THE COMPRESSIBILITY OF THE EARTH'S CORE AND THE ANTI-DYNAMO THEOREMS

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by

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Abstract

Anti-dynamo theorems are proofs that certain combinations of magnetic and velocity fields cannot produce the dynamo action needed to sustain the magnetic field. They can be divided into two classes. One class applies only to magnetic fields that are constant in time. The second is concerned with the more general case of magnetic fields that are allowed to vary in time.

The previously accepted proofs of this second class are not generally valid in a compressible fluid. An anti-dynamo theorem can be applied in a particular case only if the parameter R_{mc} is much less than one. This parameter is given by

$$R_{mc} = C R_m$$

where R_m is the magnetic Reynolds number or the ratio of the importance of transport processes to ohmic diffusion and C is the Smylie-Rochester compressibility number which gives the fractional compression of material

For large scale processes involving substantial radial motion in the core of the Earth C is about 0.2 while R_m is about 200. Thus an anti-dynamo theorem for time-dependent axisymmetric velocity and magnetic fields does not apply to the core of the Earth. Another theorem for non-radial velocity fields is probably applicable as the density difference on any surface of constant radius in the core

is not likely to be large. A third theorem on two-dimensional fields is hard to apply to the Earth because the system considered in the theorem is of infinite extent along one axis.

The theorems of the first class are not affected by the introduction of compressibility.

Acknowledgements

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"What Song the Syrens sang, or what name Achilles assumed when he hid himself among women, though puzzling Questions are not beyond all conjecture."

Sir Thomas Browne

Hydriotaphia, 1683.

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Introduction

The Earth possesses a magnetic field. It is predominantly a dipole field with a surface strength of a few tenths of a gauss (1 gauss = 10^{-4} T). The dipole is aligned almost but not quite along the Earth's axis of rotation. The field undergoes minor changes from year to year. The fossil magnetisation of rocks shows that the field has existed for over two billion years but in that time has fluctuated greatly, even reversing in sign. Throughout its history, however, the field seems to have been a dipole field lined up with the rotation axis.

The source of the field surely qualifies as a "puzzling Question". It cannot be permanent magnetisation. Aside from the changes over geologic time, the interior temperature of the Earth is far above the critical Curie point at which ferromagnetic behaviour disappears. The field is thought to originate with the motion of the conducting molten metal of the Earth's core. Dynamo theory, a branch of magnetohydrodynamics (MHD), is concerned with the details.

In 1919 Sir Joseph Larmor asked "How could a rotating body such as the Sun become a magnet?". (The Sun too has a magnetic field and the objections to permanent magnetisation for the Earth were not yet clearly established.) As a possible answer he proposed the fundamental idea of dynamo theory: that fluid motion through a magnetic field might generate electric currents in the fluid which could provide the self-same magnetic field. In the absence of such motion, any magnetic field in the conductor decays away.

Interestingly, Larmor realized that the extension of this idea to the Earth would require the existence of deep-seated fluid material in the Earth not believed at that time to exist.

One of the earliest results of dynamo theory was, in a sense, unfortunate. It was the first and most celebrated anti-dynamo theorem (ADT), that of Cowling (1933). The concept of an anti-dynamo theorem is quite simple. It is a statement that for a given combination of velocity and magnetic fields no dynamo action will take place so that the magnetic field decays with time. Cowling's theorem showed that a velocity field symmetric about an axis could not maintain a magnetic field that was everywhere steady in time and symmetric about the same axis. The theorem was surprising and cast doubt on the soundness of Larmor's idea.

It was feared that a general theorem forbidding all dynamo action lurked undiscovered. In investigating possible dynamos Elsasser (1946) advocated the expansion of both velocity and magnetic fields in spherical harmonics. As carried out by Bullard and Gellman (1954), this results in infinite sets of coupled ordinary differential equations for even the simplest configuration of velocity fields. This made actual computation difficult and left the question of the existence of dynamo action open. The discovery of two working, if physically unrealistic, dynamo models (Herzenberg, 1958; Backus, 1958) was invaluable as reassurance that no general theorem would be found. However quite a number of special ADT's have been found and more conjectured. Table 1 collects a number of these.

Table 1 Anti-dynamo Theorems

Velocity Field	Magnetic Field	Reference
1. ADT's unchanged by compressibility		
Axisymmetric	Axisymmetric Steady	Cowling, 1933; Lortz, 1968.
Radial	Arbitrary Steady	Namikawa and Matsushita, 1970
Two dimensional	Two dimensional Steady	Lortz, 1968
11. ADT's changed by compressibility		
Axisymmetric	Axisymmetric Time dependent	Backus and Chandrasekhar, 1956; Backus, 1957; Cowling, 1957b; Braginskii, 1964a
Non-radial	Arbitrary Time dependent	Bullard and Gellman, 1954 Cowling, 1957b; Backus, 1958
Two dimensional	Two dimensional Time dependent	Cowling, 1957b; Moffatt, 1978
111. Conjectured ADT		
Arbitrary	Poloidal Steady	Childress, 1968
Arbitrary	Toroidal Steady	Childress, 1968
Arbitrary	Zero helicity	Moffatt, 1978

The established ADT's can be divided into two classes, each with three members. The first class applies to magnetic fields that are everywhere constant in time. They include Cowling's original theorem, a theorem forbidding dynamo action when the velocity field is purely radial (in a sphere), and a similar theorem on two dimensional motion.

The second class was recognized as a class by Cowling (1957b). They apply to general magnetic fields, that is, for magnetic fields that are allowed to vary in time. The members of this group are: axisymmetric velocity and magnetic fields; velocity fields without a radial component and any magnetic field; and two dimensional velocity and magnetic fields.

The theory of astrophysical dynamos is of formidable complexity. It is understandable that simplifying assumptions will be made in working it out. A common assumption is that the fluid is incompressible. This thesis examines some of the results of relaxing this requirement. This point is of geophysical significance.

The material making up the outer core of the Earth is not incompressible. This is shown by the fact that it transmits P-waves. The density difference between the top and bottom of the liquid core due to compression is some 20%. Smylie and Rochester (1979) have shown by analysis of the equations governing the dynamics of the liquid core that compressibility is important for large scale motions.

This raises the question of whether compressibility might not be of importance in dynamo theory as well. Dimensional arguments show that the effects of compressibility are not always small and can be substantial. A striking result concerns the second class of ADT's. The proofs of this group all rely on flow in the fluid being divergenceless. This follows in the proofs from the assumption that the fluid is incompressible. Flow in an incompressible fluid is solenoidal, that is, has a divergence of zero. Solenoidal flow in a compressible fluid is, of course, possible but not necessary.

This is not a statement that dynamos violating the conditions laid down by the ADT's exist; there are no existence proofs here. The non-existence proofs are however nullified.

The compressibility of the core is of importance chiefly when material rises through the hydrostatic pressure gradient. The key parameter is what I have called (very much for want of anything better) 'the compressible part of the magnetic Reynolds number', R_{mc}. It is the product of the conventional magnetic Reynolds number, R_m, and the Smylie-Rochester compressibility number, C. The necessary condition for the ADT's to fail is

$$R_{mc} > 1$$

This is likely fulfilled for the axisymmetric and two dimensional cases. However the ADT for non-radial velocity fields is not affected by the purely radial hydrostatic pressure gradient.

The effects of compressibility on the first class of ADT's are much less marked. Indeed, Namikawa and Matsushita (1970) remark that compressibility is likely to be of importance for dynamo theory; The assumption of steady magnetic fields is quite a strict condition. Any fluctuation anywhere is forbidden.

The failure of the second class of ADT is important as their effects on the history of dynamo theory have been great. This is especially so for the axisymmetric theorem. The magnetic field is observed to be highly axisymmetric while rotation is expected to make axial symmetry likely for the velocity field as well.

A way out of this difficulty was proposed by Parker (1955). He realized that a system not axisymmetric in detail could still be axisymmetric in the mean. This important concept was pursued by Steenbeck, Krause, and Radler (English translation in Roberts and Stix, 1971) who separated the velocity field into two parts having two different scales of length, one large-scale mean part and a smaller-scale turbulent or random part. Much progress has been made along this road (see e.g. Moffatt, 1978).

A different approach is the nearly axisymmetric dynamo of Braginskii (1964a,b). The circulation of the core is conceived as being large scale. It, and the magnetic field, are represented by a predominant axisymmetric part and a smaller non-axisymmetric part. Solutions are sought by a perturbation technique. This model and its derivatives are the leading examples of the one-scale method (Gubbins, 1974).

Both schools grew out of the necessity of avoiding Cowling's theorem on axial symmetry. This is the importance of the failure of the time dependent version of the theorem in a compressible fluid. The failure of the theorems on non-radial and two dimensional velocity fields are also of interest for reasons that will be discussed.

If results as important as these can be changed by relaxing the assumption of incompressible flow, perhaps the rest of dynamo theory needs to be examined with that in mind. The intractable nature of the subject makes this difficult without extended analysis but some starts can be made.

Before turning to mathematical physics we must first look at the nature of the core of the Earth and other such matters.

2. The Core of the Earth

This chapter is a brief exposition of some of the properties of the core. As the core is shielded from us by a great thickness of rock our knowledge of it is indirect; sometimes exceedingly so.

Seismology reveals that the core can be divided into two parts: an outer core that fails to transmit shear waves and an inner one that does. The solid inner core has a radius of some 1200 km while the fluid outer core extends to 3500 km. The thickness of the outer core is thus about 2300 km. This will be taken as the typical length for processes involving the whole (outer) core.

The average density and moment of inertia of the Earth together indicate a high central density. The zero pressure density of the outer core is perhaps 6.3×10^3 kg m^{-3} (Stacey, 1972) The cosmic abundance of the elements makes iron the most likely main constituent. The density of molten iron is 7.0×10^3 kg m^{-3}. A lighter component must be present, silicon, sulphur, and oxygen all being possible (Loper, 1978).

The lighter component may have important consequences as a possible source of the energy needed to power the geo-dynamo. As the Earth cools the solid inner core grows from the melt. Since the solid is more metallic than the melt, the layer above the inner core becomes enriched in the lighter non-metallic component which naturally moves towards the top of the core, driven by buoyancy. This mechanical stirring of the core is said to be a highly effective means of driving circulation there. Thermal convection driven by radioactive decay, perhaps of potassium, is much less efficient as the conduction of heat up the adiabatic temperature gradient would be large (Loper, 1978).

If the circulation is vigorous enough, the core is well-mixed; that is, chemically homogeneous and adiabatically stratified. Another possibility is that the core is thermally stably stratified (Higgins and Kennedy, 1971). In this case radial motion would be inhibited, though oscillatory radial motion would still be possible. However, the Higgins-Kennedy hypothesis rests on the extrapolation of experiment at modest pressures to very high ones, and on theoretical arguments of uncertain validity so that the evidence for it is not compelling.

The anti-dynamo theorem for non-radial motion must be considered an argument against this idea. This matter will be discussed below.

A chemically homogeneous outer core can account for the observed variation of density with depth when the effects of pressure are considered (Dziewonski et al., 1975). Figure 1 shows that the density of the outer core varies from 12.1×10^3 kg m^{-3} at the bottom to 9.9 at the top, a difference of about 20%. Material moving large distance radially will expand and contract by considerable amounts.

A quantity that will be of interest is the radial derivative of density divided by the density. Figure 2 shows

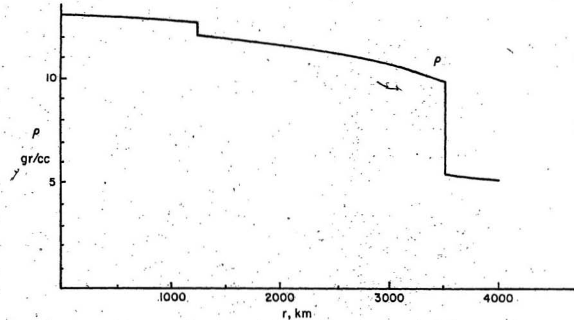

Figure 1. The variation of density, ρ , with radius, r , in the Earth.

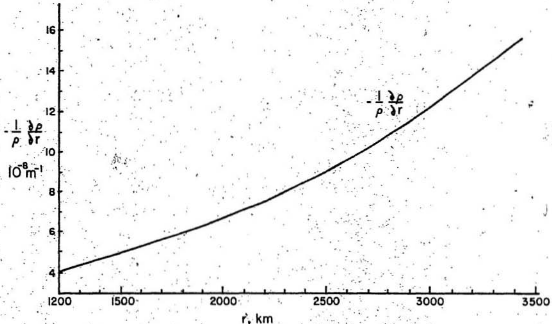

Figure 2. The radial derivative of density divided by the density, $-\frac{1}{\rho}\frac{\partial\rho}{\partial r}$, in the outer core.

that this quantity varies from $4 \times 10^{-8}\ m^{-1}$ at the bottom of the liquid core to about $16 \times 10^{-8}\ m^{-1}$ at the top. Averaged over the core, a typical value is $11 \times 10^{-8}\ m^{-1}$.

An important parameter in dynamo theory is the electrical conductivity of the core. Extrapolation of laboratory data suggests a value of around $5 \times 10^{5}\ S\ m^{-1}$ (Gardiner and Stacey, 1971).

The speed and pattern of flow, i.e. the velocity field, is poorly known. However, it may be possible to get some idea of a typical speed. A theorem of MHD says that a magnetic field in a highly conducting fluid tends to move with that fluid; the field is "frozen in". When the major features of the Earth's magnetic field are mapped year by year, they show a slow westward drift of some 11 minutes of arc a year. If this change is caused by the motion of core material then the corresponding speed is about $10^{-4}\ m\ s^{-1}$. This might be described as a plausible estimate of the typical speed (Bullard et al, 1950).

The pattern of flow is even more unclear. However, using the typical velocity, the typical length scale and the angular velocity of the Earth's rotation, we can express the importance of rotation by the Rossby number (the typical velocity divided by the typical length times the angular velocity, see Greenspan, 1968, p.7) which for the core is about 3×10^{-7}. This means that rotation is very important; a high degree of axial symmetry would not be surprising.

The flow in the core may be said to be magneto-geostrophic; that is, the Coriolis forces are balanced by the Lorentz forces. The fields required, perhaps 10^{-2} Teslas (100 gauss) are not unreasonable (Bullard and Gellman, 1954).

Observations in the Solar System indicate that rotation must be a major factor in the generation of magnetic fields. The way the dipole field of the Earth aligns with the rotation axis is one point. Table 2 collects some facts about the inner five planets. An interesting pattern exists in the first three which all have about the same density. Earth has the highest rotation rate and the strongest magnetic field. Venus rotates most slowly and has a very weak field, if any. Mercury is in between. Mars and our own Moon spoil this progression. However their density is smaller so that any metallic core that may be present may be reduced in size or solidified. Jupiter has a very strong magnetic field and a short day.

Besides leading us to expect some degree of axial symmetry, the effects of rotation may make another anti-dynamo theorem important. A theorem of fluid mechanics makes two dimensional motion very likely in any case where rotation dominates. This too will be discussed below.

In preparation for some work with equations, Table 3 collects some of the parameters from above.

Table 2. Planetary Magnetic Fields.

	Density gr/cc	Rotation Period Earth days	Typical B Field at surface, gauss
Mercury	5.4	59	3.3×10^{-3}
Venus	5.2	243	1.8×10^{-4}
Earth	5.5	1.0	3.11×10^{-1}
Mars	3.9	1.026	6.36×10^{-4}
Moon	3.34	27.3	4×10^{-4}
Jupiter	1.33	0.41	3.61

From Moffatt (1978) p. 76
and Hartmann (1972) p. 265

Table 3. Core Parameters

Radius of inner core		1200 km
Radius of outer core		3500 km
Thickness of outer core		2300 km
Density of outer core	$\bar{\rho}$	10.9×10^3 kg/m^3
Fractional density derivative	$\left\lvert \frac{1}{\rho} \frac{\partial \rho}{\partial r} \right\rvert$	11×10^{-8} m^{-1}
Electrical conductivity	σ	5×10^5 S/m
Typical velocity	V	10^{-4} m/s

3. Introduction to Dynamo Theory

Dynamo theory might be called the astrophysical branch of magnetohydrodynamics (MHD). It is divided from laboratory MHD by the large typical length scale of the processes with which it is concerned. MHD is itself separated from plasma physics in that it deals with fields that vary only slowly with time. We consider an electrically conducting fluid obeying Ohm's law.

The equations of motion of the fluid are just the normal hydrodynamic ones with the addition of the Lorentz force. Here we must make a distinction between kinematic dynamo theory and the full hydromagnetic problem. In kinematic dynamo theory we take the velocity field as known and ask whether it is capable of sustaining or increasing a magnetic field. The forces that drive the flow, in particular the Lorentz force, are ignored. Hydromagnetic or dynamic dynamo theory introduces the forces and the reaction of the magnetic field on the velocity field. This is clearly a more difficult task. Anti-dynamo theorems fortunately belong to the kinematic branch so that we may restrict ourselves to the simpler of the two theories.

Even in the kinematic theory, the velocity field must be a possible one. Mass must be conserved; the flow must obey the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \qquad 3.1$$

where ρ is the density, t is time, and $\vec{v}$ the velocity field. We may write this as

$$\frac{D\rho}{Dt} + \rho(\vec{\nabla} \cdot \vec{v}) = 0 \qquad 3.2$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \qquad 3.3$$

is the Lagrangian or material derivative. It is the rate of change in a quantity over a small parcel of fluid as that parcel moves in the velocity field.

Now the density of a parcel of incompressible fluid will not change. Thus the equation of continuity becomes

$$\vec{\nabla} \cdot \vec{v} = 0 \qquad 3.4$$

A velocity field obeying this equation is said to be solenoidal. Flow in an incompressible fluid is divergenceless.

Flow in a compressible fluid may be solenoidal but is not generally so.

Let us write down, in the MKSA system, Maxwell's equations for an isotropic medium with the permeability of free space

$$\vec{\nabla} \cdot \epsilon \vec{E} = q \qquad 3.5$$

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \epsilon \frac{\partial \vec{E}}{\partial t} \qquad 3.6$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad 3.7$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad 3.8$$

where $\vec{E}$ is the electric field, ϵ the dielectric constant, q the charge density, $\vec{B}$ the magnetic field, μ_o the permeability of free space, and $\vec{J}$ the current density.

We have limited ourselves to fields varying slowly with time. This is equivalent to the neglect of the displacement current in Ampere's law (3.6). If a typical time T is much longer than the time light takes to cross a typical length L, i.e if

$$\frac{L}{T} << c$$

then

$$\frac{|\mu_0 \epsilon\, \partial \vec{E}/\partial t|}{|\vec{\nabla} \times \vec{B}|} \cong \frac{1}{c^2}\left(\frac{L}{T}\right)^2 << 1$$

where we have used $\mu_0 \epsilon \cong 1/c^2$. If a typical velocity V is just L/T then this is equivalent to

$$\frac{V^2}{c^2} << 1$$

which will always be the case.

We can also ignore the current caused by the convective motion of charge, as

$$\frac{qV}{|\vec{J}|} \cong V \epsilon \mu_0 \frac{L}{T} \cong \frac{V^2}{c^2}$$

With the neglect of the displacement current Coulomb's law (3.5) is only needed to determine the charge density from the fields which are determined by the other three equations.

The three equations are

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} \tag{3.9}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3.10}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{3.11}$$

From (3.9) it follows that the equation of continuity for charge is

$$\vec{\nabla} \cdot \vec{J} = 0 \tag{3.12}$$

We have remarked that Ohm's law will be obeyed. For quantities (primed) fixed with respect to a moving medium

$$\vec{J}' = \sigma \vec{E}' \tag{3.13}$$

where σ is the conductivity.

A Lorentz transformation of the fields neglecting terms in $\frac{V^2}{c^2}$ yields

$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B}, \quad \vec{B}' = \vec{B}, \quad \vec{J}' = \vec{J}$$

where the last follows from the second.

Substituting the effective electric field into (3.13) gives Ohm's law in a moving medium.

$$\vec{J} = \sigma (\vec{E} + \vec{v} \times \vec{B}) \tag{3.14}$$

We are now in a position to derive the induction equation of magnetohydrodynamics.

Putting (3.14) into (3.9) gives

$$\vec{\nabla} \times \vec{B} = \mu_o \sigma \left(\vec{E} + \vec{v} \times \vec{B} \right) \qquad 3.15$$

Taking the curl of this and using (3.10)

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = -\mu_o \sigma \frac{\partial \vec{B}}{\partial t} + \mu_o \sigma \vec{\nabla} \times (\vec{v} \times \vec{B})$$

Using a vector identity and (3.11) gives the induction equation

$$\frac{\partial \vec{B}}{\partial t} = D_m \nabla^2 \vec{B} + \vec{\nabla} \times (\vec{v} \times \vec{B}) \qquad 3.16$$

where D_m is equal to $1/\mu_o \sigma$ and is called the magnetic diffusivity.

Equation (3.11) means that we can express $\vec{B}$ in terms of $\vec{A}$ the vector potential:

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad 3.17$$

By choosing the Coulomb gauge we can assure that

$$\vec{\nabla} \cdot \vec{A} = 0 \qquad 3.18$$

We will also require that

$$\vec{A} \to 0 \quad , \quad |r| \to \infty \qquad 3.19$$

This will fix the vector potential.

Putting (3.17) into (3.10) gives

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}$$

showing that $\vec{E}$ differs from $-\partial A/\partial t$ by at most a gradient of some function Φ :

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi$$

Putting this and (3.17) into (3.15) yields the 'uncurled' induction equation.

$$\frac{\partial \vec{A}}{\partial t} = \vec{v} \times (\vec{\nabla} \times \vec{A}) - \vec{\nabla}\Phi + D_m \nabla^2 \vec{A} \qquad 3.20$$

Suppose the velocity field in a conducting body is zero. Suppose also that at some initial time a magnetic field is present. The induction equation becomes

$$\frac{\partial \vec{B}}{\partial t} = D_m \nabla^2 \vec{B}$$

We may look for natural modes $\vec{B}_i$ decaying exponentially

$$\vec{B}_i(\vec{x},t) = \vec{B}_i(\vec{x}) \exp(p_i t)$$

Such functions form a complete set so that we may express any total magnetic field as a sum of the eigenfunctions.

All the eigenvalues p_i are real and negative (Moffatt, 1978, pp. 36–42). Therefore each mode has its typical decay time. For a sphere of radius R the slowest decaying mode is a dipole. It has a decay time $R^2/D_m\pi^2$. For the

Earth this is about 25,000 years. The field has of course been present for very much longer.

When the conductivity of the fluid is high, Alfven's theorem applies. Simply stated, lines of magnetic force behave as if they were frozen into the fluid and move with it. The flux F through a surface bounded by a material curve varies in time with the integral of the effective electric field around the curve by Faraday's law

$$\frac{dF}{dt} = \oint_C (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{l}$$

By Ohm's law this is

$$\frac{dF}{dt} = \oint_C \frac{\vec{J}}{\sigma} \cdot d\vec{l}$$

If σ goes to infinity while $\vec{J}$ remains finite, the flux does not change. Since this holds for each and every curve in the fluid the flux in each fluid element is conserved. By appropriate motion the magnetic field can be increased.

The balance between magnetic (or ohmic) diffusion and the effects of fluid motion can be expressed by the magnetic Reynolds number R_m. Recall the induction equation

$$\frac{\partial \vec{B}}{\partial t} = D_m \nabla^2 \vec{B} + \vec{\nabla} \times (\vec{v} \times \vec{B}) \qquad 3.16$$

Suppose that L is a typical length and v a typical velocity. Then the first term on the right hand side of (3.16) is of the order

$$| D_m \nabla^2 \vec{B} | = \frac{D_m |B|}{L^2}$$

and the second term, describing the effects of transport, is of the order

$$| \vec{\nabla} \times (\vec{v} \times \vec{B}) | \cong \frac{v |\vec{B}|}{L}$$

The magnetic Reynolds number R_m is the ratio of the second to the first, that is, of the effects of mechanical transport to those of ohmic diffusion

$$R_m = \frac{vL}{D_m} \qquad 3.21$$

A large magnetic Reynolds number indicates the predominance of transport over diffusion which is needed for dynamo action.

Substituting from Chapter 2 the radius of the core, its conductivity, and the velocity from the westward drift, we find that for whole core problems R_m is about 200 and we expect transport to dominate.

4. Compressible MHD

To this point our analysis has been standard. We will now extend the treatment to take into account the effects of compressibility.

Suppose that σ is large so that we may neglect the diffusion term in (3.16). Then the induction equation becomes

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B})$$

$$= (\vec{B} \cdot \vec{\nabla})\vec{v} - (\vec{v} \cdot \vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla} \cdot \vec{v}) \tag{4.1}$$

since $\vec{\nabla} \cdot \vec{B} = 0$

Now the equation of mass continuity (3.2) is

$$\frac{D\rho}{Dt} + \rho (\vec{\nabla} \cdot \vec{v}) = 0$$

so that

$$\vec{\nabla} \cdot \vec{v} = -\frac{1}{\rho} \frac{D\rho}{Dt}$$

Substituting into (4.1)

$$\frac{\partial \vec{B}}{\partial t} = (\vec{B} \cdot \vec{\nabla})\vec{v} - (\vec{v} \cdot \vec{\nabla})\vec{B} + \frac{\vec{B}}{\rho} \frac{D\rho}{Dt} \tag{4.2}$$

Suppose a typical length L and a typical velocity V exist. Then a typical time is just L/V. Then all the terms in (4.2) are of the order

$$\left|\frac{\partial \vec{B}}{\partial t}\right| \cong \left|(\vec{B}\cdot\vec{\nabla})\vec{v}\right| \cong \left|(\vec{v}\cdot\vec{\nabla})\vec{B}\right| \cong \frac{|B|v}{L}$$

except the last term on the RHS is of the order

$$\left|\frac{\vec{B}}{\rho}\frac{D\rho}{Dt}\right| \cong \frac{\Delta\rho}{\bar{\rho}}\frac{|B|v}{L} .$$

where $\Delta\rho$ is a typical change in density and $\bar{\rho}$ a typical density.

Then the relative importance of compressibility compared with the other parts of the transport term in the induction equation is given by

$$C = \frac{\Delta\rho}{\rho} \qquad 4.3$$

In astrophysical situations the density is a function of radius only. Then

$$\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{\rho} v_r \frac{\partial\rho}{\partial r}$$

and

$$\frac{\partial \vec{B}}{\partial t} = (\vec{B}\cdot\vec{\nabla})\vec{v} - (\vec{v}\cdot\vec{\nabla})\vec{B} + \frac{\vec{B}v_r}{\rho}\frac{\partial\rho}{\partial r} \qquad 4.4$$

where V_r is the radial velocity.

Then

$$C = \left| \frac{1}{\rho} \frac{\partial \rho}{\partial r} \right| L \qquad 4.5$$

where L is the radial component of the typical length.

We have remarked that the fractional density derivative in the Earth's outer core is approximately $11 \times 10^{-8}\ m^{-1}$. Then

$$C = 11 \times 10^{-8} L \qquad 4.6$$

when L is in metres.

The parameter C defined by (4.5) and (4.6) is essentially the Smylie-Rochester (1979) compressibility number given by

$$C = \frac{\bar{\rho}\,\bar{g}}{\lambda} L$$

where $\bar{\rho}$ is a typical density, $\bar{g}$ a typical value of the acceleration due to gravity and λ is the bulk modulus. It is clearly equal to the fraction material is compressed by its own weight in the radial distance L. The parameter was introduced to reflect the importance of compressibility in core dynamics.

Here V_r has been assumed to be the same as V. It is very hard to say what V_r might in fact be. On the other hand, the typical velocity from the westward drift is only barely justified, mostly by being the only candidate in the field. While acknowledging the uncertainty, we will nevertheless assume that V_r is equal to V.

Then C is a function of the length scale L. Figure 3 is a plot of C against L for the Earth. As can be seen, for L equal to the core radius, C is about 0.4. A more reasonable typical length might be the thickness of the outer core, for which C equals about 0.25.

In either case, the term involving compressibility is of the same order as the entire term involving transport. This makes its neglect in anything other than a first approximation hard to accept.

This does not hold for flows with a typical length under, say, 900 kilometres (for which C is 0.1). At a typical length of 100 km. the contribution of compressibility, in this analysis, is about 1%, surely negligible.

Let us now consider the induction equation with the diffusion term in place. Then another comparison can be made between the effects of that part of the transport term arising from compressibility and the effects of diffusion. I call this, somewhat apologetically, the 'compressibility

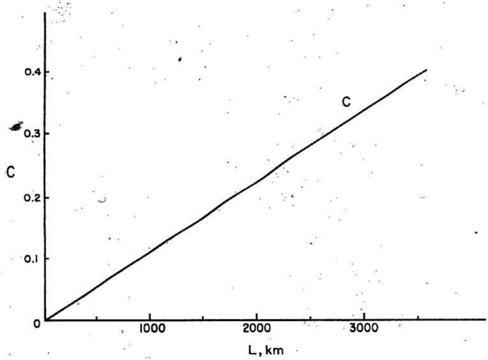

Figure 3. The Smylie-Rochester compressibility number, C , as a function of typical length, L , in the outer core.

part of the magnetic Reynolds number', R_{mc}. It is given by

$$R_{mc} = \frac{\Delta\rho}{\rho}\,\frac{vL}{D_m} = C\,R_m \qquad 4.7$$

For a radial distribution of density

$$R_{mc} = \left|\frac{1}{\rho}\frac{\partial\rho}{\partial r}\right|\frac{\bar{v}_r L^2}{D_m} \qquad 4.8$$

This quadratic dependence on the length scale contrasts with the ordinary magnetic Reynolds number where the dependence is linear.

This is well conveyed by figure 4 comparing R_m and R_{mc} for the Earth's outer core. While R_m is greater than 1 for lengths as small as 20 km, R_{mc} is less than 1 for lengths smaller than about 400 km. Still, for a typical length on the whole core scale R_{mc} is quite large, around 40.

This is of interest when the rest of the transport term fails us: when there is an anti-dynamo theorem that neglects compressibility. If R_{mc} is large enough, then the theorem may fail.

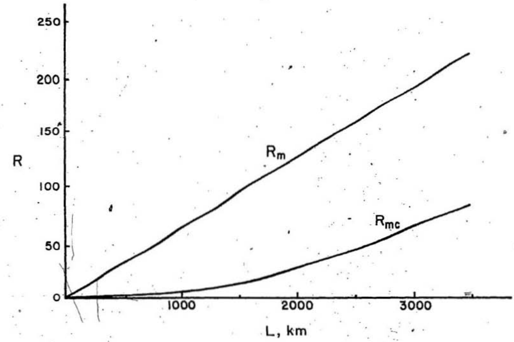

Figure 4. Comparison of the magnetic Reynolds number, R_m, and the compressibility part of the magnetic Reynolds number, R_{mc}, as functions of length L .

5. The Stationary Axisymmetric Dynamo

The first anti-dynamo theorem (ADT) was Cowling's (1933) theorem that an axisymmetric velocity field could not sustain an axisymmetric magnetic field. A proof is quite simple.

Let us introduce cylindrical co-ordinates s, φ, and z. Because of axial synmetry we can write the s and z components of the magnetic field as

$$B_s = -\frac{\partial A_\varphi}{\partial z}, \quad B_z = \frac{1}{s}\frac{\partial (s A_\varphi)}{\partial s} \qquad 5.1$$

Now $s A_\varphi$ must equal zero at the origin and, by (3.19), at infinity. Except for the trivial case, $s A_\varphi$ must therefore go through a maximum or minimum at some distance from the z-axis for any given value of z.

Similarly A_φ is zero for $z = \pm \infty$. It must likewise go through a maximum or minimum. At some pair of co-ordinates (s_0, z_0)

$$B_s = -\frac{\partial A_\varphi}{\partial z}\bigg|_{s_0, z_0} = B_z = \frac{1}{s}\frac{\partial (A_\varphi s)}{\partial s}\bigg|_{s_0, z_0} = 0 \qquad 5.2$$

Using the subscript m to designate the meridional parts of vectors, we write

$$\vec{B}_m = B_s \hat{s} + B_z \hat{z}.$$

We say that $\vec{B}_m$ has a neutral point at this pair of co-ordinates. There may be more than one such point but there must be at least one.

Recall the 'uncurled' induction equation (3.20).

$$\frac{\partial \vec{A}}{\partial t} = \vec{v} \times (\vec{\nabla} \times \vec{A}) - \vec{\nabla} \Phi + D_m \nabla^2 \vec{A}$$

Taking the φ -component yields

$$\frac{\partial A_\varphi}{\partial t} = \vec{v}_m \times \vec{B}_m - \frac{\partial \Phi}{\partial \varphi} - D_m \vec{\nabla} \times \vec{B}_m \qquad 5.3$$

Now because of the axisymmetry of the velocity and magnetic fields, $\vec{\nabla} \Phi$ must be a constant for a given pair (s_0, z_0). Since Φ must be single-valued,

$$\frac{\partial \Phi}{\partial \varphi} = 0$$

For a steady field (5.3) gives

$$\vec{V}_m \times \vec{B}_m = D_m \vec{\nabla} \times \vec{B}_m \tag{5.4}$$

Consider the circle C of radius r drawn about the neutral point (s_0, z_0) in some plane φ = constant. By (5.4)

$$\int \vec{V}_m \times \vec{B}_m \cdot d\vec{S} = D_m \int \vec{\nabla} \times \vec{B} \cdot d\vec{S} \tag{5.5}$$

where the integrals are over the area of the circle. By Stokes' theorem

$$\int \vec{\nabla} \times \vec{B}_m \cdot dS = \oint \vec{B}_m \cdot d\hat{\ell}$$

where the line integral is around the circumference of the circle.

Suppose that the average value of $\vec{B}_m$ on the circle is B. Then

$$\oint \vec{B}_m \cdot d\hat{\ell} \cong 2\pi r B$$

Suppose that the maximum value of $\vec{v}$ on the surface is v. Because $\vec{B}_m$ is zero at (s_0, z_0), for a small enough circle the mean value of $\vec{B}_m$ over the surface must be smaller than B. Then

$$\int \vec{v}_m \times \vec{B}_m \cdot d\vec{S} \leq \pi r^2 v B .$$

But by (5.5) this means

$$2 \pi r D_m B \leq \pi r^2 v B$$

or

$$D_m \leq r v .$$

For finite values of D_m and v this is impossible as r can be shrunk indefinitely. Thus no steady axisymmetric dynamo can exist.

The physical interpretation of this is clear. Around the neutral point the inductive effects represented by the

LHS of (5.5) cannot overcome the ohmic diffusion represented by the RHS.

Lortz (1968) claimed to have extended the theorem to arbitrary velocity fields. However his proof depends on the assumption that both $\vec{B}$ and $\vec{\nabla}\Phi$ are axisymmetric, and it follows from (3.20) that this assumption is equivalent to assuming axisymmetric velocity fields, as well.

The continuity equation has not been invoked in any way. The proof is not affected by whether the fluid is compressible or not.

Suppose for a moment that equation (3.20) has a solution of the form

$$\bar{A}(\vec{r},t) = \vec{A}(\vec{r}) \exp \lambda t$$

Then (3.20) becomes

$$\lambda \vec{A}(\vec{r}) = \vec{\nabla}\Phi + [\vec{v} \times (\vec{\nabla} \times \vec{A}(\vec{r}))] - D_m \vec{\nabla} \times (\vec{\nabla} \times A(r))$$

This is an eigenvalue problem. We know that zero is not an eigenvalue by the argument above. Futhermore if any given velocity field is multiplied by a constant γ which is allowed to go to zero, we recover the problem of the decay of a magnetic field in a solid conductor, for which all the eigenvalues are negative. It would seem that in 'turning up' the velocity field so that an eigenvalue becomes positive we must pass through zero which is impossible.

This argument for extending the theorem to the time dependent case has two flaws (pointed out by Backus, 1958). They stem from the fact that the right hand side of (3.20) is not self-adjoint (Cowling, 1976, pp. 91-92). This has the consequences that the eigenvalues λ need not be real and that the eigenfunctions need not form complete sets. (In this of course lies the difficulty of dynamo theory.) Therefore the path in the complex plane followed by λ in 'turning up' the velocity field need not pass through the origin. If all the eigenvalues are real, however, the lack of completeness does not preclude the existence of some solution increasing with time. Thus a more general anti-dynamo theorem must run on other lines.

6. The Time Dependent Axisymmetric Dynamo

The clearest exposition of the theorem that even a time dependent axisymmetric dynamo is impossible (if the flow is solenoidal) is that of Braginskii (1964a) whom we will follow. We will not, however, assume solenoidal flow so that our conclusions will be different.

We start with the 'uncurled' induction equation (3.20)

$$\frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\Phi + [\vec{v}\times\vec{B}] - D_m\,\vec{\nabla}\times\vec{B}.$$

and the induction equation itself (3.16)

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times[\vec{v}\times\vec{B}] - D_m\,\vec{\nabla}\times(\vec{\nabla}\times\vec{B})$$

We assume there are no sources at infinity; that is

$$|\vec{B}| \propto \frac{1}{r^3} \quad , \vec{r} \to \infty \qquad 6.1$$

We shall also assume that D_m is constant throughout the conducting fluid.

Consider a homogeneous conducting fluid contained in a volume V_1 symmetric about the z -axis. As before we use the subscript m to designate the meridional parts of vectors thus

$$\vec{B} = B_z \hat{z} + B_s \hat{s} + B_\varphi \hat{\varphi} = \vec{B}_m + B_\varphi \hat{\varphi} \tag{6.2}$$

Because of axial symmetry

$$\vec{B}_m = \vec{\nabla} \times (A_\varphi \hat{\varphi}) \tag{6.3}$$

Thus the magnetic field is given completely by two variables A_φ and B_φ.

Let us first consider A_φ.

Take the φ - component of (3.20)

$$\frac{\partial A_\varphi}{\partial t} = -\frac{\partial \Phi}{\partial \varphi} + \hat{\varphi} \cdot [\vec{v} \times \vec{\nabla} \times \vec{A}] - D_m \hat{\varphi} \cdot \vec{\nabla} \times \vec{\nabla} \times \vec{A} \tag{6.4}$$

Because of axial symmetry

$$\frac{\partial \Phi}{\partial \varphi} = 0$$

and

$$\hat{\varphi}\cdot[\vec{v}\times\vec{\nabla}\times\vec{A}] = -\frac{1}{s}\vec{v}_m\cdot\vec{\nabla}(sA_\varphi)$$

Putting these into (6.4) gives

$$\frac{\partial A_\varphi}{\partial t} = -\frac{1}{s}\vec{v}_m\cdot\vec{\nabla}(sA_\varphi) + D_m\Delta_1 A_\varphi \qquad 6.5$$

where

$$\Delta_1 f(s,z) = -\hat{\varphi}\cdot\vec{\nabla}\times\vec{\nabla}\times(f\hat{\varphi}) \qquad 6.6$$

In V_2, the space outside V_1, we have from (6.3) and (3.9)

$$\Delta_1 A_\varphi = 0 \qquad 6.7$$

Multiply equation (6.5) by $s^2 A_\varphi$

$$s^2 A_\varphi \frac{\partial A_\varphi}{\partial t} = -s A_\varphi [\vec{v}_m \cdot \vec{\nabla}(s A_\varphi)] + D_m s^2 A_\varphi \Delta_1 A_\varphi$$

Integrate this over V_1. We may integrate the second term on the RHS over all space, i.e. over $V_1 + V_2$ as the integrand is zero in V_2.

$$\int_{V_1} s^2 A_\varphi \frac{\partial A_\varphi}{\partial t} dV = -\int_{V_1} s A_\varphi (\vec{v}_m \cdot \vec{\nabla}(s A_\varphi)) dV + D_m \int_{V_1+V_2} s^2 A_\varphi \Delta_1 A_\varphi dV \qquad 6.8$$

Let us deal with this term by term.

The LHS of (6.8) is

$$\int_{V_1} s^2 A_\varphi \frac{\partial A_\varphi}{\partial t} dV = \frac{d}{dt} \int_{V_1} \frac{s^2 A_\varphi^2}{2} dV \qquad 6.9$$

The first term on the RHS is

$$\int_{V_1} s A_\varphi (\vec{v}_m \cdot \vec{\nabla} A_\varphi) dV = \int_{V_1} \vec{v}_m \cdot \vec{\nabla} \frac{(s^2 A_\varphi^2)}{2} dV$$

$$= \int_{V_1} \left(\vec{\nabla} \cdot \frac{\vec{v}_m s^2 A_\varphi^2}{2} \right) dV - \int_{V_1} \frac{s^2 A_\varphi^2}{2} (\vec{\nabla} \cdot \vec{v}) dV$$

By the divergence theorem, for S_1 bounding V_1

$$\int_{V_1}\left(\vec{\nabla}\cdot\vec{v}_m\,\frac{s^2A_\varphi^2}{2}\right)dV = \int_{S_1}\frac{s^2A_\varphi^2}{2}\,\vec{v}_m\cdot d\vec{S} = 0$$

because the normal component of $\vec{v}$ on the surface S_1 is zero. So the first term is

$$\int_{V_1} sA_\varphi\,(\vec{v}_m\cdot\vec{\nabla} sA_\varphi)\,dV = -\int_{V_1}\frac{s^2A_\varphi^2}{2}\,(\vec{\nabla}\cdot\vec{v})\,dV \qquad 6.10$$

The second term on the RHS of (6.8) is

$$D_m\int_{V_1+V_2} s^2A_\varphi\,\Delta_1 A_\varphi\,dV = D_m\int_{V_1+V_2}\vec{\nabla}\cdot(sA_\varphi\vec{\nabla}sA_\varphi - \hat{s}\,sA_\varphi^2)\,dV$$

$$- D_m\int_{V_1+V_2}|\vec{\nabla}sA_\varphi|^2\,dV$$

By the divergence theorem

$$\int_{V_1+V_2}\vec{\nabla}\cdot(sA_\varphi\vec{\nabla}sA_\varphi - \hat{s}\,sA_\varphi^2)\,dV = \int_{S_\infty}(sA_\varphi\,\vec{\nabla}sA_\varphi - \hat{s}\,sA_\varphi^2)\cdot d\vec{S}$$

where S_∞ is the surface at infinity. By (3.19)

$$A_\varphi \propto \frac{1}{r^2}\quad , r\to\infty$$

Therefore the surface integral is zero and

$$D_m\int_{V_1+V_2} s^2A_\varphi\,\Delta_1 A_\varphi\,dV = -D_m\int_{V_1+V_2}|\vec{\nabla}sA_\varphi|^2\,dV \qquad 6.11$$

Putting (6.9), (6.10), and (6.11) into (6.8) gives

$$\frac{d}{dt}\int_{V_1} \frac{s^2 A_\varphi^2}{2}\, dV = -D_m \int_{V_1+V_2} |\vec{\nabla} s A_\varphi|^2 dV + \int_{V_1} \frac{s^2 A_\varphi^2}{2} (\vec{\nabla}\cdot\vec{v}) dV \qquad 6.12$$

This is one of two equations we need.

We must derive a similar equation concerning B_φ.

Take the φ- component of the induction equation (3.16)

$$\frac{\partial B_\varphi}{\partial t} = \hat{\varphi}\cdot \vec{\nabla}\times[\vec{v}\times\vec{B}] + D_m \Delta_1 B_\varphi \qquad 6.13$$

Now

$$\hat{\varphi}\cdot\vec{\nabla}\times[\vec{v}\times\vec{B}] = -s\,\vec{v}_m\cdot\vec{\nabla}\left(\frac{B_\varphi}{s}\right) - B_\varphi(\vec{\nabla}\cdot\vec{v}) + \vec{v}_m(\vec{\nabla}\cdot\vec{B}) + \hat{\varphi}\cdot\left[\vec{\nabla}\left(\frac{v_\varphi}{s}\right)\times\vec{\nabla} s A_\varphi\right]$$

But $\vec{\nabla}\cdot\vec{B} = 0$ so (6.13) becomes

$$\frac{\partial \vec{B}_\varphi}{\partial t} = -s\vec{v}_m\cdot\vec{\nabla}\left(\frac{B_\varphi}{s}\right) - B_\varphi(\vec{\nabla}\cdot\vec{v}) + \hat{\varphi}\cdot\left[\vec{\nabla}\left(\frac{v_\varphi}{s}\right)\times\vec{\nabla} s A_\varphi\right] + D_m \Delta_1 B_\varphi \qquad 6.14$$

From (3.9), in V_2

$$\vec{\nabla}\times(B_\varphi\hat{\varphi}) = -\frac{\partial B_\varphi}{\partial z}\hat{s} + \frac{1}{s}\frac{\partial s B_\varphi}{\partial s}\hat{z} = 0$$

But by (6.1) $\vec{B}=0$ at ∞ . This means that in V_2

$$B_\varphi = 0 \tag{6.15}$$

Multiply (6.14) by $\frac{B_\varphi}{s^2}$ and integrate over V_1 .

$$\int_{V_1} \frac{B_\varphi}{s^2}\frac{\partial B_\varphi}{\partial t}\,dV = -\int_{V_1}\frac{B_\varphi}{s}\,\vec{v}_m\cdot\vec{\nabla}\left(\frac{B_\varphi}{s}\right)dV - \int_{V_1}\frac{B_\varphi^2}{s^2}(\vec{\nabla}\cdot\vec{v})\,dV$$

$$+ \int_{V_1}\frac{B_\varphi}{s^2}\,\hat{\varphi}\cdot\left[\vec{\nabla}\left(\frac{v_\varphi}{s}\right)\times\vec{\nabla}sA_\varphi\right]dV + D_m\int_{V_1}\frac{B_\varphi}{s^2}\,\Delta_1 B_\varphi\,dV \tag{6.16}$$

We will deal with this term by term.

The LHS is

$$\int_{V_1}\frac{B_\varphi}{s^2}\frac{\partial B_\varphi}{\partial t}\,dV = \frac{d}{dt}\int_{V_1}\frac{B_\varphi^2}{2s^2}\,dV \tag{6.17}$$

The first term on the RHS is

$$-\int_{V_1}\frac{B_\varphi}{s}\,\vec{v}_m\cdot\vec{\nabla}\left(\frac{B_\varphi}{s}\right)dV = -\int_{V_1}\vec{\nabla}\cdot\vec{v}_m\frac{B_\varphi^2}{2s^2}\,dV + \int_{V_1}\frac{B_\varphi^2}{2s^2}(\vec{\nabla}\cdot\vec{v})\,dV$$

By the divergence theorem

$$\int_{V_1}\vec{\nabla}\cdot\frac{\vec{v}_m B_\varphi^2}{2s^2}\,dV = \int_{S_1}\vec{v}_m\frac{B_\varphi^2}{2s^2}\cdot d\vec{S} = 0$$

because there is no normal component of velocity across S_1 .

So

$$-\int_{V_1} \frac{B_\varphi}{s} \vec{v}_m \cdot \vec{\nabla}\left(\frac{B_\varphi}{s}\right) dV = \int_{V_1} \frac{B_\varphi^2}{2s^2} (\vec{\nabla}\cdot\vec{v}) dV \qquad 6.18$$

We leave the next two terms in (6.16) as they stand.

The last term is

$$D_m \int_{V_1} \frac{B_\varphi}{s^2} \Delta_1 B_\varphi dV = D_m \int_{V_1} \vec{\nabla}\cdot\left(\frac{B_\varphi}{s}\vec{\nabla}\frac{B_\varphi}{s} + \hat{s}\frac{B_\varphi^2}{s^3}\right) dV$$

$$- D_m \int_{V_1} \left|\vec{\nabla}\left(\frac{B_\varphi}{s}\right)\right|^2 dV$$

By the divergence theorem

$$\int_{V_1} \vec{\nabla}\cdot\left(\frac{B_\varphi}{s}\vec{\nabla}\frac{B_\varphi}{s} + \hat{s}\frac{B_\varphi^2}{s^3}\right) dV = \int_{S_1}\left(\frac{B_\varphi}{s}\vec{\nabla}\frac{B_\varphi}{s} + \hat{s}\frac{B_\varphi^2}{s^3}\right)\cdot d\vec{S} = 0$$

as $B_\varphi = 0$ on S_1 by (6.15). So the last term on the RHS of (6.16) is

$$D_m \int_{V_1} \frac{B_\varphi}{s^2} \Delta_1 B_\varphi dV = -D_m \int_{V_1} \left|\vec{\nabla}\frac{B_\varphi}{s}\right|^2 dV \qquad 6.19$$

Putting (6.17), (6.18), and (6.19) into (6.16) gives

$$\frac{d}{dt}\int_{V_1} \frac{B_\varphi^2}{2s^2} dV = -\frac{1}{2}\int_{V_1} \frac{B_\varphi^2}{s^2}(\vec{\nabla}\cdot\vec{v}) dV - D_m \int_{V_1} \left|\vec{\nabla}\frac{B_\varphi}{s}\right|^2 dV$$

$$+ \int_{V_1} \frac{B_\varphi}{s^2} \hat{\varphi}\cdot\left[\vec{\nabla}\left(\frac{v_\varphi}{s}\right) \times \vec{\nabla} s A_\varphi\right] dV \qquad 6.20$$

Equations (6.12) and (6.20) correspond to equations (2.9a) and (2.9b) of Braginskii (1964a) except that we have retained the terms in $\vec{\nabla}\cdot\vec{v}$.

If we discard them (6.12) becomes

$$\frac{d}{dt}\int_{V_1}\frac{s^2A_\varphi^2}{2}dV = -D_m\int_{V_1+V_2}|\vec{\nabla}sA_\varphi|^2 dV \tag{6.21}$$

and (6.20) becomes

$$\frac{d}{dt}\int_{V_1}\frac{B_\varphi^2}{2s^2}dV = -D_m\int_{V_1}\left|\vec{\nabla}\frac{B_\varphi}{s}\right|^2 dV + \int_{V_1}\frac{B_\varphi}{s^2}\hat{\varphi}\cdot\left[\vec{\nabla}\left(\frac{v_\varphi}{s}\right)\times\vec{\nabla}sA_\varphi\right]dV \tag{6.22}$$

These two equations form the basis of the anti-dynamo theorem.

Consider (6.21). The integrand on the RHS is always positive; therefore the integral on the left must decrease with time. As the integrand on the left is also always positive, this means that A_φ must eventually go to zero.

Now consider (6.22). So long as A_φ is not equal to zero, the second term on the RHS can cause the integral on the left, and with it B_φ, to increase with time. However we know from the first part of the theorem that A_φ must go to zero. But the integrand of the other term on the RHS is always positive so that once A_φ goes to zero the integral on the LHS must decrease with time and B_φ vanishes as time goes on.

This means that no axisymmetric time dependent dynamo is possible in an incompressible fluid. The trouble comes when we keep the discarded terms.

Recall (6.12).

$$\frac{d}{dt}\int_{V_1}\frac{s^2A_\rho^2}{2}\,dV = -D_m\int_{V_1+V_2}|\vec{\nabla}sA_\rho|^2 dV + \int_{V_1}\frac{s^2A_\rho^2}{2}(\vec{\nabla}\cdot\vec{v})\,dV$$

Apparently the integral on the LHS will increase with time if

$$\int_{V_1}\frac{s^2A_\rho^2}{2}(\vec{\nabla}\cdot\vec{v})\,dV > D_m\int_{V_1+V_2}|\vec{\nabla}sA_\rho|^2 dV \qquad 6.23$$

If A_ρ does not go to zero then the second term in (6.22) does not go to zero and the argument above about B_ρ is invalid. Even if $A_\rho=0$ the full equation (6.20) now has a term containing $\vec{\nabla}\cdot\vec{v}$ of uncertain sign that still invalidates the conclusion. Thus, given (6.23), the anti-dynamo theorem breaks down.

We must now inquire whether the condition (6.23) might not be fulfilled in the core. Let us suppose that the circulation in the core is large-scale, with radial velocities not differing much from the horizontal velocities inferred from westward drift. Because of Alfven's theorem the magnetic field must also be large-scale.

Because of the continuity equation we may write

$$\int_{V_1}\frac{s^2A_\rho^2}{2}(\vec{\nabla}\cdot\vec{v})\,dV = -\int_{V_1}\frac{s^2A_\rho^2}{2}\frac{v_r}{\rho}\frac{\partial\rho}{\partial r}\,dV$$

$$\simeq -\left|\frac{1}{\rho}\frac{\partial\rho}{\partial r}\right|\int_{V_1}\frac{s^2A_\rho^2}{2}v_r\,dV$$

The integral still contains two unknown fields $A_{\varphi s}$ and V_r .. The upward and downward parts of V_r would tend to cancel were they not weighted against the vector potential. This may lead to a non-cancelling part which we can express with the aid of the parameter α . Supposing $\bar{V}_r$ to be a typical value

$$\int_{V_1} \frac{s^2 A_\varphi^2}{2} v_r \, dV = \alpha \bar{v}_r \int_{V_1} \frac{s^2 A_\varphi^2}{2} \, dV$$

where α is presumably small and may be negative. We may rephrase the question about whether (6.23) is fulfilled to ask how large α may be allowed to get before the theorem breaks down. The condition (6.23) becomes

$$\alpha \left| \frac{1}{\rho} \frac{\partial \rho}{\partial r} \right| \frac{V_r}{D_m} \left| \int_{V_1} \frac{s^2 A_\varphi^2}{2} dV \right| \Big/ \left| \int_{V_1+V_2} |\vec{\nabla} s A_\varphi|^2 dV \right| > 1 .$$

The ratio of the integrals is of the order R^2 and the condition becomes

$$\alpha \left| \frac{1}{\rho} \frac{\partial \rho}{\partial r} \right| \frac{V_r}{D_m} R^2 > 1$$

or

$$\alpha R_{mc} > 1$$

For the Earth's outer core R_{mc} is perhaps 40. Then α must be smaller than 0.025 for the anti-dynamo theorem to apply. This is not a very large number. As it arises from the product of two unknown fields there is nothing certain that can be said about the true value of α. But since the possibility exists that α is large enough for the discarded term in the anti-dynamo theorem to be as large as the term that is kept, it is probably not wise to discard it. Then, however, there is no longer an anti-dynamo theorem about time dependent axisymmetric fields that applies to the core of the Earth.

This is not a statement that axisymmetric solutions to the dynamo equations exist; it is not an existence theorem. It is calling into doubt of a non-existence theorem.

Suppose that

$$\int_{V_1} \frac{s^2 A_\rho^2}{2} (\vec{\nabla}\cdot\vec{v})\,dV = D_m \int_{V_1+V_2} |\vec{\nabla} s A_\rho|^2 dV$$

Then by (6.12)

$$\frac{d}{dt}\int_{V_1} \frac{s^2 A_\rho^2}{2}\,dV = 0$$

At first glance this might seem to contradict Cowling's theorem on the stationary axisymmetric dynamo.

This is not so. Only the value of the integral is constant, while A_ρ may be changing locally. Cowling's theorem requires the magnetic field to be constant everywhere at once.

7. Non-radial Velocity Fields

We now turn to another anti-dynamo theorem. The work below follows Moffatt (1978, p.118) except that the fluid is not assumed to be incompressible. Consider a sphere, V_1, containing a homogeneous conducting fluid. Suppose that $\vec{v}$ has no radial component.

Recall the induction equation (3.16)

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}) - D_m \vec{\nabla} \times \vec{\nabla} \times \vec{B}$$

Let us turn our attention to the radial component of the magnetic field B_r. Multiply equation (3.16) by $r^2 B_r \hat{r}$ and integrate over the sphere.

$$\int_{V_1} r^2 B_r \frac{\partial B_r}{\partial t} dV = \int_{V_1} r^2 B_r \hat{r} \cdot (\vec{\nabla} \times (\vec{v} \times \vec{B})) dV - D_m \int_{V_1} r^2 B_r \hat{r} \cdot (\vec{\nabla} \times \vec{\nabla} \times \vec{B}) dV \qquad 7.1$$

We will deal with this term by term.
The LHS of (7.1) is

$$\int_{V_1} r^2 B_r \frac{\partial B_r}{\partial t} dV = \frac{d}{dt} \int_{V_1} \frac{r^2 B_r^2}{2} dV \qquad 7.2$$

The first term on the RHS of (7.1) is

$$\int_{V_1} r^2 B_r \, \hat{r} \cdot (\vec{\nabla} \times (\vec{v} \times \vec{B})) dV = \int_{V_1} \vec{\nabla} \cdot \{ r^2 B_r \, \hat{r} \times (\vec{v} \times \vec{B}) \} dV + \int_{V_1} \vec{v} \times \vec{B} \cdot (\vec{\nabla} \times (r^2 B_r \hat{r})) dV$$

The integral involving a divergence goes to zero when transformed into a surface integral over S_1 as, of course, $\hat{r} \times (\vec{v} \times \vec{B})$ has no radial component.

Now

$$\int_{V_1} \vec{v} \times \vec{B} \cdot (\vec{\nabla} \times (r^2 B_r \hat{r})) dV = \int_{V_1} \vec{\nabla} \cdot \left(\frac{\vec{v} \, r^2 B_r^2}{2} \right) dV - \int_{V_1} \frac{r^2 B_r^2}{2} (\vec{\nabla} \cdot \vec{v}) dV$$

Again the first integral goes to zero when transformed into a surface integral as $\vec{v}$ has no normal component on S_1. Thus

$$\int_{V_1} r^2 B_r \, \hat{r} \cdot (\vec{\nabla} \times (\vec{v} \times \vec{B})) dV = - \int_{V_1} \frac{r^2 B_r^2}{2} (\vec{\nabla} \cdot \vec{v}) dV \qquad 7.3$$

The remaining term in (7.1) is

$$\int_{V_1} r^2 B_r \, \hat{r} \cdot (\vec{\nabla} \times (\vec{\nabla} \times \vec{B})) dV = \int_{V_1} \vec{\nabla} \cdot \{ r^2 B_r \, \hat{r} \times (\vec{\nabla} \times \vec{B}) \} dV + \int_{V_1} |\vec{\nabla} r B_r|^2 dV \qquad 7.4$$

The integral involving a divergence goes to zero when transformed into a surface integral as $\hat{r}\times\vec{\nabla}\times\vec{B}=0$ on the sphere. Putting (7.2), (7.3), and (7.4) into (7.1) yields

$$\frac{d}{dt}\int_{V_1}\frac{r^2B_r^2}{2}\,dV = -\int_{V_1}\frac{r^2B_r^2}{2}(\vec{\nabla}\cdot\vec{v})\,dV - D_m\int_{V_1}|\vec{\nabla}(rB_r)|^2\,dV \tag{7.5}$$

Now if $\vec{\nabla}\cdot\vec{v}=0$ then (7.5) means that the radial component of the magnetic field must decay away, as the RHS of (7.5) is then always negative.

Before turning to the other part of $\vec{B}$ we must digress for a moment on the decomposition of vector fields.

It is well known that any vector field $\vec{Q}$ can be divided into curl-free and divergence-free parts:

$$\vec{Q} = -\vec{\nabla}L + \vec{\nabla}\times\vec{A} \tag{7.6}$$

The field $\vec{A}$ may also be expanded (Roberts, 1967, p. 80):

$$\vec{A} = \vec{\nabla}R + T\vec{r} + \vec{\nabla}\times P\vec{r}$$

where L, R, T, and P are scalar fields. By expanding in spherical harmonics we can show that the mean values of R, T and P over a spherical surface may, without loss of generality, be taken to vanish.

$$\therefore\ \vec{Q} = -\vec{\nabla}L + \vec{\nabla}\times(T\vec{r}) + \vec{\nabla}\times\vec{\nabla}\times(P\vec{r}) \tag{7.7}$$

The three parts are called respectively lamellar or scaloidal, toroidal, and poloidal. The toroidal part does not have a radial component; the poloidal part in general does.

The magnetic field can be expressed as a sum of toroidal and poloidal parts

$$\vec{B} = \vec{\nabla} \times (T\hat{r}) + \vec{\nabla} \times \vec{\nabla} \times (P\hat{r}) \qquad 7.8$$

From Ampere's law

$$\vec{J} = -\vec{\nabla} \times (\hat{r}\, \vec{\nabla}^2 P) + \vec{\nabla} \times \vec{\nabla} \times (T\hat{r}) \qquad 7.9$$

When $\vec{J} = 0$

$$\nabla^2 P = T = 0 \qquad 7.10$$

Returning to the anti-dynamo theorem, if the radial component of $\vec{B}$ is zero, as eventually required by (7.5) when the flow is solenoidal, then $\vec{B}$ is purely toroidal and

$$\vec{B} = \vec{\nabla} \times (T\hat{r}) = -\vec{r} \times \vec{\nabla} T \qquad 7.11$$

From (7.10) $\vec{B} = 0$ outside V_1.

From the induction equation

$$\vec{r} \times \vec{\nabla} \frac{\partial T}{\partial t} = \vec{\nabla} \times \vec{v} \times (\vec{r} \times \vec{\nabla} T) + D_m \nabla^2 (\vec{r} \times \vec{\nabla} T)$$

$$= \vec{r} \times \vec{\nabla} (\vec{v} \cdot \vec{\nabla}) T + \vec{r} \times \vec{\nabla} (D_m \nabla^2 T)$$

Therefore

$$\frac{\partial T}{\partial t} = (\vec{v} \cdot \vec{\nabla}) T + D_m \nabla^2 T + f(r) \qquad 7.12$$

where f is some function of r alone.

Multiply (7.12) by T and integrate over V_1.

$$\int_{V_1} T\frac{\partial T}{\partial t}\,dV = \int_{V_1} T(\vec{v}\cdot\vec{\nabla})T\,dV + D_m\int_{V_1} T\nabla^2 T\,dV + \int_{V_1} T f(r)\,dV \tag{7.13}$$

We will deal with this term by term.

The LHS is

$$\int_{V_1} T\frac{\partial T}{\partial t}\,dV = \frac{d}{dt}\int_{V_1}\frac{T^2}{2}\,dV \tag{7.14}$$

The first term on the LHS is

$$\int_{V_1}\frac{\vec{v}\cdot\vec{\nabla}T^2}{2}\,dV = \int_{V_1}\vec{\nabla}\cdot\frac{\vec{v}T^2}{2}\,dV - \int_{V_1}\frac{T^2}{2}(\vec{\nabla}\cdot\vec{v})\,dV \tag{7.15}$$

The first integral goes to zero when transformed to a surface integral, as $\vec{v}$ has no radial component on S_1.

The second term on the RHS is

$$D_m\int_{V_1} T\nabla^2 T\,dV = -D_m\int_{V_1}|\vec{\nabla}T|^2\,dV + D_m\int_{V_1}\vec{\nabla}\cdot(T\vec{\nabla}T)\,dV \tag{7.16}$$

The integral involving a divergence goes to zero when transformed into a surface integral as $T = 0$ on S_1 by (7.10).

The remaining term is

$$\int_{V_1} T\, f(r)\, dV = 0 \qquad 7.17$$

as the mean value of T is zero over the surface of all spheres. Putting (7.14), (7.15), (7.16), and (7.17) into (7.13) gives

$$\frac{d}{dt}\int_{V_1} T^2 dV = -\int_{V_1} T^2(\vec{\nabla}\cdot\vec{v})dV - D_m\int_{V_1} |\vec{\nabla}T|^2 dV \qquad 7.18$$

If the flow is solenoidal then T must go to zero with time.

Thus we see that velocity fields without a radial component cannot sustain a dynamo in an incompressible fluid. Such a velocity field can be expressed by (7.7) as a toroidal field.

Toroidal velocity fields cannot sustain a dynamo. However, if the terms containing $\vec{\nabla}\cdot\vec{v}$ in (7.5) and (7.18) are large enough and of the right sign then the anti-dynamo theorem fails for compressible fluids.

By the Higgins-Kennedy Hypothesis (Higgins and Kennedy, 1971) the core is stably stratified and radial motion is strongly inhibited. This and the anti-dynamo theorem on toroidal velocity fields are in apparent contradiction and much effort has been spent on reconciling the two (Busse, 1975). The Higgins-Kennedy hypothesis is unproven, but the results above may be of interest in this connection.

However, non-radial velocity fields cannot involve the large radial density change in the earth's core. This means that the density differences appearing in

$$R_{mc} = \frac{\Delta\rho}{\rho} R_m$$

must arise from pressure gradients along surfaces of constant radius. As R_{mc} must be greater than one for the anti-dynamo theorem to fail and R_m is at best a few hundred, the excess pressure needed would be high. While it is difficult to be dogmatic, the existence of such pressures is unlikely. Thus non-radial motion is probably not able to sustain the Earth's magnetic field.

Small amplitude oscillatory motion with a radial component is possible in a stably stratified core. This motion might be able to take part in driving the dynamo as, of course, the anti-dynamo theorem on non-radial motion would not apply. If the radial wavelength of such an oscillation were large, the effects of compressibility on the motion would doubtless have to be considered.

8. The Two Dimensional Dynamo

The anti-dynamo theorem for two dimensional dynamos has two forms. One concerns steady velocity and magnetic fields that do not depend on one Cartesian co-ordinate, z, say.

That is

$$\frac{\partial \vec{B}}{\partial t} = \frac{\partial \vec{v}}{\partial t} = \frac{\partial \vec{B}}{\partial z} = \frac{\partial \vec{v}}{\partial z} = 0 \tag{8.1}$$

Lortz (1968) showed that velocity and magnetic fields obeying (8.1) could not satisfy the dynamo equations. The proof involves properties of elliptic partial differential equations and will not be reproduced here. The proof is unaffected by the compressibility of the fluid. This is reminiscent of Cowling's 1933 theorem on time dependent axisymmetric dynamos which can also be proved on the basis of properties of elliptic partial differential equations. (Backus and Chandrasekhar, 1956; Lortz, 1968).

The other theorem concerns time dependent fields. It was put forward by Cowling (1957b). Suppose that the velocity and magnetic fields do not depend on z, and that the velocity field has no z component. That is

$$\frac{\partial \vec{B}}{\partial z} = \frac{\partial \vec{v}}{\partial z} = \vec{v}\cdot\hat{z} = 0 \tag{8.2}$$

Consider an infinitely long volume, V_1, of constant cross-section perpendicular to z containing a homogeneous conducting fluid.

Recall the induction equation (3.16).

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}) - D_m \, \vec{\nabla} \times \vec{\nabla} \times \vec{B}$$

Multiply this by $B_z \hat{z}$ and integrate over a unit length of V_1, over V_1'. This is to avoid the divergence of integrals.

$$\int_{V_1'} B_z \frac{\partial B_z}{\partial t}\, dV = \int_{V_1} B_z \, \hat{z} \cdot \vec{\nabla} \times (\vec{v} \times \vec{B})\, dV - D_m \int_{V_1'} B_z \, \hat{z} \cdot \vec{\nabla} \times \vec{\nabla} \times \vec{B}\, dV \tag{8.3}$$

Now because of (8.2)

$$\int_{V_1'} B_z \, \hat{z} \cdot \vec{\nabla} \times (\vec{v} \times \vec{B})\, dV = -\int_{V_1'} \vec{\nabla} \cdot \frac{\vec{v} \, B_z^2}{2}\, dV + \int_{V_1'} \frac{B_z^2}{2} (\vec{\nabla} \cdot \vec{v})\, dV$$

where the first integral goes to zero when transformed to a surface integral; and

$$\int_{V_1'} B_z \, \hat{z} \cdot \vec{\nabla} \times \vec{\nabla} \times \vec{B}\, dV = \int_{V_1'} |\vec{\nabla} B_z|^2 \, dV$$

so that (8.3) becomes

$$\frac{d}{dt} \int_{V_1'} \frac{B_z^2}{2}\, dV = \int_{V_1'} \frac{B_z^2}{2} (\vec{\nabla} \cdot \vec{v})\, dV - D_m \int_{V_1'} |\vec{\nabla} B_z|^2 \, dV \tag{8.4}$$

We see from (8.4) that in an incompressible fluid B_z must go to zero with time, but that this is not necessarily the case if the flow is not solenoidal.

Now suppose that $B_z = 0$. Then

$$\vec{B} = \vec{\nabla} \times (A_z \hat{z}) \tag{8.5}$$

Recall the uncurled induction equation (3.20)

$$\frac{\partial \vec{A}}{\partial t} = \vec{v} \times (\vec{\nabla} \times \vec{A}) - \vec{\nabla}\Phi + D_m \vec{\nabla} \times \vec{\nabla} \times \vec{A}$$

From (8.2)

$$\frac{\partial \Phi}{\partial z} = 0$$

Multiply (3.20) by $A_z \hat{z}$ and integrate over V_1' .

$$\int_{V_1'} A_z \frac{\partial A_z}{\partial t} dV = \int_{V_1'} A_z \hat{z} \cdot \vec{v} \times (\vec{\nabla} \times \vec{A}) dV + D_m \int_{V_1'} A_z \hat{z} \cdot \vec{\nabla} \times \vec{\nabla} \times \vec{A} \, dV \tag{8.6}$$

Now given the assumed conditions

$$\int_{V_1'} A_z \hat{z} \cdot \vec{v} \times (\vec{\nabla} \times \vec{A}) dV = - \int_{V_1'} \vec{\nabla} \cdot \frac{\vec{v} A_z^2}{2} dV + \int_{V_1'} \frac{A_z^2}{2} (\vec{\nabla} \cdot \vec{v}) dV \tag{8.7}$$

where the first integral on the RHS goes to zero when transformed into a surface integral as $\vec{V}$ has no component normal to the surface of V_1 or to a plane z = constant. The last integral on the RHS of (8.6) can be taken over a unit length of all space, over $V_1' + V_2'$, as $\vec{\nabla} \times \vec{B} = 0$ outside V_1 .

Then

$$\int_{V_1'} A_z \hat{z}\cdot\vec{\nabla}\times\vec{\nabla}\times\vec{A}\,dV = \int_{V_1'+V_2'}\vec{\nabla}\cdot(A_z\vec{\nabla}A_z)dV - \int_{V_1'+V_2'}|\vec{\nabla}A_z|^2 dV \tag{8.8}$$

Now the first integral on the RHS vanishes when transformed into a surface integral as $\frac{\partial A_z}{\partial z}$ and

$$A_z \propto \frac{1}{r},\ r\to\infty\,.$$

Therefore (8.6) becomes

$$\frac{d}{dt}\int_{V_1'}\frac{A_z^2}{2}d\tau = \int_{V_1'}\frac{A_z^2}{2}(\vec{\nabla}\cdot\vec{v})dV - D_m\int_{V_1'+V_2'}|\vec{\nabla}A_z|^2 dV \tag{8.9}$$

Again we see that in an incompressible fluid A_z and therefore the magnetic field must eventually vanish, while this need not be so if the flow is not solenoidal.

A slightly different anti-dynamo theorem was derived by Moffatt (1978, p. 121). The theorem is supposed to apply to arbitrary magnetic fields; however, in going from his equation 6.52 to 6.53 it is necessary to assume $\frac{\partial A_z}{\partial z}=0$. When $B_z=0$ as is the case under study, this means that $\frac{\partial\vec{B}}{\partial z}$ must also be zero.

What is the geophysical significance of these theorems? After all they only apply to infinitely long dynamos. However, it is possible that they are not as irrelevant as might at first be thought.

The Taylor-Proudman theorem states that for steady flow in an inviscid, incompressible fluid rotating about the z-axis having typical flow speeds negligible with respect to the speeds of rotation, the flow does not depend on z.

A proof is simple. Let us write down the momentum equation for such flow (Greenspan, 1968, p. 5).

$$\frac{1}{2}\vec{\nabla}(\vec{v}\cdot\vec{v}) + (\vec{\nabla}\times\vec{v})\times\vec{v} + 2\vec{\Omega}\times\vec{v} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = -\frac{1}{\rho}\vec{\nabla}P + \vec{F} \quad 8.10$$

where $\vec{\Omega}$ is the angular velocity, P the pressure, and $\vec{F}$ the body force. If $\vec{F}=\vec{\nabla}U$, i.e. if $\vec{F}$ is conservative we may form the reduced pressure p

$$p = P + \rho U - 1/2\,\rho(\vec{\Omega}\times\vec{r})\cdot(\vec{\Omega}\times\vec{r})$$

and write

$$\vec{v}\cdot\vec{\nabla}\vec{v} + 2\vec{\Omega}\times\vec{v} = -\frac{1}{\rho}\vec{\nabla}p \quad 8.11$$

Neglecting the term of order v^2 and taking the curl yields

$$\vec{\nabla}\times(\vec{\Omega}\times\vec{v}) = 0 \quad 8.12$$

If the fluid is incompressible this is equivalent to

$$(\vec{z}\cdot\vec{\nabla})\vec{v} = 0$$

which is the required result.

The Taylor-Proudman theorem is expected to apply only approximately to the core as all the conditions for the theorem to work are at best only approximately fulfilled.

Busse (1978) has shown that the preferred mode of convection in a spherical shell of incompressible fluid consists of long counter-rotating cylinders parallel to the rotation axis arranged as a belt outside the inner boundary of the shell. A small component of velocity exists along the cylinder but the rotating of the cylinders is the main feature.

Near the middle of length of one of these cylinders the anti-dynamo theorem might well forbid dynamo action for a unit length. It would be interesting to know if relaxing the assumption of incompressibility would have any effect in this event. However, it is not clear what effects the relaxation would have on the basic model in the first place. The discussion must remain imprecise.

9. A Class of Anti-dynamo Theorems

The three anti-dynamo theorems on time dependent fields are all very similar. Cowling (1957b) has shown that they are all special cases of a more general theorem. This general anti-dynamo theorem can be modified to include the effects of compressibility. The modification involves the invalidation of the anti-dynamo theorem.

We start by seeking a transformation of Ohm's law

$$\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \qquad 9.1$$

Take a vector field $\vec{a}$ so that $\vec{a}$ equals zero outside the conducting fluid. Clearly

$$\frac{1}{\sigma} \int_{V_1} \vec{J} \cdot \vec{a} \, dV = \int_{V_1} (\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{a} \, dV \qquad 9.2$$

As before, V_1 is the space containing the fluid. As $\vec{a}$ equals zero outside V_1 the integrals could have been taken over all space $V_1 + V_2$.

Suppose we have another field $\vec{b}$ so that

$$\vec{\nabla} \times \vec{b} = \vec{a} \qquad 9.3$$

and $\vec{b}$ goes to zero at infinity at least as quickly as $1/r$. To a constant multiplier the two fields might be a current density and a magnetic field, or a magnetic field and a vector potential, or some components of such a pair.

Substituting into (9.3) and observing that $\vec{\nabla}\times\vec{b}=0$ outside V_1

$$\int_{V_1} \vec{E}\cdot\vec{a}\,dV = \int_{V_1+V_2} \vec{E}\cdot(\vec{\nabla}\times\vec{b})\,dV$$

$$= \int_{V_1+V_2} \{\vec{\nabla}\cdot(\vec{b}\times\vec{E}) + \vec{b}\cdot(\vec{\nabla}\times\vec{E})\}\,dV \qquad 9.4$$

Now by the divergence theorem

$$\int_{V_1+V_2} \vec{\nabla}\cdot(\vec{b}\times\vec{E})\,dV = \int_{S_2} (\vec{b}\times\vec{E})\cdot\hat{n}\,dS = 0$$

since S_2 is at infinity. As well

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad 9.5$$

So (9.4) becomes

$$\int_{V_1} \vec{E}\cdot\vec{a}\,dV = -\int_{V_1+V_2} \vec{b}\cdot\frac{\partial\vec{B}}{\partial t}\,dV$$

and (9.2) becomes

$$\int_{V_1+V_2} \vec{b}\cdot\frac{\partial\vec{B}}{\partial t}\,dV = \int_{V_1} \vec{v}\cdot(\vec{B}\times\vec{a})\,dV - \frac{1}{\sigma}\int_{V_1} \vec{a}\cdot\vec{J}\,dV \qquad 9.6$$

This is our transformation.

The anti-dynamo theorem will work for combinations of $\vec{B}$ and $\vec{v}$ for which $\vec{a}$ can be found so that

$$\vec{v}\cdot(\vec{B}\times\vec{a}) = \vec{v}\cdot\vec{\nabla}\psi \qquad 9.7$$

where Ψ is some scalar function. Then

$$\int_{V_1} \vec{v}\cdot(\vec{B}\times\vec{a})\,dV = \int_{V_1} \vec{\nabla}\cdot(\vec{v}\Psi)\,dV - \int_{V_1} \Psi(\vec{\nabla}\cdot\vec{v})\,dV$$

By the divergence theorem

$$\int_{V_1} \vec{\nabla}\cdot(\vec{v}\Psi)\,dV = \int_{S_1} \vec{v}\,\Psi\cdot\hat{n}\,dS = 0$$

as $\vec{v}$ has no normal component at the surface of V_1.
Then for $\vec{a}$ satisfying (9.7), equation (9.6) becomes

$$\int_{V_1+V_2} \vec{b}\cdot\frac{\partial\vec{B}}{\partial t}\,dV = -\frac{1}{\sigma}\int_{V_1} \vec{a}\cdot\vec{J}\,dV - \int_{V_1} \Psi(\vec{\nabla}\cdot\vec{v})\,dV \qquad 9.8$$

Of course for an incompressible fluid the last term is zero, and

$$\int_{V_1+V_2} \vec{b}\cdot\frac{\partial\vec{B}}{\partial t}\,dV = -\frac{1}{\sigma}\int_{V_1} \vec{a}\cdot\vec{J}\,dV \qquad 9.9$$

An anti-dynamo theorem will arise if the left hand side of (9.8) can be identified with the time derivative of some measure of the strength of the magnetic field, and the right hand side is negative. Then $\vec{B}$ will decrease with time and eventually go to zero.

For example, with $\vec{b} = \vec{B}$, (9.9) becomes

$$\frac{d}{dt}\int_{V_1+V_2} \frac{|\vec{B}|^2}{2}\,dV = -\frac{1}{\mu_0\sigma}\int_{V_1} |\vec{J}|^2\,dV$$

That is, for $\vec{v}\cdot(\vec{B}\cdot\vec{J}) = \vec{v}\cdot\vec{\nabla}\Psi$, no dynamo effect in an incompressible fluid exists. But in a compressible fluid we have the extra term in (9.8) and for our example

$$\frac{d}{dt}\int_{V_1+V_2}\frac{|\vec{B}|^2}{2}dV = -D_m\int_{V_1}|\vec{J}|^2 dV - \int_{V_1}\Psi(\vec{\nabla}\cdot\vec{v})dV$$

If

$$-\int_{V_1}\Psi(\vec{\nabla}\cdot\vec{v})dV > D_m\int_{V_1}|\vec{J}|^2 dV$$

then the magnetic field will grow with time. We no longer have an anti-dynamo theorem. The presence of the extra term for a compressible fluid makes the task of obtaining this sort of anti-dynamo theorem more difficult if not impossible.

Table 4 lists $\vec{a}$, $\vec{b}$, and Ψ for the three special anti-dynamo theorems.

Cowling remarks that "The suggestion appears plausible that the only complete proofs of steady-state dynamo maintenance must be of the type considered above." He suggests that, in general, equation (9.7) is hard to satisfy and that on this account general theorems are hard to find. It seems that for a compressible fluid such theorems are even harder to find.

Table 4. Fields for the generalized time-dependent anti-dynamo theorem

Theorem	Field	$\vec{a}$	$\vec{b}$	ψ
Axisymmetric	B_φ	$\vec{\nabla} \times \frac{B_\varphi}{s}\hat{\varphi}$	$\frac{B_\varphi}{s}\hat{\varphi}$	$\frac{B_\varphi^2}{2s^2}$
	A_φ	$s^2 A_\varphi \hat{\varphi}$	$\vec{\nabla} \times \vec{b} = s A_\varphi \hat{\varphi}$	$\frac{s^2 A_\varphi^2}{2}$
Non-radial	B_r	$-\vec{r} \times (\vec{\nabla}_r B_r)$	$\vec{r}(r B_r)$	$\frac{r^2 B_r^2}{2}$
	B_T	$a_r = \frac{T}{r}$	$\vec{\nabla} \times \vec{b} = \frac{T}{r}\hat{r}$	$\frac{T^2}{2}$
Two dimensional	B_z	$\vec{\nabla} \times B_z \hat{z}$	$B_z \hat{z}$	$\frac{B_z^2}{2}$
	A_z	$A_z \hat{z}$	$\vec{\nabla} \times \vec{b} = A_z \hat{z}$	$\frac{A_z^2}{2}$

10. Conclusion

The material making up the liquid outer core of the Earth is about 20% less dense at the top of the core than at its bottom. This density gradient can be explained in terms of a homogeneous compressible fluid. The question arises as to whether the core can be treated as an incompressible fluid in dynamo theory.

Dimensional arguments indicate that for magnetic and velocity fields having typical lengths of more than 1000 km the effects of compressibility in the induction equation were roughly comparable to the total effects of transport. The ratio of the two is given by the Smylie-Rochester compressibility number

$$C = \frac{\Delta \rho}{\rho}$$

For a radial density distribution, this becomes

$$C = \left| \frac{1}{\rho} \frac{\partial \rho}{\partial r} \right| L$$

For the Earth's outer core the fractional density derivative is about $11 \times 10^{-8}\ m^{-1}$.

These arguments also suggest that the effects of compressibility are comparable to the effects of ohmic diffusion in the core for typical lengths of a few hundred kilometres or more: The ratio of the former to the latter is given by

$$R_{mc} = R_m C$$

Anti-dynamo theorems can be divided into two classes: theorems applying for steady magnetic fields only, and theorems applying to magnetic fields that are allowed to be time-dependent. In general the proofs or the theorems dealing with time dependent fields may not be valid in a compressible fluid. In particular the theorem on axisymmetric fields can only be applied with care and possibly not at all.

This is not to say that dynamos violating the theorems can exist. There are no existence proofs, and the theorems might be re-established on other lines.

If the extension cannot be made, then the commonly made statement "that nearly all velocity fields can give rise to dynamo action if the magnetic Reynolds number is high enough" (Busse, 1978) can be extended a little further than before.

It is interesting to speculate on the role of compressibility in the wider confines of dynamo theory as a whole. In the 'kinematic' theory we assume a velocity field and let it work on the magnetic field through the induction equation. In a compressible fluid, the transport term of the induction equation can be broken down into three terms rather than two. This is not really going to be much more difficult to deal with.

Very likely the major effect will be on the task of choosing a velocity field $\vec{V}$. For instead of being solenoidal $\vec{V}$ must satisfy the more difficult condition

$$\vec{\nabla}\cdot\vec{v} = -\frac{1}{\rho}\,\vec{v}\cdot\vec{\nabla}\rho \cong -\frac{1}{\rho}\,v_r\frac{\partial\rho}{\partial r}$$

The radial density derivative is admittedly a known function but its inclusion must make the problem harder.

Turbulent or mean field models will presumably be less affected than 'one-scale' or whole-core models. A length scale of a hundred kilometres or so will likely mean that compressibility can be disregarded.

The dynamic problem will also suffer from the more complicated continuity equation. One might think that the expression of viscous forces will be made more complex. However, though the viscosity of the core is poorly known (Gans, 1972), it is generally felt that viscous effects can be neglected in comparison with the Coriolis and Lorentz forces.

An insight into the hydrodynamics of the core has been gained recently (Busse, 1978). In a laboratory model consisting of a rotating spherical shell filled with water the circulation pattern driven by a temperature difference between the inner and outer boundaries of the shell takes the form of long counter-rotating cylinders aligned with the rotation axis. The spherical boundary at the ends of the cylindrical rollers causes upward and downward flow in alternate rollers. These motions are capable of generating the magnetic field of the Earth.

It is easy to criticize the application of this model to the Earth. There is no Lorentz force in the experimental model, while viscous forces are large. In the real Earth, the roles are reversed.

Even so perhaps some comments on the effects of compressibility could be made. The laboratory model is quite clearly using an incompressible fluid. We will use the notion of the typical length.

In the experiment the rollers had a thickness about one tenth the radius of the sphere. Other things being equal the more viscous the liquid the thicker the rollers. The effects of a larger magnetic field might be the same. Taking 300 kilometres for a typical length means that compressibility could well be neglected in the induction equation.

On the other hand the circulation along the axes of the cylinders is of whole core dimensions. The induction equation ought to include the compressibility then.

No simple analysis can decide the matter. However the fact that the rollers extend virtually the whole diameter of the core suggests that the continuity equation in its fuller form should be used in a fuller analysis. The effects on the hydrodynamics and magnetohydrodynamics of the core of compressibility await further study.

References

Backus, G. E. (1957), The axisymmetric self-excited fluid dynamo. Astrophys. J. 125, 500-524.

Backus, G. E. (1958), A class of self-sustaining dissipative spherical dynamos. Ann. Phys. 4, 372-447.

Backus, G. E. & Chandrasekhar, S. (1956) On Cowling's theorem on the impossibility of self-maintained axisymmetric homogeneous dynamos. Proc. Nat. Acad. Sci. 42, 105-109.

Braginskii, S. I. (1964a), Self excitation of a magnetic field during the motion of a highly conducting fluid. Sov. Phys. JETP 20, 726-735.

Braginskii, S. I. (1964b), Theory of the hydromagnetic dynamo. Sov. Phys. JETP 20, 1462-1471.

Browne, T. (1658) Hydriotaphia, Henry Brome, London.

Bullard, E. C., Freedman, C., Gellman, H. & Nixon, J. (1950) The westward drift of the Earth's magnetic field. Phil. Trans. Roy. Soc. A243, 67-92.

Bullard, E. C. & Gellman, H. (1954), Homogeneous dynamos and terrestrial magnetism. Phil. Trans. Roy. Soc. A247, 213-278.

Busse, F.H. (1975), A necessary condition for the geodynamo. J. Geophys. Res. 80, 278-280.

Busse, F.H. (1978), Magnetohydrodynamics of the Earth's dynamo. Ann. Rev. Fluid Mech. 10, 435-462.

Childress, S. (1968), Théorie Magnetohydrodynamique de l'Effet Dynamo. Lecture notes for a course given at Institut Henri Poincaré.

Cowling, T. G. (1933), The Magnetic field of sunspots. Mon. Not. Roy. Astr. Soc. 94, 39-48.

Cowling, T. G. (1957a), Magnetohydrodynamics, Interscience, New York.

Cowling, T. G. (1957b), The dynamo maintenance of steady magnetic fields. Q. J. Mech. Appl. Math. 10, 129-136.

Cowling, T. G. (1976), Magnetohydrodynamics, 2nd ed., Adam Hilger, Bristol.

Dziewonski, A. M., Hales, A. L. & Lapwood, E. R. (1975) Parametrically simple Earth models consistent with geophysical data. Phys. Earth Plan. Int. 10, 12-48.

Elsasser, W. M. (1946), Induction effects in terrestrial magnetism, I. Theory. Phys. Rev. 69, 106-116.

Gardiner, R. B. & Stacey, F. D. (1971), Electrical resistivity of the core. Phys. Earth Plan. Int. 4, 406-410.

Greenspan, H. (1968), The Theory of Rotating Fluids, Cambridge University Press, London.

Gubbins, D. (1974). Theories of the geomagnetic and Solar Dynamos. Revs. Geophys. Space Phys. 12, 137-154.

Hartmann, W.K. (1972). Moons and Planets, Wadsworth, Belmont, Cal.

Herzenberg, A. (1958), Geomagnetic dynamos. Phil. Trans. Roy. Soc. A250, 543-583.

Higgins, G. H. & Kennedy, G. C. (1971), The adiabatic gradient and the melting point gradient in the core of the Earth. J. Geophys. Res. 76, 1870-1878.

Larmor, J. (1919), How could a rotating body such as the Sun become a magnet? Rep. Brit. Assoc. Adv. Sci. 1919, 159-160.

Loper, D. E. (1978), The gravitationally powered dynamo. Geophys. J. Roy. Astr. Soc. 54, 389-404.

Lortz, D. (1968), Impossibility of steady dynamos with certain symmetries. Phys. Fluids 11, 913-915.

Moffatt, H.K. (1978), Magnetic Field Generation in Electrically Conducting Fluids, Cambridge University Fluids, Cambridge.

Namikawa, T. & Matsushita, S. (1970), Kinematic dynamo problem. Geophys. J. Roy. Astr. Soc. 19, 395-415.

Parker, E. N. (1955), Hydromagnetic dynamo models. Astrophys. J. 122, 293-314.

Roberts, P. H. (1967), An Introduction to Magnetohydrodynamics, American Elsevier, New York.

Roberts, P. H. & Stix, M. (1971), The Turbulent Dynamo; A Translation of a Series of Papers by F. Krause, K. H. Radler, and M. Steenbeck. Tech. Note 60. NCAR, Boulder, Colorado.

Smylie, D. E. & Rochester, M. G. (1979), Compressibility and Core Dynamics. J. Geophys. Res., in press.