ON SPACES OF THE SAME HOMOTOPY TYPE AS A CW-COMPLEX

CHRIS MORGAN

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ON SPACES OF THE SAME HOMOTOPY TYPE AS A CW-COMPLEX

BY

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INTRODUCTION

Since the introduction of CW-complexes by J.H.C. Whitehead almost thirty years ago, various questions have been raised concerning the homotopy type of such complexes and their relation to homotopy theory. The structure of such complexes is simple. They are built in stages, each stage being obtained from the preceding by adjoining cells of a given dimension. In a sense, the topology of a CW-complex is simple too, and this simplicity is often reflected in topological invariants of homotopy type that can be described algebraically. Even for such spaces as polyhedra, it is useful to consider representations of them as CW-complexes, because such complexes frequently require fewer cells than a simplicial triangulation. So it became quite apparant that the category of CW-complexes (or the category of spaces of the same homotopy type as a CW-complex) would be a most useful category in which to do homotopy theory. Thus the question immediately arose of what kinds of spaces could be represented, up to homotopy type, by CW-complexes. This led to Milnor's paper [14], "On Spaces Having the Homotopy Type of a CW-Complex." The purpose of this thesis is to examine closely the content of this paper by Milnor.

Chapters I and II give the necessary background for the discussion of Milnor's work. In Chapter I we show how to construct CW-complexes and give some of their properties and results. In Chapter II we introduce the concept of a semisimplicial complex in the category of sets. We define two adjoint functors, $S:\underline{Top} \longrightarrow \underline{SSC}$ and $||:\underline{SSC} \longrightarrow \underline{Top}$, and examine some of their properties. The functor S is the singular complex functor and $||$ is the geometric realization functor whose construction is due to Milnor [15]. However, the prime motivation for Chapter II lies in §2.

Here we give a complete proof of the result due to Barratt [1] that the realization of any semisimplicial complex can be triangulated. This result plays a crucial role in Milnor's paper.

Chapters III and IV are devoted entirely to Milnor's paper with the exception of the last part of Chapter IV in which we show that the category of spaces of the homotopy type of based-CW-complexes is closed under the formation of mapping cones of its morphisms. Section one of Milnor's paper is treated in Chapter III and section two in Chapter IV.

We give the following notation:
By a "map" we mean a "continuous function". By $\rightarrowtail$ we mean an injection and by $\twoheadrightarrow$ a surjection.

In conclusion I would like to express my deep gratitude to my supervisor Dr. R. Piccinini for his help, his encouragement and the ideas he gave me in writing this thesis. I would also like to thank Ed Campbell and Dr. S. Nanda for their encouragement and helpful suggestions and the typists Elaine Boone and Sandra Crane for their hard work.

CHAPTER I

CW-COMPLEXES

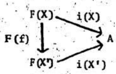
CW-complexes were first introduced by J.H.C. Whitehead in [18]; these spaces, because of their topological and homotopical properties, proved immediately to be extremely useful in Algebraic Topology.* We develop here a more categorical approach than that originally taken by Whitehead. More precisely, we construct these spaces as colimits of "convenient" diagrams in Top. In §1 we introduce this categorical concept of colimit, giving examples and some related results to be used, in §2, in the development of some of the properties and results of CW-complexes.

Much of the material in this chapter can be found in [16].

§1 - Colimits

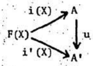
(1.1.1) In an arbitrary category $\underline{A}$, let $F:\underline{X} \to \underline{A}$ be a given diagram, that is, $\underline{X}$ is small and F is a covariant functor, and form the category $I(\underline{A},F)$ defined as follows:

Objects $\{F(X) \xrightarrow{i(X)} A\}$ where X varies in $\underline{X}$ and $A \in Ob\underline{A}$ (read - A is an object of $\underline{A}$) is fixed for each set of morphisms, such that $(\forall\ f \in \underline{X}(X,X'))$ the following diagram commutes



* Indeed the importance of these spaces was realized by Whitehead himself.

<u>Morphisms</u> $\bar{u}:\{F(X)\xrightarrow{i(X)}A\}\longrightarrow\{F(X)\xrightarrow{i'(X)}A'\}$ given by $u\in\underline{A}(A,A')$ such that $(\forall X\in Ob\underline{X})$ the following diagram commutes



We define a <u>colimit</u> of F in $\underline{A}$, denoted colim F, to be an initial object of the category $I(\underline{A},F)$. Whenever colimits exist they are unique up to isomorphism.

<u>Examples</u> 1) Given a set of objects $\{A_j\}_{j\in J}$ of a category $\underline{A}$, form the discrete category $\underline{X}$ with objects A_j, $j\in J$, and define $F:\underline{X}\longrightarrow\underline{A}$ to be the functor which takes A_j to A_j for each $j\in J$. If colim F exists, we say that the set $\{A_j\}_{j\in J}$ has a <u>coproduct</u> and write colim $F=\{A_j\longrightarrow\coprod_{j\in J}A_j\}$, where $\coprod_{j\in J}A_j$ denotes the coproduct object of the set $\{A_j\}_{j\in J}$.

2) Given $A, B\in Ob\underline{A}$ and $f, g\in\underline{A}(A,B)$, form the category $\underline{X}$ with objects A, B and morphisms 1_A, 1_B, f and g. Define $F:\underline{X}\longrightarrow\underline{A}$ to be the functor which takes A to A and B to B. A colimit of such a diagram, if it exists, is called a <u>coequalizer</u>.

Given categories $\underline{C}$ and $\underline{D}$ and functors $S:\underline{C}\longrightarrow\underline{D}$, $T:\underline{D}\longrightarrow\underline{C}$, we say that S is <u>left adjoint</u> to T (written $S\dashv T$) if the functors $\underline{D}(S-,-)$ and $\underline{C}(-,T-)$ from $\underline{C}^{opp}\times\underline{D}$ to <u>Set</u> are naturally equivalent.

The following proposition gives an important property of left adjoint functors; namely, they preserve colimits.

(1.1.2) Proposition: Let $F: \underline{X} \longrightarrow \underline{C}$ be a given diagram and let $S: \underline{C} \longrightarrow \underline{D}$, $T: \underline{D} \longrightarrow \underline{C}$ be functors with $S \dashv T$. If colim F exists, $S(\text{colim } F) \cong \text{colim}(SF)$.

Proof: Let $\{F(X) \xrightarrow{i(X)} C\}$ be an initial object of the category $I(\underline{C},F)$. We must show that $\{SF(X) \xrightarrow{Si(X)} S(C)\}$ is an initial object of $I(\underline{D},SF)$.

Given $\{SF(X) \xrightarrow{j(X)} D\}$ is an object of $I(\underline{D},SF)$ and $\theta(F(X),D)$ is the natural isomorphism $\underline{D}(SF(X),D) \cong \underline{C}(F(X),T(D))$, we have that the set $\{F(X) \xrightarrow{\theta(F(X),D)j(X)} T(D)\}$ is an object of $I(\underline{C},F)$. Since $\{F(X) \xrightarrow{i(X)} C\}$ is an initial object, there exists a unique morphism $\alpha: C \longrightarrow T(D)$ such that $(\forall X \in \text{Ob}\underline{X})$ the following diagram commutes

$$\begin{array}{ccc} & & C \\ & \nearrow^{i(X)} & \downarrow \alpha \\ F(X) & \xrightarrow[\theta(F(X),D)j(X)]{} & T(D) \end{array}$$

By naturality

$$\begin{array}{ccc} \underline{D}(SF(X),D) & \xrightarrow{\theta(F(X),D)} & \underline{C}(F(X),T(D)) \\ \uparrow {-\cdot Si(X)} & & \uparrow {-\cdot i(X)} \\ \underline{D}(S(C),D) & \xrightarrow{\theta(C,D)} & \underline{C}(C,T(D)) \end{array}$$

commutes.

Hence, $\theta(F(X),D)j(X) = \alpha i(X) = \theta(F(X),D)[\theta(C,D)^{-1}\alpha \cdot Si(X)]$ and so, for every $X \in \text{Ob}\underline{X}$ the following diagram commutes

$$\begin{array}{ccc} & \xrightarrow{Si(X)} & S(C) \\ SF(X) & & \downarrow \; \theta(C,D)^{-1}\alpha \\ & \xrightarrow{j(X)} & D \end{array}$$

//

Given a category $\underline{A}$, $\underline{A}$ is said to be <u>cocomplete</u> if any diagram in $\underline{A}$ has a colimit.

Let $\underline{A}$ and $\underline{X}$ be categories with $\underline{X}$ small. Form the functor category $\underline{A}^{\underline{X}}$ which has for objects, functors $\underline{X} \longrightarrow \underline{A}$, and for morphisms, natural transformations between these functors. If $\underline{A}$ is cocomplete, then for each $F \in Ob\underline{A}^{\underline{X}}$ we can choose an object $A_F \in Ob\underline{A}$ such that $\{F(X) \xrightarrow{i(X)} A_F\}$ is an initial object of $I(\underline{A},F)$. This gives rise to a functor $\underset{\underline{X}}{colim}: \underline{A}^{\underline{X}} \longrightarrow \underline{A}$. The next result says, roughly speaking, that colimits commute.

(1.1.3) <u>Proposition</u>: Let $\underline{A}$, $\underline{X}$, $\underline{Y}$ be categories such that $\underline{X}$, $\underline{Y}$ are small and $\underline{A}$ is cocomplete, If $F \in Ob(\underline{A}^{\underline{X}})^{\underline{Y}}$ such that $\underset{\underline{Y}}{colim}\, F$ exists, then

$$\underset{\underline{X}}{colim}(\underset{\underline{Y}}{colim}\, F) \cong \underset{\underline{Y}}{colim}(\underset{\underline{X}}{colim} \circ F)$$

<u>Proof</u>: We define a functor $C: \underline{A} \longrightarrow \underline{A}^{\underline{X}}$ in such a way that $\underset{\underline{X}}{colim} \dashv C$. Then by (I.1.2) the result follows.

$C: \underline{A} \longrightarrow \underline{A}^{\underline{X}}$ is defined as follows:

for every object A in $\underline{A}$, $C(A): \underline{X} \longrightarrow \underline{A}$ such that $C(A)(X) = A$, for every $X \in Ob\underline{X}$, and $(\forall f \in \underline{X}(X,X'))$. $C(A)(f) = 1_A$.

for every morphism g in $\underline{A}$, $C(g) = g$.

We leave to the reader to show that $\underset{\underline{X}}{\text{colim}} \dashv C$. //

§2 - CW-complexes

We now show how to construct CW-complexes.

Consider the following diagram in $\underline{\text{Top}}$

$$(1.2.1) \qquad D : K^0 \xrightarrow{\bar{i}_0} K^1 \xrightarrow{\bar{i}_1} K^2 \xrightarrow{\bar{i}_2} \cdots \xrightarrow{\bar{i}_{n-1}} K^n \xrightarrow{\bar{i}_n} \cdots$$

where K^0 is any discrete space and the maps $\bar{i}_n$ are 1-1 and closed. We assume K^{n-1} has been constructed and show how to construct K^n. Let Λ_n be any given indexing set and to each $\lambda \in \Lambda_n$ we associate a sphere S^{n-1}_λ and a map $f^{n-1}_\lambda : S^{n-1}_\lambda \longrightarrow K^{n-1}$. By the universal property of coproducts, this gives rise to a unique map $f^{n-1} : \coprod_{\lambda \in \Lambda_n} S^{n-1}_\lambda \longrightarrow K^{n-1}$. For each $\lambda \in \Lambda_n$, S^{n-1}_λ is a closed subspace of CS^{n-1}_λ and so $\coprod_{\lambda \in \Lambda_n} S^{n-1}_\lambda$ is a closed subspace of $\coprod_{\lambda \in \Lambda_n} CS^{n-1}_\lambda$. We now define K^n to be the space obtained by adjunction of $\coprod_{\lambda \in \Lambda_n} CS^{n-1}_\lambda$ to K^{n-1} via the map f^{n-1}, as shown in the following diagram.

$$(1.2.2) \qquad \begin{array}{ccc} \coprod_{\lambda \in \Lambda_n} CS^{n-1}_\lambda & \xrightarrow{\bar{f}^{n-1}} & K^n \\ {\scriptstyle i_{n-1}} \uparrow & & \uparrow {\scriptstyle \bar{i}_{n-1}} \\ \coprod_{\lambda \in \Lambda_n} S^{n-1}_\lambda & \xrightarrow{f^{n-1}} & K^{n-1} \end{array}$$

Since $\coprod_{\lambda\in\Lambda_n} S_\lambda^{n-1}$ is a closed subspace of $\coprod_{\lambda\in\Lambda_n} CS_\lambda^{n-1}$ we have that the map $\bar{i}_{n-1}$ is 1-1 and closed. (see [16;1.3.2]).

We now define a CW-complex K to be the topological space, unique up to homeomorphism, defined by a colimit of the diagram (1.2.1). The spaces K^n, $n = 0,1,\ldots$, are called the n-skeletons of the CW-complex K. If there exists an integer $n_0 \geq 0$ such that $(\forall n \geq n_0)\ K^n = K^{n_0}$, we say that K is of finite dimension n_0. Furthermore, if besides being finite dimensional, all sets Λ_n used in the construction of the K^n's are finite, we say that K is a finite CW-complex. If all the sets Λ_n are countable, we say that K is a countable CW-complex.

Instead of viewing a CW-complex K as a colimit of the diagram (1.2.1) in Top, there is a more useful form of K; namely, $K = \bigcup_{n\geq 0} K^n$ with the weak topology. By the weak topology we mean that a set $F \subset K$ is closed if and only if $(\forall n \geq 0)\ F \cap K^n$ is closed in K^n. This is just the final topology with respect to all inclusions $K^n \hookrightarrow K$. Proof that K can indeed be viewed in this form amounts to showing that $\{K^n \hookrightarrow \bigcup_{n\geq 0} K^n\}$ is an initial object of the category I(Top,D).

We now give some examples of CW-complexes. Examples (2) and (3) can be found in [17;2.4.1, 2.4.2].

(1.2.3) Examples.

(1) Let K be a simplicial complex and $|K|$, its geometric realization in $\mathbb{R}^n$.

Starting with the discrete space $|K|^0$, the collection of points in $\mathbb{R}^n$ realized from the vertices of K, it is clear that we can construct the following diagram

$$|K|^0 \longrightarrow |K|^1 \longrightarrow |K|^2 \longrightarrow \cdots \longrightarrow |K|^n$$

where $|K|^i = |K^i|$ and $|K|^i$ is obtained from $|K|^{i-1}$ by adjoining cones over spheres.

Hence, giving $|K|$ the weak topology makes it a CW-complex.

We note that the usual topology on $|K|$ is the metric topology, that is, the topology induced by the standard metric:

$d(\alpha,\alpha') = \sqrt{\sum_{v\in K} (\alpha(v)-\alpha'(v))^2}$, $\alpha, \alpha' \in |K|$. With this topology, $|K|$ is not, in general, a CW-complex. However, if K is a locally finite simplicial complex, that is, each vertex of K belongs to only finitely many simplices of K, then on $|K|$ the weak and metric topologies coincide.

(2) The n-sphere S^n is a CW-complex for all $n = 0,1,2,\ldots$

Starting with the two-point discrete space S^0 we construct the diagram

$$S^0 \overset{\bar{i}_0}{\hookrightarrow} S^1 \overset{\bar{i}_1}{\hookrightarrow} S^2 \overset{\bar{i}_2}{\hookrightarrow} \cdots \overset{\bar{i}_{n-1}}{\hookrightarrow} S^n \overset{\bar{i}_n}{\hookrightarrow} \cdots$$

where S^n is obtained from S^{n-1} as the pushout of the diagram

$$\begin{array}{ccc} CS^{n-1} \amalg CS^{n-1} & & \\ \uparrow {\scriptstyle i \amalg i} & & \\ S^{n-1} \amalg S^{n-1} & \xrightarrow{1 \amalg 1} & S^{n-1} \end{array}$$

We can then regard S^n, $n=0,1,\ldots,$ as a CW-complex by taking it as a colimit of the diagram

$$S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow \cdots \hookrightarrow S^n \xrightarrow{=} S^n \xrightarrow{=} \cdots$$

Thus, we must show that for $n = 0,1,2,\ldots$, S^n is homeomorphic to the pushout space of the above diagram.

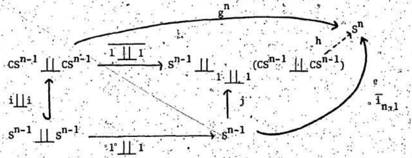
Define maps $g_1^n, g_2^n : CS^{n-1} \longrightarrow S^n$ by

$$g_1^n(x_0,x_1,\ldots,x_{n-1}) = (x_0,x_1,\ldots,x_{n-1},\sqrt{1-\sum_{i=0}^{n-1} x_i^2})$$

and

$$g_2^n(x_0,x_1,\ldots,x_{n-1}) = (x_0,x_1,\ldots,x_{n-1},-\sqrt{1-\sum_{i=0}^{n-1} x_i^2})$$

This gives rise to a unique map $g^n = g_1^n \coprod g_2^n : CS^{n-1} \coprod CS^{n-1} \longrightarrow S^n$ which makes the following diagram commute



By the universal property of pushouts, there exists a unique map h making the triangles commute. Now, $g^n \mid CS^{n-1} \coprod CS^{n-1} \setminus S^{n-1} \coprod S^{n-1}$ is a bijection and so h is a bijection. Also, S^{n-1} and $CS^{n-1} \coprod CS^{n-1}$ are compact and so $S^{n-1} \coprod_{1 \coprod 1} (CS^{n-1} \coprod CS^{n-1})$ is compact. S^n is Hausdorff and hence h is a homeomorphism, being a continuous bijection from a compact space to a Hausdorff space.

(3) Let $K = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ be the field of real, complex or quaternionic numbers, respectively. Since $\mathbb{H}$ is non-commutative we will

consider only multiplication on the left. We define on $K^{n+1} \setminus \{0\}$ the following equivalence relation $\sim$.

for all $\underline{x} = (x_0, \ldots, x_n)$ and $\underline{y} = (y_0, \ldots, y_n) \in K^{n+1} \setminus \{0\}$

$$\underline{x} \sim \underline{y} \Longleftrightarrow (\text{there exists } \lambda \in K \setminus \{0\})(\forall\, i = 0,1, \ldots, n)\ \ x_i = \lambda y_i$$

We then define the projective n-space, denoted $P_n(K)$, to be the space $K^{n+1} \setminus \{0\} / \sim$ with the quotient topology. To show that $P_n(K)$ $(n \geq 0)$ is a CW-complex we will show, analogous to example (2), that $P_n(K)$ is homeomorphic to the pushout space of the following diagram

$$\begin{array}{ccc} CS^{nk-1} & & \\ \uparrow{\scriptstyle i_{n-1}} & & \\ S^{nk-1} & \xrightarrow[f^{n-1}]{} & P_{n-1}(K) \end{array}$$

where $f^{n-1} : S^{nk-1} \longrightarrow P_{n-1}(K)$ is defined by $f^{n-1}(x_0, \ldots, x_{n-1}) = [(x_0, x_1, \ldots, x_{n-1})]$ for $x_i \in K$ and k is the dimension of K as a vector space over $\mathbb{R}$ and then take $P_n(K)$ as the colimit of the diagram

$$P_0(K) \xrightarrow{\bar{i}_0} P_1(K) \xrightarrow{\bar{i}_1} \cdots \xrightarrow{\bar{i}_{n-1}} P_n(K) \xrightarrow{=} P_n(K) \xrightarrow{=} \cdots$$

Notice that the inverse image by f^{n-1} of a point in $P_{n-1}(K)$ is homeomorphic to S^{k-1}.

Given the pushout diagram

$$\begin{array}{ccc} CS^{nk-1} & \longrightarrow & P_{n-1}(K) \coprod_{f^{n-1}} CS^{nk-1} \\ \uparrow{\scriptstyle i_{n-1}} & & \uparrow \\ S^{nk-1} & \xrightarrow[f^{n-1}]{} & P_{n-1}(K) \end{array}$$

consider the following commutative diagram

$$\begin{array}{ccc} CS^{nk-1} & \xrightarrow{g_{n-1}} & P_n(K) \\ i_{n-1} \Big\uparrow & & \Big\uparrow \bar{i}_{n-1} \\ S^{nk-1} & \xrightarrow[f^{n-1}]{} & P_{n-1}(K) \end{array}$$

where g_{n-1} and $\bar{i}_{n-1}$ are defined as follows:

$$g_{n-1}(x_0, x_1, \ldots, x_{n-1}) = [(x_0, x_1, \ldots, x_{n-1}, \sqrt{1 - \sum_{i=0}^{n-1} |x_i|^2})]$$

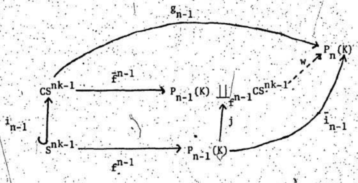
and

$$\bar{i}_{n-1}([(y_0, \ldots, y_{n-1})]) = [(y_0, y_1, \ldots, y_{n-1}, 0)]$$

By the universal property of pushouts, there exists a unique map

$$w : P_{n-1}(K) \coprod_{f^{n-1}} CS^{nk-1} \to P_n(K)$$

such that the following diagram commutes.



We show that w is a bijection, for then, since $P_{n-1}(K)$ and CS^{nk-1} are compact; $P_{n-1}(K) \coprod_{f^{n-1}} CS^{nk-1}$ is compact and as a continuous bijection from a compact space to a Hansdorff space, w is a homeomorphism.

In order to show w is a bijection, it is sufficient to show that $g_{n-1} : CS^{nk-1} \setminus S^{nk-1} \to P_n(K) \setminus P_{n-1}(K)$ and $j_{n-1} : P_{n-1}(K) \to P_{n-1}(K) \subset P_n(K)$ are bijections. This is clear for j_{n-1}. For g_{n-1}, we define a map $\bar{g}_{n-1} : P_n(K) \setminus P_{n-1}(K) \to CS^{nk-1} \setminus S^{nk-1}$ as follows:

for all $[(y_0, y_1, \ldots, y_n)] \in P_n(K) \setminus P_{n-1}(K)$

$$\bar{g}_{n-1}[(y_0, \ldots, y_n)] = \left(\frac{y_0\bar{y}_n}{|y_n| r}, \frac{y_1\bar{y}_n}{|y_n| r}, \ldots, \frac{y_{n-1}\bar{y}_n}{|y_n| r}\right)$$

where $\bar{y}_n$ denotes the conjugate of y_n and $r = \sqrt{\sum_{i=0}^{n} |x_i|^2}$

It then follows that $g_{n-1}\,\bar{g}_{n-1} = 1$ and $\bar{g}_{n-1}\,g_{n-1} = 1$. //

We now develop some of the 'nice' properties of CW-complexes. We start by showing that these spaces satisfy separation axioms T_0 to T_4.

in Kelly's notation [11]. To this end, we begin by proving

(1.2.4) Lemma: Let X be a space obtained from a space Y by adjoining n-cells. Let C be a closed subset of X and let $g : Y \longrightarrow I$, $h : C \longrightarrow I$ $(I = [0, 1])$ be maps such that $g|_{Y \cap C} = h|_{Y \cap C}$. Then there exists a map $g' : X \longrightarrow I$ such that $g'|_C = h$ and $g'|_Y = g$.

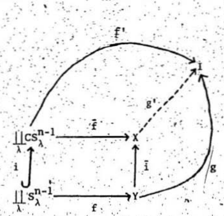
Proof: Let X be given by the following diagram

$$\begin{array}{ccc} \coprod_\lambda CS_\lambda^{n-1} & \xrightarrow{\bar{f}} & Y \coprod_f (\coprod_\lambda CS_\lambda^{n-1}) = X \\ \uparrow i & & \uparrow \bar{i} \\ \coprod_\lambda S_\lambda^{n-1} & \xrightarrow{f} & Y \end{array}$$

Take $x \in \bar{f}^{-1}(C) \cap \coprod_\lambda S_\lambda^{n-1}$. Then $\bar{f}(x) = f(x)$ because the above diagram commutes. Now, $\bar{f}(x) \in C$ and $f(x) \in Y$. Thus, $\bar{f}(x) = f(x) \in Y \cap C$ and since $g|_{Y \cap C} = h|_{Y \cap C}$, we have that $gf|_{\bar{f}^{-1}(C) \cap \coprod_\lambda S_\lambda^{n-1}} = h\bar{f}|_{\bar{f}^{-1}(C) \cap \coprod_\lambda S_\lambda^{n-1}}$.

This gives rise to a map $h\bar{f} \cup gf : \bar{f}^{-1}(C) \cup \coprod_\lambda S_\lambda^{n-1} \to I$. Since C is closed in X, $\bar{f}^{-1}(C)$ is closed in $\coprod_\lambda CS_\lambda^{n-1}$ and so $\bar{f}^{-1}(C) \cup \coprod_\lambda S_\lambda^{n-1}$ is closed in the normal space $\coprod_\lambda CS_\lambda^{n-1}$.

Therefore, by the Tietze Extension Theorem, the map $h\bar{f} \cup gf$ extends to a map $f' : \coprod_\lambda CS_\lambda^{n-1} \to I$. By the universal property of pushouts, there exists a unique map $g' : X \to I$ such that the following diagram commutes



Thus, we have $g'\bar{i} = g'|_Y = g$ and $g'\bar{f} = f'$, from which we get $g'|_C = h$. //

(1.2.5) Theorem: Every CW-complex is a normal space.

Proof: Let $K = \bigcup_{n \geq 0} K^n$ be a CW-complex and suppose C_0, C_1 are disjoint closed subsets of K. Define a map $h : C_0 \bigcup C_1 \to I$ by $h(C_0) = 0$ and $h(C_1) = 1$. Construct a sequence of maps $g^n : K^n \to I$ such that $g^n|_{K^n \cap (C_0 \bigcup C_1)} = h$ and $g^n|_{K^{n-1}} = g^{n-1}$, $n > 0$. By (1.2.4), there exists a map $g : K \to I$ such that $g|_{C_0 \bigcup C_1} = h$ and $g|_{K^n} = g^n$. Consider the sets $U_0 = g^{-1}([0, \frac{1}{4}))$ and $U_1 = g^{-1}((\frac{3}{4}, 1])$. U_0 and U_1 are clearly disjoint. They are also open sets since $U_0 = K \setminus g^{-1}([\frac{1}{4}, 1])$ and $U_1 = K \setminus g^{-1}([0, \frac{3}{4}])$. Since $h(C_0) = 0$ and $h(C_1) = 1$, $U_0 \supset C_0$ and $U_1 \supset C_1$. //

(1.2.6) Theorem. Every CW-complex is T_1.

Proof: Let $K = \bigcup_{n \geq 0} K^n$ be a CW-complex and take $x \in K$. If $x \in K^0$, then $\{x\}$ is closed (since K^0 is discrete) and hence, closed in K^n, $n > 0$, because the maps $\bar{i}_n : K^n \to K^{n+1}$ are closed maps. So suppose, now, that $x \in K^n$, $n > 0$, and $x \notin K^{n-1}$, where K^n is given by (1.2.2). Since $x \notin K^{n-1}$, there exists $\lambda' \in \Lambda_n$ and a unique $y \in CS^{n-1}_{\lambda'} \setminus S^{n-1}_{\lambda'}$ such that $\bar{f}^{n-1}(y) = x$. Consider the open set $U = (\coprod_{\lambda \neq \lambda'} CS^{n-1}_{\lambda}) \coprod (CS^{n-1}_{\lambda'} \setminus \{y\})$ and form the adjunction space $V = K^{n-1} \coprod_{f^{n-1}} U$.

Now, K^n has the final topology with respect to the maps $\bar{i}_{n-1}$ and $\bar{f}^{n-1}$. Thus, $V \subset K^n$ is open if and only if $(\bar{f}^{n-1})^{-1}(V)$ is open in $\coprod_{\lambda} CS^{n-1}_{\lambda}$ and $(\bar{i}_{n-1})^{-1}(V)$ is open in K^{n-1}. But $(\bar{f}^{n-1})^{-1}(V) = U$ and $(\bar{i}_{n-1})^{-1}(V) = K^{n-1}$. So V is open in K^n. But $V = K^n \setminus \{x\}$. Hence, $\{x\}$ is closed in K^n and consequently in any K^j, $j > n$, and ultimately in K. //

Let $\mathcal{A} = \{A_\beta | \beta \in \Lambda\}$ and $\mathcal{B} = \{B_\alpha | \alpha \in \Lambda'\}$ be open covers of a topological space X. $\mathcal{B}$ is said to be a refinement of $\mathcal{A}$ if, for each $\alpha \in \Lambda'$, $B_\alpha \subset A_\beta$, for some $\beta \in \Lambda$. Moreover, the refinement $\mathcal{B}$ is said to be locally finite if each point of X has a neighbourhood which intersects only finitely many members of $\mathcal{B}$.

A topological space X is said to be paracompact if and only if it is regular and each open cover of X has a locally finite refinement.

We remark that the definition of paracompactness as given in [11], requires that each open cover of X have an open locally finite refinement. However, both statements are equivalent for a regular topological space as shown in [13; 2.1].

We are going to show that every CW-complex is paracompact. To do this, we will first show that the n-skeletons of a CW-complex are paracompact and then the CW-complex, itself.

(1.2.7) <u>Lemma</u>: Let $K = \bigcup_{n \geq 0} K^n$ be a CW-complex. Then $(\forall n \geq 0)$ K^n is paracompact.

<u>Proof</u>: We proceed by induction on the n-skeletons of K. For $n = 0$, K^0 is a discrete space and, hence, paracompact. Assume K^i, $i = 1, \ldots, n-1$ is paracompact. We show K^n is paracompact.

By (1.2.5) and (1.2.6), we have that K^n is regular. So we must show that every open cover of K^n has a locally finite refinement. Let K^n be given by (1.2.2) and suppose $\mathcal{O} = \{O_\alpha | \alpha \in \Lambda\}$ is an open cover of K^n. Then $\mathcal{O}' = \{(\bar{f}^{n-1})^{-1}(O_\alpha) | O_\alpha \in \mathcal{O}\}$ and $\mathcal{O}'' = \{O_\alpha \cap K^{n-1} | O_\alpha \in \mathcal{O}\}$ are open covers of $\coprod_\lambda CS_\lambda^{n-1}$ and K^{n-1}, respectively. Now, $\coprod_\lambda CS_\lambda^{n-1}$ is paracompact and K^{n-1} is paracompact by the induction hypothesis. Hence, there exist locally finite refinements $\mathcal{A} = \{A_i | i \in I\}$ and $\mathcal{B} = \{B_j | j \in J\}$ of $\mathcal{O}'$ and $\mathcal{O}''$, respectively.

For each $x \in \coprod_\lambda S_\lambda^{n-1}$, choose a set $A_x \in \mathcal{A}$ such that $x \in A_x$ and, similarly, for $f^{n-1}(x) \in f^{n-1}(\coprod_\lambda S_\lambda^{n-1})$, choose a set $B_{f^{n-1}(x)} \in \mathcal{B}$ such that $f^{n-1}(x) \in B_{f^{n-1}(x)}$.

Consider the following pushout

$$\begin{array}{ccc}
A_x & \xrightarrow{\ \bar{f}^{n-1}|_{\{x\}}\ } & B_{f^{n-1}(x)} \coprod_{f^{n-1}|\{x\}} A_x \\
{\scriptstyle i_{n-1}|_{\{x\}}} \uparrow & & \uparrow {\scriptstyle \bar{I}_{n-1}|_{B_{f^{n-1}(x)}}} \\
\{x\} & \xrightarrow[\ f^{n-1}|_{\{x\}}\ ]{} & B_{f^{n-1}(x)}
\end{array}$$

and form the collection $\{(B_{f^{n-1}(x)} \coprod_{f^{n-1}|\{x\}} A_x) \cap 0_\alpha \mid 0_\alpha \in \mathcal{O}$, $x \in \coprod_\lambda S_\lambda^{n-1}\}$. Let V be the collection consisting of the sets $\bar{f}^{n-1}(A_i \cap (\coprod_\lambda CS_\lambda^{n-1} \setminus \coprod_\lambda S_\lambda^{n-1}))$, $(B_{f^{n-1}(x)} \coprod_{f^{n-1}|\{x\}} A_x) \cap 0_\alpha$ and $\bar{i}_{n-1}(B_j \cap (K^{n-1} \setminus f^{n-1}(\coprod_\lambda S_\lambda^{n-1})))$. Then V is clearly a refinement of $\mathcal{O}$. We claim that V is also locally finite. If $x \in \coprod_\lambda CS_\lambda^{n-1} \setminus \coprod_\lambda S_\lambda^{n-1}$, then, because $\coprod_\lambda CS_\lambda^{n-1} \setminus \coprod_\lambda S_\lambda^{n-1}$ is open, there exists a neighbourhood of $\bar{f}^{n-1}(x) \in K^n$ which is contained entirely in $\bar{f}^{n-1}(\coprod_\lambda CS_\lambda^{n-1} \setminus \coprod_\lambda S_\lambda^{n-1})$ and meets only finitely many members of V, because A is locally finite. Similarly, if $x \in K^{n-1} \setminus f^{n-1}(\coprod_\lambda S_\lambda^{n-1})$, there exists a neighbourhood of $\bar{i}_{n-1}(x) \in K^n$ which is contained entirely in $\bar{i}_{n-1}(K^{n-1} \setminus f^{n-1}(\coprod_\lambda S_\lambda^{n-1}))$ and meets only finitely many members of V, because B is locally finite.

Suppose, now, that $x \in \coprod_\lambda S_\lambda^{n-1}$ and consider $\bar{f}^{n-1}(x) = \bar{i}_{n-1} f^{n-1}(x)$ in K^n. As a point of $\bar{f}^{n-1}(\coprod_\lambda CS_\lambda^{n-1})$, there exists a neighbourhood N_x of $x \in \coprod_\lambda CS_\lambda^{n-1}$ which meets only finitely many $A_i \in A$. As a point of $\bar{i}_{n-1}(K^{n-1})$, there exists a neighbourhood $M_{f^{n-1}(x)}$ of $f^{n-1}(x) \in K^{n-1}$ which meets only finitely many $B_j \in B$. Form the adjunction space $M_{f^{n-1}(x)} \coprod_{f^{n-1}|\{x\}} N_x$. Then clearly, $M_{f^{n-1}(x)} \coprod_{f^{n-1}|\{x\}} N_x$ is a neighbourhood of the point $\bar{f}^{n-1}(x) = \bar{i}_{n-1} f^{n-1}(x)$ in K^n and intersects only finitely many members of V. //

(1.2.8) Lemma: $(\forall n \geq 0)$ $K^{n+1} \setminus K^n$ is paracompact.

Proof: It is sufficient to show that every open cover of $K^{n+1} \setminus K^n$ has a locally finite refinement.

Let $\mathcal{O} = \{0_\alpha \mid \alpha \in \Lambda\}$ be an open cover of $K^{n+1} \setminus K^n$. Since K^n is closed in K^{n+1}, $K^{n+1} \setminus K^n$ is open in K^{n+1} and so $\mathcal{O}$ is a collection of sets open in K^{n+1}. Let G be any cover of K^n by open sets of K^{n+1}. Form the open cover $G \cup \mathcal{O} = \{V \mid \text{either } V \in \mathcal{O} \text{ or } V \in G\}$ of K^{n+1}. Since K^{n+1} is paracompact, there exists a locally finite refinement $\mathcal{B} = \{B_j \mid j \in J\}$ of $G \cup \mathcal{O}$. Form the cover $\{B_j \cap (K^{n+1} \setminus K^n) \mid B_j \in \mathcal{B}\}$ of $K^{n+1} \setminus K^n$. Clearly, this collection forms a locally finite refinement of $\mathcal{O}$. //

(1.2.9) <u>Theorem</u>: Every CW-complex K is paracompact.

<u>Proof</u>: Once again, it is sufficient to show that every open cover of K has a locally finite refinement.

Let $A = \{A_i \mid i \in I\}$ be an open cover of K. Then, for each $i \in I$, $A_i \cap K^n$ is open in K^n $(n \geq 0)$. Since, $(\forall n \geq 0)$ $K^{n+1} \setminus K^n$ is open in K^{n+1}, the sets $G_n = \{A_i \cap (K^{n+1} \setminus K^n) \mid A_i \in A\}$ form covers of $K^{n+1} \setminus K^n$ $(n \geq 0)$ by open sets of K^{n+1}. By (1.2.8), $K^{n+1} \setminus K^n$ is paracompact $(\forall n \geq 0)$ and hence, $(\forall n \geq 0)$ there exist locally finite refinements B_n of G_n. Take $\mathcal{B} = (\bigcup_{n \geq 0} B_n) \cup B_{-1}$ where B_{-1} is a locally finite refinement of the open cover $\{A_i \cap K^0 \mid A_i \in A\}$ of K^0. Clearly, $\mathcal{B}$ is a locally finite refinement of A. //

Let $K = \bigcup_{n \geq 0} K^n$ be a CW-complex where K^n is given by (1.2.2). Set $\bar{f}_\lambda^{n-1} = \bar{f}^{n-1}|_{CS_\lambda^{n-1}}$. Then $\bar{f}_\lambda^{n-1}(CS_\lambda^{n-1} \setminus S_\lambda^{n-1}) = \sigma_\lambda{}^n$ is an open subset of K^n (see [16; 1.3.2]). Also, $\bar{f}_\lambda^{n-1}(CS_\lambda^{n-1}) = \bar{\sigma}_\lambda{}^n$ is closed in K^n (and hence in K) as a compact subset of a Hausdorff space. We call $\sigma_\lambda{}^n$ and $\bar{\sigma}_\lambda{}^n$ an <u>open n-cell</u> and <u>closed n-cell</u>, respectively, of K.

(1.2.10) <u>Theorem</u>: Let X be a compact subset of a CW-complex K. Then X intersects only a finite number of open cells of K.

<u>Proof</u>: Let $K = \bigcup_{n \geq 0} K^n$ where K^n is given by (1.2.2). Let $A_n = \{\lambda \in \Lambda_n | X \cap \sigma_\lambda^n \neq \emptyset\}$. For each $\lambda \in A_n$, choose a point $x_\lambda \in X \cap \sigma_\lambda^n$. Since the σ_λ^n's are open and disjoint, the set $G_n = \{x_\lambda | \lambda \in A_n\}$ has the discrete topology and thus is finite, being contained in the compact space X. Thus $(\forall n \geq 0)$ A_n is finite.

We claim there exists an integer $N \geq 0$ such that $X \subset K^N$. Then, $(\forall n > N)$ $A_n = \emptyset$ and hence $\bigcup_{n \geq 0} A_n$ is finite. Thus X intersects only finitely many open cells of K.

We prove the claim by contradiction. Suppose that $(\forall n \geq 0)$ $X \cap (K \setminus K^n) \neq \emptyset$. Then choose $x_n \in X \cap (K \setminus K^n)$, $n \geq 0$, and let P be the set of these points. If P is finite, then there exists a positive integer n such that $P \subset K^n$. But $x_n \in P$. Hence, P must be infinite. Now, if $n > p$, $K \setminus K^n \subset K \setminus K^p$ and so $x_n \notin K^p$. Then $P \cap K^p$ contains at most p elements; that is, $P \cap K^p$ is finite. Similarly, if $Q \subset P$, $Q \cap K^p$ is finite. But K^p is T_1 and so $Q \cap K^p$ is closed in K^p and hence in K. Similarly for P. Thus any subset of P is closed in P and so P is discrete. But $P \subset X$ and X is compact. Hence, P must be finite, a contradiction. //

As an immediate consequence of (1.2.10), we have that any closed n-cell $\bar{\sigma}_\lambda^n$ of a CW-complex K intersects only a finite number of open cells of K. This property is the "closure-finite" property of J.H.C. Whitehead.

Let $A = \bigcup_{n \geq 0} A^n$ and $X = \bigcup_{n \geq 0} X^n$ be CW-complexes. A is said to be a <u>sub-CW-complex</u> (abbreviated subcomplex) of X if and only if $(\forall n \geq 0)$ A^n is a closed subset of X^n and $X^n \cap A = A^n$.

<u>Examples</u>: (1) Let $X = \bigcup_{n \geq 0} X^n$ be a CW-complex. We can regard X^n, $n \geq 0$, as a CW-complex by taking it as a colimit of the diagram

$$X^0 \hookrightarrow X^1 \hookrightarrow X^2 \hookrightarrow \ldots \hookrightarrow X^n \overset{=}{\hookrightarrow} X^n \overset{=}{\hookrightarrow} \ldots$$

Clearly, X^n, $n \geq 0$, is a subcomplex of X.

(2) Let X and Y be CW-complexes. By (1.1.3), $X \coprod Y$ is a CW-complex and clearly, X and Y are subcomplexes of $X \coprod Y$.

(3) The path components of a CW-complex are subcomplexes. Let $K = \bigcup_{n \geq 0} K^n$ be a CW-complex where K^n is given by (1.2.2) and let X be a path component of K. For each $\lambda \in \Lambda_n$, $\bar{f}_\lambda^{n-1}(CS_\lambda^{n-1})$ is path connected, being the continuous image of the path connected set CS_λ^{n-1}. For each $n \geq 0$, define $X^n = \bigcup_{\lambda \in A_n} \bar{f}_\lambda^{n-1}(CS_\lambda^{n-1})$ where $A_n = \{\lambda \in \Lambda_n | X \cap \bar{f}_\lambda^{n-1}(CS_\lambda^{n-1}) \neq \emptyset\}$.

Now, X^n is a closed subset of K^n if and only if $(\bar{f}^{n-1})^{-1}(X^n)$ is closed in $\coprod_{\lambda \in \Lambda_n} CS_\lambda^{n-1}$ and $(\bar{i}_{n-1})^{-1}(X^n)$ is closed in K^{n-1}. But $(\bar{f}^{n-1})^{-1}(X^n) = \coprod_{\lambda \in A_n} CS_\lambda^{n-1}$ and $(\bar{i}_{n-1})^{-1}(X^n) = \coprod_{\lambda \in A_n} f^{n-1}(S_\lambda^{n-1})$. Hence, $(\forall n \geq 0)$ X^n is a closed subset of K^n. Clearly, $K^n \cap X = X^n$ and so the path components of K are subcomplexes.

(4) Analogously to (3), we have that the connected components of a CW-complex are subcomplexes.

(5) The union and intersection of subcomplexes are subcomplexes.

Let $\{X_\alpha | \alpha \in \Lambda\}$ be a collection of subcomplexes of a CW-complex K and let $X = \bigcap_{\alpha \in \Lambda} X_\alpha$. For each $n \geq 0$, $X^n = \bigcap_{\alpha \in \Lambda} X^n_\alpha$. Since, for each $\alpha \in \Lambda$, X_α is a subcomplex of K, X^n_α is closed in K^n and so $X^n = \bigcap_{\alpha \in \Lambda} X^n_\alpha$ is closed in K^n for each n. Also, $K^n \cap X = K^n \cap (\bigcap_{\alpha \in \Lambda} X_\alpha) = \bigcap_{\alpha \in \Lambda} (K^n \cap X_\alpha) = \bigcap_{\alpha \in \Lambda} X^n_\alpha = X^n$. Hence, the intersection of subcomplexes is a subcomplex.

Similarly, take the collection $\{X_\alpha | \alpha \in \Lambda\}$ and let $X = \bigcup_{\alpha \in \Lambda} X_\alpha$. Then, for each $n \geq 0$, $X^n = \bigcup_{\alpha \in \Lambda} X^n_\alpha$. Since, for each $\alpha \in \Lambda$, X_α is a subcomplex of K, X^n_α is closed in K^n and hence $K^n \setminus X^n_\alpha$ is open in K^n. Thus, $\bigcup_{\alpha \in \Lambda} (K^n \setminus X^n_\alpha) = K^n \setminus \bigcup_{\alpha \in \Lambda} X^n_\alpha$ is open in K^n and so $(\forall n \geq 0)$ $X^n = \bigcup_{\alpha \in \Lambda} X^n_\alpha$ is closed in K^n. Also, $K^n \cap X = K^n \cap (\bigcup_{\alpha \in \Lambda} X_\alpha) = \bigcup_{\alpha \in \Lambda} (K^n \cap X_\alpha) = \bigcup_{\alpha \in \Lambda} X^n_\alpha = X^n$. Hence, the union of subcomplexes is a subcomplex.

Note that if L is a subcomplex of a CW-complex K, then L is closed in K since $L \cap K^n = L^n$ is closed in K^n for each $n \geq 0$.

(1.2.11) <u>Theorem</u>: Let K be a CW-complex. Then K is connected if and only if it is path connected.

<u>Proof</u>: Path connectedness implies connectedness is clear. Assume K is connected but not path connected. Write $K = \bigcup_{\alpha \in \Lambda} X_\alpha$ where, for each $\alpha \in \Lambda$, X_α is a path component of K. Now, the path components of K are subcomplexes and so closed in K. Also, the union of subcomplexes is again a subcomplex. So choose $\alpha' \in \Lambda$ and rewrite K as $K = X_{\alpha'} \cup (\bigcup_{\alpha \neq \alpha'} X_\alpha)$.

Then K is the union of two disjoint closed subsets and hence is disconnected, contrary to our assumption. //

The reader should now recall that a space A is said to be dominated by a space X if there exist maps $f : A \to X$ and $g : X \to A$ such that $gf \simeq 1_A$.

(1.2.12) Theorem: If a space A is dominated by a CW-complex X, then the path components of A are open.

Proof: Let $f : A \to X$, $g : X \to A$ be maps such that $gf \simeq 1_A$. Let U be a path component of A and, for each $x \in U$, consider $f(x) \in X$. Since X is locally path connected, there exists an open neighbourhood V of $f(x)$ in X such that V is path connected. By the continuity of f, $f^{-1}(V)$ is an open neighbourhood of x in A. We show that $f^{-1}(V) \subset U$. Then, for each $x \in U$, U is a neighbourhood of x in A and hence, U is open in A.

Take $y \in f^{-1}(V)$ and consider $f(y) \in V$. Since V is path connected, there exists a map $\lambda : I \to V$ $(I = [0, 1])$ such that $\lambda(0) = f(x)$, $\lambda(1) = f(y)$. Form the composite map $g\lambda : I \to g(V) \subset A$. Then $g\lambda(0) = gf(x)$, $g\lambda(1) = gf(y)$. Now, $gf \simeq 1_A$ and so, there exists $H : A \times I \to A$ such that $H(-, 0) = gf$, $H(-, 1) = 1_A$. Define $h : I \to A$ by $h(t) = H(x, t)$. Then $h(0) = H(x, 0) = gf(x)$ and $h(1) = H(x, 1) = x$. Similarly, define $k : I \to A$ by $k(t) = H(y, t)$ and so $k(0) = gf(y)$, $k(1) = y$. Now, define $r : I \to A$ as follows:

$$r(t) = \begin{cases} h(1 - 3t) & 0 \le t \le \frac{1}{3} \\ g\lambda(3t - 1) & \frac{1}{3} \le t \le \frac{2}{3} \\ k(3t - 2) & \frac{2}{3} \le t \le 1 \end{cases}$$

Clearly, r is continuous, and hence there exists a path in A joining x to y. Thus, $y \in U$. //

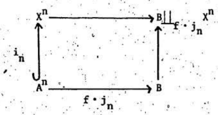
Given CW-complexes $A = \bigcup_{n \geq 0} A^n$ and $B = \bigcup_{n \geq 0} B^n$, a map $f : A \to B$ is said to be cellular if $(\forall n \geq 0)\ f(A^n) \subset B^n$.

(1.2.13) Theorem: Let $B = \bigcup_{n \geq 0} B^n$ and $X = \bigcup_{n \geq 0} X^n$ be CW-complexes and let $A = \bigcup_{n \geq 0} A^n$ be a subcomplex of X. If $f : A \to B$ is a cellular map, then $B \coprod_f X$ is a CW-complex.

Proof: (The following is just an outline. For a more detailed proof, see [16; 1.5.7].)

Let $j_n : A^n \hookrightarrow A$, $i_n : A^n \hookrightarrow X^n$ and $j_{n,m} : A^n \hookrightarrow A^m$ $(m \geq n)$ be the inclusion maps.

Consider the following pushout for each $n \geq 0$.



from which we get the following diagram

$$D : B\coprod_{f \cdot j_0} X^0 \xrightarrow{\bar{i}_0} B\coprod_{f \cdot j_1} X^1 \xrightarrow{\bar{i}_1} \cdots \xrightarrow{\bar{i}_{n-1}} B\coprod_{f \cdot j_n} X^n \xrightarrow{\bar{i}_n} \cdots$$

where, for each $n \geq 0$, $\bar{i}_n$ is the obvious inclusion map. By (1.1.3), $B\coprod_f X$ is homeomorphic to a colimit of diagram D. From a series of 'appropriate' pushout diagrams involving the cellular structures of X and A, one can show that $(\forall n \geq 0)$ $B\coprod_{f\cdot j_n} X^n$ is obtained from $B\coprod_{f\cdot j_{n-1}} X^{n-1}$ by the adjunction of n-cells. Now, let $K_n = B\coprod_{f\cdot j_n} X^n$ and let K_n^r stand for the r-skeleton of K_n. Since the map f is cellular, we have the following diagram:

$$
\begin{array}{ccccccccc}
K_0^0 & \longrightarrow & K_0^1 & \longrightarrow & K_0^2 & \longrightarrow \cdots \longrightarrow & K_0^r & \longrightarrow & \cdots \\
\| & & \cap & & \cap & & \cap & & \\
K_1^0 & \longrightarrow & K_1^1 & \longrightarrow & K_1^2 & \longrightarrow \cdots \longrightarrow & K_1^r & \longrightarrow & \cdots \\
\| & & \| & & \cap & & \cap & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \\
\| & & \| & & \| & & \| & & \\
K_n^0 & \longrightarrow & K_n^1 & \longrightarrow & K_n^2 & \longrightarrow \cdots \longrightarrow & K_n^r & \longrightarrow & \cdots \\
\| & & \| & & \| & & \| & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

where $K_n^i = K_{n+1}^i$ if $i \leq n$ and each square is commutative. Now, K_0^0 is a discrete space and K_n^n is obtained from K_{n-1}^{n-1} by the adjunction of n-cells. Hence, a colimit of

$$K_0^0 \longrightarrow K_1^1 \longrightarrow \cdots \longrightarrow K_n^n \longrightarrow \cdots$$

is a CW-complex. By (1.1.3), this colimit is homeomorphic to $B\coprod_f X$. //

CHAPTER II

Semisimplicial Complexes

The concept of a semisimplicial complex (abbreviated ssc) has meaning in any category $\underline{C}$. However, we will be interested only in semisimplicial sets. In this category a ssc can be regarded as a generalization of an ordered simplicial complex. Each element of a ssc has a dimension; one of dimension n is called an n-simplex and, like an ordered n-simplex, has $n + 1$ faces of dimension $n - 1$ and $n + 1$ degeneracies of dimension $n + 1$. Milnor has associated with each ssc X, a CW-complex $|X|$ called its geometric realization, which is a generalization from the case of an ordered simplicial complex. For each non-degenerate n-simplex x of X we associate an n-cell $|x|$ of $|X|$. However, in contrast to the situation for simplicial complexes, the cells of $|X|$ need not be homeomorphic to E^n, because of the equivalence relation put on the underlying set of $|X|$.

We will see that some of the semisimplicial theory developed in this chapter enters into the study of CW homotopy type in Chapter IV.

§1 - There are several equivalent definitions of a ssc but we will give only two. Both definitions will be used interchangeably throughout this chapter.

(A) Let $\underline{\Delta}$ be the category which has for objects the set of integers from 0 to n inclusive, $n = 0,1,2, \ldots$; (we denote the set $\{0,1, \ldots , n\}$ by [n]) and for morphisms monotonic functions $\alpha : [p] \longrightarrow [q]$; that is, whenever $0 \leq i \leq j \leq p$, $\alpha(i) \leq \alpha(j)$. We shall hereafter refer to montonic functions as operators.

We define a semisimplicial complex X to be a contravariant functor $X: \underline{\Delta} \longrightarrow \text{Set}$ and a semisimplicial map (ss map) $f: X \longrightarrow Y$ to be a natural transformation from the contravariant functor X to the contravariant functor Y. Elements of the set $X[n] = X_n$ are called the n-simplices of X. An element $x \in X_n$ is called degenerate if and only if there exists an operator $\beta \in \underline{\Delta}([n],[q])$, $q \leq n$, $\beta \neq 1$ and $y \in X_q$ such that $x = \beta^* y$ ($\beta^* = X(\beta)$). One not of this form is called nondegenerate.

(B) A semisimplicial complex X consists of a sequence of disjoint sets $X_0, X_1, \ldots,$ together with a collection of functions in each dimension n

$d_i : X_{n+1} \longrightarrow X_n \qquad i = 0,1, \ldots, n+1 \qquad$ the ith face operator

$s_i : X_n \longrightarrow X_{n+1} \qquad i = 0,1, \ldots, n \qquad$ the ith degeneracy operator

subject to the identities

(1) $d_i d_j = d_{j-1} d_i \qquad i < j$

(2) $s_i s_j = s_{j+1} s_i \qquad i \leq j$

(3)
$$d_i s_j = \begin{cases} s_{j-1} d_i & i < j \\ 1 & i = j, j+1 \\ s_j d_{i-1} & i > j+1 \end{cases}$$

The elements of X_n are called the n-simplices of X. If X and Y are ssc's, a semisimplicial map $f: X \longrightarrow Y$ is a sequence of maps

$f_n : X_n \longrightarrow Y_n$, commuting with the face and degeneracy operators. A simplex of the form $s_i x$ is degenerate; one not of this form is non-degenerate.

To see that (A) is equivalent to (B) we first notice that in $\underline{\Delta}$ we have the following distinguished morphisms

$$\lambda^i_{n+1} = \lambda^i : [n] \longrightarrow [n+1] \qquad 0 \leq i \leq n+1$$

defined by

$$\lambda^i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}$$

and $$\mu^i_{n+1} = \mu^i : [n+1] \longrightarrow [n] \qquad 0 \leq i \leq n$$

defined by

$$\mu^i(j) = \begin{cases} j & j \leq i \\ j-1 & j > i \end{cases}$$

subject to the identities

$$(1) \quad \lambda^j \lambda^i = \lambda^i \lambda^{j-1} \qquad i < j$$

$$(2) \quad \mu^j \mu^i = \mu^i \mu^{j+1} \qquad i \leq j$$

$$(3) \quad \mu^j \lambda^i = \begin{cases} \lambda^i \mu^{j-1} & i < j \\ 1 & i = j,\ j+1 \\ \lambda^{i-1} \mu^j & i > j+1 \end{cases}$$

Any operator in $\underline{\Delta}$ is composed of these morphisms λ^i, μ^j (see [17;1.1.4]). In fact, any operator α in $\underline{\Delta}$ can be written uniquely as $\alpha = v.u$ where u is a surjective operator and v is an injective operator.

If $X:\underline{\Delta} \longrightarrow \underline{Set}$ is a contravariant functor, then by defining

$$X_0 = X[0], \; X_1 = X[1], \; \ldots$$

and

$$d_i^{n+1} = X(\lambda_{n+1}^i) : X_{n+1} \longrightarrow X_n, \quad i = 0, \ldots, n+1$$

$$s_i^{n+1} = X(\mu_{n+1}^i) : X_n \longrightarrow X_{n+1}, \quad i = 0, \ldots, n$$

it is easy to see that this construction yields a ssc. Conversely, starting with (B) we get a contravariant functor $X:\underline{\Delta} \longrightarrow \underline{Set}$ in the obvious way.

We denote by SSC, the category of all semisimplicial complexes and semisimplicial maps.

Examples: (1) Standard n-simplex $\Delta[n]$

For each integer $n \geq 0$, $\Delta[n]$ is the ssc defined as follows. A q-simplex of $\Delta[n]$ is an operator $\sigma:[q] \longrightarrow [n]$. For each operator $\beta:[p] \longrightarrow [q]$, the p-simplex $\sigma\beta$ is defined as the composite

$$[p] \xrightarrow{\beta} [q] \xrightarrow{\sigma} [n]$$

Given the ssc's $\Delta[m]$ and $\Delta[n]$, for each operator $\alpha:[m] \longrightarrow [n]$ we define $\Delta\alpha:\Delta[m] \longrightarrow \Delta[n]$ to be the ss map which assigns to each q-simplex $\tau \in \Delta[m]$ the composite

$$[q] \xrightarrow{\tau} [m] \xrightarrow{\alpha} [n]$$

In particular we define the ss maps

$$d_i^* : \Delta[n] \longrightarrow \Delta[n+1]$$

$$s_i^* : \Delta[n] \longrightarrow \Delta[n-1]$$

by means of the operators

$$\lambda^i : [n] \longrightarrow [n+1] \quad , \qquad i = 0,1, \ldots ,n+1$$

$$\mu^i : [n] \longrightarrow [n-1] \quad , \qquad i = 0,1, \ldots ,n$$

If $\sigma:[p] \longrightarrow [n]$ is an operator of $\Delta[n]$, then $d_i^*(\sigma) = \lambda^i\sigma$.

Similarly, $s_i^*(\sigma) = \mu^i\sigma$.

We call d_i^* the ith face map and s_i^* the ith degeneracy map. One should note that the corresponding categorical construction of $\Delta[n]$ requires that $\Delta[n]$ be a covariant functor from $\underline{\Delta}^{OPP}$ to Set.

(2) Singular Complex

Let $\mathbf{R}^{n+1}$ be the (n+1)-dimensional real vector space with orthogonal basis $A_i = (0, \ldots ,0,1,0, \ldots ,0)$ ith vertex = 1, $i = 0, \ldots ,n$.

Define $\Delta_n = \{u = \sum_{i=0}^{n} u_iA_i \mid u_i \geq 0, \sum_{i=0}^{n} u_i = 1\}$. Δ_n is called the geometric n-simplex.

Given an operator $\alpha:[n] \longrightarrow [q]$, α induces a linear map $|\alpha|:\Delta_n \longrightarrow \Delta_q$ defined by

$$|\alpha|\left(\sum_{i=0}^{n} u_iA_i\right) = \sum_{i=0}^{n} u_iA_{\alpha(i)}$$

It then follows quite easily that $|\alpha\beta| = |\alpha||\beta|$ and $|1| = 1$.

Given a topological space X, we define the singular complex of X, denoted SX, to be the ssc defined as follows. A q-simplex of SX is a map $x_q:\Delta_q \longrightarrow X$. If $\alpha:[n] \longrightarrow [q]$ is an operator, we define

$SX(\alpha) : \underline{Top}(\Delta_q,X) \longrightarrow \underline{Top}(\Delta_n,X)$ by $SX(\alpha)(x_q) = x_q|\alpha|$.

If $f:X \longrightarrow Y$ is a continuous function, then f induces a semisimplicial map $Sf:SX \longrightarrow SY$ given by $Sf(x_q) = fx_q$, where $x_q:\Delta_q \longrightarrow X$. It is clear from the above definitions that S is a functor from Top to SSC.

In fact, this functor S has some very nice properties, as we will see shortly.

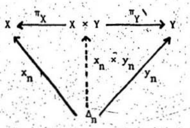
Given $X, Y \in \text{Ob}\underline{SSC}$, we define the cartesian product $X \times Y$ to be the semisimplicial complex defined as follows. For each $n \geq 0$, $(X \times Y)_n = X_n \times Y_n$. If $\alpha:[q] \longrightarrow [n]$ is an operator, then $(X \times Y)(\alpha)$: $(X \times Y)_n \longrightarrow (X \times Y)_q$ is defined by $X(\alpha) \times Y(\alpha):X_n \times Y_n \longrightarrow X_q \times Y_q$.

(2.1.1) **Proposition:** $S(X \times Y) = SX \times SY$

Proof: Using the categorical definition of a ssc, it is sufficient to show that both contravariant functors coincide on the objects and morphisms of Δ.

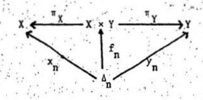
For each $n \geq 0$, $S(X \times Y)([n]) = S(X \times Y)_n = \underline{Top}(\Delta_n, X \times Y)$ and $(SX \times SY)([n]) = (SX \times SY)_n = (SX)_n \times (SY)_n = \underline{Top}(\Delta_n,X) \times \underline{Top}(\Delta_n,Y)$

But, by the universal property of products, there exists for each $x_n:\Delta_n \longrightarrow X$ and $y_n:\Delta_n \longrightarrow Y$ a unique map $x_n \times y_n:\Delta_n \longrightarrow X \times Y$ making the following diagram commute



Hence, both functors coincide on the objects of $\underline{\Delta}$.

Now, let $\alpha:[q] \longrightarrow [n]$ be an operator. Then α gives rise to the linear map $|\alpha|:\Delta_q \longrightarrow \Delta_n$. Let $f_n:\Delta_n \longrightarrow X \times Y$. Then by the universal property of products there exist unique maps $x_n:\Delta_n \longrightarrow X$, $y_n:\Delta_n \longrightarrow Y$ such that the following diagram commutes.



We can write f_n as $f_n = x_n \times y_n$. Then $S(X \times Y)(\alpha)(f_n) = f_n|\alpha| = (x_n \times y_n)|\alpha| = x_n|\alpha| \times y_n|\alpha| = (SX)(\alpha)(x_n) \times (SX)(\alpha)(y_n) = (SX \times SY)(\alpha)(x_n \times y_n) = (SX \times SY)(\alpha)(f_n)$.

Hence, both functors coincide on the morphisms of $\underline{\Delta}$ //

(2.1.2) Proposition: For each $n \geq 0$, there is a semisimplicial map $i:\Delta[n] \longrightarrow S\Delta_n$.

Proof: Using the categorical definition of a ssc we show that i is a natural transformation.

For each $[q] \in \mathrm{Ob}\underline{\Delta}$ define $i_q:\Delta[n]_q \longrightarrow (S\Delta_n)_q$ as follows:

If $\sigma:[q] \longrightarrow [n]$, then $i_q(\sigma) = |\sigma|:\Delta_q \longrightarrow \Delta_n$. We must show that for every $[q] \in \mathrm{Ob}\underline{\Delta}$ and operator $\alpha:[q] \longrightarrow [r]$ the following diagram commutes:

$$\begin{array}{ccc}
\Delta[n]_q & \xrightarrow{\ i_q\ } & (S\Delta_n)_q = \underline{Top}(\Delta_q, \Delta_n) \\
\Big\uparrow{\scriptstyle \Delta[n](\alpha)} & & \Big\uparrow{\scriptstyle (S\Delta_n)(\alpha)} \\
\Delta[n]_r & \xrightarrow{\ i_r\ } & (S\Delta_n)_r = \underline{Top}(\Delta_r, \Delta_n)
\end{array}$$

Consider $\tau:[r] \longrightarrow [n]$ belonging to $\Delta[n]_r$. Then $[i_q \Delta[n](\alpha)](\tau) = i_q(\tau\alpha) = |\tau\alpha| = |\tau||\alpha|$

But $[(S\Delta_n)(\alpha) i_r](\tau) = (S\Delta_n)(\alpha)|\tau| = |\tau||\alpha|$. Hence, the above diagram commutes and so i is a natural transformation //

Let $X, Y \in Ob\underline{SSC}$ and let $f, g: X \longrightarrow Y$ be semisimplicial maps. We say that f and g are <u>semisimplicially homotopic</u> if there exists a semisimplicial map $F: X \times \Delta[1] \longrightarrow Y$ such that for each $[n] \in Ob\underline{\Delta}$, $F_n \mid X_n \times 0 = f_n$ and $F_n \mid X_n \times 1 = g_n$, where $0:[n] \longrightarrow [1]$ is defined by $0(i) = 0$ $(\forall\, 0 \leq i \leq n)$ and $1:[n] \longrightarrow [1]$ is defined by $1(i) = 1$ $(\forall\, 0 \leq i \leq n)$.

(2.1.3) <u>Proposition</u>: Let X and Y be topological spaces and let $f, g: X \longrightarrow Y$. If f and g are homotopic, then $Sf, Sg: SX \longrightarrow SY$ are semisimplicially homotopic.

<u>Proof</u>: First notice that the geometric 1-simplex, Δ_1, can be identified with the unit interval I.

Let $h: X \times I \longrightarrow Y$ be a homotopy between f and g. Now, $SX \times S\Delta_1 = S(X \times \Delta_1) = S(X \times I)$ and by (2.1.2) there exists a semisimplicial map $i:\Delta[1] \longrightarrow S\Delta_1$. Consider the following composition of semisimplicial maps.

$$SX \times \Delta[1] \xrightarrow{1\times i} SX \times S\Delta_1 = S(X \times I) \xrightarrow{Sh} SY$$

This is a semisimplicial homotopy between Sf and Sg. //

We now define what we mean by the geometric realization of a semi-simplicial complex. This concept, as earlier stated, is due to Milnor [15].

Let $X \in \text{Ob}\underline{SSC}$. If $\alpha:[n] \longrightarrow [q]$ is an operator, we denote by α^*, the function $X(\alpha):X_q \longrightarrow X_n$. Let $\overline{X} = \coprod_n X_n \times \Delta_n$ and let $\sim$ be the equivalence relation on $\overline{X}$ generated by the following relation R:

$$(\alpha^* x, u) \; R \; (x, |\alpha| u)$$

where $x \in X_q$, $u \in \Delta_n$, $|\alpha|:\Delta_n \longrightarrow \Delta_q$

Thus $(x,u) \sim (y,v)$ if there is a finite chain of such relations given above, beginning at (x,u) and ending at (y,v).

We define the <u>geometric realization</u> of X, denoted $|X|$, to be the quotient $\overline{X}/\sim$. We denote the elements of $|X|$ by $|x,u|$. Let $\pi:\overline{X} \longrightarrow |X|$ be the quotient function defined by $\pi(x,u) = |x,u|$. Then, giving to each X_n the discrete topology and to each Δ_n the subspace topology of $\mathbb{R}^{n+1}$, $|X|$ becomes a topological space with the quotient topology, that is, the final topology with respect to π.

Given a semisimplicial map $f:X \longrightarrow Y$, let $\overline{f}:\overline{X} \longrightarrow \overline{Y}$ be the map defined by $\overline{f}(x,t) = (f(x),t)$. Then this induces a function $|f| : |X| \longrightarrow |Y|$ on the quotients, defined by $|f||x,u| = |f(x),u|$, such that the following diagram commutes

$$\begin{array}{ccc} \overline{X} & \xrightarrow{\overline{f}} & \overline{Y} \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ |X| & \overset{|f|}{\dashrightarrow} & |Y| \end{array}$$

Since $\pi_Y \bar{f}$ is continuous and π_X' is an identification map, $|f|$ is continuous.

Thus $||$ is a covariant functor from SSC to Top, called the geometric realization functor. In fact, $||$ is a covariant functor from SSC to CW (see [17,4.2.5]) and $||$ is left adjoint to our functor S earlier defined (see [17;4.2.3]).

From the definition of $||$, we have

(2.1.4) Proposition: $(\forall n \geq 0)$ $|\Delta[n]| = \Delta_n$ [17;4.2.7]

(2.1.5) Proposition: If $X \in$ ObSSC, $|X| = \coprod \text{In}x$ where x runs over all nondegenerate simplices of X and for each $x \in X_n$, $\text{In}x = \{|x,t| \in |X| \mid t \in \text{In}\Delta_n\}$ ($\text{In}\Delta_n$ = interior of Δ_n). [17;4.1.9]

Let $X, Y \in$ ObSSC and let $X \times Y$ be the cartesian product. Let $p: X \times Y \longrightarrow X$ and $p': X \times Y \longrightarrow Y$ be the projection maps. Then $|p| : |X \times Y| \longrightarrow |X|$ and $|p'| : |X \times Y| \longrightarrow |Y|$. Define $\eta : |X \times Y| \longrightarrow |X| \times |Y|$ by $\eta = |p| \times |p'|$.

(2.1.6) Theorem: $\eta : |X \times Y| \longrightarrow |X| \times |Y|$ is a bijection.

Proof: By (2.1.5) $|X \times Y| = \coprod \text{In}(x \times y)$ where $x \times y$ runs over all nondegenerate simplices of $X \times Y$. To show that η is a bijection we must show that η is bijective on all simplices $|x \times y, t|$ where $x \in X_n$, $y \in Y_n$, $t \in \text{In}\Delta_n$ and $x \times y$ is nondenerate.

Now x and y can be written uniquely as $x = \alpha^* x'$ and $y = \beta^* y'$ where α, β are surjective operators, $x' \in X_r$, $y' \in Y_s$, $r, s \leq n$ and x', y' are nondenerate. Then

$$
\begin{aligned}
n|x \times y,t| &= |p| \times |p'|\,|x \times y,t| \\
&= |x,t| \times |y,t| \\
&= |\alpha^* x',t| \times |\beta^* y',t| \\
&= |x',|\alpha|t| \times |y',|\beta|t| .
\end{aligned}
$$

Now, let $|x,t| \in |X|$, $|y,s| \in |Y|$ with $x \in X_r$, $t \in \mathrm{In}\Delta_r$, $y \in Y_m$, $s \in \mathrm{In}\Delta_m$, $t = (t_0, t_1, \ldots, t_r)$, $s = (s_0, s_1, \ldots, s_m)$. Assume $t_0 \leq s_0$. Then define $w \in \Delta_{r+m}$ by

$$
w = \left(t_0, t_1, \ldots, t_{p_0}, s_0 - \sum_{i=0}^{p_0} t_i, s_1, s_2, \ldots, s_{p_1}, \sum_{i=0}^{p_0+1} t_i - \sum_{i=0}^{p_1} s_i, t_{p_0+2}, \ldots, t_{p_2}, \sum_{i=0}^{p_1+1} s_i - \sum_{i=0}^{p_2} t_i, \ldots \right)
$$

where $p_0 < p_2 < p_4 < \ldots ; p_1 < p_3 < p_5 < \ldots$

and
$$
\sum_{i=0}^{p_j+1} t_i - \sum_{i=0}^{p_j+1} s_i > 0, \quad \sum_{i=0}^{p_j+1} t_i - \sum_{i=0}^{p_{j+1}+1} s_i < 0 \quad \text{for each } j = \text{even}
$$

$$
\sum_{i=0}^{p_j+1} s_i - \sum_{i=0}^{p_j+1} t_i > 0, \quad \sum_{i=0}^{p_j+1} s_i - \sum_{i=0}^{p_{j+1}+1} t_i < 0 \quad \text{for each } j = \text{odd}.
$$

Clearly there exists $\alpha : [r+m] \longrightarrow [r]$, $\beta : [r+m] \longrightarrow [m]$ such that $t = |\alpha| w$ and $s = |\beta| w$.

Now, define $\bar{n} : |X| \times |Y| \longrightarrow |X \times Y|$ by

$$
\bar{n}(|x,t| \times |y,s|) = |\alpha^* x \times \beta^* y, w| .
$$

Then

$$\overline{n}n(|x,t| \times |y,s|) = n|\alpha^*x \times \beta^*y,w|$$
$$= |\alpha^*x,w| \times |\beta^*y,w|$$
$$= |x,|\alpha|w| \times |y,|\beta|w|$$
$$= |x,t| \times |y,s|$$

Thus

$\overline{n}n = 1_{|X| \times |Y|}$ and similarly $n\overline{n} = 1_{|X \times Y|}$. Hence, n is a bijection. //

Given two ssc's X and Y, we know that their geometric realizations $|X|$ and $|Y|$, respectively, are CW-complexes. But the cartesian product of two CW-complexes need not be a CW-complex (see [5]). However, if $|X| \times |Y|$, in the above theorem, is a CW-complex, then in fact $\overline{n}$ is continuous (see [17;p.81]) and so n is a homeomorphism. We then have as an immediate consequence

(2.1.7) Corollary: A semisimplicial homotopy $h:K \times \Delta[1] \longrightarrow K'$ induces an ordinary homotopy $|K| \times [0,1] \longrightarrow |K'|$.

Proof: By (2.1.4) $|\Delta[1]| = \Delta_1 \cong [0,1]$. Since $[0,1]$ is compact, $|K| \times [0,1]$ is a CW-complex. The homotopy is now given by the following composition.

$$|K| \times [0,1] \xrightarrow{\overline{n}} |K \times \Delta[1]| \xrightarrow{|h|} |K'| \quad //$$

§2 - Barratt's Result

The remainder of this chapter will be devoted to the development of the result that "the realization of any ssc can be triangulated". This result is due to M. Barratt [1]. Subsequent papers on this subject were written by S. Weingram [22], R. Fritsch [8] and R. Fritsch and D. Puppe [9]. We rely heavily on the proofs given by Fritsch and Puppe.

The technique of proof is as follows.

We define, on the category of ssc's, a functor Sd which we call the barycentric subdivision functor. We show that, for any ssc X, SdX belongs to a class of ssc's, which we call regulated ssc's, with the property that the realization is a regular CW-complex. By a regular CW-complex, we mean that each closed n-cell is homeomorphic to E^n. We then show that for any regular CW-complex, and in particular for $|SdX|$, there exists a semisimplicial complex K and a homeomorphism $k : |K| \longrightarrow |SdX|$. Using the proof by Fritsch and Puppe, we show that, for any ssc X, there is a homeomorphism $h : |SdX| \longrightarrow |X|$. Composing h with k, we get the desired result.

Let $\Delta[n]$ be the standard n-simplex. We define the barycentric subdivision $Sd\Delta[n]$, denoted $\Delta'[n]$, to be the ssc given as follows.

A q-simplex of $\Delta'[n]$ is a sequence $(\sigma_0, \ldots, \sigma_q)$ where σ_i's are non-degenerate simplices of $\Delta[n]$ (that is, the operator $\sigma_i : [\dim \sigma_i] \longrightarrow [n]$ is a monomorphism) and $\sigma_i = \sigma_{i+1}\alpha$, for some α. For each operator $\beta : [p] \longrightarrow [q]$ and q-simplex $(\sigma_0, \ldots, \sigma_q)$ we define the p-simplex $\beta^*(\sigma_0, \ldots, \sigma_q) = (\sigma_{\beta(0)}, \ldots, \sigma_{\beta(p)})$. For any $\alpha : [m] \longrightarrow [n]$, the subdivision of $\Delta\alpha$ is the ss map $\Delta'\alpha : \Delta'[m] \longrightarrow \Delta'[n]$ given by $\Delta'\alpha(\tau_0, \ldots, \tau_q) = (\sigma_0, \ldots, \sigma_q)$ where σ_i is the unique non-degenerate simplex of $\Delta[n]$ for which there exists an epimorphism $\gamma_i : [\dim \tau_i] \longrightarrow [\dim \sigma_i]$ such that the following diagram commutes:

$$\begin{array}{ccc} [\dim \tau_i] & \overset{\tau_i}{\rightarrowtail} & [m] \\ \gamma_i \big\downarrow & & \big\downarrow \alpha \\ [\dim \sigma_i] & \overset{\sigma_i}{\rightarrowtail} & [n] \end{array}$$

In particular this defines

$$\mathrm{Sd}\, d_i^* : \Delta'[n] \longrightarrow \Delta'[n+1]$$

and

$$\mathrm{Sd}\, s_i^* : \Delta'[n+1] \longrightarrow \Delta'[n]$$

where $d_i^*:\Delta[n] \longrightarrow \Delta[n+1]$ is the ith face map and $s_i^*:\Delta[n+1] \longrightarrow \Delta[n]$ is the ith degeneracy map defined earlier.

(2.2.1) Remark: Notice that we can also define $\Delta'[n]$ with a reordered structure; that is, a q-simplex of $\Delta'[n]$ would now be a sequence $(\sigma_q, \ldots, \sigma_0)$ where σ_i is still a face of σ_{i+1} but with the appropriate changes in the morphisms. Both definitions are equivalent. For convenience we will adopt this reordered structure on $\Delta'[n]$ when we come to talk about regulated simplices.

Given a ssc X, we define the ssc $\mathrm{Sd}X$ as follows: Let $\overline{X} = \coprod_{x \in X} x \times \Delta'[\dim x]$. Then a q-simplex of $\overline{X}$ is a pair (x,σ) where $x \in X$ and $\sigma \in \Delta'[\dim x]$ such that $\dim \sigma = q$, that is, $\sigma = (\sigma_0, \ldots, \sigma_q)$. Given the operator $\beta:[p] \longrightarrow [q]$, $\overline{\beta}(x,\sigma)$ is the p-simplex defined by $\overline{\beta}(x,\sigma) = (x, \beta^*\sigma)$. If (x,σ) and $(y,\tau) \in \overline{X}$, we define $(x,\sigma) \sim (y,\tau)$ if there exists an operator $\alpha:[\dim y] \longrightarrow [\dim x]$ such that $y = \alpha^* x$ and $\sigma = \Delta'\alpha\tau$. We then define $\mathrm{Sd}X = \overline{X}/\sim$. The ss operations are given by $d_i(x,\sigma) = (x,d_i\sigma)$ and $s_j(x,\sigma) = (x,s_j\sigma)$ where we let (x,σ) stand for the equivalence class of the element $x \times \sigma$ of $x^{\sigma} \times \Delta'[\dim x]$.

If $f:X \longrightarrow Y$ is a ss map, then $Sdf:SdX \longrightarrow SdY$ is defined by $Sdf(x,\sigma) = (fx,\sigma)$.

It is clear from the above definitions that Sd is a functor from the category of ssc to itself. Sd is called the barycentric subdivision functor.

Given any simplex $\sigma \in \Delta'[n]$, σ has a unique representation $\Delta'\alpha\tau$ where $\alpha:[p] \longrightarrow [n]$ is some injective operator and τ is an interior simplex of $\Delta'[p]$. By an interior simplex of $\Delta'[p]$ we mean a simplex which has for its last vertex, the vertex corresponding to the simplex $\Delta[p]$ itself. For example if the q-simplex $\tau = (\tau_0, \ldots, \tau_q)$ is an interior simplex of $\Delta'[p]$, then $\tau_q = 1:[p] \longrightarrow [p]$. If we view $\Delta'[p]$ with its reordered structure, then $1:[p] \longrightarrow [p]$ would appear as the zeroth vertex.

In the equivalence class of each element (x,σ) of $\overline{X}$ there exists a unique irreducible representative (y_p,τ) where τ is interior to $\Delta'[p]$ and y_p is a nondegenerate simplex of X_p. We determine this irreducible representative as follows:

Represent σ uniquely as $\Delta'\alpha\tau'$ where α is some injective operator and τ' is an interior simplex. Then

$$(x,\sigma) = (x,\Delta'\alpha\tau') \sim (\alpha^*x,\tau')$$

Now we can write α^*x uniquely as β^*y_p where β is a surjective operator (possibly the identity) and y_p is nondegenerate. Then

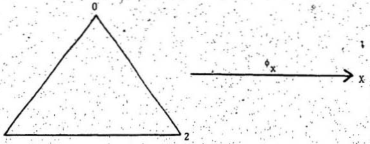
$(\alpha^*x,\tau') = (\beta^*y_p,\tau') \sim (y_p,\Delta'\beta\tau')$ Set $\tau = \Delta'\beta\tau'$. Since β is surjective, $\Delta'\beta$ is a simplicial map and so maps the interior simplex τ' to an interior simplex. Therefore (y_p,τ) is an irreducible point of $\overline{X}$.

Notice that any simplex (x,τ) where x is a nondegenerate simplex of X_n and τ is an interior simplex of $\Delta'[n]$ is irreducible and any two such simplices represent distinct equivalence classes in SdX. We call such simplices <u>nondegenerate simplices</u> of SdX.

For any n-simplex x of a ssc X, there is a ss map $\phi_x:\Delta[n] \longrightarrow X$ defined by $\phi_x(1_{[n]}) = x$. ϕ_x is called the <u>characteristic map</u> of the simplex x. Note that ϕ_x is completely determined by its action on the fundamental n-simplex $1:[n] \longrightarrow [n]$; for if $\alpha:[p] \longrightarrow [n]$ belongs to $\Delta[n]$, then $\phi_x(\alpha) = \alpha^*(\phi_x(1_{[n]})) = \alpha^*x$.

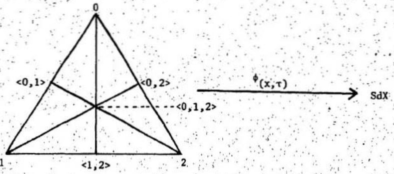
Using this characteristic map ϕ_x, we can define the corresponding characteristic map for the pair (x,σ) of SdX. In this case the characteristic map $\phi_{(x,\sigma)}$ of (x,σ) is the composition of the inclusion of σ into $\Delta'[n]$, followed by the ss map $Sd\phi_x$. $\phi_{(x,\sigma)}$ carries the simplex σ of $\Delta'[n]$ to the equivalence class of the irreducible representative of (x,σ). Notice, that if x is nondegenerate and τ,τ' are interior simplices of $\Delta'[n]$ with $\tau \neq \tau'$, then $Sd\phi_x(\tau) \neq Sd\phi_x(\tau')$; that is, $Sd\phi_x$ is bijective on the interior simplices of $\Delta'[n]$.

Given x an n-simplex of the ssc X and $\phi_x:\Delta[n] \longrightarrow X$ its characteristic map, we say that x is <u>regulated</u> if the restriction of ϕ_x to $\Delta[n] \setminus d_0^*(\Delta[n-1])$ is injective. X is said to be <u>regulated</u> if each nondegenerate simplex of X is regulated. Geometrically, this concept of a regulated simplex means that whenever you have two distinct faces σ, σ' of $\Delta[n]$, which both contain the zeroth vertex, $\phi_x(\sigma) \neq \phi_x(\sigma')$. For example, consider the characteristic map $\phi_x : \Delta[2] \longrightarrow X$, where x is a 2-simplex of X.



Then x is regulated if ϕ_x is injective on all faces of the triangle except the faces <1>, <2>, <1,2>.

Consider $\Delta'[2]$ and order its simplices as shown in Remark (2.2.1). Let $\phi_{(x,\tau)}$ be the characteristic map for the simplex (x,τ) of SdX, where $\tau = (1_{[2]}, \ldots, \tau_0)$ is an interior simplex of $\Delta'[2]$ and ϕ_x is the characteristic map given above.



Note that the barycentre <0,1,2> corresponds to the operator $1:[2] \longrightarrow [2]$.

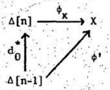
Now, the simplex (x,τ) is regulated if, whenever two faces

σ, σ' of $\Delta'[2]$ which both contain the barycentre as the zeroth vertex, that is $\sigma = (1_{[2]}, \ldots, \sigma_0)$, $\sigma' = (1_{[2]}, \ldots, \sigma'_0)$, $\phi_{(x,\tau)}(\sigma) \neq \phi_{(x,\tau)}(\sigma')$; that is, $\phi_{(x,\tau)}$ must be injective on all faces except those on the boundary. But, then σ and σ' are interior simplices of $\Delta'[2]$ and so if x is nondegenerate then, since $Sd\phi_x$ is bijective on the interior simplices of $\Delta'[2]$, (x,τ) is regulated. Now, $Sd\phi_x$ is bijective on the interior simplices of $\Delta'[n]$, for any nondegenerate n-simplex x, and so SdX is regulated. The property of being regulated is the only property of SdX we will need.

(2.2.2) <u>Lemma</u>: Let X be a ssc and x a nondegenerate n-simplex of X such that the subcomplex <x> of X which is generated by x is regulated. Let $\phi_x : \Delta[n] \longrightarrow X$ denote the characteristic map of X. Then there exists an integer p and a face operator F such that $|\phi_x|$ is injective on all open cells outside the p-dimensional face $F^*\Delta_p$ of Δ_n. On this face there is a face $F'^*\Delta_q$ such that the restriction of $|\phi_x|$ to $F^*\Delta_p$ is $|\phi'|\cdot D^*$, where D is the identity operator or a suitable degeneracy operator, F'x a nondegenerate simplex of X and ϕ' its characteristic map.

<u>Proof</u>: Since x is regulated, ϕ_x is injective on $\Delta[n] \setminus d_0^*(\Delta[n-1])$. Now, if d_0x is non-degenerate then it is also regulated since it belongs to <x>, which by assumption is regulated. Thus if ϕ' is the characteristic map for d_0x, then ϕ' is injective on $\Delta[n-1] \setminus d_0^*(\Delta[n-2])$. But $d_0^*(1_{[n-1]}) = \lambda_n^0 : [n-1] \longrightarrow [n]$ and $\phi'(1_{[n-1]}) = d_0x$. Thus

$\phi_x d_0^*(1_{[n-1]}) = \phi_x(\lambda_n^0) = X(\lambda_n^0)(x) = d_0 x = \phi'(1_{[n-1]})$ and so the following diagram commutes.



Since ϕ_x is injective on $\Delta[n] \setminus d_0^*(\Delta[n-1])$ and ϕ' is injective on $\Delta[n-1] \setminus d_0^*(\Delta[n-2])$, ϕ_x must be injective on $\Delta[n] \setminus (d_0)^*(\Delta[n-2])$. This conclusion can be continued until either

(1) ϕ_x is injective on all simplices of $\Delta[n]$ - then there is nothing to prove

OR (2) there is an integer p' such that $d_0^{p'} x$ is degenerate.

In this case let $\overline{p}$ denote the smallest such integer and let $p = n - \overline{p}$ and $F = d_0^{\overline{p}}$. Since $d_0^{\overline{p}-1} x$ is nondegenerate, ϕ_x is injective on $\Delta[n] \setminus F^*(\Delta[p])$ and hence $|\phi_x|$ is injective on $\Delta_n \setminus F^*\Delta_p$. Now, Fx is degenerate and so there exists a unique nondegenerate simplex $y \in X_q$, $q \leq p$ and a unique degeneracy operator D such that $Fx = Dy$. Let F'' be any face operator such that $F''D = 1$ and define $F' = F''F$. Then $F'x = F''Fx = F''Dy = y$. Let ϕ' be the characteristic map of $F'x = y$ and let $t \in \Delta_p$. Then $|\phi_x|.F^*t = (x, F^*t)$ and $|\phi'|.D^*t = (F'x, D^*t)$. But $(x, F^*t) \sim (Fx, t) = (Dy, t) \sim (y, D^*t) = (F'x, D^*t)$. Hence $|\phi_x|.F^* = |\phi'|.D^*$. //

By iteration of the above lemma we have

(2.2.3) Corollary: If x is a nondegenerate n-simplex of X such that the subcomplex of X generated by x is regulated, then $|\phi_x|:\Delta_n \longrightarrow |X|$ makes the following identifications (and no others):

There is a sequence of faces of Δ_n

$$\Delta_n = \tau_0 \supset \sigma_1 \supset \tau_1 \supset \sigma_2 \supset \tau_2 \supset \cdots \supset \sigma_r \supset \tau_r$$

of dimensions $\dim \sigma_i = p_i$, $\dim \tau_i = q_i$, and degeneracy maps D_i such that

(1) $|\phi_x||_{\tau_i}$ is bijective on all open cells outside of σ_{i+1}, $0 \leq i \leq r-1$

(2) $|\phi_x||_{\sigma_{i+1}} = (|\phi_x||_{\tau_{i+1}})D^*_{i+1}$, $0 \leq i \leq r-1$

(3) $|\phi_x|$ is bijective on the interior of τ_i, $0 \leq i \leq r-1$.

We now prove that if x is a nondegenerate simplex such that the subcomplex of X generated by x is regulated, then x realizes to a regular n-cell of the CW-complex $|X|$.

(2.2.4) Lemma: Let $\tau \subset \sigma \subset \Delta_n$ be proper faces, let $D^*:\sigma \longrightarrow \tau$ be a degeneration map, let L be the quotient of Δ_n by the identifications of D^* and let $\phi:\Delta_n \longrightarrow L$ be the quotient map. Then there is a homeomorphism $h:\Delta_n \longrightarrow L$ such that $h|\tau = \phi|\tau$.

Proof: Let σ' be the face of Δ_n opposite σ; that is, if Δ_n has basis $\{e_0, e_1, \ldots, e_n\}$ and σ is generated by the set $\{e_0, \ldots, e_p\}$ then σ' is generated by the set $\{e_{p+1}, \ldots, e_n\}$. If P is a point of Δ_n, then P can be written uniquely as $P = \sum_{i=0}^{n} \lambda_i e_i$ where $\lambda_i \geq 0$,

$\sum_{i=0}^{n} \lambda_i = 1$. Let $\lambda_0 + \lambda_1 + \ldots + \lambda_p = 1-t$ and $\lambda_{p+1} + \ldots + \lambda_n = t$.

Then $P = (\lambda_0 e_0 + \lambda_1 e_i + \ldots + \lambda_p e_p) + (\lambda_{p+1} e_{p+1} + \ldots + \lambda_n e_n)$

$$= \frac{1-t}{1-t}(\lambda_0 e_0 + \ldots + \lambda_p e_p) + \frac{t}{t}(\lambda_{p+1} e_{p+1} + \ldots + \lambda_n e_n)$$

$$= (1-t)\left(\left(\frac{\lambda_0}{1-t}\right)e_0 + \ldots + \left(\frac{\lambda_p}{1-t}\right)e_p\right) + t\left(\left(\frac{\lambda_{p+1}}{t}\right)e_{p+1} + \ldots + \left(\frac{\lambda_n}{t}\right)e_n\right)$$

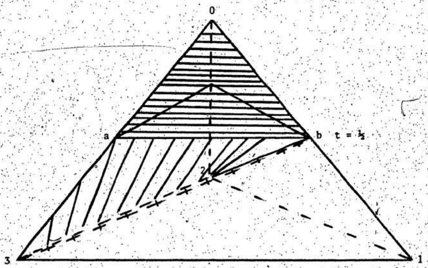
Notice, this is only true if $t \neq 0$, $t \neq 1$; that is, if $P \notin \sigma$, $P \notin \sigma'$. Since $\sum_{i=0}^{p} \frac{\lambda_i}{1-t} = 1$ and $\sum_{i=p+1}^{n} \frac{\lambda_i}{t} = 1$, $\left(\frac{\lambda_0}{1-t}\right)e_0 + \ldots + \left(\frac{\lambda_p}{1-t}\right)e_p \in \sigma$ and $\left(\frac{\lambda_{p+1}}{t}\right)e_{p+1} + \ldots + \left(\frac{\lambda_n}{t}\right)e_n \in \sigma'$. Therefore, if $P \notin \sigma$, $P \notin \sigma'$ then P can be written uniquely as $P = (1-t)Q + tQ'$ where $Q \in \sigma$, $Q' \in \sigma'$, $t \in [0,1]$.

Define a function $\rho: \Delta_n \longrightarrow \Delta_n$ as follows:

$$\begin{aligned} \rho(P) &= P & &\text{if } \tfrac{1}{2} \leq t \leq 1 \\ &= t(Q+Q') + (1-2t)D^*Q & &\text{if } 0 \leq t \leq \tfrac{1}{2} \end{aligned}$$

At $t = \frac{1}{2}$, $P = \frac{1}{2}(Q+Q') = \frac{1}{2}(Q+Q') + (1-1)D^*Q$ and so ρ is continuous.

The following diagram is meant to give some insight into how this map ρ acts on a particular subset of Δ_3.



Let $\sigma = \langle 1,2,3 \rangle$, $\tau = \langle 2,3 \rangle$ and $\sigma' = \langle 0 \rangle$. Above $t = \frac{1}{2}$, ρ is the identity. This region is shaded by horizontal lines. Consider the trapezium in the face $\langle 0,1,3 \rangle$ determined by the points a, b, 1 and 3. The image of the trapezium under ρ is shaded by the vertical lines. Notice that this plane cuts through the interior of the tetrahedron meeting the face $\langle 0,1,3 \rangle$ in the line segment determined by a and b. Notice also that the image of the trapezium under ρ is not convex since the interior points of the crossed line segment joining the points 3 and b do not belong to image ρ.

ρ has the following properties:

(i) Suppose $\rho(P) = \rho(P')$ where $P, P' \in \Delta_n$. If $P \in \sigma'$, then $P' \in \sigma'$ and so $P = \rho(P) = \rho(P') = P'$. Thus $\phi(P) = \phi(P')$. If $P \in \sigma$, then $P' \in \sigma'$ and so $D^*(P) = \rho(P) = \rho(P') = D^*P'$. Thus, again, $\phi(P) = \phi(P')$. Suppose now that $P, P' \notin \sigma$ and $P, P' \notin \sigma'$. Let $P = (1-s)R + sR'$ where $R \in \sigma$, $R' \in \sigma'$, $0 < s < 1$, and $P' = (1-t)Q + tQ'$ where $Q \in \sigma$, $Q' \in \sigma'$, $0 < t < 1$. Then $\rho(P) = s(R+R') + (1-2s)D^*R$ and $\rho(P') = t(Q+Q') + (1-2t)D^*Q$. But $\rho(P) = \rho(P')$ and so

$$sR + sR' + (1-2s)D^*R = tQ + tQ' + (1-2t)D^*Q$$

that is, $(1-s)\left(\frac{s}{1-s} R + \left(\frac{1-2s}{1-s}\right) D^*R\right) + sR' = (1-t)\left(\frac{t}{1-t} Q + \left(\frac{1-2t}{1-t}\right) D^*Q\right) + tQ'$

where $\frac{s}{1-s} R + \left(\frac{1-2s}{1-s}\right) D^*R \in \sigma$, $\frac{t}{1-t} Q + \left(\frac{1-2t}{1-t}\right) D^*Q \in \sigma$

Since $t \neq 0,1$ and $s \neq 0,1$, the above expressions are defined and unique. Hence, $s = t$, $R' = Q'$ and $\frac{s}{1-s} R + \left(\frac{1-2s}{1-s}\right) D^*R = \frac{t}{1-t} Q + \left(\frac{1-2t}{1-t}\right) D^*Q$ from which it follows that $R = Q$. Thus $P = P'$ and so $\phi(P) = \phi(P')$. Conversely, if $\phi(P) = \phi(P')$, $P, P' \in \Delta_n$, then either $P = P'$, in which case $\rho(P) = \rho(P')$, or, $D^*P = D^*P'$ where $P, P' \in \sigma$. In this case we again have $\rho(P) = \rho(P')$.

Hence, $\rho(P) = \rho(P')$ if and only if $\phi(P) = \phi(P')$.

(ii) Image ρ is a compact subset of Δ_n.

(iii) If $P \in \tau$, then $P = 1.P$ and $\rho(P) = D^*P$. But D^* is the identity on τ. So $\rho(P) = P$ for all $P \in \tau$.

Properties (i) and (ii) imply that image ρ is homeomorphic to L by means of a homeomorphism h':image $\rho \longrightarrow L$.

Property (iii) gives us that $h'|\tau = \phi|\tau$.

For each triple (Q,Q',t), where $Q \in \sigma$, $Q' \in \sigma'$, $t \in [0,1]$, let $w(Q,Q',t) = \{P = (1-t')Q + t'Q' \mid t \leq t' \leq 1\}$. Notice that if $P \in w(Q,Q',\frac{1}{2})$, then $\rho(P) = P$. Thus $w(Q,Q',\frac{1}{2}) \subset$ image ρ and so $w(Q,Q',\frac{1}{2}) \subset w(Q,Q',0) \cap$ image ρ. We show that for all $Q \in \sigma$, $Q' \in \sigma'$, $w(Q,Q',0) \cap$ image ρ is connected.

We can write $w(Q,Q',0) \cap$ image ρ as $w(Q,Q',\frac{1}{2}) \cup$ (image $\rho \cap X$) where $X = \{P = (1-t)Q + tQ' \mid 0 \leq t \leq \frac{1}{2}\}$. Now, $w(Q,Q',\frac{1}{2}) \cap$ (image $\rho \cap X$) $\neq \phi$ and $w(Q,Q',\frac{1}{2})$ is connected. So, it is sufficient to show that image $\rho \cap X$ is connected. Notice that image $\rho \cap X = \{P \in X \mid P = \rho(Y)$ for some $Y \in \Delta_n\}$. Let t_0 be the smallest point of $[0,1]$ such that $P_0 = (1-t_0)Q + t_0Q' \in$ image ρ. Since $\frac{1}{2}(Q+Q') \in$ image $\rho \cap X$ and $Q \notin$ image $\rho \cap X$, we have that $0 < t_0 \leq \frac{1}{2}$. We show that for all $t' \in [0,1]$ such that $t_0 < t' \leq \frac{1}{2}$, $P' = (1-t')Q + t'Q' \in$ image ρ. Since $P_0 \in$ image ρ, $P_0 = s(R+R') + (1-2s)D^*R$ for some $R \in \sigma$, $R' \in \sigma'$, $s \in [0,1]$. Thus $(1-t_0)Q + t_0Q' = sR + (1-2s)D^*R + sR'$

$$= (1-s)\left(\frac{s}{1-s}R + \left(\frac{1-2s}{1-s}\right)D^*R\right) + sR'$$

Since $t_0 \neq 0,1$, the representation $(1-t_0)Q + t_0Q'$ is unique and so $Q' = R'$, $t_0 = s$ and $Q = \frac{t_0}{1-t_0}R + \left(\frac{1-2t_0}{1-t_0}\right)D^*R$.

Applying D^* to Q we have that

$$D^*Q = \frac{t_0}{1-t_0}D^*R + \left(\frac{1-2t_0}{1-t_0}\right)D^*R = D^*R.$$

Take $H \in \sigma$ such that H belongs to the line segment in σ determined by the points D^*R, R and such that

$$Q = \frac{t'}{1-t'} H + \left(\frac{1-2t'}{1-t'}\right) D^*R$$

This is possible since $t_0 < t' \leq \frac{1}{2}$. If $t' = \frac{1}{2}$, then $w(Q,Q',0) \cap$ image $\rho = w(Q,Q',\frac{1}{2})$, and we are done.

So assume $t_0 < t' < \frac{1}{2}$. Then

$$D^*Q = \frac{t'}{1-t'} D^*H + \left(\frac{1-2t'}{1-t'}\right) D^*R$$

But $D^*Q = D^*R$; so $D^*H = D^*R$. Hence, $Q = \frac{t'}{1-t'} H + \left(\frac{1-2t'}{1-t'}\right) D^*H$

and so $P' = (1-t')Q + t'Q'$

$$= (1-t')\left(\frac{t'}{1-t'} H + \left(\frac{1-2t'}{1-t'}\right) D^*H\right) + t'Q'$$

$$= t'H + (1-2t')D^*H + t'Q'$$

$$= \rho(Y) \quad \text{where } Y = (1-t')H + t'Q'$$

Thus $P' \in$ image ρ and so image $\rho \cap X$ is connected.

Now, let H denote the convex hull of image ρ. Then H is a compact subset of Δ_n, being closed in Δ_n. We claim that $H \cap \sigma = \tau$.

Clearly $\tau \subset H \cap \sigma$. Take $P \in H \cap \sigma$. As a point of H, P can be written uniquely as $P = \sum_{i=1}^{n} \lambda_i P_i$ where $P_i \in$ image ρ, $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$. Since $P \in \sigma$, $\sum_{i=1}^{n} \lambda_i P_i \in \sigma$. Now, if $P_i \notin \sigma$ for some i, then $\lambda_i = 0$; otherwise $P \notin \sigma$. Rewrite P as $P = \sum_{i=1}^{m} \lambda_i P_i$ where

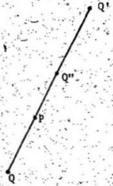
$P_i \in \sigma$, $\lambda_i > 0$, $\sum_{i=1}^{m} \lambda_i = 1$. Since $P_i \in \sigma$, $P_i \in \text{image } \rho$, for each $i = 1,2, \ldots, m$, $P_i = t(Q+Q') + (1-2t)D^*Q$ for some $Q \in \sigma$, $Q' \in \sigma'$, $t \in [0,1]$. If the coefficient t of Q' is non-zero, then $P_i \notin \sigma$. So $t = 0$ and so, for each $i = 1, \ldots, m$, $P_i = D^*Q \in \tau$. Hence $P \in \tau$.

For each $(Q,Q') \in \sigma \times \sigma'$ we define

$$h_{Q,Q'} : w(Q,Q',0) \cap H \longrightarrow w(Q,Q',0) \cap \text{image } \rho$$

to be the homeomorphism which maps its domain linearly onto its range. Since $w(Q,Q',0) \cap \text{image } \rho$ is connected, this is clearly possible.

Consider the following diagram.



Suppose the line segment determined by Q', P represents $w(Q,Q',0) \cap H$ and the line segment determined by Q', Q'' represents $w(Q,Q',0) \cap \text{image } \rho$. As a point of $w(Q,Q',0)$, P can be written as $P = (1-t)Q + tQ'$, $t \in [0,1]$. Then $h_{Q,Q'}(P) = (1-t)Q'' + tQ'$. The homeomorphism $h_{Q,Q'}$ is actually shrinking the line segment $\overline{PQ'}$ continuously to the line segment $\overline{Q''Q'}$. Notice that under this homeomorphism Q' remains fixed. Also, if $Q \in \tau$, then

$$w(Q,Q',0) \cap H = w(Q,Q',0) = w(Q,Q',0) \cap \text{image } \rho$$

Thus $h_{Q,Q'} = 1$ for all $Q \in \tau$.

We can now define a map $h_H : H \longrightarrow \text{image } \rho$ by

$$h_H(P) = h_{Q,Q'}(P) \quad \text{if} \quad P \in \text{domain } h_{Q,Q'}$$

It is clear that h_H defines a homeomorphism provided it is well-defined. We know that if $P \notin \sigma$ or $P \notin \sigma'$ then P has a unique representation as $P = (1-t)Q + tQ'$, $Q \in \sigma$, $Q' \in \sigma'$, $t \in [0,1]$. The only trouble that could occur is when $P \in \sigma$ or $P \in \sigma'$.

If $P \in \sigma'$, then $h_{Q,P}(P) = P$ for all $Q \in \sigma$ and so h_H is single-valued on points of σ'. If $P \in \sigma$, then $P \in \tau$ since $P \in H$ and $H \cap \sigma = \tau$. But $h_{P,Q'}(P) = P$ for all $Q' \in \sigma'$ and so h_H is simple-valued on points of σ. Hence, h_H is a well-defined map. Furthermore, for all $Q \in \tau$, $h_H(Q) = Q$.

Let $h'_H : \Delta_n \longrightarrow H$ be any homeomorphism which is constructed by a radial contraction to the barycentre of σ'. Each such homeomorphism is the identity on τ. Form the composite $h'' = h_H \cdot h'_H$. Then $h'' : \Delta_n \longrightarrow \text{image } \rho$ is a homeomorphism and furthermore, $h'' |_\tau = \rho |_\tau$.

Let $h = h' \cdot h''$. Then h is a homeomorphism from Δ_n onto L such that $h|_\tau = \phi|_\tau$. //

(2.2.5) <u>Proposition</u>: Let x be a nondegenerate n-simplex of the ssc X such that the subcomplex generated by x is regulated. Then $|x|$ is a regular n-cell of the CW-complex $|X|$. Hence, if X is a regulated ssc, then $|X|$ is a regular CW-complex.

<u>Proof</u>: Suppose x is the simplex for which ϕ_x has the form described in (2.2.3). Let L_i be the quotient of Δ_n by the identifications of $D_1^*, \ldots, D_i^*$ and let $\phi_i : \Delta_n \longrightarrow L_i$ be the quotient map for each i. Suppose for some k we have a homeomorphism $h_k : \Delta_n \longrightarrow L_k$ such that $\phi_k \mid \tau_{k+1} = h_k \mid \tau_{k+1}$. This is true for $k = 1$, by (2.2.4). Consider the following diagram.

$$\begin{array}{ccc}
\sigma_{k+1} & \xrightarrow{\;\cong\;} & h_k(\sigma_{k+1}) \subset L_k \\
\Big\downarrow D_{k+1}^* & & \Big\downarrow \\
\tau_{k+1} & \xrightarrow{\;\cong\;} & h_k(\tau_{k+1}) \subset L_k
\end{array}$$

Since h_k is a homeomorphism, the map induced by D_{k+1}^* on $h_k(\sigma_{k+1}) \subset L_k$ corresponds to D_{k+1}^* on $\sigma_{k+1} \subset \Delta_n$. Thus by (2.2.4) there is a homeomorphism $\overline{h} : L_k \longrightarrow L_{k+1}$ such that if $\overline{\phi} : L_k \longrightarrow L_{k+1}$ is the quotient map, then $\overline{h} \mid \tau_{k+2} = \overline{\phi} \mid \tau_{k+2}$. Set $h_{k+1} = \overline{h} h_k$ and $\phi_{k+1} = \overline{\phi}\phi_k$. Then $h_{k+1} \mid \tau_{k+2} = \phi_{k+1} \mid \tau_{k+2}$ and so after r steps we arrive at the homeomorphism $h_r : \Delta_n \longrightarrow L_r$. But, from (2.2.3), $L_r = |x|$.

Hence, $|x|$ is a regular n-cell of $|X|$. //

Let X be a regular CW-complex. Then each closed cell of X is a subcomplex. Moreover, each closed n-cell of X is homeomorphic to E^n. Define $T(X)$ to be the simplicial complex whose vertices are the cells of X and whose simplices are defined as follows: A finite collection of cells of X form the vertices of a simplex of $T(X)$ if and only if the cells of the collection can be arranged in order so that each is a proper face of the next. We topologize $|T(X)|$ by giving it the weak topology with respect to the closed simplices. Notice that $T(X) = \bigcup_n T(X^n)$.

(2.2.6) Lemma:- If σ is a cell of a regular complex, then $|T(\bar{\sigma})|$ is the join of the vertex $|\sigma|$ with the subcomplex $|T(\dot{\sigma})|$.

Proof: Let τ be a cell of $|T(\bar{\sigma})|$. Then $\tau = |<\sigma_0, \ldots, \sigma_k>|$ where σ_i's are cells of X and σ_i is a proper face of σ_{i+1}. Now, either $\sigma_k = \sigma$ or τ is a cell of $|T(\dot{\sigma})|$. If $\sigma_k = \sigma$, then $\bar{\tau}$, as a closed cell, is the join of the vertex $|\sigma|$ with the cell $|<\sigma_0, \ldots, \sigma_{k-1}>|$, unless $k = 0$. If $k = 0$, then $\tau = |\sigma|$. Thus the cells of $|T(\bar{\sigma})|$ are those of the join of the vertex $|\sigma|$ with the subcomplex $|T(\dot{\sigma})|$. //

(2.2.7) Theorem: If X is a regular CW-complex, then X is homeomorphic to $|T(X)|$.

Proof: We define a homeomorphism $h: |T(X)| \longrightarrow X$ by step-wise extension over the subcomplexes $|T(X^k)|$. For $k = 0$, $X^0 = T(X^0)$ and so we have the obvious homeomorphism $h_0: |T(X^0)| \longrightarrow X^0$. Suppose we have extended h_0 to a homeomorphism $h_{k-1}: |T(X^{k-1})| \longrightarrow X^{k-1}$. Let σ be a k-cell of X. Then, by (2.2.6), $|T(\bar{\sigma})| = |\sigma| \cdot |T(\dot{\sigma})|$. Choose a homeomorphism $f: E^k \longrightarrow \bar{\sigma} \cdot E^k$ is homeomorphic to the join of the origin with S^{k-1}. By hypothesis, $h^{\sigma}_{k-1}: |T(\dot{\sigma})| \longrightarrow \dot{\sigma}$ is a homeomorphism and so the map $f^{-1}h^{\sigma}_{k-1}: |T(\dot{\sigma})| \longrightarrow S^{k-1}$ is a homeomorphism. We extend $f^{-1}h^{\sigma}_{k-1}$ to a homeomorphism g mapping $|T(\bar{\sigma})|$ onto E^k which sends the vertex $|\sigma|$ into the origin

$$
\begin{array}{ccc}
|T(\bar{\sigma})| = |\sigma| \cdot |T(\dot{\sigma})| & \overset{g}{\dashrightarrow} & E^k \approx 0 \cdot S^{k-1} \\
\uparrow i & & \uparrow i \\
|T(\dot{\sigma})| & \xrightarrow[f^{-1}h^{\sigma}_{k-1}]{} & S^{k-1}
\end{array}
$$

We define h^{σ}_k on $|T(\bar{\sigma})|$ to be fg. Then on $|T(\dot{\sigma})|$ we have that $h^{\sigma}_k = fg = ff^{-1}h^{\sigma}_{k-1} = h^{\sigma}_{k-1}$ and so h^{σ}_k extends h^{σ}_{k-1}. We proceed in this manner for every k-cell of X. The resulting map $h_k = \coprod_{\sigma \in X^k} h^{\sigma}_k$ clearly extends h_{k-1} and is a homeomorphism because its inverse h^{-1}_k is continuous. Continuity of h^{-1}_k follows from the fact that X has the weak topology and h^{-1}_k is continuous on each closed k-cell. The collection of maps $\{h_k\}$ defines a bijective function $h: |T(X)| \longrightarrow X$. h^{-1} is continuous because it is continuous on each skeleton of X. h is also continuous because it is continuous, in the weak topology, on each subcomplex $|T(X^k)|$. Hence, h is a homeomorphism. //

(2.2.8) Theorem: For each ssc X there is a homeomorphism $h:|SdX|\longrightarrow|X|$.

Proof: Recall that $|X|$ is formed by taking the coproduct

$$\overline{X} = \coprod_q X_q \times \Delta_q \qquad (X_q \text{ discrete})$$

modulo the identification $(\alpha^*y,u) \sim (y,|\Delta\alpha|u)$ for all $y \in X_p$, $u \in \Delta_q$ and operators $\alpha:[q]\longrightarrow[p]$. Corresponding we form $|SdX|$ from $\overline{X}$ modulo the identification

$$(\alpha^*y,u) \sim (y,|\Delta'\alpha|u)$$

For each $x \in X_q$ and each $q \geq 0$, we construct a map $h_x:\Delta_q\longrightarrow\Delta_q$ such that the following hold:

(A) If $\alpha:[q]\longrightarrow[p]$ is an operator and $x = \alpha^*y$, then the following diagram commutes:

$$\begin{array}{ccc} \Delta_q & \xrightarrow{h_x} & \Delta_q \\ {\scriptstyle |\Delta'\alpha|}\downarrow & & \downarrow{\scriptstyle |\Delta\alpha|} \\ \Delta_p & \xrightarrow{h_y} & \Delta_p \end{array}$$

(B) If x is nondegenerate, then h_x maps the interior $I_n\Delta_q = \{u = (u_0,u_1,\ldots,u_q) \in \Delta_q \mid u_i > 0 \text{ for all } i\}$ of Δ_q bijectively onto itself.

Then the system (h_x) yields a map of $\overline{X}$ into itself defined as follows:

$$\text{if } (x,u) \in X_q \times \Delta_q, \ (h_x)(x,u) = (x,h_x(u))$$

This gives rise to a function $h: |SdX| \longrightarrow |X|$ in the obvious manner. We must check that h is well-defined. It is sufficient to show that if two elements of $|SdX|$ differ by a single elementary equivalence, then h maps them to the same equivalence class in $|X|$.

Consider $(\alpha^*y,u) \sim (y,|\Delta'\alpha|u)$ in $|SdX|$. Now $h|\alpha^*y,u| = |\alpha^*y,h_{\alpha^*y}(u)|$ and $h|y,|\Delta'\alpha|u| = |y,h_y(|\Delta'\alpha|u)|$. But $(\alpha^*y,h_{\alpha^*y}(u)) \sim (y,|\Delta_\alpha|h_{\alpha^*y}(u))$ and from (A), $(y,h_y|\Delta'\alpha|u) = (y,|\Delta\alpha|h_{\alpha^*y}(u))$. Hence, $h|\alpha^*y,u| = h|y,|\Delta'\alpha|u|$ and so h is well-defined.

Consider the following commutative diagram:

$$\begin{array}{ccc} \overline{X} & \xrightarrow{(h_x)} & \overline{X} \\ \downarrow{\scriptstyle \pi_{SdX}} & & \downarrow{\scriptstyle \pi_X} \\ |SdX| & \xrightarrow{h} & |X| \end{array}$$

Since π_{SdX} is an identification map and $\pi_X(h_x)$ is continuous, h is continuous.

From (2.1.5), $|X| = \coprod_{x \in X} \operatorname{In} x$ where x runs over all nondegenerate simplices of X and, by (B), h_x takes $\operatorname{In}\Delta_q$ bijectively onto itself, for each nondegenerate simplex x of X. Hence, h is a bijection. Because of (B), h_x is a surjection and thus an identification map, as a continuous surjection from a compact space to a Hausdorff space. We claim that h is an identification map. Then h being injective, we have immediately that h is a homeomorphism.

Since h is a continuous bijection it is sufficient to show that $|X|$ has the final topology with respect to the map h; that is, for all spaces Z and maps $g:|X|\longrightarrow Z$, g is continuous if and only if gh is continuous.

g is continuous implies gh is continuous is obvious. So suppose gh is continuous. Then $gh|_{In_x}$ is continuous for each nondegenerate simplex x in X. But $gh|_{In_x} = gh'_x$ and for each nondegenerate x in X, h_x is an identification map. Hence, $g|_{In\,x}$ is continuous, for each nondegenerate simplex x, and so g is continuous.

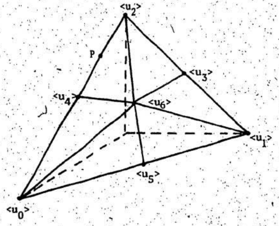
Construction of h_x.

Each point $u \in \Delta_q = |\Delta'[q]|$ can be written in the form

(2.2.9) $$u = \sum_{j=0}^{n} t_j \langle u_j \rangle$$

where $t_j \geq 0$, $\sum_{j=0}^{n} t_j = 1$ and the u_j's are injective operators with range $[q]$ which define an n-simplex $(u_0, u_1, \ldots, u_n)$ of $\Delta'[q]$. $\langle u_j \rangle$ is the barycentre of the face $|u_j|$ of Δ_q. (It is also the 0-cell which corresponds to the 0-simplex (u_j) of $\Delta'[q]$).

For example, consider $\Delta_2 = |\Delta'[2]|$, which is a full triangle in $\mathbb{R}^3$.



Using the form (2.2.9), we can write the point P above as $P = t_0\langle u_2\rangle + t_1\langle u_4\rangle$ where $t_0, t_1 > 0$, $t_0 + t_1 = 1$. But we could also write P as $P = t_0'\langle u_2\rangle + t_1'\langle u_2\rangle + t_2'\langle u_4\rangle$, where $t_0' + t_1' = t_0$, $t_2' = t_1$, or $P = t_0''\langle u_2\rangle + t_1''\langle u_4\rangle + t_2''\langle u_6\rangle$ where $t_0'' = t_0$, $t_1'' = t_1$, $t_2'' = 0$. Thus one can see that in the general dimension q, a point of Δ_q may have many distinct representations of the form (2.2.9). However, it is clear that we can go from one representation to the other by leaving out $t_j\langle u_j\rangle$ whenever $t_j = 0$ and replacing $t_j\langle u_j\rangle + t_{j+1}\langle u_{j+1}\rangle$ by $(t_j+t_{j+1})\langle u_j\rangle$ whenever $u_j = u_{j+1}$.

We define injective operators u_{kj}, $\overline{u}_{kj}$; surjective operators

ρ_j, ρ_{kj}; and nondegenerate simplices z_j of X by means of the following formulas.

$$\begin{array}{lll} & (1) & u_j^* x = \rho_j^* z_j \\ (2.2.10) & (2) & u_j = u_k u_{kj} \qquad (j \leq k) \\ & (3) & \rho_k u_{kj} = \overline{u}_{kj} \rho_{kj} \end{array}$$

Since u_j and u_k are injective operators, we have from (2) that u_{kj} is also injective.

From (3) we have the following commutative diagram:

$$\begin{array}{ccc} [\dim u_j] & \xrightarrow{\ \rho_{kj}\ } & [r_{kj}] \\ {\scriptstyle u_{kj}}\downarrow & & \downarrow{\scriptstyle \overline{u}_{kj}} \\ [\dim u_k] & \xrightarrow{\ \rho_k\ } & [r_k] \end{array}$$

Notice that $\dim u_k \geq r_k \geq r_{kj}$ and $\dim u_k \geq \dim u_j \geq r_{kj}$. Also, if $j = k$, then $u_{jj} = 1$ and so $\rho_j = \overline{u}_{jj}\rho_{jj}$. But ρ_j is surjective and so $\overline{u}_{jj}$ must also be surjective. Since $\overline{u}_{jj}$ is injective we conclude that $\overline{u}_{jj} = 1$ and so $\rho_j = \rho_{jj}$.

Now, given a surjective operator $\rho:[p] \longrightarrow [r]$ we define a right inverse $\beta:[r] \longrightarrow [p]$ as follows:

$$\beta(i) = \text{Max}\ \rho^{-1}(i).$$

We then define

$$h_x(u) = \sum_{0\leq j\leq n} t_j(1-t_n - \dots - t_{j+1}) \langle u_j\beta_{jj}\rangle + \sum_{0\leq j<k\leq n} t_j t_k \langle u_j\beta_{kj}\rangle$$

(For $j = n$ we set $1-t_n - \dots - t_{j+1} = 1$)

We must check that the sum of the coefficients of $h_x(u)$ is 1.

$$\sum_{0\leq j\leq n} t_j(1-t_n - \dots - t_{j+1}) + \sum_{0\leq j<k\leq n} t_j t_k = \sum_{0\leq j\leq n} t_j(1-t_n - \dots - t_{j+1})$$

$$+ t_0(t_1 + \dots + t_n) + \dots + t_{n-1}t_n$$

$$= t_0t_0 + t_1(t_0+t_1) + \dots + t_{n-1}(t_0 + \dots + t_{n-1}) + t_n + t_0(t_1 + \dots + t_n) + \dots + t_{n-1}t_n$$

$$= t_0(t_0+t_1 + \dots + t_n) + t_1(t_0+t_1 + \dots + t_n) + \dots + t_{n-1}(t_0 + \dots + t_n) + t_n$$

$$= t_0 + t_1 + \dots + t_n = 1.$$

So $h_x(u)$ is indeed a point of Δ_q. Now we know that each $u \in \Delta_q$ can have more than one distinct representation of the form (2.2.9). However, by going from one representation to the other in the way earlier described, it is clear that the value of $h_x(u)$ remains unchanged and so h_x is well-defined. In fact, $h_x(u)$ is uniquely determined and is clearly continuous.

Proof of (A): Let $u \in \Delta_q$ be represented as in (2.2.9). Given an operator $\alpha:[q] \longrightarrow [p]$, this gives rise to the ss map $\Delta'\alpha:\Delta'[q] \longrightarrow \Delta'[p]$ defined by $\Delta'\alpha(u_0, \dots, u_n) = (v_0, \dots, v_n)$, where v_j is the unique nondegenerate simplex of $\Delta[p]$ for which there exists a surjective operator $\tau_j:[\dim u_j] \longrightarrow [\dim v_j]$ such that the following diagram commutes:

$$\begin{array}{ccc} [\dim u_j] & \xrightarrow{\;u_j\;} & [q] \\ {\scriptstyle \tau_j}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ [\dim \nu_j] & \xrightarrow{\;\nu_j\;} & [p] \end{array}$$

We now construct injective operators ν_{kj}, $\overline{\nu}_{kj}$; surjective operators σ_j, σ_{kj}; and nondegenerate simplices z_j' of X from y and $(\nu_0, \ldots, \nu_n)$ in the same way we constructed ρ_{kj}, etc., from x and $(u_0, \ldots, u_n)$, according to (2.2.10). Thus we have

$$\begin{aligned} u_j^* x &= u_j^* \alpha^* y \\ (u_k u_{kj})^* x &= (\alpha u_j)^* y \\ u_{kj}^* u_k^* x &= (\nu_j \tau_j)^* y \\ u_{kj}^* \rho_k^* z_k &= \tau_j^* \nu_j^* y \\ (\rho_k u_{kj})^* z_k &= \tau_j^* (\nu_k \nu_{kj})^* y \qquad\qquad (j \leq k) \\ (\overline{u}_{kj} \rho_{kj})^* z_k &= \tau_j^* \nu_{kj}^* \nu_k^* y \\ &= \tau_j^* \nu_{kj}^* \sigma_k^* z_k' \\ &= \tau_j^* (\sigma_k \nu_{kj})^* z_k' \\ &= \tau_j^* (\overline{\nu}_{kj} \sigma_{kj})^* z_k' \\ &= (\overline{\nu}_{kj} \sigma_{kj} \tau_j)^* z_k' \end{aligned}$$

Since z_k and z_k' are nondegenerate simplices of X, it follows that $(\overline{u}_{kj} \rho_{kj})^* = (\overline{\nu}_{kj} \sigma_{kj} \tau_j)^*$ and so $\overline{u}_{kj} \rho_{kj} = \overline{\nu}_{kj} (\sigma_{kj} \tau_j)$. But $\overline{u}_{kj}$, $\overline{\nu}_{kj}$ are injective operators and ρ_{kj}, $\sigma_{kj} \tau_j$ are surjective operators. Since the composition of a surjective operator followed by an injective operator is unique, we have that

$$\bar{u}_{kj} = \bar{v}_{kj} \qquad (j \leq k)$$

$$\rho_{kj} = \sigma_{kj}\tau_j$$

From this we obtain

$$\alpha u_j\beta_{kj} = \alpha u_j\hat{\tau}_j\theta_{kj} = \nu_j\tau_j\hat{\tau}_j\theta_{kj} = \nu_j\theta_{kj}$$

(Notice that $\nu_j\theta_{kj}$ is injective).

Thus, $|\Delta\alpha| \quad <u_j\beta_{kj}> = <\alpha u_j\beta_{kj}> = <\nu_j\theta_{kj}>$

On the other hand

$$|\Delta'\alpha|\sum_{j=0}^{n} t_j<u_j> = \sum_{j=0}^{n} t_j|\Delta'\alpha| <u_j> = \sum_{j=0}^{n} t_j <\nu_j>$$

Hence, for each $u = \sum_{j=0}^{n} t_j <u_j> \in \Delta_q$

$$|\Delta\alpha|h_x(u) = |\Delta\alpha|\Big(\sum_{0\leq j\leq n} t_j(1-t_n-\dots-t_{j+1})<u_j\beta_{jj}> + \sum_{0\leq j<k\leq n} t_jt_k<u_j\beta_{kj}>\Big)$$

$$= \sum_{0\leq j\leq n} t_j(1-t_n-\dots-t_{j+1})|\Delta\alpha|<u_j\beta_{jj}> + \sum_{0\leq j<k\leq n} t_jt_k|\Delta\alpha|<u_j\beta_{kj}>$$

$$= \sum_{0\leq j\leq n} t_j(1-t_n-\dots-t_{j+1})<\nu_j\theta_{jj}> + \sum_{0\leq j<k\leq n} t_jt_k<\nu_j\theta_{kj}>$$

$$= h_y\Big(\sum_{j=0}^{n} t_j<\nu_j>\Big)$$

$$= h_y|\Delta'\alpha|\left(\sum_{j=0}^{n} t_j\langle u_j\rangle\right)$$

$$= h_y|\Delta'\alpha|u$$

Proof of (B): For each $u \in \Delta_q$, fix u to be of the form (2.2.9) with $n = q$ and u_j to have domain $[j]$, for each j. In this case, u_q is the identity on $[q]$. It is always possible to write u in this form because we allow the possibility for t_j to be zero. With u in this form, there exists a permutation ϕ of $[q]$ so that

$$\text{image } u_j = \{\phi(0), \phi(1), \ldots, \phi(j)\}$$

Suppose that $x \in X_q$ is nondegenerate. From (2.2.10) we have that $u_q^*x = \rho_q^* z_q$. But $u_q^* = 1_{X_q}$ and since x, z_q are nondegenerate and u_q, ρ_q are surjective we have that

$$\rho_q^* = u_q^* = 1_{X_q} \quad \text{and so} \quad \rho_q = \rho_{qq} = 1_{[q]}$$

Now, $\rho_q u_{qj} = \overline{u}_{qj}\rho_{qj}$, by (2.2.10). So $1_{[q]}u_{qj} = u_{qj}1_{[j]} = \overline{u}_{qj}\rho_{qj}$. But u_{qj}, $\overline{u}_{qj}$ are injective operators and $1_{[j]}$, ρ_{qj} are surjective operators. Hence, $u_{qj} = \overline{u}_{qj}$ and $\rho_{qj} = 1_{[j]}$, from which we get that ρ_{qj} is the identity on $[j]$, for each j.

We denote the ith co-ordinate of a point of $\mathbb{R}^{q+1}$ by the subscript i.

Let $u = \sum_{j=0}^{q} t_j \langle u_j \rangle$ and recall

$$h_x(u) = \sum_{0\le j\le q} t_i(1-t_q-\dots-t_{j+1})\langle u_j\beta_{jj}\rangle + \sum_{0\le j<k\le q} t_j t_k \langle u_j\beta_{kj}\rangle.$$

If $t_j = 0$, for all $j = 0,1,\dots,q-1$, then $h_x(u) = t_q \langle u_q\beta_{qq}\rangle$.

Thus $h_x(u)_i \ge t_q \langle u_q\beta_{qq}\rangle_i = \frac{t_q}{q+1}$ for all $i = 0,1,\dots,q$.

But $u \in \mathrm{In}\Delta_q$ and so $t_q > 0$. Hence, $h_x(u) \in \mathrm{In}\Delta_q$.

To show that h_x is injective on $\mathrm{In}\Delta_q$, take $u = \sum_{j=0}^{q} t_j \langle u_j\rangle$ and $u' = \sum_{j=0}^{q} t'_j \langle u'_j\rangle$ belonging to $\mathrm{In}\Delta_q$ and suppose that $h_x(u) = h_x(u')$.

We show by decreasing induction on $j = q,\dots,0$ that

(2.2.11) either $t_j = t'_j = 0$ or $u_j = u'_j$ and $t_j = t'_j$

<u>For $j = q$</u>: If $i \notin$ image u_j, then $\langle u_j\beta_{kj}\rangle_i = 0$. So we have that

$$\frac{t_q}{q+1} = h_x(u)_{\phi(q)} = h_x(u')_{\phi(q)} \ge \frac{t'_q}{q+1}$$

and so $t_q \ge t'_q$. Similarly, if ϕ' is a permutation of $[q]$ such that image $u'_j = \{\phi'(0),\phi'(1),\dots,\phi'(j)\}$, then

$$\frac{t'_q}{q+1} = h_x(u')_{\phi'(q)} = h_x(u)_{\phi'(q)} \ge \frac{t_q}{q+1}$$

and so $t'_q \ge t_q$. Hence, $t_q = t'_q$.

Suppose now that $\ell < q$ and assume (2.2.11) to be true for all $j > \ell$. Then we can write u' as

$$u' = \sum_{j=0}^{\ell} t_j' \langle u_j' \rangle + \sum_{j=\ell+1}^{q} t_j \langle u_j \rangle$$

and so $h_x(u') = \left(\sum_{0 \leq j \leq \ell} t_j'(1 - t_q' - \dots - t_{j+1}') \langle u_j' \beta_{jj}' \rangle + \sum_{\substack{0 \leq j \leq \ell \\ j < k \leq q}} t_j' t_k' \langle u_j' \beta_{kj}' \rangle \right)$

$$+ \left(\sum_{\ell+1 \leq j \leq q} t_j(1 - t_q - \dots - t_{j+1}) \langle u_j \beta_{jj} \rangle + \sum_{\substack{\ell+1 \leq j \leq q-1 \\ j < k \leq q}} t_j t_k \langle u_j \beta_{kj} \rangle \right)$$

If we let $T = \sum_{0 \leq j \leq \ell} t_j(1 - t_q - \dots - t_{j+1}) \langle u_j \beta_{jj} \rangle + \sum_{\substack{0 \leq j \leq \ell \\ j < k \leq q}} t_j t_k \langle u_j \beta_{kj} \rangle$

and T', the corresponding expression for u', then, since $h_x(u) = h_x(u')$, $T = T'$.

If $u_\ell \neq u_\ell'$, then there exists an $i \in$ image u_ℓ which does not belong to image u_ℓ'. So $\langle u_j' \beta_{kj}' \rangle_i = 0$ for $j = 0, \dots, \ell$ and thus $T_i' = 0$. But $T = T'$ and so $T_i = T_i' = 0$.

Thus we have that

$$\frac{t_\ell t_q}{\ell+1} = t_\ell t_q \langle u_\ell \rangle_i = t_\ell t_q \langle u_\ell \beta_{q\ell} \rangle_i \leq T_i = T_i' = 0$$

But $u \in \text{In}\Delta_q$ and so $t_q > 0$. Hence, $t_\ell = 0$ and by symmetry, $t_\ell' = 0$.

Suppose now that $u_\ell = u'_\ell$. Then, by the induction hypothesis, the expression

$$S = (1 - t_q - \dots - t_{\ell+1}) \langle u_\ell \beta_{\ell\ell} \rangle + \sum_{\ell+1 \le k \le q} t_k \langle u_\ell \beta_{k\ell} \rangle$$

is equal to the corresponding expression S' for u'. Let $i = \phi(\ell)$. Then, for each $0 \le j < \ell$, $\langle u_j \beta_{kj} \rangle_i = 0$ and so

$$T_i = t_\ell (1 - t_q - \dots - t_{\ell+1}) \langle u_\ell \beta_{\ell\ell} \rangle_i + \sum_{\ell+1 \le k \le q} t_\ell t_k \langle u_\ell \beta_{k\ell} \rangle_i.$$

So we have that

$$t_\ell S_i = T_i = T'_i \geq t'_\ell S'_i = t'_\ell S_i$$

But $S_i \geq t_q \langle u_\ell \beta_{q\ell} \rangle_i = t_q \langle u_\ell \rangle_i = \frac{t_q}{\ell+1} > 0$

Hence $\frac{t_\ell t_q}{\ell+1} \geq \frac{t'_\ell t_q}{\ell+1}$ and so $t_\ell \geq t'_\ell$.

By symmetry, $t'_\ell \geq t_\ell$ and thus $t_\ell = t'_\ell$. Hence (2.2.11) is true for $j = \ell$ and so for all $j = q, \dots, 0$. Therefore, $u = u'$ and h_x is injective.

We now show that $h_x(\mathrm{In}\Delta_q) = \mathrm{In}\Delta_q$.

Since h_x is injective on $\mathrm{In}\Delta_q$, h_x is a bijection from $\mathrm{In}\Delta_q$ onto $h_x(\mathrm{In}\Delta_q)$. But $\mathrm{In}\Delta_q$ is compact and $h_x(\mathrm{In}\Delta_q)$ is Hausdorff. Thus $h_x : \mathrm{In}\Delta_q \longrightarrow h_x(\mathrm{In}\Delta_q)$ is a homeomorphism. Now, $\mathrm{In}\Delta_q \cong S^n \setminus *$ and

$\mathrm{In}\Delta_q$ is open. Hence, by the "theorem of invariance of domain" (see [7;p.303]), $h_x(\mathrm{In}\Delta_q)$ is open in S^n and hence in $\mathrm{In}\Delta_q$. But $h_x(\mathrm{In}\Delta_q) = \mathrm{In}\Delta_q \cap h_x(\Delta_q)$ and $h_x(\Delta_q)$ is closed in Δ_q, as a compact subset of a Hausdorff space. Thus $h_x(\mathrm{In}\Delta_q)$ is closed in $\mathrm{In}\Delta_q$. Because it is non-empty, $h_x(\mathrm{In}\Delta_q)$ must be equal to $\mathrm{In}\Delta_q$. //

(2.2.12) <u>Theorem</u> (Barratt): The realization of any ssc X can be triangulated.

<u>Proof:</u> Composing the homeomorphism of (2.2.8) with the homeomorphism of (2.2.7) for $|SdX|$ gives us the required result. //

Recall that $|\;|:\underline{SSC} \longrightarrow \underline{Top}$ is left adjoint to $S:\underline{Top} \longrightarrow \underline{SSC}$, where S is the functor earlier defined. Let $\theta:\underline{SSC}(-,S-) \overset{\cdot}{\longrightarrow} \underline{Top}(|-|,-)$ be the natural equivalence. For every $X \in \mathrm{Ob}\underline{Top}$, let $j_X:|SX| \longrightarrow |X|$ be the map defined by

$$j_X = \theta(SX,X)(1_{SX})$$

As a consequence of (2.2.12) we have that the map j_X induces isomorphisms of homotopy groups in all dimensions. This result is needed in Chapter IV and so its proof will be given here.

(2.2.13) <u>Lemma:</u> Let $K \in \mathrm{Ob}\underline{SSC}$, $X \in \mathrm{Ob}\underline{Top}$ and $f:|K| \longrightarrow X$ be a map. Then there exists a unique ss map $f':K \longrightarrow SX$ such that $j_X|f'| = f$.

<u>Proof</u>: Let $f':K \longrightarrow SX$ be the unique morphism of <u>SSC</u> such that $\theta(K,X)(f') = f$. The naturality of θ and the morphism f' give rise to the commutative diagram

$$\begin{array}{ccc} \underline{SSC}(K,SX) & \xrightarrow{\theta(K,X)} & \underline{Top}(|K|,X) \\ \uparrow{\scriptstyle -\cdot f'} & & \uparrow{\scriptstyle -\cdot |f'|} \\ \underline{SSC}(SX,SX) & \xrightarrow{\theta(SX,X)} & \underline{Top}(|SX|,X) \end{array}$$

Hence, $\theta(SX,X)(1_{SX})|f'| = \theta(K,X)(1_{SX}f') = \theta(K,X)(f') = f$ and so $j_X|f'| = f$. //

Notice that it is possible to give the actual form of the map j_X; namely,

$$(\forall |x,t| \in |SX|) \quad j_X|x,t| = x(t) \quad \text{where} \quad (x,t) \in SX_n \times \Delta_n$$

(2.2.14) <u>Theorem</u>: For every topological space X and every integer $n \geq 0$, the induced homomorphism $(j_X)_n : \pi_n(|SX|) \longrightarrow \pi_n(X)$ is an isomorphism.

<u>Proof</u>: Let $x_0 \in X$ be the base point of X. Because of the way in which j_X is defined there is one and only one $s_0 \in |SX|$ such that $j_X(s_0) = x_0$. (namely, if x_0 is the map $x_0:\Delta_0 \longrightarrow X$ such that $x_0(0) = x_0$, take $|x_0,0| \in |SX|$). Take s_0 to be the base vertex of $|SX|$.

$\underline{(j_X)_n \text{ is epic}}$: Let $[f] \in \pi_n(X)$ be the base-homotopy class of the map $f:(S^n,*) \longrightarrow (X,x_0)$, where $*$ is the base point of S^n. We can regard S^n as $|K|$, where K is a convenient simplicial complex. By (2.2.13), there is a unique ss map $f':K \longrightarrow SX$ such that $j_X \cdot |f'| = f$. Notice that $j_X \cdot |f'|(*) = f(*) = x_0$ and so $|f'|(*) \in j_X^{-1}(x_0) \Longrightarrow |f'|(*) = s_0$. Then $[|f'|] \in \pi_n(|SX|)$ and $(j_X)_n[|f'|] = [f]$.

$\underline{(j_X)_n \text{ is monic}}$: Let $g:(|K|,*) \longrightarrow (|SX|,s_0)$ be such that $j_X g \simeq c(x_0)$ where $c(x_0):|K| \longrightarrow X$ is such that $c(x_0)(s) = x_0$, for every $s \in |K|$. By (2.2.12), there exists a simplicial complex K_{SX} whose geometric realization is homeomorphic to $|SX|$. Let $\theta:|SX| \longrightarrow |K_{SX}|$ be such a homeomorphism. On the other hand, by the simplicial approximation theorem, there is a simplicial map of a convenient barycentric subdivision $K^{(r)}$ of K into K_{SX}, say $g':K^{(r)} \longrightarrow K_{SX}$, such that $|g'| \simeq \theta g$ (recall that $|K^{(r)}| \approx |K|$). Hence $j_X\theta^{-1}(\theta g) \simeq j_X\theta^{-1}|g'| \Longrightarrow$ $j_X\theta^{-1}|g'| \simeq c(x_0)$; in other words, there is a homotopy $H:|K| \times I \cong |K \times \Delta[1]| \longrightarrow X$ such that $H\big|_{|K| \times 0} = j_X\theta^{-1}|g'|$, $H\big|_{* \times I} = c(x_0)$ and $H\big|_{|K| \times 1} = c(x_0)$. By (2.2.13) there exists a unique ss map $H':K \times \Delta[1] \longrightarrow SX$ such that $H = j_X|H'|$. Thus

$$H\big|_{|K| \times 0} = j_X\theta^{-1}|g'| = j_X|H'|\big|_{|K| \times 0}$$

$\Longrightarrow \theta^{-1}|g'| = H'\big|_{K \times 0}$ by uniqueness (see (2.2.13)).

On the other hand, $c(x_0) = H\big|_{|K| \times 1} = j_X|H'|\big|_{|K| \times 1}$.

Since $c(x_0) = j_X c(s_0)$, where $c(s_0):|K| \longrightarrow |SX|$ is the constant map over $s_0 \in |SX|$, again by uniqueness we have $|H'|\big|_{|K| \times 1} = c(s_0)$. Thus $\theta^{-1}|g'| \sim c(s_0) \Longrightarrow g \sim c(s_0)$ and so $(j_X)_n$ is monic. //

CHAPTER III

MILNOR'S WORK [14] - PART I

Milnor's paper, "On Spaces of the Same Homotopy Type of a CW-complex", is a classical paper in homotopy theory which is frequently referred to. In this chapter, and the one to follow, we give a detailed analysis and clarification of that work. Conditions for a space to be of the same homotopy type of a countable CW-complex is the topic for this chapter. This is section one of Milnor's paper. The more general situation for CW-n-ads is discussed in Chapter IV.

We start by giving four examples of spaces which do not have the homotopy type of a CW-complex. The fourth example is an interesting example due to Borsuk [4]. It is a locally contractible, compact metric space whose homology groups are nontrivial for every integer $n \geq 0$.

(1) Cantor Set

Let C be a cantor set and suppose C is of the same homotopy type as a CW-complex X. Then, in particular, X dominates C and hence, by (1.2.12), the path components of C are open. Now the path components of C are the singleton sets. But C is a T_1-space and so the singleton sets are also closed. Hence, C is a discrete space, a contradiction. Therefore, C cannot be of the same homotopy type as a CW-complex.

(2) Let X = graph of $\sin\frac{1}{x}$, $0 < x \leq 1$, in $\mathbb{R}^2$ and $A = \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$. Form the set $B = X \cup A$ and give to it the subspace topology in $\mathbb{R}^2$. Notice that B is connected but not path connected. Suppose B is of the same homotopy type as a CW-complex K. Since connectedness is a homotopy type invariant, K is connected and thus path connected by (1.2.11). But path connectedness is also a homotopy type invariant and so B must be path connected. This is a contradiction to the fact that B is not path connected and so B cannot be of the same homotopy type as a CW-complex.

(3) Let X be the subspace of $\mathbb{R}$ consisting of the points 0 and $\frac{1}{n}$, for all integers $n \geq 1$. Since each point $\frac{1}{n}$ is both open and closed, the path components of X are just the single points. So if X was of the same homotopy type as a CW-complex K, then K would have to have an infinite number of path components. This is because, under a homotopy equivalence, the path components are in a 1-1 correspondence (see [6 ; Ch. 18, 2.2.1]). But, if $f : X \to K$ were a homotopy equivalence, $f(X)$ would be compact, since X is, and so, by (1.2.10), would be contained in a finite subcomplex of K. Thus, $f(X)$ would be contained in the union of a finite number of path components, contradicting the assumption that f is a homotopy equivalence.

(4) Borsuk has constructed the following space.

Let $Q = \prod_{n=1}^{\infty} [0, \frac{1}{n}]$ with the metric $d(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$

Note that this metric is well-defined since the infinite sum converges. Let. $A_0 = \{x \in Q | x_1 = 0\}$ and $A_k = \{x \in Q | \frac{1}{k+1} \leq x_1 \leq \frac{1}{k}\}$. For $k \geq 1$, let C_k be the subspace of $\mathbb{R}^\infty$ defined by $\{x \in \mathbb{R}^\infty | x_i = 0, i > k\}$. Then the boundary of A_k in C_k is the $(k - 1)$-sphere, S^{k-1}. Borsuk has shown that the space $B = A_0 \bigcup_{k=2}^{\infty} S^{k-1}$ is connected, compact metric and locally contractible. Furthermore, for every $k \geq 2$, there exists a retraction of B onto S^{k-1}; that is, there exists a map $r : B \to S^{k-1}$ such that $ri = 1_{S^{k-1}}$, where $i : S^{k-1} \to B$ is the inclusion map. Thus, on the homology level, we have that $r_* i_* = 1_{H_n(S^{k-1}, \mathbb{Z})}$, $(\forall n \geq 0)$, where r_* and i_* are the induced homomorphisms

$$H_n(S^{k-1}, \mathbb{Z}) \xrightarrow{i_*} H_n(B, \mathbb{Z}) \xrightarrow{r_*} H_n(S^{k-1}, \mathbb{Z})$$

Since i_* is injective and $H_0(S^{k-1}, \mathbb{Z})$ and $H_{k-1}(S^{k-1}, \mathbb{Z})$ are non-zero for all $k \geq 2$, we have that $H_n(B, \mathbb{Z})$ is non-zero for all $n \geq 0$. We will see shortly why this space B cannot be of the same homotopy type as a CW-complex (page 82).

We denote by W_0 the category of all spaces which have the homotopy type of a countable CW-complex. We will see that this category contains a wide variety of spaces, including absolute neighbourhood retracts.

An absolute neighbourhood retract (abbreviated, ANR) is a separable metric space X such that whenever X is imbedded as a closed subset of another separable metric space Z, it is a retract of some neighbourhood in Z.

We remark that the above definition of an ANR is Kuratowski's modification [12; p. 270] of Borsuk's original definition [3; p. 222] in that it requires the added condition of separability. The proof of Milnor's first result depends partially on some results on ANR's, found in Hanner's paper [10]. Here, Hanner uses Kuratowski's definition of ANR.

(3.1) Theorem: For a space A, the following are equivalent:

(1) A belongs to W_0.

(2) A is dominated by a countable CW-complex.

(3) A has the homotopy type of a countable, locally finite simplicial complex.

(4) A has the homotopy type of an absolute neighbourhood retract.

Proof: The implication (1) $\Rightarrow$ (2) is obvious. The implication (3) $\Rightarrow$ (1) follows immediately from the fact that a locally finite simplicial complex is a CW-complex (see example 1 of a CW-complex).

(2) $\Rightarrow$ (3): Suppose A is dominated by a countable CW-complex X. If A is path connected, then the result follows from the following theorem due to Whitehead [19; Theorem 24]:

(3.2) A path connected space A, which is dominated by a countable CW-complex, is of the same homotopy type as some locally finite polyhedron.

We claim that it is sufficient to consider the path connected case. So suppose A is not path connected and let P be a path component of A. Since X dominates A, there exist maps $f : A \to X$, $g : X \to A$ such that $gf = 1_A$. Now P is path connected and hence f(P) is path connected, and thus contained in some path component C of X. We claim that C dominates P. Clearly, this is true if $g(C) \subset P$. So let $a \in P$ and take x to be an arbitrary point of C. Consider x and $f(a)$ in C. Since C is path connected, there exists a map $\lambda : I \to C$ such that $\lambda(0) = x$, $\lambda(1) = f(a)$ (Here, $I = [0, 1]$). Form the composite map $g\lambda : I \to g(C)$. Then $g\lambda(0) = g(x)$ and $g\lambda(1) = gf(a)$. Since $gf \simeq 1_A$, there exists a homotopy $H : A \times I \to A$ with $H(-, 0) = gf$, $H(-, 1) = 1_A$. Define $h : I \to A$ by $h(t) = H(a, t)$. Then $h(0) = gf(a)$ and $h(1) = a$. Now, define $r : I \to A$ by

$$r(t) = \begin{cases} g\lambda(2t), & 0 \leq t \leq \frac{1}{2} \\ h(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Clearly, r is continuous and defines a path in A joining $g(x)$ to a. Hence, $g(x) \in P$ and thus $g(C) \subset P$. As a consequence of the above argument, we have that the path components of A are in a 1-1 correspondence with the path components of X. Since X is a countable CW-complex, it can have only countably many path components and, hence, the same for A. By (1.2.12), the path components of A are open and so, we can write A as $A = \bigcup_{i=1}^{\infty} P_i$ where, for each $i = 1, 2, \ldots$, P_i is an open path component of A. Applying (3.2) to each path component of A, we get a countable collection of locally finite simplicial complexes K_i, $i = 1, 2, \ldots$, such that for each i, K_i has the same homotopy type as P_i. Let $K = \coprod_{i=1}^{\infty} K_i$. Then K is a countable locally finite simplicial complex. We claim that A is of the same homotopy type as K.

For each $i = 1, 2, \ldots,$ let $h_i : P_i \to K_i$, $k_i : K_i \to P_i$ be maps such that $k_i h_i = 1_{P_i}$ and $h_i k_i = 1_{K_i}$. Define $h : A \to K$ and $k : K \to A$ in the obvious way: If $a \in P_i$, define $f(a) = h_i(a)$. Similarly, if $b \in K_i$, define $k(b) = k_i(b)$. It is clear that if h and k are continuous, then $kh = 1_A$ and $hk = 1_K$. So it remains to show continuity of these functions.

Consider $h : A \to K$ and let U be an open set in K. Since K has the weak topology, $U \cap K_i$ is open in K_i, for each $i = 1, 2, \ldots$. Thus, $h_i^{-1}(U \cap K_i)$ is open in P_i, for each $i = 1, 2, \ldots$. But, for each $i = 1, 2, \ldots$, P_i is open in A. Hence, $h_i^{-1}(U \cap K_i)$ is open in A, for each $i = 1, 2, \ldots$, and so $h^{-1}(U) = \bigcup_{i=1}^{\infty} h_i^{-1}(U \cap K_i)$ is open in A. Thus, h is continuous and in a similar fashion, one can show that k is continuous.

<u>(3) ⇒ (4)</u>: Follows from Hanner's result [10; Cor. 3.5] that every locally finite polyhedron is an ANR.

<u>(4) ⇒ (2)</u>: Follows from Hanner's result [10; Theorem 6.1] that every ANR is dominated by a (countable) locally finite simplicial complex. //

Recall that a <u>topological n-manifold</u> is a Hausdorff space X such that for each $x \in X$, there is an open neighbourhood V_x of x which is homeomorphic to an open subset of $\mathbb{R}^n$.

As a consequence of (3.1), we have

(3.3) <u>Corollary</u>: Every separable topological n-manifold belongs to W_0.

Proof: Let X be a separable topological n-manifold. Then $X = \bigcup_{x \in X} V_x$ where, for each $x \in X$, V_x is an open neighbourhood of x homeomorphic to some open subset of $\mathbb{R}^n$. Now, $\mathbb{R}^n$ is an ANR and hence, every open subset of $\mathbb{R}^n$ is an ANR (see [10; Lemma 3.1]). Thus, for each $x \in X$, V_x is an ANR. But the union of open ANR's is an ANR (see [10; Theorem 3.3]). Hence, X is an ANR and, by (3.1), X belongs to W_0. //

Let A, X be spaces and form the function space $A^X = \{f : X \to A \mid f \text{ is continuous}\}$. For each subset K of X and each subset U of A, denote by $W(K, U)$, the set of all maps $f : X \to A$ such that $f(K) \subset U$. The family of all sets of the form $W(K, U)$, for K a compact subset of X and U open in A, is a subbase for the compact-open topology for A^X.

As another consequence of (3.1), we have that certain function spaces belong to W_0. More precisely

(3.4) Corollary: If A belongs to W_0 and C is compact metric, then the function space A^C (with the compact-open topology) belongs to W_0.

Proof: By (3.1), we may assume that A is an ANR. Let C_0 be a subset of the compact metric space C and let a_0 be a point of A. We show that the function space $(A, a_0)^{(C, C_0)}$ is an ANR. The proof is due to Borsuk [2; 4.5.1] and is based upon the following result:

(3.5) In order that a metrizable space X be an ANR, it is necessary that X be an r-image of an open subset of a convex set lying in a normed linear space; it is sufficient that X be an r-image of an open subset of a convex set lying in a locally convex linear space.

By an r-map between two spaces X and Y, we mean a map $f : X \to Y$ such that there exists a map $g : Y \to X$ with $fg = 1_Y$. The definition of ANR used in (3.5) does not require the condition of separability. However, we will prove that A^C is an ANR in the sense of Borsuk and then show that the added condition of separability is carried through to the function space.

Since A is an ANR, there exists, by (3.5), an r-map $f : U \to A$ where U is an open subset of some convex set Q, lying in a normed linear space Z. Let $g : A \to U$ be a right inverse of f and set $z_0 = g(a_0)$. Given $\phi, \psi \in (Z, z_0)^{(C, C_0)}$, define $\lambda\phi + \mu\psi$, $\lambda, \mu \in \mathbb{R}$, as follows:

$$(\lambda\phi + \mu\psi)(c) = \lambda\phi(c) + \mu\psi(c) + (1 - \lambda - \mu)z_0$$

This is possible since Z is linear. Since C is compact, we can define for each $\phi \in (Z, z_0)^{(C, C_0)}$ its norm $|\phi|$ as follows:

$$|\phi| = \sup_{c \in C} d(\phi(c), z_0)$$

Thus, under these definitions, $(Z, z_0)^{(C, C_0)}$ becomes a normed linear space. Consider the function space $(Q, z_0)^{(C, C_0)}$ and let $\phi, \psi : (C, C_0) \to (Q, z_0)$. Then, for each $t \in [0, 1]$, $c \in C$

$$(t\phi + (1-t)\psi)(c) = t\phi(c) + (1-t)\psi(c) + (1 - t - (1-t))z_0$$
$$= t\phi(c) + (1-t)\psi(c)$$

But Q is convex and so $t\phi(c) + (1-t)\psi(c) \in Q$. Hence, $(Q, z_0)^{(C, C_0)}$ is a convex subset of $(Z, z_0)^{(C, C_0)}$. Also, since $U \subset Q$ is open, $(U, z_0)^{(C, C_0)}$ is an open subset of $(Q, z_0)^{(C, C_0)}$. Define $\Phi_f : (U, z_0)^{(C, C_0)} \to (A, a_0)^{(C, C_0)}$ by $\Phi_f(\phi) = f\phi$, for all $\phi \in (U, z_0)^{(C, C_0)}$. Similarly, define $\Phi_g : (A, a_0)^{(C, C_0)} \to (U, z_0)^{(C, C_0)}$ by $\Phi_g(\psi) = g\psi$, for all $\psi \in (A, a_0)^{(C, C_0)}$. We claim that Φ_f, Φ_g are continuous. We show continuity for Φ_f. The proof for Φ_g is analogous.

(3.6) Let $\phi \in (U, z_0)^{(C, C_0)}$ and let $V \subset A$ be an open set containing $a_0 \in A$ such that $f\phi(C) \subset V$. Consider the element $W((C, C_0), (V, a_0))$ of the subbasis of $(A, a_0)^{(C, C_0)}$. Since $\phi(C_0) = z_0$ and $z_0 = g(a_0)$, $\Phi_f(\phi)(C_0) = f\phi(C_0) = f(z_0) = fg(a_0) = a_0$ and so $\Phi_f(\phi) \in W((C, C_0), (V, a_0))$. Since f is continuous, there exists an open neighbourhood V' of $\phi(C) \subset U$ such that $f(V') \subset V$. Consider the set $G = \{h \in (U, z_0)^{(C, C_0)} \mid h(C) \subset V'\}$. Then G is an element of the subbasis of $(U, z_0)^{(C, C_0)}$ and $\phi \in G$. To show Φ_f is continuous, it is sufficient to show that Φ_f takes G into $W((C, C_0), (V, a_0))$. If $h \in G$, then $\Phi_f(h) = fh$. Now $fh(C) = f(h(C)) \subset f(V') \subset V$ and $fh(C_0) = f(z_0) = fg(a_0) = a_0$. Hence $\Phi_f(h) \in W((C, C_0), (V, a_0))$ as required.

Now, if $\phi \in (A, a_0)^{(C, C_0)}$, then $(\Phi_f \cdot \Phi_g)(\phi) = fg\phi = \phi$. Thus, $(A, a_0)^{(C, C_0)}$ is an r-image of the open subset $(U, z_0)^{(C, C_0)}$ of the convex set $(Q, z_0)^{(C, C_0)}$, lying in the normed linear space $(Z, z_0)^{(C, C_0)}$ and so, by (3.5), $(A, a_0)^{(C, C_0)}$ is an ANR in the sense of Borsuk. Setting $C_0 = \emptyset$, we have that A^C is an ANR.

Since C is compact, the space A^C becomes separable metric by defining the distance between its elements as follows

$$|f - g| = \max_{x \in X} |f(x) - g(x)|$$

where $|f(x) - g(x)|$ denotes the distance in the separable metric space A. The result now follows from (3.1). //

We remark that the condition of compactness on C in (3,4) is essential, as can be seen by the following example.

Let A be any two-point discrete space and let C be a countable discrete space, which is certainly not compact. Then the function space A^C is a Cantor set, which we saw earlier, is not of the same homotopy type as a CW-complex.

We now turn our attention to compact spaces and spaces which have the Lindelöf property. By the Lindelöf property, we mean that every open covering of a space has a countable sub-covering.

(3.7) Proposition: If a compact space A has the homotopy type of a CW-complex X, then A is dominated by a finite CW-complex.

Proof: Let $f : A \to X$, $g : X \to A$ be maps such that $gf \simeq 1_A$ and $fg \simeq 1_X$. Since A is compact, $f(A)$ is compact and, hence, by (1.2.10), is contained in some finite subcomplex K of X. Let $h = g|_K$. Then $hf \simeq 1_A$ and so K dominates A. //

As an observation of (3.7), one could ask: Under what conditions will a space, which is dominated by a finite complex, have the homotopy type of a finite complex? From (3.1), all we can say is that such a space has the homotopy type of a countable CW-complex. It turns out, that in the simply connected case any space dominated by a finite complex, has the homotopy type of a finite complex. The complete solution to this problem can be found in Wall [20].

We now return to our space B of example (4) and show why it cannot be of the same homotopy type as a CW-complex.

Assume the contrary; that is, assume B is of the same homotopy type as a CW-complex. Now B is compact and hence, (by 3.7), is dominated by a finite CW-complex K. Let $f : B \to K$, $g : K \to B$ be maps such that $gf \simeq 1_B$. Then, on the homology level, we have that $g_*f_* = 1_{H_n(B,\mathbb{Z})}$ $(\forall n \geq 0)$. Now, K is a finite CW-complex and so $(\forall q > n)$, $H_q(K, \mathbb{Z}) = 0$, where n = dimension of K. But f_* is injective at all dimensions and, hence, $(\forall q > n)$ $H_q(B, \mathbb{Z}) = 0$. This is a contradiction to the fact that the homology groups of B are non-trivial for all $n \geq 0$. Hence, B cannot be of the homotopy type of a CW-complex.

(3.8) <u>Proposition</u>: If a space A has the Lindelöf property and if A has the same homotopy type as a CW-complex, then A belongs to W_0.

Proof: Let $f : A \to K$ be a homotopy equivalence, where K is a CW-complex and let L be the smallest subcomplex of K which contains $f(A)$. Clearly, L dominates A. We claim that L is a countable subcomplex of K. The result then follows from (3.1).

To show L is countable, we show that $f(A)$ meets only a countable number of open cells of K. Let K^n be given by (1.2.2) and let $A_n = \{\lambda \in \Lambda_n \mid f(A) \cap \sigma_\lambda^n \neq \emptyset\}$. For each $\lambda \in A_n$, choose a point $x_\lambda \in f(A) \cap \sigma_\lambda^n$. Since the σ_λ^n's are open and disjoint, the set $G_n = \{x_\lambda \mid \lambda \in A_n\}$ has the discrete topology and, hence, is closed in $f(A)$. But $f(A)$ is Lindelöf, being the continuous image of a Lindelöf space. Thus, G_n is Lindelöf and so, being discrete, must be countable. Hence, A_n is countable $(\forall n \geq 0)$; that is, $f(A)$ meets only countably many open cells of $f(A)$. //

CHAPTER IV

MILNOR'S WORK [14] - PART II

By an n - ad $\underline{A} = (A; A_1, \ldots, A_{n-1})$, we mean an n-tuple consisting of a space A and n - 1 subspaces $A_1, \ldots, A_{n-1}$. For example, by a CW-n-ad $\underline{K} = (K; K_1, \ldots, K_{n-1})$, we mean a CW-complex K together with n - 1 subcomplexes $K_1, \ldots, K_{n-1}$. If $\underline{A} = (A; A_1, \ldots, A_{n-1})$ and $\underline{B} = (B; B_1, \ldots, B_{n-1})$ are n-ads, then a n-ad map $\underline{f} : \underline{A} \to \underline{B}$ is given by a map $f : A \to B$ such that $(\forall 1 \leq i \leq n-1)\ f(A_i) \subset B_i$. The product of a space C with an n-ad $\underline{A} = (A; A_1, \ldots, A_{n-1})$ is the n-ad $\underline{A} \times C = (A \times C; A_1 \times C, \ldots, A_{n-1} \times C)$ and an n-ad homotopy $\underline{H} : \underline{A} \times \underline{I} \to \underline{B}$ is a homotopy $H : A \times I \to B$ that restricts to a homotopy on each A_i; that is, $(\forall 1 \leq i \leq n-1)\ H_i : A_i \times I \to B_i$ is given by $H_i = H|_{A_i \times I}$. Retraction, deformation retraction and deformation are defined analogously.

We denote by W^n, the category of all n-ads which have the homotopy type of a CW-n-ad. We are going to examine conditions for an n-ad to belong to W^n. In particular, our basic objective will be to show that certain function space constructions, which are important in homotopy theory, do not lead us outside the category W. For example, we know that if X is a CW-complex, then the space of loops in X based at x_0, denoted ΩX_{x_0}, is not in general a CW-complex. However, we will see that there does exist a CW-complex K such that K and ΩX_{x_0} have the same homotopy type.

We start with a characterization theorem for the category W^n.

(4.1) Theorem: For an n-ad $\underline{A} = (A; A_1, \ldots, A_{n-1})$, the following are equivalent:

(1) $\underline{A}$ belongs to W^n.

(2) $\underline{A}$ is dominated by a CW-n-ad.

(3) $\underline{A}$ has the homotopy type of a simplicial n-ad in the weak topology.

(4) $\underline{A}$ has the homotopy type of a simplicial n-ad in the metric topology.

Note that the metric topology on a simplicial complex K is the same as the coarsest topology on K for which the barycentric co-ordinates, considered as functions from K to [0, 1], are continuous. This is what Milnor calls the "strong topology" and is just the initial topology with respect to the barycentric co-ordinate functions.

Proof of (4.1): The implication (1) ⇒ (2) is clear. For (3) ⇒ (1), recall from example (1) of a CW-complex, given in Chapter I, that every simplicial complex and hence every simplicial n-ad, in the weak topology, is a CW-complex, respectively, CW-n-ad.

(3) ⇔ (4): Let $\underline{K} = (K; K_1, \ldots, K_{n-1})$ be a simplicial n-ad. We denote by $\underline{K}_m$, the simplicial n-ad $\underline{K}$ in the metric topology and by $\underline{K}_w$, the same n-ad in the weak topology. We show that:

(i) The identity map $i : \underline{K}_w \to \underline{K}_m$ is continuous.

(ii) $\underline{i}$ is a homotopy equivalence.

Proof of (i): Let $f_\beta : K_m \to [0, 1]$ denote the β^{th} barycentric co-ordinate function, where β is a vertex of K. Since K_m has the initial topology with respect to all the barycentric co-ordinate functions, i is continuous $\Longleftrightarrow$ the composite $K_w \xrightarrow{i} K_m \xrightarrow{f_\beta} [0, 1]$ is continuous for every barycentric co-ordinate function f_β, $\beta \in K$.

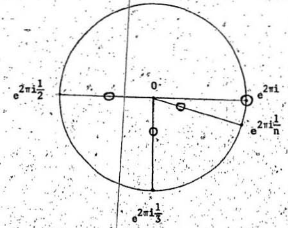
$\Longleftrightarrow K_w \xrightarrow{f_\beta} [0, 1]$ is continuous for all f_β, $\beta \in K$.

$\Longleftrightarrow$ the composite $K^n \xrightarrow{i_n} K_w \xrightarrow{f_\beta} [0, 1]$ is continuous for all f_β, $\beta \in K$ and all i_n, $n = 0, 1, \ldots$.

$\Longleftrightarrow K^n \xrightarrow{f_\beta} [0, 1]$ is continuous for all f_β, $\beta \in K$ and all $n = 0, 1, \ldots$.

Since the barycentric co-ordinate functions are continuous on a simplicial complex and hence on any finite dimensional subcomplex, i is continuous and so the n-ad map $\underline{i}$ is continuous. //

We remark that if K is a locally finite simplicial complex, then i is, in fact, a homeomorphism since the weak and metric topologies coincide. However, if K is not locally finite, then i is not a homeomorphism as can be seen by the following example (see [21; p. 30]).



Take a 1-complex whose 1-simplices have one vertex at the centre 0 of a unit circle and the other vertex at the point $e^{2\pi i\frac{1}{n}}$, $n = 1, 2, \ldots$. On the 1-simplix with vertices 0 and $e^{2\pi i\frac{1}{n}}$, take a point at distance $\frac{1}{n}$ from the centre of the unit circle. The set of these points is called C. They are denoted in the diagram by small circles. In the weak topology, the set C is closed since it meets each cell in a single point. Whereas, in the metric topology, C is not closed since 0 is a limit point of C which does not belong to C. Hence, the weak topology on this 1-complex is finer than the metric topology.

Proof of (ii): Given β a vertex of K, we define the star of β, denoted St_β, to be $\{\alpha \in K_m | \alpha(\beta) \neq 0\}$. For every vertex β of K, St_β is open in K_m because $St_\beta = f_\beta^{-1}((0, 1]) = K_m \setminus f_\beta^{-1}(0)$. Thus, $\{St_\beta\}_{\beta \in K}$ forms an open cover of K_m, indexed by the collection $\{\beta\}$ of vertices of K. Now, K_m is metrizable and hence paracompact. Thus, there exists an open locally finite refinement $\{U_\gamma | \gamma \in \Lambda\}$ of $\{St_\beta\}_{\beta \in K}$ and a partition of unity subordinate to the cover $\{U_\gamma | \gamma \in \Lambda\}$. By a partition of unity subordinate to $\{U_\gamma | \gamma \in \Lambda\}$, we mean a collection $\{p_\gamma | \gamma \in \Lambda\}$ of continuous functions $p_\gamma : K_m \to [0, 1]$ such that

(a) $(\forall \alpha \in K_m)$ there exists a neighbourhood N_α of α such that $p_\gamma(N_\alpha) = 0$ for all but finitely many p_γ's.

(b) $\sum_{\gamma \in \Lambda} p_\gamma(\beta) = 1$ for all vertices $\beta \in K$.

(c) for each $\gamma \in \Lambda$, $p_\gamma(K_m \setminus U_\gamma) = 0$.

We claim that we can choose this open locally finite refinement so that the indexing set $\Lambda = \{\beta\}$ of vertices of K and, for each $\beta \in \{\beta\}$,

U_β is contained in St_β. In fact, for each vertex $\beta \in K$, define $U_\beta = \{\alpha \in K_m | \alpha(\beta) > \frac{1}{2} \operatorname{Max}_v \alpha(v)\}$. We then claim that $\{U_\beta\}_{\beta \in K}$ has the required properties.

(A) If $\alpha \in U_\beta$, then $\alpha(\beta) \neq 0$ because $\alpha(\beta) > \frac{1}{2} \operatorname{Max}_v \alpha(v)$. Therefore, for each vertex $\beta \in K$, $U_\beta \subset St_\beta$ and so $\{U_\beta\}_{\beta \in K}$ is a refinement of $\{St_\beta\}_{\beta \in K}$.

(B) Since, for every vertex $\beta \in K$, St_β is open in K_m, U_β is open in K_m if and only if $St_\beta \setminus U_\beta$ is closed in St_β. Note that $\delta \in St_\beta \setminus U_\beta$ if and only if $\delta(\beta) \neq 0$ and $\delta(\beta) \leq \frac{1}{2} \operatorname{Max}_v \delta(v)$.

Suppose U_β is not open in K_m; that is, $St_\beta \setminus U_\beta$ is not closed in St_β. Then there exists a sequence of points $\{\alpha_i\}$ in $St_\beta \setminus U_\beta$ such that $\{\alpha_i\}$ converges to $\alpha \in U_\beta$. Now, $\{\alpha_i\}$ converges to $\alpha \iff \sqrt{\sum_{v \in K} [\alpha_i(v) - \alpha(v)]^2} \to 0$ as $i \to \infty$,

$\iff |\alpha_i(v) - \alpha(v)| \to 0$ as $i \to \infty$ for all vertices $v \in K$.

Set $\alpha(\beta) - \frac{1}{2} \operatorname{Max}_v \alpha(v) = t > 0$ ($t \in [0, 1]$) and choose $\epsilon > 0$ such that $\epsilon < \frac{t}{2}$. Now, since $|\alpha_i(v) - \alpha(v)| \to 0$ as $i \to \infty$ for all vertices $v \in K$, there exists some positive integer N such that $(\forall i \geq N)$ $\{v \in K | \alpha_i(v) \neq 0\} = \{v \in K | \alpha(v) \neq 0\}$. Hence, there exists some positive integer $N_1 \geq N$ such that $(\forall i \geq N_1)$ $|\frac{1}{2} \operatorname{Max}_v \alpha_i(v) - \frac{1}{2} \operatorname{Max}_v \alpha(v)| < \epsilon$.

Thus, $(\forall i \geq N_1)$ $\alpha(\beta) > \frac{1}{2} \operatorname{Max}_v \alpha_i(v)$, since, if $\alpha(\beta) \leq \frac{1}{2} \operatorname{Max}_v \alpha_i(v)$, then $|\frac{1}{2} \operatorname{Max}_v \alpha_i(v) - \frac{1}{2} \operatorname{Max}_v \alpha(v)| \geq |\alpha(\beta) - \frac{1}{2} \operatorname{Max}_v \alpha(v)| = t > \epsilon$. Now, for each i, $\alpha_i \in St_\beta \setminus U_\beta$ and so $(\forall i \geq N_1)$ $\alpha_i(\beta) \leq \frac{1}{2} \operatorname{Max}_v \alpha_i(v) < \alpha(\beta)$, from which we get $\alpha(\beta) - \alpha_i(\beta) \geq \alpha(\beta) - \frac{1}{2} \operatorname{Max}_v \alpha_i(v)$. Hence, $(\forall i \geq N_1)$.

$$\begin{aligned}|\alpha(\beta) - \alpha_i(\beta)| &\geq |\alpha(\beta) - \tfrac{1}{2}\operatorname*{Max}_v \alpha_i(v)| \\ &= |(\alpha(\beta) - \tfrac{1}{2}\operatorname*{Max}_v \alpha(v)) + (\tfrac{1}{2}\operatorname*{Max}_v \alpha(v) - \tfrac{1}{2}\operatorname*{Max}_v \alpha_i(v))| \\ &= |t - (\tfrac{1}{2}\operatorname*{Max}_v \alpha_i(v) - \tfrac{1}{2}\operatorname*{Max}_v \alpha(v))| \\ &\geq t - |\tfrac{1}{2}\operatorname*{Max}_v \alpha_i(v) - \tfrac{1}{2}\operatorname*{Max}_v \alpha(v)| \\ &\geq t - \epsilon \\ &> \epsilon\end{aligned}$$

This contradicts the fact that $\{\alpha_i\} \to \alpha$. Hence, for each $\beta \in K$, U_β is open in K_m.

(C) For any infinite subcollection $\{U_{\beta_i}\}_{\beta_i \in K}$, $i = 0, 1, \ldots,$ of $\{U_\beta\}_{\beta \in K}$, $\bigcap_{i=0}^{\infty} U_{\beta_i} = \phi$, since, if $\alpha \in \bigcap_{i=0}^{\infty} U_{\beta_i}$, then $\alpha(\beta_i) \neq 0$ for infinitely many vertices β_i of K which is impossible. Hence, the only non-empty intersections of members of $\{U_\beta\}_{\beta \in K}$ are finite intersections. For each $\alpha \in K_m$, let $U_{\beta_0}, U_{\beta_1}, \ldots, U_{\beta_n}$ be the maximum number of members of $\{U_\beta\}_{\beta \in K}$ such that $\alpha \in U_{\beta_i}$ and $\bigcap_{i=0}^{n} U_{\beta_i} \neq \phi$. Since, for each $i = 0, 1, \ldots, n$, U_{β_i} is open in K_m, $\bigcap_{i=0}^{n} U_{\beta_i}$ is open in K_m and, hence, is an open neighbourhood of α, intersecting only finitely many members of $\{U_\beta\}_{\beta \in K}$. Thus, $\{U_\beta\}_{\beta \in K}$ is locally finite.

Let $\{p_\beta\}_{\beta \in K}$ be a partition of unity subordinate to $\{U_\beta\}_{\beta \in K}$ and define $\underline{p} : \underline{K}_m \to \underline{K}_w$ as follows:

$$(\forall \beta \in K)(\forall \alpha \in K_m)\ p(\alpha)(\beta) = p_\beta(\alpha)$$

We claim that $\underline{p}$ is continuous.

It is sufficient to show that, for each $\alpha \in K_m$, there exists a neighbourhood V_α of α on which p is continuous.

Since $\{p_\beta\}_{\beta \in K}$ is a partition of unity subordinate to $\{U_\beta\}_{\beta \in K}$, there exists, for each $\alpha \in K_m$, a neighbourhood V_α of α such that $p_\beta(V_\alpha) = 0$ for all but finitely many p_β's. Let $p_{\beta_1}, \ldots, p_{\beta_n}$ be the members of $\{p_\beta\}_{\beta \in K}$ such that $p_{\beta_i}(V_\alpha) \neq 0$. Then $p(V_\alpha)$ lies in the finite subcomplex of K with vertices $\beta_1, \ldots, \beta_n$. This finite subcomplex is locally finite and, hence, on this subcomplex, the weak and metric topologies coincide. Thus

$p|_{V_\alpha} : V_\alpha \to K_w$ is continuous $\Leftrightarrow$ the composite $V_\alpha \xrightarrow{p|_{V_\alpha}} K_w \xrightarrow{f_{\beta_i}} [0, 1]$ is continuous for each barycentric co-ordinate function f_{β_i}, $i = 1, \ldots, n$. But, for each $i = 1, \ldots, n$, $f_{\beta_i} \cdot p|_{V_\alpha} = p_{\beta_i}$ and the p_{β_i}'s are continuous. Hence, $p|_{V_\alpha}$ is continuous and so p is continuous.

We must now show that, for each $j = 1, \ldots, n - 1$, p carries $(K_j)_m$ into $(K_j)_w$, for then, $\underline{p}$ will be an n-ad map.

Let $\alpha \in (K_j)_m$ and let $v_1, v_2, \ldots, v_n$ be the vertices of K on which α is non-zero. So, α lies in the n-simplex with vertices $v_1, \ldots, v_n$. Since, for each vertex $\beta \notin \{v_1, \ldots, v_n\}$, $\alpha(\beta) = 0$, $\alpha \notin U_\beta$ and hence $p_\beta(\alpha) = 0$. Thus, $p(\alpha)(\beta) = p_\beta(\alpha) = 0$, for all vertices $\beta \notin \{v_1, \ldots, v_n\}$ and $p(\alpha)$ lies in the n-simplex with vertices $v_1, \ldots, v_n$. Thus, p maps each simplex into itself and hence $(K_j)_m$ into $(K_j)_w$, for each $j = 1, \ldots, n - 1$.

Since ip and pi map each simplex into itself, any convex combination of points of K is again a point of K and so there exists a linear homotopy $t\alpha + (1 - t)\, p(\alpha)$ of p with the identity i which is continuous in either the metric or weak topologies. Hence, $\underline{i}$ is a homotopy equivalence. //

Given spaces A and B, a map $f : A \to B$ is called a <u>singular homotopy equivalence</u> if $f_* : \Pi_k(A, a) \to \Pi_k(B, f(a))$ is an isomorphism for all $n \geq 0$ and for all $a \in A$. Here, Π_0 denotes the set of path-components. We now generalize this notion of singular homotopy equivalence to n-ads.

Consider the n-ad $\underline{A} = (A; A_1, \ldots, A_{n-1})$ and for each non-empty set S of integers between 1 and $n - 1$, define $A_S = \bigcap_{i \in S} A_i$. For the empty set, define $A_\phi = A$. Then a map $\underline{f} : \underline{A} \to \underline{B}$ of n-ads induces 2^{n-1} maps $f_S : A_S \to B_S$. We define $\underline{f}$ to be a <u>singular homotopy equivalence</u> if each f_S is a singular homotopy equivalence.

The proof of the implication (2) $\Rightarrow$ (3) will now be based upon the following lemma:

(4.2) <u>Lemma</u>: If $\underline{A}$ and $\underline{B}$ belong to W^n, then every singular homotopy equivalence $\underline{f} : \underline{A} \to \underline{B}$ is an ordinary homotopy equivalence.

<u>Proof</u>: We may assume that $\underline{A}$ and $\underline{B}$ are CW-n-ads and, by the cellular approximation theorem, that f is cellular. Form the mapping cylinder of $\underline{f}$

$$\begin{array}{ccc} \underline{A} \times I & \longrightarrow & \underline{B} \coprod_{\underline{f}} (\underline{A} \times I) = \underline{M} \\ \uparrow & & \uparrow \\ \underline{A} & \xrightarrow[\underline{f}]{} & B \end{array}$$

By (1.2.13), $\underline{M}$ is a CW-n-ad. (Here, $\underline{M} = (M; M_1, \ldots, M_{n-1})$ where for each $1 \leq i \leq n - 1$, M_i is the mapping cylinder of $f_i : A_i \to B_i$.)

To show that $\underline{f}$ is a homotopy equivalence, it is sufficient to show that $\underline{A}$ is a strong deformation retract of $\underline{M}$. Thus, we will construct a homotopy $\underline{H} : \underline{M} \times I \to \underline{M}$ satisfying

$$H(x, 0) = x \quad \text{for all } x \in M$$

$$H(x, 1) \in A \quad \text{for all } x \in M$$

$$H(a, t) = a \quad \text{for all } a \in A, \text{ for all } t \in I$$

Consider the following pushout

$$\begin{array}{ccc} A_S \times I & \longrightarrow & M_S \\ \uparrow & & \uparrow j_S \\ A_S & \xrightarrow{f_S} & B_S \end{array}$$

where, for each set S, M_S is the mapping cylinder of f_S. Now, $j_S : B_S \to M_S$ is a homotopy equivalence and, hence, a singular homotopy equivalence. But from the hypothesis, each $f_S : A_S \to B_S$ is a singular homotopy equivalence. Hence, for each set S, the inclusion $i_S : A_S \to M_S$ is a singular homotopy equivalence. Consider the exact homotopy sequence of the triple (M_S, A_S, a).

$$\cdots \to \Pi_{k+1}(M_S, A_S, a) \to \Pi_k(A_S, a) \xrightarrow{(i_S)_*} \Pi_k(M_S, a) \to \Pi_k(M_S, A_S, a) \to \cdots$$

Since i_S is a singular homotopy equivalence, $(i_S)_*$ is an isomorphism for all $k \geq 0$ and for all $a \in A_S$. Hence, $\Pi_k(M_S, A_S, a) = 0$, for all $a \in A_S$ and for all $k \geq 0$.

Now, $\pi_k(M_S, A_S, a) = 0$ if and only if every map of the pair $(E^k, \dot{E}^k)$ into (M_S, A_S) can be extended to a map of $(E^k \times I, E^k \times 1)$ into (M_S, A_S). But $(\forall k \geq 0)$ $(E^k, \dot{E}^k)$ is homeomorphic to $(E^k \times 0 \cup \dot{E}^k \times I, \dot{E}^k \times 1)$ via the map $\theta : (E^k \times 0 \cup \dot{E}^k \times I, \dot{E}^k \times 1) \to (E^k, \dot{E}^k)$ defined as follows:

If $(x, 0) \in E^k \times 0$, then $\theta(x, 0) = \frac{1}{2}x$.

If $(x, t) \in \dot{E}^k \times I$, then $\theta(x, t) = \frac{1}{2}(t + 1)x$

Hence,

$\pi_k(M_S, A_S, a) = 0$ if and only if every map of the pair

(4.3) $(E^k \times 0 \cup \dot{E}^k \times I, \dot{E}^k \times 1)$ into (M_S, A_S) can be extended to a map of $(E^k \times I, E^k \times 1)$ into (M_S, A_S).

We now construct our homotopy $\underline{H} : \underline{M} \times I \to \underline{M}$ by induction on the k-skeletons of M. Let $\underline{F}_k = (\underline{M} \times 0) \cup (\underline{A} \times I) \cup (\underline{M}^{k-1} \times I)$. For $k = 0$, define $\underline{H}_0 : \underline{F}_0 \to \underline{M}$ to be the identity on $\underline{M} \times 0$ and the projection map of $\underline{A} \times I$ onto $\underline{A}$. Assume that $\underline{H}_n$ has been defined on $\underline{F}_n$ and that $\underline{H}_n$ extends $\underline{H}_{n-1}$. Let σ^k be any k-cell of $M \setminus A$ and let S be the set of integers $1 \leq i \leq n - 1$ such that $\sigma^k \subset M_S$ and M_S is maximal that is, there does not exist any other set S' such that $M_{S'} \subset M_S$. Now $\underline{H}_n$ has already been defined on $(\bar{\sigma}^k \times 0) \cup (\dot{\sigma}^k \times I)$ and maps this set into M_S. Furthermore, $\underline{H}_n$ maps $\dot{\sigma}^k \times 1$ into $A \cap M_S = A_S$. Hence, by (4.3), $\underline{H}_n$ can be extended over $\bar{\sigma}^k \times I$ so as to map this set into M_S and $\bar{\sigma}^k \times 1$ into A_S. Continuing in this manner, we extend $\underline{H}_n$ over all k-cells of M_S and then over all k-cells of all maximal intersections. We then extend $\underline{H}_n$ over the next largest intersections. After a finite number of steps, we will have extended $\underline{H}_n$ over all proper intersections. Then we extend it over the remaining k-cells of M^k. The resulting map $\underline{H}_{n+1}$ will then map $(M \times 0) \cup (A \times I) \cup (M^k \times I)$ into M and will send $M^k \times 1$ into A and each $M_S \times 1$ into the corresponding A_S. Continuing by induction, we get our required map $\underline{H}$. //

(2) $\Rightarrow$ (3): Suppose $\underline{A}$ is dominated by a CW-n-ad $\underline{K}$. Let $\underline{f} : \underline{A} \to \underline{K}$, $\underline{g} : \underline{K} \to \underline{A}$ be n-ad maps such that $\underline{gf} \simeq 1_{\underline{A}}$.

Let $|SA|$ and $|SK|$ denote the geometric realizations of the singular complexes of A and K, respectively, and consider the following commutative diagram

$$\begin{array}{ccccc} |SA| & \xrightarrow{|Sf|} & |SK| & \xrightarrow{|Sg|} & |SA| \\ \downarrow j & & \downarrow j' & & \downarrow j \\ A & \xrightarrow{f} & K & \xrightarrow{g} & A \end{array}$$

where $j : |\underline{SA}| \to \underline{A}$ is defined by $j|x, t| = x(t)$. Since $gf \simeq 1$, by (2.1.3), $S(gf) \simeq S(1) = 1$ and hence, by (2.1.7), $|S(gf)| \simeq |1| = 1$. Now, $S(gf) = SgSf$ and $|SgSf| = |Sg||Sf|$. Thus $|Sg||Sf| \simeq 1$.

Since $\underline{K}$ is a CW-n-ad, (4.2) together with (2.2.14) implies that j' is a homotopy equivalence. Let k be a homotopy inverse to j'. Then

$$(|Sg|kf)j \simeq |Sg|(kj')|Sf| \simeq |Sg||Sf| \simeq 1$$

and

$$j(|Sg|kf) = g(j'k)f \simeq gf \simeq 1.$$

Hence, j is a homotopy equivalence. But, by (2.2.12), $|SA|$ can be viewed as a simplicial complex in the weak topology. Hence, the result follows. //

It should be noted that Barratt's result, that the realization of any semisimplicial complex can be triangulated, is essential to the proof of the implication (2) $\Rightarrow$ (3) above. Thus, it is essentially this result that enables us to link up statement (4) with statement (1) of (4.1). We will see shortly that it is the replacement of (1) by (4) that enables us to show that certain function space constructions do not lead us outside the class W.

Let $\underline{A} = (A; A_1, \ldots, A_{n-1})$ and $\underline{C} = (C; C_1, \ldots, C_{n-1})$ be n-ads. We denote by $\underline{A^C}$ the subspace of the function space A^C consisting of all maps $\underline{f} : \underline{C} \to \underline{A}$. By $\underline{A}^{\underline{C}}$, we mean the n-ad

$$(\underline{A^C}; (A, A_1)^{(C, C_1)}, \ldots, (A, A_{n-1})^{(C, C_{n-1})})$$

(4.4) Theorem: If $\underline{A}$ belongs to W^n and $\underline{C}$ is a compact n-ad, then the function space $\underline{A^C}$ belongs to W. Moreover, the n-ad $\underline{A}^{\underline{C}}$ belongs to W^n.

Assuming (4.4), we have as an immediate consequence:

(4.5) Corollary: If the pair (A, a_0) belongs to W^2, then the pair $(\Omega A_{a_0}, w_0)$ belongs to W^2. (Here, w_0 denotes the constant loop at a_0.)

Proof: Since the pair (A, a_0) belongs to W^2, the triad $\underline{A} = (A; a_0, a_0)$ belongs to W^3. Let $\underline{C}$ be the compact triad $(I; \dot{I}, I)$ where $I = [0, 1]$

and $\dot{I} = \{0, 1\}$. By (4.4), the triad $\underline{A}^C = (A^I;\ (A, a_0)^{(I, \dot{I})},\ (A, a_0)^{(I, I)})$ belongs to W^3. Now, $(A, a_0)^{(I, \dot{I})} = \{f : I \to A \mid f(0) = a_0,\ f(1) = a_0\}$

$$= \Omega A_{a_0}$$

and $(A, a_0)^{(I, I)} = \{f : I \to A \mid f(t) = a_0, \text{ for all } t \in I\}$.

$$= w_0.$$

So $\underline{A}^I = (A^I;\ \Omega A_{a_0},\ w_0)$. Now, $\underline{A}^C$ has the homotopy type of some CW-triad $(K;\ K_1,\ K_2)$. In particular, ΩA_{a_0} has the homotopy type of K_1 and w_0, the homotopy type of K_2. But $w_0 \in \Omega A_{a_0}$. Hence, the pair $(\Omega A_{a_0},\ w_0)$ has the homotopy type of the pair $(K_1;\ K_1 \cap K_2)$. //

(4.6) A topological space A is said to be equi-locally convex, written ELCX, if there exists a neighbourhood U of the diagonal in $A \times A$, a continuous function $\lambda : U \times I \to A$ and an open covering $\{V_\beta\}$ of A such that

(1) $\lambda(a, b, 0) = a$, $\lambda(a, b, 1) = b$, for all $(a, b) \in U$.

(2) $\lambda(a, a, t) = a$ for all $a \in A$, for all $t \in I$.

(3) for all β, $V_\beta \times V_\beta \subset U$ and $\lambda(V_\beta \times V_\beta \times I) \subset V_\beta$.

We now generalize this notion of ELCX to n-ads.

(4.7) An n-ad $\underline{A} = (A;\ A_1, \ldots, A_{n-1})$ is said to be ELCX if, for each $1 \leq i \leq n - 1$, A_i is a closed subset of A; if conditions (1), (2) and (3) of (4.6) are satisfied for the space A; and if the following condition holds

(4) if $a, b \in A_i$ with $(a, b) \in U$, then $\lambda(a, b, t) \in A_i$, for all $t \in I$.

We now prove (4.4) using the following lemmas, the proofs of which will be given later.

(4.8) Lemma: Every simplicial n-ad in the metric topology is ELCX.

(4.9) Lemma: If $\underline{A} = (A; A_1, \ldots, A_{n-1})$ is ELCX and $\underline{C} = (C; C_1, \ldots, C_{n-1})$ is compact, then the n-ad $\underline{A}^{\underline{C}} = (A^C; (A, A_1)^{(C, C_1)}, \ldots, (A, A_{n-1})^{(C, C_{n-1})})$ is ELCX.

(4.10) Lemma: Every paracompact ELCX n-ad belongs to W^n.

Proof of (4.4): We claim that it is sufficient to show only the second assertion; namely, the n-ad $\underline{A}^{\underline{C}}$ belongs to W^n.

Suppose $\underline{A}^{\underline{C}}$ has the homotopy type of the CW-n-ad $\underline{K} = (K; K_1, \ldots, K_{n-1})$. In particular, each $(A, A_i)^{(C, C_i)}$ has the homotopy type of K_i, $i = 1, \ldots, n-1$. But $\underline{A}^{\underline{C}} = (A, A_1)^{(C, C_1)} \cap \ldots \cap (A, A_{n-1})^{(C, C_{n-1})}$. Hence, $\underline{A}^{\underline{C}}$ has the homotopy type of $K_1 \cap \ldots \cap K_{n-1}$ and so belongs to W.

If $\underline{A}$ belongs to W^n, then, by (4.1), $\underline{A}$ has the homotopy type of a simplicial n-ad $\underline{K} = (K; K_1, \ldots, K_{n-1})$ in the metric topology. By (4.8), $\underline{K}$ is ELCX and so, by (4.9), the n-ad $\underline{K}^{\underline{C}}$ is ELCX. Now the metric topology on K is metrizable and since C is compact, the function space K^C is metrizable (see [6, p. 270]) and hence paracompact. Therefore, the n-ad $\underline{K}^{\underline{C}}$ is paracompact, which together with ELCX, implies by (4.10), that $\underline{K}^{\underline{C}}$ belongs to the category W^n. We claim that $\underline{A}^{\underline{C}}$ is of the same homotopy type as $\underline{K}^{\underline{C}}$.

Let $\underline{f} : \underline{A} \to \underline{K}$ and $\underline{g} : \underline{K} \to \underline{A}$ be n-ad maps such that $\underline{gf} = 1_{\underline{A}}$ and $\underline{fg} = 1_{\underline{K}}$. Define $\underline{\phi} : \underline{A}^{\underline{C}} \to \underline{K}^{\underline{C}}$ by $\phi(h) = fh$, for all maps $h \in A^C$.

Similarly, define $\underline{\psi} : \underline{K}^C \to \underline{A}^C$ by $\psi(h') = gh'$, for all maps $h' \in \underline{K}^C$. Then ϕ and ψ are continuous (see (3.6)) and clearly, for each $i = 1, \ldots, n-1$, $\phi((A, A_i)^{(C, C_i)}) \subset (K, K_i)^{(C, C_i)}$ and $\psi((K, K_i)^{(C, C_i)}) \subset (A, A_i)^{(C, C_i)}$. Hence, $\underline{\phi}$ and $\underline{\psi}$ are n-ad maps. Now, $\underline{gf} = 1_{\underline{A}}$ and so, there exists an n-ad homotopy $\underline{H} : \underline{A} \times I \to \underline{A}$ such that $\underline{H}(-, 0) = 1_{\underline{A}}$ and $\underline{H}(-, 1) = \underline{gf}$. Define $\underline{G} : \underline{A}^C \times I \to \underline{A}^C$ by

$$G(h, t)(c) = H(h(c), t) \quad \text{for all } c \in C, \text{ for all } t \in I.$$

Then $\underline{G}$ is an n-ad homotopy because $\underline{H}$ is. Also, $\underline{G}(-, 0) = 1_{\underline{A}^C}$ and $\underline{G}(-, 1) = \underline{\psi\phi}$. Hence, $\underline{\psi\phi} = 1_{\underline{A}^C}$ and similarly, we can show that $\underline{\phi\psi} = 1_{\underline{K}^C}$. Therefore, $\underline{A}^C$ is of the same homotopy type as $\underline{K}^C$ and so $\underline{A}^C$ belongs to W^n. //

By the product of an n-ad $\underline{A} = (A; A_1, \ldots, A_{n-1})$ with an m-ad $\underline{B} = (B; B_1, \ldots, B_{m-1})$, we mean the $(n + m - 1)$ - ad

$$\underline{A} \times \underline{B} = (A \times B; A_1 \times B, \ldots, A_{n-1} \times B, A \times B_1, \ldots, A \times B_{m-1})$$

(4.11) **Lemma**: If the n-ad $\underline{A}$ and the m-ad $\underline{B}$ are ELCX, then the $(n + m - 1)$ - ad $\underline{A} \times \underline{B}$ is ELCX.

Proof: Since $\underline{A}$ and $\underline{B}$ are ELCX, there exist neighbourhoods U and V of the diagonals in $A \times A$ and $B \times B$, respectively, maps $\lambda_1 : U \times I \to A$ and $\lambda_2 : V \times I \to B$, and open covers $\{U_\beta\}$ and $\{V_\alpha\}$ of A and B, respectively, satisfying (1), (2) and (3) of (4.6). Also, for each $i = 1, \ldots, n-1$, and each $j = 1, \ldots, m-1$, A_i is a closed subset of A, B_j is a closed subset of B and condition (4) of (4.7) holds for both spaces A and B.

Let $W = \{((a, b), (a', b')) \in (A \times B) \times (A \times B) \mid (a, a') \in U$ and $(b, b') \in V\}$. Then W is clearly a neighbourhood of the diagonal in $(A \times B) \times (A \times B)$. Define $\lambda : W \times I \to A \times B$ as follows:

$$\text{for all } ((a, b), (a', b')) \in W,$$
$$\lambda((a, b), (a', b'), t) = (\lambda_1(a, a', t), \lambda_2(b, b', t))$$

Since λ_1 and λ_2 are continuous, λ is continuous. It is easily verified that λ satisfies (1) and (2) of (4.6). Now, let $\mathcal{O} = \{U_\beta \times V_\alpha \mid U_\beta \in \{U_\beta\}, V_\alpha \in \{V_\alpha\}\}$. Then $\mathcal{O}$ is an open cover of $A \times B$ and clearly satisfies (3) of (4.6). Thus $A \times B$ is ELCX. Now, for all $i = 1, \ldots, n - 1$ and for all $j = 1, \ldots, m - 1$, A_i and B_j are closed subsets of A and B, respectively. Hence, for all $i = 1, \ldots, n - 1$ and for all $j = 1, \ldots, m - 1$, $A_i \times B$ and $A \times B_j$ are closed subsets of $A \times B$. Also, if (a, b) and (a', b') belong to $A_i \times B$, $i = 1, \ldots, n - 1$, with $((a, b), (a', b')) \in W$, then $\lambda((a, b), (a', b'), t) \in A_i \times B$, since $\lambda_1(a, a', t) \in A_i$. Similarly for $A \times B_j$. Hence, $\underline{A} \times \underline{B}$ is ELCX. //

Using (4.8) and (4.10), we can now prove the following.

(4.12) Proposition: If $\underline{A}$ belongs to W^n and $\underline{B}$ belongs to W^m, then $\underline{A} \times \underline{B}$ belongs to W^{n+m-1}.

Proof: By (4.1), $\underline{A}$ has the homotopy type of a simplicial n-ad $\underline{K}$ in the metric topology. Similarly, $\underline{B}$ has the homotopy type of a simplicial m-ad $\underline{L}$ in the metric topology. By (4.8), $\underline{K}$ and $\underline{L}$ are ELCX and hence, by (4.11), $\underline{K} \times \underline{L}$ is ELCX. Now, $\underline{K}$ and $\underline{L}$ are metrizable and so, $\underline{K} \times \underline{L}$ is metrizable and, hence, paracompact. By (4.10),

$\underline{K} \times \underline{L}$ belongs to W^{n+m-1}. It remains to show that $\underline{A} \times \underline{B}$ is of the same homotopy type as $\underline{K} \times \underline{L}$.

Let $\underline{f} : \underline{A} \to \underline{K}$, $\underline{g} : \underline{K} \to \underline{A}$ and $\underline{f}' : \underline{B} \to \underline{L}$, $\underline{g}' : \underline{L} \to \underline{B}$ be n-ad maps such that $\underline{gf} = 1_{\underline{A}}$, $\underline{fg} = 1_{\underline{K}}$ and $\underline{g}'\underline{f}' = 1_{\underline{B}}$, $\underline{f}'\underline{g}' = 1_{\underline{L}}$. Consider the n-ad maps

$$\underline{f} \times \underline{f}' : \underline{A} \times \underline{B} \to \underline{K} \times \underline{L} \text{ and } \underline{g} \times \underline{g}' : \underline{K} \times \underline{L} \to \underline{A} \times \underline{B}$$

Then $(\underline{f} \times \underline{f}')(\underline{g} \times \underline{g}') = \underline{fg} \times \underline{f}'\underline{g}' = 1_{\underline{K}} \times 1_{\underline{L}} = 1_{\underline{K} \times \underline{L}}$ and $(\underline{g} \times \underline{g}')(\underline{f} \times \underline{f}') = \underline{gf} \times \underline{g}'\underline{f}' = 1_{\underline{A}} \times 1_{\underline{B}} = 1_{\underline{A} \times \underline{B}}$. //

We now prove (4.8), (4.9) and (4.10).

Proof of (4.8): Let $\underline{K}$ be a simplicial n-ad in the metric topology and let $\{St_\beta\}_{\beta \in K}$ be the covering of K_m by the open star neighbourhoods of the vertices $\beta \in K$. Let $U = \bigcup_{\beta \in K} St_\beta \times St_\beta$. Then U is a neighbourhood of the diagonal in $K_m \times K_m$. Define $\mu : U \to K_m$ as follows:

$$\text{for all } (\alpha_1, \alpha_2) \in U, \quad \mu(\alpha_1, \alpha_2)(\beta) = \frac{\mathrm{Min}(\alpha_1(\beta), \alpha_2(\beta))}{\sum_{v \in K} \mathrm{Min}(\alpha_1(v), \alpha_2(v))}$$

that is, for each pair $(\alpha_1, \alpha_2) \in U$, $\mu(\alpha_1, \alpha_2)$ is the point of K_m with barycentric co-ordinates $\varepsilon_\beta = \dfrac{\mathrm{Min}(\alpha_1(\beta), \alpha_2(\beta))}{\sum_{v \in K} \mathrm{Min}(\alpha_1(v), \alpha_2(v))}$

$$\begin{aligned}
\text{Now, } (\alpha_1, \alpha_2) \in U &\Rightarrow \alpha_1, \alpha_2 \in St_\beta \text{ for some vertex } \beta \in K \\
&\Rightarrow \alpha_1(\beta) \neq 0, \; \alpha_2(\beta) \neq 0 \\
&\Rightarrow \mathrm{Min}(\alpha_1(\beta), \alpha_2(\beta)) \neq 0 \\
&\Rightarrow \sum_{v \in K} \mathrm{Min}(\alpha_1(v), \alpha_2(v)) \neq 0
\end{aligned}$$

Thus, the denominator of the quotient is never zero. Also, since there are only a finite number of vertices at which each $\alpha \in K_m$ is non-zero, the sum $\sum_{v \in K} \text{Min}(\alpha_1(v), \alpha_2(v))$ is well-defined. Hence, μ is well-defined.

Now, μ is continuous $\iff$ the composite $U \xrightarrow{\mu} K_m \xrightarrow{f_\beta} I$ is continuous for every barycentric co-ordinate function f_β.

But $f_\beta \cdot \mu(\alpha_1, \alpha_2) = \dfrac{\text{Min}(\alpha_1(\beta), \alpha_2(\beta))}{\sum_{v \in K} \text{Min}(\alpha_1(v), \alpha_2(v))}$

for every vertex $\beta \in K$ and pair $(\alpha_1, \alpha_2) \in U$. Since the barycentric co-ordinate functions are continuous and the operations of taking min and sum in $\mathbf{R}$ are continuous, the quotient is continuous and hence, μ is continuous.

Notice that if $\{v_1, \ldots, v_n\}$ is the set of vertices of K on which α_1 is non-zero and $\{w_1, \ldots, w_m\}$ is the set of vertices of K on which α_2 is non-zero, then, since $\text{Min}(\alpha_1(v), \alpha_2(v)) \neq 0 \iff \alpha_1(v) \neq 0$ and $\alpha_2(v) \neq 0$, the set of vertices of K on which $\mu(\alpha_1, \alpha_2)$ is non-zero is the set $\{v_1, \ldots, v_n\} \cap \{w_1, \ldots, w_m\}$; that is, $\mu(\alpha_1, \alpha_2)$ lies in the intersection of the smallest simplex containing α_1 and the smallest simplex containing α_2. Note, further, that $\{v_1, \ldots, v_n\} \cap \{w_1, \ldots, w_m\}$ is never empty since $\alpha_1, \alpha_2 \in St_\beta$, for some β. Thus, we can define $\lambda : U \times I \to K_m$ as follows:

for all $t \in [0, 1]$, $(\alpha_1, \alpha_2) \in U$

$$\lambda(\alpha_1, \alpha_2, \tfrac{1}{2}t) = (1 - t)\alpha_1 + t\mu(\alpha_1, \alpha_2)$$

$$\lambda(\alpha_1, \alpha_2, \tfrac{1}{2} + \tfrac{1}{2}t) = (1 - t)\mu(\alpha_1, \alpha_2) + t\alpha_2$$

Then λ is a linear path from α_1 to $\mu(\alpha_1, \alpha_2)$ and then to α_2, covering the first part during the time interval $0 \leq t' \leq \frac{1}{2}$ and the second during the time interval $\frac{1}{2} \leq t' \leq 1$. λ is clearly continuous and satisfies (1) and (2) of (4.6).

Now, if $(\alpha_1, \alpha_2) \in St_\beta \times St_\beta$, then $\mu(\alpha_1, \alpha_2)(\beta) \neq 0$ and so, for all $t \in I$, the β^{th} co-ordinates of the points $(1 - t)\alpha_1 + t\mu(\alpha_1, \alpha_2)$ and $(1 - t)\mu(\alpha_1, \alpha_2) + t\alpha_2$ are non-zero. Therefore, the whole path λ lies in St_β; that is, $\lambda(St_\beta \times St_\beta \times I) \subset St_\beta$. Hence, (3) of (4.6) is satisfied.

Suppose $\alpha_1, \alpha_2 \in K_i$, $i = 1, \ldots, n - 1$, and $(\alpha_1, \alpha_2) \in U$. Then, since α_1 and $\mu(\alpha_1, \alpha_2)$ lie in a closed simplex of K_i, any convex combination of the points α_1 and $\mu(\alpha_1, \alpha_2)$ again lies in K_i. Similarly for α_2 and $\mu(\alpha_1, \alpha_2)$. Hence, for all $t \in I$, $\lambda(\lambda_1, \lambda_2, t) \in K_i$, $i = 1, \ldots, n - 1$, and (4) of (4.7) is satisfied.

For each $i = 1, \ldots, n - 1$, K_i is closed in K, being a subcomplex. Hence, $\underline{K}$ is ELCX. //

Proof of (4.9): Let U be a neighbourhood of the diagonal in $A \times A$, $\lambda : U \times I \to A$ a map and $\{V_\beta\}$ an open cover of A, satisfying (1), (2) and (3) of (4.6) and (4) of (4.7).

Let $W = \{(f, g) \in A^C \times A^C \mid (f(c), g(c)) \in U$, for every $c \in C\}$. Since U is a neighbourhood of the diagonal in $A \times A$, there exists an open set 0×0 in $A \times A$ such that $\Delta \subset 0 \times 0 \subset U$. Now, $0^C \times 0^C$ is an open set in $A^C \times A^C$ (in the compact-open topology) which clearly contains the diagonal in $A^C \times A^C$ and is such that $0^C \times 0^C \subset W$. Hence, W is a neighbourhood of the diagonal in $A^C \times A^C$.

Define the map $\lambda' : W \times I \to A^C$ by

$$\lambda'(f, g, t)(c) = \lambda(f(c), g(c), t) \text{ for every } c \in C.$$

λ' clearly satisfies (1) and (2) of (4.6).

Since C is compact, there exist finitely many compact subsets covering C and so, for every $f \in A^C$, $f \in (A; V_{\beta_1}, \ldots, V_{\beta_k})^{(C; D_1, \ldots, D_k)}$ for some $V_{\beta_i} \in \{V_\beta\}$ and compact subsets $D_1, \ldots, D_k$ covering C. We denote the neighbourhood $(A, V_{\beta_1}, \ldots, V_{\beta_k})^{(C; D_1, \ldots, D_k)}$ of f by Z_f. Then $\{Z_f\}_{f \in A^C}$ forms an open cover of A^C. Now, for all $c \in D_i$, $i = 1, \ldots, k$, and $(g, g') \in Z_f \times Z_f$, $(g(c), g'(c)) \in V_{\beta_i} \times V_{\beta_i} \subset U$. Because $D_1, \ldots, D_k$ cover C, this is true for all $c \in C$ and so, $Z_f \times Z_f \subset W$. Also, $(\forall 1 \le i \le k)(\forall (g, g') \in Z_f \times Z_f)(\forall c \in D_i)$.

$$\lambda'(g, g', t)(c) = \lambda(g(c), g'(c), t) \in V_{\beta_i}$$

Thus, $\lambda'(Z_f \times Z_f \times I) \subset Z_f$ and so (3) of (4.6) is satisfied. It is easily verified that (4) of (4.7) is satisfied, and so it remains to show that, for all $i = 1, \ldots, n-1$, $(A, A_i)^{(C, C_i)}$ is closed in A^C.

For each $x \in C_i$, $\{x\}$ is compact. Let $H_x = \{f \in A^C \mid f(x) \in A_i\}$. Consider the set $A^C \setminus H_x = \{f \in A^C \mid f(x) \in A \setminus A_i\}$. Since A_i is closed in A, $A \setminus A_i$ is open in A and so the set $A^C \setminus H_x$ is open in A^C, being an element of the subbasis of A^C. Hence, H_x is closed in A^C for each $x \in C_i$. But $(A, A_i)^{(C, C_i)} = \bigcap_{x \in C_i} H_x$. Therefore, $(A, A_i)^{(C, C_i)}$ is closed in A^C for each $i = 1, \ldots, n-1$. //

If $\mathcal{U}$ is an open cover of a space X and $x \in X$, then the star at x of $\mathcal{U}$ is the union of the members of $\mathcal{U}$ to which x belongs.

Furthermore, a cover $\mathcal{V}$ is a star refinement of $\mathcal{U}$ if the family of stars of $\mathcal{V}$ at points of X is a refinement of $\mathcal{U}$.

Recall that a space X is fully normal if and only if each open cover of X has an open star refinement.

The following proof can be found in [21; 4.5.3].

Proof of (4.10): Let $\underline{X} = (X; X_1, \ldots, X_{n-1})$ be a paracompact ELCX n-ad. Let V be a neighbourhood of the diagonal in $X \times X$, $\phi : V \times I \to X$ a map, and $\mathcal{U} = \{U_\alpha | \alpha \in \Lambda\}$ an open cover of X, satisfying (1), (2) and (3) of (4.6) and (4) of (4.7). Now, every paracompact space is fully normal (see [11, p. 170]). So let $\mathcal{U}^* = \{U^*_\beta | \beta \in \Lambda'\}$ be a star refinement of $\mathcal{U}$ with the added property that if a set $U^*_\beta \in \mathcal{U}^*$ meets each of the subspaces $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$ of X, then $U^*_\beta \cap (\bigcap X_i | j \in \{i_1, \ldots, i_k\}) \neq \phi$. This is possible since if $\mathcal{U}'^*$ is any star refinement of $\mathcal{U}$, we let $\mathcal{U}^*$ be the collection of all sets $U'^* \setminus \bigcup\{X_j | j \in \{i_1, \ldots, i_k\}\}$ where $U'^* \cap \{\bigcap X_j | j \notin \{i_1, \ldots, i_k\}\} \neq \phi$. Let $\{p_\beta\}_{\beta \in \Lambda'}$ be a partition of unity subordinate to $\mathcal{U}^*$. Form the simplicial complex $N(\mathcal{U}^*) = \{t_0 U^*_{\beta_0} + \ldots + t_n U^*_{\beta_n} | t_i \geq 0, \sum_{i=0}^{n} t_i = 1, \bigcap_{i=0}^{n} U^*_{\beta_i} \neq \phi\}$, called the nerve of the covering $\mathcal{U}^*$, and let N_i, $i = 1, \ldots, n-1$ be the subcomplexes of $N(\mathcal{U}^*)$ with simplices $\langle U^*_{i_0}, \ldots, U^*_{i_n} \rangle$ where $X_i \cap \{\bigcap U^*_j | j \in \{i_0, \ldots, i_n\}\} \neq \phi$. Let $\underline{N} = (N(\mathcal{U}^*); N_1, \ldots, N_{n-1})$. The partition $\{p_\beta\}_{\beta \in \Lambda'}$ determines a canonical n-ad map $\underline{p} : \underline{X} \to \underline{N}$ according to the formula $p(x) = \sum_\beta p_\beta(x) U^*_\beta$. Giving $N(\mathcal{U}^*)$ the weak topology makes p continuous (see [21; p. 130]). We will construct an n-ad map $\underline{q} : \underline{N} \to \underline{X}$ such that $\underline{qp} = 1_{\underline{X}}$. The result then follows from (4.1).

Choose an ordering of the simplicial complex $N(\mathcal{U}^*)$ and, for each $U^*_\beta \in \mathcal{U}^*$, choose a point u^*_β such that, if $U^*_\beta \cap X_i \neq \phi$, then $u^*_\beta \in X_i$. Define $q : N(\mathcal{U}^*) \to X$ as follows:

For each vertex U^*_β of $N(\mathcal{U}^*)$, $q(U^*_\beta) = u^*_\beta$. In the n-simplex $< U^*_{\beta_0}, \ldots, U^*_{\beta_n} >$ where $U^*_{\beta_0} < \ldots < U^*_{\beta_n}$, we define q inductively on $< U^*_{\beta_0} >, \ldots, < U^*_{\beta_0}, \ldots, U^*_{\beta_k} >$. Suppose q has been defined on $< U^*_{\beta_0}, \ldots, U^*_{\beta_k} >$ and let $x \in < U^*_{\beta_0}, \ldots, U^*_{\beta_{k+1}} >$. Now, x can be written uniquely as $x = t\, U^*_{\beta_{k+1}} + (1 - t)x_1$, where $x_1 \in < U^*_{\beta_0}, \ldots, U^*_{\beta_k} >$ and $t \in [0, 1]$.

Define $q(x) = \phi(q(U^*_{\beta_{k+1}}), q(x_1), t) = \phi(u^*_{\beta_{k+1}}, q(x_1), t)$. Notice, that in order for q to make sense, the pair $(u^*_{\beta_{k+1}}, q(x_1))$ must belong to V. We show that there exists a set $U_\alpha \in \mathcal{U}$ such that $(u^*_{\beta_{k+1}}, q(x_1)) \in U_\alpha \times U_\alpha \subset V$.

Let $y \in \bigcap_{i=0}^{k+1} U^*_{\beta_i}$. Since $\mathcal{U}^*$ is a star refinement of $\mathcal{U}$, $\bigcup_{i=0}^{k+1} U^*_{\beta_i} \subset \operatorname{Star} y \subset U_\alpha \in \mathcal{U}$, for some $\alpha \in \Lambda$. In particular, $u^*_{\beta_{k+1}} \in U_\alpha$. We show that $q(x_1) \in U_\alpha$.

Now, $q(U^*_{\beta_0}) = u^*_{\beta_0} \in U^*_{\beta_0} \subset U_\alpha$. If $x \in < U^*_{\beta_0}, U^*_{\beta_1} >$, then $x = t\, U^*_{\beta_1} + (1 - t)U^*_{\beta_0}$, and so $q(x) = \phi(u^*_{\beta_1}, u^*_{\beta_0}, t) \in U_\alpha$ since $(u^*_{\beta_1}, u^*_{\beta_0}) \in U_\alpha \times U_\alpha$. Similarly, if $x \in < U^*_{\beta_0}, U^*_{\beta_1}, U^*_{\beta_2} >$, then $x = t\, U^*_{\beta_2} + (1 - t)x_1$, where $x_1 \in < U^*_{\beta_0}, U^*_{\beta_1} >$. But $q(x_1) \in U_\alpha$ and so $q(x) = \phi(u^*_{\beta_2}, q(x_1), t) \in U_\alpha$. Continuing in this way, we have that, if $x \in < U^*_{\beta_0}, \ldots, U^*_{\beta_k} >$, then $q(x) \in U_\alpha$. So q is well-defined and also continuous. We must now show that, for each $1 \leq i \leq n - 1$, q maps N_i into X_i.

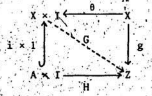
Let $< U^*_{\beta_0}, \ldots, U^*_{\beta_k} >$ be a simplex of N_i, $1 \leq i \leq n-1$. Then $X_i \cap \{\cap U^*_j \mid j \in \{\beta_0, \ldots, \beta_k\}\} \neq \phi$ and so $u^*_{\beta_0}, \ldots, u^*_{\beta_k}$ belong to X_i. If $x \in < U^*_{\beta_0}, \ldots, U^*_{\beta_k} >$ then $x = t\,U^*_{\beta_k} + (1-t)x_1$ where $x_1 \in < U^*_{\beta_0}, \ldots, U^*_{\beta_{k-1}} >$. Using the same argument as the one used in showing q is well-defined, we have that $q(x_1) \in X_i$ and so $q(x) = \phi(u^*_{\beta_{k+1}}, q(x_1), t) \in X_i$ by (4) of (4.7). Hence, for each $1 \leq i \leq n-1$, q maps N_i into X_i.

Since U^* is a star refinement of U, there exists for each $x \in X$, a set $U_\alpha \in U$ such that the pair $(x, qp(x)) \in U_\alpha \times U_\alpha \subset V$. Hence, the map $H(x, t) = \phi(x, qp(x), t)$ is a homotopy of qp with 1_X and so $\underline{X}$ belongs to W^n. //

We conclude this chapter with a result on mapping cones.

Let $\underline{Top}_*$ be the category of based topological spaces and base-point preserving maps. We denote by W_*, the category of all objects of $\underline{Top}_*$ having the homotopy type of based CW-complexes. We are going to show that W_* is closed under the construction of mapping cones of its morphisms.

Given a pair of spaces (X, A), where $A \subset X$, (X, A) is said to have the <u>homotopy extension property</u>, written $(X, A) = HEP$, if and only if $(\forall Z \in \text{Ob } \underline{Top})(\forall H : A \times I \to Z)(\forall g : X \to Z$ such that $g|_A = H(-, 0))$, there exists $G : X \times I \to Z$ making the following triangles commute.



where $\theta(x) = (x, 0)$, for every $x \in X$.

A useful example of such a pair is (CY, Y), for every $Y \in \text{Ob}\ \underline{\text{Top}}$. In fact, identify Y to $Y \times 1$ and take $H : Y \times I \to Z$ and $g : CY \to Z$ so that $H(y, 0) = g(y, 1)$, for every $y \in Y$. Now define $G : CY \times I \to Z$ by

$$(4.13) \qquad G((y, s), t) = \begin{cases} g(y, ts + s) & 0 \leq s \leq \dfrac{1}{t+1} \\ H(y, ts + s - 1) & \dfrac{1}{t+1} \leq s \leq 1 \end{cases}$$

Recall that if $f : X \to Y$ is a morphism between objects of $\underline{\text{Top}}_*$, the mapping cone C_f of f is given by the pushout diagram

$$\begin{array}{ccc} CX & \xrightarrow{\ \bar{f}\ } & CX \coprod_f Y = C_f \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle Pf} \\ X & \xrightarrow[\ f\ ]{} & Y \end{array}$$

where, here, CX is the reduced cone of X.

Since Pf and $\bar{f}|_{CX \setminus i(X)}$ are $1 - 1$, we shall denote the elements of C_f just by $y \in Y$ or $(x, s) \in CX$ (rather than by $Pf(y)$ and $\bar{f}(x, s)$)

Let

$$(4.14) \qquad \begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \psi} \\ X' & \xrightarrow[\ f'\ ]{} & Y' \end{array}$$

be a homotopy commutative diagram of $\underline{\text{Top}}_*$. Let $\Phi : X \times I \to Y'$ be such that $\Phi(-, 0) = f'\phi$ and $\Phi(-, 1) = \psi f$. Since $Pf'\Phi(-, 0) = \bar{f}'C\phi i$ and (CX, X) = HEP, the homotopy $Pf'\Phi: X \times I \to C_{f'}$ can be extended to a homotopy $H : CX \times I \to C_{f'}$ such that $H(-, 0) = \bar{f}C\phi$ and $H(i \times 1) = Pf'\Phi$. By the universal property of pushouts, there exists a unique map $C(\phi, \psi, \Phi) \in Top_*(C_f, C_{f'})$ such that $C(\phi, \psi, \Phi)\bar{f} = H(-, 1)$ and $C(\phi, \psi, \Phi)Pf = Pf'\psi$. From (4.13), we can give a precise form to $C(\phi, \psi, \Phi)$, namely

$$(\forall y \in Y)\ C(\phi, \psi, \Phi)(y) = \psi(y)$$

(4.15) $$(\forall (x, s) \in CX)\ C(\phi, \psi, \Phi)(x, s) = \begin{cases} (\phi(x), 2s) & 0 \le s \le \frac{1}{2} \\ \Phi(x, 2s - 1) & \frac{1}{2} \le s \le 1 \end{cases}$$

Suppose now that ϕ and ψ are homotopy equivalences in (4.14), with homotopy inverses ϕ^{-1}, ψ^{-1}, respectively. Let $\Phi' : X' \times I \to Y$ be such that $\Phi'(-, 0) = f\phi^{-1}$ and $\Phi'(-, 1) = \psi^{-1}f'$. Define $\Phi' \cdot \Phi : X \times I \to Y$ as follows:

$$\Phi' \cdot \Phi(x, t) = \begin{cases} \Phi'(\phi(x), 2t) & 0 \le t \le \frac{1}{2} \\ \psi^{-1}\Phi(x, 2t - 1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Then, from (4.15), $C(\phi^{-1}\phi, \psi^{-1}\psi, \Phi' \cdot \Phi) : C_f \to C_f$ is defined by

$$(\forall y \in Y)\ C(\phi^{-1}\phi, \psi^{-1}\psi, \Phi' \cdot \Phi)(y) = \psi^{-1}\psi(y)$$

$$(\forall (x, s) \in CX)\ C(\phi^{-1}\phi, \psi^{-1}\psi, \Phi' \cdot \Phi)(x, s) = \begin{cases} (\phi^{-1}\phi(x), 2s) & 0 \le s \le \frac{1}{2} \\ \Phi'(\phi(x), 4s - 2) & \frac{1}{2} \le s \le \frac{3}{4} \\ \psi^{-1}(\Phi(x, 4s - 3)) & \frac{3}{4} \le s \le 1 \end{cases}$$

and $C(\phi^{-1}, \psi^{-1}, \Phi') \, C(\phi, \psi, \Phi) : C_f \to C_f$ by

$$(\forall y \in Y) \; C(\phi^{-1}, \psi^{-1}, \Phi') \, C(\phi, \psi, \Phi)(y) = \psi^{-1}\psi(y)$$

$$(\forall (x, s) \in CX) \; C(\phi^{-1}, \psi^{-1}, \Phi') \, C(\phi, \psi, \Phi)(x, s) = \begin{cases} (\phi^{-1}\phi(x), 4s) & 0 \le s \le \frac{1}{4} \\ \Phi'(\phi(x), 4s - 1) & \frac{1}{4} \le s \le \frac{1}{2} \\ \psi^{-1}\Phi(x, 2s - 1) & \frac{1}{2} \le s \le 1 \end{cases}$$

(4.16) Lemma: $C(\phi^{-1}, \psi^{-1}, \Phi') \, C(\phi, \psi, \Phi) \simeq C(\phi^{-1}\phi, \psi^{-1}\psi, \Phi' \circ \Phi)$.

Proof: By (1.1.3), $C_f \times I \approx (CX \times I) \coprod_{f \times 1} (Y \times I)$. So, define $H : C_f \times I \to C_f$ as follows:

$$(\forall (y, t) \in Y \times I) \; H(y, t) = \psi^{-1}\psi(y)$$

$$(\forall ((x, s), t) \in CX \times I) \; H((x, s), t) = H_x((x, s), t)$$

where for each $x \in X$

$$H_x((x, s), t) = \begin{cases} (\phi^{-1}\phi(x), \frac{4s}{t + 1}) & 0 \le s \le \frac{t + 1}{4} \\ \Phi'(\phi(x), 4s - 1 - t) & \frac{t + 1}{4} \le s \le \frac{t + 2}{4} \\ \psi^{-1}(\Phi(x, \frac{4s - t - 2}{2 - t}) & \frac{t + 2}{4} \le s \le 1 \end{cases}$$

One easily checks that this is the required homotopy. //

Let $H : X \times I \to X$ and $H' : Y \times I \to Y$ be maps such that $H(-, 0) = \phi^{-1}\phi$, $H(-, 1) = 1_X$ and $H'(-, 0) = \psi^{-1}\psi$, $H'(-, 1) = 1_Y$. Define $\underline{\psi} : X \times I \to Y$ by

$$\underline{\psi}(x, t) = \begin{cases} fH(x, 1 - 4t) & 0 \le t \le \frac{1}{4} \\ \Phi'\cdot\Phi(x, \frac{4t - 1}{2}) & \frac{1}{4} \le t \le \frac{3}{4} \\ H'(f(x), 1 - 4 + 4t) & \frac{3}{4} \le t \le 1 \end{cases}$$

Then $\underline{\psi}$ is a homotopy between f and f. Let $\underline{\psi}' : X \times I \to Y$ be such that $\underline{\psi}'(x, t) = \underline{\psi}(x, 1 - t)$.

Define $\theta : (X \times I) \times I \to Y$ by

$$\theta((x, s), t) = \begin{cases} fH(x, s - 4t) & 0 \le t \le \frac{s}{4} \\ \Phi'\cdot\Phi(x, \frac{4t - s}{4 - 2s}) & \frac{s}{4} \le t \le 1 - \frac{s}{4} \\ H'(f(x), s - 4 + 4t) & 1 - \frac{s}{4} \le t \le 1 \end{cases}$$

(4.17) <u>Lemma</u>: $C(\phi^{-1}\phi, \psi^{-1}\psi, \Phi'\cdot\Phi) \simeq C(1_X, 1_Y, \underline{\psi})$.

<u>Proof</u>: Define $G : C_f \times I \to C_f$ as follows:

$$(\forall (y, t) \in Y \times I)\ G((y, t)) = H'(y, t)$$

$$(\forall ((x, s), t) \in CX \times I)\ G((x, s), t) = \begin{cases} (H(x, t), 2s) & 0 \le s \le \frac{1}{2} \\ \theta((x, t), 2s - 1) & \frac{1}{2} \le s \le 1 \end{cases}$$

One easily checks that this gives the required homotopy. In fact, G is precisely the map $C(H, H', \theta)$. //

(4.18) <u>Lemma</u>: $C(\phi, \psi, \Phi)$ has a left homotopy inverse.

<u>Proof</u>: By (4.16), $C(\phi^{-1}, \psi^{-1}, \Phi')\,C(\phi, \psi, \Phi) \simeq C(\phi^{-1}\phi, \psi^{-1}\psi, \Phi'\cdot\Phi)$ and by (4.17), $C(\phi^{-1}\phi, \psi^{-1}\psi, \Phi'\cdot\Phi) \simeq C(1_X, 1_Y, \underline{\psi})$. Hence, $C(\phi^{-1}, \psi^{-1}, \Phi')\,C(\phi, \psi, \Phi) \simeq C(1_X, 1_Y, \underline{\psi})$ and so $C(1_X, 1_Y, \underline{\psi}')$ $C(\phi^{-1}, \psi^{-1}, \Phi')\,C(\phi, \psi, \Phi) \simeq C(1_X, 1_Y, \underline{\psi}')\,C(1_X, 1_Y, \underline{\psi})$. We claim that

$C(1_X, 1_Y, \underline{\psi}') C(1_X, 1_Y, \psi) = 1_{C_f}$. Then $C(1_X, 1_Y, \underline{\psi}') C(\phi^{-1}, \psi^{-1}, \Phi')$ is a left homotopy inverse for $C(\phi, \psi, \Phi)$.

Define $F : C_f \times I \to C_f$ as follows:

$$(\forall (y, t) \in Y \times I) \quad F(y, t) = y$$

$$(\forall ((x, s), t) \in CX \times I) \quad F((x, s), t) = \begin{cases} (x, \frac{4s}{4 - 3t}) & 0 \leq s \leq \frac{4 - 3t}{4} \\ \underline{\psi}'(x, 4s + 3t - 4) & \frac{4 - 3t}{4} \leq s \leq \frac{2 - t}{2} \\ \underline{\psi}(x, 2s - 1) & \frac{2 - t}{2} \leq s \leq 1 \end{cases}$$

One easily checks that F has the required properties. //

One should note that $C(\phi, \psi, \Phi)$ is, in fact, a homotopy equivalence (see [16; p. 56]). However, we need only the weaker version of (4.18).

(4.19) Theorem: Let $X, Y \in \text{Ob } W_*$ and let $f : X \to Y$ be a base-point preserving map. Then $C_f \in \text{Ob } W_*$.

Proof: Let $\phi : X \to K(X)$, $\psi : Y \to K(Y)$ be homotopy equivalences with $K(X)$, $K(Y)$ based CW-complexes. Let x_0 be the base 0-cell of $K(X)$. By the cellular approximation theorem $\psi f \phi^{-1}$ is homotopic rel$\{x_0\}$ to a cellular map $f' : K(X) \to K(Y)$. By (1.2.13), $C_{f'}$ is a based CW-complex and by (4.18), $C(\phi, \psi, \Phi)$ has a left homotopy inverse. Thus, $C_{f'}$ dominates C_f and so by (4.1), $C_f \in W_*$. //

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