

$$\delta = (3w_1 + w_2)(w_1 + 3w_2). \tag{3.11}$$

When $\delta = 0$ the solution becomes (ST eqn. (3.9)). When $\delta > 0$ the polynomial has a double root on the right or left, (one is the mirror image of the other, so assume the double root is on the right, $w_1 < w$). Solutions with double roots on the left can be obtained by reflecting ζ across zero. When $\delta < 0$ the double root is between the two single roots, $w_2 < w < w_1$. The analysis of this section is divided by the sign of δ as well as by the sign of D. The common argument to each is:

$$\zeta_3 = \frac{1}{4}\sqrt{|\delta|}\sqrt{\frac{B}{|D|}}((\xi - \xi_0)). \tag{3.12}$$

The first general solution is, for $\delta/D > 0$:

$$W(\zeta_3) = w + \frac{\delta/2}{\mu_1(w_1 - w_2)\cosh(\zeta_3 + \theta) - 4w}, \tag{3.13}$$

such that when

$$\begin{aligned}
D > 0: \quad & \mu_1 = 1,\ W \le w_2 \ \text{ or } \ w \le W \ \text{ singular} \\
& \mu_1 = -1,\ w_1 \le W \le w \ \text{ bump} \\
D < 0: \quad & \mu_1 = 1,\ w_2 \le W \le w \ \text{ bump} \\
& \mu_1 = -1,\ w \le W \le w_1 \ \text{ bump}
\end{aligned}$$

and where

$$\theta = -\ln(w_1 - w_2).$$

a) When $D > 0$ and $\delta > 0$, the solution (eqn. (3.13)) is shown in (Fig. 5). For $\mu_1 = 1$, this solution is singular at $\zeta_3 = \ln(4w \pm \sqrt{\delta})$ and, having removed the singularities, represents a magnetic 'tri-layer'. The $\lim_{\zeta_3 \to \pm\infty} W(\zeta_3) = w$. When $\mu_1 = -1$, the solution is a bump, with an extremum at $\zeta_3 = \ln(w_1 - w_2)$ and is a magnetic order nucleation center. This solution is equivalent to the one in [9]. The maximum height of this bump is $w - w_1$. There are no real solutions between w_2 and w_1. Increasing the magnetic field increases the 'gap' between w_2 and w_1, and moves the constant magnetization up the graph. The width of the portion between the two singularities is $\Delta\zeta_3 = \ln|\frac{4w+\sqrt{\delta}}{4w-\sqrt{\delta}}|$.

b) When $D < 0$ and $\delta < 0$, the solution (eqn. (3.13)) is shown in (Fig. 6). These are two bumps with extremums at $\zeta_3 = \ln(w_1 - w_2)$ and respective heights $w_1 - w$, $w - w_2$, $\lim_{\zeta_3 \to \pm\infty} W(\zeta_3) = w$. The magnetization is limited to values between w_2 and w_1, and has nucleation centers of magnetic order at the same point but of different magnitude and orientation. Increasing the external field further distorts the bumps from the symmetric position they take with no external field, and moves the limiting values farther away from the zero position.

The second general solution, for $\delta/D < 0$, is:

$$W(\zeta_3) = w + \frac{\delta/2}{\mu_2(w_1 - w_2)\sin\zeta_3 - 4w}, \tag{3.14}$$

such that when

$$D > 0: \quad \mu_2 = 1,\ W \leq w_2 \text{ or } w_1 \leq W \ \text{ periodic, singular}$$

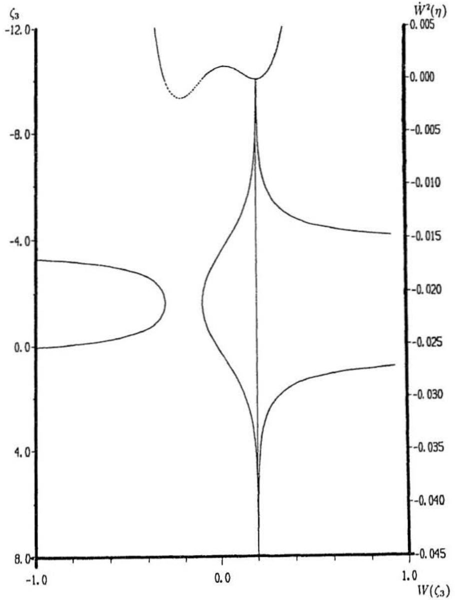

Figure 5: W^4 2SD Two distinct single and a double real root, $D > 0$, $\delta > 0$, $w_1 = -0.1$, $w_2 = -0.3$ and $w = 0.2$

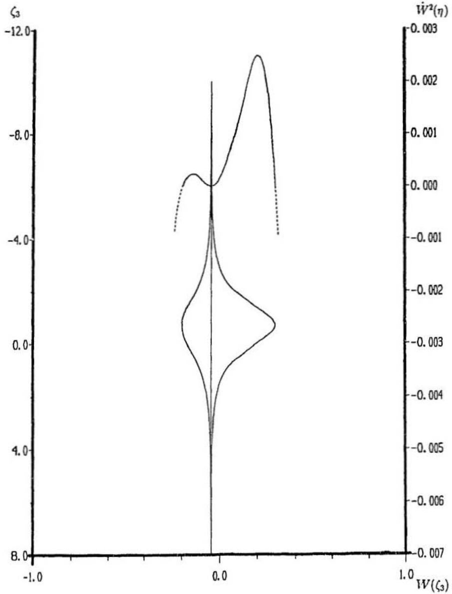

Figure 6: W^4 2SD Two distinct single and a double real root, $D < 0$, $\delta < 0$, $w_1 = 0.3$, $w_2 = -0.2$, and $w = -0.05$

$$D < 0: \quad \mu_2 = -1,\ w_2 \leq W \leq w_1 \text{ periodic}$$

c) When $D > 0$ and $\delta < 0$, the solution (eqn. (3.14)) is shown in (Fig. 7). This solution is periodically singular at $\zeta_3 = \arcsin \frac{4w}{w_1 - w_2}$, and has relative minimums at w_1 and maximums at w_2. It represents a periodic arrangement of magnetic double 'layers' once the singularities have been removed. The width of the lower 'layer', between singularities is $\pi - 2\arcsin[4w/(w_1 - w_2)]$ and of the upper 'layer' is $\pi + 2\arcsin[4w/(w_1 - w_2)]$. Either of these going to zero is equivalent to $\delta \to 0$. As the field increases, the 'layers' move apart, and away from $W = 0$.

d) When $D < 0$ and $\delta > 0$, the solution (eqn. (3.14)) is shown in (Fig. 8). This solution oscillates between the values w_1 and w_2 with a period of 2π and is non-singular. It represents a spin wave (changing between various stable and metastable phases). Increasing the field increases the amplitude of these waves.

3.1.3 Elliptic solutions

The remaining integrals (the most general cases) are elliptic, and thus contain only periodic (and singular) solutions. Each integral, all from [26], has either an upper or lower integration limit; this does not mean a boundary condition has been lost.

The solutions are expressed in terms of the Jacobian elliptic functions $\mathrm{tn}(\zeta, k)$, $\mathrm{cn}(\zeta, k)$ and $\mathrm{sn}(\zeta, k)$. (*N.B.* $\mathrm{tn}(\zeta, k) = \mathrm{sn}(\zeta, k)/\mathrm{cn}(\zeta, k)$.) The parameter, k, is called the modulus; the complementary modulus is $k' = \sqrt{1 - k^2}$. Generally $0 <$

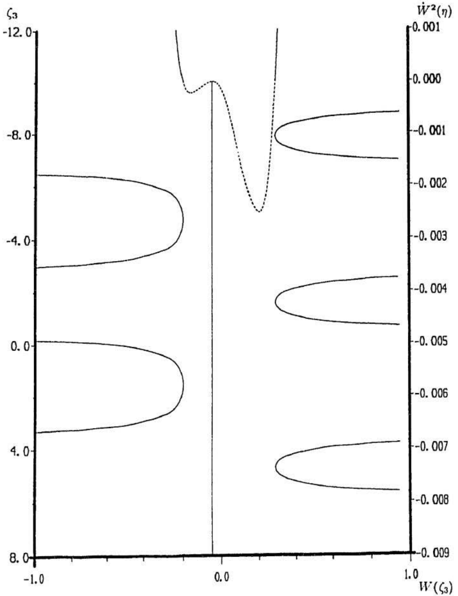

Figure 7: W^4 2SD Two distinct single and a double real root, $D > 0$, $\delta < 0$, $w_1 = 0.3$, $w_2 = -0.2$, and $w = -0.05$

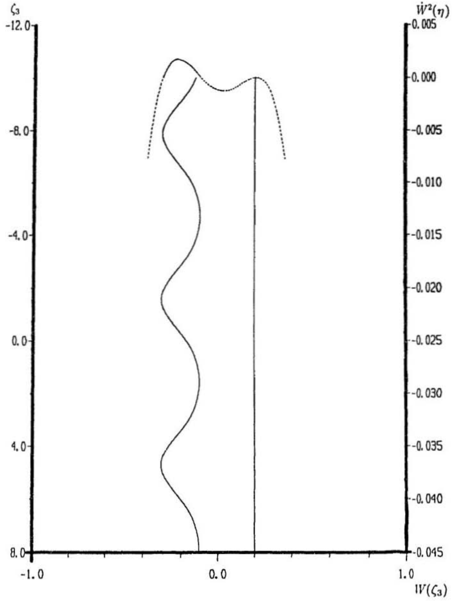

Figure 8: W^4 2SD Two distinct single and a double real root, $D < 0$, $\delta > 0$, $w_1 = -0.1$, $w_2 = -0.3$, and $w = 0.2$

$k < 1$, if not, a transformation is employed to make it so. Here, this inequality is true by definition. These functions are one valued functions of ζ, and are doubly periodic, having one real period and one complex period. The periods are, respectively, $(2\mathcal{K}, 4i\mathcal{K}')$, $(4\mathcal{K}, 2\mathcal{K} + 2i\mathcal{K}')$ and $(4\mathcal{K}, 2i\mathcal{K}')$. (*N.B.* The function $\mathrm{sn}^2(\zeta, k)$ has period $(2\mathcal{K}, 2i\mathcal{K}')$.) The periodic function is defined by the complete elliptic integral (of the first kind):

$$\int_0^{\frac{\pi}{2}} \frac{d\vartheta}{\sqrt{1 - k^2 \sin^2 \vartheta}} = \mathcal{K}(k) \equiv \mathcal{K} \tag{3.15}$$

and $\mathcal{K}(k') \equiv \mathcal{K}'$.

The graphs have not been drawn in this section because it would require three dimensional graphing, and because of some technical problems. The two basic elliptic functions $\mathrm{sn}(\zeta, k)$ and $\mathrm{cn}(\zeta, k)$ have as limiting functions:

$$\mathrm{sn}(\zeta, 0) = \sin\zeta\,, \quad \mathrm{sn}(\zeta, 1) = \tanh\zeta\,,$$

$$\mathrm{cn}(\zeta, 0) = \cos\zeta\,, \quad \mathrm{cn}(\zeta, 1) = \mathrm{sech}\zeta\,;$$

and limiting values:

$$-1 \leq \mathrm{sn}(\zeta, k) \leq 1, \quad -1 \leq \mathrm{cn}(\zeta, k) \leq 1,$$

$$-\infty \leq \mathrm{tn}(\zeta, k) \leq \infty\,.$$

These are all periodic functions, so the graphs are expected to be either periodically singular (as in 2SD ($D > 0, \delta < 0$) Fig. 7), but with modulations on the curve;

or periodic and bounded (as in 2SD ($D < 0, \delta > 0$) Fig. 8), and again with modulations on the curve. There are no multiple roots in this section, so there will be no constant solutions and no bumps. Essentially, it is not necessary to see the details of these solutions as it is possible to deduce the general shape and nature of each graph from the work done earlier.

Two distinct complex conjugate pairs of roots (2C)

The roots are:

$$w_1 = w_2^* = p + iq_1,\ w_3 = w_4^* = -p + iq_2\,,$$

$$p, q_1, q_2 \in \Re;\ p, q_1, q_2 \neq 0,\ q_1^2 \neq q_2^2.$$

The field is $h = p_1(q_2^2 - q_1^2)B/2$, and $s_0 = (p^2 + q_1^2)(p^2 + q_2^2)B/4 > 0$, and $A = (q_1^2 + q_2^2 - 2p^2)B/2$. These solutions are in terms of $\mathrm{tn}(\zeta_4)$ with the real period $2K$. The only real solution is obtained when $D > 0$:

$$W(\zeta_4) = p + \frac{q_1}{\alpha_1}\left[1 - \frac{\alpha_1^2 + 1}{\alpha_1 \mathrm{tn}(\zeta_4, k_1) + 1}\right], \tag{3.16}$$

where the argument is

$$\zeta_4 = \frac{(\mathcal{A} + \mathcal{B})}{4}\sqrt{\frac{B}{D}}(\xi - \xi_0)\,; \tag{3.17}$$

and

$$\mathcal{A}^2 = 4p^2 + (q_1 + q_2)^2,\ \ \mathcal{B}^2 = 4p^2 + (q_1 - q_2)^2,$$

$$k_1^2 = \tfrac{4\mathcal{A}\mathcal{B}}{(\mathcal{A}+\mathcal{B})^2} < 1\,, \qquad \alpha_1^2 = \frac{4q_1^2 - (\mathcal{A} - \mathcal{B})^2}{(\mathcal{A} + \mathcal{B})^2 - 4q_1^2}.$$

It is periodically singular at $\text{tn}(\zeta_4, k_1) = -1/\alpha_1$. These solutions represent a periodic arrangement of magnetic double 'layers' of differently magnetized regions. These solutions can reduce to the CD case. The graph will be similar to (Fig. 7), but without the 'gap' between 'layers' because there are no real roots to create an inaccessible region.

A complex conjugate pair and two distinct real roots (C2S)

The roots are:

$$w_1 > w_2,\ w_3 = w_4^* = p + iq,\ p = -\frac{1}{2}(w_1 - w_2)\,;$$

$$w_1, w_2, p, q \in \Re;\ w_1, w_2, p, q \neq 0,\ w_1^2 \neq w_2^2\,.$$

The field is $h = -Bp(p^2 + q^2)/2$, and $s_0 = w_1 w_2(p^2 + q^2)B/4 > 0$ and $A = -(3p^2 - w_1 w_2 - q^2)B/2$. These solutions are in terms of $\text{cn}(\zeta_5)$ which has real period $4\mathcal{K}$. The argument is:

$$\zeta_5 = \frac{\sqrt{\mathcal{AB}}}{2}\sqrt{\frac{B}{D}}(\xi - \xi_0)\,, \tag{3.18}$$

with these defined constants:

$$\mathcal{A}^2 = \tfrac{1}{4}(3w_1 + w_2)^2 + q^2, \qquad \mathcal{B}^2 = \frac{1}{4}(w_1 + 3w_2)^2 + q^2,$$

$$k_2^2 = \tfrac{1}{4\mathcal{AB}}\left[(\mathcal{A} + \mathcal{B})^2 - (w_1 - w_2)^2\right]\ ,\ \ k_2'^2 = 1 - k_2^2\,.$$

a) When $D > 0$, two periodically singular solutions, valid for $W(\zeta_5) > w_1$ and $w_2 > W(\zeta_5 + 2\mathcal{K}(k_2))$, are found:

$$W(\zeta_5) = \frac{w_1\mathcal{B} + w_2\mathcal{A}}{\mathcal{A}+\mathcal{B}} + \frac{2(w_1 - w_2)\mathcal{A}\mathcal{B}}{(\mathcal{A}+\mathcal{B})^2\mathrm{cn}(\zeta_5, k_2) - \mathcal{A}^2 + \mathcal{B}^2}. \tag{3.19}$$

The graph will be similar to (Fig. 7) including the 'gap' of width $w_1 - w_2$.

b) When $D < 0$, two bounded periodic solutions, valid for $w_1 \geq W(\zeta_5) > w_2$ and $w_1 > W(\zeta_5 + 2\mathcal{K}(k_2')) \geq w_2$, are obtained:

$$W(\zeta_5) = \frac{w_1\mathcal{B} - w_2\mathcal{A}}{\mathcal{B}-\mathcal{A}} + \frac{2(w_1 - w_2)\mathcal{A}\mathcal{B}}{(\mathcal{A}-\mathcal{B})^2\mathrm{cn}(\zeta_5, k_2') + \mathcal{A}^2 - \mathcal{B}^2}. \tag{3.20}$$

These solutions oscillate between w_1 and w_2. There are two solutions for one region in this case because of the integration limit mentioned earlier. The graph will be similar in nature to Fig. 8. They will reduce to the simpler cases (CD) and (2SD).

Four distinct real roots (4S)

The roots are:

$$w_1 > w_2 > w_3 > w_4 \in \Re,$$

and distinct from each other and their negatives, $w_1 > 0 > w_4$. The field is the general case:

$$h = \frac{C}{6}\frac{(w_1+w_2)(w_2+w_3)(w_3+w_4)(w_2+w_4)(w_1+w_3)}{(w_1+w_2+w_3+w_4)}.$$

These solutions are all in terms of the elliptic function $\mathrm{sn}^2(\zeta_6, k_3^{(\prime)})$ with real period $2\mathcal{K}(k_3)$. The argument is:

$$\zeta_6 = \frac{1}{4}\sqrt{(w_1 - w_3)(w_2 - w_4)}\sqrt{\frac{B}{D}}(\xi - \xi_0), \tag{3.21}$$

with the modulus given by:

$$0 < k_3^2 = \frac{(w_2 - w_3)\,(w_1 - w_4)}{(w_1 - w_3)\,(w_2 - w_4)} < 1, \;\; 0 < k_3'^2 = 1 - k_3^2 < 1.$$

The argument ζ_6 can be real or imaginary, since $\mathrm{sn}^2(i\zeta, k) = -\mathrm{tn}^2(\zeta, k')$. All of these solutions can be found from any other, by the appropriate choice of argument; *e.g.* (3.23) can be generated by using $\zeta_6 + \mathcal{K} + i\mathcal{K}$ in (3.22).

a) When $D > 0$, there are four regions, two of which produce singular periodic solutions, and two of which produce bounded periodic solutions.

For the regions $w_1 < W(\zeta_6)$ and $W(\zeta_6 + \mathcal{K}) < w_4$, the solution is:

$$W(\zeta_6) = w_2 + \frac{(w_1 - w_2)}{1 - \alpha_2^2 \mathrm{sn}^2(\zeta_6, k_3)}, \tag{3.22}$$

$$1 < \alpha_2^2 = \frac{(w_1 - w_4)}{(w_2 - w_4)};$$

and it is singular at $\mathrm{sn}^2(\zeta_6, k_3) = \alpha_2^{-2} < 1$. The graph is similar to Fig. 7.

For the regions $w_3 \leq W(\zeta_6) < w_2$ and $w_3 < W(\zeta_6 + \mathcal{K}) \leq w_2$, the solution is:

$$W(\zeta_6) = w_1 - \frac{(w_1 - w_2)}{1 - \alpha_3^2 \mathrm{sn}^2(\zeta_6, k_3)}, \tag{3.23}$$

$$0 < \alpha_3^2 = \frac{(w_2 - w_3)}{(w_1 - w_3)} < 1.$$

These solutions are non-singular and oscillate between w_2 and w_3. The graph is similar to Fig. 8.

b) When $D < 0$, all the solutions are periodic, non-singular and oscillate between their respective bounds. The graphs are similar to Fig. 8.

For the regions $w_2 < W(\zeta_6) \leq w_1$ and $w_2 \leq W(\zeta_6 + K') < w_1$, the solution is:

$$W(\zeta_6) = w_3 + \frac{(w_2 - w_3)}{1 - \alpha_4^2 \mathrm{sn}^2(\zeta_6, k_3')}, \tag{3.24}$$

$$0 < \alpha_4^2 = \frac{(w_1 - w_2)}{(w_1 - w_3)} < 1 .$$

For the regions $w_4 < W(\zeta_6) \leq w_3$ and $w_4 \leq W(\zeta_6 + K') < w_3$, the solution is:

$$W(\zeta_6) = w_1 - \frac{(w_1 - w_4)}{1 - \alpha_5^2 \mathrm{sn}^2(\zeta_6, k_3')}, \tag{3.25}$$

$$\alpha_5^2 = \frac{-(w_3 - w_4)}{(w_1 - w_3)} < 0 .$$

All of these solutions reduce appropriately to the solutions given for simpler cases in (2SD).

3.2 Solutions of the W^6 equation

The elementary solutions of the equation (3.5):

$$
\begin{aligned}
W^6: \ \dot{W}^2(\eta) &= \epsilon(\lambda + \alpha W + \beta W^2 + \gamma W^4 + W^6) \\
&= \epsilon(W - w_1)(W - w_2)(W - w_3) \times \\
&\quad (W - w_4)(W - w_5)(W - w_6) \qquad (3.26) \\
\epsilon = \pm 1, \ \eta = \kappa^{-1}\xi; \qquad & \kappa^{-2} = \frac{\epsilon C}{6D}, \ (\lambda, \alpha, \beta, \gamma) = \frac{6}{C}(s_0, -h, \frac{A}{2}, \frac{B}{4}),
\end{aligned}
$$

which corresponds to first order phase transitions, are found when at least a double root exists. There are fifteen cases, which can be divided into four categories: constant, five elementary, five elliptic and four hyper-elliptic solutions. The elliptic and hyper-elliptic solutions are not included here. The Landau coefficient B, in all but one case (CDC), is shown to be explicitly negative in the elementary solutions. All solutions are expressed as $\zeta = \zeta(W)$, since it is no longer possible to invert them. This means it is necessary to be very careful about any $\pm$ signs associated with radicals and periodicity (branches) of functions in the implicit solution. The basic solution is 1-1 and will give only one portion of the solution. Proper consideration of the $\pm$ sign and $n\pi$ will give the complete solution. This is particularly important for bumps and periodic solutions. The roots can be real or complex, but any complex roots must occur in conjugate pairs, since the coefficients are real. No other assumptions are made concerning the roots. Since there is no quintic term (W^5) the roots must satisfy:

$$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 = 0 .$$

There is also no cubic term (W^3), thus two roots can be written:

$$w_{5,6} = -\frac{(w_1 + w_2 + w_3 + w_4)}{2} \pm \frac{Q}{2} ,$$

where

$$\begin{aligned} Q^2 &= \left[w_4^3 - (w_1 + w_2 + w_3) w_4^2 - (w_1 + w_2 + w_3)^2 w_4 + w_3^3 \right. \\ &\quad \left. -(w_1 + w_2) w_3^2 - (w_1 + w_2)^2 w_3 + w_2^3 - w_1 w_2 (w_2 + w_1) + w_1^3 \right] \\ &\quad /(w_1 + w_2 + w_3 + w_4) , \end{aligned}$$

($(w_1 + w_2 + w_3 + w_4) \neq 0$ is equivalent to $w_5 \neq -w_6$, and preserves a non-zero field). Using this condition, the coefficients can be written:

$$\begin{aligned} \lambda &= \frac{w_1 w_2 w_3 w_4}{4} \left[(w_1 + w_2 + w_3 + w_4)^2 - Q^2 \right] , \\ \alpha &= -\frac{(w_1 + w_2)(w_1 + w_3)(w_1 + w_4)(w_2 + w_3)(w_2 + w_4)(w_3 + w_4)}{(w_1 + w_2 + w_3 + w_4)} \neq 0 , \\ \beta &= \frac{1}{4} \left[(w_1^3 + w_2^3 + w_3^3 + w - 4^3)(w_1 + w_2 + w_3 + w_4) \right. \\ &\quad +(w_1^2 + w_2^2 + w_3^2 + w_4^2)^2 - 2(w_1^4 + w_2^4 + w_3^4 + w_4^4) \qquad (3.27) \\ &\quad +(w_1^2 - \frac{Q}{2})(w_2 w_3 + w_3 w_1 + w_4 w_1) + (w_2^2 - \frac{Q}{2})(w_3 w_4 + w_4 w_1 + w_1 w_3) \\ &\quad \left. +(w_3^2 - \frac{Q}{2})(w_4 w_1 + w_1 w_2 + w_2 w_3) + (w_4^2 - \frac{Q}{2})(w_1 w_2 + w_2 w_3 + w_3 w_1) \right] , \\ \gamma &= w_1 w_2 + w_3 (w_1 + w_2) + w_4 (w_1 + w_2 + w_3) \\ &\quad -\frac{3}{4}(w_1 + w_2 + w_3 + w_4)^2 - \frac{Q^2}{2} , \end{aligned}$$

with λ generally non-zero. Note, again, $\alpha \propto h$ being non-zero generally requires each root $\{w_i\}$ to be different from the negative of any other $i \neq j$, $w_i \neq -w_j$. The field can be expressed in terms of the roots:

$$h = C\frac{(w_1+w_2)(w_1+w_3)(w_1+w_4)(w_2+w_3)(w_2+w_4)(w_3+w_4)}{6(w_1+w_2+w_3+w_4)} \neq 0\,. \quad (3.28)$$

3.2.1 Constant solutions

The five constant solutions are given by the roots of (3.1) and the integration constant s_0 can then be found by substituting those solutions in (3.2). Naturally, the roots must all be real, since a real solution is sought. These represent homogeneous fields. Again, the symmetry variable on the light cone, gives piece-wise continuous solutions.

3.2.2 Elementary solutions

A complex conjugate pair and a quadruple real root (CQ)

The roots are:

$$w_1 = w_2 = w_3 = w_4 = w, \quad w_5 = w_6^* = w(2+i); 0 \neq w \in \Re\,.$$

The coefficients associated with W^6 require $s_0 = 5Cw^6/6$, $A = 5Cw^4 > 0$ (above the transition temperature), while $B = -10Cw^2/3 < 0$ and the field is $h = -8Cw^5/3$. The only real solution occurs when $D > 0$, (Fig. 9). The argument is:

$$\zeta_7 = 10w^2\sqrt{\frac{C}{6D}}(\xi - \xi_0)$$

$$= \frac{\sqrt{(W-2w)^2+w^2}}{W+w} + \mu\frac{3}{\sqrt{10}}\text{arcsinh}\left[\frac{7w-3W}{W+w}\right], \quad (3.29)$$

such that

$$-w \leq W: \ \mu = -1$$

$$W \leq -w: \ \mu = 1.$$

This solution is singular at $W = -w$. Truncating the singularities leaves a homogeneous magnetic field, proportional to the external field.

Single, double and triple real roots (SDT)

The roots are:

$$w_1 = 5w, \quad w_2 = w_3 = -4w, \quad w_4 = w_5 = w_6 = w; \ 0 \neq w \in \Re.$$

The coefficients of W^6 require $s_0 = 40Cw^6/3$, $A = 55Cw^5 > 0$ (above the transition temperature), while $B = -20Cw^2 < 0$ and the field is $h = 36Cw^5 > 0$, since w is chosen to be positive. The common argument is:

$$\zeta_8 = 5w^2\sqrt{\frac{C}{6|D|}}(\xi - \xi_0). \quad (3.30)$$

a) When $D > 0$, the solution, (Fig. 10), is:

$$\zeta_8 = \frac{\mu}{3\sqrt{5}}\ln\left|\frac{2w}{W+4w}\left(17w - 7W + \mu 3\sqrt{5}\sqrt{(W-5w)(W-w)}\right)\right|$$
$$-\frac{\mu_1}{2}\sqrt{\frac{W-5w}{W-w}}, \mu, \mu_1 = \pm 1; \quad (3.31)$$

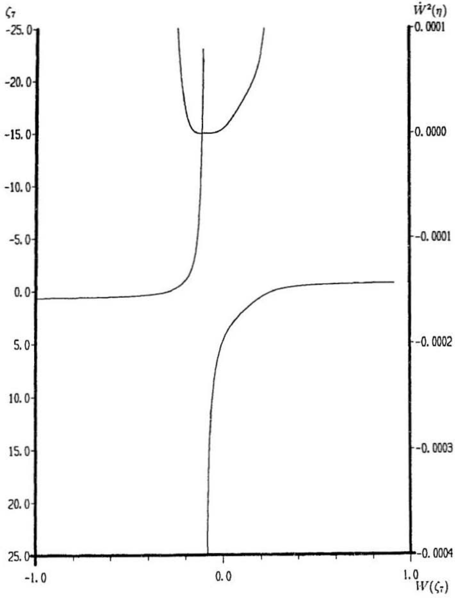

Figure 9: W^6 CQ Complex conjugate pair and a quadruple real root, $D > 0$, $\mu = \pm 1$, $w = 0.1$

where each region is given by:

$$W \leq -4w : \quad \mu_1 = 1$$

$$-4w \leq W \leq w : \quad \mu_1 = 1$$

$$5w \leq W : \quad \mu_1 = -1 \,,$$

and $\mu = \pm 1$ gives the two halves of each solution. There are three areas of solutions; the two outside which are singular and the middle region $-4w \leq W \leq w$ which has kinks. The height of the kinks increases as the field increases.

b) When $D < 0$, the only real solution (Fig. 11) is:

$$\begin{aligned} \zeta_8 \;=\; & \frac{1}{3\sqrt{5}} \arcsin\left(\frac{7}{2} - \frac{45w}{2(W+4w)}\right) \\ & + \frac{n\pi}{3\sqrt{5}} + \frac{1}{2}\sqrt{\frac{5w - W}{W - w}} \end{aligned} \tag{3.32}$$

where n is an integer. The two halves of the solution are $\zeta_8(n=0)$ and $-\zeta_8(n=-1)$, the solid and dotted lines, respectively in Fig. 11. It is valid in the region $w \leq W \leq 5w$ and describes a bump with height $4w$. This represents a nucleation site of magnetic order.

A complex pair and a double complex pair of roots (CDC)

The roots are:

$$w_1 = w_2 = w_3^* = w_4^* = p + iq, \quad w_3 = w_4^* = 2p + i\sqrt{p^2 + q^2}\,,$$

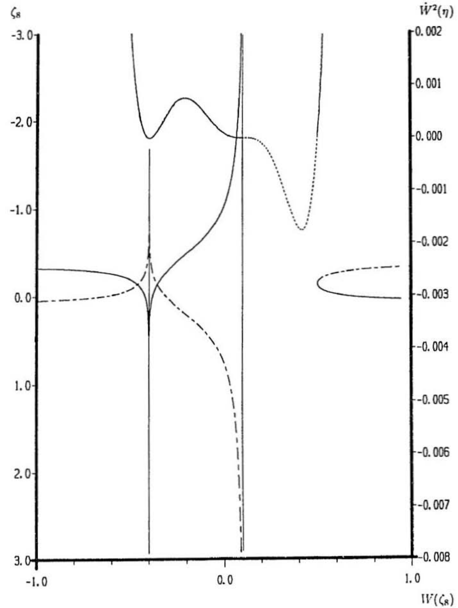

Figure 10: W^6 SDT Single, double, triple real roots, $D > 0$, $w = 0.1$

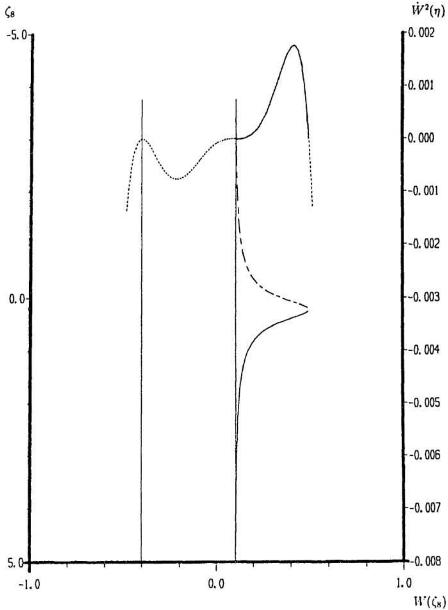

Figure 11: W^{c6} SDT Single, double, triple real roots, $D < 0$, $w = 0.1$, $n = 0, -1$

$$0 \neq p, q \in \Re .$$

The coefficients of W^6 require $s_0 = C(p^2+q^2)^2(q^2+5p^2)/6 > 0$, and $A = C(3q^4 + 2p^2q^2 + 15p^4)/3 > 0$ (above the transition temperature), $B = 2C(3q^2 - 5p^2)/3$ and the field is $h = 8Cp^3(q^2+p^2)/3$. (*N.B.* Here it is not clear whether B is negative.) The only real solution (Fig. 12), necessarily singular, occurs for $D > 0$. The argument is:

$$\begin{aligned}
\zeta_9 &= 2qr\sqrt{\frac{pC}{6D}}(\xi - \xi_0),\ D > 0 \\
&= \frac{1}{2}\sqrt{r-5p}\ln\left|\frac{4(M^2+N^2)}{(W-p)^2+q^2}\right| \\
&\quad -\sqrt{r+5p}\left(\arctan\left(\frac{N(W-p)+Mq}{M(W-p)-Nq}\right)+n\pi\right). \qquad (3.33)
\end{aligned}$$

$$(3.34)$$

where n is an integer, and

$$\begin{aligned}
r &= \sqrt{25p^2+9q^2} > 5p > 0\,, \\
M &= p(3W+7p)+q^2+\sqrt{p(r+5p)R_1(W)}\,, \\
N &= q(W+2p)+\sqrt{p(r-5p)R_1(W)}\,, \\
R_1 &= (W+2p)^2+p^2+q^2\,.
\end{aligned}$$

This solution is periodically singular at

$$\begin{aligned}
\zeta_9 &= \frac{1}{2}\sqrt{r-5p}\ln\left|4[(3p+\sqrt{p(r+5p)})^2+(q+\sqrt{p(r-5p)})^2]\right| \\
&\quad -\sqrt{r+5p}\left(\arctan\left(\frac{q+\sqrt{p(r-5p)}}{3p+\sqrt{p(r+5p)}}\right)-n\pi\right).
\end{aligned}$$

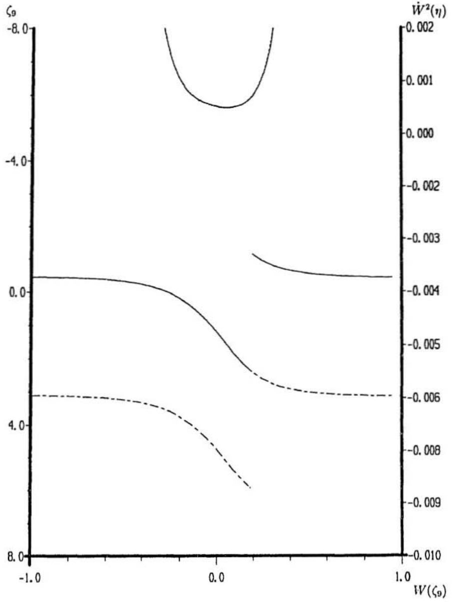

Figure 12: W^6 CDC Complex pair and a double complex pair of roots, $D > 0$, $p = 0.1$, $q = 0.2$, and $n = 0, 1$

Removing the singularities, this solution represents a periodically inhomogeneous (lumpy) magnetic field.

A complex pair and two distinct double roots (C2D)

The roots are:

$$w_1 = w_2 = w, \quad w_3 = w_4 = rw, \quad w_5 = w_6^* = -w(1 + r + i\sqrt{r}),$$

$$0 \neq r, w \in \Re, r > 0, w > rw.$$

The coefficients of W^6 require $s_0 = Cr^2w^6(r^2+3r+1)/6 > 0$, $A = Cw^4(r^4+3r^3+7r^2+3r+1)/3 > 0$ (above the transition temperature), $B = -2Cw^2(2r^2+r+2)/3 < 0$ and the field is $h = Crw^5(1+r)^3/3 > 0$, since w is chosen to be positive. The only real solution (Fig. 13) occurs for $D > 0$:

$$\begin{aligned}
\zeta_{10} &= w^2(1-r)\sqrt{1+r}\sqrt{\frac{C}{6D}}(\xi - \xi_0) \\
&= \frac{\mu_1}{\sqrt{r+4}}\operatorname{arcsinh}\left(\frac{2+r}{\sqrt{r}} + \frac{w(1+r)(4+r)}{\sqrt{r}(W-w)}\right) \\
&\quad + \frac{\mu_2}{\sqrt{4r+1}}\operatorname{arcsinh}\left(\frac{1+2r}{\sqrt{r}} + \frac{w(1+r)(1+4r)}{\sqrt{r}(W-rw)}\right), \qquad (3.35)
\end{aligned}$$

which is valid in these regions:

$$\begin{aligned}
W \leq rw: &\quad \mu_1 = 1,\ \mu_2 = -1 \\
rw \leq W \leq w: &\quad \mu_1 = 1,\ \mu_2 = 1 \\
w \leq W: &\quad \mu_1 = -1,\ \mu_2 = 1.
\end{aligned}$$

There is a kink between rw and w, and singular solutions outside that region. The kink represents two stable magnetic regions, one at w and the other at $-rw$.

Two single and two double real roots (2S2D)

The roots are:

$$w_1 = w_2 = w, \quad w_3 = w_4 = -rw,$$

$$w_5 = w(r - 1 + \sqrt{r}) > w_6 = w(r - 1 - \sqrt{r});$$

with the conditions:

$$0 < r, w \in \Re, \; r \neq \frac{1}{4}, 1, \frac{(3 \pm \sqrt{5})}{2}, 4.$$

The excluded values of r are simpler cases already dealt with ($r = 1/4, 4 \Rightarrow$ SDT), more restricted solutions ($r = (3 \pm \sqrt{5})/2 \Rightarrow s_0 = 0$) or values not allowed because it implies no field ($r = 0, 1, \infty \Rightarrow h = 0$). The Landau coefficients are $s_0 = r^2 w^6((r-1)^2 - r)C/6$, $A = w^4((r-1)^4 + r(r+1)^2 - r^2)C/3$, $B = 2w^2(r - 2(1 + r^2))C/3 < 0$ and the field $h = rw(r-1)^3 C/6$. The analysis is divided by regions of r; $r \in (0, 1/4)$ and $r \in (1/4, 1)$, the solutions for the remaining two regions $r \in (4, \infty)$ and $r \in (1, 4)$ (topologically equivalent, respectively to the first two) can be found by reflecting the ζ axis. The common argument is:

$$\zeta_{11} = w^2(1 + r)\sqrt{1 - r}\sqrt{\frac{C}{6|D|}}(\xi - \xi_0). \tag{3.36}$$

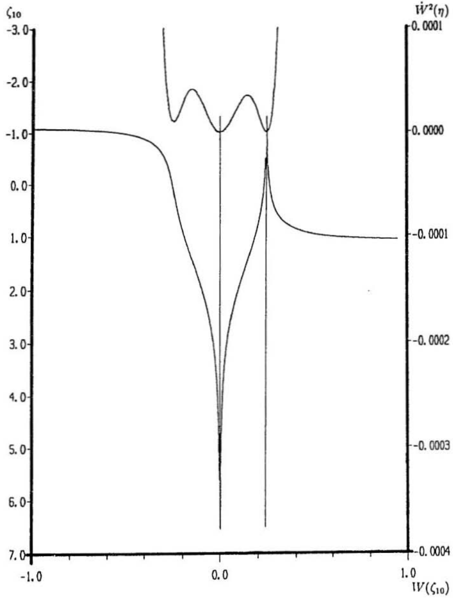

Figure 13: W^6 C2D Complex pair and two double real roots, $D > 0$, $r = 0.01$, $w = 0.25$

a) When $D > 0$ and $r \in (0, 1/4)$, the solution (Fig. 14) is:

$$
\begin{aligned}
\zeta_{11} &= \frac{-\mu_1}{\sqrt{4-r}} \ln \left| \frac{2w}{W-w} \left(w(r-4)(r-1) \right.\right. \\
&\quad \left.\left. +(2-r)(W-w) + \mu_1 \sqrt{(r-4)(r-1)R_2(W)} \right) \right| \\
&\quad + \frac{\mu_2}{\sqrt{1-4r}} \ln \left| \frac{2w}{W+rw} \left(w(4r-1)(r-1) \right.\right. \\
&\quad \left.\left. +(1-2r)(W+rw) + \mu_2 \sqrt{(4r-1)(r-1)R_2(W)} \right) \right| , \qquad (3.37) \\
&\quad \mu_1, \mu_2 = \pm 1
\end{aligned}
$$

where

$$
R_2(W) = (W - w_5)(W - w_6)\,,
$$

and the regions are:

$$
W \le w_6, w_5 \le W \le -rw, w \le W : \ \mu = 1
$$

$$
-rw \le W \le w : \ \mu = -1\,.
$$

The values μ_1 and μ_2 give the halves of two different solutions when matched properly. Fig. 14 shows solutions obtained with $\mu_1\mu_2 = -1$. The second solutions are found using $\mu_1\mu_2 = 1$. Each half ($e.g.$ ζ_{11}, $\mu_1 = 1$, $\mu_2 = -1$) is paired with the negative of it's other half ($e.g.$ $-\zeta_{11}, \mu_1 = -1$, $\mu_2 = 1$). There is a kink between $-rw$ and w, and a bump between $-rw$ and w_5. The kink is the boundary between two stable regions of constant magnetization, and the bump is a nucleation site of magnetic order.

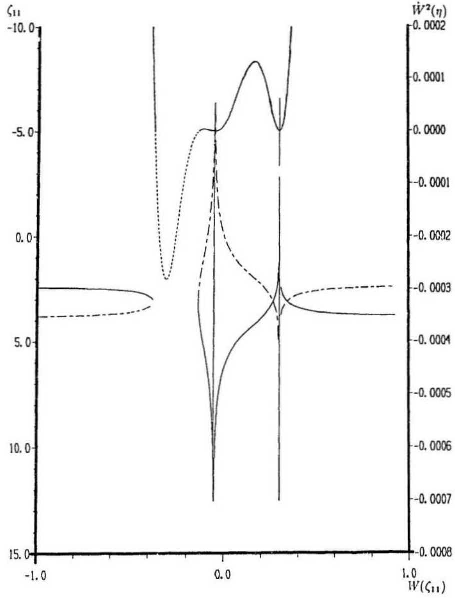

Figure 14: W^6 2S2D Two single and two double real roots, $D > 0$, $r = 0.16$, $w = 0.3$, $\mu_1\mu_2 = -1$

When $D > 0$ and $r \in (1/4, 1)$, the solution (Fig. 15) is:

$$
\begin{aligned}
\zeta_{11} &= \frac{\mu}{\sqrt{4-r}} \ln \left| \frac{2w}{W-w} \left(w(4-r)(1-r) \right. \right. \\
&\quad \left. \left. +(2-r)(W-w) + \mu\sqrt{(4-r)(1-r)R_2(W)} \right) \right| \\
&\quad + \frac{\mu_1}{\sqrt{4r-1}} \arcsin\left(\frac{w(1-4r)(1-r)}{\sqrt{r}(W+rw)} + \frac{1-2r}{\sqrt{r}} \right) \\
&\quad + \frac{n\pi}{\sqrt{4r-1}}, \qquad (3.38)
\end{aligned}
$$

$$(3.39)$$

where

$$
\begin{aligned}
W \leq w_6 &: \quad \mu_1 = -1 \\
w_5 \leq W \leq w &: \quad \mu_1 = 1 \\
w \leq W &: \quad \mu_1 = 1 .
\end{aligned}
$$

The value $\mu = \pm 1$ gives the two halves of the solution in the following way: pair ζ_{11}, $\mu = 1$, $n = 0$ with $-\zeta_{11}$, $\mu = -1$, $n = 1$. The bump between w_3 and w represents a nucleation site of magnetic order.

b) When $D < 0$ and $r \in (0, 1/4)$, the solution (Fig. 16) is:

$$
\begin{aligned}
\zeta_{11} &= \frac{1}{\sqrt{4-r}} \arcsin\left(\frac{w(4-r)(1-r)}{\sqrt{r}(W-w)} + \frac{2-r}{\sqrt{r}} \right) \\
&\quad - \frac{1}{\sqrt{1-4r}} \arcsin\left(\frac{w(1-4r)(1-r)}{\sqrt{r}(W+rw)} + \frac{1-2r}{\sqrt{r}} \right) \\
&\quad + n\pi \left\{ \frac{1}{\sqrt{4-r}} - \frac{1}{\sqrt{1-4r}} \right\}, \qquad (3.40)
\end{aligned}
$$

$$(3.41)$$

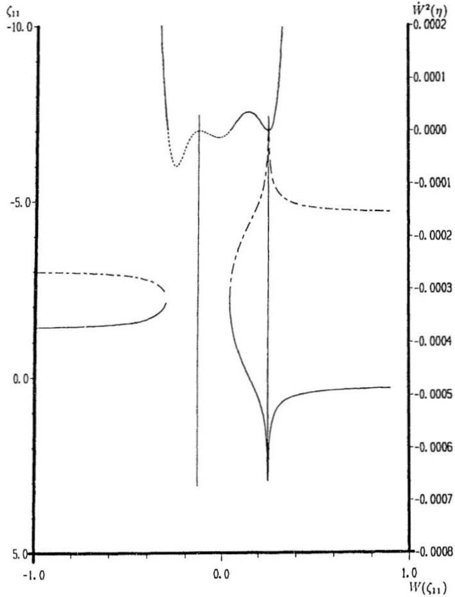

Figure 15: W^{-6} 2S2D Two single and two double real roots. $D > 0$, $r = 0.48$, $w = 0.25$, $n = 0, 1$

the periodic solutions are constructed as follows: $-\zeta_{11}$, $n = 1$; ζ_{11}, $n = 0$; $-\zeta_{11}$, $n = -1$; and ζ_{11}, $n = -2$; respectively, the dashed, solid, solid, dashed lines of Fig. 16, reading from negative to positive ζ_{11}.

When $D < 0$ and $r \in (1/4, 1)$, the solution (Fig. 17) is:

$$\begin{aligned}
\zeta_{11} \;=\; & \frac{1}{\sqrt{4-r}} \arcsin\left(\frac{w(4-r)(1-r)}{\sqrt{r}(W-w)} + \frac{2-r}{\sqrt{r}}\right) + \frac{n\pi}{\sqrt{4-r}} \\
& + \frac{\mu}{\sqrt{4r-1}} \ln\left|\frac{2w}{W+rw}\left(w(1-4r)(1-r)\right.\right. \\
& \left.\left. +(2r-1)(W+rw) + \mu\sqrt{(1-4r)(1-r)R_3(W)}\right)\right|, \qquad (3.42)
\end{aligned}$$

$$(3.43)$$

where

$$R_3 = (w_5 - W)(W - w_6),$$

and $\mu = \pm 1$ gives the two halves of the solution. The solutions by region are:

$$w_6 \le W \le -rw: \quad \zeta_{11},\ \mu = 1,\ n = 0 \text{ with } -\zeta_{11},\ \mu = -1,\ n = -1$$

$$-rw \le W \le w_5: \quad \zeta_{11},\ \mu = 1,\ n = 1 \text{ with } -\zeta_{11},\ \mu = -1,\ n = 0.$$

The graph (Fig. 17) shows two bumps.

3.2.3 Elliptic solutions

There are five distinct cases of elliptic solutions each with one multiple root. This mulptiple root may have a bump solution, and will always have a constant solution associated with it.

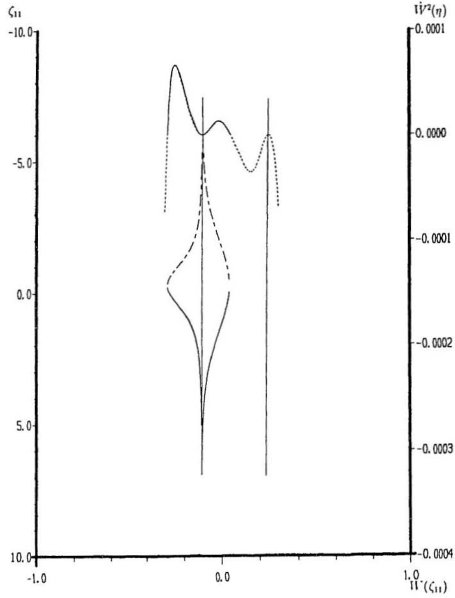

Figure 16: W^6 2S2D Two single and two double real roots, $D < 0$, $r = 0.16$, $w = 0.3$, $n = -2, -1, 0, 1$

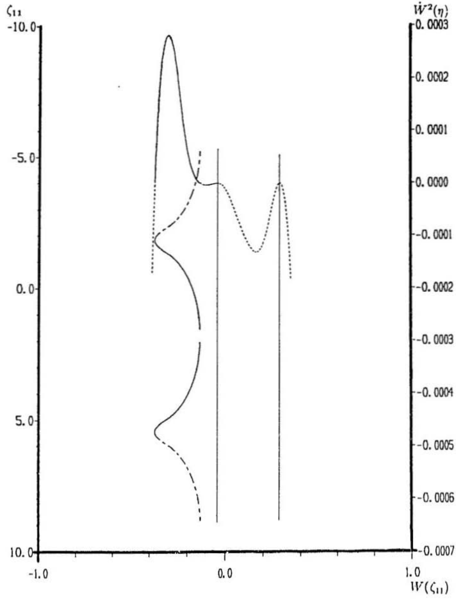

Figure 17: W^6 2S2D Two single and two double real roots, $D < 0$, $r = 0.48$, $w = 0.25$

- A complex pair, real single and a triple root (CST): This case will have solutions very similar to the ST case (see Figs. 2,3). The complex conjugate pair will only change the specific nature of the graph, but not the general shape and boundaries. So for $D > 0$, there will be singular solutions, and for $D < 0$ there will be a bump.

- Three distinct single and a triple root (3ST): This case will have periodic solutions between two single roots, bump solutions between a single and a triple root, and singular solutions outside the real roots. The graphs will look very similar to the 2SD case, since the multiplicity of the root will not change the general shape. So, with the triple root on the outside ($D > 0$), there will be a periodic elliptic solution and a singular periodic elliptic solution. When the triple root is between two single roots ($D > 0$), there will be a bump, and corresponding singular solutions as in Figs. 5,15. For $D < 0$, both situations will have a similar graph, that of a bump and a bounded periodic elliptic solution.

- Two distinct complex pairs and a double root (2CD): This case will have a solution very similar to the CD case (see Fig. 4); the extra complex conjugate pair of roots will not change the general shape of the graph. As in the CD case there is only a solution for $D > 0$. This solution is also similar to CQ (see Fig. 9).

- A complex pair, two distinct single and a double root (C2SD): This case will have solutions very similar to the 2SD case (see Figs. 5-8); again the complex conjugate pair of roots will not change the essence of the graph and solutions.

- Four distinct single and a double root (4SD): This case has three situations each for $D > 0$ and $D < 0$. For $D > 0$: with the double root on the outside, there will be a bump and corresponding singular solution, a periodic elliptic solution and corresponding singular (probably periodic) solution; with three single roots on one side of the double root, there will be one periodic elliptic solution and two singular periodic solutions on the outside; with the double root in the middle there will be a pair of bumps, and the corresponding singular solutions on the outside. For $D < 0$: with the double root in the middle or on the outside, there will be two bounded periodic elliptic solutions; with the double root btween three single roots and one single root, there will be a pair of bump solutions and a bounded periodic elliptic solution.

The integrals and solutions of each can be found in detail in Byrd and Friedman [26]. The integrals are of the form:

$$(\eta - \eta_0) = \int \frac{dW}{(W - w_1)\sqrt{\epsilon(W - w_2)(W - w_3)(W - w_4)(W - w_5)}} \, .$$

3.2.4 Hyper-elliptic solutions

The remaining cases are the most general, but as in the W^4 case, do not have any interesting solitary waves or constant solutions, only periodic solutions. Between two real roots there will be bounded periodic elliptic solutions, and outside the real roots there will be singular periodic elliptic solutions.

- Three complex conjugate pairs (3C): This case will have solutions very similar to CDC (see Fig. 12) and will have only a solution for $D > 0$. It is also similar to the (not graphed) 2C case.
- Two complex conjugate pairs and two distinct real roots (2C2S): This case will have solutions similar to the (not graphed) case C2S.
- One complex conjugate pair and four distinct real roots (C4S): This case will have solutions similar to the (not graphed) case 4S.
- Six distinct real roots (6S): This case will have solutions similar to 4S, but more general in that there are more bounded periodic solutions.

These integrals belong to the class of hyperelliptic integrals, and are of the form:

$$(\eta - \eta_0) = \int \frac{dW}{\sqrt{\epsilon(W - w_1)(W - w_2)(W - w_3)(W - w_4)(W - w_5)(W - w_6)}}$$

and where no two roots are the same.

4 DISCUSSION OF THE SOLUTIONS

In the previous chapter, exact solutions of the equation (3.2) were presented. Some discussion of the effects of changing the parameter h were also discussed. The general interpretation follows. Tables 4,5 summarise these solutions and their interpretations. Table 6 postulates the kinds of solutions of the last two sections of W^6, elliptic and hyper-elliptic solutions. Figures 18, 19, and 20 summarise graphically the solutions in Chapter 3. Moving the horizontal axis up and down corresponds to changing the integration constant s_0, this shows how the solutions between regions connect.

The role of the field has been discussed with respect to each solution. Note that the presence of the field has broken the symmetry of the solutions. Consider the simple case of no external field and no gradient term. The polynomial is cubic or quadratic in M^2. This results in single, double or triple wells. Adding the external field skews those wells, and allows for single and multiple roots. It is the variety of roots that gives rise to the different kinds of solutions. Adding the gradient term gave rise to a differential equation, whereas previously there had been a simple algebraic relationship between the temperature, external field and the order parameter.

It is now possible to make several general statements about the types of solutions that can be expected from considering the shape of the potential $V(W)+s_0 =$

Table 4: Summary of W^4 solutions

Case			Solution Type and Comments
Constant			constant, homogeneous field
ST	$D > 0$		singular, 'tri-layer', two defect planes
	$D < 0$		bump, nucleation center of magnetic order
CD	$D > 0$		singular, mostly homogeneous field one defect plane
2SD	$D > 0$	$\delta > 0$	singular, mostly homogeneous field, 'tri-layer', two defect planes
			bump, nucleation center
		$\delta < 0$	periodically singular periodic defect planes
	$D < 0$	$\delta > 0$	periodic, spin waves
		$\delta < 0$	two bumps, (opposing) nucleation centers
2C	$D > 0$		periodically singular branches similar to W^6 CDC
C2S	$D > 0$		periodically singular similar to W^4 2SD $\delta D < 0$
	$D < 0$		periodic, spin waves similar to W^4 $\delta D < 0$
4S	$D > 0$		periodically singular similar to W^4 $\delta D < 0$
			periodic, spin waves similar to W^4 $\delta D < 0$
	$D < 0$		periodic, spin waves similar to W^4 $\delta D < 0$

Table 5: Summary of W^6 solutions

Case			Solution Type	Comments
Constant			constant	homogeneous field
CQ	$D > 0$		singular	one defect plane
SDT	$D > 0$		singular	unphysical?
			kink	Bloch domain wall
	$D < 0$		bump	nucleation center
CDC	$D > 0$		periodically singular	inhomogeneous field (?) periodic defect planes
C2D	$D > 0$		singular	unphysical?
			kink	Bloch domain wall
2S2D	$D > 0$	$r \in (0, 1/4)$	singular	
			kink	Bloch domain wall
		$r \in (1/4, 1)$	singular	unphysical?
			bump	nucleation center
	$D < 0$	$r \in (0, 1/4)$	periodic	spin waves
		$r \in (1/4, 1)$	bumps	nucleation center

Table 6: Expected elliptic and hyper-elliptic solutions

Case	Solution Types	Similar Cases
CST	singular, bump	W^4 ST
3ST	singular, periodic, bump	W^4 2SD
2CD	singular	W^4 CD, W^6 CQ
C2SD	singular, bump(s), constant periodically singular, periodic	W^4 2SD
4SD	singular, periodic, bump(s)	W^4 2SD
3C	singular	W^4 2C, W^6 CDC
2C2S	singular, periodic	W^4 C2S
C4S	singular, periodic	W^4 4S
6S	singular, periodic	W^4 4S

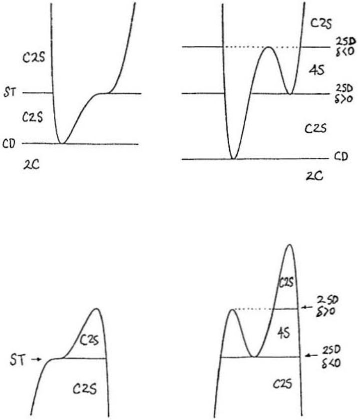

Figure 18: Graphical summary of W^4 solutions

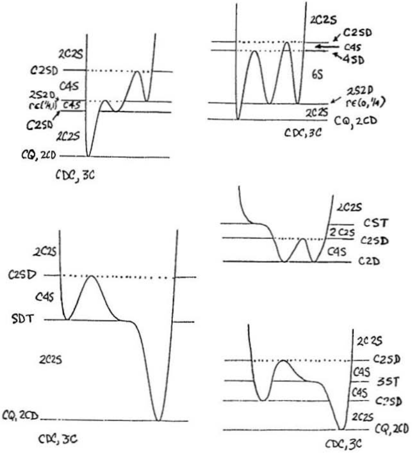

Figure 19: Graphical summary of W^6 $(D > 0)$ solutions

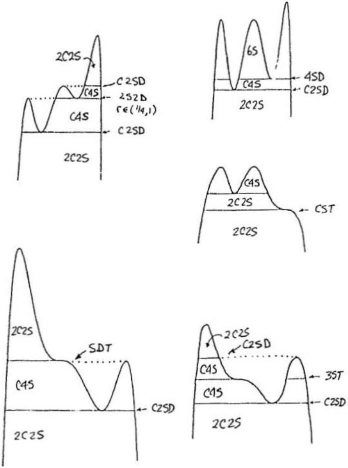

Figure 20: Graphical summary of W^6 ($D < 0$) solutions

$D\dot{W}^2(\xi)$. Where the potential crosses the real axis there is a boundary or root of the solution. Between two single real roots, there are periodic solutions. These may be simply trigonometric, or elliptic. Between a single real root and a multiple real root there is a bump. Between two real multiple roots there is a kink. Between plus/minus infinity and a boundary, there must be singular solutions. The nature of the singularity is determined by the other boundaries (roots) of the potential. A periodic solution 'next door' between two other real roots mirrors its periodicity in the singular solution. A bump between a single root and a multiple root, is also mirrored by the singular solutions next to it. Similarly, kinks have corresponding singular solutions. Complex roots play no part in determining the bounds of a solution, but do affect the exact shape. These statements apply to any potential that satisfies the differential equation given.

In particular, the presence of the external field in the W^4 case prevents any kink solutions. Thus the only solutions possible, and found, are bumps and periodic solutions, together with their corresponding singular solutions. Physically, this means that near a second order phase transition, in the presence of an external field, there cannot be multiply degenerate ground states. This makes sense, the system now has a preferential ground state. Allowing the field to go to zero does not recover the solutions possible with no field because that symmetry has been lost. Adiabatically switching off the field does not leave the system with multiply degenerate ground states. In the W^6 case there are kink solutions, and again this is characteristic of a first order phase transition. Near that transition, with an external field it is possible to have two different ground states.

There are two physical considerations to be made. Free energies with absolute minima are easily dealt with; the system has a natural mode. But in this thesis, negatively unbounded free energy densities occur when $D < 0$. To deal with this, it is necessary to introduce the lattice size a, and require a minimum wavelength of a periodic solution $\lambda_{\min} = 2a$. This is a high frequency cutoff. The singular solutions also present a problem. Physically, the magnetization cannot increase without bound. From one lattice site to the next the most a spin vector can change is from spin up ($+S$) to spin down ($-S$), or vice versa. This can be expressed as a maximum gradient of the magnetization $|\nabla M| \leq 2\gamma S/a$, where γ is the gyromagnetic ratio. This problem occurs because continuity assumes the change in spin from site to site will stay constant as the distance between the sites goes to zero. This is unphysical, so a singular solution is truncated, and the equation gives no useful information about how the solutions behaves closer to the singularity.

4.1 Interpretation of the solutions

Constant solutions

These correspond to homogeneous magnetic fields. It is the trivial solution for W^4 and W^6 cases. There is one symmetry variable $x_0 \pm x_1$ corresponding to a mixed spatial signature whose only solutions are constant. The symmetry variable is a plane, and since W can take only one of five values, it is a piece-wise continuous solution.

Singular solutions

Singular solutions occur outside all the real roots and under complex roots and only when $D > 0$. The singularity is removed by truncating the solution at the maximum value of the gradient, and farther away from the singular points, the solutions can be interpreted as magnetic layers. Periodically singular solutions can be interpreted as a series of periodically spaced double layers. The singularities can be considered as a series of defect planes in the sample. Connecting the solutions before the cutoff would give something like a spin wave, but it may not be justifiable.

Periodic solutions

Spin waves are bounded periodic solutions that oscillate between a stable and a metastable state. It is the most common type of solution. The few examples that were graphed give a good idea of type of solution to be found in the W^4 elliptic cases, and the W^6 hyper-elliptic cases, though not in much detail. The more exotic solutions come from particular combinations of roots.

Bump solutions

Bump solutions reflect a nucleation center of magnetic order in an otherwise homogeneous field. Burt [9] found bump type solutions in his calculations. In the W^6 case it is necessary to carefully consider each piece of a solution to obtain the expected bump. In an actual magnetic system, it may be possible to pin together bumps. The local portion of the bump need only be far enough away on a microscopic scale not to notice or be aware of other bumps in the region.

Kink solutions

Kink solutions are domain walls with two homogeneous phases. The localized interface separates the neighbouring equilibrium phases. Gordon [11] found kink type solutions, with the interface between a ferro-electric phase and a para-electric phase. Kink type solutions only occur in the elementary W^6 cases. The height of the kink solution is directly proportional to the external field, and thus these kinks will disappear when the field goes to zero.

4.2 Calculations based on the solutions

4.3 Stability

The first calculation that needs to be done is that of stability. The condition to be satisfied is given by the second variation of the free energy with respect to the order parameter. The only term that survives in bulk is:

$$\frac{\delta^2 F}{\delta M^2} = \int \left[A + 3BM^2 + 5CM^4\right] d\vec{r} > 0 \tag{4.1}$$

It must be positive to guarantee the solutions found minimise the free energy. Each of the solutions need to be tested to see which are stable, and where the stability condition is satisfied.

Energy

It is not immediately apparent which of the regional solutions graphed will appear in a magnetic system. In the W^4 case, the singular solutions on either side match up asymptotically, but they may not be accessible as a 'double' or 'triple' layer with one or two defect planes at the singularities. It is shown below that the energy of a solution is proportional to the area under the curve of $\dot{W}^2$. This is easily seen to be infinite in the areas that have singular solutions, thus any non-singular solution must have a lower energy and be more attainable.

Having found a large class of exact solutions, it is now possible to do several calculations. First, it is interesting to consider the total free energy of the solution.

$$F = \int_V f(M(\vec{x}), |\nabla M(\vec{x})|) dV. \tag{4.2}$$

The solutions are all in terms of a symmetry variable ξ, so it makes sense to consider a sample in the shape of a right solid with arbitrary base area $S_\perp$ and height ℓ parallel to ξ. The free energy then becomes:

$$F = S_\perp \int_{\xi_0}^{\xi_0+\ell} f(W(\xi), |\dot{W}|)d\xi. \tag{4.3}$$

The free energy density can be expressed as:

$$f = f_0 - hW + \frac{A}{2}W^2 + \frac{B}{4}W^4 + \frac{C}{6}W^6 + D\dot{W}^2. \tag{4.4}$$

Using equation (3.2) the free energy density can be expressed as follows:

$$f = f_0 - s_0 + 2D\dot{W}^2(\xi). \tag{4.5}$$

Now the free energy integral becomes:

$$F = S_\perp \left\{ (f_0 - s_0)\ell + 2D\int_{\xi_0}^{\xi_0+\ell} \dot{W}^2(\xi)d\xi \right\}. \tag{4.6}$$

For periodic solutions, use $\ell = \mathrm{T}\lambda_0$, where T is the period, and λ_0 is an integer. In principle, this calculation can now be performed, though it is difficult.

An example energy calculation for the simplest case in W^4 (ST $D < 0$) gives this result:

$$F = S_\perp \left\{ (f_0 - s_0)\ell - 8w^4 B\ell \left[\frac{(3+z^2)(3z^2-1)}{3(1+z^2)^3} + \arctan z \right] \right\} \tag{4.7}$$

where

$$z^2 = -w^2 D\ell^2 > 0.$$

Susceptibility

Another calculation it is possible to make is that of susceptibility:

$$\chi = \frac{\partial M(x)}{\partial h} \tag{4.8}$$

which becomes, with the symmetry variable,

$$\chi = \frac{\partial W(\xi)}{\partial h}. \tag{4.9}$$

It is necessary to integrate the susceptibility found over space to give an overall susceptibility. This can be difficult to calculate. The expressions for h are generally in terms of more than one root. In principle, though this calculation is now possible. This kind of calculation would give experimentally verifiable results.

5 CONCLUSIONS

This thesis models a three dimensional magnetic system near second and first order phase transitions in the presence of an external field, while including nearest neighbour interactions using a Landau-Ginzburg free energy density of the form:

$$f = f_0 - hM + \frac{1}{2}AM^2 + \frac{1}{4}BM^4 + \frac{1}{6}CM^6 + \nabla M \cdot \mathbf{D} \cdot \nabla M .$$

A variational calculation yields a steady state PDE:

$$2D \left\{ \begin{array}{c} \Delta M \\ \\ \Box M \end{array} \right\} = -h + AM + BM^3 + CM^5 .$$

This equation is a generalization of previous work done in the absence of an external field. Sophisticated group theory is used to find the symmetries of the equation, and reduce it to an ODE:

$$D\dot{W}^2(\xi) = \left\{ \begin{array}{l} s_0 - hW + \frac{1}{2}AW^2 + \frac{1}{4}BW^4 + \frac{1}{6}CW^6 \\ \\ \frac{1}{2}[f - f_0 + s_0] \end{array} \right.$$

which could be related back to the free energy density. The symmetry variables found show more than the usual translational and rotatioanl spatial invariances. The ODE is solved by considering the root structure of the polynomial. Particularly interesting solutions are the bumps and kinks. The most common type of solution are periodic and bounded, either trigonmetric or elliptic spin waves. Two physical cutoffs were imposed: a high frequency cutoff and a maximum change in magnetization per lattice spacing, to deal with negatively unbounded free energies and singular solutions, respectively. In addition, exact solutions have been found to an equation that does not have the Painlevé property. It is obvious from the presentation in the previous chapters that, in spite of having calculated some exact solutions, this is an ongoing project and there is much more to calculate before a complete picture is available. The solutions calculated correspond to the stationary points of the free energy density; it is necessary to show which of these minimise the free energy density and thus will be stable solutions. It is now possible to calculate the energies and susceptibilities. An obvious extension would be to consider how this system changes with time. The equation of motion for $M(t, \vec{x})$, derived using the Onsager relation [5], is:

$$M_t + D\nabla^2 M = (h + AM + BM^3 + CM^5).$$

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