THE "+" CONSTRUCTION

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PREFACE

The main goal of this thesis is to give a complete and explicit description of Quillen's "+"construction which yields for each connected based CW-complex $(X,*)$ and perfect subgroup $G \subset \pi_1(X,*)$ a connected based CW-complex $(X^+,*)$ containing $(X,*)$ as a subcomplex such that the inclusion $i:(X,*) \hookrightarrow (X^+,*)$ has the following two properties

(i) $\pi_1 i: \pi_1(X,*) \longrightarrow \pi_1(X^+,*)$ is an epimorphism with kernel the normal closure of G in $\pi_1(X,*)$.

(ii) For all $\pi_1 X^+$-modules A, i induces isomorphisms in all homology groups with local coefficients determined by A.

Studies of the "+"construction make a fairly thorough understanding of homology with local coefficients desirable. Unfortunately, there seems to be no source available which really suits the purpose of introducing a student into the material needed here.

Thus, chapter I is concerned with a careful development of the theory of homology with local coefficients and, wherever possible, its relation to ordinary singular homology. In particular, it is shown that homology with local coefficients can be described by functors having properties which are analogous to the Eilenberg Steenrod axioms for ordinary homology.

We develop two main applications of homology with local coefficients both of which can be viewed as consequences of a fundamental theorem due to Eilenberg saying that for a universal covering the equivariant homology groups of the total space with coefficients in the Π-module A (Π denotes the fundamental group of the base space) are isomorphic to the homology groups of the base space with coefficients in the system of local coefficients determined by A.

If in this situation the base space is a $K(\Pi,1)$, the total space will be contractible which implies that the singular chain complex of the total space forms a free resolution of the integers regarded as a Π-module with trivial action. But then the n-th homology group of $K(\Pi,1)$ with local coefficients in the system determined by the Π-module A will be isomorphic to $Tor_n^{\pi}(Z,A)$.

The functors Tor can be defined in a purely Algebraic manner. Thus, homology with local coeffi -

cients in a $K(\Pi,1)$ establishes a strong bridge between certain questions in Algebraic Topology and Homological Algebra.

This Bridge deserves even more attention in the light of a theorem of Kan and Thurston which gives for every connected CW-complex X a group Π and a map $j: K(\Pi,1) \to X$ which induces isomorphisms in all homology groups with local coefficients. Furthermore, X can be reobtained, up to homotopy type, by applying the "+"construction to the perfect kernel of the homomorphism induced by j on fundamental groups.

Our second main application is a Whitehead type theorem in terms of homology with local coefficients: A map between CW-complexes inducing isomorphisms in all homology groups with local coefficients and in fundamental groups is a homotopy equivalence.

It is the combination of these results, which allows Baumslag, Dyer and Heller to reconstruct Topology in a purely Algebraic manner within a suitable category of fractions. Applications in concrete cases are still open to further investigations.

In chapter II, we give a full description of the "+"construction. Certain details are separated from the main flow of thoughts and postponed until later in the text. It is hoped that this serves well the more advanced reader who would not need the details anyway and the not quite so experienced reader who might wish to find some supplementary information to the main presentation.

For the convenience of the reader, I also include four appendices on material in category theory ("general abstract nonsense", if we follow N. Steenrod's terminology) and Homological Algebra, the fundamental groupoid, covering spaces and CW-complexes. These are designed to establish a link between the background a student should ideally have after one course in Algebraic Topology according to J. F. Adams' suggestion [A] and the background needed in the text. Also some notation is stipulated there.

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Also, I am indebted to K. Varadarajan who managed to straighten out my understanding of homology with local coefficients during a short visit as a guest speaker.

CONTENTS

CHAPTER I

HOMOLOGY WITH LOCAL COEFFICIENTS

Introduction

Homology with local coefficients is a generalization of singular homology with (trivial/ordinary) coefficients in an abelian group G obtained by taking into account the structure of the fundamental groupoid of a given space X in a suitable way. This is due to N. Steenrod [St].

Instead of working with a fixed coefficient group, we work with a system of coefficient groups in X, which, for practical purposes, may roughly be visualized as follows: Above each $x \in X$, we have an abelian group and each homotopy class of paths in X gives rise to an isomorphism between the groups at its end points. Formally, such a system is most advantageously described by a functor from the fundamental groupoid of X into the category of abelian groups.

In section 2, we show how singular simplices in X can be used to construct a chain complex that also reflects information contained in the structure of a given system of local coefficients in X.

Section 3 is dedicated to systematic studies of fundamental properties of the resulting homology groups with local coefficients. The relationship between homology with ordinary and local coefficients is well illustrated by Theorem 3.10 which enables us to understand homology with local coefficients as functors from a category whose objects are triples (X, A, G), (X, A) being a pair of spaces, G a system of coefficients, into the category of abelian groups. These functors are shown to satisfy properties analogous to the Eilenberg-Steenrod axioms for ordinary homology. Indeed, these functors reduce to ordinary homology functors on suitable subcategories.

In section 4, we deal with results of Eilenberg [E] establishing a link between the homology with local coefficients of the base space of a covering projection with the equivariant homology of the total space induced by the group of covering transformations. This link has deep consequences in view of more recent results by Kan-Thurston [K-T] and Baumslag, Dyer, Heller [B-D-H]. Using Eilenberg's theorem, these results allow us to reconstruct homology of spaces by algebraic means within a suitable category.

In section 5, we study homology with local coefficients in CW-complexes. In the first part, we show that standard theorems concerning the ordinary homology of n-cell adjunctions can be generalized to analogous theorems concerning the homology with local coefficients of n-cell adjunctions. In the second part, we prove a Whitehead type theorem for homology with local coefficients, which also shows that homology with local coefficients provides more information than ordinary homology. An instructive example for this is given.

Unless otherwise specified, I followed more or less closely the splendid exposition of homology with local coefficients in G. Whitehead's book, "Elements of Homotopy Theory" [Wh1]. The treatment here, however, is much more detailed and explicit. Furthermore, parts of sections 1,4,5 go a bit beyond the material covered in [Wh1].

According to our main task: namely, to provide background for the "+"construction of Quillen, the material of Chapter I has been arranged so as to facilitate the results of sections 4,5. Apart from that, there are at least two main applications for the (co)-homology derived out of local coefficients which are not treated here:

(i) The homotopy/homology of (locally trivial) fibre bundles, which actually motivated Steenrod [St]. (The homotopy motivation for local coefficients is obvious.)

(ii) Obstruction theory: Let (Y,X) be an n-cell adjunction with attaching maps $\varphi_\lambda : S^{n-1}_\lambda \to X$;

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$f : X \to Z$ a map. Using the homotopy extension property of (B^n, S^{n-1}), it is easy to see (II-7.1) that f can be extended to a map $F: Y \to Z$, $F_{|X} \equiv f$ iff for all λ, $f\circ\varphi_\lambda : S_\lambda^{n-1} \to Z$ is homotopically trivial. Let * be a base point for S_λ^{n-1}. If $f\circ\varphi_\lambda(*) = z_o$ for all λ, then $f\circ\varphi_\lambda$ represents an element of $\pi_{n-1}(Z,z_o)$, and we obtain a well defined element

$$c(f) \in \text{Hom}\,(H_n(Y,X),\ \pi_{n-1}(Z,z_o))$$

whose vanishing is a necessary and sufficient condition for f to be extendable.

In general, however, $f\circ\varphi_\lambda$ will represent an element of $\pi_{n-1}(Z,z_\lambda)$, the z_λ's not all being the same. So we get

$$c(f) \in \text{Hom}\,(H_n(Y,X),\ \bigoplus_{z\in Z} \pi_{n-1}(Z,z))$$

which is an element of the n-th cochain group of (Y,X) with coefficients in the system $\pi_{n-1}(Z,fx)$.

1. Systems of Local Coefficients

In this section, we study systems of local coefficients in a space X. Starting with elementary properties, we work our way up to the point where we see that such a system is already uniquely determined (up to natural equivalence) by an abelian group Gx_0 and an action of $\pi_1(X,x_0)$ on Gx_0 for any $x_0 \in X$, provided that X is path connected. In turn, this information is utilized to derive criteria for systems of local coefficients in a 0-connected space to be natural equivalent.

Let X be a topological space, ΠX its fundamental groupoid and *Ab* the category of abelian groups.

(1.1) Definition: A contravariant functor $G : \Pi X \to Ab$ is called a system of local coefficients in X.

If φ is a homotopy class of paths in X connecting $x, y \in X$, and $\overline{\varphi}$ is the inverse of φ in ΠX, then $G\varphi$ is invertible in *Ab* since

$$1_{Gx} = G(1_x) = G(\varphi\overline{\varphi}) = (G\varphi)(G\overline{\varphi})$$

Denote by $G(\Pi X)$ the subcategory of *Ab* whose objects are the abelian groups Gx for $x \in X$, and whose morphisms are the group homomorphisms $G\varphi \in Ab(Gy, Gx)$ for $\varphi \in \Pi X(x,y)$. As a consequence of the previous observation, we obtain:

(1.2) Remark: $G(\Pi X)$ is a groupoid. In particular, if $x, y \in X$ are elements of the same path component of X, then Gx, Gy are isormorphic abelian groups, and for each homotopy class φ of paths connecting x to y

$$G\varphi : Gy \to Gx$$

is an isomorphism.

(1.3) Example: For $n > 1$ ($n \geq 1$ if $\pi_1(X,x)$ is abelian for all $x \in X$), let $G_n : \Pi X \to Ab$ be defined by

$$G_n x = \pi_n(X, x)$$

on objects and on morphisms in the following way. For a path α connecting x with y (i.e. $\alpha(0) = x$, $\alpha(1)=y$) and $f : (B^n, S^{n-1}) \to (X, y)$, we define a homotopy

$$\tilde{f} : S^{n-1} \times I \ni (z, t) \to \alpha(1-t) \in X$$

satisfying $f|_{S^{n-1}} \equiv \tilde{f}|_{S^{n-1} \times \{0\}}$. Since the pair (B^n, S^{n-1}) has the homotopy extension property, $\tilde{f}$ can be extended to a map $F : B^n \times I \to X$ such that $F|_{S^{n-1} \times I} \equiv \tilde{f}$. Then

$$\alpha_* f := F|_{B^n \times \{1\}} : (B^n, S^{n-1}) \to (X, x).$$

Elementary considerations show that for $\alpha' \approx \alpha$, $f' \approx f$ we obtain

$$\alpha_* f \approx \alpha_* f' \approx \alpha'_* f'.$$

Thus, if φ is a path class connecting x with y, we obtain a function $\varphi_* : \pi_n(X, y) \to \pi_n(X, x)$ by picking a representative α for φ, and f for $[f] \in \pi_n(X, y)$, and defining

$$\varphi_*[f] := [\alpha_* f]$$

In fact, $\varphi_* : \pi_n(X, y) \to \pi_n(X, x)$ is a group homomorphism, which can be seen by using a similar version of the HEP argument given above. Moreover,

(i) $(1_x)_*[f] = [f]$ for all $[f] \in \pi_n(X, x)$

(ii) $(\varphi\psi)_* = \varphi_* \circ \psi_*$ for $\varphi \in \Pi X(x, y)$, $\psi \in \Pi X(y, z)$.

Thus, setting $G_n \varphi := \varphi_*$ for a morphism φ in ΠX

makes G_n a contravariant functor on ΠX. Example (1.3) is of crucial importance if it comes to defining the obstruction cocycle of a map $f : X \to Z$ of an n-cell adjunction (Y, X) with a non simply connected space Z and $f \circ \chi^n_\lambda \big|_{S^{n-1}_\lambda}$ is not base point preserving for at least one attaching map χ^n_λ. However, since we do not intend to develop obstruction theory, we shall make no further use of Example (1.3) and the explanation just given.

(1.4) <u>Example/Definition:</u> Let A be an abelian group. We define the constant system G_A in X determined by A by

$$G_A x := A \quad \text{on objects}$$
$$G_A \varphi := 1_A \quad \text{on morphisms.}$$

More examples of systems of local coefficients can be found in [Whl], pp. 257-263. Now let X be a topological space and G a system of local coefficients in X.

(1.5) <u>Definition:</u> G is simple iff there exists $A \in |Ab|$ such that the functors G and G_A (constant system determined by A) are naturally equivalent.

(1.6) <u>Proposition:</u> Let $G : \Pi X \to Ab$ be a system of local coefficients in X; then,

$$X \text{ 1-connected} \Rightarrow G \text{ simple.}$$

<u>Proof:</u> Select an arbitrary element $x_o \in X$ and denote by H the constant system in X induced by Gx_0. To construct a natural equivalence τ between G and H, we proceed as follows. X is path connected. Consequently, for each $x \in X$, we find a path α connecting x with x_0 (i.e.: $\alpha(0) = x$, $\alpha(1) = x_0$) Since X is 1-connected, α represents the unique path class $\varphi := [\alpha]$ connecting x with x_o. Thus,

$$\tau x := G\varphi : Gx_o = Hx \to Gx$$

is well defined. τ is a natural equivalence between G and H, for if $x, x' \in X$ and α, α' are paths connecting x, x' with x_o, then $\alpha'\bar{\alpha}$ represents the unique path class ψ connecting x' with x.

$$\begin{array}{cccc}
x & Gx & \xleftarrow{\tau x = G[\alpha]} & Gx_o = Hx \\
\psi \uparrow & \downarrow G\psi & & \downarrow 1_{Gx_o} \\
x' & Gx' & \xleftarrow[\tau x' = G[\alpha']]{} & Gx_o = Hx
\end{array}$$

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Because G is a contravariant functor, we have

$$G\psi = G[\alpha'\bar{\alpha}] = G[\alpha'] \circ G[\bar{\alpha}] = G[\alpha'] \circ G[\alpha]^{-1}$$

which proves the commutativity of the diagram. Thus, $\tau : H \dot{\to} G$ is a natural transformation. The same type of argument shows: $\tau^{-1} : G \dot{\to} H$ is also a natural transformation, and our claim follows. □

Let X,Y be topological spaces, and $f : X \to Y$ a continuous map. If α, β are homotopic paths in X connecting x_1 with x_2 then $f\circ\alpha$, $f\circ\beta$ are homotopic paths in Y connecting fx_1 with fx_2 (homotopy is an equivalence relation). Thus f induces a functor $F : \Pi X \to \Pi Y$. We can use F to pull a given system of local coefficents G in Y back to a system of local coefficients

$$f^*G := G\circ F : \Pi X \to \mathit{Ab}$$

in X, and call it the system in X induced by f. Evidently

(1.7) $f : X \to Y$, $g : Y \to Z$ continuous; H a system of local coefficients in Z, then

$$(g \circ f)^* H = f^* (g^* H)$$

In the situation where A is a subspace of X, $i : A \hookrightarrow X$ the inclusion and G a system of local coefficients in X, we also write $G_{|A}$ for i^*G.

The rest of this section has been placed here for systematic reasons and, except for theorem 1.9, will not be needed until the end of section 3. Accordingly the reader may proceed to section 2 now and come back to the material here as we refer to it.

If $p : E \to B$ is a universal cover of the path connected space B and G a system of local coefficients in B, then (1.6) shows that p^*G is a simple system in E. Using a different argument, we could also get this as a special case of the following more general situation.

Let B be 0-connected, locally path connected and semilocally 1-connected, and let $b_o \in B$. Let G be a system of local coefficients in B, and let $K := \{\zeta \in \pi_1(B,b_o) : G\zeta = 1_{Gb_o}\}$. Then K is a subgroup of $\pi_1(B,b_o)$. These hypotheses guarantee the existence (see the appendix) of a covering map $p : \tilde{B} \to B$ and a point $\tilde{b}_o \in p^{-1}\{b_o\}$ such that

$$- \quad p_* \pi_1(\tilde{B}, \tilde{b}_o) = K$$

(1.8) <u>Proposition:</u> p^*G is simple

<u>Proof:</u> The proof of (1.6) carries over to verify the following statement: If $p^*G(\Pi\tilde{B})$ has precisely one morphism between each pair of objects, then p^*G is simple. Hence, we show that $p^*G(\Pi\tilde{B})$ has precisely one morphism between each pair of objects. There exists at least one, because $\tilde{B}$ is 0-connected. There exists at most one:

Let φ, ψ be path classes connecting x with x' for $x, x' \in \tilde{B}$ and μ a path class connecting $\tilde{b}_o$ with x. Then $\mu\varphi\bar{\psi}\bar{\mu} \in \pi_1(\tilde{B}, \tilde{b}_o)$, and

$$^1Gb_o = G[p(\mu\varphi\bar{\psi}\bar{\mu})] = p^*G(\mu\varphi\bar{\psi}\bar{\mu})$$
$$= (p^*G\bar{\mu}) \circ (p^*G\varphi) \circ (p^*G\bar{\psi}) \circ (p^*G\bar{\mu})$$

This is an equation between group isomorphisms. Hence,

$$(p^*G\bar{\mu})^{-1} = (p^*G\mu) = (p^*G\mu) \circ (p^*G\varphi) \circ (p^*G\bar{\psi})$$

and so

$$1_{G_{px}} = (p^*G\varphi) \circ (p^*G\overline{\psi}).$$

Thus : $(p^*G\overline{\psi})^{-1} = p^*G\psi = p^*G\varphi$. □

The preceding observations enable us to get some information on a given system of local ceofficients in particular cases. We now turn to a method of contstructing such systems in a given 0-connected space.

First of all, note that, given a space X , a system of local coefficients G in X, $x_o \in X$, then $\pi_1(X, x_o)$ acts on the abelian group Gx_o by automorphisms:

$$\alpha.a := (G\alpha)a, \quad \alpha \in \pi_1(X,x_o), \; a \in Gx_o$$

Accordingly, the abelian group Gx_o is at the same time a $Z\pi_1(X,x_o)$-module. Conversely, we have:

(1.9) Theorem: Let X be 0-connected and A an abelian group on which $\pi_1(X,*)$ acts by automorphisms (i.e. a $\pi_1(X,*)$-module). Then there exists a system of local coefficients G in X such that

$$G* = A$$

and which induces the given operation of $\pi_1(X,*)$ on A. Furthermore, all such systems are naturally equivalent.

Proof: Let

$$\xi : X \to \{\Pi X(*,x)\}_{x\in X}, \quad \xi(x) \in \Pi X(*,x)$$

be a choice function such that $\xi(*)$ is the neutral element of the group $\pi_1(X,*)$. Now define $G : \Pi X \to Ab$ by

$$Gx = A \quad \text{on objects}$$

$$G(\varphi \in \Pi X(x,x'))(a) := (\xi(x)\varphi\overline{(\xi(x'))}).a \quad \text{on morphisms}$$

G is a contravariant functor on ΠX. To see this, let $\varphi \in \Pi X(x_o, x_1)$ and $\psi \in \Pi X(x_1, x_2)$. Then

$$\begin{aligned} G(\varphi\psi)(a) &= [\xi(x_o)\varphi\psi\overline{\xi(x_2)}].a \qquad \text{for } a \in A \\ &= [\xi(x_o)\varphi(\overline{\xi(x_1)})(\xi(x_1))\psi(\overline{\xi(x_2)}).a \\ &= [\xi(x_o)\varphi\overline{(\xi(x_1))}]((\xi(x_1)\psi(\overline{\xi(x_2)})).a \\ &= (G\varphi) \circ (G\psi)(a) \end{aligned}$$

If c_x denotes the class of the constant path at x, then

$$(Gc_x)(a) = (\xi(x)c_x\overline{\xi(x)}).a = c_*.a = a$$

Furthermore, if $\varphi \in \Pi X(*,*) = \pi_1(X,*)$

$$(G\varphi)(a) = [\xi(*)\varphi\overline{\xi(*)}].a = \varphi.a$$

as required.

Notice that the construction of a system of local coefficients G in the proof of (1.9) depends crucially on the choice function ξ. Thus, it seems plausible that we could get many different systems of local coefficients in X satisfying the requirements given in the statement of (1.9). However, this is not so. In fact, if $H : \Pi X \to Ab$ were any other system satisfying these requirements, then

$$\tau : G \to H \; ; \quad X \ni x \to \tau(x) := H\overline{\xi(x)} \in Ab(A, Hx)$$

is a natural equivalence between G and H:

$$\begin{array}{ccccc} x & & A & \xrightarrow{H\overline{\xi(x)}} & Hx \\ \alpha \uparrow & & G\alpha \downarrow & & \downarrow H\alpha \\ x' & & A & \xrightarrow[H\overline{\xi(x')}]{} & Hx' \end{array}$$

By assumption, we have $H* = A$ and

$$(G\alpha)(a) = (\xi(x')\alpha\overline{\xi(x)}).a =$$

$$= H(\xi(x')\alpha\overline{\xi(x)})(a) = (H\xi(x')\circ(H\dot{\alpha})\circ(H\overline{\xi(x)})(a)$$

Furthermore, since $\xi(x')\overline{\xi(x')}$ is contractible,

$$(H\xi(x'))^{-1} = H\overline{\xi(x')}$$

whence the above diagram commutes.

Thus, all systems of local coefficients in our consideration are naturally equivalent to G, and consequently pairwise naturally equivalent. □

Given two systems of local coefficients in a space X, we can use Theorem (1.9) to reduce the question of whether they are equivalent or not to a question of purely group theoretic nature. More precisely:

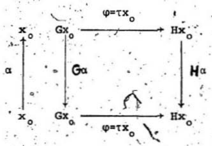
(1.10) <u>Corollary:</u> Let G,H be systems of local coefficients in the 0-connected space X, $x_o \in X$. Let $g : \pi_1(X, x_o) \to \mathrm{Aut}\, Gx_o$, $h : \pi_1(X, x_o) \to \mathrm{Aut}\, Hx_o$ be the induced actions of $\pi_1(X, x_o)$ on Gx_o, Hx_o ; then:

G, H are naturally equivalent iff

(i) There exists an isomorphism $\phi : Gx_o \to Hx_o$ and, consequently, an isomorphism $\phi' : \mathrm{Aut}\, Gx_o \ni f \mapsto \varphi f \varphi^{-1} \in \mathrm{Aut}\, Hx_o$.

(ii) The restriction of ϕ to $g(\pi_1(X, x_o))$ is an isomorphism $\phi' : g(\pi_1(X, x_o)) \to h(\pi_1(X, x_o))$ and $\phi\, g = h$.

<u>Proof:</u> "⇒" Let $\tau : G \dot{\to} H$ be a natural equivalence. Then $\varphi := \tau x_o : Gx_o \to Hx_o$ is an isomorphism. On the other hand, the defining property of τ shows



that $H\alpha = \phi' G\alpha$, and $G\alpha = \phi'^{-1} H\alpha$.

"<=" As in (1.9) let $\xi : X \to \{\Pi X(x_o, x)\}_{x \in X}$ be a choice function and denote by G', H' the systems of local coefficients in X induced by g, h and ξ. By (1.9) G is equivalent to G' and H is equivalent to H'. But $\tau' : G' \to H'$ defined by $\tau'(x) = \varphi$ is a natural equivalence, as the following diagram shows.

$$\begin{array}{ccccc}
X & & Gx = Gx_o & \xrightarrow{\varphi} & Hx_o = Hx \\
\uparrow \alpha & & \downarrow g(\xi(x')\alpha\overline{\xi(x)}) & & \downarrow h(\xi(x')\alpha\overline{\xi}(x)) \\
X' & & Gx' = Gx_o & \xrightarrow{\varphi} & Hx_o = Hx'
\end{array}$$

Theorem (1.9) and Corollary (1.10) may be interpreted as follows:

(1.11) <u>Corollary:</u> Given a 0-connected space X, $x_o \in X$ and an abelian group A, then:

(i) Each system G of local coefficients in X with $Gx_o \cong A$ can be represented by an <u>equivalent</u> system G' such that $G'x_o = A$.

(ii) Let G, H be local systems in X such that $Gx_o = Hx_o = A$, and let g, h be as in (1.10). Then, G is equivalent to H iff $g(\pi_1(X,x_o))$ and $h(\pi_1(X, x_o))$ are conjugate subgroups of $\operatorname{Aut} A$.

(iii) Let S be the class whose elements are equivalence classes of systems of local coefficients in X having local group A at x_o. If $g, h : \pi_1(X,x_o) \to \operatorname{Aut} A$ are homomorphisms, call g, h conjugate iff there exists $\varphi \in \operatorname{Aut} A$ such that for all $\alpha \in \pi_1(X,x_o)$, $h\alpha = \varphi^{-1}(g\alpha)\varphi$ holds. Denote by $[g]$ the conjugacy class of g and define

$$K := \{[g] : g : \pi_1(X,x_o) \to \operatorname{Aut} A \text{ is a homomorphism}\}.$$

Then there is a 1-1 correspondece between K and S. In particular, S is a proper set.

<u>Proof:</u> (i), (ii) follow directly from (1.10). (iii) is a restatement of (ii). □

(1.12) <u>Example:</u> Let X be 0-connected, $x_o \in X$.

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Let A be an abelian group such that $\text{Aut}\, A$ is abelian; then, each conjugacy class of K consists of precisely one element, and (1.11) (iii) tells us that the set S is in 1-1 correspondence with the set of homomorphisms of $\pi_1(X, x_o)$ into Aut A.

Note that, since $\text{Aut}\, \mathbb{Z} \simeq \mathbb{Z}_2$ is abelian, (1.12) applies in particular in the case $A \simeq \mathbb{Z}$. We use theorem (1.9) one more time to give a local condition for the existence of a natural transformation of one local system into another.

(1.13) <u>Corollary:</u> Let X be 0-connected, $x_o \in X$, G, H systems of local coefficients in X; then, a homomorphism $\varphi : Gx_o \to Hx_o$ which makes the diagram

$$\begin{array}{ccc} Gx_o & \xrightarrow{\varphi} & Hx_o \\ {\scriptstyle G\alpha}\downarrow & & \downarrow{\scriptstyle H\alpha} \\ Gx_o & \xrightarrow{\varphi} & Hx_o \end{array}$$

commute for each $\alpha \in \pi_1(X,x_o)$, determines a natural transformation $\tau : G \to H$.

Proof: Proceed as in (1.10) (ii) → (i) and define $\tau(x) := \varphi$. □

Remark: Taking φ to be the 0-map, we see that natural transformations always exist.

2. Homology Groups with Local Coefficients

As when constructing ordinary singular homology theory, we define

$$\Delta_p := \{(x_o,\ldots,x_p)\in\mathbb{R}^{p+1} : \sum_{i=o}^{p} x_i = 1,\ x_i \geq 0\} \subset \mathbb{R}^{p+1}$$

to be the standard p-simplex. For $p \geq 1$ the subset

$$\Delta_p^i := \{(x_o,\ldots,x_p)\in\Delta_p : x_i = 0\} \quad 0 \leq i \leq p$$

of Δ_p is the i-th face of Δ_p. Δ_p^i is linearly homeomorphic to Δ_{p-1} by the following map

$$f_p^i : \Delta_{p-1} \ni (x_o,\ldots,x_{p-1}) \mapsto (x_o,\ldots,x_{i-1},0,x_i,\ldots,x_{p-1}) \in \Delta_p^i$$

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For the applications we have in mind, it will be convenient to call $e_p := (1,0,\dots,0) \in \Delta_p$ the preferred vertex of Δ_p and $f_p^i(e_{p-1})$ the preferred vertex of the i-th face of Δ_p. Note that

$$f_p^o(e_{p-1}) = (0,1,0,\dots,0) \quad \text{and} \quad f_p^i(e_{p-1}) = e_p \qquad i \geq 1$$

Now let X be a space, G a system of local coefficients in X. Let $\sigma_p := \{u : \Delta_p \to X \text{ continuous}\}$, and let

$$C_p(X;G) := \bigoplus_{u \in \sigma_p} G u(e_p)$$

The abelian group $C_p(X;G)$ can be thought of as the set of all functions $c : \sigma_p \to \prod_{x \in X} Gx$ such that

(1) for every singular p-simplex $u : \Delta_p \to X$, $c(u) \in Gu(e_p)$

(2) for all but finitely many singular simplices, $c(u) = 0$.

A relationship between $C_p(X;G)$ and $C_pX := \bigoplus_{u \in \sigma_p} Z$

the free abelian group generated by σ_p is given by

$$C_p(X;G) = \bigoplus_{u \in \sigma_p} (Z \otimes Gu(e_p))$$

The elements of $C_p(X;G)$ are called singular p-chains with coefficients in G. We call $c \in C_p(X;G)$ elementary iff $c(u) \neq 0$ for at most one $u \in \sigma_p$, in which case we may write an arbitrary $c \in C_p(X;G)$ as a finite sum

$$c = \sum_k c(u_k) \cdot u_k$$

where $c(u_k) \in Gu_k(e_p)$.

The groups $C_p(X;G)$ form a graded group, which we make into a chain complex by defining a boundary operator $\partial = \{\partial_p : C_p(X;G) \to C_{p-1}(X;G)\}$. By the univeral property of weak sums, it suffices to define $\partial_p(g \cdot u)$ for all elementary p-chains $g \cdot \mu$.

Proceeding as in ordinary singular homology, we might attempt to define $\partial_p(g \cdot u) = \Sigma\ (-1)^i g \cdot u \circ f_p^i$. This is a futile attempt because in general $g \notin Gu \circ f_p^o(e_{p-1})$, so that the righthand term in this equation is not an element of $C_{p-1}(X;G)$. This obstacle may be surmounted, however, by observing that the edge of Δ_p determined by $f_p^o(e_{p-1})$ and e_p can be interpreted as the path

$$a : [o,1] \ni t \mapsto (f_p^o(e_{p-1}) + t(e_p - f_p^o(e_{p-1})) \in \Delta_p$$

Taking the composite with u, a yields a path class $\alpha_u := [u \circ \alpha] \in \Pi X(u \circ f_p^o(e_{p-1}), u(e_p))$ which induces an isomorphism

$$G\alpha_u : Gu(e_p) \rightarrow Gu \circ f_p^o(e_{p-1})$$

This gives rise to the following modification of our original attempt

$$\partial_p(gu) := (G\alpha_u)g \cdot u \circ f_p^o + \sum_{i=1}^{p} (-1)^i g \cdot u \circ f_p^i$$

We want to show that $\partial_{p-1}\partial_p(g \cdot u) = 0$. So we must find out where the preferred vertices of faces of faces of Δ_p are.

$$f_p^i \cdot f_{p-1}^j(e_{p-2}) = \begin{cases} f_p^i(e_{p-1}) & j \geq 1 \quad = \begin{cases} e_p & i \geq 1 \\ (0,1,0,\ldots,0) & i=0 \end{cases} \\ f_p^i(0,1,0,\ldots0) & j=0 \quad = \begin{cases} (0,1,0,\ldots0) & i \geq 2 \\ (0,0,1,0,\ldots,0) & i=0 \vee i=1. \end{cases} \end{cases}$$

Thus

$$\bar{\bar{\partial}}_{p-1}\partial_p(g\cdot u) =$$

$$= \partial_{p-1}((G\alpha_u)g\cdot uof_p^o + \sum_{i=1}^{p}(-1)^i g\cdot uof_p^i) =$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{p-1}(-1)^{i+j} g\cdot uof_p^i of_{p-1}^j +$$

$$+ \sum_{i=2}^{p}(-1)^i (G\alpha_u) g\cdot uof_p^i of_{p-1}^o \qquad \text{(o-faces of } uof_p^i \ i\geq 2)$$

$$+ (-1)^1 (G\alpha_{uof_p^1}) g\cdot uof_p^1 of_{p-1}^o \qquad \text{(o-face of } uof_p^1)$$

$$+ \sum_{j=1}^{p-1}(-1)^j (G\alpha_u) g\cdot uof_p^o of_{p-1}^j \qquad \text{(j-face of } uof_p^o \ j\geq 1)$$

$$+ (G\alpha_{uof_p^o}) o (G\alpha_u) g\cdot uof_p^o of_{p-1}^o \qquad \text{(o-face of } uof_p^o)$$

Now, the preferred vertices of $uof_p^1 of_{p-1}^o$ and $uof_p^o of_{p-1}^o$ are the same, as we saw above. Furthermore, the edge path $[e_p, f_p^1 of_{p-1}^o(e_{p-2}), f_p^o(e_{p-1}), ep]$ is contractible in Δ_p.

Hence its image under u is contractible in X. This shows that

$$(G\alpha_{uof_p^1})^{-1} \circ (G\alpha_{uof_p^o}) \circ (G\alpha_u) = 1_{Gu(e_p)}$$

and consequently

$$G\alpha_{uof_p^1} = (G\alpha_{uof_p^o}) \circ (G\alpha_u)$$

Furthermore, observe that

$$(x_o,\ldots,x_{p-2}) \xrightarrow{f_{p-1}^i} (x_o,\ldots,x_{i-1},\overset{i+1}{\overset{\downarrow}{0}},x_i,\ldots,x_{p-2}) \xrightarrow{f_p^j}$$

$$\xrightarrow{f_p^j} \begin{cases} (x_o,\ldots,x_{j-1},\overset{j+1}{\overset{\downarrow}{0}},x_j,\ldots,x_{i-1},\overset{i+2}{\overset{\downarrow}{0}},x_i,\ldots,x_{p-2}) & o\leq j\leq i\leq p\leq p-1 \\ (x_o,\ldots,x_{i-1},\underset{i+1}{\underset{\uparrow}{0}},x_i,\ldots,x_{j-2},\underset{j+1}{\underset{\uparrow}{0}},x_{j-1},\ldots,x_{p-2}) & o\leq i\leq j\leq p \end{cases}$$

which yields

$$f_p^j \circ f_{p-1}^i = f_p^i \circ f_{p-1}^{j-1} \qquad o\leq i\leq j\leq p$$

Writing

$$\sum_{i=2}^{p} (-1)^{i} (G\alpha_u) g \cdot u o f_p^i o f_{p-1}^o = \sum_{j=1}^{p-1} (-1)^{j+1} (G\alpha_u) g \cdot u o f_p^{j+1} f_{p-1}^o$$

and using $f_p^{j+1} o f_{p-1}^o = f_p^o o f_{p-1}^j$ $(o \leq j \leq p)$ we see that the lower four lines in the equation $\partial_{p-1}\partial_p(g \cdot u) = \ldots,$ cancel out. Thus we are left with

$$\partial_{p-1}\partial_p(g \cdot u) =$$

$$= g \cdot \sum_{j=1}^{p} \sum_{i=1}^{p-1} (-1)^{i+j} u o f_p^j o f_{p-1}^i$$

$$= g \cdot \left[\sum_{j=2}^{p} \sum_{i=1}^{j-1} (-1)^{i+j} u o f_p^j o f_{p-1}^i + \sum_{i=1}^{p-1} \sum_{j=1}^{i} (-1)^{i+j} u o f_p^j o f_{p-1}^i \right]$$

$$= g \cdot \left[\sum_{j=2}^{p} \sum_{i=1}^{j-1} (-1)^{i+j} u o f_p^i o f_{p-1}^{j-1} + \sum_{i=1}^{p-1} \sum_{j=1}^{i} (-1)^{i+j} u o f_p^j f_{p-1}^i \right]$$

$$= g \cdot \left[\sum_{j=1}^{p-1} \sum_{i=1}^{j} (-1)^{i+j+1} u o f_p^i o f_{p-1}^j + \sum_{j=1}^{p-1} \sum_{i=1}^{j} (-1)^{i+j} u o f_p^i f_{p-1}^j \right]$$

$$= g \cdot 0 = 0$$

This proves

(2.1) Lemma: $(C_*(X;G),\partial)$ is a chain complex. □

It is now a standard procedure in homological algebra to derive homolgy groups out of this chain complex. (See for instance [H-S], pp 117-118)

(2.2) Definition: For $p \geq 0$ we call the p-th homology group of the chain complex $(C_*(X;G),\partial)$

$$H_p(X;G) := H_p(C_*(X;G))$$

the p-th homology group of X with local coefficients in G. The following proposition shows that homology groups with local coefficients are a true generalisation of ordinary homology groups.

(2.3) Proposition: If X is a 0-connected topological space and G a simple system of local coefficients in X, then

$$H_*(X;G) \cong H_*(X;G_o)$$

where $G_o := Gx_o$ for some $x_o \in X$.

Proof: G is simple if and only if the functor $G:\Pi X \to Ab$ is naturally equivalent to a constant functor $F:\Pi X \to Ab$. By the uniqueness part of (1.9), we may pick any $x_o \in X$ and assume that F equals the constant system in X induced by $G_o := Gx_o$. Let

$$\tau : G \to F$$

be a natural equivalence.

To show that $H_*(X;G) \cong H_*(X;G_o)$, it is sufficient to construct a chain equivalence between $(C_*(X;G);\partial)$ and $(C_*X \otimes G_o;\partial')$. Let $g \cdot u$ be an elementary p-chain in $C_p(X;G)$ and define (for each $u \in \sigma_p$, G_o^u denotes a copy of G_o indexed by u)

$$\begin{array}{ccc} \varphi_p : C_p(X;G) & \to & (C_pX) \otimes G_o \\ \| & & \wr\| \\ \bigoplus_{u \in \sigma_p} Gu(e_p) & & \bigoplus_{u \in \sigma_p} G_o^u \end{array}$$

on $g \cdot u$ by

$$\varphi_p(g\cdot u) := [\tau u(e_p)]g\cdot u$$

By the universal property of the weak sum this extends to the homomorphism φ_p, which is easily seen to be an isomorphism between abelian groups. It remains to check that the φ_p combine to define a chain map φ. To do this, we compute

$$\begin{array}{ccc}
g\cdot u & \overset{\partial_p}{\longmapsto} & G(\alpha_u)g\cdot u\circ f_p^0 + \sum_{i=1}^{p}(-1)^i g\cdot u\circ f_p^i \\
\Big\downarrow \varphi_p & & \Big\downarrow \varphi_{p-1} \\
 & & [\tau u\circ f_p^0(e_{p-1})]\cdot G(\alpha_u)g\cdot u\circ f_p^0 + \sum_{i=1}^{p}(-1)^i[\tau u(e_p)]g\cdot u\circ f_p^i \\
[\tau u(e_p)]g\cdot u & \underset{\partial_p'}{\longmapsto} & [\tau u(e_p)]g\cdot \sum_{i=0}^{p}(-1)^i u\circ f_p^i
\end{array}$$

The defining property of τ shows that the two terms in the right bottom corner of the previous computation are equal:

$$\begin{array}{ccccc}
u(e_p) & & Gu(e_p) & \xrightarrow{\tau u(e_p)} & G_o \\
\Big\uparrow \alpha_u & & \Big\downarrow G\alpha_u & & \Big\downarrow 1_{G_o} \\
u\circ f_p^o(e_{p-1}) & & Gu\circ f_p^o(e_{p-1}) & \xrightarrow[\tau u\circ f_p^o(e_{p-1})]{} & G_o
\end{array}$$

We conclude that φ is a chain equivalence. □

(2.4) A subspace $A \subset X$ gives rise to relative homology groups with local coefficients: Let $G: \Pi X \to Ab$ be a system of local coefficients in X and $i: A \hookrightarrow X$; then

$$i_{\#}: C_p(A;G|_A) \ni \Sigma g_j u_j \mapsto \Sigma g_j (i \circ u_j) \in C_p(X;G)$$

is a well defined monomorphism. This allows us to interpret $C_p(A;G|_A)$ as a subgroup of $C_p(X;G)$. Since the boundary operator ∂' of the chain complex $(C_*(A;G|_A), \partial')$ is the restriction of ∂, we may regard the chain complex $(C_*(A;G|_A), \partial')$ as a subcomplex of $(C_*(X;G), \partial)$.

Defining

$$C_p(X,A;G) := C_p(X;G) \big/ i_{\#} C_p(A;G|_A)$$

we obtain a short exact sequence

$$0 \to C_*(A;G|_A) \to C_*(X;G) \to C_*(X,A;G) \to 0$$

of chain complexes. As usual in homological algebra, we define

$$H_p(X,A;G) := H_p(C_*(X,A;G)),$$

and obtain a long exact sequence

$$\ldots H_{p+1}(X,A;G) \xrightarrow{\partial_{p+1}(X,A;G)} H_p(A;G|_A) \to H_p(X;G) \to H_p(X,A;G) \to \ldots$$

□

3. Functorial Properties of Homology with Local Coefficients

The reader will doubtless have realized that homology groups with local coefficients behave analogously to ordinary singular homology groups with respect to subspaces. For practical purposes the most important property of ordinary singular homology groups is that they can be described by functors which satisfy the Eilenberg-Steenrod axioms. Thus we may ask whether homology groups with local coefficients can be described by functors, which, in turn, poses the question for a suitable domain category. We shall exhibit such a category $\mathcal{L}$. Contrary to ordinary singular homology it will not suffice to pick a subcategory of pairs of topological spaces and coefficients. When we define morphisms in $\mathcal{L}$, we want them to respect both of these structures and they should give rise to chain maps between the corresponding singular chain complexes with local coefficients. This will enable us to interpret $H_*: \mathcal{L} \to Ab$ as functors. After defining the notion of homotopy in $\mathcal{L}$ we shall be in a position to state analogues of the Eilenberg-Steenrood axioms for homology with local coefficients and proceed to verify that the H_*'s satisfy these axioms.

Let $\mathcal{L}$ be the category whose objects are triples $(X,A;G)$ with (X,A) a pair in Top and G a system of local coefficients in X. A morphism $\phi:(X,A;G)\to(Y,B;H)$ is a pair (ϕ_1,ϕ_2) such that

$$\phi_1:(X,A) \to (Y,B)$$

is a continuous map of pairs, and

$$\phi_2: G \dot{\to} \phi_1^* H$$

is a natural transformation. We define composites of morphisms in $\mathcal{L}$ in the following way:

Functorial Properties of Homology with Local Coefficients

Let $\phi:(X,A;G) \to (Y,B;H)$, $\psi:(Y,B;H) \to (Z,C;K)$, then

$$(\psi\phi)_1 := \psi_1\phi_1$$

is the composite of maps, and $(\psi\phi)_2$ is obtained from the following consideration. If $x,x' \in X, \alpha \in \Pi X(x',x)$, then the diagram

$$\begin{array}{ccccccc}
x & & Gx & \xrightarrow{\phi_2 x} & (\phi_1^* H)x = H(\phi_1 x) & \xrightarrow{\psi_2(\phi_1 x)} & \psi_1^* K(\phi_1 x) = (\phi_1^*\psi_1^* K)x \\
\uparrow \alpha & & \downarrow G\alpha & & \downarrow & & \downarrow \\
x' & & Gx' & \xrightarrow{\phi_2 x'} & (\phi_1^* H)x' = H(\phi_1 x') & \xrightarrow[\psi_2(\phi_1 x')]{} & \psi_1^* K(\phi_1 x') = (\phi_1^*\psi_1^* K)x'
\end{array}$$

commutes and, since $(\phi_1^*\psi_1^*)K = (\psi\phi)_1^* K$, we may define

$$(\psi\circ\phi)_2\ x := \psi_2(\phi_1 x) \circ \phi_2 x$$

We also write $(\phi_1^*\psi_2)(x)$ for $\psi_2(\phi_1 x)$. With this notation: $(\psi\phi)_2 = (\phi_1^*\psi_2)\circ\phi_2$.

(3.1) Proposition : $\mathcal{L}$ is a category.

Proof: Identity morphisms on objects of $\mathcal{L}$ obviously exist. We show that composition of morphisms in $\mathcal{L}$ is associative. Let $(X,A;G) \xrightarrow{\phi} (Y,B;H) \xrightarrow{\psi} (Z,C;K) \xrightarrow{\Gamma} (U,D;L)$ be objects and morphisms in $\mathcal{L}$. We want to show that

$$(\Gamma\psi)\phi = \Gamma(\psi\phi).$$

For the first coordinate this follows from associativity of compositions of maps. For the second coordinate we calculate

$$[\Gamma(\psi\circ\phi)]_2=[(\psi\circ\phi)_1^*\Gamma_2]\circ\phi_1^*\psi_2\circ\phi_2$$
$$=[\phi_1^*(\psi_1^*\Gamma_2)\circ\phi_1^*\psi_2]\circ\phi_2=\phi_1^*[\psi_1^*\Gamma_2\circ\psi_2]\circ\phi_2=[(\Gamma\circ\psi)\phi]_2$$

□

Now let $\phi : (X,A;G) \to (Y,B;H)$ be a morphism in $\mathcal{L}$. We want to show

(3.2) Proposition: ϕ gives rise to a chain morphism $\phi_\# : C_*(X,A;G) \to C_*(Y,B;H)$.

Proof: We first derive a chain morphism $\hat{\phi}_{\#}: C_*(X;G) \to C_*(Y;H)$ and then use $\hat{\phi}_{\#}$ to define $\phi_{\#}$ in a purely algebraic manner. Define

$$\hat{\phi}_{\#} : C_*(X;G) \to C_*(Y;H)$$

on elementary singular p-chains $g.u$ by

$$\hat{\phi}_{\#_p}(g.u) := (\phi_2 u(e_p)g)\cdot\phi_1 \circ u$$

Since

$$\phi_2 u(e_p) : Gu(e_p) \to (\phi_1^* H)u(e_p) = H(\phi_1 u(e_p))$$

is a homomorphism, $\hat{\phi}_{\#_p}$ is a homomorphism on the direct summands $Gu(e_p)$ of $C_p(X,A;G)$. By the universal property of the weak direct sum $\hat{\phi}_{\#_p}$ is a homomorphism. $\hat{\phi}_{\#} : C_*(X;G) \to C_*(Y;H)$ defined as above is actually a chain morphism. To see this, let $g.u \in C_p(X;G)$ be an elementary singular p-chain. We compute

$$
\begin{array}{ccc}
g\cdot u & \overset{\partial_p}{\longmapsto} & G(\alpha_u)g\cdot u\circ f_p^0 + \sum_{i=1}^{p}(-1)^i g\cdot u\circ f_p^i \\
\Big\downarrow \phi_{\#p} & & \Big\downarrow \phi_{\#p-1} \\
 & & \phi_2(u\circ f_p^0(e_{p-1}))G(\alpha_u)g\cdot\phi_1\circ u\circ f_p^0 + \sum_{i=1}^{p}(-1)^i\phi_2(u\circ f_p^i(e_{p-1}))g\cdot\phi_1\circ u\circ f_p^i \\
(\phi_2 u(e_p))g\cdot\phi_1\circ u & \underset{\partial_p'}{\longmapsto} & H(\alpha_{\phi_1\circ u})(\phi_2 u(e_p))g\cdot\phi_1\circ u\circ f_p^0 + \sum_{i=1}^{p}(-1)^i(\phi_2 u(e_p))g\cdot\phi_1\circ u\circ f_p^i
\end{array}
$$

and we wish to see why the two terms in the bottom right corner are equal. For the summands of index $1\le i\le p$ we need merely observe that $u(e_p) = u\circ f_p^i(e_{p-1})$. For the summands of index $i=0$, we apply the natural transformation property of $\phi_2 : G \to \phi_1^*H$ as in the following diagram.

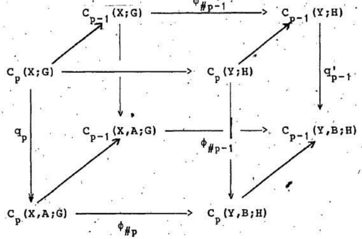
$$
\begin{array}{ccccc}
u(e_p) & & G(u(e_p)) & \xrightarrow{\ \phi_2 u(e_p)\ } & \phi_1^*H(u(e_p)) \\
\uparrow{\scriptstyle \alpha_u} & & \downarrow{\scriptstyle G(\alpha_u)} & & \downarrow{\scriptstyle \phi_1^*H(\alpha_u)=H(\phi_1\circ\alpha_u)} \\
u\circ f_p^o(e_{p-1}) & & G(u\circ f_p^o(e_{p-1})) & \xrightarrow[\ \phi_2 u\circ f_p^o(e_{p-1})\ ]{} & \phi_1^*H(u\circ f_p^o(e_{p-1}))
\end{array}
$$

Now observe that $\phi_1 \circ \alpha_u = \alpha_{\phi_1 \circ u}$ by definition of α_u.

From this it follows that $\phi_{|A\#} : C_*(A;G_{|A}) \to C_*(B;H_{|B})$ is a chain morphism. Since $\phi_1(A) \subset B$ we see that the composition $q'_p \circ \bar{\phi}_{\#p}$ maps $C_p(A;G_{|A})$ into 0. We now define $\phi_{\#p}$ to be the unique map making the following diagram commute.

$$
\begin{array}{ccc}
C_p(X;G) & \xrightarrow{\ \bar{\phi}_{\#p}\ } & C_p(Y;H) \\
\downarrow{\scriptstyle q_p} & & \downarrow{\scriptstyle q'_p} \\
C_p(X,A;G) & \xrightarrow[\ \phi_{\#p}\ ]{} & C_p(Y,B;H)
\end{array}
$$

We must show that $\phi_{\#p}$ is indeed a chain morphism. All faces, except the bottom face in the following cube diagram are known to commute by what has been done. Since q_p is onto, we can use some element $c \in C_p(X;G)$ to trace its residue class $\bar{c}$ along these commuting faces to see that the bottom face also commutes. But this means that $\phi_{\#p}$ is a chain morphism.



Let $\mathcal{M}^{\mathbb{Z}}$ be the category of chain complexes of abelian groups. Then, for each $(X,A;G) \in \mathcal{L}$, $(C_*(X,A;G), \partial)$ can be regarded as an element of $\mathcal{M}^{\mathbb{Z}}$ by defining $C_q(X,A;G) := 0$ for $q < 0$.

Functorial Properties of Homology with Local Coefficients

(3.3) Let $C_* : \mathcal{L} \longrightarrow M^{\mathbb{Z}}$ be defined by

$$(X,A;G) \longmapsto (C_*(X,A;G),\partial) \quad \text{on objects}$$

$$[\phi:(X,A;G) \to (Y,B;H)] \longmapsto C_*\phi := \phi_\# \quad \text{as in} \quad (3.2)$$

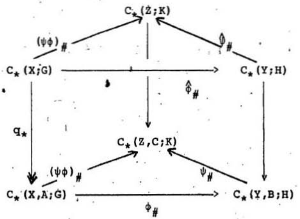
then C_* is a functor.

Proof: $id_{(X,A;G)} \longrightarrow id_{C_*(X,A;G)}$ is obvious.

Now consider

$$(X,A;G) \xrightarrow{\phi} (Y,B;H) \xrightarrow{\psi} (Z,C;K)$$

in $\mathcal{L}$. We want to show that $(\psi\phi)_\# = \psi_\#\phi_\#$. Using the same epimorphism argument as in (3.2), the following diagram with commuting side faces shows, that it suffices to establish commutativity of the top triangle, which we proceed to do now.



Since $(\psi\phi)_{\#}$, $\hat{\phi}_{\#}$, $\hat{\psi}_{\#}$ are homomorphisms, it suffices to verify this for elementary singular p-chains. So let $g.u \in C_p(X;G)$. We obtain:

$$\begin{aligned}\hat{\psi}_{\#p}\circ\hat{\phi}_{\#p}(g.u) &= \hat{\psi}_{\#p}(\phi_2(u(e_p))g.\phi_1 o u)\\ &\hat{=} [\psi_2(\phi_1 o u(e_p))\phi_2(u(e_p))]g.\psi_1(\phi_1 u)\\ &= [(\phi_1^*\psi_2)o\phi_2]u(e_p)g.(\psi_1\phi_1)u\\ &= (\psi\phi)_2 u(e_p)g.(\psi\phi)_1 u\\ &= \widehat{(\psi\phi)}_{\#p}(g.u)\end{aligned}$$

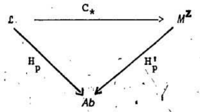
□

(3.4) <u>Corollary:</u> For all $p \in \mathbb{N}_0$, $H_p : \mathcal{L} \longrightarrow Ab$ is a covariant functor.

Proof: We adopt the following fact from homological algebra (see e. g. [R] p. 169): The assignation $H_p' : M^{\mathbb{Z}} \longrightarrow Ab$, defined on objects by

$$(C,\partial) \longmapsto \text{p-th homology group of } (C,\partial)$$

is a covariant functor. Recalling how we constructed H_p, we see that H_p is the composite of two covariant functors:



We proceed by defining the notion of homotopy in $\mathcal{L}$. If $\phi, \psi : (X,A;G) \longrightarrow (Y,B;H)$ are morphisms in $\mathcal{L}$, a homotopy λ of ϕ to ψ (if it exists), should firstly be a homotopy of pairs of ϕ_1 to ψ_1 and secondly, it should respect the natural transformations $\phi_2 : G \dot{\to} \phi_1^* H$ and $\psi_2 : G \dot{\to} \psi_1^* H$ in a way which allows us to establish the equality of the homomorphisms

$$\phi_* = \psi_* : H_*(X,A;G) \longrightarrow H_*(Y,B;H)$$

As an auxiliary concept we define

(3.5) <u>Definition:</u> The prism over $(X,A;G) \in \mathcal{L}$ is

$$(X,A;G)\times I := (X\times I, A\times I; p^*G) \in |\mathcal{L}|$$

where $p : X\times I \longrightarrow X$ is the projection.

We also need the following technical observation. Every map $i_t : X \ni x \mapsto (x,t) \in X\times I$ gives rise to a morphism $(i_t, j_t) : (X,A;G) \dot{\to} (X,A;G)\times I$ in $\mathcal{L}$ by letting $j_t : G \dot{\to} i_t^*(p^*G)$ be the identity transformation. This makes sense because $p \circ i_t$ is the identity on X.

Again, let $\phi, \psi : (X,A;G) \longrightarrow (Y,B;H)$ be morphisms in $\mathcal{L}$.

(3.6) Definition: $\lambda:(X,A;G)\times I \longrightarrow (Y,B;H)$ is a homotopy of ϕ to ψ iff

$$\lambda o(i_0,j_0) = \phi \quad \text{and} \quad \lambda o(i_1,j_1) = \psi$$

This means that λ_1 is a homotopy of pairs of ϕ_1 to ψ_1, and $\lambda_2 : p^*G \to \lambda_1^*H$ is a natural transformation such that $i_0^*\lambda_2 = \phi_2$ and $i_1^*\lambda_2 = \psi_2$. As usual, we say that ϕ is homotopic to ψ $(\phi \simeq \psi)$ iff there exists a homotopy λ of ϕ to ψ.

Of course, we should like to know how to construct homotopies in $\mathcal{L}$ out of the familiar construction of homotopies between continuous maps. We shall not tackle this question in its entire generality. However, we prove the following.

(3.7) Lemma: Let $(X,A;G)$ be an object in $\mathcal{L}$, $\phi:(X,A;G) \longrightarrow (X,A;G)$ a morphism, $\psi_1:(X,A) \longrightarrow (X,A)$ a continuous map and

$$\lambda_1 : (X\times I, A\times I) \longrightarrow (X,A)$$

a homotopy of ϕ_1 to ψ_1. Then there exist natural transformations

$$\psi_2 : G \dot{\to} \psi_1^* G \quad \text{and} \quad \lambda_2 : p^* G \dot{\to} \lambda_1^* G$$

such that $\psi := (\psi_1, \psi_2) : (X,A;G) \longrightarrow (X,A;G)$ is a morphism in $\mathcal{L}$ and

$$\lambda := (\lambda_1, \lambda_2) : (X,A;G) \times I \longrightarrow (X,A;G)$$

is a homotopy of ϕ to ψ.

Proof: For $(x,t) \in X \times I$, let $\mu(x,t)$ be the path class in $X \times I$ represented by $\mu(x,t) : I \ni s \longmapsto (x,(1-s)t) \in X \times I$. For $\alpha \in \Pi(X \times I)[(x',t'),(x,t)]$, consider

$$\begin{array}{ccccc}
Gx & \xrightarrow{\phi_2 x} & G\phi_1(x) & \xrightarrow{G\lambda_1(\mu(x,t))} & G\lambda_1(x,t) \\
\downarrow{\scriptstyle G(p\alpha)} & & \downarrow{\scriptstyle G\phi_1 o p(\alpha)} & & \downarrow{\scriptstyle G\lambda_1(\alpha)} \\
Gx' & \xrightarrow[\phi_2 x']{} & G\phi_1(x') & \xrightarrow[G\lambda_1(\mu(x',t'))]{} & G\lambda_1(x',t')
\end{array}$$

The left rectangle commutes, because $\phi_2 : G \dot{\to} \phi_1^* G$ is a natural transformation. The right rectangle commutes, because $\phi_1 o p(\alpha) = \lambda_1(p(\alpha),0)$ and $p(\alpha) = \overline{\mu(x',t')}\alpha\mu(x,t)$. Furthermore, all maps in the above diagram are group isomorphisms. Hence, we may define:

$$\lambda_2(x,t) := [G\lambda_1(\mu(x,t))]o\phi_2 x$$
$$\psi_2 x := \lambda_2(x,1)$$

Clearly ψ_2, λ_2 satisfy the required properties. □

Homotopy in $\mathcal{L}$ has the following two properties:

(3.8) <u>Proposition:</u> Homotopy between morphisms in $\mathcal{L}$ is an equivalence relation.

(3.9) <u>Proposition:</u> If $\phi \approx \psi : (X,A;G) \longrightarrow (Y,B;H)$ and $\phi' \approx \psi' : (Y,B;H) \longrightarrow (Z,C;K)$, then $\phi' o \phi \approx \psi' o \psi : (X,A;G) \longrightarrow$ $\longrightarrow (Z,C;K)$.

Proof of (3.8): Reflexivity is trivial. To check symmetry let $\phi \approx \psi : (X,A;G) \longrightarrow (Y,B;H)$ and λ a homotopy of ϕ to ψ. Define a homotopy μ of ψ to ϕ by

$$\mu_1 : X \times I \ni (x,t) \longmapsto \lambda_1(x,1-t) \in Y$$
$$\mu_2 : X \times I \ni (x,t) \mapsto \lambda_2(x,1-t) \in Ab(Gx, H\lambda_1(x,1-t))$$

In order to show that μ_2 is the desired natural

transformation, we observe the following:
If $\varphi = (\varphi_1, \varphi_2) : I \to X \times I$ is a path connecting (x,t) to (x',t') and $\varphi' = (\varphi_1, 1-\varphi_2) : I \to X \times I$, then

$$\begin{array}{ccc} p^*G(x,t) & = & p^*G(x,1-t) \\ {\scriptstyle p^*G[\varphi]}\uparrow & & \uparrow{\scriptstyle p^*G[\varphi']} \\ p^*G(x',t') & = & p^*G(x',1-t') \end{array}$$

commutes, i.e. the two isomorphisms are the same. From this, it is immediate that

$$\begin{array}{cccc} (x,t) & Gx & \xrightarrow{\mu_2(x,t)} & \lambda_1^*H(x,1-t) = \mu_1^*H(x,t) \\ \alpha\uparrow & {\scriptstyle p^*G\alpha'}\downarrow{\scriptstyle = p^*G\alpha} & & \downarrow{\scriptstyle \lambda_1^*H\alpha' = \mu_1^*H\alpha} \\ (x',t') & Gx' & \xrightarrow[\mu_2(x',t')]{} & \lambda_1^*H(x',1-t') = \mu_1^*H(x',t') \end{array}$$

commutes. Furthermore

$$i_o^*\mu_2 = i_1^*\lambda_2 = \psi_2 \quad \text{and} \quad i_1^*\mu_2 = i_o^*\lambda_2 = \phi_2$$

Transitivity: Let $\phi,\psi,\Gamma : (X,A;G) \to (Y,B;H)$, and let λ be a homotopy of ϕ to ψ and μ a homotopy of ψ to Γ. We define a homotopy ν of ϕ to Γ by setting ν_1 to be

$$\nu_1 : X\times I \ni (x,t) \mapsto \begin{cases} \lambda_1(x,2t) & 0\le t\le\frac{1}{2} \\ \mu_1(x,2t-1) & \frac{1}{2}\le t\le 1 \end{cases} \in Y$$

and a natural transformation $\nu_2 : p^*G \dot{\to} \nu_1^*H$ by

$$\nu_2(x,t) := \begin{cases} \lambda_2(x,2t) & 0\le t\le\frac{1}{2} \\ \mu_2(x,2t-1) & \frac{1}{2}\le t\le 1 \end{cases}$$

We certainly have

$$i_o^*\nu_2 = i_o^*\lambda_2 = \phi_2 \quad \text{and} \quad i_1^*\nu_2 = i_1^*\mu_2 = \Gamma_2$$

In order to check that ν_2 is the desired natural transformation, we observe the following:
If $\varphi = (\varphi_1,\varphi_2) : I \to X\times I$ is a path connecting (x,t) to (x',t'), then φ is homotopic to a path $\varphi' = (\varphi_1',\varphi_2')$

whose second coordinate function is linear. A suitable homotopy is given by

$$f : I\times I \ni (s,\sigma) \mapsto (\varphi_1(s), \varphi_2(s)+\sigma(t+s(t'-t)-\varphi_2(s))) \in X\times I$$

From this, using the naturality of λ_2 (respectively μ_2), the naturality of ν_2 in the cases $0\leq t,t'\leq\frac{1}{2}$ (respectively $\frac{1}{2}\leq t,t'\leq 1$) is immediate. In all remaining cases, the above homotopy shows that $\alpha \in \Pi(X\times I)((x,t),(x',t'))$ is the composite $\alpha_1\alpha_2 = \alpha$ of two classes in $X\times I$ such that

α_1 joins (x,t) to $(x",\frac{1}{2})$

α_2 joins $(x",\frac{1}{2})$ to (x',t')

and α_1 is represented by a path β_1 in $X\times[o,\frac{1}{2}]$ (or $X\times[\frac{1}{2},1]$)

α_2 is represented by a path β_2 in $X\times[\frac{1}{2},1]$ (or $X\times[o,\frac{1}{2}]$)

We then obtain (suppose $o\leq t\leq\frac{1}{2}\leq t'\leq 1$)

$$
\begin{array}{ccccc}
(x',t') & Gx' & \xrightarrow[\nu_2(x',t')]{\mu_2(x',2t'-1)} & \mu_1^*H(x',2t'-1) & = \nu_1^*H(x',t') \\
\uparrow \alpha_2 & p^*G\alpha_2 \Big\downarrow p^*G\hat{\alpha}_2 & & \Big\downarrow \mu_1^*H\hat{\alpha}_2 = \nu_1^*H\alpha_2 & \Big\downarrow \\
(x'',\frac{1}{2}) & Gx'' & \xrightarrow[\lambda_2(x'',1)]{\mu_2(x'',0)} & \begin{cases}\mu_1^*H(x'',0)\\ \lambda_1^*H(x'',1)\end{cases} & = \nu_1^*H(x',\frac{1}{2}) \\
\uparrow \alpha_1 & p^*G\alpha_1 \Big\downarrow p^*G\hat{\alpha}_1 & & \Big\downarrow \mu_1^*H\hat{\alpha}_1 = \nu_1^*H\alpha_1 & \Big\downarrow \\
(x,t) & Gx & \xrightarrow[\nu_2(x,t)]{\lambda_2(x,2t)} & \lambda_1^*H(x,2t) & = \nu_1^*H(x,t)
\end{array}
$$

Where $\hat{\alpha}_i$ is represented by $\hat{\beta}_1(s) := (x_{\beta_1}(s), 2t_{\beta_1}(s))$, with x_{β_1}, t_{β_1} the coordinate functions of the previously mentioned function β_1. Similarly, $\hat{\alpha}_2$ is represented by $\beta_2(s) := (x_{\beta_2}(s), 2t_{\beta_2}(s)-1)$, $\beta_2 = (x_{\beta_2}, t_{\beta_2})$.

Commutativity of the whole diagram follows now from commutativity of the upper and lower rectangles and the indicated equalities.

□

Proof of (3.9): Let λ be a homotopy of ϕ to ψ and λ' a homotopy of ϕ' to ψ'. As in the case of homotopic maps of pairs, we show that

(i) $\phi'\phi \approx \phi'\psi$

(ii) $\phi'\psi \approx \psi'\psi$

and, using the transitivity of "$\approx$", we infer that $\phi'\phi \approx \psi'\psi$.

Proof of (i): $\phi' \circ \lambda : (X,A;G) x I \to (Z,C;K)$ is a homotopy of $\phi'\phi$ to $\phi'\psi$.

Proof of (ii): We observe that ψ gives rise to a prism morphism $(\psi x 1_I) : (X,A;G) x I \to (Y,B;H) x I$ in the following way:

$$(\psi x 1_I)_1 : (XxI, AxI) \ni (x,t) \mapsto (\psi_1(x),t) \in (YxI, BxI)$$

$$(\psi x 1_I)_2 \, (x,t) := \psi_2(x)$$

This is indeed a morphism in $\mathcal{L}$ as the following diagram shows

$$\begin{array}{ccccc}
(x,t) & p^*G(x,t) = Gx & \xrightarrow{\psi_2 x} & \psi_1^*Hx = H(p(\psi_1 x,t)) & \\
\alpha \uparrow & G(p\alpha) \downarrow \quad \downarrow & & \downarrow \psi_1^*H(p\alpha) = H(p\psi_1\alpha,t)) & \downarrow \\
(x',t') & p^*G(x',t') = Gx' & \xrightarrow[\psi_2 x']{} & \psi_1^*Hx' = H(p(\psi_1 x',t')) &
\end{array}$$

Now $\lambda' o(\psi \times 1_I)$ is a homotopy of $\phi'\psi$ to $\psi'\psi$. □

In the following theorem, we record some fundamental properties of homology with local coefficients. These are in analogy with the Eilenberg-Steenrod axioms for ordinary homology. More precisely:

Each abelian group G gives rise to an imbedding of Top^2, the category of pairs of topological spaces, into a full subcategory of $\mathcal{L}$. We define an imbedding functor E_G on objects by $(X,A) \longrightarrow (X,A;G_X)$ the system of local coefficients G_X in X being defined path componentwise to be the constant system determined by G. The resulting system is unique up to natural equivalence by (1.9). We define E_G on morphisms

$$\left[f:(X,A) \mapsto (Y,B)\right] \longrightarrow \left[(f,\varphi):(X,A;G_X) \to (Y;B;G_Y)\right]$$

where φ is the unique natural equivalence between the simple systems G_X, f^*G_Y in X,Y respectively.

The following properties of homology in $\mathcal{L}$ reduce, when restricted to any of the subcategories $E_G(Top^2)$, just to the Eilenberg-Steenrod axioms for ordinary homology (use 2.3 to see this). For a statement and development of fundamental consequences of these axioms, we refer the reader to [Hu].

First some notation. An inclusion $i_1: (X_1, A_1) \hookrightarrow (X, A)$ in Top^2 and a system of local coefficients G in X give rise to an inclusion $i: (X_1, A_1; i_1^* G) \hookrightarrow (X, A; G)$ in $\mathcal{L}$ by setting $i_2 x := 1_{Gx}$.

(3.10) Theorem: (i) $H_p: \mathcal{L} \to Ab$ is a covariant functor for every $p \in \mathbb{N}_0$.

(ii) Exactness: For $(X,A;G) \in |\mathcal{L}|$; $i: (A; G_{|A}) \hookrightarrow (X;G)$, $j: (X;G) \hookrightarrow (X,A;G)$ inclusions, the sequence

$$\dots \to H_{p+1}(X,A;G) \xrightarrow{\partial_{p+1}} H_p(A;G_{|A}) \xrightarrow{H_p i} H_p(X;G) \xrightarrow{H_p j} H_p(X,A;G) \to \dots$$

is exact, where $\partial_{p+1} = \partial_{p+1}(X,A;G)$ is as in (2.4).

(iii) Commutativity: For $p \in \mathbb{N}$, $\partial_p: H_{p+1} \xrightarrow{\cdot} H_p \circ R$ is a natural transformation, where $R: \mathcal{L} \to \mathcal{L}$ is the functor defined on objects by $(Y,B;H) \to (B,\emptyset;H_{|B})$ and on morphisms by

$$\left[\phi: (Y,B;H) \to (Y',B';H')\right] \longmapsto \left[R\phi: (B,\emptyset;H_{|B}) \to (B',\emptyset;H'_{|B'})\right]$$

with $(R\phi)_1 := \phi_{1|B}$ and $(R\phi)_2 := \phi_{2|B}$.

(iv) Homotopy: For homotopic morphisms

$$\phi, \phi' : (X,A;G) \to (Y,B;H) \quad \text{in } \mathcal{L} :$$

$$H_p\phi = H_p\phi' : H_p(X,A;G) \to H_p(Y,B;H)$$

for all p .

(v) Excision: Let $(X,A;G)$ be an object in $\mathcal{L}$, and let $B \subset A$ such that $\bar{B} \subset \mathring{A}$. Then the inclusion $i : (X-B, A-B; G_{|X-B}) \to (X,A;G)$ induces isomorphisms on relative homology groups with local coefficients

$$i_* : H_*(X-B, A-B; G_{|X-B}) \xrightarrow{\cong} H_*(X,A;G)$$

(vi) Dimension: Let $X = \{x\}$ be the singleton space, $(X;G) \in |\mathcal{L}|$. Then

$$H_p(X;G) = 0 \qquad p \geq 1$$
$$H_o(X;G) \cong Gx$$

(vii) Additivity: Let X be the union of a family of mutually disjoint sets X_λ such that for all $x_1 \in X_{\lambda_1}$, $x_2 \in X_{\lambda_2}$, $\Pi X(x_1,x_2) \neq \emptyset$ implies

$\lambda_1 = \lambda_2$. Let $A \subset X$, $A_\lambda := A \cap X_\lambda$; G a system of local coefficients in X , $G_\lambda := G_{|X_\lambda}$. Then the injections $H_*(X_\lambda, A_\lambda; G_\lambda) \to H_*(X,A;G)$ give rise to an isomorphism

$$\bigoplus_\lambda H_*(X_\lambda, A_\lambda; G_\lambda) \longrightarrow H_*(X,A;G)$$

Proof: (i) has been shown in (3.4) □

(ii) has been derived in (2.4) □

(iii) Let $\phi : (X,A;G) \to (Y,B;H)$ be a morphism in $\mathcal{L}$. We want to show that the following diagram commutes.

$$\begin{array}{ccc} H_{p+1}(X,A;G) & \xrightarrow{\partial_{p+1}(X,A;G)} & H_p(A;G_{|A}) \\ \downarrow{\scriptstyle \phi_{*p+1}} & & \downarrow{\scriptstyle (R\phi)_{*p}} \\ H_{p+1}(Y,B;H) & \xrightarrow{\partial_{p+1}(Y,B;H)} & H_p(B,H_{|B}) \end{array}$$

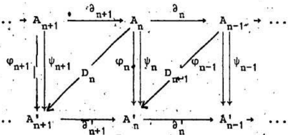
We know already that each of the maps in this diagram is a homomorphism and furthermore that each map is obtained by applying the corresponding chain operator to an arbitrary representative of the homology class in consideration. For example

$$\partial_{p+1}(X,A;G)((z+C_{p+1}(A;G_{|A}))+B_{p+1}C_*(X,A;G))=\partial_{p+1}z+B_pC_*(A;G_{|A})$$

where ∂_{p+1} on the right hand side is the differential operator of the chain complex $C_*(X;G)$. This observation reduces the commutativity calculation in the above diagram to the calculation on page 37. □

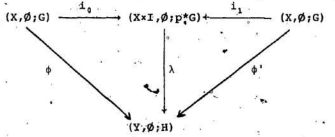
(iv) We imitate the verification of the homotopy property in the case of ordinary singular homology. (See e.g. [V] pp 15–18). We shall treat the non relative case $\phi,\phi' : (X,\emptyset;G) \to (Y,\emptyset;H)$ first and then derive the relative result from that one.

We remind ourselves of a concept in homological algebra. Let $(A,\partial),(A',\partial')$ be chain complexes of Λ-modules and $\varphi,\psi : (A,\partial) \to (A',\partial')$ chain maps. A chain homotopy of φ to ψ is a collection of module homomorphisms $\{D_n : A_n \to A'_{n+1}\}_{n\in\mathbb{Z}}$ such that



for all $n \in \mathbb{Z}$: $\varphi_n - \psi_n = \partial'_{n+1} D_n + D_{n-1} \partial_n$. It can be shown (see e.g. [H-S] pp 124-126) that $\varphi_{*n} = \psi_{*n} : H_n A \to H_n A'$, if φ, ψ are chain homotopic.

We now construct a chain homotopy D of $\phi_\#$ to $\phi'_\#$, where $\phi_\#, \phi'_\# : C_*(X;G) \to C_*(Y;H)$. Let λ be a homotopy of ϕ to ϕ'. As in (3.5), let $i_t : X \ni x \mapsto (x,t) \in X\times I$, for $t \in [0,1]$. Then $(i_t, 1_G) : (X,\emptyset;G) \to (X\times I,\emptyset;p^*G)$ is a morphism in $\mathcal{L}$ which we denote again by i_t. Then, by definition of λ as a homotopy, the following diagram commutes



By (3.3), the corresponding diagram of chain complexes commutes. We show that $i_{0\#}, i_{1\#} : C_*(X,\emptyset;G) \to C_*(X\times I,\emptyset;p^*G)$ are chain homotopic, and it is then straightforward to see that $\lambda_\# i_{0\#} = \phi_\#$ and $\lambda_\# i_{1\#} = \phi'_\#$ are chain homotopic.

Let $u : \Delta_n \to X$ be a singular n-simplex and consider the following diagram.

$$\begin{array}{ccccc}
C_n(\Delta_n;u^*G) & \xrightarrow{u_\#} & C_n(X;G) & & \\
\downarrow j_{0\#} & & \downarrow i_{0\#} & \searrow \phi_\# & \\
C_n(\Delta_n\times I;p_1^*u^*G) & \xrightarrow{(u\times 1)_\#} & C_n(X\times I;p^*G) & \xrightarrow{\lambda_\#} & C_n(Y;H) \\
\uparrow j_{1\#} & & \uparrow i_{1\#} & \nearrow \phi'_\# & \\
C_n(\Delta_n;u^*G) & \xrightarrow{u_\#} & C_n(X;G) & &
\end{array}$$

First of all observe that $p_1^*u^*G = (u\times 1)^*p^*G$, where $p_1:\Delta_n\times I \to \Delta_n$, $p:X\times I \to X$ are projections, so that $(u\times 1, 1_{p_1^*u^*G})$ is indeed a morphism in $\mathcal{L}$. Since Δ_n, $\Delta_n\times I$ are contractible, hence 1-connected, the corresponding systems of local coefficients u^*G, $p_1^*u^*G$ are simple (1.6), and the proof of (2.3) shows that the ordinary singular chain complexes of Δ_n, $\Delta_n\times I$ with trivial coefficients in u^*G_*, $p_1^*u^*G_*$ $(* \in \Delta_n)$ are chain isomorphic to the corresponding singular chain complexes with local coefficients in u^*G; $p_1^*u^*G$. Since j_0, j_1 are homotopic (in Top), there exists a chain homotopy D'' of j_{0+} to j_{1+} (the chain morphism for trivial coefficients induced by j_0, j_1) which, making use of our chain isomorphism, carries over to a chain homotopy D' of $j_{0\#}$ to $j_{1\#}$. To get a chain homotopy D' of $i_{0\#}$ to $i_{1\#}$, we define

$$D_n(g.u) := (u\times 1)_{\#n+1} D'_n(g.1_{\Delta_n})$$

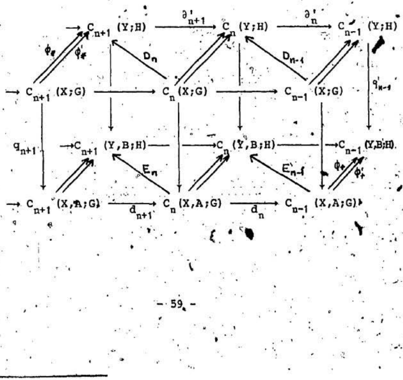
where 1_{Δ_n} is the identity map on Δ_n. A quick check shows that the coefficient groups in $u(e_p)$ and e_p are the same, so the above equation is well defined. By the universal property of "weak sum", this extends to a homomorphism $D_n: C_n(X;G) \to C_{n+1}(X\times I; p^*G)$. D_n is a chain homotopy of $i_{0\#}$ to $i_{1\#}$. It suffices to check this on generators $g.u$ (dimension indices are n if not otherwise indicated).

$$\begin{aligned} i_{0\#}g.u - i_{1\#}g.u &= i_{0\#}u_{\#}(g.1_{\Delta_n}) - i_{1\#}u_{\#}(g.1_{\Delta_n}) \\ &= (u\times 1)_{\#}j_{0\#}(g.1_{\Delta_n}) - (u\times 1)_{\#}j_{1\#}(g.1_{\Delta_n}) \\ &= (u\times 1)_{\#}\left[(j_{0\#} - j_{1\#})g.1_{\Delta_n}\right] \\ &= (u\times 1)_{\#}\left[(d'_{n+1}D'_n + D'_{n-1}d_n)g.1_{\Delta_n}\right] \\ &= \partial'_{n+1}(u\times 1)_{\#n+1}D'_n(g.1_{\Delta_n}) + \\ &\quad + (u\times 1)_{\#}D'_{n-1}d_n(g.1_{\Delta_n}) \\ &= \partial'_{n+1}D_n(g.u) + D_n\partial_n(g.u) \end{aligned}$$

Here d_n, d'_{n+1}, ∂_n, ∂'_{n+1} are the differential operators on $C_n(\Delta_n, u^*G)$, $C_{n+1}(\Delta_n\times I, p_1^*u^*G)$, $C_n(X;G)$, $C_{n+1}(X\times I; p^*G)$ respectively. By our previous remark we infer that in the nonrelative situation $\phi_\#$ and $\phi'_\#$ are chain homotopic.

It is now a purely algebraic matter to establish a chain homotopy in the relative situation. But let us first recollect where we stand and, in doing this, simplify notation a little bit.

Let ∂, ∂', d, d' be the differential operators of the chain complexes $C_*(X;G)$, $C_*(Y;H)$, $C_*(X,A;G)$, $C_*(Y,B;H)$. We have a chain homotopy D of $\phi_\#$ to $\phi'_\#$, $\phi_\#, \phi'_\#: C_*(X;G) \to C_*(Y;H)$. By (3.2), we also have chain morphisms $\phi_+, \phi'_+: C_*(X,A;G) \to C_*(Y,B;H)$ induced by ϕ, ϕ' and want a chain homotopy E of ϕ_+ to ϕ'_+. Since $C_*(X,A;G)$, $C_*(Y,B;H)$ are obtained by taking the quotient of $C_*(X;G)$, $C_*(Y;H)$ over the subcomplexes $C_*(A;G_{|A})$, $C_*(B;H_{|B})$, (see 2.4) we obtain the following diagram:



Here $q_* : C_*(X;G) \to C_*(X,A;G)$, $q'_* : C_*(Y,H) \to C_*(Y,B;H)$ are quotient maps. If we chase a generator $g.u$ of $C_*(A;G_{|A})$ through the various chain groups, we find that $\phi_\#$, $\phi'_\#$ and D map it into $C_*(B;H_{|B})$. Thus $C_*(A;G_{|A})$ is contained in the kernels of $q'\phi_\#$, $q'\phi'_\#$, $q'D$. In (3.2) we used this observation to derive the existence of unique maps ϕ_+ , ϕ'_+ making the side rectangles of the above diagram commute; it also guarantees us the existence of a chain homotopy.

Since q_n is onto, for $z \in C_n(X,A;G)$ there exists some $w \in C_n(X;G)$ such that $q_n w = z$.

Thus

$$\begin{aligned}(\phi_{+n} - \phi'_{+n})z &= (\phi_{+n} - \phi'_{+n})q_n w \\ &= q'_n(\phi_{\# n} - \phi'_{\# n})w \\ &= q'_n(\partial'_{n+1}D_n + D_{n+1}\partial_n)w \\ &= d'_{n+1}q'_{n+1}D_n w + E_{n-1}q_{n-1}\partial_n w \\ &= d'_{n+1}E_n q_n w + E_{n-1}d_n q_n w \\ &= (d'_{n+1}E_n + E_{n-1}d_n)z\end{aligned}$$

This concludes the proof of (iv) . □

Proof of (v) (excision): Again parts of the proof of excision in ordinary singular homology are of help in the present situation. In this we follow Vick [V] pp54,55 and appendix I. Denote by $\mathfrak{U},\mathfrak{U}'$ the coverings of X,A given by $\{X-B,\mathring{A}\},\{A-B,\mathring{A}\}$. Note that, by hypothesis, we have $A,X \subset \overline{X-B}^{\circ} \cup \mathring{A} = (X-\overline{B}) \cup \mathring{A}$. Denote by

$$C_*^{\mathfrak{U}}(X;G) :=$$
$$=\{\Sigma g_i . u_i \in C_*(X;G) : u_i \text{ elementary and } u_i(\Delta_*) \subset X-B \text{ or } u_i(\Delta_*) \subset \mathring{A}\}$$

the subcomplex of $C_*(X;G)$ all of whose generating singular simplices map Δ_* entirely in at least one of the sets of $\mathfrak{U}$. Similary we define $C_*^{\mathfrak{U}'}(A;G_{|A}) \subset C_*(A;G_{|A})$. Let

$$j:C_*^{\mathfrak{U}}(X;G) \to C_*(X;G) \text{ and } j':C_*^{\mathfrak{U}'}(A;G_{|A}) \to C_*(A;G_{|A})$$

be the inclusions. Both j,j' induce isomorphisms in homology, which we are going to show later. For now, we assume that j,j' induce isomorphisms to verify the excision property. Now, j,j' are obviously chain maps, and we get a commutative diagram of long exact homology sequences:

$$\begin{array}{ccccccccccc}
\cdots \to & H_n^{\mathfrak{U}'}(A;G_{|A}) & \to & H_n^{\mathfrak{U}}(X;G) & \to & H_n^{\mathfrak{U},\mathfrak{U}'}(X,A;G) & \to & H_{n-1}^{\mathfrak{U}'}(A;G_{|A}) & \to & H_{n-1}^{\mathfrak{U}}(X;G) & \to \cdots \\
 & \downarrow j'_* & & \downarrow j_* & & \downarrow h & & \downarrow j'_* & & \downarrow j_* & \\
\cdots & H_n(A;G_{|A}) & \to & H_n(X;G) & \to & H_n(X,A;G) & \to & H_{n-1}(A;G_{|A}) & \to & H_{n-1}(X;G) & \to \cdots
\end{array}$$

Since $j_*^{!}$, j_* are isomorphisms, the five-lemma tells us that h^{*} is an isomorphism.

We can write $C_*^{\mathfrak{U}}(X;G)$ as the sum of two subgroups

$$C_*^{\mathfrak{U}}(X;G) = C_*(X-B;\ G_{|X-B}) + C_*(\mathring{A};G_{|\mathring{A}})$$

but not necessarily as a direct sum. Similarly

$$C_*^{\mathfrak{U}'}(A;G_{|A}) = C_*(A-B;\ G_{|A-B}) + C_*(\mathring{A};G_{|\mathring{A}})$$

Accordingly we may regard $C_*(X-B;G_{|X-B})$ as a subcomplex of $C_*^{\mathfrak{U}}(X;G)$ and $C_*(A-B;G_{|A-B})$ as a subcomplex of $C_*^{\mathfrak{U}'}(A;G_{|A})$. This yields an isomorphism between the relative homology groups

$$H_*^{\mathfrak{U},\mathfrak{U}'}(X,A;G) \cong H_*(X-B,A-B;G_{|X-B})$$

in the following way (we delete coefficients for simplicity).

$$Z_*^{\mathfrak{U},\mathfrak{U}'}(X,A) \cong \frac{\partial^{-1}C_{*-1}(A-B) + C_*\mathring{A}}{C_*(A-B) + C_*\mathring{A}} \cong \frac{\partial'^{-1}C_{*-1}(A-B)}{C_*(A-B)}$$

$$= Z_*(X-B, A-B)$$

$$B_*^{\mathfrak{U},\mathfrak{U}'}(X,A) \cong \frac{B_*(X-B) + B_*\mathring{A}}{C_*(A-B) + C_*\mathring{A}} \cong \frac{B_*(X-B)}{C_*(A-B)} = B_*(X-B, A-B)$$

Here ∂ is the differential operator on $C_*^{\mathfrak{U}}X$ and ∂' its restriction to $C_*(X-B)$. The indicated isomorphisms follow upon little meditation of elementary group and set theoretic nature which is left to the reader. Now combine this isomorphism with h to get the desired isomorphism

$$H_n(X-B, A-B; G_{|X-B}) \xrightarrow{\cong} H_n(X,A;G)$$

We now verify that $j_*: H_*^{\mathfrak{U}}(X;G) \to H_*(X;G)$ and $j_*': H_*^{\mathfrak{U}'}(A;G_{|A}) \to H_*(A;G_{|A})$ are isomorphisms. We actually prove a slightly more general result:

(3.11) Lemma: Let X be a space, G a system of local coefficients in X. Let $\mathfrak{U}$ be a family of subsets of X such that

$$\bigcup_{U \in \mathfrak{U}} \mathring{U} = X$$

then the chain map $j: C_*^{\mathfrak{U}}(X;G) \to C_*(X;G)$ induces isomorphisms

$$j_*: H_*^{\mathfrak{U}}(X;G) \longrightarrow H_*(X;G)$$

Again, $C_*^{\mathfrak{U}}(X;G)$ is the subcomplex of $C_*(X;G)$ whose generating singular simplices map Δ_* entirely into some element of $\mathfrak{U}$, and $H_n^{\mathfrak{U}}(X;G)$ is the n-th homology group of this subcomplex.

Proof: Compare the analogous proof for ordinary singular homology in Vick [V] appendix I.

We construct a chain map $Sd:C_*(X;G) \to C_*(X;G)$ of degree 0, which is chain homotopic to the identity, by sending each elementary p-chain into a p-chain whose singular simplices have smaller range. Applying Sd to each elementary p-chain separately finitely often, we eventually obtain a chain map $\gamma:C_*(X;G) \to C_*^{\mathfrak{U}}(X;G)$ which we show to be a homotopy inverse of j. Furthermore, γj will be the identity. Thus j is a chain homotopy equivalence, and our claim follows.

Preliminaries for the proof of (3.11): For a bounded subset $A \subset \mathbb{R}^n$, we define

$$\text{mesh } A := \sup\{|x-y| : x,y \in A\}$$

As a consequence of Lebesgue's number lemma we obtain (see e.g. [M] pp 179, 180)

(A) Given an open covering of the compact metric space Δ_n, there exists a Lebesgue number $\lambda > 0$ such that each subset $A \subset \Delta_n$ satisfies: If mesh $A \leq \lambda$, then A is entirely contained in one of the covering sets of Δ_n.

(B) "Barycentric subdivision" divides an affine p-simplex in $\mathbb{R}^p$ into $p!$ simplices $\Delta^1, \dots, \Delta^{p!}$ of dimension p, and we have the estimate

$$\text{mesh } \Delta^i \leq \frac{p}{p+1} \text{ mesh } \Delta$$

for all $i = 1, \dots, p!$.

(C) Subdividing barycentrically the domains of singular p-simplices in Δ_n, we obtain a chain map of degree zero (G_0 a given coefficient group)

$$Sd'_{G_0} : C_*(\Delta_n; G_0) \longrightarrow C_*(\Delta_n; G_0)$$

which is chain homotopic to the identity. Denote this chain homotopy by D'.

For (B), (C) see Vick [V], appendix I.

We use Sd' to construct $Sd: C_*(X;G) \to C_*(X;G)$. Using the same technique as in (3.10)(iv), let $u: \Delta_p \to X$ be a singular p-simplex, and let

$$u_{\#}: C_p(\Delta_p; u^*G) \longrightarrow C_p(X;G)$$

be the induced homomorphism.

Now define

$$Sd(g.u) = Sd(u_{\#}g.1_{\Delta_p}) := u_{\#}Sd'_{Gu(e_p)}(g.1_{\Delta_p})$$

Again we surmount the difficulty of assigning the 'right' coefficients to the newly appearing preferred vertices by going back to the simple pull back system of local coefficients in Δ_p. Similarly set

$$D(g.u) = Du_{\#}(g.1_{\Delta_p}) := u_{\#}D'(g.1_{\Delta_p})$$

To be precise, when defining D, we really should compose D' with the chain equivalence induced by the isomorphism in $\mathfrak{L}$, $(1_{\Delta_p},\tau):(\Delta_p,\emptyset;Gu(e_p)) \to (\Delta_p,\emptyset;u^*G)$ where $\tau:Gu(e_p) \to u^*G$ is a natural equivalence between the constant system $Gu(e_p)$ on Δ_p and the simple pull back system with respect to u, according to (1.6). We ignore this fact in our notation.

Let us prove next that Sd is a chain map, and that D is a chain homotopy of Sd to the identity. In fact, since $u_{\#}$ is a chain map, we see that Sd is a chain map on elementary p-chains and consequently also on $C_*(X;G)$ by the universal property of weak direct sums. To see that D is a chain homotopy of Sd to the iden - tity, we evaluate

$$\begin{aligned}(Sd-1)g.u &= (Sd-1)u_{\#}g.1_{\Delta p}\\ &= u_{\#}(Sd'_{Gu(e_p)}(g.1_{\Delta p}) - g.1_{\Delta p})\\ &= u_{\#}(D'_{p-1}\partial'_p + \partial'_{p+1}D'_p)g.1_{\Delta p}\\ &= D_{p-1}u_{\#}\partial'_{p-1}(g.1_{\Delta p}) + \partial_{p+1}u_{\#}D'_p(g.1_{\Delta p})\\ &= D_{p-1}\partial_p u_{\#}(g.1_{\Delta p}) + \partial_{p+1}u_{\#}D'_p(g.1_{\Delta p})\\ &= (D_{p-1}\partial_p + \partial_{p+1}D_p)g.1_{\Delta p},\end{aligned}$$

Here ∂, ∂' are differential operators on $C_*(X;G)$, $C_*(\Delta_p;Gu(e_p))$, and 1 is the identity on $C_*(X;G)$. we are now ready to constuct γ.

Let $u:\Delta_p \to X$ be a singular p-simplex, and let $\mathfrak{V}:=\{u^{-1}(U):U\in\mathfrak{U}\}$. By assumption X is covered by interiors of elements of $\mathfrak{U}$, and by continuity of u

$$\Delta_p = \bigcup_{V\in\mathfrak{V}} \mathring{V}$$

According to (A), let $\lambda > o$ be a Lebesque number for this open cover. Using (B), we see that, because $(\frac{n}{n+1})^{\nu}$ converges to o as ν approaches ∞, there exists a least integer $m := m(u)$ such that all simplices obtained by the m-th iterated barycentric subdivision of Δ_p have mesh less than or equal λ :

$$\text{mesh } Sd'^{m(u)}\Delta_p \le \lambda$$

By construction of Sd, this implies

$$Sd^{m(u)}u \in C_p^{\mathfrak{U}}(X;G)$$

Since $\partial D + D\partial = Sd - 1$, we have for $k \in \mathbb{N}$

$$\partial D Sd^k + D\partial Sd^k = Sd^{k+1} - Sd^k.$$

and consequently

$$\partial_{p+1} D_p (1+\ldots+Sd_p^k) + D_{p-1}\partial_p (1+\ldots+Sd_p^k) = Sd_p^{k+1} - 1$$

Since Sd is a chain map, this is equivalent to

$$\partial_{p+1} D_p (1+\ldots+Sd_p^k) = Sd_p^{k+1} - 1 - D_{p-1}(1+\ldots+Sd_{p-1}^k)\partial_p$$

For an elementary p-chain $g.u \in C_p(X;G)$, define

$$T_p(g.u) := D_p(1+\ldots+Sd_p^{m(u)})(g.u) \in C_p'(X;G)$$

We obtain

$$\begin{aligned}(\partial_{p+1}T_p + T_{p-1}\partial_p)(g.u) &= \partial_{p+1}D_p(1+\ldots Sd_p^{m(u)})g.u + \\ &+ D_{p-1}(1+\ldots+Sd_{p-1}^{m(u\circ f_p^o)})G(\alpha_{u(\dot{o})})g.u\circ f_p^o + \\ &+ \sum_{i=1}^{p}(-1)^i D_{p-1}(1+\ldots+Sd_{p-1}^{m(u\circ f_p^i)})g.u\circ f_p^i\end{aligned}$$

$$= (Sd^{m(u)+1} - 1)g.u - D_{p-1}(1+\ldots+Sd^{m(u)}_{p-1})\partial_p(g.u) + \ldots$$

Since for $o \leq i \leq p$, $m(u) \geq m(uof^i_p)$, we obtain (if $m(uof^i_p) = m(u)$, the expression $Sd^{m(uof^o_p)+1}+Sd^{m(u)}_{p-1} = 0$ - homomorphism)

$$(\partial_{p+1}T_p + T_{p-1}\partial_p)g.u = (Sd^{m(u)+1}_p - 1)g.u$$

$$- D_{p-1}(Sd^{m(uof^o_p)+1}_{p-1}+\ldots+Sd^{m(u)}_{p-1})G(\alpha_{u(o)})g.uof^o_p$$

$$- \sum_{i=1}^{p}(-1)^i D_{p-1}(Sd^{m(uof^i_p)+1}_{p-1}+\ldots+Sd^{m(u)}_{p-1})g.uof^i_p$$

Hence we may define

$$\gamma(g.u) := (\partial_{p+1}T_p + T_{p-1}\partial_p)g.u - g.u$$

which is an element of $C^{\mathfrak{U}}_p(X;G)$, as inspection of exponents shows. By construction, T is a chain homotopy of $j\circ\gamma$ to the identity. On the other hand, if $g.u \in C^{\mathfrak{U}}_p(X;G)$, then $m(u) = 0$ and $\gamma\circ j$ is the identity.

This concludes the proof of (3.10)(v). □

To verify the validity of the dimension axiom (3.10)(vi) we need merely observe that a system of local coefficients in a one point space is constant, which by (2.3) reduces homology with local coefficients to ordinary singular homology. □

The proof of "additivity" (3.10)(vii) is the same as in ordinary singular homology: By assumption a singular p-simplex in X maps the 0-connected space Δ_p and its boundary into precisely one X_λ. By definition of the chain complex of X with local coefficients in G, we obtain the following direct sum decompositions.

$$C_*(X;G) \cong \bigoplus_\lambda C_*(X;G_\lambda)$$

$$Z_p(X;G) \cong \bigoplus_\lambda Z_p(X,G_\lambda)$$

$$B_p(X;G) \cong \bigoplus_\lambda B_p(X;G_\lambda)$$

Our claim follows since

$$\frac{\bigoplus_\lambda Z_p(X;G_\lambda)}{\bigoplus_\lambda B_p(X;G_\lambda)} \cong \bigoplus_\lambda \frac{Z_p(X_\lambda;G_\lambda)}{B_p(X_\lambda;G_\lambda)}$$

This completes the proof of (3.10). □

(3.12) Corollary (homology ladder of a triple) :
let $(X,A;G)$, $(X',A';G')$, be objects in $\mathcal{L}$,
$\phi = (\phi_1,\phi_2) : (X,A;G) \to (X',A';G')$ a morphism in $\mathcal{L}$.
Let $B \subset A$, $B' \subset A'$ be such that $\phi_1(B) \subset B'$, then , by taking suitable restrictions of ϕ , we obtain morphisms in $\mathcal{L}$:

$$\phi' : (X,B;G) \to (X',B';G')$$
$$\psi : (A,B;G_{|A}) \to (A',B';G'_{|A'})$$

These and the inclusions

$$i : (A,B;G_{|A}) \to (X,B;G) \ , \quad i' : (A',B';G'_{|A'}) \to (X',B';G')$$
$$j : (X,B;G) \to (X,A;G) \ , \quad j' : (X',B';G') \to (X',A';G')$$

give rise to a commutative ladder of long exact homology sequences

$$\begin{array}{ccccccccc}
\cdots \to & H_n(A,B;G_{|A}) & \xrightarrow{i_*} & H_n(X,B;G) & \xrightarrow{j_*} & H_n(X,A;G) & \xrightarrow{\partial} & H_{n-1}(A,B;G_{|A}) & \to \cdots \\
& \downarrow \psi_* & & \downarrow \phi'_* & & \downarrow \phi_* & & \downarrow \psi_* & \\
\cdots \to & H_n(A',B';G_{|A'}) & \xrightarrow{i'_*} & H_n(X',B';G') & \xrightarrow{j'_*} & H_n(X',A';G') & \xrightarrow{\partial'} & H_{n-1}(A',B';G_{|A'}) & \to \cdots
\end{array}$$

Proof: This can be shown using the axioms only. Since the proof of the analogous statement in ordinary homology presented by Hu [Hu] pp 31-38 carries over word for word, we omit it here. □

We also have a Mayer-Vietoris sequence for homology with local coefficients.

(3.13) Theorem: Let $(X;G)$ be an object in $\mathcal{L}$ and $A,B \subset X$ subspaces such that

$$i : (A, A \cap B; G|_A) \longrightarrow (A \cup B, B; G|_{A \cup B})$$
$$j : (B, A \cap B; G|_B) \longrightarrow (A \cup B, A; G|_{A \cup B})$$

induce isomorphisms

$$i_*: H_*(A, A \cap B; G|_A) \longrightarrow H_*(A \cup B, B; G|_{A \cup B})$$
$$j_*: H_*(B, A \cap B; G|_B) \longrightarrow H_*(A \cup B, A; G|_{A \cup B})$$

in homology with local coefficients. Then there is a long exact sequence

$$\cdots \to H_p(A \cap B; G|_{A \cap B}) \longrightarrow H_p(A; G|_A) \oplus H_p(B; G|_B) \longrightarrow$$
$$\longrightarrow H_p(X; G) \longrightarrow H_{p-1}(A \cap B; G|_{A \cap B}) \longrightarrow \cdots$$

Proof: Again, this can be derived from the axioms as in [Hu] pp 105-112. □

Remark: Using "excision" (3.10)(v), we see that the assumptions of (3.13) are in particular satisfied if $X = \mathring{A} \cup \mathring{B}$ and $\mathring{A} \cap \mathring{B} \neq \emptyset$.

As another corollary of (3.10) we obtain

(3.14) Corollary: Let $(X,A;G)$, $(X,A;H)$ be objects in $\mathcal{L}$ with equivalent systems of local coefficients G, H, then

$$H_*(X,A;G) \cong H_*(X,A;H)$$

More precisely, since G, H are naturally equivalent, there exists a morphism with natural equivalence $\phi_2 : G \to H$, $\phi = (1_X, \phi_2) : (X,A;G) \to (X,A;H)$, which induces isomorphisms in homology with local coefficients.

Proof: Define $\phi^{-1} := (1_X, \phi_2^{-1}) : (X,A;H) \to (X,A;G)$. Then ϕ, ϕ^{-1} are inverse to each other in $\mathcal{L}$. Now use the fact that H_* is a functor on $\mathcal{L}$.

Corollary (3.14) poses the question: How can we decide whether two systems of local coefficients in X are equivalent? Theorem (1.9) and its corollaries give the general key how to reduce this problem to a problem of purely group theoretic nature. In particular, we refer the reader back to corollary (1.10) now.

Functorial Properties of Homology with Local Coefficients

The information of (1.9) can also be utilized to compute H_0 for path connected spaces.

(3.15) Theorem: Let X be 0-connected, $x_0 \in X$; G a system of local coefficients in X. Let $h:\pi_1(X,x_0) \to \mathrm{Aut}\, Gx_0$ describe the action of $\pi_1(X,x_0)$ on Gx_0 induced by G, and let $H := \mathrm{im}\, h$. Then

$$H_0(X;G) \cong Gx_0 / \langle\{\phi g - g : g \in Gx_0, \phi \in H\}\rangle$$

Proof: As in (1.9), let $\xi : X \to \{\Pi X(x_0,x)\}_{x \in X}$ be a choice function, and let G' be the system of local coefficients in X determined by h and ξ. By (1.9), G and G' are naturally equivalent functors. By (3.14), $H_*(X;G) \cong H_*(X;G')$. Consequently, we may verify our claim using the system G'.

Now let $x_1, x_2 \in X$, and consider representing paths for the path classes $\overline{\xi(x_2)}\alpha\xi(x_1)$, $\alpha \in \pi_1(X,x_0)$. Using a suitable homeomorphism of $I \to \Delta_1$, we may interpret such a representing path as a singular 1-simplex with x_1 being the image of the preferred vertex e_1 of Δ_1. Then

$$\partial_1(g.u) = h(\xi(x_2)\overline{\xi(x_2)}\alpha\xi(x_1)\overline{\xi(x_1)})g.x_2 - g.x_1$$
$$= (h\alpha)g.x_2 - g.x_1$$

Furthermore, if u' is a singular 1-simplex in X such that $u'(1,0) = x_1$, $u(0,1) = x_2$, then $\partial_1 u' = (h\alpha)g.x_2 - g.x_1$ for some $\alpha \in \pi_1(X,x_0)$. Consequently,

$$B_0(X;G') = \langle\{\phi g.x_2 - g.x_1 : \phi \in H;\ g \in Gx_0;\ x_1, x_2 \in X\}\rangle$$

The map

$$\Gamma : Z_0(X;G') \ni \sum_{\text{finite}} g_i.x_i \longrightarrow \Sigma g_i \in Gx_0$$

is an epimorphism. Now consider the diagram

$$\begin{array}{ccc} Z_0(X;G') & \xrightarrow{\ \Gamma\ } & Gx_0 \\ \Big\downarrow q_1 & & \Big\downarrow q_2 \\ \dfrac{Z_0(X;G')}{B_0(X;G')} = H_0(X;G) & \dashrightarrow^{\gamma} & Gx_0/_T \end{array}$$

Here $T := \langle\{\phi g - g : g \in Gx_0, \phi \in H\}\rangle$ and q_1, q_2 are the canonical quotient epimorphisms. It is then a straight-forward procedure to check that

$$\gamma : \frac{Z_0(X;G')}{B_0(X;G')} \ni z + B_0(X;G') \longmapsto q_2\Gamma z \in Gx_0/_T$$

is a well defined isomorphism. □

(3.16) Example: Let G be a system of local coefficients in S^1, then there are, up to natural equivalence, exactly two possibilities for G (see 1.12). Furthermore, using (3.15), we compute $(G_0 := G(1,0))$:

(i) $H_0(S^1;G) \cong G_0$ iff G is simple

(ii) $H_0(S^1;G) \cong G_0/2G_0$ iff G is not simple

In particular, if $G_0 \cong \mathbb{Z}$, we obtain

$$H_0(S^1;G) \cong \mathbb{Z}_2 \quad \text{iff } G \text{ is not simple.}$$

4. Equivariant Homology and Homology with Local Coefficients.

The key result of this section is due to Samuel Eilenberg and published in his paper "Homology of spaces with operators" [E] in 1944.

Let X be a space, Π a group.

(4.1) Definition: Given a homomorphism

$$\varphi : \Pi \to \text{Homeo } X$$

of Π into the group of homeomorphisms of X, we say that Π acts on X (by homeomorphisms).

Let X be a space on which Π acts by $\varphi:\Pi \to \text{Homeo } X$.

(4.2) Observation: φ turns (C_*X,∂) into a chain complex of $Z\Pi$ - modules.

Proof: Since for each $\alpha\in\Pi$,

$$(\varphi\alpha)_\# : C_*X \to C_*X$$

is a chain isomorphism, we see that Π acts on C_nX $(n\in N_o)$ and furthermore for $z\in C_nX$

$$\partial_n(\alpha.z) = \partial_n[(\varphi\alpha)_{\# n}z] = (\varphi\alpha)_{\# n-1}[\partial_n z] = \alpha.\partial_n z$$

which shows that ∂ is a $Z\Pi$ - module morphism. □

We shall continue to interpret this action on C_*X as a left action. If G is a right $Z\Pi$ - module, we may form the chain complex $G\otimes_{Z\Pi}C_*X$ with boundary operator $1_G\otimes\partial$.

(4.3) Definition: The resulting homology groups

$$E_p^{\Pi}(X;G) := H_p(G\otimes_{Z\Pi}C_*X)$$

are called equivariant homology groups with coefficients in the $Z\Pi$ - module G. If $p:E \to B$ is a covering map, then $\pi_1 B$ acts on E by covering transformations. If E is a universal covering space, then the group of covering transformations is isomorphic to $\pi_1 B$ (See the appendix).

Now let X be a 0 - connected locally path connected and semilocally 1 - connected space with base point x_o, and let $G:\Pi X \to Ab$ be a system of local coefficients in X. Then there exists a universal covering $p:\tilde{X} \to X$. Let $\tilde{x}_o \in p^{-1}\{x_o\}$ be the base point of $\tilde{X}$, $\Pi := \pi_1(X,x_o)$. The action $\varphi:\Pi \to \text{Homeo}\,\tilde{X}$ can be chosen to be a homomorphism such that, if $\alpha \in \Pi$ and $u:I \to \tilde{X}$ is a path from $\tilde{x}_o$ to $\varphi(\alpha)\tilde{x}_o$, then $p\cdot u : (I,\partial I)\to(X,x_o)$ is a representative of α.

We convert the left action of Π on $G_0 := Gx_0$ into a right action by $g.\alpha := (G\alpha)^{-1}g$.

(4.4) Theorem (Eilenberg): $H_*(X;G) \cong E_*^{\Pi}(\widetilde{X};G_0)$

Proof: A complete proof of this theorem can be found in G. Whitehead's book "Elements of Homotopy Theory" [Wh1] pp 278-280, so we content ourselves with a mere outline of the idea.

Since $\widetilde{X}$ is simply connected, a choice function as in (1.9), is a uniquely defined bijection

$$\xi:\widetilde{X} \ni \widetilde{x} \mapsto \pi\widetilde{X}(\widetilde{x}_0,\widetilde{x}) \in \left\{\pi\widetilde{X}(\widetilde{x}_0,\widetilde{x})\right\}_{\widetilde{x}\in\widetilde{X}}$$

Let $w:\Delta_p \to \widetilde{X}$ be a singular simplex, $g \in G_0$, $\alpha \in \Pi$ and define a map

$$g.\alpha\otimes_{Z\Pi} w \mapsto G[p_*\bar{\xi}(\varphi\alpha(w(e_p)))]g.pw$$

the image being an element of $C_p(X;G)$. This map can be proven to extend to a chain isomorphism

$$G_0\otimes_{Z\Pi}C_*\widetilde{X} \to C_*(X;G)$$ □

We use theorem (4.4) to gain some insight in the homology of path connected spaces. We need the following lemma.

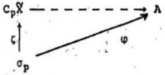
(4.5) Lemma: Let $p:(\tilde{X},*) \to (X,*)$ be a universal cover - ing, with X path connected. Then $C_*\tilde{X}$ is a chain com - plex of free $\mathbb{Z}\Pi_1(X,*)$ - modules.

Proof: For $p \in \mathbb{N}_0$, let $\sigma_p := \{u:\Delta_p \to X \text{ continuous}\}$. We claim that σ_p is a basis for $C_p\tilde{X}$ as a $\mathbb{Z}\Pi$-module, $\Pi := \pi_1(X,*)$.

Let $u \in \sigma_p$, $\tilde{x}_0 \in p^{-1}\{u(e_p)\}$. Since Δ_p is 1-connect- ed, the lifting lemma for covering projections gives us a unique lifting $\tilde{u}:\Delta_p \to \tilde{X}$ of u such that $u = p\tilde{u}$ and $\tilde{u}(e_p) = \tilde{x}_0$. Denote by $p_*^{-1}\{u\}$ the set of liftings obtained as $\tilde{x}_0$ ranges through $p^{-1}\{u(e_p)\}$, and let $\tilde{\sigma}_p$ be the set of singular p-simplices in $\tilde{X}$. Then $\tilde{\sigma}_p$ is the disjount union

$$\tilde{\sigma}_p = \bigcup_{u \in \sigma_p} p^{-1}\{u\}$$

Now let $\zeta:\sigma_p \longrightarrow \{p^{-1}\{u\}:u \in \sigma_p\}$, $\zeta(u) \in p^{-1}\{u\}$ be a choice function, A a Π-module, and $\varphi:\sigma_p \to A$ a function. In order to show that $C_p\tilde{X}$ is a free Π-module, we must find a unique module homomorphism such that the following triangle commutes.



This is accomplished by defining

$$\sum_{\text{finite}} n(u)\alpha(u).\zeta(u) \longrightarrow n(u)\alpha(u).\varphi(u)$$

where $\alpha(u) \in \pi_1(X,*) = \Pi$, $n(u) \in \mathbb{Z}$, $u \in \sigma_p$. □

Now let Π be a group, $X := K(\Pi,1)$ the Eilenberg-Mac Lane CW-complex satisfying

$$\pi_n X = \begin{cases} 0 & \text{if } n \neq 1 \\ \Pi & \text{if } n = 1 \end{cases}$$

Then (X is 0-connected) the long exact homotopy sequence of the Serre-fibration $\tilde{X} \to X$ yields $\pi_n \tilde{X} = 0$ for all $n \in \mathbb{N}_0$. The Hurewicz-theorem implies

$$H_n\tilde{X} = \begin{cases} \mathbb{Z} & \text{iff } n = 0 \\ 0 & \text{iff } n \geq 1 \end{cases}$$

Thus $C_*\tilde{X}$ is a projective resolution of $\mathbb{Z}$ regarded as a Π-module with trivial action. For a Π-module G_0 we then obtain

$$E_p^{\Pi}(\tilde{X};G_0) = \mathrm{Tor}_p^{\Pi}(\mathbb{Z}, G_0)$$

(For a development of the functors Tor, the reader is referred to Hilton, Stammbach [H-S] IV,11)

Consequently theorem (4.4) yields immediately

(4.6) Theorem: If Π is a group and G a system of local coefficients in $K(\Pi,1)$, $G_o := Gx_o$, $x_o \in K(\Pi,1)$, then:

$$H_p(K(\Pi,1);G) \cong \mathrm{Tor}_p^{\Pi}(\mathbb{Z},G_o)$$ □

From this theorem we conclude the following.

(a) Computation of homology with local coefficients of $K(\Pi,1)$'s can be reduced to purely group theoretic considerations.

(b) The resulting homology groups depend solely on G_o and the action of Π on G_o.

Note that (b) is reminiscent of (1.10), for we saw there that a system of local coefficients in a 0-connected space is, up to natural equivalence, uniquely determined by G_o and the action of Π on G_o. Furthermore, we showed in (3.14) that $H_*(X,A;G) \cong H_*(X,A;G')$ for equivalent systems G,G' in X.

As we just saw, this is exactly the information one needs to compute homology of $K(\Pi,1)$'s. In order to transfer this convenient information to 0-connected spaces other than $K(\Pi,1)$'s, we might attempt to pursue one of the following two possibilities:

Starting with X 0-connected, we ask:

(A) Is there a group Π and a map $f:X \to K(\Pi,1)$ such that each system in X is naturally equivalent to a pullback system f^*G and the morphisms

$$f_*:H_*(X;f^*G) \xrightarrow{\cong} H_*(K(\Pi,1);G)$$

are isomorphisms?

(B) Is there a group Π' and a map $f':K(\Pi',1) \to X$ such that

$$f'_*:H_*(K(\Pi',1);f'^*G) \xrightarrow{\cong} H_*(X;G).$$

are isomorphisms for all systems of local coefficients G in X ?

I know of no answer to question (A). Question (B), however, has been answered positively for 0-connected spaces of the homotopy type of a CW-complex in 1976 by D.M.Kan and W.P.Thurston [K-T].

5 Local Coefficients in CW-Complexes

As in ordinary singular homology it is possible to make use of the cell structure of CW-complexes in a systematic way.

(5.1) <u>Theorem:</u> Let $(Y,X;G)$ be an object in $\mathcal{L}$ such that (Y,X) is an n-cell adjunction $(n>0)$ with n-cells $\{e^n_\iota\}_{\iota\in J}$ and characteristic maps $\chi_\iota:(B^n_\iota,S^{n-1}_\iota)\to(Y,X)$, $\chi:=\coprod_{\iota\in J}\chi_\iota:\coprod_{\iota\in J}(B^n_\iota,S^{n-1}_\iota)\to(Y,X)$. Then

(i) $H_p(Y,X;G)=0$ for $p\neq n$

(ii) For all $\iota\in J$: $\chi_{\iota *}:H_n(B^n_\iota,S^{n-1}_\iota;\chi^*_\iota G)\to H_n(Y,X;G)$ is a monomorphism.

(iii) $\chi_* = \bigoplus_{\iota\in J}\chi_{\iota *}:\bigoplus_{\iota\in J}H_n(B^n_\iota,S^{n-1}_\iota;\chi^*_\iota G)\xrightarrow{\cong}H_n(Y,X;G)$ is an isomorphism.

Proof: Compare Massey [Ma2] pp 78,79.

Let $B:=\coprod_{\iota\in J}B^n$ $\qquad S:=\coprod_{\iota\in J}S^{n-1}$

$D^n_\iota:=\{x\in B^n_\iota:|x|\le\frac{1}{2}\}$ $\qquad D:=\coprod_{\iota\in J}D^n_\iota$

$E:=\coprod_{\iota\in J}\{0_\iota:0_\iota\in D_\iota\}$

and consider the following diagram.

$$\begin{array}{ccc}
\bigoplus_{\iota\in J} H_n(D^n_\iota, D^n_\iota-\{0_\iota\};\chi^*_\iota G) & \xrightarrow[7]{\oplus\chi_{\iota *}} & \bigoplus_{\iota\in J} H_n(\chi_\iota(D^n_\iota),\chi_\iota(D^n_\iota-\{0_\iota\});G) \\
\downarrow 1 & & \downarrow 2 \\
H_n(D,D-E;\chi^*G) & \xrightarrow[8]{\chi_*} & H_n(\chi(D),\chi(D-E);G) \\
\downarrow 3 & & \downarrow 4 \\
H_n(B,B-E;\chi^*G) & \xrightarrow[9]{\chi_*} & H_n(Y,Y-\chi(E);G) \\
\uparrow 5 & & \uparrow 6 \\
H_n(B,S;\chi^*G) & \xrightarrow[10]{\chi_*} & H_n(Y,X;G)
\end{array}$$

In order to simplify notation, we did not write down the obvious restrictions of the maps χ_ι, χ and the various local systems.

The diagram commutes because the corresponding diagram of maps in $\mathcal{L}$ commutes. We claim that all maps are isomorphisms.

For the maps 1,2, this follows from "additivity" (3.10)(vii). The map 7 is an isomorphism because $\chi_\iota|D^n_\iota$ is a homeomorphism and thus induces isomorphisms on each direct summand. Maps 3,4 are isomorphisms by "excision"(v) (3.10)(v). Maps 5,6 are isomorphisms because the pair $(B,B-E;\chi^*G)$ is homotopically equivalent to $(B,S;\chi^*G)$, and $(Y,Y-\chi(E);G)$ is homotopically equivalent to $(Y,X;G)$. This is clear in Top and follows in $\mathcal{L}$ from lemma (3.7).

This proves (ii) and (iii). To see (i), we need merely observe that B_1^n is simply connected. Hence χ_1^*G is a simple system of local coefficients in B_1^n. By (2.3), we may pick any element $x_0 \in B_1^n$, and get

$$H_p(B_1^n,S_1^n;G\chi_1(x_0)) \cong H_p(B_1^n,S_1^n;\chi_1^*G)$$

the group on the left hand side being an ordinary singular homology group with trivial coefficients in the group $G\chi_1(x_0)$. This implies (i). □

Since the only non trivial part of the long exact homology sequence of the pair $(Y,X;G)$ is the following:

$$0 \to H_n(X;G_{|X}) \to H_n(Y;G) \to H_n(Y,X;G) \to H_{n-1}(X;G_{|X}) \to H_{n-1}(Y;G) \to 0$$

we obtain:

(5.2) <u>Corollary:</u> The map $i_*:H_p(X;G_{|X}) \to H_p(Y;G)$ is an isomorphism except possibly for $p=n$ or $p=n-1$. □

(5.3) <u>Corollary:</u> Let $(X;G)$ be an object in $\mathcal{L}$, X being a CW-complex with n-skeleton X^n $(n \geq 0)$. Then for $p>n$: $H_p(X^n;G_{|X^n}) = 0$.

Proof: Induction on n. For $n = 0$, this follows from "dimension" and "additivity" (3.10(vi),(vii)), because the 0-skeleton is discrete. For $n > 0$, use the long exact homology sequence of the pair $(X^n, X^{n-1}; G_{|X^n})$ and (5.2). □

(5.4) Corollary: Let $(X;G)$ be as in (5.3). Then for $p > n \geq m \geq 0$ (or for $n \geq m \geq p \geq 0$)

$$H_p(X^n, X^m; G_{|X^n}) = 0 .$$

Proof: By induction on $0 \leq k := n-m < n$ (or $0 \leq k := n-m < n-p$) and the long exact sequence of the triple $(X^n, X^{n-k}, X^{n-k-1})$. □

(5.5) Theorem: Let $(X;G)$ be as in (5.3). Then for $n \geq p \geq 0$, the inclusion $X^n \to X$ induces an isomorphism

$$H_p(X^n; G_{|X^n}) \longrightarrow H_p(X;G)$$

Proof: Using the axioms (3.10), we can use a method due to Milnor, [Mi1] lemma 1, to show that

$$H_p(X;G) \cong \lim\left(H_p(X^1;G_{|X^1}) \to H_p(X^2;G_{|X^2}) \to H_p(X^3;G_{|X^3}) \to \dots\right)$$

the maps being induced by inclusions. By (5.4), $H_p(X^n; G_{|X^n}) \to H_p(X^{n+k}; G_{|X^{n+k}})$ is an isomorphism for $n \geq p$ and $k \geq 0$, which yields what we want. (See e.g. [G] Proposition (15.4)). □

(5.6) <u>Corollary</u>: Let $(X;G)$ be as in (5.3). Then for $n \geq p$: $H_n(X,X^n;G) = 0$.

Proof: Use (5.5) and the long exact sequence of $(X,X^n;G)$. □

I owe statement and a sketch of the proof of the following Whitehead type theorem to K. Varadarajan (oral communication).

(5.7) <u>Theorem:</u> Let X,Y be based connected CW-complexes, $f:(X,*) \to (Y,*)$ a continuous map such that $\pi_1 f$ is an isomorphism and f induces isomorphisms

$$f_*: H_*(X; f^*\mathbb{Z}\pi_1(Y,*)) \longrightarrow H_*(Y;\mathbb{Z}\pi_1(Y,*))$$

then f is a homotopy equivalence.

Proof: Let $\tilde{X},\tilde{Y}$ be universal covering spaces of X,Y and $\tilde{f}:\tilde{X} \to \tilde{Y}$ be the map induced by f such that

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

Using Whitehead's theorem (CW 17), it suffices to show that f is a weak homotopy equivalence. By assumption, we are left to verify that f induces isomorphisms in homotopy groups of dimension ≥ 2. Using the exact sequence of a Serre fibration, we see that $\pi_n p$ and $\pi_n q$ are isomorphisms for $n \geq 2$. Consequently, $\pi_n f$ is an isomorphism $(n \geq 2)$ iff $\pi_n \tilde{f}$ is an isomorphism. But $\tilde{X}, \tilde{Y}$ are simply connected, so (CW 18) it suffices to show that

$$\tilde{f}_*: H_*\tilde{X} \longrightarrow H_*\tilde{Y}$$

are isomorphisms.

To do this, consider

$$\begin{array}{ccc}
\tilde{f}_*: H_n(C_*\tilde{X}) & \longrightarrow & H_n(C_*\tilde{Y}) \\
\| & & \| \\
H_n(C_*\tilde{X} \underset{\pi_1 X}{\otimes} Z\pi_1 X) & & \\
\| & & H_n(C_*\tilde{Y} \underset{\pi_1 Y}{\otimes} Z\pi_1 Y) \\
H_n(C_*\tilde{X} \underset{\pi_1 X}{\otimes} Z\pi_1 Y) & & \\
\| & & \| \\
E_n^{\pi_1 X}(\tilde{X}; Z\pi_1 Y) & & E_n^{\pi_1 Y}(\tilde{Y}; Z\pi_1 Y) \\
\| & & \| \\
f_*: H_n(X; f^* Z\pi_1 Y) & \xrightarrow{\cong} & H_n(Y; Z\pi_1 Y)
\end{array}$$

which yields what we want. A few minor things must be justified.

(a) The above diagram commutes. This can be seen easily by looking at the corresponding diagram of chain complexes and using the chain isomorphism in the proof of Eilenberg's theorem (4.4).

(b) If R is a ring and M a right R-module, then we have an isomorphism

$$M \otimes_R R \ni m \otimes_R r \to m.r \in M$$

of abelian groups.

(c) If <C,∂> is a complex of right R-modules, and R' is a ring isomorphic to R via h:R → R', then R' is a left R-module by r.r' := h(r)r'. Furthermore the chain complexes $C \otimes_R R$ and $C \otimes_R R'$ are isomorphic. The proof is straight forward. □

Since (2.3), we know that ordinary singular homology is a special case of homology with local coefficients. But up to here we have not seen that we really gain information by paying attention to the extra structure of systems of local coefficients. We shall eliminate this deficiency now.

First of all, the reader is reminded that a weak homotopy equivalence f:X → Y between CW-complexes is actually a homotopy equivalence. If we assume f to be a (ordinary) homology equivalence, then f will in general not be a homotopy equivalence. One way to find counterexamples is to construct spaces X,Y with non iso-morphic perfect fundamental groups, (which implies

$H_1X = H_1Y = 0$) and still permit an $f: X \to Y$ which is a homology equivalence.

Inspired by the hypotheses of theorem (5.7), we might wonder whether requiring in addition that $\pi_1 f$ be an isomorphism, is exactly the thing to do to fill the gap we just observed. However, an example which I owe to K. Varadarajan shows that this is still not sufficient.

(5.8) <u>Example:</u> Without giving a proof here, we assume the existence of the following two spaces

$$X := K(\mathbb{Q},1) = M(\mathbb{Q},1)$$
$$Y := M(\mathbb{Z}_2,3)$$

which are connected CW-complexes satisfying the following properties:

$$H_0X = \mathbb{Z}, \quad H_1X = \pi_1X = \mathbb{Q}, \quad \text{for } n > 1: H_nX = \pi_nX = 0$$
$$H_0Y = \mathbb{Z}, \quad H_3Y = \mathbb{Z}_2, \quad \pi_1Y = \pi_2Y = 0, \quad \text{for } n \notin \{0,3\}: H_nY = 0$$

Furthermore, X, Y have precisely one 0-cell x_0, y_0 which we use as base points to define

$$i: (X,x_0) \vee (Y,y_0) \ni \;\longmapsto \begin{cases} (\zeta, y_0) & \text{if } \zeta \in X \\ (x_0, \zeta) & \text{if } \zeta \in Y \end{cases} \in X \times Y$$

Recall from the definition of the wedge product that we may regard i as an inclusion.

Claim: (i) $\pi_1 i$ is an isomorphism.

(ii) For any abelian group A, i induces isomorphisms in ordinary homology with coefficients in A.

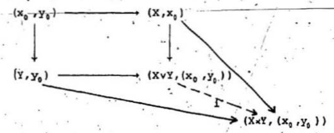
(iii) i is not a homotopy equivalence.

We conclude that there exist systems of local coefficients G in $X \times Y$ such that for some $n \in \mathbb{N}$

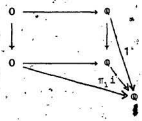
$$i_*: H_n(X \times Y; i^*G) \to H_n(X \times Y; G)$$

is not an isomorphism. This follows from (5.7).

Proof: (i) Using the Seifert-Van Kampen theorem (see the appendix on the fundamental groupoid), we see that the following adjunction square of based path connected spaces



gives rise to the following diagram of fundamental groups:



From the proof of the fact that $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$, we infer that homomorphism 1 is an isomorphism. By commutativity: $\pi_1 i$ is an isomorphism.

(ii) We first compute the homology groups of $X \vee Y$, $X \times Y$ with integral coefficients separately. $X \vee Y$ is a CW-complex, hence has a contractible neighbourhood U of $\{x_0, y_0\}$. Define $X' := X \cup U$, $Y' := Y \cup U$. Then X, Y are strong deformation retracts of X', Y'. The Mayer-Vietoris sequence of the triple $(X \vee Y, X', Y')$ yields the following exact sequence in reduced homology:

$$\ldots \to \tilde{H}_{n+1} X \vee Y \to \tilde{H}_n U \to \tilde{H}_n X' \oplus \tilde{H}_n Y' \to \tilde{H}_n X \vee Y \to \tilde{H}_{n-1} U \to \ldots$$

Since U is contractible, $\tilde{H}_n X' \oplus \tilde{H}_n Y' \to \tilde{H}_n X \vee Y$ is an isomorphism for all $n \in \mathbb{N}_0$. Consequently

$$H_0 X \vee Y \cong \mathbb{Z} \qquad H_1 X \vee Y \cong \mathbb{Q}$$
$$H_3 X \vee Y = \mathbb{Z}_2 \qquad H_n X \vee Y = 0 \quad n \in \mathbb{N} - \{1, 3\}$$

The homology groups of $X \times Y$ can be calculated using the split exact sequences of the Künneth theorem

$$0 \longrightarrow \sum_{i+j=n} H_i X \otimes H_j Y \longrightarrow H_n X\times Y \longrightarrow \sum_{i+j=n-1} \mathrm{Tor}(H_i X, H_j Y) \longrightarrow 0$$

which yields

$$H_n X\vee Y \cong H_n X\times Y$$

for all $n \in \mathbb{N}_0$. In order to check that i indeed induces such isomorphisms, it only remains to consider dimensions 1 and 3. Let $p: X\times Y \to X$ be the projection and consider

$$\begin{array}{ccccccc} X & \xrightarrow{1} & X' & \xrightarrow{2} & X\vee Y & \xrightarrow{i} & X\times Y \\ & \searrow^{1_X} & & & & & \downarrow p \\ & & & & & & X \end{array}$$

We have seen before that the maps 1,2 induce iso - morphisms in homology in dimension 1. Hence, applying the homology functor H_1 to the above diagram, we are left with

$$\begin{array}{ccc} H_1 X\vee Y \cong \mathbb{Q} & \longrightarrow & \mathbb{Q} \cong H_1 X\times Y \\ & \searrow^{\cong} & \downarrow p_* \\ & & \mathbb{Q} \cong H_1 X \end{array}$$

From this, we see that i_* is injective. But a homomorphism $\mathbb{Q} \to \mathbb{Q}$ is either the 0-homomorphism or an isomorphism. Thus i_* is an isomorphism in dimension 1. The same type of argument, applied to the sequence $Y \to Y' \to X\vee Y \to X\times Y \to Y$, shows that i induces also an isomorphism in homology dimension 3. Thus i induces isomorphisms in integral homology.

Now let A be an abelian group and consider the long exact sequence

$$\ldots \to H_{n+1}(Z, X\vee Y;A) \to H_n(X\vee Y;A) \xrightarrow{j_*} H_n(Z;A) \to H_n(Z, X\vee Y;A) \to \ldots$$

where Z is the mapping cylinder of $i: X\vee Y \to X\times Y$ and j_* is induced by $j: X\vee Y \ni \zeta \to (\zeta, 0) \in Z$.

It suffices to show that j_* is an isomorphism, because in the following commuting triangle

$$\begin{array}{ccc} X\vee Y & \xrightarrow{\ i\ } & X\times Y \\ & {\scriptstyle j}\searrow & \uparrow {\scriptstyle r} \\ & & Z \end{array}$$

the retraction

$$r: Z \ni \zeta \longrightarrow \left\{ \begin{array}{ll} \zeta & \zeta \in X\times Y \\ i(s) & \zeta = (s,t) \in X\vee Y\times I \end{array} \right\} \in X\times Y$$

can easily be extended to a strong deformation retraction of Z into $X\times Y$. Hence i induces isomorphisms in ordinary homology with coefficients in A iff j induces isomorphisms.

To see this, we show that $H_n(Z, X\vee Y;A) = 0$ for all

$n \in \mathbb{N}$. For $A = \mathbb{Z}$, this follows from the exact sequence and the fact that i induces isomorphisms in integral homology. Now apply the universal coefficient theorem.

(iii) $X \vee Y, X \times Y$ are CW-complexes (although we don't show it here), and consequently have universal covering spaces $\widetilde{X \vee Y}, \widetilde{X \times Y}$. But since $\widetilde{Y} = Y$ and $\widetilde{X}$ is contractible, we have

$$\widetilde{X \times Y} = \widetilde{X} \times \widetilde{Y} = \widetilde{X} \times Y \approx Y$$

Up to homotopy type, we can also simplify the space $\widetilde{X \vee Y}$ by the following consideration. Let $p: \widetilde{X} \to X$ be the universal covering projection. Then, using simple connectedness of Y, we see that a universal covering of $X \vee Y$ can be constructed by attaching a copy of Y with its base point y_0 to each element of $p^{-1}\{x_0\} \subset \widetilde{X}$. Since $\widetilde{X}$ is contractible, and since $p^{-1}(x_0)$ is in 1-1 correspondence with $\pi_1(X, x_0) \cong \mathbb{Q}$, $\widetilde{X \vee Y}$ is homotopically equivalent to a wedge of countably many copies of Y.

To see this, we first observe that the Moore space Y is finite dimensional (see e.g. [G] p159) and, therefore, we may apply the technique illustrated in J. Milnor's "Morse Theory", [Mi2] pp 20-22, to verify the stated homotopy equivalence (the facts of interest are collected in the appendix on CW-complexes).

Using the Mayer-Vietoris argument of the proof of (5.8)(ii) once again, in combination with a simple category theoretical limiting argument, we see that

$$H_3\widetilde{X\vee Y} \cong \bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{2,k} \ncong \mathbb{Z}_2 \cong H_3\widetilde{X\times Y}.$$

where $\mathbb{Z}_{2,k}$ denotes the k-th element in the family $\{\mathbb{Z}_{2,k}\}_{k\in\mathbb{N}}$ of copies of $\mathbb{Z}_2$.

We use this information to contradict the assumption that $i: X\vee Y \to X\times Y$ be a homotopy equivalence in the following way. Consider the commutative square

$$\begin{array}{ccc} \widetilde{X\vee Y} & \xrightarrow{\ \tilde{i}\ } & \widetilde{X\times Y} \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 2} \\ X\vee Y & \xrightarrow{\ i\ } & X\times Y \end{array}$$

The long exact sequence of Serre fibrations tells us that the covering projections 1,2 induce isomorphisms in homotopy in dimensions ≥ 2. If i is a homotopy equivalence, $\pi_n\tilde{i}$ are also isos for $n \geq 2$, by commutativity. $\widetilde{X\vee Y}, \widetilde{X\times Y}$ are simply connected. Consequently, $\tilde{i}$ is a weak homotopy equivalence and hence a homotopy equivalence. But then $H_3\widetilde{X\vee Y} \cong H_3\widetilde{X\times Y}$, a contradiction. □

CHAPTER II

The "+" Construction

In this chapter we give a detailed proof for the existence and uniqueness (up to homotopy type) of the "+" construction due to D. Quillen [Q]. In doing this, we follow a very condensed suggestion in a paper by J. B. Wagoner [Wa].

In order not to disrupt the main outline of ideas behind this construction too often, we first give a semi condensed version, deferring details and references till later. Thus numbers assigned to statements in this semi condensed version indicate the number under which the reader can find details and/or further references if he feels inclined to do so.

Throughout this section we shall be working in the category of path connected topological spaces having the homotopy type of a CW-complex. All spaces are based, and all maps and homotopies are base point preserving. In the proofs of the following four statements, however, we shall tacitly assume that the spaces we are working with are actually CW-complexes. More general statements, concerning spaces that are only of the homotopy type of a CW-complex will follow easily from arguments that are at least similar to the one given in the appendix on CW-complexes (19).

Theorem A: Let X be a space, $H \subset \pi_1 X$ a perfect sub-group (H is equal to its commutator subgroup). Then there exists a space X^+ and a map $i:X \to X^+$ such that

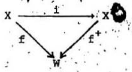
(i) $\pi_1 X^+ \cong \pi_1 X/\bar{H}$ where $\bar{H}$ denotes the normal closure of H in $\pi_1 X$ (the intersection of all normal subgroups of $\pi_1 X$ containing H).

(ii) For any $\mathbb{Z}\pi_1 X^+$-module A, the map $i:X \to X^+$ induces isomorphisms

$$i_*:H_*(X;i^*G) \to H_*(X^+;G)$$

of homology groups with local coefficients in i^*G, G. Here G denotes the system of local coefficients in X^+ induced by the action of $\mathbb{Z}\pi_1 X^+$ on A.

Theorem B: Let $i:X \to X^+$ be as in theorem A, $f:X \to W$ a map such that $H \subset \ker \pi_1 f$, then there exists a map $f^+:X^+ \to W$ such that the following triangle commutes.



Corollary C: In the situation of theorem A, the space X^+ is unique up to homotopy type.

Theorem D: Let $f:X \to Z$ be a map such that $\pi_1 f(G) \subset H$ where G, H are perfect subgroups of $\pi_1 X, \pi_1 Z$. Let $i:X \to X^+$, $j:Z \to Z^+$ satisfy (i), (ii) of theorem A. Then there is a map $f^+:X^+ \to Z^+$, that gives a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Z \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ X^+ & \xrightarrow[\ f^+\ ]{} & Z^+ \end{array}$$

Proof of theorem A: The construction is explicit. Under the functor π_1 an adjunction square of path connected based spaces gives rise to a pushout square of fundamental groups

$$\begin{array}{ccc} W & \longrightarrow & W' \\ {\scriptstyle p}\downarrow & \text{adj} & \downarrow \\ Z & \longrightarrow & Z' \end{array} \qquad \begin{array}{ccc} \pi_1(W,*) & \longrightarrow & \pi_1(W',*) \\ {\scriptstyle p_*}\downarrow & \text{push} & \downarrow \\ \pi_1(Z,*) & \longrightarrow & \pi_1(Z',*) \end{array}$$

We compute

(1) $\pi_1(W',*) = 0 \Longrightarrow \pi_1(Z',*) \cong \pi_1(Z,*)/\operatorname{im} p_*$

This observation gives us a clue how to take care of requirement (i): There exists a path connected covering projection $p:(Y,*) \to (X,*)$ such that $\pi_1 p(\pi_1 Y) = H$. (See the appendix on CW-complexes)

(2) Now attach 2-cells to Y to kill $\pi_1 Y$.

Denote the resulting space by Y^2. Since (Y^2,Y) is a 2-cell adjunction, the inclusion $Y \to Y^2$ is a closed cofibration. Consequently the space X^2 in the following pushout diagram satisfies requirement (i) of theorem A.

$$\begin{array}{ccc} Y & \xrightarrow{\quad k \quad} & Y^2 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \dashrightarrow_{k'} & X^2 \end{array}$$

We take care of requirement (ii) as follows. Since Y^2 is simply connected, the pull back system P'^*G in Y^2 is simple (I.1.6), consequently the system $k^*p'^*G$ in Y is also simple. We shall attach 3-cells to Y^2 to obtain a map $j:Y^2 \to Y^3$ such that

$$(jk)_*: H_*(Y;\underset{\sim}{Z}) \xrightarrow{\cong} H_*(Y^3;Z)$$

are isomorphisms in all dimensions. Then define X^+ by the following pushout diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\ jk\ } & Y^3 \\
{\scriptstyle p}\downarrow & & \downarrow \\
X & \dashrightarrow_{i} & X^+
\end{array}
$$

According to (1), we have $\pi_1(X^+,*) \cong \pi_1(X^2,*)$ and the commuting homology ladders of the triples (Y^3,Y^2,Y) respectively (X^+,X^2,X), combined with a universal coef - ficient theorem, will help us to verify the isomorphism properties of i_*.

So let's analyse the homology situation of the map $k_*: H_pY \to H_pY^2$. According to (I.5.2), k induces isomor - phisms except possibly in dimensions 1 and 2. The only non trivial part of the long exact homology sequence of the pair (Y^2,Y) being

$$0 \to H_2Y \to H_2Y^2 \to H_2(Y^2,Y) \to H_1Y \to H_1Y^2 \to 0$$

Using the Hurewicz isomorphism theorem and the fact

(3) $\quad \pi_1 Y \cong H \quad$ which is perfect,

we see that $H_1Y \cong H_1Y^2 = 0$.

Turning our attention to homology in dimension 2, we observe that $H_2(Y^2,Y)$ is free abelian (by I.5.1) which gives us a split exact portion of the above sequence

$$0 \to H_2Y \to H_2Y^2 \to H_2(Y^2,Y) \to 0$$
$$\|$$
$$H_2Y \oplus H_2(Y^2,Y) \xleftarrow[h]{\cong} \pi_2 Y^2$$

Since $\pi_1 Y^2 = 0$, the Hurewicz homomorphism h is an isomorphism. As in (2), attach 3-cells to kill the direct summand $h^{-1}H_2(Y^2,Y)$ of $\pi_2 Y^2$. Denote the resulting space by Y^3 and $j:Y^2 \to Y^3$ the resulting inclusion. Then another exact sequence calculation shows:

(4) $(jk)_*:H_*Y \to H_*Y^3$ are isomorphisms.

As announced, we define X^+ to be the pushout of $X \xleftarrow{p} Y \xrightarrow{jk} Y^3$, and proceed to verify that X^+ and $i:X \to X^+$ satisfy conditions (i) and (ii) of theorem A.

(i) has been justified above.

(ii) Let $\pi_1(X^+,*)$ act on the abelian group A, i.e. A is a $Z\pi_1(X^+,*)$-module. According to (I.1.9), there is a system $G:\Pi X^+ \to Ab$ of local coefficients in X^+, unique up to natural equivalence, such that $G_* = A$ and G induces the action of $\pi_1(X^+,*)$ on A. We want to show that $i_*:H_*(X;i^*G) \to H_*(X^+;G)$ are isomorphisms.

The systems q^*G, $(jk)^*q^*G$ are simple in Y^3, Y. By (I.2.3) and the universal coefficient theorem, we have

(5) $$H_*(Y^3, Y; q^*G) \cong H_*(Y^3, Y; A) = 0$$

Now look at the following portion of the commuting exact homology ladder of triples (I.3.12):

$$\begin{array}{ccccccccc}
0 = H_{n+1}(Y^3,Y;q^*G) & \rightarrow & H_{n+1}(Y^3,Y^2;q^*G) & \xrightarrow{\cong} & H_n(Y^2,Y;q^*G) & \rightarrow & H_n(Y^3,Y;q^*G) = 0 \\
\downarrow & & \downarrow 1 & & \downarrow 2 & & \downarrow \\
\rightarrow H_{n+1}(X^+,X;G) & \longrightarrow & H_{n+1}(X^+,X^2;G) & \xrightarrow[3]{} & H_n(X^2,X;G) & \rightarrow & H_n(X^+,X;G) \rightarrow
\end{array}$$

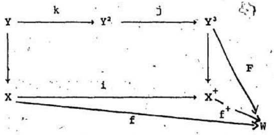
(6) Arrows 1 and 2 are isomorphisms.

By commutativity, arrow 3 is an isomorphism, which shows that $H_*(X^+,X;G) = 0$. Now the long exact sequence of $(X^+,X;G)$ shows that

$$i_*: H_*(X; i^*G) \longrightarrow H_*(X^+; G)$$

are isomorphisms. □

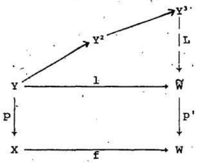
Proof of theorem B: According to the proof of theorem A, we know that $i: X \to X^+$ is part of the following pushout diagram



where $p: Y \to X$ is the covering projection satisfying $\pi_1 p(\pi_1 Y) = H$.

As indicated in the above diagram, we shall construct a map $F: Y^3 \to W$ such that $fp = Fjk$, which implies the existence of f^+ using the pushout property.

Since $\operatorname{im} \pi_1 p \subset \ker \pi_1 f$, it follows that $\pi_1(fp)$ is the 0-map, whence there exists a lifting $l: Y \to \widetilde{W}$ such that the rectangle in the following diagram commutes.



In this situation, we shall employ a piece of classical obstruction theory to extend l over the 2- and 3-skeleton of Y^3 to a map $L: Y^3 \to \widetilde{W}$ such that the above diagram commutes. Defining $F := p'L$, we see that this indeed satisfies the required commutativity properties.

Turning to the question whether $l: Y \to \widetilde{W}$ can be extended to $L:Y^3 \to \widetilde{W}$, we first observe that

(7) because $\widetilde{W}$ is simply connected, l can be extended to a map $l':Y^2 \to \widetilde{W}$.

(8) l can be extended over Y^3 if the obstruction cochain $c^3(l')$ is a coboundary in the cochain complex $(\mathrm{Hom}(H_{n+1}(Y^{n+1},Y^n),\pi_n\widetilde{W}))$, where $Y^1 := Y$, $Y^k := Y^3$ for $k \geq 3$.

To see that $c^3(l')$ is a coboundary, remember that $c^3(l') = (\pi_2 l')\partial h^{-1}$ in the following diagram

$$\begin{array}{ccccccc} H_3(Y^3,Y^2) & \xleftarrow[\cong]{h} & \pi_3(Y^3,Y^2) & \xrightarrow{\partial} & \pi_2 Y^2 & \xrightarrow{\pi_2 l'} & \pi_2\widetilde{W} \\ d\big\downarrow\cong & & & & & & \\ H_2(Y^2,Y) & & & & & & \end{array}$$

Here h denotes the appropriate Hurewicz homomorphism, ∂ the differential operator of the exact homotopy sequence of the pair (Y^3,Y^2) and d the boundary operator of the chain complex

$$\cdots\ 0 \to H_3(Y^3,Y^2) \xrightarrow{\quad d \quad} H_2(Y^2,Y) \to 0$$
$$H_3(Y^3,Y^2) \searrow H_2Y^2 \nearrow H_2(Y^2,Y)$$

which coincides with the isomorphism d in the long exact sequence of the triple (Y^3, Y^2, Y):

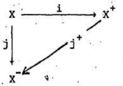
$$0 = H_3(Y^3, Y) \to H_3(Y^3, Y^2) \xrightarrow{d} H_2(Y^2, Y) \to H_2(Y^3, Y) = 0$$

Consequently, $c^3(1')d^{-1} \in C^2(Y^2, Y; \pi_2 \widetilde{W})$

and $\quad \delta_2(c^3(1')d^{-1}) = c^3(1')d^{-1}d = c^3(1')$

which shows that $c^3(1')$ is a coboundary, and the existence of $L: Y^3 \to \widetilde{W}$, such that $L_{|Y} = 1$ follows. □

Proof of Corollary C: Let $j: X \to X^-$ be a map satisfying the requirements (i), (ii) of theorem A, and let $i: X \to X^+$ be as in the construction in the proof of theorem A. By theorem B there exists a map $j^+: X^+ \to X^-$ such that the following triangle commutes.



But then we have from the hypotheses in theorem A

$$\pi_1 j^+ : \pi_1 X^+ \to \pi_1 X^-$$
$$j^+_* : H_*(X^+; j^*G) \to H_*(X^-; G)$$

are isomorphisms, where G is an arbitrary system of local coefficients in X^-. Now a Whitehead type theorem in terms of homology with local coefficients (I.5.7) implies that j^+ is a homotopy equivalence. □

Proof of theorem D: Construct $i:X \to X^+$ as in the proof of theorem A. Now apply theorem B to the map jf. □

Before we proceed to supplement the indicated parts in the preceeding discussion, let us digress momentarily to high light a certain technique in the proof of theorem A.

There we started with a space Y, killed its fundamental group by attaching 2-cells and compensated for the resulting change in homology dimension 2 by attaching 3-cells, which did not take any influence on the homology in dimension 3.

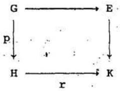
This turns out to be a special case of considerations due to K. Varadarajan [Va] 1966, concerning the existence of Moore spaces $M(\Pi,1)$ (a CW-complex X is called a Moore space $M(\Pi,n)$ iff $\pi_n X = \Pi$, $\pi_k X = 0$ for $k < n$, $H_k X = 0$ for $k > n$). In this paper it is shown that, given an abelian group Π, $M(\Pi,1)$ exists if and only if all homology classes of the 2-skeleton $K(\Pi,1)^2$ of the Eilenberg-Mac Lane space $K(\Pi,1)$ $(\pi_1 K(\Pi,1) = \Pi,$

$\pi_k K(\Pi,1) = 0$ for $k \neq 1$) are images of the Hurewicz homomorphism $\pi_2 K(\Pi,1)^2 \to H_2 K(\Pi,1)^2$. It is exactly then possible to attach 3-cells to $K(\Pi,1)^2$ to obtain $M(\Pi,1)$.

In our case, when proving theorem A, this presupposition was granted by the Hurewicz isomorphism theorem.

We shall now give details to the proofs of the preceeding discussion in the indicated ordering.

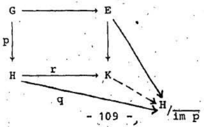
(1) Let G,H be groups; $p:G \to H$ a homomorphism. According to the appendix on general nonsense, the following pushout diagram exists in the category of groups and group homomorphisms.



Here E denotes the 1-element group.

Claim: $K \cong H/_{\text{im } p}$

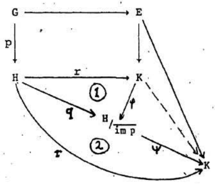
Proof: Consider the following diagram:



Here q is defined to be the canonical quotient homomorphism. Since E is the 1-element group, the unlabelled arrows are unique and, by commutativity of the square, we infer

$$\text{im } p \subset \overline{\text{im } p} = \ker q \subset \ker r$$

It follows that qp is trivial; the outer diagram commutes and the pushout property yields a unique homomorphism $\varphi: K \to {}^{H}/_{\overline{\text{im } p}}$ making the whole diagram commute. q is onto, hence φ is onto. Furthermore $\ker r \subset \ker q$ and thus $\ker r = \ker q = \overline{\text{im } p}$. A standard theorem of group theory gives us an isomorphism $\psi: {}^{H}/_{\overline{\text{im } p}} \to \text{im } r \subset K$ such that $\psi q = r$. Now consider the following diagram.



The outer diagram clearly commutes, giving rise to a unique map $\gamma: K \to K$ such that $\gamma r = r$. It follows $\gamma = 1_K$.

On the other hand, we know that triangles 1 and 2 commute, which implies

$$q = \varphi r \quad \text{and} \quad r = \psi q$$

thus

$$(\psi\varphi) r = r$$

and by uniqueness, we have $\psi\varphi = \gamma = 1_K$. But then φ is a monomorphism.

We have shown that φ is mono and epi. □

(2) Let $(Y,*)$ be 0-connected, $K \subset \pi_n(Y,*)$ a sub-group $(n \geq 1)$.

<u>Claim:</u> If $n = 1$, we can attach 2-cells to Y to obtain a space $(Y',*)$ such that $\pi_1(Y',*) \cong \pi_1(Y,*)/\overline{K}$, where $\overline{K}$ denotes the normal closure of K in $\pi_1(Y,*)$.

If $n \geq 2$ and $(Y,*)$ simply connected, we can attach $(n+1)$-cells to Y to obtain a space $(Y',*)$ such that $\pi_n(Y',*) \cong \pi_n(Y,*)/K$.

Furthermore, in either case, $(Y',*)$ contains $(Y,*)$ as a closed subspace. If $\overline{1}:(Y,*) \to (Y',*)$ denotes the inclusion, then $\pi_n\overline{1}$ is an epimorphism with kernel $\overline{K}$.

In the construction of Y', we make essential use of the following fact.

(2.1) Let $\varphi:(S^n,*) \to (Y,*)$ be a map and define $(Y',*)$ by the following adjunction square.

$$\begin{array}{ccc}
(S^n,*) & \xrightarrow{\ i\ } & (B^{n+1},*) \\
\varphi \downarrow & & \downarrow \chi \\
(Y,*) & \xrightarrow[\ \bar{i}\ ]{} & (Y',*)
\end{array}$$

Then $\bar{i}:(Y,*) \to (Y',*)$ is an inclusion and $\bar{i}\circ\varphi$ is homotopically trivial.

Proof: Since i is homotopically trivial, $\chi\circ i = \bar{i}\circ\varphi$ is homotopically trivial. □

Accordingly, we can kill K as follows. Let S be a set of generators for K. For each $s \in S$ choose a map $\varphi_s:(S^n,*) \to (Y,*)$ representing the homotopy class s. Now define $(Y',*)$ by the following adjunction square.

$$\begin{array}{ccc}
\bigvee_{s\in S}(S^n_s,*) & \xrightarrow{\ i\ } & \bigvee_{s\in S}(B^{n+1}_s,*) \\
\bigvee\varphi_s \downarrow & & \downarrow \\
(Y,*) & \xrightarrow[\ \bar{i}\ ]{} & (Y',*)
\end{array}$$

We claim that $\pi_n(Y',*) \cong \pi_n(Y,*)/\bar{K}$

For $n = 1$, this follows from (1) above, because

$\vee(B^{n+1}_s, *)$ is contractible.

For $n \geq 2$, consider the following commuting diagram:

$$\begin{array}{ccccccccc}
H_{n+1}(Y',Y) & \xleftarrow{h} & \pi_{n+1}(Y',Y) & \xrightarrow{\partial} & \pi_n Y & \xrightarrow{\pi_n \bar{i}} & \pi_n Y' & \to & \pi_n(Y',Y) \\
 & & & & \cup & & & & \\
 & & & & K & & & & \\
\uparrow \cong & & k \uparrow \cong & & \uparrow 1 & & \uparrow & & \\
H_{n+1}(\vee B^{n+1}_s, S^n_s) & \underset{h'}{\overset{\cong}{\longleftarrow}} & \pi_{n+1}(\vee B^{n+1}_s, S^n_s) & \underset{\partial'}{\overset{\cong}{\longrightarrow}} & \pi_n \vee S^n_s & \to & \pi_n \vee B^{n+1}_s = 0 & &
\end{array}$$

The top row is exact, being a portion of the exact sequence of the pair (Y',Y). By the cellular approximation theorem, $\pi_n \bar{i}$ is an epimorphism and consequently:

$$\pi_n(Y',*) \cong \pi_n(Y,*) \big/ \ker \pi_n \bar{i}$$

By construction, the image of homomorphism 1 is K. By commutativity of the middle square:

$$K \subset \ker \pi_n \bar{i} = \operatorname{im} \partial$$

The horizontal arrows in the left square are isomorphisms by the Hurewicz isomorphism theorem (here we use the assumption that Y is simply connected). Since the left vertical arrow is also an isomorphism, we infer by commutativity that k is an isomorphism. But then $\partial = 1 \circ \partial' k^{-1}$, which implies $K = \operatorname{im} \partial = \ker \pi_n \bar{i}$ and our claim follows. □

(3) By construction, $\mathrm{im}(\pi_1 p) = H$. Using the long exact sequence of a Serre fibration (see the appendix on covering spaces), $\pi_1 p$ is a monomorphism. This shows that $\pi_1 p$ is an isomorphism. □

(4) Let us recollect our situation more precisely. According to (2), we chose a basis S of homotopy classes for the free abelian direct summand $h^{-1}(H_2(Y^2,Y))$ of $\pi_2 Y^2$ due to the diagram

$$H_2 Y \oplus H_2(Y^2,Y) \cong H_2 Y^2 \xleftarrow{h} \pi_2 Y^2$$

where h denotes the Hurewicz isomorphism (Y^2 is simply connected). In turn we used for each $s \in S$ a representative $\varphi_s : (S^2,*) \to (Y^2,*)$ as attaching map for a 3-cell to kill s. Now, we want to show that the inclusion $jk: (Y,*) \to (Y^3,*)$ induces isomorphisms in integral homology in all dimensions.

Since we only attached 2- and 3-cells to Y, a theorem concerning the homology of a cell adjunction (I.5.1) tells us that we need only check dimensions 1,2,3.

Dimension 1 is trivial since $H_1 Y = H_1 Y^3 = 0$.

Dimension 2: We know that

$$k: H_2 Y \to H_2 Y \oplus H_2(Y^2, Y) \cong H_2 Y^2$$

is an isomorphism on the direct summand $H_2 Y$ of $H_2 Y^2$. To see the effect of $j:(Y^2,*) \to (Y^3,*)$, consider the following commuting diagram of the Hurewicz isomorphism:

$$\begin{array}{ccc} \pi_2(Y^2,*) & \xrightarrow{\pi_2 j} & \pi_2(Y^3,*) \\ h \downarrow \cong & & \downarrow \cong \\ H_2 Y^2 & \longrightarrow & H_2 Y^3 \end{array}$$

The result in (2) tells us that $\pi_2 j$ is an epimorphism with kernel $h^{-1}(H_2(Y^2,Y))$, which implies that jk induces an isomorphism in dimension 2.

Dimension 3: We know that $k_*: H_3 Y \to H_3 Y^2$ is an isomorphism and are left to show that $j_*: H_3 Y^2 \to H_3 Y^3$ is an isomorphism. To do this, consider the following commuting diagram.

$$\begin{array}{ccccccccc} 0 \to H_3 Y^2 & \xrightarrow{j_*} & H_3 Y^3 & \xrightarrow{1} & H_3(Y^3, Y^2) & \xrightarrow{\partial} & H_2 Y^2 \to \cdots \\ \uparrow & & \uparrow & & \uparrow \cong & & \uparrow 2 \\ 0 \to H_3 \bigvee_{s \in S} (S^2_s,*) & \to & H_3 \bigvee_{s \in S} (B^3_s,*) & \to & H_3(\bigvee_{s \in S} B^3, \bigvee_{s \in S} S^2) & \xrightarrow{\cong} & H_2 \bigvee_{s \in S} S^2 \to 0 \\ & & \| & & & & \\ & & 0 & & & & \end{array}$$

In this diagram the vertical arrows are induced by the attaching maps φ_s and the resulting characteristic maps and we have:

j_* is an isomorphism <-> arrow 1 is the 0-map
<-> ∂ is a monomorphism
<-> arrow 2 is a monomorphism.

We show that arrow 2 is a monomorphism by considering the following commuting Hurewicz diagram:

$$\begin{array}{ccc} \pi_2 Y^2 & \xrightarrow[\cong]{h} & H_2 Y^2 \\ {\scriptstyle 1}\uparrow & & \uparrow{\scriptstyle 2} \\ \pi_2 \bigvee_{s\in S} S^2_s & \xrightarrow{\cong} & H_2 \bigvee_{s\in S} S^2_s \end{array}$$

Again the vertical arrows are induced by attaching maps. By construction, arrow 1 is induced by a bijection between the bases of the free abelian groups $\pi_2 \bigvee S^2_s$ and $h^{-1}(H_2(Y^2,Y))$. Hence 1, and consequently 2, are monomorphisms. □

(5) The indicated equivalence follows from I.2.3. To see $H_*(Y^3,Y;A) = 0$, we first consider the long exact sequence

of the pair (Y^3,Y) with integral coefficients. Since the inclusion $Y \to Y^3$ induces isomorphisms in all dimensions, $H_*(Y^3,Y) = 0$. Now the universal coefficient theorem implies $H_*(Y^3,Y;A) = 0$. □

(6) We show that arrow 1 is an isomorphism, then arrow 2 will be an isomorphism for exactly the same reason.

Now, the pair (Y^2,Y) is a 2-cell adjunction, and from the following diagram we see that (X^2,X) is also a 2-cell adjunction. For the lower rectangle is a pushout diagram by construction. In this situation lemma 4 in the appendix on general nonsense implies that the outer rectangle is also a pushout diagram and hence an adjunction square.

$$\begin{array}{ccc}
\coprod_{\lambda\in\Lambda} S^1_\lambda & \hookrightarrow & \coprod_{\lambda\in\Lambda} B^2_\lambda \\
\varphi\downarrow & & \downarrow\chi \\
Y & \xrightarrow{k} & Y^2 \\
p\downarrow & & \downarrow q' \\
X & \xrightarrow[\bar{k}]{} & X^2
\end{array}$$

Here φ denotes the coproduct of attaching maps, χ the coproduct of characteristic maps. But this means that the coproduct of characteristic maps for the 2-cell adjunction (X^2,X) is $q'\circ\chi$, giving rise to the commuting triangle of morphisms in $\mathcal{L}$:

$$\begin{array}{ccc} & & (Y^2,Y;q'^*G_{|X^2}) \\ & \overset{1}{\nearrow}\cong & \\ \left(\coprod_{\lambda\in\Lambda} B^2_\lambda,\ \coprod_{\lambda\in\Lambda} S^1_\lambda\ ;(q'\chi)^*G_{|X^2}\right) & & \downarrow (q',1) \\ & \underset{2}{\searrow}\cong & \\ & & (X^2,X;G_{|X^2}) \end{array}$$

By I.5.1, arrows 1 and 2 induce isomorphisms in homology with local coefficients, and our claim follows by commutativity. □

(7) This is an application of the fundamental idea of obstruction theory, which is most advantageously explained by verifying the following

(7.1) Lemma: Let X be a space to which an $(n+1)$-cell is attached $(n\geq 0)$ by the map $\varphi:S^n\to X$ and characteristic map $\chi:B^{n+1}\to X\cup_\varphi e^{n+1}$. Let $f:X\to Z$ be a map. Then:

f can be extended to a map $F: X \cup_\varphi e^{n+1} \to Z$ satisfying $F_{|X} \equiv f$ iff $f \circ \varphi$ is homotopically trivial.

Proof: "=>" Let F be an extension, then

$$\phi: S^n \times I \ni (s,t) \mapsto F \circ \chi((1-t)s) \in Z$$

is a trivializing homotopy.

"<=" Given a trivializing homotopy $\phi: S^n \times I \to Z$, define

$$F: X \cup_\varphi e^{n+1} \ni \zeta \mapsto \left\{ \begin{matrix} f(\zeta) & \zeta \in X \\ \phi\left(\frac{\chi^{-1}(\zeta)}{|\chi^{-1}(\zeta)|}, 1 - |\chi^{-1}(\zeta)|\right) & \zeta \in e^{n+1},\ \chi^{-1}(\zeta) \neq 0 \\ \phi(.,1) & \zeta \in e^{n+1},\ \chi^{-1}(\zeta) = 0 \end{matrix} \right\} \in Z$$

which is easily seen to be a continuous extension of f. □

Returning to the situation in the main text, we are facing the question whether $1: Y \to \widetilde{W}$ can be extended to a map $1': Y^2 \to \widetilde{W}$. By the previous lemma, this is reduced to the question as to whether or not $1 \circ \varphi: S^1 \to \widetilde{W}$ is contractible, where φ denotes an attaching map for some 2-cell of Y^2. But this is granted because $\widetilde{W}$ is 1-connected.

(8) The idea just explained can be formalized as follows. Let (Y,X) be an $(n+1)$-cell adjunction with characteristic maps $\{\varphi_\lambda: (S^n_\lambda,*) \to (X,*)\}_{\lambda \in \Lambda}$ and let

$f:(X,*) \to (Z,*)$ be a map. Then $\varphi_\lambda \mapsto f\circ\varphi_\lambda$ defines a function from the basis of the free abelian group $H_{n+1}(Y,X)$ into $\pi_n(Z,*)$ which extends to a homomorphism

$$c^{n+1}(f):H_{n+1}(Y,X) \to \pi_n(Z,*)$$

We interpret $c^{n+1}(f)$ as a cochain (the obstruction cochain to extending f over the $(n+1)$-cell adjunction) of

$$\Gamma^{n+1}(Y,X;\pi_n(Z,*)) = \mathrm{Hom}(\Gamma_{n+1}(Y,X),\pi_n(Z,*))$$

Here we make use of the fact that (Y^k = k-skeleton)

$$\begin{array}{ccccc}
\Gamma^{k+1}(Y^{k+1},Y^k) & & \Gamma^k(Y^k,Y^{k-1}) & & \Gamma^{k-1}(Y^{k-1},Y^{k-2}) \\
\| & & \| & & \| \\
H^{k+1}(Y^{k+1},Y^k) & \xrightarrow{d_{k+1}} & H^k(Y^k,Y^{k-1}) & \xrightarrow{d_k} & H^{k-1}(Y^{k-1},Y^{k-2}) \\
 & \searrow \quad \nearrow & & \searrow \quad \nearrow & \\
 & H^kY^k & & H^{k-1}Y^{k-1} &
\end{array}$$

is a chain complex satisfying

$$H^k(Y,X) \cong H^k(\Gamma^*(Y,X),d^*)$$

For more information on this, see G. Whitehead "Elements of Homotopy Theory" [Wh1] chap.II sec. 2 .

With this information, we can understand the theorem that was used in the main text:

(8.1) Let (Y,X) be a relative CW-complex; $f:Y^n \to Z$ a map. Then $f_{|Y^{n-1}}$ can be extended over Y^{n+1} iff $c^{n+1}(f)$ is a coboundary.

For more information on obstruction theory, as well as a proof of (8.1), see [Wh1] chap. V sec. 5 .

□

APPENDIX I

General Nonsense and Homological Algebra

We assume the reader to know what a category and a functor is and to be familiar with elementary concepts of Homological Algebra, such as chain complexes and the (co)homology groups of a given chain complex with coefficients in an abelian group G. On this basis we collect material out of Category Theory and Homological Algebra that is needed in the main text. For more detailed information, we refer the reader to. S.MacLane "Categories for the Working Mathematician" [ML1], S.MacLane "Homology" [ML2] and P.Hilton, U.Stammbach "Homological Algebra" [H-S].

First some notation. We write

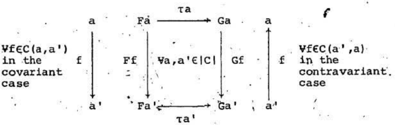
$$a \in |C|$$

to denote an object a in the category C, and

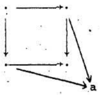
$$C(a,a')$$

to denote the class of morphisms with domain $a \in |C|$ and codomain $a' \in |C|$.

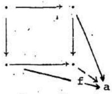
If C,D are categories and $F,G:C \to D$ functors of the same type (i.e. either both covariant or both contravariant), we denote by $\tau:F \dot{\to} G$ a natural transformation of F into G . Frequently we express the defining property of a natural transformation $\tau:F \dot{\to} G$ by commativity requirements of the following diagram:



(1) Definition: Let C be a category. A commuting square of objects (in vertices) and morphisms (connecting the vertices) in C is called a pushout diagram (in C) iff for all $a \in |C|$ and pairs of arrows such that



... commutes, there exists a unique arrow f such that ...



commutes.

(2) Proposition: Pushouts exist in Top, the category of topological spaces and continuous maps.
Proof: Starting with $Y \xleftarrow{f} A \xrightarrow{g} X$ in Top, we define $P \in |\mathrm{Top}|$ by

$$P := X \sqcup Y \Big/ \xi \sim \xi' \iff \xi = \xi' \vee \exists a \in A : (fa = \xi \wedge ga = \xi') \vee (ga = \xi \wedge fa = \xi')$$

with the quotient topology. If $q : X \sqcup Y \to P$ is the quotient map, then

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q|_X} \\ Y & \xrightarrow[q|_Y]{} & P \end{array}$$

is a pushout diagram in Top. □

(3) Proposition: Pushouts exist in Grp, the category of groups and group homomorphisms.
Proof: Starting with $H_1 \xleftarrow{h_1} G \xrightarrow{h_2} H_2$ in Grp, we define $P := H_1 *_G H_2$ (the amalgamated product of H_1, H_2 along G) as follows.

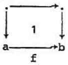
Let F be the free group over the disjoint union of the sets H_1, H_2 , and let N be the intersection of all normal subgroups of F containing all elements of the following types

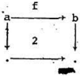
a) $(\alpha\beta)\beta^{-1}\alpha^{-1}$ if $\alpha, \beta \in H_1$

b) $(\mu\nu)\nu^{-1}\mu^{-1}$ if $\mu, \nu \in H_2$

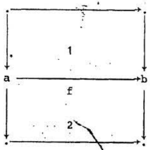
c) $h_1(g)\, h_2(g^{-1})$ $\forall$ $g \in G$

Now define $P := F/N$. □

(4) Lemma: Let 

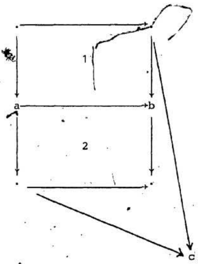
, 

be pushout diagrams in C , then the outer rectangle of

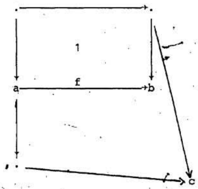


is also a pushout diagram in C.

Proof: Suppose

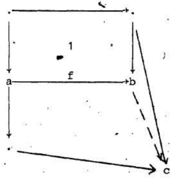


commutes, then automatically



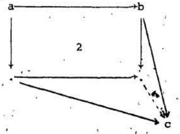
commutes. Consequently the pushout property of the upper rectangle yields a unique arrow b → c such that

(*)



commutes. The existence of an arrow . → c , such that

(**)



commutes follows. This arrow also makes the whole diagram commute, so we are left to show the uniqueness of it. But each such arrow gives rise to an arrow b → c making diagram (*) commute. By uniqueness this must be the arrow, we already found. Then $\to c$ makes diagram (**) commute, hence must be the arrow we constructed above. This shows uniqueness. □

(5) Definition: Let C be a category. C is a groupoid iff all arrows in C are invertible. As an immediate consequence, we observe: If $f \in C(a,b)$, for a groupoid C, then f is an isomorphism.
Furthermore: If $C(a,a)$ is a proper set, then $(C(a,a),\circ)$ is a group, where "$\circ$" denotes composition of morphisms.

We turn to some material in homological algebra.

(6) The integral group ring: Let G be a group written multiplicatively. The integral group ring $\mathbb{Z}G$ of G is defined as follows. Use the set G as the basis for the free abelian group $\mathbb{Z}G$ which has a multiplication, derived from the multiplication of its basis elements, turning it into a ring:

$$\left(\sum_{g\in G} m(g)g\right)\cdot\left(\sum_{h\in G} n(h)h\right) := \sum_{g,h\in G} [m(g)n(h)]gh$$

According to the definition of a free abelian group, these sums are actually finite. The group ring is characterized by the following universal property.

(6.1) Let R be a ring, $f: G \to R$ such that $f(gg') = f(g)f(g')$ and $f(1) = 1_R$. Then, there exists a unique ring homomorphism $f': \mathbb{Z}G \to R$ such that the following triangle commutes

$$\begin{array}{ccc} \mathbb{Z}G & \overset{f'}{\dashrightarrow} & R \\ \uparrow & \nearrow_{f} & \\ G & & \end{array}$$

where the inclusion $G \to \mathbb{Z}G$ maps g into $1g$. For more details, see also [H-S] VI.1 .

(7) Actions of a group on a set: Let G be a group, X a set. A function $G \times X \ni (g,x) \to g.x \in X$ is called a left action of G on X iff

$$1.x = x \quad \text{and} \quad g.(g'.x) = (gg').x$$

for all $x \in X$, $g, g' \in G$. Equivalently, such an action can be described by a group homomorphism $\varphi: G \to S(X)$ (where $S(X)$ denotes the symmetric group of X), by setting

$$g.x := [\varphi(g)]x$$

Certain structures on X give rise to certain subgroups of $S(X)$, such as the group of homeomorphisms if X is a topological space or the group of automorphisms if X is an algebraic system. Often one restricts the image of φ to such a subgroup to act on X.

Similarly, we define a right action of G on X. Note that a left action can be converted into a right action by setting $x.g^{-1} := g.x$.

(8) Modules: Let M be an abelian group; R a ring with unity. M is a left R-module iff there is a scalar multiplication $R \times M \to M$ on the left satisfying

$$(r + r')m = rm + r'm \qquad r(m + m') = rm + rm'$$

$$(rr')m = r(r'm)$$

Similarly, right R-modules are defined by a scalar multiplication $M \times R \to M$.

One way to visualize modules, is as generalized vector spaces. The scalars are ring elements rather than field elements. Yet, there is still another way which occurs frequently.

Let G be a group acting on the abelian group A by automorphisms (on the left), then A is a (left)

$\mathbb{Z}G$-module in a canonical way:

$$\Big[\sum_{g\in G} m(g)g\Big]a := \sum_{g\in G} m(g)[g.a]$$

(8.1) Let M, M' be left R-modules. A function $h:M \to M'$ is called a module homomorphism (or linear) iff

$$h(rm + r'm') = rh(m) + r'h(m')$$

for all $r,r' \in R$, $m,m' \in M$. Thus, left R-modules and module homomorphisms form a category.

(9) Tensor Products: Let N be a right R-module, M a left R-module; A an abelian group.

(9.1) A function $f:N\times M \to A$ is called a middle map iff

$$f(n+n',m) = f(n,m) + f(n',m)$$
$$f(n,m+m') = f(n,m) + f(n,m')$$
$$f(nr,m) = f(n,rm)$$

for all $n,n' \in N$; $m,m' \in M$; $r \in R$.

(9.2) Let $F(N,M)$ be the free abelian group having the set $N\times M$ as a basis. Let $R(N,M)$ be the subgroup of $F(N,M)$ generated by the following elements

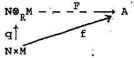
$$(n+n',m) - (n,m) - (n',m) \qquad (n,m+m') - (n,m) - (n,m')$$
$$(n,rm) - (nr,m)$$

for $n, n' \in N$; $m, m' \in M$; $r \in R$. The tensor product of N, M over R, denoted by $N \otimes_R M$, is defined to be the quotient

$$N \otimes_R M := F(N,M) / R(N,M)$$

The canonical map $N \times M \to F(N,M) \to N \otimes_R M$ is then a middle map.

(9.3) With respect to the above map $q: N \times M \to N \otimes_R M$, the tensor product has the following universal property. Given an abelian group A and a middle map $f: N \times M \to A$, there exists a unique group homomorphism $F: N \otimes_R M \to A$ such that the following triangle commutes.



(9.4) Let N, N' be right R-modules, M, M' left R-modules; $f: N \to N'$, $g: M \to M'$, $q': N' \times M' \to N' \otimes_R M'$.

$$q' \circ (f \times g): N \times M \ni (n,m) \mapsto fn \otimes_R gm \in N' \otimes_R M'$$

is a middle map giving rise to a unique group homomorphism

$$f \otimes_R g: N \otimes_R M \to N' \otimes_R M'$$

This follows from (9.2) and (9.3) and the fact that the composite of a middle map with the product of two module homomorphisms is again a middle map.

(10) Homology with (left) R-modules: A chain complex (M,∂) of (left) R-modules is a sequence

$$\cdots \to M_{p+1} \xrightarrow{\partial_{p+1}} M_p \xrightarrow{\partial_p} M_{p-1} \to \cdots$$

of (left) R-modules M_p and module homomorphisms ∂_p satisfying $\partial_{p+1}\partial_p = 0$. The definition of the n-th homology group of such a chain complex is formally the same as the definition of the n-th homology group in a chain complex of abelian groups. Furthermore, the homology of (M,∂) with coefficients in the right R-module N is the homology of the chain complex

$$\cdots \to N\otimes_R M_{p+1} \xrightarrow{1_N \otimes_R \partial_{p+1}} N\otimes_R M_p \xrightarrow{1_N \otimes_R \partial_p} N\otimes_R M_{p-1} \to \cdots$$

(11) The functor Tor_n^R: From (9.3) and (9.4) we can deduce that $\otimes_R$ is a bifunctor from the product of the category of right R-modules with the category of left R-modules into the category of abelian groups which is covariant in both variables. It is of crucial interest in homological algebra that $N\otimes_R\cdot$ does not necessarily yield exact sequences of abelian groups out of exact sequences of left R-modules.

In the special case of applying $N\otimes_R\cdot$ to a positive exact sequence of free left R-modules M_p,

$$\ldots \to M_p \to \ldots \to M_1 \to M_0 \to \overline{M} \to 0 \to 0 \ldots$$

it can be shown that the deviation from exactness of the resulting sequence depends only on N and $\overline{M}$ and is measured by

$$\mathrm{Tor}_n^R(N,\overline{M}) := H_n(N \otimes_R M)$$

where M denotes the chain complex $\ldots M_p \to \ldots \to M_0 \to 0$.

The sequence M is also called a free resolution of $\overline{M}$. If M' is another free resolution of $\overline{M}$, the above statement says that Tor_n^R still yields the same groups.

APPENDIX II

The Fundamental Groupoid

In this section, we briefly introduce the concept of fundamental groupoid which can be used to formulate a very sharp version of the Seifert-Van Kampen theorem. This material and much more, is extensively treated in R. Brown "Elements of Modern Topology" [B].

Let X be a space; $x,x' \in X$; $\alpha, \alpha' : I \to X$ such that $\alpha(0) = \alpha'(0) = x$ and $\alpha(1) = \alpha'(1) = x'$.

(1) Convention: We call α homotopic to α' iff there is a homotopy rel ∂I of α to α'.
Thus two paths α, α' in X are homotopic iff the start point of α is the start point of α', the end point of α is the end point of α' and α can be deformed into α' the boundary points being fixed throughout the homotopy.

(2) Notation: For a space X and $x,x' \in X$, we denote the set of homotopy classes of paths starting at x and ending at x' by $\pi_1(X,x,x')$ or also by $\Pi X(x,x')$. If $x = x'$, we write for short $\pi_1(X,x) := \pi_1(X,x,x')$, which is nothing but the fundamental group of X based at x.
Note: $\pi_1(X,x,x') = \emptyset$ iff x,x' belong to different path components of X.

(3) Observation: If $x, x', x'' \in X$, we have a function

$$\pi_1(X,x,x') \times \pi_1(X,x',x'') \to \pi_1(X,x,x'')$$

by taking composites in the following way. For $\zeta \in \pi_1(X,x,x')$, $\zeta' \in \pi_1(X,x',x'')$, represented by $\alpha, \alpha' : I \to X$, let $\zeta\zeta' \in \pi_1(X,x,x'')$ be represented by

$$\alpha\alpha' = \gamma : I \ni t \longrightarrow \begin{Bmatrix} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \alpha'(2t-1) & \frac{1}{2} \le t \le 1 \end{Bmatrix} \in X$$

In doing so, we actually hold the key for defining a (small) category ΠX whose objects are the elements of X and whose morphisms are homotopy classes of paths in X. Clearly, the identity morphism on an object $x \in |\Pi X|$ is just the identity element of the fundamental group of X based at x.

(4) Observation: Each morphism in ΠX is invertible. For, let $\zeta \in \Pi X(x,y)$ be represented by $\alpha : I \to X$, then

$$\overline{\alpha} : I \ni t \mapsto \alpha(1-t) \in X$$

represents a morphism $\overline{\zeta} \in \Pi X(y,x)$, which is an inverse of ζ. This means that the category ΠX is a groupoid.

(5) ΠX is called the fundamental groupoid of X.

Making systematic use of the notion of fundamental groupoid enables one to obtain the fundamental groupoid of a pushout of spaces as a pushout of fundamental groupoids, provided some reasonable assumptions are satisfied. See [B] sec. 8.4. As a special case of these results, we obtain a sharpened version of the classical Seifert-Van Kampen theorem.

(6) Seifert-Van Kampen Theorem: Let $(A,*)$, $(X,*)$, $(Y,*)$ be based path connected spaces; $i:(A,*) \to (X,*)$ a closed cofibration, $f:(A,*) \to (Y,*)$ a map. Then the following pushout of spaces and base point preserving maps gives rise to a pushout of fundamental groups under the functor π_1.

$$\begin{array}{ccc} (A,*) & \xrightarrow{\;i\;} & (X,*) \\ {\scriptstyle f}\downarrow & \text{push} & \downarrow \\ (Y,*) & \longrightarrow & (X \sqcup_f Y,*) \end{array}$$

yields

$$\begin{array}{ccc} \pi_1(A,*) & \xrightarrow{\;\pi_1 i\;} & \pi_1(X,*) \\ {\scriptstyle \pi_1 f}\downarrow & \text{push} & \downarrow \\ \pi_1(Y,*) & \longrightarrow & \pi_1(X \sqcup_f Y,*) \end{array}$$

APPENDIX III

Covering Spaces

We assume the reader to be familiar with the concept of covering space. Here we merely collect some of the material that is used frequently in the main body of the text. For more information, the reader is referred to the appropriate sections in R. Brown "Modern Topology" [B], J. Munkres "Topology" [Mu] or E. Spanier "Algebraic Topology" [Sp].

Let E, B be topological spaces, $p: E \to B$ a continuous surjective map.

(1) Definition: (i) The open set $U \subset B$ is evenly covered by p iff the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_λ in E such that for each λ the restriction of p to V_λ is a homeomorphism of V_λ onto U.

(ii) E is called a covering space for B and p a covering map (or covering projection) iff every point $b \in B$ has an open neighborhood $U \subset B$ such that U is evenly covered by p.

(iii) E is a universal covering space for B if E is simply connected.

In order to formulate conditions for the existence of covering projections, we need the following two connectedness concepts.

(2) Definition: A space X is locally path connected iff given any point $x \in X$ and open neighborhood $U \subset X$ of x, there exists a pathconnected open neighborhood $V \subset X$ of x such that $x \in V \subset U$ holds.

(3) Definition: A space X is semilocally simply connected iff every $x \in X$ has an open neighborhood U of x such that the homomorphism $\pi_1(U,x) \to \pi_1(X,x)$ induced by the inclusion is trivial.

(4) Theorem: Let B be path connected and locally path connected. Let $b_0 \in B$. Let $H \subset \pi_1(B,b_0)$ be a subgroup. Then the following two statements are equivalent

(i) B is semilocally simply connected.

(ii) There exists a path connected covering space $p: E \to B$, and $e_0 \in p^{-1}(b_0)$ such that $\pi_1 p(\pi_1(E,e_0)) = H$.

(5) "Lifting Lemma": Let $p: E \to B$ be a covering map; $p(e_0) = b_0$. Let $f: (X,*) \to (B,b_0)$ be a based map, X a path connected and locally path connected space. Then the following two statements are equivalent.

(i) $\pi_1 f(\pi_1(X,*)) \subset \pi_1 p(\pi_1(E,e_0))$

(ii) There exists a lifting of f, $\tilde{f}:(X,*) \to (E,e_0)$ such that $p\tilde{f} = f$.

Furthermore, if such a lifting $\tilde{f}$ exists, it is unique.

Of particular interest is the case of $p:\tilde{X} \to X$ being a universal covering. If $p\tilde{x}_1 = p\tilde{x}_2 = x \in X$, using the simply connectedness of $\tilde{X}$, the lifting lemma gives us the existence of a unique map $\tilde{f}:(\tilde{X},\tilde{x}_1) \to (\tilde{X},\tilde{x}_2)$ such that $p\tilde{f} = p$.

This observation gives rise to the following

(6) Definition: Let $p:E \to B$ be a covering map, B path connected. Let $f:E \to E$ be a map. We call f a covering transformation iff $pf = p$.

(7) Theorem: Let $p:\tilde{X} \to X$ be a universal covering, $\tilde{X}$ path connected and locally path connected. Let T denote the set of covering transformations. Then

(i) T is a group whose elements are homeomorphisms of $\tilde{X}$.

(ii) $T \cong \pi_1 X$

The lifting lemma implies in particular that a covering map $p:E \to B$, B path connected is a Serre fibration (i.e. p has the homotopy lifting property with respect to the cubes I^n, $n \in \mathbb{N}$). Thus the long exact sequence of a Serre fibration yields:

(8) Theorem: Let $p:(E,e_0) \to (B,b_0)$ be a covering map, B 0-connected, then the following sequence is exact.

$$\cdots \to \pi_{n+1}(B,b_0) \to 0 \to \pi_n(E,e_0) \xrightarrow{\pi_n p} \pi_n(B,b_0) \to$$
$$\to 0 \to \pi_{n-1}(E,e_0) \to \cdots \to \pi_1(B,b_0)$$

For more information on this matter, see B. Gray "Homotopy Theory", [G] pp 79 - 85.

APPENDIX IV

CW Complexes

The notion of a CW-complex is due to J. H. C. Whitehead and was first published in his famous paper "Combinatorial Homotopy" in 1949 [Wh2]. Here, we give a more modern definition (still describing the same spaces) and give a collection of definitions and theorems that are needed in the main body of the text. For more information on this subject the reader is referred to J. H. C. Whithehead's original paper [Wh2], B. Gray "Homotopy Theory" [G] or a book to appear by R.Fritsch and R. Piccinini on CW-complexes.

It is a pleasure to use the opportunity to express my appreciation to the latter author, R. Piccinini. Much of my understanding of CW-complexes has its provenance in a course taught by him at Memorial University of Newfoundland, fall 1982.

An n-cell e^n $(n \geq 0)$ is a space homeomorphic to $\mathring{B}^n$, the interior of the compact ball

$$B^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$$

with respect to the surrounding space $\mathbb{R}^n$. B^n, $\mathring{B}^n$ have the (n-1)-sphere S^{n-1} as a common boundary.

"Attaching n-cells to a space X", means the following: Given a space X, a family $\{S_\lambda^{n-1}\}_{\lambda\in\Lambda}$ of copies of S^{n-1} $(n \geq 1)$ and a family of continuous maps $\{\varphi_\lambda : S_\lambda^{n-1} \to X\}_{\lambda\in\Lambda}$ ("attaching maps"), we define a space Y, obtained from X by attaching a family of n-cells $\{e_\lambda^n\}_{\lambda\in\Lambda}$, to be the quotient space

$$Y := X \coprod (\coprod_{\lambda\in\Lambda} B_\lambda^n) \Big/ \sim$$

where "$\coprod$" denotes the disjoint union of spaces with the appropriate topology and "$\sim$" denotes the following equivalence relation on $X \coprod (\coprod B_\lambda^n)$ generated by the relation

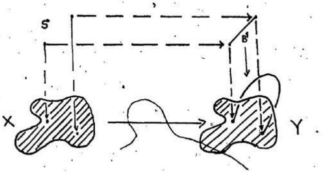
$$x \sim y \quad \text{if there exists } \lambda\in\Lambda \text{ such that } x \in S_\lambda^{n-1} \text{ and } y = \varphi_\lambda(x)$$

Thus, Y can be visualized as the space X with n-cells e_λ^n attached to it by gluing the boundary points S_λ^{n-1} of B_λ^n to X by means of the attaching maps φ_λ.

The space Y, so defined, makes the following diagram into an adjunction square, i.e. a pushout diagram in the category of topological spaces such that the top horizon -

tal arrow is a closed cofibration.

$$\begin{array}{ccc} \coprod_{\lambda\in\Lambda} S^{n-1}_\lambda & \hookrightarrow & \coprod_{\lambda\in\Lambda} B^n_\lambda \\ \downarrow{\scriptstyle \coprod\varphi_\lambda} & & \downarrow{\scriptstyle \chi} \\ X & \xrightarrow{\ i\ } & Y \end{array}$$



The continuous maps i, χ are the obvious restrictions of the quotient map $X \coprod (\coprod B^n_\lambda) \to Y$. The picture describes the situation in the case of attaching one cell of dimension 1. The pair (Y,X) is also called a n-cell adjunction. Denote

$$\chi_\lambda := \chi|_{B^n_\lambda}$$

then χ_λ is also called the characteristic map for the n-cell $\overline{e^n_\lambda}$. $\chi_\lambda|_{\overset{o}{B}{}^n_\lambda} : \overset{o}{B}{}^n_\lambda \to e^n_\lambda$ is a homeomorphism (See.e.g. [D] chap IV, sec.6).

Informally speaking, a relative CW-complex is a pair (Y,X) of topological spaces, where Y is obtained from X by iterated cell adjunctions of strictly increasing dimension. To make this idea precise, even in the case of an infinite iteration of cell adjunctions, we define:

(1) Definition: (i) A relative CW-complex is a pair (Y,X) of topological spaces such that Y is the colimit of a filtration of Y of the following type

$$X=Y^0 \hookrightarrow Y^1 \hookrightarrow Y^2 \hookrightarrow \dots \hookrightarrow Y^n \hookrightarrow Y^{n+1} \hookrightarrow \dots$$

where each pair (Y^{n+1},Y^n) is an $(n+1)$-cell adjunction. We allow $Y^{n+1}=Y^n$, Y^n is called the n-skeleton of Y.

(ii) A space Y is a CW-complex iff there exists a discrete space X such that (Y,X) is a relative CW-complex. In this case the elements of X are also called the 0-cells of Y. We collect some of the fundamental properties of CW-complexes. So let Y be a CW-complex.

(2) Each $y \in Y$ is an element of precisely one cell of Y.

(3) "Closure finite": If e^n_λ is a cell of Y and χ^n_λ its characteristic map, then $\chi^n_\lambda(S^{n-1}_\lambda)$ is contained in finitely many cells of dimension at most n-1.

(4) "Weak topology": Let Y be the colimit of

$$Y_0 \hookrightarrow Y_1 \hookrightarrow Y_2 \hookrightarrow \ldots$$

then Y is homeomorphic to the union $Y_0 \cup Y_1 \cup Y_2 \cup \ldots$ having the weak topology with respect to the inclusions $Y_n \hookrightarrow Y$.

(5) The universal property of "weak toplogy" yields: Let $f: Y \to Z$ be a function from the CW-complex Y into the space Z, then the following three statements are equivalent.

(a) f is continuous.

(b) For all $n \in \mathbb{N}_0$, $f_{|Y^n}$ is continuous.

(c) For all $n \in \mathbb{N}_0$ and for all $\lambda \in \Lambda_n$, $f \circ \chi^n_\lambda$ is continuous, where Λ_n denotes the indexing set for the n-cells of Y.

(6) Y is T_1, normal and paracompact.

(7) The path components of Y are the same as the connected components with respect to the usual covering condition.

(8) Y is locally contractible.

Since local contractibility implies local connectedness and semilocal simple connectedness, we have

(9) If Y is connected, $H \subset \pi_1(Y,y_0)$ a subgroup, then there exists a covering projection $p:(E,e_0) \to (Y,y_0)$ such that

$$\pi_1 p(\pi_1(E,e_0)) = H$$

Furthermore, E can be given the structure of a CW-complex such that p maps any n-cell of E onto an n-cell of B. This holds for all dimensions.

(10) Definition: Let Y,Z be CW-complexes, $f:Y \to Z$ continuous. Then f is called cellular iff for all $n \in \mathbb{N}_0$

$$f(Y^n) \subset Z^n$$

Thus, the map p in (9) is in particular a cellular map.

(11) Cellular approximation theorem: Let Y, Z be CW-complexes, $f: Y \to Z$ a map. Then f is homotopic to a cellular map.

(12) Let (Y,X) be an n-cell adjunction, X a CW-complex. Then Y is homotopically equivalent to a CW-complex which contains X as a subcomplex.

This yields by induction:

(13) Let (Y,X) be a relative finite dimensional CW-complex, X a CW-complex. Then Y is homotopically equivalent to a CW-complex which contains X as a subcomplex.

Statements (12),(13) can be derived from the following two lemmas in J. Milnor's "Morse Theory" [Mi2] pp20-22 and the cellular approximation theorem.

(14) Let $\varphi_1, \varphi_2 : S^{n-1} \to X$ $(n \geq 1)$ be attaching maps for n-cells e_1^n, e_2^n such that φ_1 is homotopic to φ_2. Then the identity map of X extends to a homotopy equivalence

$$H: X \cup_{\varphi_1} e_1^n \longrightarrow X \cup_{\varphi_2} e_2^n$$

(15) Let $\varphi:S^{n-1} \to X$ be an attaching map; $f:X \to Y$ a homotopy equivalence. Then f extends to a homotopy equivalence

$$F:X\cup_{\varphi}e^n \longrightarrow Y\cup_{f\varphi}e^n$$

One of the main features of CW-complexes is that there is an algebraic condition for a map to be a homotopy equivalence. We need

(16) Definition: Let X,Z be spaces, $f:K \to Z$ a map. Call f a weak homotopy equivalence iff for all $n \in \mathbb{N}_0$

$$\pi_n f:\pi_n(X,*) \to \pi_n(Z,f*)$$

is an isomorphism.

(17) Theorem (J.H.C. Whithehead): Let Y,Z be CW-complexes, $f:Y \to Z$ a weak homotopy equivalence. Then f is a homotopy equivalence.

(18) Theorem: Let Y,Z be simply connected CW-complexes, $f:Y \to Z$ a homology equivalence. Then f is a homotopy equivalence.

On many occasions (e.g. the section on Quillen's + construction) one works in a category whose objects are of the homotopy type of a CW-complex, and it is convenient to point out here that some results for CW-complexes can be extended to such larger categories. A typical example is

(19) Proposition: Let $(X,*),(Z,*)$ be path connected spaces having the homotopy type of CW-complexes. Let $f:(X,*) \to (Y,*)$ be a weak homotopy equivalence, then f is a homotopy equivalence.

Proof: By assumption, there exist CW-complexes X',Z' and maps

$$X \underset{\psi}{\overset{\varphi}{\rightleftarrows}} X' \qquad Z \underset{\psi'}{\overset{\varphi'}{\rightleftarrows}} Z'$$

which are mutual homotopy inverses. Now consider

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Z \\ \psi \uparrow\downarrow \varphi & & \psi' \uparrow\downarrow \varphi' \\ X' & \xrightarrow[\varphi' f \psi]{} & Z' \end{array}$$

Since φ',f,ψ are weak homotopy equivalences and π_n is a functor, $\varphi'f\psi$ is a weak homotopy equivalence between CW-complexes. By (17), $\varphi'f\psi$ is a homotopy equivalence. Thus X,Z are of the same homotopy type. With little extra consideration, we see that f is indeed a homotopy equivalence. □

We proceed to give a proof for the local contractibility of CW-complexes - not because this proof is exceptionally difficult, but because such a proof seems to be hard to come by in the English literature. As a side effect, it provides various opportunities to observe, how the cell structure of CW-complexes can be used efficiently.

The following lemma gives insight into the structure of open sets in CW-complexes.

(20) Lemma: Let Y be a CW-complex, $U \subset Y$. Let Λ_n be an indexing set for all n-cells; χ^n_λ the characteristic map for e^n_λ. Then

(i) U open in Y iff for all $n \in \mathbb{N}_0$, $U \cap Y^n$ is open in Y^n.

(ii) U open in Y^{n+1} iff

(a) $U \cap Y^n$ is open in Y^n

(b) For all $s_\lambda \in (\chi^{n+1}_\lambda|S^n_\lambda)^{-1}(U)$ there exists an open subset $V(s_\lambda)$ of B^{n+1}_λ such that $s_\lambda \in V(s_\lambda)$ and $\chi^{n+1}_\lambda(V(s_\lambda)) \subset U$.

(c) There exists an open subset $V \subset (\bigcup_{\lambda \in \Lambda_{n+1}} e^{n+1}_\lambda)$

such that

$$U \cap Y^{n+1} = [U \cap Y^n] \cup W^{n+1} \cup V$$

where $W^{n+1} = \bigcup_{\substack{\lambda \in \Lambda \\ s_\lambda \in (\chi_\lambda^{n+1}|S_\lambda^n)^{-1}(U)}} \chi_\lambda^{n+1}(V(s_\lambda))$

Proof: Use (4) □

Proof of (8): Compare also H. Schubert "Topologie" [S] III.3.6 .

Let $y_0 \in Y$; $Y \supset U$ be open such that $y_0 \in U$. We want an open neighborhood V of y_0 such that $y_0 \in V \subset U \subset Y$ and such that y_0 is a strong deformation retract of V inside U.

By (2), $y_0 \in e_\lambda^n$ for precisely one $n \in \mathbb{N}_0$, $\lambda \in \Lambda_n$. According to (20), we are going to construct V recur - sively in Y^{n+k} for $k \geq 0$. If $n = 0$, define $V_1 := \{y_0\}$, a 0-cell of Y. If $n > 1$, there exists (by (20)) an open ball $E_\lambda^n \subset B_\lambda^n$ such that $y_0 \in V_1 := \chi_\lambda^n(E_\lambda^n) \subset U$. In either case, there is a strong deformation retraction H_1 of V_1 to $\{y_0\}$ in $U \cap Y^n$ (constant if $n = 0$).

Now assume we have constructed $V_1 \subset \ldots \subset V_k$, where V_i is open in Y^{n+i} and $V_i \subset U \cap Y^n$ such that V_i is a strong deformation retract of V_{i+1} inside $U \cap Y^{n+i+1}$ with homotopy H_{i+1}. We are going to construct V_{k+1}, H_{k+1}

with analogous properties.

By (20.b), for each $s_\lambda \in (\chi_\lambda^{n+k+1})^{-1}(V_k)$ there exists an open half ball $E(s_\lambda) \subset B_\lambda^{n+k+1}$ centered at s_λ such that

$$\chi_\lambda^{n+k+1}(E(s_\lambda)) \subset U \cap Y^{n+k+1}$$
$$\chi_\lambda^{n+k+1}(E(s_\lambda) \cap S_\lambda^{n+k+1}) \subset V_k$$

Now define

$$V_{k+1} := V_k \cup \left(\bigcup_{\lambda, s_\lambda} \chi_\lambda^{n+k+1}(E(s_\lambda)) \right)$$

By (20), V_{k+1} is open in Y^{n+k+1} and by construction, $V_{k+1} \cap Y^{n+k} = V_k$, $V_{k+1} \subset U$. Furthermore, S_λ^{n+k} is a strong deformation retract of $B_\lambda^{n+k+1} - \{0\}$. With no harm, we may assume that the open half balls $E(s_\lambda)$ are small enough not to contain 0 of B_λ^{n+k+1}. Then, by restriction, $(\chi_\lambda^{n+k+1})^{-1}(V_k)$ is a strong deformation retract of $(\chi_\lambda^{n+k+1})^{-1}(V_{k+1})$. By (5), these strong deformation retractions composed with the χ_λ^{n+k+1}'s combine to a continuous strong deformation retraction H_{k+1} of V_{k+1} into V_k inside U and constant on V_k.

This completes the recursion. Define $V := \cup V_k$. We shall now turn to the construction of a strong deformation retraction H of V to $\{y_0\}$ inside U.

Define

$$\phi_1 : V_1 \times I \ni (y,t) \mapsto \begin{Bmatrix} y & \text{for} & t \in [0,\frac{1}{2}] \\ H_1(y,2t) & \text{for} & t \in [\frac{1}{2},1] \end{Bmatrix} \in V_1$$

and assume that ϕ_k contracts V_k to $\{y_0\}$ being constant on $[0,\frac{1}{k+1}]$. Now define

$$\phi_{k+1} : V_{k+1} \times I \ni (y,t) \mapsto \begin{cases} y & t \in [0,\frac{1}{k+2}] \\ H_{k+1}(y,(k+1)(k+2)(t-\frac{1}{k+2})) & t \in [\frac{1}{k+2},\frac{1}{k+1}] \\ \phi_k(H_{k+1}(y,1),t) & t \in [\frac{1}{k+1},1] \end{cases}$$

ϕ_k is well defined and continuous by (5). Note also that

$$\phi_k | V_i \times I \equiv \phi_j | V_i \times I$$

for $i \leq k,j$. Using (5) again, we see that

$$H : V \times I \ni (y,t) \mapsto \begin{Bmatrix} y & \text{for} & t = 0 \\ \phi_k(y,t) & \text{for} & t \geq \frac{1}{k+1} \end{Bmatrix} \in V_{k+1}$$

is also well defined and continuous. From the construction it is clear that H contracts V to $\{y_0\}$ inside V and hence also inside U. □

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