ON THE CONVERGENCE OF THE SEQUENCE OF ITERATES

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ABSTRACT

The main objective of this thesis is to study the convergence of sequences of various iterates to a fixed point in certain Banach spaces.

The preliminaries of metric spaces, normed linear spaces, and basic fixed point concepts are covered in the first chapter.

In the second chapter, the first section gives a brief statement of the problem of convergence of sequences of iterates.

The second section traces the development of some of the more important results on the convergence of sequences of iterates. The concept of a densifying mapping is then introduced and a similar development of theorems for these mappings is given.

The last section covers a more general sequence of iterates. An extension of a theorem by Senter [19] is presented and a corrected version of his original theorem is given. The corrected version is then shown to be a corollary of the new theorem. It is also shown that a condition Z of Rhoades [17] is a subset of one of the conditions of the new theorem. The final theorem in this section is an extension of one by Petryshyn and Williamson [15] to this more general sequence of iterates.

CHAPTER I

BANACH CONTRACTION PRINCIPLE AND FIXED POINTS

1.1 Introductory Concepts

In this chapter definitions and examples are given which have been used in the subsequent work.

Definition 1.1.1 Let X be a set and d be a function from $X \times X$ into $\mathbb{R}^+$ such that for every x, y and $z \in X$, we have

i) $d(x,y) \geq 0$,

ii) $d(x,y) = 0$ if and only if $x = y$,

iii) $d(x,y) = d(y,x)$,

iv) $d(x,z) \leq d(x,y) + d(y,z)$.

Then d is called a metric or distance function and the pair (X,d) is a metric space. A metric space may be denoted by X if the metric d is understood.

Definition 1.1.2 A sequence of points $\{x_n\}$ of a metric space X is said to converge to a point z if given $\varepsilon > 0$. There exists a positive integer N_ε such that

$$d(x_n,z) < \varepsilon \text{ for } n > N_\varepsilon.$$

We denote convergence by $x_n \to z$ or $\lim d(x_n,z) = 0$.

Definition 1.1.3 A sequence $\{x_n\}$ is a Cauchy sequence if for each

$\varepsilon > 0$, there is a positive integer N_ε such that

$$d(x_n, x_m) < \varepsilon \quad \text{for all } n, m \geq N_\varepsilon.$$

A convergent sequence is always a Cauchy sequence but the converse is not true.

<u>Definition 1.1.4</u> A metric space X is <u>complete</u> if every Cauchy sequence in X converges to a point in X.

<u>Definition 1.1.5</u> The <u>diameter</u> $\delta(A)$ of a non-empty subset A of a metric space X is defined by

$$\delta(A) = \sup\{d(x,y) \mid x,y \in A\}.$$

If the diameter of A is finite, then A is said to be bounded, otherwise unbounded.

<u>Definition 1.1.6</u> A subset A of a metric space X is <u>totally bounded</u> if given $\varepsilon > 0$, there exists a finite number of subsets $A_1, A_2, \ldots, A_n$ of X such that

$$\delta(A_i) < \varepsilon \quad \text{for } i = 1, 2, \ldots, n \text{ and } A \subset \bigcup_1^n A_i.$$

<u>Remark 1.1.7</u> A totally bounded set is bounded but the converse is not true. However, in $\mathbb{R}$ bounded and totally bounded are equivalent.

The following theorem is stated without proof.

<u>Theorem 1.1.8</u> A subset A of a metric space X is totally bounded if and only if every sequence of points of A contains a Cauchy subsequence. [8].

<u>Definition 1.1.9</u> A metric space X is <u>compact</u> if every open covering of X has a finite subcover.

<u>Definition 1.1.10</u> A <u>linear space</u> over a field K is a quadruple $(X,K,+,\cdot)$ where X is a non-empty set,

$+$ is a mapping $(x,y) \to x+y$ of $X \times X$ into X,

$\cdot$ is a mapping $(\alpha,x) \to \alpha\cdot x$ of $K \times X$ into X,

such that the following are satisfied for all $x,y,z \in X$ and $\alpha, \beta \in K$:

i) $x+y = y+x$,

ii) $x+(y+z) = (x+y)+z$,

iii) there exists $0 \in X$ such that $x + 0 = x$,

iv) for each $x \in X$, there exists $-x \in X$ such that $x + (-x) = 0$,

v) $(\alpha+\beta) \cdot x = \alpha\cdot x+\beta\cdot x$,

vi) $\alpha\cdot(x+y) = \alpha\cdot x+\alpha\cdot y$,

vii) $\alpha\cdot(\beta\cdot x) = (\alpha\beta)\cdot x$,

viii) $1\cdot x = x$.

We will write the linear space X rather than $(X,K,+,\cdot)$.

<u>Definition 1.1.11</u> Let X be a linear space over K. A mapping $x \to ||x||$ of X into the set $\mathbb{R}^+$ is called a <u>norm</u> on X if the following are satisfied for all $x, y \in X$ and $\lambda \in K$.

i) $||x|| \geq 0$,

ii) $||x|| = 0$ if and only if $x = 0$,

iii) $||\lambda x|| = |\lambda|\ ||x||$,

iv) $||x+y|| \leq ||x|| + ||y||$.

A <u>normed linear space</u> over K is a pair $(X, ||\cdot||)$ where X is a linear space over K and $||\cdot||$ is a norm on X. We will write a normed linear space X rather than $(X, ||\cdot||)$.

<u>Remark 1.1.12</u> Every normed linear space is a metric space with a metric d defined on X by

$$d(x,y) = ||x-y|| \text{ for all } x, y \in X.$$

<u>Definition 1.1.13</u> A complete normed linear space is called a <u>Banach space</u>.

<u>Example 1.1.14</u> $C[a,b]$, the space of continuous functions on $[a,b]$ with

$$||f|| = \sup\{|f(t)| \mid t \in [a,b]\},$$

is a Banach space.

<u>Example 1.1.15</u> ℓ^p for $p \geq 1$, the space of sequences $x = \{x_n\}$ for which $\Sigma|x_i|^p < \infty$ with $||x|| = (\Sigma|x_i|^p)^{1/p}$, is a Banach space.

<u>Definition 1.1.16</u> A set C in a Banach space X is <u>convex</u> if

$$\alpha x+(1-\alpha)y \in C \text{ whenever } x,y \in C \text{ and } 0 \leq \alpha \leq 1.$$

<u>Definition 1.1.17</u> The intersection of all closed, convex sets containing a set G is a closed, convex set which contains G and which is contained in every closed, convex set containing G. This set is the <u>closed convex hull</u>, or the <u>convex closure</u> of G and is denoted by $\overline{CO}(G)$.

<u>Definition 1.1.18</u> A Banach space X is called <u>uniformly convex</u> if for

every $\varepsilon > 0$, there exists a $\delta > 0$, depending on ε, such that if $||x|| = ||y|| = 1$ and $||x-y|| \geq \varepsilon$ then $\left|\left|\frac{x+y}{2}\right|\right| \leq 1-\delta$.

Remark 1.1.19 X is uniformly convex if for any two points x, y on the unit sphere S, the midpoint of the segment joining x and y can be close to but not on the sphere only if x and y are sufficiently close together. $(S = \{x \in X \mid ||x|| = 1\})$.

Definition 1.1.20 A Banach space X is strictly convex if from $x, y \in X$ and $||x+y|| = ||x|| + ||y||$, it follows that $x = \lambda y$ where $\lambda > 0$.

Remark 1.1.21 X is a strictly convex Banach space if whenever $||x|| = ||y|| = 1$ and $x \neq y$ then $\left|\left|\frac{x+y}{2}\right|\right| < 1$. Thus, a uniformly convex space is strictly convex but not conversely.

1.2 Fixed point Concepts.

Fixed point theorems are useful mainly in existence theory of differential equations, partial differential equations, random differential equations, and in related areas. They have a very fruitful application in eigenvalue problems and boundary value problems.

Definition 1.2.1 Let $T: X \to X$ be a function on set X. A point $x_0 \in X$ is a fixed point of T if $Tx_0 = x_0$, that is a point which remains invariant under a transformation T. We denote the set of fixed points of T by $F(T)$.

Example 1.2.2 Let $T: [0,1] \to [0,1]$ be defined by $Tx = x/10$.

Then $T(0) = 0$ and 0 is a fixed point of T.

<u>Definition 1.2.2</u> A mapping $T:X \to X$ is a <u>contraction</u> map if

$$d(Tx,Ty) \leq kd(x,y) \text{ for all } x,y \in X \text{ and } 0 \leq k < 1.$$

The following well-known theorem is due to Banach [1]. We include the proof for the sake of completeness.

<u>Theorem 1.2.3</u> <u>Banach Contraction Principle</u>.

Let $T:X \to X$ be a contraction mapping on a complete metric space (X,d). Then T has a unique fixed point.

<u>Proof</u> Let x_0 be any point in X and let

$$x_n = Tx_{n-1} = T^n x_0.$$

<u>We wish to show $\{x_n\}$ is a Cauchy sequence.</u>

Because T is a contraction mapping, we have

$$d(x_1,x_2) = d(Tx_0,Tx_1) \leq \alpha d(x_0,x_1)$$

$$d(x_2,x_3) = d(Tx_1,Tx_2) \leq \alpha d(x_1,x_2) \leq \alpha^2 d(x_1,x_0)$$

and in general

$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) \leq \alpha^n d(x_0,x_1).$$

Now,

$$d(x_n,x_m) = d(T^n x_0, T^m x_0) \leq \alpha^n d(x_0, x_{m-n}), \text{ for } m > n$$

$$\leq \alpha^n \{d(x_0,x_1) + d(x_1,x_2) + \ldots + d(x_{m-n-1}, x_{m-n})\}$$

$$\leq \alpha^n \{d(x_0,x_1) + d(x_0,x_1) + \ldots + \alpha^{m-n-1} d(x_0,x_1)\}$$

$$\leq \alpha^n d(x_0,x_1)\{1 + \alpha + \alpha^2 + \ldots + \alpha^{m-n-1}\}$$

$$\leq \frac{\alpha^n d(x_0,x_1)}{1-\alpha}$$

Since $\alpha < 1$, $d(x_n,x_m)$ is arbitrarily small for large n. Thus, $\{x_n\}$ is a Cauchy sequence.

<u>Show T has a fixed point.</u>

Set the $\lim_{n\to\infty} x_n = x$. Since T is continuous

$$Tx = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x.$$

Thus, T has at least one fixed point.

<u>Show T has only one fixed point.</u>

Assume T has two distinct fixed points; say $Tx = x$ and $Ty = y$ where $x \neq y$.

Then $d(x,y) = d(Tx,Ty) \leq \alpha d(x,y)$.

That is $\alpha \geq 1$, a contradiction to the fact that $\alpha < 1$.

Hence

$$d(x,y) = 0$$

and

$$x = y.$$

Thus, T has one and only one fixed point.

Q.E.D.

Remark 1.2.4 Both conditions of the Theorem are necessary.

1. The mapping $T:(0,1] \to (0,1]$ defined by $Tx = x/2$ is a contraction map but has no fixed point, since $(0,1]$ is not a complete space.

2. The mapping $T:\mathbb{R} \to \mathbb{R}$ defined by $Tx = x+5$ is not a contraction and has no fixed points although $\mathbb{R}$ is complete.

Remark 1.2.5 The construction of the sequence $\{x_n\}$ and the study of its convergence is known as the method of successive approximations.

Definition 1.2.6 A mapping $T:X \to X$ is contractive if

$$d(Tx,Ty) < d(x,y) \quad \text{for all } x, y \in X \text{ where } x \neq y.$$

Remark 1.2.7 A contractive mapping is continuous and clearly is more general than a contraction mapping.

Remark 1.2.8 Completeness of a space and a contractive mapping are not enough to ensure the existence of a fixed point.

Let $T:\mathbb{R} \to \mathbb{R}$ be defined by $Tx = x + \frac{\pi}{2} - \arctan x$. Then T is contractive, but T does not have a fixed point, since $\arctan x < \frac{\pi}{2}$ for all x.

Therefore, fixed point theorems for contractive mappings require further restrictions on the space or extra conditions on the mapping or its range.

<u>Definition 1.2.9</u> A mapping $T:X \to X$ is said to be <u>non-expansive</u> if

$$d(Tx,Ty) \leq d(x,y) \quad \text{for all} \quad x,y \in X.$$

<u>Example 1.2.10</u> A translation map, identity map, and isometry are examples of non-expansive mappings.

<u>Definition 1.2.11</u> A mapping $T:X \to X$ is said to be <u>quasi-non-expansive</u> provided T has at least one fixed point $(F(T) \neq \phi)$ and if $p \in F(T)$ then

$$||Tx-p|| \leq ||x-p|| \quad \text{for all} \quad x \in X.$$

<u>Definition 1.2.12</u> A mapping $T:X \to X$ is said to be <u>asymptotically regular</u> if

$$||T^{n+1}x - T^n x|| \quad \text{converges to } 0 \text{ as } n \to \infty, \quad \text{for all } x \in X.$$

<u>Definition 1.2.13</u> The <u>set $L(x)$</u> will denote the set of subsequential limit points of the sequence of iterates $\{T^m x\}_{m=1}^{\infty}$.
That is the set of all $y \in X$ such that

$y = \lim T^{m_i} x$ for some subsequence

$\{T^{m_i} x\}_{i=1}^{\infty}$ of the sequence $\{T^m x\}_{m=1}^{\infty}$.

CHAPTER II

CONVERGENCE OF THE SEQUENCE OF ITERATES

2.1 Introduction

Let T be a non-expansive mapping of a Banach Space X into itself. The question we are concerned with is finding the fixed points of T by an iteration method

$$x_{n+1} = Tx_n, \; n = 0, 1, 2 \ldots \tag{1}$$

where x_0 is a given initial approximation.

The basic question is whether or not the sequence in (1) converges. If T is a contraction mapping then the sequence $\{x_n\}$ converges to the unique fixed point of T. However, if T is non-expansive then there is no guarantee that the sequence will converge and generally it does not.

Consider the following simple examples of non-expansive mappings which do not have convergent sequences.

<u>Example 2.1.1</u> If T is a translation from $\mathbb{R} \to \mathbb{R}$.

<u>Example 2.1.2</u> If T is a rotation of the plane around the origin.

<u>Remark 2.1.3</u> In the second example the sequence $\{x_n\}$ does not converge in general. It converges only if the initial approximation is the origin.

We will investigate this concept of convergence of iterates for more general forms of iterations and various types of non-expansive mappings.

2.2 Convergence of a sequence of iterates

Consider mappings of the form

$$T_\lambda = \lambda I + (I-\lambda)T, \quad 0 < \lambda < 1,$$

for $T:X \to X$, and the sequence of iterates $\{T_\lambda^n x_0\}$. Under certain restrictions this sequence of iterates converges.

The first result of this form was given by Krasnoselskii [11] for $\lambda = \frac{1}{2}$ and is stated below without proof.

Theorem 2.2.1 Let X be a uniformly convex Banach space, C be a closed convex subset of X and $T:C \to C$ be a non-expansive mapping such that $T(C)$ is contained in a compact subset of C. Then for an arbitrary $x_0 \in C$ the sequence of iterates

$$x_{n+1} = \tfrac{1}{2}(x_n + Tx_n)$$

converges to a fixed point of T in C.

This result was extended by Schaeffer [18] for λ, $0 < \lambda < 1$ where the sequence of iterates was

$$x_{n+1} = \lambda x_n + (I-\lambda)Tx_n.$$

Edelstein [7] further extended Theorem 2.2.1 to a strictly convex Banach space.

Diaz and Metcalf [4] derived the following more general result which contained the result of Edelstein as a corollary.

<u>Theorem 2.2.2</u> Let $T:X \to X$ be a continuous function, and suppose that

i) $F(T) \neq \phi$,

ii) for each $x \in X$ with $x \notin F(T)$, $p \in F(T)$,

$$d(Tx,p) < d(x,p).$$

Let $x \in X$. Then either $\{T^n x\}$ has no convergent subsequence or $\lim_{n\to\infty} T^n x$ exists and belongs to $F(T)$.

<u>Proof</u>

Assume $L(x) \neq \phi$, otherwise there is nothing to prove.

<u>Show lim exists</u>. If $T^n x \in F(T)$ for some integer $n \geq 1$, then $\lim T^n x$ exists and belongs to $F(T)$.

Assume, therefore, that $T^n x \notin F(T)$ for $n = 0,1,2, \ldots$. Then for any $y \in F(T)$

$$d(T^{n+1}x,y) < d(T^n x,y) \quad n = 1, 2, 3 \ldots .$$

Thus, the sequence of reals is decreasing and a limit exists.

<u>Show $L(x) \subseteq F(T)$</u>. Let $T^{n_i}x \to p$ and we wish to show $p \in F(T)$.

Assume $p \notin F(T)$. Then for $y \in F(T)$,

$$\lim d(T^{n_{i+1}}x,y) = d(\lim T^{n_{i+1}}x,y) < d(\lim T^{n_i}x,y)$$
$$= \lim d(T^{n_i}x,y).$$

This is a contradiction since the sequence $\{d(T^n x,y)\}$ converges.

Therefore $L(x) \subseteq F(T)$.

Show $L(x)$ contains at most one point.

Assume to the contrary that $L(x)$ contains at least two distinct points p and q. Then these are strictly increasing sequences of positive integers $\{m_i\}_{i=1}^{\infty}$ and $\{n_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} T^{m_i}x = p$ and $\lim_{i\to\infty} T^{n_i}x = q$. Now take a subsequence $\{m_i'\}_{i=1}^{\infty}$ of $\{m_i\}_{i=1}^{\infty}$ so that $m_i' > n_i$ for $i = 1, 2 \ldots$ Then $\lim_{i\to\infty} T^{m_i'}(x) = P$.

Furthermore, because $p \neq q$ we have $T^m(x) \notin F(T)$ for $m = 1, 2 \ldots$.

But condition (ii) implies

$$d(T^{m_i'}x,q) < d(T^{m_{i-1}'}x,q) < \ldots < d(T^{n_i}x,q)$$

and since $\lim_{i\to\infty} d(T^{n_i}x,q) = 0$. This implies $\lim_{i\to\infty} T^{m_i}x = q$.

This is a contradiction of the assumption $p \neq q$ and therefore $L(x)$ contains at most one point.

Show $\{T^m x\}$ converges to $y \in F(T)$.

Since $T^{n_i}x \to y$ therefore, for $m > n_i$

$$d(T^m x,y) < d(T^{m-1}x,y) < \ldots < d(T^{n_i}x,y) < \epsilon.$$

This implies $T^m x \to y$.

Q.E.D.

Using the above Theorem Diaz and Metcalf [4] proved the following theorem which we state without proof.

Theorem 2.2.3 Let C be a closed, convex subset of a strictly convex

Banach space X, $T:C \to C$ be a mapping satisfying

i) $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in C$,

ii) $T(C)$ is contained in a compact subset C' of C.

Then for $x \in C$, the sequence of iterates $\{T^n x\}$ converges to a fixed point of T.

Remark 2.2.4 We note that the earlier results are special cases. In the case that $\lambda = \frac{1}{2}$, we get the result due to Edelstein [7]. In the case where X is a uniformly convex Banach space, we get the result of Schaeffer [18]. Finally, in the case where $\lambda = \frac{1}{2}$ and X is a uniformly convex Banach space we get the result of Krasnoselskii [11].

The following result is also due to Diaz and Metcalf [4], it introduces a quasi non-expansive mapping.

Theorem 2.2.5 Let X be a Banach space and T be a continuous mapping of X into itself such that

i) T is asymptotically regular,

ii) T is quasi non-expansive; i.e. whenever $p \in F(T)$,

$$||Tx - p|| \leq ||x - p|| \text{ for all } x \in X.$$

Suppose that for some $x_0 \in X$ the sequence $\{T^n x_0\}$ has a convergent subsequence

$$\{T^{n_i} x_0\} \to z;$$

then z is a fixed point of T and $\{T^n x_0\} \to z$.

The following well-known theorem is due to Browder and Petryshyn [3].

<u>Theorem 2.2.6</u> Let $T:X \to X$ be a non-expansive, asymptotically regular mapping in a Banach space X. Let $F(T)$ be non-empty, and let T satisfy the following condition.

(A) $(I - T)$ maps bounded closed sets into closed sets.

Then for each $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to some point in $F(T)$.

<u>Proof</u> If $y \in F(T)$ then

$$||T^{n+1}x_0 - y|| \leq ||T^n x_0 - y||, \; n = 1, 2 \ldots,$$

and the sequence $\{T^n x_0\}$ is bounded. Let G be the closure of $\{T^n x_0\}$. By condition (A) it follows that $(I-T)G$ is closed. This along with the fact that T is asymptotically regular gives 0 in $(I-T)G$. Thus, there exists a $Z \in G$ such that $(I-T)Z = 0$; i.e. $Z = TZ$.

Now this implies either $Z = T^n x_0$ for some n, or there exists a subsequence $\{T^{n_i} x_0\}$ converging to Z. Since Z is a fixed point, it can be concluded that the sequence $\{T^n x_0\}$ converges to Z.

Q.E.D.

Diaz and Metcalf [4] proved the following.

<u>Theorem 2.2.7</u> Let $T:X \to X$ be a continuous, quasi non-expansive mapping of a Banach space X into itself such that

i) T is asymptotically regular,

ii) the continuous real valued function f, defined by

$$f(x) = ||(I-T)x||$$

for $x \in X$, maps bounded, closed subsets of X into closed sets of real numbers.

Then, for $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to some point in $F(T)$.

<u>Proof</u> The function f maps bounded, closed subsets of X into closed sets of real numbers.

Let D be a bounded, closed set in X. Then condition (ii) implies that

$$f(D) = \{||(I-T)x|| \mid x \in D\}$$

is a closed set of real numbers. The norm function is continuous and therefore the inverse image of the set $f(D)$, with respect to the norm function, namely the set $(I-T)D$ is a closed subset of X.

Thus, every bounded, closed set D is mapped by $I-T$ into a closed set.

Therefore, condition (ii) implies (A) of Theorem 2.2.6. Hence, the proof follows on the same lines as that of Theorem 2.2.6.

Q.E.D.

Diaz and Metcalf proved the following theorem.

<u>Theorem 2.2.8</u> Let $T:X \to X$ be a continuous, quasi non-expansive

asymptotically regular mapping of a Banach space X into itself. Suppose T satisfies

(A) $(I-T)$ maps bounded, closed subsets of X into closed subsets of X.

Then for $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to a fixed point of T.

Dotson proved the following [5].

<u>Theorem 2.2.9</u> Let X be a uniformly convex Banach space and T a quasi-non-expansive mapping of X into itself. Define the mapping T_λ by

$$T_\lambda = \lambda I + (I-\lambda)T, \quad 0 < \lambda < 1.$$

Then the mapping T is quasi non-expansive and asymptotically regular. (See also Yadav's thesis for the proof).

<u>Theorem 2.2.10</u> Let T be a continuous, quasi non-expansive mapping of a uniformly convex Banach space X into itself. Also, if T satisfies

(A) $(I-T)$ maps closed, bounded subsets of X into closed subsets of X.

Define $T_\lambda : X \to X$ by

$$T_\lambda = \lambda I + (1-\lambda)T, \quad 0 < \lambda < 1.$$

Then for $x_0 \in X$, the sequence $\{T_\lambda^n x_0\}$ converges to some point in $F(T)$.

<u>Proof</u> From Theorem 2.2.9 it follows that T_λ is a continuous, quasi non-expansive asymptotically regular mapping. Also $F(T) = F(T_\lambda)$. Moreover, T_λ satisfies condition (A).

Thus, T_λ satisfies all the hypothesis of Theorem 2.2.8.

Q.E.D.

<u>Definition 2.2.11</u> Let D be a bounded subset of a metric space X. We define the <u>measure of non-compactness</u>

$$\alpha(D) = \inf\{\varepsilon > 0 \mid D \text{ admits a finite covering consisting of subsets of diameter } \leq \varepsilon\}.$$

<u>Remark 2.2.12</u> The following properties of α are very useful. See Nussbaum [12] for a proof.

1. $\alpha(A) \leq \delta(A)$ where $\delta(A)$ = diameter of A.
2. If $A \subset B$, then $\alpha(A) \leq \alpha(B)$.
3. $\alpha(\overline{A}) = \alpha(A)$ where $\overline{A}$ is the closure of A.
4. $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

<u>Definition 2.2.13</u> Let T be a continuous mapping of a Banach space X into itself. Then T is <u>densifying</u> if

1. for all bounded sets $A \subset X$, we have $T(A)$ bounded and
2. $\alpha(TA) < \alpha(A)$ for $\alpha(A) > 0$.

The following Theorem due to Nussbaum [13] is stated without proof.

<u>Theorem 2.2.14</u> Let $T: C \to C$ be a densifying mapping defined on a closed, bounded convex subset C of a Banach space X.

Then T has at least one fixed point. This result was also given

by Furi and Vignoli: Fixed points for densifying mappings, Accad. Naz. Lincei 47(1969), 465-467.

The following Theorem is due to Singh [20].

<u>Theorem 2.2.15</u> Let X be a Banach space and let C be a closed, bounded, convex subset of X. Let $T:C \to C$ be a densifying mapping. Define a mapping

$$T_\lambda = \lambda I + (I-\lambda)T, \quad 0 < \lambda < 1$$

such that $T_\lambda : C \to C$ and let T_λ satisfy

(A) whenever $p \in F(T_\lambda)$,
$||T_\lambda x - p|| < ||x - p||$ for all
$x \in C - T(T_\lambda)$.

Then for each $x_0 \in C$, the sequence of iterates $\{T_\lambda^n x_0\}$ converges to a fixed point of T.

<u>Proof</u> $F(T) \neq \phi$, from Theorem 2.2.14. Because T is densifying, T_λ is also densifying and $F(T_\lambda) \neq \phi$.

If we show that $\{T_\lambda^n x_0\}$ has a convergent subsequence the proof will follow from Theorem 2.2.2.

For each $x_0 \in C$, the sequence

$$A = \{T_\lambda^n x_0 \mid n = 0, 1, 2, \ldots\}$$

is bounded and it is transformed to

$$B = T_\lambda A = \{T^n x_0 \mid n = 1, 2, \ldots\}.$$

Now $\alpha(A) = \alpha(B \cup \{x_0\})$

$= \max\{\alpha(B), \alpha\{x_0\}\}$

$= \alpha(B)$.

Therefore, $\alpha(A) = 0$ since T_λ is densifying.

Thus, $\overline{A}$ is compact and $\{T_\lambda^n x_0\}$ has a convergent subsequence.

Thus the result follows from Theorem 2.2.2.

Q.E.D.

The following result due to Petryshyn [14] follows as a corollary to Theorem 2.2.15.

<u>Theorem 2.2.16</u> Let X be a strictly convex Banach space, C be a closed, bounded, convex subset of X, and $T:C \to C$ be a densifying, non-expansive mapping.

Let $T_\lambda = \lambda I + (I-\lambda)T$, $0 < \lambda < 1$.

Then, for each $x_0 \in C$, the sequence

$$x_{n+1} = T_\lambda^n x_0$$

converges to a fixed point of T.

In the following theorem due to Yadav [24] quasi non-expansive mapping is used in place of non-expansive.

<u>Theorem 2.2.17</u> Let X be a strictly convex Banach space, C be a closed, bounded, convex subset of X, and $T:C \to C$ be a densifying mapping. Define $T_\lambda:C \to C$ by

$$T_\lambda = \lambda I + (I-\lambda)T, \quad 0 < \lambda < 1,$$

such that

$$||T_\lambda x-p|| < ||x-p||$$

for all $x \in C$ and $p \in F(T_\lambda)$. Then, for each $x_0 \in C$, the sequence $\{T_\lambda^n x_0\}$ converges to a fixed point of T.

<u>Proof</u> Because T_λ is quasi non-expansive and X is a strictly convex Banach space, we have

$$||T_\lambda x-p|| < ||x-p|| \quad \text{for all}$$
$$p \in F(T_\lambda) \quad \text{and} \quad x \in C-F(T_\lambda).$$

The proof follows from Theorem 2.2.15.

The following result was given by Singh and Yadav [23].

<u>Theorem 2.2.18</u> Let X be a Banach space, C be a closed, bounded, convex subset of X, and $T:C \to C$ be a densifying mapping. Define $S:C \to C$ by

$$S = a_0 I + a_1 T + a_2 T^2 + \ldots + a_K T^K,$$

where $a_i \geq 0$, $a_1 > 0$, and $\sum_{i=0}^{K} a_i = 1$.

Let S satisfy

$$||Sx-p|| < ||x-p|| \quad \text{for} \quad p \in F(S)$$

and $x \in C - F(S)$. Then, for each $x_0 \in C$, the sequence $\{S^n x_0\}$

converges to a fixed point of T.

Proof By Theorem 2.2.14, we can say

$$F(T) = F(S) \neq \phi.$$

Since T is densifying, S is also densifying and therefore, the sequence of iterates $\{S^n x_0\}$ has a convergent subsequence (as in the proof of Theorem 2.2.15).

Then the proof follows from Theorem 2.2.15.

Q.E.D.

Remark 2.2.19 By putting appropriate values of λ and the a_i's, it is possible to get the results of Diaz and Metcalf [4], Edelstein [7], Kirk [10], Krasnoselskii [11], Petryshyn [14], Reinermann [16], Schaeffer [18], and Singh [20] as corollaries.

Singh and Riggio [22] gave the following Theorem which is an extension of a result given by Barbutti and Guerra [2]

Theorem 2.2.20 Let X be a strictly convex Banach space, C be a closed, bounded, convex, subset of X, and $T:C \to C$ be a densifying mapping satisfying

$$(B') \quad ||Tx-Ty|| \leq a||x-y|| + b(||Tx-x|| + ||Ty-y||), \quad x, y \in X.$$

Let $T_\lambda : C \to C$,

$$T_\lambda = \lambda I + (I-\lambda)T, \quad 0 < \lambda < 1.$$

Then, for each $x_0 \in C$, the sequence of iterates $\{T^n x_0\}$ converges to a fixed point of T.

Proof By Theorem 2.2.14

$$F(T) = F(T_\lambda) \neq \phi .$$

Since T satisfies (B′) and X is a strictly convex Banach space, we get

$$||T_\lambda x-p|| < ||x-p|| \quad \text{for} \quad p \in F(T_\lambda)$$

and for all $x \in C - F(T_\lambda)$.

Then the proof follows from Theorem 2.2.15.

Q.E.D.

The following Theorem was given by Singh [21].

Theorem 2.2.21 Let X be a strictly convex Banach space, C be a closed, bounded, convex subset of X, and $T:C \to C$ be a densifying mapping satisfying

(A′) $||Tx-Ty|| \leq a||x-y|| + b(||x-Tx|| + ||y-Ty||) + c(||x-Ty|| + ||y-Tx||)$

where $a \geq 0$, $b \geq 0$, $c \geq 0$ and $a + 2b + 2c \leq 1$.

Then for each $x_0 \in C$ the sequence of iterates $\{T^n x_0\}$ converges to a fixed point of T.

<u>Proof</u> We wish to show that $||T_\lambda x-p|| < ||x-p||$ for $p \in F(T_\lambda)$ and for all $x \in X - F(T_\lambda)$. Then the proof will follow from Theorem 2.2.15.

Take $p \in F(T)$, then from (A´) we have

$$||Tx-p|| \leq a||x-p|| + b(||x-Tx||) + c(||x-p|| + ||p-Tx||)$$
$$(1-c)||Tx-p|| \leq (a+c)||x-p|| + b||x-Tx||.$$

However,

$$||x-Tx|| = ||x-p+p-Tx|| \leq ||x-p|| + ||p-Tx||$$

and substituting we get

$$(1-c)||Tx-p|| \leq (a+c)||x-p|| + b||x-p|| + b||p-Tx||$$
$$(1-b-c)||Tx-p|| \leq (a+b+c)||x-p||.$$

$$||Tx-p|| \leq \frac{a+b+c}{1-b-c}||x-p||.$$

Thus, $||Tx-p|| \leq ||x-p||$.

In the case $x \neq p$, we have

$$\begin{aligned} ||T_\lambda x-p|| &= ||T_\lambda x - T_\lambda p|| \\ &= ||\lambda(Tx-p) + (I-\lambda)(x-p)|| \\ &= ||x-p|| \left|\left| \frac{\lambda(Tx-p)}{||x-p||} + \frac{(I-\lambda)(x-p)}{||x-p||} \right|\right|. \end{aligned}$$

If strict inequality holds, we get

$$\left\| \frac{\lambda(Tx-p)}{||x-p||} + \frac{(I-\lambda)(x-p)}{||x-p||} \right\|$$

$$\leq \frac{\lambda||Tx-p||}{||x-p||} + (1-\lambda) < 1.$$

Hence,

$$||T_\lambda x-p|| < ||x-p||.$$

In the case equality holds, since $(Tx-p)/||x-p||$ and $(x-p)/||x-p||$ have unit norm and since X is a strictly convex Banach space, we get

$$\left\|\frac{\lambda(Tx-p)}{||x-p||} + \frac{(I-\lambda)(x-p)}{||x-p||}\right\| < 1.$$

Thus, $||T_\lambda x-p|| < ||x-p||$ for $x \neq p$.

Q.E.D.

<u>Remark 2.2.22</u> In case $b = c = 0$, $a = 1$, $\lambda = \frac{1}{2}$ we get a result due to Edelstein [7]. In case $c = 0$, we get Theorem 2.2.20 due to Singh and Riggio [22].
In case $c = 0$ and T is completely continuous, we get the result due to Barbutti and Guerra [2].

The following result is due to Petryshyn and Williamson [15].

<u>Theorem 2.2.23</u> Let C be a closed subset of a Banach space X and let $T:C \to X$ be continuous and satisfy

i) $F(T) = \phi$

ii) for each $x \in C$ and $p \in F(T)$,

$||Tx-p|| \leq ||x-p||$

iii) there exists an $x_0 \in C$ such that

$$x_n = T^n(x_0) \in C \quad \text{for} \quad n \geq 1.$$

Then $\{x_n\}$ converges to a fixed point of T in C if and only if

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

<u>Proof</u> The condition $\lim_{n\to\infty} d(x_n, F(T)) = 0$ is clearly necessary. To show that it is sufficient we shall assume it is true and show that $\{x_n\}$ is a Cauchy sequence.

Given $\varepsilon > 0$ there is an $n_1 > N$ such that $n \geq n_1$, then $d(x_n, F(T)) < \varepsilon/2$.

For $\ell, k \geq n_1$

$$||x_\ell - x_k|| = ||x_\ell - p + p - x_k|| \leq ||x_\ell - p|| + ||x_k - p||$$

and for $p \in F(T)$ we have the following from quasi non-expansive (ii)

$$||x_\ell - p|| \leq ||x_{n_1} - p||,$$

$$||x_k - p|| \leq ||x_{n_1} - p||,$$

$$||x_\ell - x_k|| \leq 2||x_{n_1} - p||.$$

Taking the infimum over $p \in F(T)$ gives

$$||x_\ell - x_k|| \leq 2d(x_n, F(T)) < \varepsilon.$$

Thus $\{x_n\}$ is a Cauchy sequence.

Now we know that $\{x_n\}$ converges to a point in C since C is closed. Furthermore, because T is continuous, $F(T)$ is closed, then

$$\lim_{n\to\infty} d(x_n, F(T)) = 0,$$

implies that $\{x_n\}$ converges to a point in $F(T)$.

Q.E.D.

The following Theorem, also due to Petryshyn and Williamson [15], is a restatement of the last theorem for the mapping T_λ.

<u>Theorem 2.2.24</u> Let C be a closed convex subset of a Banach space X and let $T:C \to X$ be a continuous mapping satisfying

(i) $F(T) \neq \phi$,

(ii) T is quasi-nonexpansive,

(iii) there exists an $x_0 \in C$ such that

$$x_n = T_\lambda^n(x_0) \in C \text{ for } n \geq 1,$$

and some $\lambda \in (0,1)$.

Then, $\{x_n\}$ converges to a fixed point of T in C if and only if

$$d(T_\lambda^n(x_0), F(T)) \to 0 \text{ as } n \to \infty.$$

<u>Proof</u> We wish to show that T_λ satisfies the conditions of Theorem 2.2.23.

Since C is convex, T_λ is well defined on C and $F(T) = F(T_\lambda)$. Condition (i) ensures the existence of a fixed point. Therefore,

for $p \in F(T)$, $\lambda \in (0,1)$, and $x \in C$ we have

$$||T_\lambda(x)-p|| = ||\lambda x+(1-\lambda)Tx-\lambda p-(1-\lambda)p||$$
$$\leq \lambda||x-p|| + (1-\lambda)||Tx-p|| \leq ||x-p|| .$$

Thus, T_λ is quasi non-expansive. From condition (ii), there is a $x_0 \in C$ such that $T_\lambda^n x \in C$ for $n \geq 1$.

Hence, the conditions of Theorem 2.2.23 are satisfied.

Q.E.D.

2.3 Iterates of the form $t_n Tx_n + (1-t_n)x_n$.

The sequence of iterates discussed in section 2.2 is not a very general one and therefore the study of the sequence of iterates given as

$$x_{n+1} = t_n Tx_n + (1-t_n)x_n$$

with suitable restrictions on t_n, has been of considerable interest in recent years.

The following definitions are taken from Senter [19].

Definition 2.3.1 Let P denote the set of positive integers and suppose C is a convex non-empty subset of a Banach space X. For arbitrary $x_1 \in X$,

$$m(x_1, t_n, T)$$

will denote the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-t_n)x_n + t_n Tx_n$$

where $t_n \in [a,b]$ for all $n \in P$ and $0 < a < b < 1$.

<u>Definition 2.3.2</u> Let X be a Banach space and C be a nonempty subset of X. A mapping $T:C \to C$ with a nonempty fixed point set F in C satisfies <u>condition I</u> if there is a nondecreasing function $f:[0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for $r \in (0,\infty)$ such that

$$|x-Tx| \geq f(d(x,F)) \qquad \text{I}$$

for all $x \in C$ where

$$d(x,F) = \inf\{|x-z| : z \in F\}.$$

<u>Definition 2.3.3</u> If $T:C \to C$ has a nonempty fixed point set F, then T will be said to satisfy <u>condition II</u> provided there exists a real number $\alpha > 0$ such that

$$|x-Tx| \geq \alpha \cdot d(x,F) \text{ for all } x \in X \qquad \text{II}$$

where $d(x,F) = \inf\{|x-z| : z \in F\}$.

The following Lemmas are due to Dotson [5]. They are stated without proof.

<u>Lemma 2.3.4</u> (Dotson [5]) For a uniformly convex Banach space X, if $\{w_n\}$ and $\{y_n\}$ are sequences in the closed unit ball of x and if

$$\{z_n\} = \{(1-t_n)w_n + t_n y_n\}$$

satisfies $|z_n| = 1$ where

$$t_n \in [a,b] \text{ for } 0 < a < b < 1,$$

then $\lim |w_n - y_n| = 0$.

Lemma 2.3.5 (Dotson [5]) Suppose C is a closed convex subset of a strictly convex Banach space X,

If $T:C \to C$ is quasi-nonexpansive then the fixed point set F of T in C is closed and convex.

The following theorem is due to Senter [19].

Theorem 2.3.6 Suppose X is a uniformly convex subset of X and T is a quasi non-expansive mapping of C into C. If T satisfies condition I where F is the fixed point set of T in C, then for arbitrary $x_1 \in X$, $M(x_1, t_n, T)$ converges to a member of F.

Proof

If $x_1 \in F$ the result is trivial.

Show the limit exists

Assume $x_1 \in C \setminus F$. For arbitrary $z \in F$ and $n \in P$ (the set of positive integers), we have from quasi non-expansiveness that

$$|Tx_n - z| \leq |x_n - z|$$

and

$$|x_{n+1} - z| \leq (1 - t_n)\ |x_n - z| + t_n\ |Tn_x - z|$$
$$\leq |x_n - z|.$$

Thus, $d(x_{n+1}, F) \leq d(x_n, F)$ for all $n \in P$.

The sequence $\{d(x_n, F)\}$ is non-increasing and bounded below, so

$\lim_{n\to\infty} d(x_n, F)$ exists.

<u>Show the limit must be zero.</u> (Using Lemma 2.3.4)

Suppose $\lim_{n\to\infty} d(x_n, F) = b > 0$.

Then for arbitrary $z_0 \in F$

$$\lim |x_n - z_0| = b' \geq b > 0.$$

Let $N > 0$ be an integer such that $|x_n - z_0| \leq 2b'$ for all $n \geq N$.

Let $y_n = \dfrac{(Tx_n - z_0)}{|x_n - z_0|}$ and $w_n = \dfrac{(x_n - z_0)}{|x_n - z_0|}$.

Then $|y_n| \leq 1$ and $|w_n| = 1$ for all $n \in P$ and for $n \geq N$,

$$|w_n - y_n| = \frac{|x_n - Tx_n|}{|x_n - z_0|} \geq \frac{f(d(x_n,F))}{|x_n - z_0|} \geq \frac{f(b)}{2b'} > 0. \quad \text{(From Condition I).}$$

Thus, $|w_n - y_n| \neq 0$.

Now, $\lim |(1-t_n)w_n + t_n y_n| = \lim |x_{n+1} - z_0|/|x_n - z_0| = \dfrac{b'}{b'} = 1$.

This contradicts Lemma 2.3.4 since $\lim |w_n - y_n| \neq 0$ implies that $\lim|(1 - t_n)w_n + t_n y_n| \neq 1$.

Therefore, $\lim_{n\to\infty} d(x_n, F) = 0$.

<u>Show $\{x_n\}$ converges to an element of F</u>

Since $\lim_{n\to\infty} d(x_n, F) = 0$, there corresponds to each $\varepsilon > 0$ a positive integer N_ε and $Z_\varepsilon \in F$ such that

$$|x_{N_\varepsilon} - Z_\varepsilon| < \varepsilon/4,$$

which implies $|x_n - Z_\varepsilon| < \varepsilon/4$ for all $n \geq N_\varepsilon$.

If $\varepsilon_k = \frac{1}{2}^k$ for $k \in P$ then for each ε_k there is an integer $N_k > 0$ and a $z_k \in F$ such that

$$|x_n - z_k| < \frac{\varepsilon_k}{4} \text{ for all } n \geq N_k.$$

We require $N_{k+1} > N_k$ for all $k \in P$.

We have for all $k \in P$

$$|z_k - z_{k+1}| = |z_k - x_{N_{k+1}} + x_{N_{k+1}} - z_{k+1}| \leq \frac{\varepsilon_k}{4} + \frac{\varepsilon_{k+1}}{4} = \frac{3\varepsilon_{k+1}}{4}.$$

Let $S(z, \varepsilon) = \{x \in X : |x-z| \leq \varepsilon\}$.

For $x \in S(z_{k+1}, \varepsilon_{k+1})$ we have

$$|z_k - x| = |z_k - z_{k+1} + z_{k+1} - x| < \frac{3\varepsilon_{k+1}}{4} + \varepsilon_{k+1} < 2\varepsilon_{k+1} = \varepsilon_k.$$

Therefore $S(z_{k+1}, \varepsilon_{k+1}) \subseteq S(z_k, \varepsilon_k)$ for all $k \in P$. Thus, $S(z_k, \varepsilon_k)$ is a nested sequence of nonvoid closed spheres with radii ε_k tending to zero. By the Cantor Intersection Theorem $\bigcap_{k \in P} S(z_k, \varepsilon_k)$ contains exactly one point, say w.

By Lemma 2.3.5, F is closed and the sequence $\{z_k\}$ in F converges to w, therefore $w \in F$.

We have

$$|z_k - w| < \varepsilon_k \text{ for all } k \in P$$

and

$$|x_n - z_k| < \frac{\varepsilon_k}{4} \text{ for all } n \geq N_k,$$

hence

$$|x_n - w| < \frac{5\varepsilon_k}{4} \text{ for } n \geq N_k.$$

Thus $\{x_n\} \to w$.

Q.E.D.

<u>Theorem 2.3.7</u> Let X be a uniformly convex Banach space and C be closed convex subset of X. Suppose $T:C \to C$ satisfies

(A) $|Tx - Ty| \leq \beta[|x - Ty| + |y - Tx|]$ where $0 \leq \beta \leq \frac{1}{2}$

OR

(B) $|Tx - Ty| \leq a|x - Tx| + a|y - Ty| + b|x - Ty| + c|y - Tx| + d|x - y|$,

where $\frac{1-b-c-d}{1+a-c} = \alpha > 0$

$2a + b + c + d \leq 1$,

and $a, b, c, d \geq 0$

$$\lim_{n\to\infty} d(x_n, F(T)) \to 0.$$

Then T satisfies condition II and for any $x_1 \in C$, $M(x_1, t_n T)$ converges to the fixed point of T.

<u>Proof</u>

<u>Show (B) is quasi non-expansive</u>

If p is a fixed point of T where T satisfies (B) we have

$$|Tx - Tp| = |Tx - p| \leq a|x - Tx| + b|x - p| + c|p - Tx| + d|x - p|$$

$$|Tx - Tp| \leq \frac{a}{1-c}|x - Tx| + \frac{b+d}{1-c}|x - p|$$

and

$$\frac{a}{1-c}|x - Tx| = \frac{a}{1-c}|x - p + p - Tx| \leq \frac{a}{1-c}|x - p| + \frac{a}{1-c}|p - Tx|.$$

Substituting,

$$|Tx - p| \leq \frac{a}{1-c}|x - p| + \frac{a}{1-c}|p - Tx| + \frac{b+d}{1-c}|x - p|.$$

Let $\alpha = \frac{a}{1-c}$. Then

$$|Tx - Tp| \leq \frac{\alpha}{1-\alpha}|x - p| + \frac{b+d}{1-c}\ 1 - \alpha\ |x - p|$$

$$= \frac{a+b+d}{1-c} \cdot \frac{1-c}{1-c-a}\ |x - p|$$

$$= \frac{a+b+d}{1-c-a}\ |x - p|.$$

The constant $\frac{a+b+d}{1-c-a} \leq 1$ because we know

$$2a + b + c + d \leq 1$$

$$a + b + d \leq 1 - c - a$$

$$\frac{a+b+d}{1-c-a} \leq 1,$$

and therefore,

$$|Tx - Tp| \leq |x - p| \quad \text{and}$$

T satisfying (B) is quasi non-expansive.

<u>Show (B) satisfies condition II.</u>

Assume T satisfies (B) and p is the fixed point; then

$$|Tx - p| = |Tx - Tp| \leq a|x - Tx| + b|x - p| + c|p - Tx| + d|x - p|$$

$$= a|x - Tx| + (b+d)|x - p| + c|p - Tx|.$$

$$|Tx - p| \leq \frac{a}{1-c}|x - Tx| + \frac{b+d}{1-c}|x - p|.$$

Now

$$|Tx - p| = |Tx-x+x-p| \geq |x - p| - |Tx - x|.$$

Substituting,

$$\frac{a}{1-c}\,|x - Tx| + \frac{b+d}{1-c}\,|x - p| \geq |x - p| - |Tx - x|$$

$$|x - Tx| \geq 1 - \frac{b+d}{1-c} \Big/ \left(1 + \frac{a}{1-c}\right) |x - p|$$

$$= \frac{1-b-c-d}{1+a-c}\,|x - p|.$$

The constant $\frac{1-b-c-d}{1+a-c}$ is positive by hypothesis.

Thus, T satisfying (B) meets condition II.

<u>Show (A) quasi non-expansive</u>

If T satisfies (A) and p is the fixed point of T, we have

$$|Tx - Tp| \leq \beta|x - p| + \beta|p - Tx|$$

$$|Tx - Tp| \leq \frac{\beta}{1-\beta}$$

and for $0 \leq \beta \leq \frac{1}{2}$ we have

$$|Tx - Tp| \leq |x - p|,$$

and T is quasi non-expansive.

<u>Show (A) meets condition II</u>

Assume T satisfies (A) and p is the fixed point of T; then

$$|Tx - Tp| = |Tx - p| \leq \beta|x - p| + \beta|p - Tx|$$

$$(1-\beta)\,|Tx - Tp| \leq \beta|x - p|.$$

Now

$$|Tx - p| = |Tx-x+x-p| \geq |Tx - x| - |x - p|.$$

Substituting,

$$(1-\beta)|Tx-x| - (1-\beta)|x-p| \leq \beta\,|x-p|$$

$$(1-\beta)|Tx-x| \leq (\beta+1-\beta)|x-p|$$

$$(1-\beta)|Tx-x| \leq |x-p|$$

$$-(1-\beta)|Tx-x| \geq -|x-p|$$

$$|Tx-x| \geq -\frac{1}{\beta-1}|x-p|$$

$$= \frac{1}{1-\beta}|x-p|$$

and for $0 \leq \beta \leq \frac{1}{2}$ the constant is positive.

Thus T satisfies condition II for (A).

Because T satisfying (A) or (B), meets condition I, is quasi non-expansive, and $\lim_{n\to\infty} d(x_n,F) \to 0$, it follows from Theorem 2.3.6 that $M(x_1, t_n, T)$ converges to the fixed point.

Q.E.D.

We will state the following Theorem given by Senter [19], show where one condition is in error, and finally show the corrected version is a corollary of Theorem 2.3.7.

<u>Theorem 2.3.8</u> Let X be a Banach space and C be a subset of X. Suppose $T:C \to C$ satisfies either

(A) $|Tx - Ty| \leq \beta[|x - Tx| + |y - Ty|]$,

$$0 < \beta \leq \frac{1}{2},$$

OR

(B) $|Tx - Ty| \leq a|x - Tx| + b|y - Ty| + c|x - y|$, where $a, b > 0$, $c \geq 0$ and $a+b+c \leq 1$, and has a (unique) fixed point in C.

Then T satisfies condition II.

If C is convex and closed and X is uniformly convex, then for any $x_1 \in C$, $M(x_1, t_n, T)$ converges to the fixed point of T.

<u>Remark 2.3.9</u> The proof for this theorem follows the same lines as Theorem 2.3.7. The error occured when (B) was assumed to be quasi non-expansive with the restrictions given. However, assuming p is the fixed point of T and T satisfies (B) we get

$$|Tx-p| \leq a|x-Tx| + c|x-p| .$$

Now,

$$|x-Tx| \leq |x-p| + |p-Tx| ,$$

hence

$$|Tx-p| \leq a|x-p| + a|p-Tx| + c|x-p| ,$$

i.e.

$$(1-a)|Tx-p| \leq (a+c)|x-p| ,$$

$$|Tx-p| \leq \frac{a+c}{1-a}|x-p| .$$

Thus, (B) is quasi non-expansive only when

$$\frac{a+c}{1-a} \leq 1 \qquad \text{i.e.} \qquad a+c \leq 1-a$$

$$2a + c \leq 1.$$

Therefore, there are not enough restrictions attached to the constants in (B).

We now restate the theorem assuming that $a = b$ in Senter's notation.

<u>Theorem 2.3.10</u> (Corrected). Let X be a Banach space and C be a subset of X. Suppose $T:C \to C$ satisfies either,

(A´) $|Tx - Ty| \leq \beta[|x - Tx| + |y - Ty|], \quad 0 \leq \beta \leq \frac{1}{2}$,

OR

(B´) $|Tx - Ty| \leq a|x - Tx| + a|y - Ty| + c|x - y|$ where $c \neq 1$ and $2a + c \leq 1$,

and has a (unique) fixed point in C.

Then T satisfies condition II.

If C is convex and closed and X is uniformly convex, then for any $x_1 \in C$, $M(x_1, t_n, T)$ converges to the fixed point of T.

<u>Remark 2.3.11</u> The proof follows the same lines as Theorem 2.3.7. We will only show that (B´) is quasi non-expansive.

Assuming p is the fixed point of T and T satisfies (B´) we have

$$|Tx - p| \leq a|x - Tx| + c|x - p| .$$

Now

$$|x - Tx| \leq |x - p| + |p - Tx| ,$$

and

$$|Tx - p| \leq a|x - p| + a|p - Tx| + c|x - p|$$

$$(1-a)\ |Tx - p| \leq (a+c)|x - p|$$

$$|Tx - p| \leq \frac{a+c}{1-a}\ |x - p| .$$

Thus, T satisfying (B´) is quasi non-expansive when

$$\frac{a+c}{1-a} \leq 1 ,$$

or

$$a+c \leq 1-a ,$$

$$2a+c \leq 1.$$

Remark 2.3.12 The corrected statement of Senter Theorem (Theorem 2.3.10) follows as a corollary of Theorem 2.3.7.

1) Taking (B) with $a = \beta$, $b, c, d = 0$ we get

$$|Tx - Ty| \leq \beta|x - Tx| + \beta|y - Ty|$$

with the restrictions

$\beta \geq 0$, $\frac{1}{1+\beta} > 0$ and $2\beta \leq 1$

or $0 \leq \beta \leq \frac{1}{2}$.

This is the same as (A′) of Senter.

2) Taking (B) with $b, c = 0$ we get

$$|Tx - Ty| \leq a|x - Tx| + a|y - Ty| + d|x - y|$$

with the restrictions

$$2a + d \leq 1$$

and

$$\frac{1-d}{1+a} > 0$$

or equivalently $d \neq 1$.

This is the same as (B′) of Senter if c of Senter is substituted for d above.

The remainder of the Theorems are equivalent and thus, Senter's Theorem follows as a corollary of Theorem 2.3.7.

The following definition is taken from Rhoades [17].

Definition 2.3.13 Let α, β, γ be real non-negative numbers satisfying $\alpha < 1$, $\beta, \gamma < \frac{1}{2}$. We say $T: X \to X$ satisfies condition Z if for each $x, y \in X$, at least one of the following is satisfied:

i) $|Tx - Ty| \leq \alpha|x - y|$,

ii) $|Tx - Ty| \leq \beta[|x - Tx| + |y - Ty|]$,

iii) $|Tx - Ty| \leq \gamma[|x - Ty| + |y - Tx|]$.

<u>Remark 2.3.14</u> We will show that condition Z is a subset of condition (B) in Theorem 2.3.7.

Condition (B) states

$$|Tx - Ty| \leq a|x - Tx| + a|y - Ty| + b|x - Ty| + c|y - Tx| + d|x - y|$$

with $a, b, c, d \geq 0$, $2a + b + c + d \leq 1$ and $\frac{1-b-c-d}{1+a-c} = \alpha > 0$.

1. Taking (B) with $a, b, c = 0$ and $\alpha = d$ we get

$$|Tx - Ty| \leq \alpha|x - y|$$

with the restrictions $1 - \alpha > 0$, $\alpha \geq 0$ or $0 \leq \alpha < 1$.

This is (i) of Rhoades definition.

2. Taking (B) with $a = 0$, $b = c = \beta$ and $d = 0$ we get

$$|Tx - Ty| \leq \beta[|x - Ty| + |y - Tx|]$$

with the restrictions $\beta \geq 0$ and $\frac{1-2\beta}{1-\beta} > 0$ or equivalently, $0 \leq \beta < \frac{1}{2}$.

This is condition (iii) of Rhoades' definition.

3. Taking (B) with $a = \beta$, $b, c, d = 0$ we get

$$|Tx - Ty| \leq \beta[|x - Tx| + |y - Ty|]$$

with the restrictions $\beta \geq 0$, $2\beta \leq 1$ and $\frac{1}{1+\beta} > 0$ or equivalently, $0 \leq \beta \leq \frac{1}{2}$.

This includes condition (ii) of Rhoades but it is to be noted that it also includes $\beta = \frac{1}{2}$.

Remark 2.3.15 Several new conditions similar to Rhoades can be derived from (B) with different combinations of the constants. For example:

1. Taking (B) with $c, d = 0$ we get

$$|Tx - Ty| \leq a|x - Tx| + a|y - Ty| + b|x - Ty|$$
$$2a + b \leq 1 \text{ and } b \neq 1.$$

2. Taking (B) with $b, d = 0$ we get

$$|Tx - Ty| \leq a|x - Tx| + a|y - Ty| + c|y - Tx|$$
$$2a + c \leq 1 \text{ and } c \neq 1.$$

We give the following theorem which extends Theorem 2.2.24 of Petryshyn and Williamson to a more general sequence of iterates; namely

$$x_{n+1} = t_n Tx_n + (1-t_n)x_n.$$

Theorem 2.3.16 Let C be a closed convex subset of a Banach space X and $T:C \to X$ be a continuous function. If the following conditions are satisfied

i) $F(T) \neq \phi$,

ii) $|Tx - p| \leq |x - p|$ for all $x \in C$ and every $p \in F(T)$,

iii) $x_{n+1} = t_n Tx_n + (1-t_n)x_n$ for all $x_0 \in C$ where $t_n \in [a,b]$ with $0 < a < b < 1$.

iv) $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Then $\{x_n\}$ converges to a fixed point of T.

Proof. We will construct a notation similar to Petryshyn and Williamson by setting

$$x_{n+1} = T_{t_n} x_n = (1-t_n)x_n + t_n T x_n .$$

We wish to show that $F(T) = F(T_{t_n})$

a) Consider

$$T_{t_n} x = t_n Tx + (1-t_n)x .$$

Let $x_0 = Tx_0$.

Then

$$\begin{aligned} T_{t_n} x_0 &= t_n Tx_0 + (1-t_n)x_0 \\ &= t_n x_0 + (1-t_n)x_0 \\ &= x_0 . \end{aligned}$$

That is T_{t_n} has a fixed point x_0.

b) Show $x_0 = Tx_0$ if $T_{t_n} x_0 = x_0$

$$\begin{aligned} x_0 &= t_n Tx_0 + (1-t_n)x_0 \\ 0 &= t_n Tx_0 - t_n x_0 \\ t_n x_0 &= t_n Tx_0 \\ x_0 &= Tx_0 . \end{aligned}$$

Therefore, $F(T) = F(T_{t_n})$.

Show T_{t_n} is quasi non-expansive

That is to show $|T_{t_n} x - p| \leq |x - p|$ for $p \in F(T_{t_n})$.

Taking the left hand side of the equation

$$|t_n Tx + (1-t_n)x - p|$$

$$= |t_n Tx - t_n p + t_n p + (1-t_n)x - p|$$

$$= |t_n Tx - t_n p + (1-t_n)(x-p)|$$

$$\leq t_n |Tx - p| + (1-t_n) |x - p|.$$

We are given that $|Tx - p| \leq |x - p|$.

Therefore

$$|T_{t_n} x - p| \leq t_n |x - p| + (1-t_n) |x - p|$$

$$= |x - p| .$$

We now have all the conditions necessary for Theorem 2.2.24 of Petryshyn and Williamson [15] to hold.

Therefore, $\{x_n\}$ converges to a fixed point of T.

Q.E.D.

BIBLIOGRAPHY

1. BANACH, S. — Sur les operations dans les ensembles abstraits et leur application aux equations integrals, Fund. Math., 3(1922), 133-181.

2. BARBUTTI, U. and GUERRA, S.G. — On an extension of a theorem of J.B. Diaz and F. T. Metcalf, Accad. Naz. Lincei, 51(1971), 29-31.

3. BROWDER, F.E. and PETRYSHYN, W.V. — The solution of iteration of non-linear functional equations in Banach spaces, Bull. Amer. Math. Soc. 72(1966), 571-575.

4. DIAZ, J.B. and METCALF, F.T. — On the set of subsequential limit points of successive approximations, Trans. Amer. Math. Soc. 135(1969), 459-485.

5. DOTSON, W. G. Jr. — Some iterative methods for approximation of fixed points and solution of functional equations in Banach spaces, Doctor of Philosophy Thesis, University of North Carolina at Chapel Hill (1968).

6. EDELSTEIN, M. — The construction of an asymptotic center with a fixed point property, Bull. Amer. Math. Soc. 78(1972), 206-208.

7. EDELSTEIN, M. — A remark on a theorem of M.A. Krasnoselskii, Amer. Math. Monthly, 73(1966).

8. GOLDBERG, R.R. — Methods of Real Analysis, Ginn-Blaisdell Co. 1963.

9. HOLDEN, B.J. — Sum of non-linear operators with fixed points, Master's Thesis, Memorial University (1972).

10. KIRK, W.A. — On successive approximations for non-expansive mappings, Glasgow Math. Jour. 12(1971), 6-9.

11. KRASNOSELSKII, M.A. — Two remarks on the method of successive approximations, Uspehi Mat. Nauk. 10(1965), 123-127.

12. NUSSBAUM, R.D. — The radius of the essential spectrum, Duke. Math. Jour. 37 9(1970), 473-478.

13. _____ — The fixed point index and asymptotic fixed point theorems for k-set contractions, Bull. Amer. Math. Soc. 75(1969), 490-495.

14. PETRYSHYN, W.V. — Structure of fixed point sets of k-set contractions, Arch. Rat. Mech. Anal. 40(1971), 312-328.

15. PETRYSHYN, W.V. and WILLIAMSON, T.E. Jr. — Strong and weak convergence of the sequence of successive approximations for quasi-non-expansive mappings. J. Math. Anal. and Appl. 43(1973), 459-497.

16. REINERMANN, J. — On a fixed point theorem of Banach type, Ann. Polon. Math. 23(1970-71), 105-107.

17. RHOADES, B.E. — Fixed point iterations using infinite matrices, Trans. Amer. Math. Soc. 196(1974), 161-176.

18. SCHAEFFER, H. — Uber die Methode subzessive approximtionen. Jber. Deutsch Math. Verein 59(1957), 131-140.

19. SENTER, H.F. — Approximations of fixed points by mean-value iterative methods in Banach spaces, Doctor of Philosophy Thesis, North Carolina State University at Raleigh (1973).

20. SINGH, S.P. — Some results on non-linear contractions, Notices, Amer. Math. Soc. 19 No. 5(1972), A-695.

21. _____ — On the convergence of sequence of iterates. Accad. Naz. Dei Lincei, LVII(1975), 502-505.

22. SINGH, S.P. and RIGGIO, M.I. — On a theorem of Diaz and Metcalf, Accad. Naz. Lincei, 53(1972), 246-249.

23. SINGH, S.P. and YADAV, R.K. — On the convergence of the sequence of iterates, Annales Soc. Sci. de Bruxelles 87(1973), 279-284.

24. YADAV, R.K. — Geometry of Banach spaces and some fixed point theorems, Master's Thesis, Memorial University (1972).