STUDIES IN THE SPACE C (X) OF REAL-VALUED CONTINUOUS FUNCTIONS ON A TOPOLOGICAL SPACE X

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ABSTRACT

In this work, we study some aspects of the ring C(X) of real-valued continuous functions defined on a topological space X. Explicitly, we consider C(X) as a ring, as a Banach space and as a Topological Vector Space. Our aim, throughout this work, is to relate the topological properties of X with appropriate properties of a suitable structure on C(X).

In Chapter 0, we develop the necessary prerequisites, and show that completely regular spaces are the right kind of topological spaces for such a study. The ideals and filters in the ring C(X) are studied in Chapter 1, and a one-to-one correspondence is established between maximal ideals in C(X) and z-ultrafilters on X. These are then employed to construct the classical Stone-Čech Compactification βX and Hewitt's realcompactification υX. This leads to the consideration of how to "recover" the underlying space X from the ring C(X), generating some "Banach-Stone" type theorems. Alternative constructions of βX and υX are furnished.

Taking X to be a compact Hausdorff space, we then study the Banach space C(X) under the uniform norm, and describe its dual space via the Riesz Representation Theorem. The necessary and sufficient conditions for an arbitrary Banach space B, to be C(X) for some X, are then obtained. A version of the Banach-Stone theorem is proved, namely, the Banach spaces C(X) and C(Y) are isometric if and only if X and Y are homeomorphic. The class of spaces C(X), for X compact Hausdorff and extremally disconnected, is precisely the class of " Hahn-Banach spaces", (i.e., they enjoy the Hahn-Banach extension property).

The final chapter is devoted to the study of C(X) as a topological vector space, when equipped with the compact-open topology. The conditions on X, necessary and sufficient for C(X) to be metrizable, barreled, and bornological, respectively, are then given. It turns out that C(X) is bornological when and only when X is realcompact.

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CHAPTER 0

General Properties of C(X)

In this chapter, we develop the prerequisites needed for the study of the space C(X) of all real-valued continuous functions defined on a topological space X. The standard reference will be Gillman and Jerison [19] and, in general, we shall follow their terminology.

0.1 Introduction

As usual, $\mathbf{R}$ will denote the set of real numbers with the usual topology. Let (X, τ) be a (nonempty) topological space, and $C(X, \mathbf{R})$ (or briefly, C(X)) denote the set of all τ-continuous real-valued functions defined on X. Clearly, C(X) contains all constant functions, and for each $r \in \mathbf{R}$, the boldface $\mathbf{r}$ denotes the constant function f such that $f(x) = r$, for all $x \in X$. For $f, g \in C(X)$, if we define $(f+g)(x) = f(x) + g(x)$, and $(f \cdot g)(x) = f(x)g(x)$ $(x \in X)$, then C(X) becomes a commutative ring with unity; the functions $\mathbf{0}$, $\mathbf{1}$, defined by $\mathbf{0}(x) = 0$, and $\mathbf{1}(x) = 1$, (for every $x \in X$), are the *zero* and *unity* in

C(X). We shall denote the function $\frac{1}{f}$ (whenever it exists) by f^{-1}. Note that the set $\{x \in X: f(x) \in A\}$ will be denoted by $f^{\leftarrow}[A]$. C(X) is also a commutative algebra over $\mathbf{R}$, and since for any $r \in \mathbf{R}$ and $f \in C(X)$, we can define $rf = \mathbf{r} \cdot f$, C(X) becomes an $\mathbf{R}$-vector space. The relation $\geq$ defined by $f \geq g$ if and only if $f(x) \geq g(x)$ for all $x \in X$, is a partial order on C(X), which is translation-invariant (i.e., $f \geq g$ if and only if $f + h \geq g + h$ for all $h \in C(X)$), and further, it is true that $f, g \geq \mathbf{0}$ implies $f \cdot g \geq \mathbf{0}$. In other words, C(X) is a *partially ordered ring*. The *absolute value function* $|f|$, defined by $|f|(x) = |f(x)|$, for all $x \in X$, clearly belongs to C(X) whenever f belongs to C(X). For $f, g \in C(X)$, define $(f \vee g)(x) = \max\{f(x), g(x)\}$, and $(f \wedge g)(x) = \min\{f(x), g(x)\}$, $(x \in X)$. It is easily seen that $f \vee g = 2^{-1}(f + g + |f - g|) \in C(X)$. Thus, C(X) becomes a *lattice ordered ring*. Note that $f \wedge g = -(-f \vee -g)$.

If A is a nonempty subset of C(X), A is *bounded above*, if there exists $f \in C(X)$, and $g \leq f$, for every $g \in A$. Similarly, A is *bounded below*, if there exists $h \in C(X)$, and $h \leq g$, for every $g \in A$. Clearly, if for any $f \in C(X)$, the set $\{n \cdot f : n \in \mathbf{N}\}$ is bounded above, then $f \leq \mathbf{0}$. Therefore C(X), is a *lattice archimedian ordered ring*.

The subring of C(X) consisting of all *bounded* functions will be denoted by $C^*(X)$. In general, C(X) is distinct from $C^*(X)$. When X is compact, they are identical. We say that X is *pseudocompact* if and only if $C(X) = C^*(X)$.

Every countably compact space is pseudocompact. For, if $f \in C(X)$, let $U_n = \{x \in X : |f(x)| < n\}$. Then, $\{U_n\}_{n \in N}$ is a countable open cover of X, and the fact that it reduces to a finite subcover shows that f is bounded. An example of a pseudocompact space that is not compact (not even countable compact) is furnished below.

0.1.1 Example -

Let X be any infinite countable set and let $p \in X$ be fixed. The topology $\tau_p = \{A \subseteq X : p \in A\} \cup \{\emptyset\}$ is referred to as the *countable particular point topology* on X. Since there are no two proper disjoint open sets in X, it is clear that all τ_p-continuous real-valued functions on X are constants, and hence bounded. In other words, $C(X) = C^*(X)$, i.e., X is pseudocompact. Let $\{x_n\}_{n=1}^{\infty}$ be an enumeration of members of X, with $x_1 = p$. Define $U_n = \bigcup_{i=1}^{n} \{x_i\}$. The family $\{U_n\}_{n \in N}$ is a (countable) open cover of X with no finite subcover. Thus, X is not (countably) compact. □

In the presence of normality and T_1-separation, pseudocompactness reduces to countable compactness. More generally, in a weakly normal space (that is, a completely regular space in which two disjoint closed sets, one of which is countable, have disjoint neighborhoods), pseudocompactness is equivalent to countable compactness, [14, p232].

0.2 Square Roots and Idempotents in C(X)

Even at this elementary stage, we are able to relate the ring-theoretical properties of $C(X)$ with the topology of the underlying space X. We give a few illustrations. A member $g \in C(X)$ is said to be a *square root* of a member $f \in C(X)$ if and only if $g \cdot g\ (= g^2) = f$. Clearly, $f \geq \mathbf{0}$, and for $f > \mathbf{0}$, we always have two square roots, namely $\pm\sqrt{f}$, with the obvious meaning. The number of square roots for the function $\mathbf{1}$ depends heavily on the nature of connectedness of the space X, as the following theorem reveals.

Theorem 0.2.1 - *For $m < \aleph_0$, the function* $\mathbf{1}$ *has exactly* 2^m *square roots in* $C(X)$ *if and only if the space* X *has* m *components.*

Proof- Any square root of $\mathbf{1}$ must take the value $+1$ on some of the components, and the value -1 on the remaining. Since there are m components, there are 2^m ways of choosing such a function. □

Corollary 0.2.2 - *The space* X *is connected if and only if the function* $\mathbf{1}$ *has exactly two square roots in* $C(X)$.

Recall that an idempotent (in the ring $C(X)$) is a member f such that $f^2 = f$. Clearly $\mathbf{0}$ and $\mathbf{1}$ are idempotents. It is easy to see that X is connected if and only if $\mathbf{0}$ and $\mathbf{1}$ are the only idempotents.

The following counterexample shows that the conclusions of Theorem 0.2.1 are not true if m is infinite.

Example 0.2.3 -

Let $N^* = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\right\}$ with the topology inherited from $\mathbf{R}$. Since $\{0\}$ itself is a component, it is clear that N^* has $\aleph_0$ components. If a square root of $\mathbf{1}$ assumes the value $+1$ at 0, it must retain the value $+1$ throughout a neighborhood of 0. Since a neighborhood of 0 contains all but a finite number of points of N^*, the value -1 is assumed at most at a finite number of points. The number of finite subsets of a countable set is $\aleph_0$. Hence, we conclude that there are only $\aleph_0$ possible square roots. □

Choosing X to be some special subspaces of the real line, we obtain interesting results. We use the standard symbols $\mathbf{N}$, $\mathbf{Q}$, to denote the subspaces of $\mathbf{R}$ consisting of all natural numbers and all rational numbers, respectively.

Theorem 0.2.4 - *For $m \leq \aleph_0$, there exist functions in $C(\mathbf{R})$, $C(\mathbf{N})$ and $C(\mathbf{N}^*)$ with exactly 2^m square roots. However, if a function in $C(\mathbf{Q})$ has more than one square root, it already has $2^{\aleph_0}$ of them.*

Proof- For $m \leq \aleph_0$, choose a function f_m in $C^*(\mathbf{R})$ with the property that $f_m(x) \neq 0$ for $x \in \bigcup_{i=1}^{m} (i-1, i)$ and 0 elsewhere. The function $g = f_m^2$

has exactly 2^m square roots, since any $f \in C^*(\mathbf{R})$, such that $f(x) = \pm f_m(x)$ on any of the intervals $(i - 1, i)$, $(i = 1, 2, \ldots, m)$ and identical with f_m elsewhere, is obviously a square root of g. There are 2^m choices for such a function.

Repeating the above construction with single points $\{i\}$ instead of open intervals $(i - 1, i)$, $(i = 1, 2, \ldots, m)$, we obtain the desired function in $C(\mathbf{N})$, actually in $C^*(\mathbf{N})$. With the additional restriction that $\lim_{x \to 0} f_m(x) = 0$, we can obtain the corresponding function which will work for $C(\mathbf{N}^*)$.

Let f_1, f_2 be two distinct square roots of $g \in C(\mathbf{Q})$. Choose an irrational ξ and define f_ξ by setting

$$f_\xi(r) = \begin{cases} f_1(r), & r \in \mathbf{Q},\ r < \xi \\ f_2(r), & r \in \mathbf{Q},\ r > \xi \end{cases}$$

Now, since f_ξ is separately continuous on two disjoint open sets whose union is $\mathbf{Q}$, it follows that $f_\xi \in C(\mathbf{Q})$, and $f_\xi^2 = g$. Thus, for each irrational ξ, we obtain a square root f_ξ of g. □

Corollary 0.2.5 - *$C(\mathbf{R})$ has just two idempotents, $C(\mathbf{N}^*)$ has exactly $\aleph_0$ idempotents, and $C(\mathbf{Q})$ and $C(\mathbf{N})$ have $2^{\aleph_0}$ idempotents.*

Proof - Trivial, from Corollary 0.2.2 and Theorem 0.2.4. □

0.3 Zero-sets and Cozero-sets

For $f \in C(X)$, the *zero-set* of f is defined to be the subset $Z_X(f) = \{x \in X: f(x) = 0\} = f^{\leftarrow}[0]$ of X. When the context is clear, we just write $Z(f)$. A *cozero-set* is the complement in X of any zero-set. A zero-set is always closed, but not every closed subset of X is a zero-set, (see Example 0.3.1 below). However, in any metric space (X, d), since the distance function d is continuous, every closed set A is a zero-set, being the set of all points a distance 0 from A. Also, note that $C(X)$ and $C^*(X)$ have the same family of zero-sets, since we can associate with each $f \in C(X)$, the function $|f| \wedge 1 \in C^*(X)$, and $Z(f) = Z(|f| \wedge 1)$. Thus, the two families $\{Z(f): f \in C(X)\}$ and $\{Z(f): f \in C^*(X)\}$ are identical, and we shall unambiguously denote this collection by $Z[X]$.

It is readily seen that $Z(f) \cup Z(g) = Z(f \cdot g)$ and $Z(f) \cap Z(g) = Z(f^2 + g^2)$. Hence, zero-sets are closed under finite unions and finite intersections. As in the case of closed sets, there exist examples of closed, countable unions of zero-sets that are not zero-sets [19, problem 6P.5]. However, they are closed under countable intersections. For, given a sequence of zero-sets $\{Z_n\} = \{Z(f_n)\}$, set $g = \sum_{n \in N} (|f_n| \wedge 2^{-n})$. By uniform convergence on X, $g \in C(X)$ and $Z(g) = \bigcap_{n \in N} Z(f_n)$.

But, an arbitrary intersection of zero-sets need not be closed, as shown by Example 0.3.1 below. We first observe that every zero-set is a G_δ-set, since $Z(f)$

$= \bigcap_{n \in N} \left\{ x \in X : |f(x)| < \frac{1}{n} \right\}$. Conversely, in any normal space, every closed G_δ-set is a zero-set, and the result is not true if the space is not normal, [19, problem 3K.6].

Example 0.3.1 -

Let X be an uncountable set, and $p \in X$ be fixed. Define a topology on X as follows: every point except p is isolated, and an open neighborhood of p consists of any subset of X containing p, whose complement is at most countable. Clearly, $\{p\}$ is a closed set. If $x \neq p$, since $\{x\}$ and $X \sim \{x\}$ are both closed and open, we can find a member $f_x \in C(X)$, such that, the zero-set $Z(f_x) = X \sim \{x\}$. Consequently, $\{p\} = \bigcap_{x \in X \sim \{p\}} Z(f_x)$. We claim that $\{p\}$ is not a zero-set. It suffices to show that $\{p\}$ is not a G_δ-set. In order for $\{p\}$ to be a G_δ-set, we must be able to express $X \sim \{p\}$ as a countable union of complements of neighborhoods of $\{p\}$, which is impossible, since X is uncountable. □

The following zero-sets are often useful.

$$\{x : f(x) \geq 0\} = Z(f \wedge 0) = Z(f - |f|)$$

$$\{x : f(x) \leq 0\} = Z(f \vee 0) = Z(f + |f|)$$

The corresponding cozero-sets are $\{x: f(x) > 0\}$ and $\{x: f(x) < 0\}$.

A zero-set that is also a neighborhood is called a *zero-set neighborhood*. Also, recall that a *neighborhood of a set* A is any set whose interior contains A.

0.4 Completely Separated Sets

Let A and B be disjoint subsets of X. An *Urysohn function* for the ordered pair (A, B) is a member $f \in C^*(X)$ such that $0 \leq f \leq 1$, $f[A] = \{0\}$ and $f[B] = \{1\}$. Whenever an Urysohn function for (A, B) exists, we say that A and B are *completely separated*. If A and B are completely separated, so are C and D, where $C \subseteq A$, $D \subseteq B$. Note that $1 - f$ is an Urysohn function for (B, A), whenever f is an Urysohn function for (A, B), and consequently, B and A are completely separated, whenever A and B are completely separated. For A and B to be completely separated, it is sufficient to demand the existence of a function $g \in C(X)$ such that $g(x) \leq 0$ on A and $g(x) \geq 1$ on B, since $(0 \vee g) \wedge 1$ is clearly an Urysohn function for (A, B). Also, we can replace 0 and 1 by any real numbers r and s, with $r < s$.

The classical Urysohn's Lemma, [26, p115], asserts that in a normal topological space, any two disjoint closed subsets are completely separated.

The following theorem characterizes completely separated sets in terms of zero-sets.

Theorem 0.4.1 - *Two sets are completely separated if and only if they are* *contained in disjoint zero-sets; furthermore, completely separated sets have disjoint zero-set neighborhoods.*

Proof - Let A and B be completely separated, and f be an Urysohn function for (A, B). Then, clearly, the sets

$Z_A = \left\{x : f(x) \le \frac{1}{3}\right\}$, $Z_B = \left\{x : f(x) \ge \frac{2}{3}\right\}$ are disjoint zero-sets containing A, B respectively, in fact, disjoint zero-set neighborhoods of A and B.

Conversely, if Z(f) and Z(g) are disjoint zero-sets, then, $|f(x)| + |g(x)|$ is nonzero for all $x \in X$, and $\frac{|f|}{|f| + |g|}$ is an Urysohn function for the pair (Z(f), Z(g)). Consequently, A and B are completely separated, where $A \subseteq Z(f)$ and $B \subseteq Z(g)$. □

Corollary 0.4.2 - *In any metric space, disjoint closed sets are completely separated.*

Corollary 0.4.3 - *Completely separated sets are contained in disjoint cozero-sets.*

Proof - For completely separated sets A and B, let Z_A, Z_B be as in Theorem 0.4.1. Choose r such that $\frac{1}{3} < r < \frac{2}{3}$. Then the sets $C = \left\{x : f(x) < r\right\}$, $D = \left\{x : f(x) > r\right\}$ are disjoint cozero-sets containing A, B respectively. □

0.5 C and C*-embedding

We say that a subspace $S \subset X$ is *C-embedded* (respectively *C*-embedded*) in X if each element of C(S) (respectively C*(S)) can be extended to an element of C(X) (respectively C*(X)). In this definition, it is sufficient to consider non-negative functions in C(S) (respectively C*(S)), since, for f in C(S) (respectively C*(S)), both $f \vee \mathbf{0}$ and $-(f \wedge \mathbf{0})$ are non-negative, and $f = (f \vee \mathbf{0}) - (-(f \wedge \mathbf{0}))$. It is easy to see that $\mathbf{N}$ is both C- and C*-embedded in $\mathbf{R}$. For, given $f \in C(\mathbf{N})$ (or in $C^*(\mathbf{N})$), the function $g \in C(\mathbf{R})$ (respectively $C^*(\mathbf{R})$) defined by

$$g(x) = \begin{cases} f(1), & x \in (-\infty, 1] \\ f(k) + [f(k+1) - f(k)](x - k), & x \in (k, k+1], \; k = 1, 2, \cdots \end{cases}$$

is the desired extension.

Proposition 0.5.1 - *If S is a C-embedded (respectively C*-embedded) subspace of X, then C(S) (respectively C*(S)) is a (ring) homomorphic image of C(X) (respectively C*(X)); if S is dense in X, this homomorphism is an isomorphism.*

Proof - The mapping which associates each member of C(X) (respectively C*(X)) with its restriction to S is the desired homomorphism, which is onto, since S is C-embedded (respectively C*-embedded). The latter part follows, since a continuous function is uniquely determined by its values on a dense subspace. □

In light of the above proposition, it turns out that if S is a dense C-embedded (respectively C^*-embedded) subspace of X, and **(P)** is any topological property of **X**, which is determined purely by the ring structure of C(X) (respectively $C^*(X)$), then we have the following pleasant result : *X has property* **(P)** *if and only if S has property* **(P)**. Connectedness and compactness are two such properties.

Proposition 0.5.2 - *A subspace $S \subseteq X$ is C^*-embedded in X if and only if every function in $C^*(S)$ can be extended to a function in C(X).*

Proof - The "if" part is obvious, since $C^*(X) \subseteq C(X)$. Conversely, if $f \in C^*(S)$ and g is its extension to C(X), then $h = (-\mathbf{n} \vee g) \wedge \mathbf{n} \in C^*(X)$, where **n** is a bound for $|f|$. Clearly, the restriction of h to C(S) is f. □

From the above proposition, it is clear that every C-embedded subspace is C^*-embedded. An example of a C^*-embedded subspace that is not C-embedded is furnished in Chapter 1 (see Example 1.2.1). Note that the subspaces **Q** and $\mathbf{R} \sim \{0\}$ are not C^*-embedded, and therefore, not C-embedded in **R**.

The following theorem, usually referred to as Urysohn's Extension Theorem (not to be confused with Urysohn's Lemma), is the basic result about C^*-embedding.

Theorem 0.5.3 (Urysohn's Extension Theorem) - *A subspace $S \subseteq X$ is C^*-embedded in X if and only if any two completely separated sets in S are completely separated in X.*

Proof - Let A and B be completely separated in S. There exists $f \in C^*(S)$ which assumes the value 0 on A and 1 on B. Clearly, the extension of f to $C^*(X)$ takes values 0 on A, and 1 on B.

Conversely, let $f \in C^*(S)$. There exists a positive integer m, such that, $|f(x)| < m$, for every $x \in S$. For simplicity of notation, let $r_n = \frac{m}{2}\left(\frac{2}{3}\right)^n$ for each $n \in \mathbf{N}$. Inductively, we shall define two sequences (f_n), (g_n) of function with the following properties:

(a) $f_n \in C^*(S)$, $|f_n| \leq 3r_n$

(b) $g_n \in C^*(X)$, $|g_n| \leq r_n$

(c) $f_{n+1} = f_n - (g_n|_S)$

Define $f_1 = f$, and $g_1 = r_1$, so that $|f_1| \leq m = 3r_1$. Let $f_2 = f_1 - g_1|_S$. For $n > 1$, assume that we have constructed $f_1, f_2, \cdots, f_n$ and $g_1, g_2, \cdots, g_{n-1}$ satisfying conditions (a), (b), (c). Now, consider subsets $A_n = \{x \in S: f_n(x) \leq - r_n\}$ and $B_n = \{x \in S: f_n(x) \geq r_n\}$. By Theorem 0.4.1, they are completely separated in S, and hence by hypothesis, completely separated in X. So, we can find an Urysohn function for (A_n, B_n); in fact a function g_n in

$C^*(X)$ such that $g_n(x) = -r_n$ for $x \in A_n$, $g_n(x) = r_n$ for $x \in B_n$ and $|g_n| \leq r_n$. Letting $f_{n+1} = f_n - g_n|_S$, conditions (b) and (c) are satisfied, and it only remains to verify that $|f_{n+1}| \leq 3r_{n+1}$. If $x \in A_n$, by the induction hypothesis, $-3r_n \leq f_n(x) \leq -r_n$ and $g_n = -r_n$. Similarly, if $x \in B_n$, $r_n \leq f_n(x) \leq 3r_n$ and $g_n = r_n$. If $x \notin A_n \cup B_n$, then, $-r_n < f_n(x) < r_n$, $-r_n < g_n(x) < r_n$. In all three cases, we have $|f_{n+1}(x)| \leq 2r_n$. Since $2r_n = 3r_{n+1}$, condition (a) follows, and the induction is complete.

Now, let $g(x) = \sum_{n=1}^{\infty} g_n(x)$. By uniform convergence and condition (b), $g \in C^*(X)$. Note that $(g_1 + g_2 + \cdots + g_n)|_S = (f_1 - f_2) + (f_2 - f_3) + \cdots + (f_n - f_{n+1}) = f_1 - f_{n+1}$. Since $r_n \to 0$, it follows that $f_{n+1}(s) \to 0$ for each $s \in S$. Thus, g and f_1 agree on S, and therefore, g is an extension of f. The proof is complete. □

The above theorem, along with Theorem 0.4.1, yields the following interesting corollary.

Corollary 0.5.4 - *If $S \subseteq X$, and every zero-set in S is a zero-set in X, then S is C^*-embedded in X.*

A necessary and sufficient condition for a C^*-embedded subspace to be C-embedded is furnished by the following theorem.

Theorem 0.5.5 - *A C^*-embedded subspace is C-embedded if and only if it is completely separated from every zero set disjoint from it.*

Proof - Let S be a C^*-embedded subspace of X for the entire proof. In addition, assume that S is C-embedded, and disjoint from a zero-set $\mathbf{Z}(g)$. Since g is nonzero on S, the function f defined by $f(s) = \frac{1}{g(s)}$, $(s \in S)$ is continuous, so it has a continuous extension h to all of X. Clearly, $g{\cdot}h$ is an Urysohn function for $(\mathbf{Z}(g), S)$.

Let $f \in C(S)$. Then, $(\arctan \circ f) \in C^*(S)$, and let g be its extension to $C(X)$. The set $Z = \{x \in X: |g(x)| \geq \frac{\pi}{2}\}$ is a zero-set disjoint from S. By hypothesis, there exists an Urysohn function h for (Z, S). The function $g{\cdot}h$ agrees with $(\arctan \circ f)$ on S, and also $|(g{\cdot}h)(x)| < \frac{\pi}{2}$, for all $x \in X$. Now, $(\tan \circ (g{\cdot}h))$ is the desired extension of f to all of X. □

Theorem 0.5.6 - *The following statements are equivalent in any topological space X:*

(a) X is normal;

(b) Any two disjoint closed sets in X are completely separated;

(c) Every closed set in X is C-embedded;

(d) Every closed set in X is C^*-embedded.

Proof - (a) implies (b) by Urysohn's lemma. Assume (b). If A and B are completely separated in S, by Theorem 0.4.1, they are contained in disjoint zero sets in S. Since closed sets in S are closed in X, it follows that A and B are completely separated in X. We invoke Theorem 0.5.3 to conclude that S is C^*-embedded. Hence (b) implies (d).

Always, (c) implies (d).

(d) implies (a): Let A and B be disjoint closed sets in X. Then, $S = A \cup B$ is a closed subspace of X, and by [26, p53, corollary 19] the function f, which assumes the value 0 on A and 1 on B belongs to $C^*(S)$. By hypothesis, it has an extension $g \in C^*(X)$, such that, $g[A] = 0$ and $g[B] = 1$, which easily implies normality of X.

Finally, to complete the chain, we show that (a) implies (c). If F is a closed set in X, by (d), it is C^*-embedded in X. If Z is a zero-set disjoint from F, by (b) it is completely separated from F, and the result follows from Theorem 0.5.5. □

The above theorem immediately gives the fact, that in any metric space, in particular in **R**, every closed set is C, as well as C^*-embedded. The following corollary is useful.

Corollary 0.5.7 - *Every C^*-embedded zero-set S in X is C-embedded in X.*

Proof - If Z is a zero-set disjoint from S, by Theorem 0.4.1, S and Z are completely separated. □

0.6 Completely Regular (Hausdorff) Spaces

In our study of C(X), it is desirable that the underlying topological space X is not too restrictive, and at the same time, will guarantee a rich supply of continuous functions. In particular, we would like C(X) to contain more than the constant functions. For this to happen, complete regularity is an adequate setting. Recall from [19], that a *completely regular space* is a Hausdorff space in which for every closed set F, and point $x \notin F$, there exists an Urysohn function for the pair $(F, \{x\})$. In this section, we bring out a characterization, and several useful properties of completely regular spaces, which are relevant to our study of C(X). We also point out that there is no gain in generality by considering spaces more general than completely regular spaces.

It is always the case that the topology on X determines all the continuous (real-valued) functions. The following theorem guarantees that in a completely regular space, the converse is also true: it's topology is determined by the family of all continuous (real-valued) functions defined on it.

Theorem 0.6.1 - *A Hausdorff space X is completely regular if and only if the family of zero-sets in X form a base for closed sets. (Equivalently, the family* *of cozero-sets form a base for open sets.)*

Proof - Let F be a closed subset of a completely regular space X. If $x \notin F$, there exists $f \in C(X)$ such that $f(x) = 1$ and $f[F] = \{0\}$. Clearly, the zero-set $Z(f)$ contains F, but misses the point x, and hence F can be obtained as the intersection of all such zero-sets.

Conversely, assume that $Z[X]$ is a base for closed sets in X. Let F be closed and $x \notin F$. There exists $g \in C(X)$, such that $F \subseteq Z(g)$ and $g(x) = r \neq 0$. Now, the function $f = g \cdot r^{-1}$ is an Urysohn function for $(F, \{x\})$. □

The rôle of zero-set neighborhoods in a completely regular space is illustrated in the following corollary.

Corollary 0.6.2 - *In a completely regular space X, the following are true:*

(i) *Every closed set F is the intersection of zero-set neighborhoods of F;*

(ii) *Every neighborhood of a point contains a zero-set neighborhood of that point.*

We are now in a position to give a result for completely regular spaces, similar to that given in part (c) of Theorem 0.5.6.

Theorem 0.6.3 - *Every compact subset S in a completely regular space X is C-embedded, and hence, C^*-embedded.*

Proof - First, we show that if A and B are disjoint closed subsets of X, and A is compact, they are completely separated in X. For each $x \in A$, choose disjoint zero-sets Z_x, W_x, with Z_x a neighborhood of x, and $B \subseteq W_x$. The open cover $\{Z_x : x \in A\}$, reduces to a finite subcover $\{Z_{x_1}, Z_{x_2}, \cdots, Z_{x_n}\}$. Clearly, A and B are respectively contained in the disjoint zero-sets $Z_{x_1} \cup Z_{x_2} \cup \cdots \cup Z_{x_n}$ and $W_{x_1} \cap W_{x_2} \cap \cdots \cup W_{x_n}$.

Now, if A and B are completely separated subsets of S, they have disjoint closures in S, and by the above argument, they are completely separated in X, since these closures are compact. By Theorem 0.5.3, S is C^*-embedded. The fact that S is C-embedded follows by using Theorem 0.5.5. □

The *weak topology* on X induced (or generated) by a family G of real-valued functions on X is the weakest topology on X which renders each function in G continuous.

Theorem 0.6.4 - *A Hausdorff space X, whose topology is the weak topology induced by a family G of real-valued functions, is completely regular.*

Proof - Clearly, $G \subseteq C(X)$, so that the weak topology induced by G on X is coarser than the weak topology induced by the family C(X). But, the latter topology is always coarser than the given topology on X. Hence G and C(X) coincide. Since closed rays form a subbase for closed sets in $\mathbf{R}$, their preimages under members of C(X), namely sets of the form $\{x \in X : f(x) \geq r\}$

$= \mathbf{Z}((f - r) \wedge \mathbf{0})$, $r \in \mathbf{R}$ and $f \in C(X)$, which are obviously zero-sets, form a subbase for closed sets in X. Also, each zero-set arises this way, since $\mathbf{Z}(f) = \{x \in X : -|f|(x) \geq 0\}$. Therefore, $\mathbf{Z}[X]$ forms a base for closed sets in X, and Theorem 0.6.1 completes the proof. □

From the details of the above proof, it follows that if X is given the weak topology induced by C(X) (or $C^*(X)$), then $\mathbf{Z}[X]$ becomes a base for the closed sets in X. Combining this with Theorem 0.6.1, we summarize these results in the following theorem.

Theorem 0.6.5 - *For a topological space X, the weak topology induced by C(X) coincides with the topology induced by* $C^*(X)$, *and if X is Hausdorff, this topology coincides with the given topology on X if and only if X is completely regular.*

The next theorem guarantees that when considering algebraic properties of C(X) we may always assume (without loss of generality) that X is completely regular.

Theorem 0.6.6 - *If X is an arbitrary topological space, there exists a completely regular space Y such that C(X) is isomorphic to C(Y).*

Proof - The relation $\sim$ defined on X by setting $a \sim b$ if and only if $f(a) = f(b)$ for all $f \in C(X)$, is an equivalence relation. Let Y denote the set of all equivalence classes under $\sim$, and τ be the map which associates each $x \in X$ with its equivalence class $[x] \in Y$. For $f \in C(X)$, define $g: Y \to \mathbf{R}$ by $g([x]) = f(x)$, i.e., $f = g \circ \tau$. Let G denote the family of all such functions g, and endow Y with the weak topology determined by G. Thus, $G \subseteq C(Y)$. We claim that τ is continuous. A subbasic closed set $F \subseteq Y$ is of the form $g^{\leftarrow}[A]$, where A is closed in $\mathbf{R}$. Also, $\tau^{\leftarrow}[F] = \tau^{\leftarrow}[g^{\leftarrow}[A]] = (g \circ \tau)^{\leftarrow}[A]$ is closed in X, since $g \circ \tau \in C(X)$. Also, Y is Hausdorff, since if $[x] \neq [y]$, then there exists $g \in G$ such that $g(x) \neq g(y)$. Hence by Theorem 0.6.4, Y is completely regular.

Finally, if $h \in C(Y)$, then $h \circ \tau \in C(X)$, which implies that $h \in G$. Therefore, $G = C(Y)$, and clearly, the passage $g \to g \circ \tau$ is an isomorphism. □

Remark - It is worth pointing out that the map τ in the above proof is not necessarily a quotient map, since the topology on Y may not be the largest, that makes τ continuous. For example, let X be $\mathbf{R}$ with discrete topology, Y be the set $\mathbf{R}$, and τ, the identity map on $\mathbf{R}$. Then G reduces to $\mathbf{R}^{\mathbf{R}}$. It is easily seen that Y becomes indiscrete, whereas the quotient topology on Y induced by τ is the discrete topology.

The isomorphism which we obtained in the above theorem is, in fact, a lattice isomorphism, and further, carries $C^*(Y)$ onto $C^*(X)$. We first show that every ring homomorphism is also a lattice homomorphism. Recall that a *lattice homomorphism* is a map which preserves the lattice operations $\vee$ and $\wedge$.

Theorem 0.6.7 - *Every (ring) homomorphism ϕ from $C(Y)$ (or $C^*(Y)$) into $C(X)$ is a lattice homomorphism.*

Proof - Since $(f \vee g) + (f \vee g) = f + g + |f - g|$, it suffices to show that ϕ preserves absolute values. Note that ϕ is order-preserving. For, if $f \geq 0$, then $f = h^2$ for some $h \in C(X)$, so that $\phi(f) = \phi(h^2) = (\phi(h))^2 \geq 0$. Next, if f is arbitrary, $(\phi|f|)^2 = \phi(|f|^2) = \phi(f^2) = (\phi(f))^2$, and since $\phi(|f|) \geq 0$, we have $\phi(|f|) = |\phi(f)|$. Similarly, we can prove that ϕ preserves $\wedge$. □

The next result shows that boundedness of a function is also determined by the ring structure of $C(X)$.

Theorem 0.6.8 - *If ϕ is a (ring) homomorphism of $C(Y)$ into $C(X)$ whose image contains $C^*(X)$, then ϕ carries $C^*(Y)$ onto $C^*(X)$*

Proof - Let $k \in C(Y)$ be such that $\phi(k) = \mathbf{1}$. Now, $\mathbf{1} = \phi(k) = \phi(k \cdot \mathbf{1}) = \phi(k) \cdot \phi(\mathbf{1}) = \phi(\mathbf{1})$, whence $\phi(\mathbf{n}) = \mathbf{n}$, for $n \in \mathbf{N}$. Let $g \in C^*(Y)$ be such that $|g| \leq \mathbf{n}$. So, $|\phi(g)| = \phi(|g|) \leq \phi(\mathbf{n}) = \mathbf{n}$, showing that $\phi(g) \in C^*(X)$. Also, if $f \in C^*(X)$, with $|f| \leq \mathbf{n}$, choose $h \in C(Y)$, such that $\phi(h) = f$. Define $g = (-\mathbf{n} \vee h) \wedge \mathbf{n}$. Clearly, $g \in C^*(Y)$ and $\phi(g) = f$. □

In light of the above results, when considering the function space $C(X)$, we shall always assume that :

the underlying topological space X is completely regular.

CHAPTER 1

The Stone-Cech Compactification of X

In this chapter, we study the ideals in the ring C(X) and filters on X. The correspondence between maximal ideals in C(X) and certain filters on X (called *z*-filters) is established. By suitably topologizing the family of all maximal ideals in C(X), we construct the classical Stone-Cech Compactification βX and the realcompactification υX of the space X. As a consequence of these constructions, we are able to characterize the maximal ideals in C(X) as well as in $C^*(X)$, and obtain certain "Banach-Stone type" theorems, as well as give a solution to the problem of how to recover the topological space X from the rings C(X) and $C^*(X)$. We shall also indicate how the above constructions could be carried out using various other methods.

1.1 Ideals and z-Filters

An *ideal* in C(X) is a subring I of C(X) which has the property that $g \cdot f \in I$ whenever $g \in I$ and $f \in C(X)$. Trivially, C(X) itself is an ideal, which will be referred to as the *improper ideal*, while all other ideals will be called proper. The intersection of any family of ideals is again an ideal. The *ideal generated by a subset* $A \subseteq C(X)$ is the smallest ideal that contains A, namely $\bigcap \{I: I$ is an ideal in $C(X)$ and $I \supseteq A\}$, and is denoted by $< A >$. Note that, sometimes, this can be improper. It is easy to show that $< A > = \{\sum_{finite} a_i f_i : a_i \in A, f_i \in C(X)\}$.

All the above concepts apply equally to $C^*(X)$, and if I is an ideal in $C(X)$, then $I \cap C^*(X)$ is an ideal in $C^*(X)$.

A proper ideal I that is not contained in any other proper ideal is called a *maximal ideal*. Zorn's lemma guarantees that any proper ideal is contained in a maximal ideal. A proper ideal I is *prime*, if whenever $f \cdot g \in I$, we have $f \in I$ or $g \in I$. As in classical ring theory, it can easily be shown that every maximal ideal is a prime ideal. An example of a prime ideal in $C(X)$ that is not maximal is given later.

For us, the word *ideal*, unmodified, will always mean a *proper* ideal. Clearly, such an ideal cannot contain any function f for which the inverse $\frac{1}{f}$ exists. Such functions are called *units* in $C(X)$, and are characterized by the property that their zero-sets are empty. Note that a similar characterization does not hold in $C^*(X)$. For example, the identity function $\mathbf{i}$ in $C(\mathbf{N})$ has an inverse $\mathbf{j}$ in $C^*(\mathbf{N})$, where $\mathbf{j}(n) = \frac{1}{n}$, $(n \in \mathbf{N})$, and $\mathbf{Z}(\mathbf{j}) = \emptyset$. However, $\mathbf{j}$ is not a unit in

$C^*(N)$, since $\mathbf{1} \notin C^*(N)$.

A *(Bourbaki) filter* on a set X is a nonempty family **F** of subsets of X that satisfy the following:

(i) $\emptyset \notin \mathbf{F}$;

(ii) $A, B \in \mathbf{F}$ implies that $A \cap B \in \mathbf{F}$;

(iii) $A \in \mathbf{F}$, $A \subseteq B$ implies that $B \in \mathbf{F}$.

A *z-filter* is a filter on X, all of whose members are zero-sets. Note that a z-filter is a topological object, while a filter is a set-theoretical one. In a discrete space, every set is a zero-set, so the two notions coincide. In any space, the intersection of a filter with $\mathbf{Z}[X]$ is obviously a z-filter. If **B** is any family of zero-sets with finite intersection property (abbreviated hereafter as F.I.P.), then **B** is contained in a z-filter, and the smallest such z-filter **F** is said to be *generated by* **B**. In addition, if **B** itself is closed under finite intersections, then **B** is said to be a *base* for **F**. A *z-ultrafilter* is a maximal z-filter. Using Zorn's lemma, it can be shown that every z-filter is contained in a z-ultrafilter.

We define a mapping **Z** on the family of all ideals in C(X) as follows: for each ideal I in C(X), $\mathbf{Z}[I] = \{Z(f): f \in I\}$. The following theorem establishes that the map **Z** is a natural correspondence between ideals and z-filters.

Theorem 1.1.1 - *The mapping* **Z** *has the following properties:*

(a) *If I is an ideal in $C(X)$, then $\mathbf{Z}[I]$ is a z-filter on X;*

(b) *If $\mathbf{F}$ is a z-filter on X, the set $\mathbf{Z}^{\leftarrow}[\mathbf{F}] = \{f \in C(X): \mathbf{Z}(f) \in \mathbf{F}\}$ is an ideal in $C(X)$.*

Proof - (a): We shall verify the three conditions. Since I contains no unit, $\emptyset \notin \mathbf{Z}[I]$. Let $\mathbf{Z}(f_1), \mathbf{Z}(f_2) \in \mathbf{Z}[I]$. Since I is an ideal, $f_1^2 + f_2^2 \in I$. Hence, $\mathbf{Z}(f_1) \cap \mathbf{Z}(f_2) = \mathbf{Z}(f_1^2 + f_2^2) \in \mathbf{Z}[I]$. Finally, if $\mathbf{Z}(f) \in \mathbf{Z}[I]$, and $\mathbf{Z}(f) \subseteq \mathbf{Z}(g)$, then $g \cdot f \in I$, since I is an ideal. Consequently, $\mathbf{Z}(g) = \mathbf{Z}(g) \cup \mathbf{Z}(f) = \mathbf{Z}(g \cdot f) \in \mathbf{Z}[I]$.

To prove (b), let $f, g \in I = \mathbf{Z}^{\leftarrow}[\mathbf{F}]$, and $h \in C(X)$. Since $\emptyset \notin \mathbf{F}$, I contains no unit. Next, $\mathbf{Z}(f - g) \supseteq \mathbf{Z}(f) \cap \mathbf{Z}(g) \in \mathbf{F}$, and $\mathbf{Z}(h \cdot f) \supseteq \mathbf{Z}(f) \in \mathbf{F}$. Thus, $f - g$ and $h \cdot f$ belong to I, proving that I is an ideal. □

Note that $\mathbf{Z}[\mathbf{Z}^{\leftarrow}[\mathbf{F}]] = \mathbf{F}$, and $\mathbf{Z}^{\leftarrow}[\mathbf{Z}[I]] \supseteq I$. In other words, the mapping $\mathbf{Z}$ is always onto. Here is a quick example to show that it need not be one-to-one.

Example 1.1.2 - Let $I = \{f \in C(\mathbf{R}): f(x) = x \cdot g(x) \text{ for some } g \in C(\mathbf{R})\}$. In fact, I can be recognized as the principal ideal generated by the identity function $\mathbf{i}$, and $0 \in \mathbf{Z}(f)$ for each $f \in I$. [An ideal is called *principal* if it is generated by a single element]. The ideal $M = \mathbf{Z}^{\leftarrow}[\mathbf{Z}[I]]$ consists of all members of $C(\mathbf{R})$ which vanish at 0, so the function $\mathbf{i}^{\frac{1}{3}} \in M$. If $\mathbf{i}^{\frac{1}{3}} = g \cdot \mathbf{i}$, for some $g \in C(\mathbf{R})$, then $g(x) = x^{-\frac{2}{3}}$ $(x \neq 0)$, so $g \notin C(\mathbf{R})$. This implies that $\mathbf{i}^{\frac{1}{3}}$ is not in I. □

We point out that part (a) of Theorem 1.1.1 is not valid for $C^*(X)$. Let J be the ideal in $C^*(\mathbf{N})$ consisting of all sequences that converge to 0. The function j defined earlier (namely, $j(n) = \frac{1}{n}$, $n \in \mathbf{N}$) is in J, but $\mathbf{Z}(j) = \emptyset$, so $\mathbf{Z}[J]$ is not a z-filter.

Next, we show that even though the map $\mathbf{Z}$ is not bijective, its restriction to maximal ideals establishes a one-to-one correspondence between maximal ideals in $C(X)$ and z-ultrafilters on X.

Theorem 1.1.3 -

(a) If I is a maximal ideal in C(X), $\mathbf{Z}[I]$ *is a z-ultrafilter on X;*

(b) If $\mathbf{F}$ *is a z-ultrafilter on X, then* $\mathbf{Z}^{-}[\mathbf{F}]$ *is an maximal ideal in C(X).*

The straightforward proof is omitted.

A property of maximal ideals (respectively, z-ultrafilters) which is frequently used in this chapter, is given in the following theorem.

Theorem 1.1.4 -

(a) Let M be a maximal ideal. If $\mathbf{Z}(f)$ *meets each member of* $\mathbf{Z}[M]$, *then* $f \in M$.

(b) Let $\mathbf{F}$ *be a z-ultrafilter on X. If a zero-set Z meets every member of* $\mathbf{F}$,

then $Z \in \mathbf{F}$.

Proof - (a) and (b) are equivalent by Theorem 1.1.3. We shall prove (b). Note that $\mathbf{F} \cup \{Z\}$ generates a *z*-filter, which contains $\mathbf{F}$. The maximality of $\mathbf{F}$ shows that $Z \in \mathbf{F}$. □

It is worth pointing out that the properties (a) and (b) above, are characteristic of maximal ideals and *z*-ultrafilters, respectively.

An ideal I for which $\mathbf{Z}^{\leftarrow}[\mathbf{Z}[I]] = I$ is called a *z-ideal.* Note that the map $\mathbf{Z}$ sets up a one-to-one correspondence between *z*-ideals and *z*-filters. For a nonempty subset $S \subseteq X$, the family of function in C(X) that vanish on S constitutes a *z*-ideal. It turns out that in $C(\mathbf{N})$, every ideal is a *z*-ideal. For, if $\mathbf{Z}(f) \in \mathbf{Z}[I]$, let $\mathbf{Z}(f) = \mathbf{Z}(g)$ for some $g \in I$. Define h on $\mathbf{N}$ by setting $h(n) = 0$ on $\mathbf{Z}(g)$, and $h(n) = \dfrac{f(n)}{g(n)}$ otherwise. Since $\mathbf{N}$ is discrete, $h \in C(\mathbf{N})$, and $h \cdot g = f \in I$.

Trivially, every maximal ideal is a *z*-ideal in view of Theorem 1.1.3. The following example shows that a *z*-ideal need not be a maximal ideal.

Example 1.1.5 - Let I denote the ideal in $C(\mathbf{R})$ consisting of all functions that vanish at 0 as well as 1. Clearly, I is a *z*-ideal and is properly contained in the ideal consisting of all functions that vanish at 0 only. Hence, it is not a maximal ideal.

This example also shows that a z-ideal is not necessarily prime. Let $\mathbf{i}$ denote the identity function in $C(\mathbf{R})$. The product $\mathbf{i}\cdot(\mathbf{i}-\mathbf{1}) \in I$, yet $\mathbf{i} \notin I$ and $(\mathbf{i}-\mathbf{1}) \notin I$. □

To complete the chain of counterexamples, we now provide a prime ideal that is not a z-ideal, and hence not a maximal ideal.

Example 1.1.6 - Choose a function f in $C(\mathbf{R})$ with the properties that $f(0) = 0$ and $\left|\frac{f^n(x)}{x}\right| \to \infty$ as $x \to 0$, for each $n \in \mathbf{N}$. For example, the function

$$f(x) = \begin{cases} \sum_{n=0}^{\infty} \frac{|x|^{\frac{1}{2n+1}}}{(2n+1)^2}, & |x| < 1 \\ \frac{\pi^2}{8}, & |x| \geq 1 \end{cases}$$

meets the requirements. Observe that for any n, $f^n \notin \langle \mathbf{i} \rangle$. Otherwise, there exists $g \in C(\mathbf{R})$ such that $x \cdot g(x) = f^n(x)$ for all x. So, for $x \neq 0$, $g(x) = \frac{f^n(x)}{x}$, which is unbounded near origin, and hence g is not continuous. Using Zorn's lemma, we can construct an ideal P, maximal for the property that $P \supseteq \langle \mathbf{i} \rangle$, and is disjoint from $\{f^n : n \in \mathbf{N}\}$. We claim that P is prime. Let $h, k \notin P$. By the maximality property of P, there exist $m, n \in \mathbf{N}$ such that $f^m \in \langle P \cup \{h\} \rangle$ and $f^n \in \langle P \cup \{k\} \rangle$. Hence there exist elements $l_i \in C(\mathbf{R})$, $p_i \in P$, $(i = 1, 2)$ such that $f^m = (l_1 h + p_1)$ and $f^n = (l_2 k + p_2)$, so that $(l_1h + p_1) \cdot (l_2k + p_2) = f^{m+n} \notin P$. Expanding, we see that $l_1l_2hk \notin P$, and therefore $hk \notin P$. Hence P is prime.

However, P is not a z-ideal, for $\mathbf{Z}(f) = \mathbf{Z}(i) \in \mathbf{Z}[P]$, but $f \notin P$. □

In conclusion, we define the notion of a prime z-filter. A z-filter, $\mathbf{F}$ is said to be *prime*, if whenever $Z_1 \cup Z_2 \in \mathbf{F}$, either $Z_1 \in \mathbf{F}$ or $Z_2 \in \mathbf{F}$. As an immediate consequence of part (b) of Theorem 1.1.4, we get the following corollary.

Corollary 1.1.7 - *Every z-ultrafilter is prime.*

Proof - Let $\mathbf{F}$ be a z-ultrafilter and $Z_1 \cup Z_2 \in \mathbf{F}$. If $Z_1 \notin \mathbf{F}$ and $Z_2 \notin \mathbf{F}$, then there exist $F_1, F_2 \in \mathbf{F}$ such that $Z_i \cap F_i = \emptyset$, $i = 1, 2$. Thus, $Z_1 \cup Z_2$ misses $F_1 \cap F_2$, and hence cannot be in $\mathbf{F}$. □

1.2 Fixed and Free Ideals

The notions of a fixed and a free ideal are introduced. We show that the space X is compact if and only if all the ideals in $C(X)$ are fixed.

An ideal I is said to be *fixed* if $\bigcap_{f \in I} Z(f) \neq \emptyset$. Otherwise, it is *free*. A z-filter is said to be *fixed* if the intersection of all its members is nonempty. Otherwise, it is called *free*. Hence, in $C(X)$, an ideal I is fixed (free) if and only if the corresponding z-filter $\mathbf{Z}[I]$ is fixed (free). Such a statement is not true in

$C^*(X)$, in view of the remark following Example 1.1.2. For any topological space X and $p \in X$, the ideal $M_p = \{f \in C(X) : f(p) = 0\}$ is always a fixed ideal, since $p \in Z(f)$ for each $f \in M_p$. A quick example of a free ideal is the ideal I in $C(\mathbf{N})$ consisting of all functions which vanish at all but a finite number of points in $\mathbf{N}$. We now use this free ideal I to construct an example (promised in section 0.5) of a space that is C^*-embedded, but not C-embedded.

Example 1.2.1 - Let $\mathbf{U}$ be the z-ultrafilter on $\mathbf{N}$ containing the z-filter $\mathbf{Z}[I]$, where I is as above. Clearly, $\mathbf{U}$ is free. Let $\Sigma = \mathbf{N} \cup \{t\}$, where $t \notin \mathbf{N}$. We topologize Σ by specifying that all points except t are isolated, and a neighborhood of t consists of sets of the form $U \cup \{t\}$, where $U \in \mathbf{U}$. Clearly, $\mathbf{N}$ is dense in Σ. Observe that for $A \subseteq \Sigma$, if $t \notin A$, then A is open, since each of its points are isolated. Therefore, if $t \in A$, then A is closed. We first show that Σ is normal. Let C and D be disjoint closed sets in Σ, and assume $t \notin D$. If $t \notin C$, then both C and D are open. If $t \in C$, then $\Sigma \sim D$ is an open set containing C, disjoint from D.

We shall show that $\mathbf{N}$ is C^*-embedded in Σ.

Let A and B be completely separated in $\mathbf{N}$. So, there exist disjoint zero-sets Z_A and Z_B (necessarily closed) in $\mathbf{N}$, containing A and B, respectively. Thus, $Z_A = F \cap \mathbf{N}$ and $Z_B = G \cap \mathbf{N}$, for closed sets F and G in Σ. By Theorem 0.5.7, if F and G are disjoint, they are completely separated in Σ, and so are A and B. The only other possibility is that $t \in F \cap G$, in which case t is in the closure of F and G, so $U \cap Z_A \neq \emptyset \neq U \cap Z_B$, for every U $\in \mathbf{U}$. Therefore, $\mathbf{N} \sim Z_A$ and $\mathbf{N} \sim Z_B$ are finite, which is a contradiction to Z_A and Z_B being disjoint. We then invoke Theorem 0.5.3 to conclude that **N** is C^*-embedded in Σ.

Note that $\{t\}$ is a zero-set; in fact, $\{t\} = \mathbf{Z}(f)$ where $f \in C(\Sigma)$ is defined by, $f(n) = \frac{1}{n}$, $n \in \mathbf{N}$, and $f(t) = 0$. Since **N** is not closed, it cannot be completely separated from the zero-set $\{t\}$, so by Theorem 0.5.5, **N** is not C-embedded in Σ. □

We point out one more interesting property of this example.

Since **N** is dense and C^*-embedded in Σ, Proposition 0.5.1 implies that $C^*(N)$ is isomorphic to $C^*(\Sigma)$, yet N is locally connected but Σ is not locally connected. It is therefore clear that unlike the case of connectedness, local connectedness of X is in general not determined by $C^*(X)$.

Note that there exist spaces in which every ideal is fixed. If X is compact, $\mathbf{Z}[I]$ is a family of closed sets with F.I.P. Therefore, every ideal I in C(X) (equivalently in $C^*(X)$) is fixed.

The following theorem yields a characterization of all fixed maximal ideals in C(X) and $C^*(X)$.

Theorem 1.2.2 - *The fixed maximal ideals in C(X) are precisely the sets*

$M_p = \{f \in C(X): f(p) = 0\}$ $(p \in X)$, *and are distinct for distinct* p.

Proof - The function $\phi_p : C(X) \to \mathbf{R}$ defined by $\phi_p(f) = f(p)$ is clearly a ring homomorphism for $p \in X$, which is onto, since each real number r is the image of the constant function $\mathbf{r}$. By the fundamental theorem of ring homomorphisms, $\dfrac{C(X)}{\mathrm{Ker}\ \phi_p} = \dfrac{C(X)}{M_p} \cong \mathbf{R}$, which is a field. Consequently, M_p is a maximal ideal, which is trivially fixed. Conversely, if M is a fixed maximal ideal, there exists a p in X such that $p \in \bigcap_{f \in M} Z(f)$. i.e., $f(p) = 0$ for all f in M, which implies that $M \subset M_p$, and by maximality, we have $M = M_p$.

To show that M_p is distinct for distinct p, we recall that X is completely regular, so for $q \in X$, $q \neq p$, there exists $f \in C(X)$, $f(q) \neq 0$, $f(p) = 0$. Then, $f \in M_p$, but $f \notin M_q$, so $M_p \neq M_q$. □

It is not hard to see that Theorem 1.2.2 holds in $C^*(X)$ as well.

We now prove a lemma which describes the compactness of a zero-set in terms of free z-filters.

Lemma 1.2.3 - *A zero-set* Z *is compact if and only if it does not belong to any free z-filter.*

Proof - Let Z be a compact zero-set. If Z is a member of a free z-filter $\mathbf{F}$, then the intersection of members of $\mathbf{F}$ with Z is a family of closed subsets of Z with F.I.P., and with empty intersection.

Let **B** be a family of closed subsets of Z with F.I.P., and note that the members of **B** are closed in X. Let **F** be the *z*-filter in X consisting of all zero-sets in X that contain finite intersections of members of **B**. Clearly, $Z \in$ **F**. Since the zero-sets form a base for closed sets, $\bigcap \mathbf{B} = \bigcap \mathbf{F} \neq \emptyset$, by assumption, so Z is compact. □

The above theorem implies that even if a single zero-set of a *z*-filter is compact, it cannot be free. However, it is not the case that if every member of a *z*-filter (even a *z*-ultrafilter) is non-compact then the *z*-filter is free.

Example 1.2.4 - Recall Example 0.3.1. The *z*-ultrafilter $\mathbf{Z}[M_p]$ is fixed, yet we will show that no member of this family is compact. Since $p \in \mathbf{Z}(f)$, for each $f \in M_p$ (since every zero-set is a G_δ), there exist $\{O_n : n \in \mathbf{N}\}$, a family of open sets containing p, such that $\mathbf{Z}(f) = \bigcap_{n=1}^{\infty} O_n$. Therefore $\mathbf{Z}(f)$ is uncountable, since $X \sim \mathbf{Z}(f) = \bigcup_{n=1}^{\infty} (X \sim O_n)$ is countable since each member of this union is countable (being neighborhoods of p). For $i = 1, 2, 3, \ldots$, pick distinct elements $x_i \in \mathbf{Z}(f) \sim \{p\}$. Then $\{x_i : i \in \mathbf{N}\}$, together with $\mathbf{Z}(f) \sim \bigcup_{i=1}^{\infty} \{x_i\}$ is an open cover of $\mathbf{Z}(f)$ with no finite subcover. □

The following theorem not only characterizes the maximal ideals in C(X), where the topological space X is compact (Hausdorff), but even shows this characterization to be unique to such spaces.

Theorem 1.2.5 - *The following are equivalent for a topological space* X:

(1) X *is compact.*

(2) Every ideal (respectively, z-filter) in $C(X)$ *is fixed;*

(3) Every ideal in $C^*(X)$ *is fixed;*

(4) Every maximal ideal (respectively, z-ultrafilter) in $C(X)$ *is fixed;*

(5) Every maximal ideal in $C^*(X)$ *is fixed.*

Proof - Since X is in every z-ultrafilter, it follows from Lemma 1.2.3 that (1) and (2) are equivalent. Also, (1) implies (3) since $C(X) = C^*(X)$ when X is compact.

(3) implies (2): Let I be a free ideal in $C(X)$. Then $I \cap C^*(X)$ is a free ideal in $C^*(X)$, a contradiction. Finally, (2) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (5), since if I is a free ideal, then $I \subseteq M$, for some maximal ideal M, and M must be free, a contradiction. □

1.3 Structure Space of a Commutative Ring with Unity:

For an arbitrary commutative ring with unity, we define a topology on its collection of maximal ideals. The resulting space is called the structure space of the ring. In this section, we obtain a necessary and sufficient condition for the structure space to be compact Hausdorff.

Let $(R, +, \cdot)$ be a commutative ring with unity. We denote the collection of all its maximal ideals by $\mathbf{M}(R)$, or simply $\mathbf{M}$, if no confusion arises. For any subcollection $\mathbf{A} \subseteq \mathbf{M}$, we define the *kernel* of $\mathbf{A}$ to be $\cap \mathbf{A}$; for an ideal I of R, the *hull* of I is the set $\{M \in \mathbf{M} : I \subseteq M\}$. Further, for each $\mathbf{A} \subseteq \mathbf{M}$, define $k(\mathbf{A}) = \{M \in \mathbf{M} : \cap \mathbf{A} \subseteq M\}$. Note that $k(\mathbf{A})$ is well-defined, since the intersection of a family of ideals is again an ideal. It is easily verified that the operator k is a Kuratowski closure operator on the subsets of $\mathbf{M}$, and hence defines a unique topology on $\mathbf{M}$ such that $k(\mathbf{A})$ is equal to the closure of $\mathbf{A}$. Since the closure $k(\mathbf{A})$ of any subset is obtained by taking the hull of the kernel of the set, this topology is called the *Hull-Kernel topology*. We refer to $\mathbf{M}$, endowed with this topology as the *structure space* of the ring R. For each $r \in R$, let $\mathbf{M}(r)$ denote the set of all maximal ideals containing the member r. This set is closed, since it is precisely $k(\{< r >\})$. Conversely, if $\mathbf{A} \subseteq \mathbf{M}$, then the closure of $\mathbf{A} = \cap \{\mathbf{M}(r) : r \in \cap \mathbf{A}\}$. Therefore the collection $\{\mathbf{M}(r) : r \in R\}$ is a base for closed sets for the Hull-Kernel topology.

We now explore some properties of the structure space of a commutative ring with unity.

Theorem 1.3.1 - *The structure space of the ring R is always a compact T_1-space.*

Proof - Let M and N be distinct maximal ideals in R. Choose $m \in M$

such that $m \notin N$. Then $\mathbf{M} \sim \mathbf{M}(m)$ is a neighborhood of N not containing M, showing that the space is T_1.

To prove compactness, let $\mathbf{F} = \{\mathbf{M}(a) : a \in A\}$ for $A \subseteq R$, be a family of (basic) closed sets with F.I.P., and let I denote the ideal generated by the set $\{a : a \in A\}$. We first show that I is not "all" of R. Assume the contrary; then $\mathbf{1} = \sum_{i=1}^{n} r_i a_i$, for $r_i \in R$, $a_i \in A$. This means that no maximal ideal could contain the collection $\{a_i : i = 1, 2, 3, \ldots, n\}$, and $\bigcap_{i=1}^{n} \mathbf{M}(a_i) = \emptyset$, contradicting the F.I.P. So, there exists a maximal ideal J containing I. Then, $J \in \mathbf{M}(a)$ for every $a \in A$ and $\mathbf{F}$ has nonempty intersection. Therefore, $\mathbf{M}$ is compact. □

Rings with exactly one maximal ideal are called *local rings* (e.g., fields), and *semi-local*, if they have only a finite number of maximal ideals.

Corollary 1.3.2 - *The structure space of a semi-local ring is always discrete.*

Proof - This follows easily since all finite T_1-spaces are discrete. □

From Corollary 1.3.2, it is clear that all fields and finite rings have discrete structure spaces.

We point out that the Hull-Kernel topology need not, in general, be Hausdorff. The next theorem gives a necessary and sufficient condition on the ring R, for the structure space to be Hausdorff.

Theorem 1.3.3 - *The structure space* $\mathbf{M}$ *of the ring* R *is a Hausdorff space if and only if for each pair* M *and* N *of distinct members of* $\mathbf{M}$, *there exist elements* $m \notin M$ *and* $n \notin N$ *such that* $m{\cdot}n \in \bigcap \mathbf{M}$.

Proof - Assume that $\mathbf{M}$ is Hausdorff. Then for distinct maximal ideals M and N, there exist $n \notin N$ and $m \notin M$, such that $\mathbf{M} \sim \mathbf{M}(m)$ and $\mathbf{M} \sim \mathbf{M}(n)$ are disjoint (basic) open neighborhoods of M and N, respectively. Therefore $\mathbf{M} \sim (\mathbf{M}(m) \cup \mathbf{M}(n)) = \emptyset$, and since maximal ideals are prime, $\mathbf{M}(m) \cup \mathbf{M}(n) = \mathbf{M}(m{\cdot}n)$, so that $m{\cdot}n \in \bigcap \mathbf{M}$.

Conversely, if M and N are distinct maximal ideals with $m \notin M$ and $n \notin N$, then, $\mathbf{M} \sim \mathbf{M}(m)$ and $\mathbf{M} \sim \mathbf{M}(n)$ are neighborhoods of M and N, respectively. Also, $(\mathbf{M} \sim \mathbf{M}(m)) \cap (\mathbf{M} \sim \mathbf{M}(n)) = \mathbf{M} \sim \mathbf{M}(m{\cdot}n)$. By assumption, $m{\cdot}n$ is in every maximal ideal, and the the fact that $\mathbf{M}(m{\cdot}n) = \mathbf{M}$ forces these neighborhoods to be disjoint. In other words, $\mathbf{M}$ is Hausdorff. □

We now observe that the structure space of the ring $\mathbf{Z}$ of integers is not Hausdorff. For any prime number p, the set $p\mathbf{Z} = \{ p{\cdot}z : z \in \mathbf{Z}\}$ is a maximal ideal in $\mathbf{Z}$, and we choose two such maximal ideals $p\mathbf{Z}$, $q\mathbf{Z}$, for distinct primes p, q. Let $x \notin p\mathbf{Z}$ and $y \notin q\mathbf{Z}$. By the fundamental theorem of arithmetic, there exists a prime s such that $x{\cdot}y \notin s\mathbf{Z}$ and therefore $x{\cdot}y$ cannot be in every maximal ideal in $\mathbf{Z}$.

1.4 The Stone-Cech Compactification

In the preceding section, we studied the structure space of an arbitrary commutative ring with unity. Here, we specialize this notion to the ring $C(X)$, and prove that the structure space of $C(X)$ is precisely the familiar Stone-Cech Compactification βX of the topological space X. Throughout this section, the symbol $\mathbf{M}$ will denote the structure space of $C(X)$, namely the collection of all its maximal ideals endowed with the Hull-Kernel topology. Recall that for each $f \in C(X)$, $\mathbf{M}(f)$ denotes the collection of members of $\mathbf{M}$ which contain f, and the family $\{\mathbf{M}(f): f \in C(X)\}$ forms a base for closed sets in the Hull-Kernel topology.

Theorem 1.4.1 - *The space* $\mathbf{M}$ *is a compact Hausdorff space.*

Proof - In view of Theorem 1.3.1, it suffices to verify the Hausdorff condition. Let M and N be distinct maximal ideals in $\mathbf{M}$. By part (a) of Theorem 1.1.4, they contain members with disjoint zero-sets $\mathbf{Z}_M$ and $\mathbf{Z}_N$, respectively, which in turn, (by Corollary 0.4.3) are contained in disjoint cozero-sets, C and D, respectively. Thus, there exist $f,g \in C(X)$, with $\mathbf{Z}(f) = X \sim C$, and $\mathbf{Z}(g) = X \sim D$. Clearly, $f \notin M$ and $g \notin N$, but $f{\cdot}g$ belongs to every maximal ideal, since $\mathbf{Z}(f) \cup \mathbf{Z}(g) = X$, so, $f{\cdot}g = \mathbf{0}$. The result now follows from Theorem 1.3.3. □

Next, we show that X is homeomorphic to a dense subspace of **M**, thus realizing **M** as a Hausdorff compactification of X.

Theorem 1.4.2 - M *is a Hausdorff compactification of X.*

Proof - We claim that the map e: $X \rightarrow \mathbf{M}$ defined by $e(x) = \{f \in C(X): f(x) = 0\}$, $(x \in X)$, is an embedding of X into a dense subspace of **M**. By Theorem 1.2.2, e is well defined. If $x \neq y$, then there exists $f \in C(X)$ such that $f(x) = 0$, $f(y) \neq 0$, so that $f \in e(x)$, but $f \notin e(y)$, implying that $e(x) \neq e(y)$ and therefore e is one-to-one.

To show that e is a homeomorphism into a subspace of **M**, we show that e carries basic closed sets in X to basic closed sets in the subspace e[X]. Let **Z**(f) be any zero-set (i.e., a basic closed set) in X. It is easily checked from definitions that $e[\mathbf{Z}(f)] = \mathbf{M}(f) \cap e[X]$.

To conclude the proof, we need to show that e[X] is dense in **M**. To see this, we apply the closure operator k (defined in section 1.3) to e[X], i.e., $k(e[X]) = \{M \in \mathbf{M} : \bigcap_{x \in X} e(x) \subseteq M\} = \mathbf{M}$, since **0** is in every ideal. □

An immediate consequence of this theorem, is that when X is compact (Hausdorff), X is homeomorphic to **M**. This yields a characterization of the maximal ideals in C(X), since in this case $e[X] = \mathbf{M}$, so every maximal ideal in C(X) is of the form $e(x) = \{f \in C(X): f(x) = 0\}$, where $x \in X$. Note that all such maximal ideals are fixed. Since $C^*(X)$ and C(X) are identical for compact X, this characterization applies to $C^*(X)$ as well.

We now identify M with the Stone–Cech compactification βX of X. To simplify notation, we assume X to be a (dense) subspace of βX, instead of simply homeomorphic to such a subspace. Of the several characterizations for the Stone–Cech compactification, we shall use the following, [43, p10]:

Any Hausdorff compactification of X is a continuous image of the Stone-Cech compactification under a mapping which leaves the points of X fixed.

We use the notation $cl_T A$ to denote the closure of a subset A in the topological space T. The subscript T will be dropped when no confusion can arise.

Since X is completely regular, βX always exists, and we obtain the following theorem.

Theorem 1.4.3 - M *is homeomorphic to βX.*

Proof - Let h be the continuous map from βX onto M such that, if $x \in X$, then $h(x) = e(x)$, (i.e., h leaves the points of X fixed). Since βX is compact and M is Hausdorff, h is a closed map. Finally, to show that h is a homeomorphism, we need only show it is one-to-one.

If p, q are distinct points of βX, since βX is completely regular, there exists $f \in C(\beta X)$ such that $f(p) = 0$, $f(q) = 1$ and $0 \leq f \leq 1$. Define $\mathbf{Z}_1 = \{x \in X : f(x) \leq \frac{1}{3}\}$ and $\mathbf{Z}_2 = \{x \in X : f(x) \geq \frac{2}{3}\}$. Clearly, these are two disjoint zero-sets of X. We first show that $p \in cl_{\beta X}\mathbf{Z}_1$. Assume the contrary. Then there exists a neighborhood N of p in βX that is disjoint from $\mathbf{Z}_1$. By continuity of f, we obtain a neighborhood N_1 of p in βX such that $f(x) < \frac{1}{3}$ for $x \in N_1$. Now, $N \cap N_1$ is a neighborhood of p disjoint from X, a contradiction, since X is dense in βX. Similarly, we can show that $q \in cl_{\beta X}\mathbf{Z}_2$.

The continuity of h implies that $h(p) \in cl_M h[\mathbf{Z}_1]$ and $h(q) \in cl_M h[\mathbf{Z}_2]$. The proof is completed by showing that $cl_M h[\mathbf{Z}_1]$ and $cl_M h[\mathbf{Z}_2]$ are disjoint. Observe that $cl_M h[\mathbf{Z}_1] \cap cl_M h[\mathbf{Z}_2] = cl_M e[\mathbf{Z}_1] \cap cl_M e[\mathbf{Z}_2] = \{M \in \mathbf{M} : \bigcap_{x \in Z_1} e(x) \subseteq M\} \cap \{M \in \mathbf{M} : \bigcap_{x \in Z_2} e(x) \subseteq M\}$. If a maximal ideal M is in this intersection, then M will contain continuous functions l and k such that $\mathbf{Z}(l) = \mathbf{Z}_1$, and $\mathbf{Z}(k) = \mathbf{Z}_2$. This is a contradiction, since $l^2 + k^2$ is a unit. □

Thus, we have identified the structure space $\mathbf{M}$ of C(X) as the Stone-Cech compactification, βX of X. Consequently, the following extension property (due to Stone) holds, and in fact, is unique to βX.

Stone Extension Property - *Any continuous map f from X into a compact Hausdorff space K has a unique extension f^β from βX into K.*

This leads to the following characterization of βX, which relates to the ring $C^*(X)$.

Theorem 1.4.4 - *βX is the Hausdorff compactification of X in which X is C^*-embedded.*

Proof - We show that the statement is equivalent to the Stone extension property. Clearly if βX has the Stone extension property, and f is any member of $C^*(X)$, then the closure (in $\mathbf{R}$) of the range of f is a compact Hausdorff space, and the result follows.

Conversely, assume that X is C^*-embedded in βX, and let f be a continuous map from X into a compact Hausdorff space K. For each $g \in C^*(K)$, $(g \circ f) \in C^*(X)$, and by assumption has a continuous extension $(g \circ f)^\beta$ to all of βX. Since K is Tychonoff, we invoke the embedding lemma, [26, p116] to obtain the embedding $\rho: K \rightarrow T$, where $T = \prod_{g \in C^*(K)} I_g$, and ρ is defined by $\rho(k)(g) = g(k)$, $(k \in K)$, where I_g denotes a closed bounded interval containing the range of g. The map $h : \beta X \rightarrow \prod_{g \in C^*(K)} I_g$, where for each $p \in \beta X$, $h(p)(g) = (g \circ f)(p)$, is continuous, since $P_g \circ h = (g \circ f)^\beta$ is continuous for each $g \in C^*(K)$, where P_g is the g^{th}-projection map onto I_g, [26, p91, Theorem 3]. Next, if $x \in X$, $h(x)(g) = (g \circ f)(x) = g(f(x))$, and $\rho(f(x)) = g(f(x))$, showing that $h|_X = f$. Finally, $h[\beta X] = h[cl_{\beta X} X] \subseteq cl_T\, h[X] = cl_T\, f[X] \subseteq cl_T \rho[K] = \rho[K]$, showing that $h[\beta X] \subseteq \rho[K]$. Since X is dense in βX, h is unique. □

The following corollary is immediate.

Corollary 1.4.5 - *$C^*(X)$ is isomorphic to $C^*(\beta X)$.*

The isomorphism associates to each function f in $C^*(X)$, it's (unique) extension f^β to all of βX. □

Note - Since βX is compact, $C(\beta X) = C^*(\beta X)$, so, trivially, we have that $C^*(X)$ is isomorphic to $C(\beta X)$.

If X and Y are homeomorphic spaces, it is trivial that the rings C(X) and C(Y) (respectively $C^*(X)$ and $C^*(Y)$) are isomorphic. The converse direction is not always true. For example, if X is not compact, $C^*(X)$ is isomorphic to $C^*(\beta X)$, whereas X is not homeomorphic to βX. Results which give conditions under which the converse is true, are known as *Banach-Stone type theorems*. One such result is obtained in the next corollary.

Corollary 1.4.6 - *For compact spaces X and Y, X is homeomorphic to Y if and only if $C^*(X)$ is isomorphic to $C^*(Y)$.*

Proof - Since $C^*(X)$ and $C^*(Y)$ have homeomorphic structure spaces, it follows that βX is homeomorphic to βY. Therefore, the result follows since X and Y are homeomorphic to βX and βY, respectively. □

This shows that $C^*(X)$ distinguishes among compact spaces. However, if X is not compact, X is not homeomorphic to βX, yet $C^*(X)$ is isomorphic to $C^*(\beta X)$, showing that $C^*(X)$ does not distinguish among non-compact spaces.

We now characterize the maximal ideals in $C^*(X)$ when X is not necessarily compact. Since the closure (in $\mathbf{R}$) of the range of any function $f \in C^*(X)$, is a compact Hausdorff space, by the Stone Extension Property, f has a unique extension f^β to all of βX.

Theorem 1.4.7 - *The maximal ideals in $C^*(X)$ are precisely the sets $M_p^* = \{f \in C^*(X) : f^\beta(p) = 0\}$, where $p \in \beta X$ and f^β is the extension of f to βX, and are distinct for distinct p.*

Proof - This is immediate since $C^*(X)$ is isomorphic to $C(\beta X)$, and maximal ideals in $C(\beta X)$ have this form, in view of Theorem 1.2.2 and part (4) of Theorem 1.2.5. That M_p^* is distinct for distinct p, follows from Theorem 1.2.2. □

We saw in Theorem 1.2.5 that when X is compact, all the maximal ideals in $C(X)$ are fixed. For X non-compact, there are also free maximal ideals. More precisely, if $p \in X$, then M_p^* is fixed and if $p \in \beta X \sim X$, then M_p^* is free.

In Theorem 1.4.3 we proved that the structure space of $C(X)$ is homeomorphic to βX, and in Theorem 1.4.7 we showed that there is a bijection between the maximal ideals in $C^*(X)$ and the points of βX. A natural question then, is whether or not the structure space of $C^*(X)$ is in fact homeomorphic to βX. The next theorem provides an answer to this question.

Note - We use M^* to denote the structure space of the ring $C^*(X)$, i.e., the collection of all the maximal ideals in $C^*(X)$, endowed with the Hull-Kernel topology.

Theorem 1.4.8 - M^* *is homeomorphic to* βX.

Proof - Since βX is completely regular it follows from Theorem 0.6.1, and corollary 1.4.5 that $\{Z(f^\beta) : f \in C^*(X)\}$ forms a base for closed sets for the topology on βX. If we define $\mathbf{M}^*(f) = \{M \in \mathbf{M}^* : f \in M\}$, then the collection $\{\mathbf{M}^*(f) : f \in C^*(X)\}$ is a base for closed sets for the Hull-Kernel topology on $\mathbf{M}^*$. For any $p \in \beta X$, and any $f \in C^*(X)$, $p \in Z(f^\beta) \iff f^\beta(p) = 0 \iff f \in M_p^*$. Therefore the bijection which sends $p \in \beta X$ to $M_p^* \in \mathbf{M}^*$, also sends basic closed sets in βX to basic closed sets in $\mathbf{M}^*$, and is therefore a homeomorphism. □

The above theorem and Theorem 1.4.3 gives us a partial solution to the problem of how to *recover* the topological space X (more precisely, a homeomorphic copy of X) from the rings C(X) and $C^*(X)$. We refer to this problem as the *recovery problem*. For both rings, constructing the structure space allows us to recover' βX. Therefore, only if X is compact, will we be able to recover it from it's function rings, C(X) and $C^*(X)$.

Next, we characterize the maximal ideals in C(X). We need the following Lemma.

Lemma 1.4.9 - *If* Z_1, *and* Z_2 *are zero-sets on* X, *then* $\mathrm{cl}_{\beta X}Z_1 \cap \mathrm{cl}_{\beta X}Z_2 = \mathrm{cl}_{\beta X}(Z_1 \cap Z_2)$.

Proof - It suffices to show that $\mathrm{cl}_{\beta X}Z_1 \cap \mathrm{cl}_{\beta X}Z_2 \subseteq \mathrm{cl}_{\beta X}(Z_1 \cap Z_2)$. If $p \in \mathrm{cl}_{\beta X}Z_1 \cap \mathrm{cl}_{\beta X}Z_2$, then for any zero-set neighborhood F of p in βX, $p \in \mathrm{cl}_{\beta X}(F \cap Z_1)$, and $p \in \mathrm{cl}_{\beta X}(F \cap Z_2)$. Therefore, these two sets are nonempty. Now, X is C^*-embedded in βX, and disjoint zero-sets are completely separated. If $F \cap Z_1$ and $F \cap Z_2$ were disjoint, they would be completely separated in βX. This would imply that they were contained in disjoint zero-sets, showing that their closures (in βX) are disjoint - a contradiction. Thus $(F \cap Z_1)$ meets $(F \cap Z_2)$ and hence $p \in \mathrm{cl}_{\beta X}(Z_1 \cap Z_2)$. □

Theorem 1.4.10 - (Gelfand and Kolmogoroff) *The maximal ideals in* $C(X)$ *are in one-to-one correspondence with the points of* βX, *and are given by the sets* $M_p = \{f \in C(X) : p \in \mathrm{cl}_{\beta X}\mathbf{Z}(f)\}$, *for*, $p \in \beta X$.

Proof - We first show that $\mathbf{Z}[M_p]$ is a z-ultrafilter on X. Clearly, $\emptyset$ cannot be a member of $\mathbf{Z}[M_p]$. If Z_1 is a zero-set containing $Z \in \mathbf{Z}[M_p]$, then $\mathrm{cl}_{\beta X}Z_1 \supseteq \mathrm{cl}_{\beta X}Z$, so $p \in \mathrm{cl}_{\beta X}Z_1$ and hence $Z_1 \in \mathbf{Z}[M_p]$. Next, if $Z_1, Z_2 \in \mathbf{Z}[M_p]$, then by lemma 1.4.9, $p \in \mathrm{cl}_{\beta X}(Z_1 \cap Z_2)$, showing that $Z_1 \cap Z_2 \in \mathbf{Z}[M_p]$. Therefore, $\mathbf{Z}[M_p]$ is a z-filter. To show it is maximal, let a zero-set Z, (of X) meet every member of $\mathbf{Z}[M_p]$, and suppose $p \notin \mathrm{cl}_{\beta X}Z$. Then there exists a zero-set neighborhood F of p in βX, which misses Z. Now, $F \cap X \in \mathbf{Z}[M_p]$, since for

any neighborhood N of p in βX, $N \cap F$ is a neighborhood of p, so it must meet X. This is a contradiction, since $F \cap X$ misses Z. Consequently, M_p is a maximal ideal.

Next we show that every maximal ideal, M is of the form M_p, for some $p \in \beta X$. By Lemma 1.4.9, disjoint zero-sets in X have disjoint closures in βX. Therefore, the collection $\{cl_{\beta X} Z : Z \in \mathbf{Z}[M]\}$ is a family of closed sets in the compact space βX with F.I.P., and thus has a nonempty intersection. To complete the proof, we show that this intersection is precisely a singleton. Suppose $p, q \in \cap \{cl_{\beta X} Z : Z \in \mathbf{Z}[M]\}$. Then there exist disjoint zero-set neighborhoods Z_p, Z_q of p, q respectively, in βX. Consequently, $Z_p \cap X$, $Z_q \cap X$ are disjoint zero-sets in X, and meet every member of the ultrafilter $\mathbf{Z}[M]$, a contradiction. Hence $p = q$, and thus $M = M_p$. □

For any $p \in X$, and Z, a zero-set on X, $p \in cl_{\beta X} Z \iff p \in Z$. Therefore if $p \in X$, then M_p is fixed and if $p \in \beta X \sim X$, then M_p is free.

By combining Theorem 1.4.3 and Theorem 1.4.8, we obtain the following corollary.

Corollary 1.4.11 - M *is homeomorphic to* $\mathbf{M}^*$

Note - For both spaces **M** and $\mathcal{M}$, X is homeomorphic to the subspace of fixed maximal ideals in the respective structure spaces.

We point out that even for simple spaces, the Stone-Cech compactification can be "large" and often pathological. We give a few illustrations.

Example 1.4.12 - Consider $\beta(0,1]$. The function $f(x) = \sin\frac{1}{x}$ is continuous from $(0,1]$ to $[-1,1]$, and therefore has a continuous extension F to all of $\beta(0,1]$. For each $t \in [-1,1]$, there is a sequence $\{x_n\}$ in $(0,1]$ converging to 0 such that $\{f(x_n)\} = \{t\}$. There is a cluster point u of $\{x_n\}$ in $\beta(0,1] \sim (0,1]$ such that $F(u) = t$. Therefore, $\beta(0,1] \sim (0,1]$ has at least one distinct member for each point in $[-1,1]$.

A few properties of $\beta\mathbf{N}, \beta\mathbf{Q}, \beta\mathbf{R}$ are mentioned below. Details can be found in [43, Chapter 3] and [19, Chapter 6].

The topological spaces $\beta\mathbf{N}$, $\beta\mathbf{Q}$, and $\beta\mathbf{R}$ are all equipotent, and have cardinality $2^{2^{\aleph_0}}$. **N** is open in $\beta\mathbf{N}$, **R** is open in $\beta\mathbf{R}$, but **Q** is not open in $\beta\mathbf{Q}$. Both $\beta\mathbf{Q}$ and $\beta\mathbf{R}$ are continuous images of $\beta\mathbf{N}$, and also $\beta\mathbf{N}$ and $\beta\mathbf{R}$ are continuous images of $\beta\mathbf{Q}$. However, neither $\beta\mathbf{N}$ nor $\beta\mathbf{Q}$, is a continuous image of $\beta\mathbf{R}$. While $\beta\mathbf{R}$ is connected, both $\beta\mathbf{Q}$ and $\beta\mathbf{N}$ are totally disconnected, and hence zero-dimensional.

$\beta\mathbf{N} \sim \mathbf{N}$ contains a copy of $\beta\mathbf{N}$, $\beta\mathbf{Q} \sim \mathbf{Q}$ is dense in $\beta\mathbf{Q}$, and $\beta\mathbf{R} \sim \mathbf{R}$ is the union of two disjoint homeomorphic connected sets. Every countable subset of $\beta\mathbf{N}$ is C*-embedded. $\beta\mathbf{Q} \sim \mathbf{Q}$ is not C*-embedded in $\beta\mathbf{Q}$.

The properties of βX developed in this section can be summarized in the following theorem. While many of the implications mentioned below are available in our discussions so far, we refer to [43, section 1.46] for the remaining.

Theorem 1.4.13 (Compactification Theorem) - *Every completely regular space X has a unique compactification βX which has the following equivalent properties:*

(1) **(Stone Extension Property):** *Every continuous map of X into a compact Hausdorff space can be extended uniquely to βX;*

(2) **(Stone-Cech)**: *X is C^*-embedded in βX;*

(3) Every point of βX is the limit of a unique z-ultrafilter on X;

(4) **(Cech)**: *Disjoint zero-sets in X have disjoint closures in βX;*

(5) For any two zero-sets Z_1, Z_2 on X,

$$cl_{\beta X} Z_1 \cap cl_{\beta X} Z_2 = cl_{\beta X}(Z_1 \cap Z_2)$$

(6) Completely separated sets in X have disjoint closures in βX;

(7) βX is maximal in the partially ordered set of Hausdorff compactifications of X.

1.5 The Realcompactification

Every $f \in C(X)$, can be viewed as a function from X to $\mathbf{R}^* = \mathbf{R} \cup \{\infty\}$, the one-point compactification of $\mathbf{R}$. Since $\mathbf{R}$ is locally compact and Hausdorff, $\mathbf{R}^*$ is a compact Hausdorff space, and hence, f admits a continuous extension f^β to all of βX. For $p \in \beta X$, $f^\beta(p)$ is either ∞, or a real number. In the latter case, f^β (restricted to $X \cup \{p\}$) can be viewed as a continuous extension of f to $X \cup \{p\}$. Therefore, if $f^\beta(p) \in \mathbf{R}$ for every $f \in C(X)$, then X is C-embedded in $X \cup \{p\}$. For each $f \in C(X)$, a *real point* of f is defined to be a member of the set $\upsilon_f(X) = \beta X \sim \{p \in \beta X : f^\beta(p) = \infty\}$. In other words, real points of f are points of βX where f^β is finite. The subspace of βX consisting of points which are real points of each member of $C(X)$ is called the *realcompactification* of X, and is denoted by υX. Thus, $\upsilon X = \bigcap\{\upsilon_f X : f \in C(X)\}$. Since $X \subseteq \upsilon X \subseteq \beta X$, X is dense in υX, and it follows from the definition, that υX is the largest subspace of βX in which X is C-embedded. This means that $C(X)$ is isomorphic to $C(\upsilon X)$, under the map which associates with each $f \in C(X)$, the function $f^\upsilon \in C(\upsilon X)$, where $f^\upsilon = f^\beta|\upsilon X$. We say that X is *realcompact* if $X = \upsilon X$.

Clearly, every compact space is realcompact. However the converse is not true. As we shall see in Example 1.5.6, the non-compact spaces $\mathbf{R}$, $\mathbf{Q}$, and $\mathbf{N}$ are realcompact.

Next, we describe υX in terms of certain maximal ideals in $C(X)$. We recall that all maximal ideals in $C(X)$ are of the form $M_p = \{f \in C(X) : p \in cl_{\beta X}Z(f)\}$, for $p \in \beta X$. We say that a maximal ideal M_p is *real* if the corresponding point $p \in \upsilon X$.

Theorem 1.5.1 - *The set of real maximal ideals in $C(X)$, endowed with the Hull-Kernel topology, is isomorphic to υX.*

Proof - The homeomorphism between the structure space $\mathbf{M}$ of $C(X)$ (restricted to the real maximal ideals) and βX (given in theorem 1.4.3) is the desired homeomorphism. □

We saw in section 1.4, that βX can always be recovered from the function ring $C(X)$. The above theorem then, gives us a better solution to this recovery problem, since we can always recover υX, (a space "smaller" than βX) from $C(X)$. Therefore, the topological space X itself, can be recovered if X is realcompact.

Our description of a real maximal ideal does not lend itself to an application to $C^*(X)$. Therefore, we refer to [19, chapter 5], for a more "algebraic" definition of a real maximal ideal which will apply to $C^*(X)$. We ask then, if υX can be recovered from $C^*(X)$. However, in general this is not possible, since every maximal ideal in $C^*(X)$ is real, [19, Theorem 5.8(a)].

In order to describe the real maximal ideals in $C(X)$, we first establish the following useful lemma.

Lemma 1.5.2 - *The zero-sets of βX are countable intersections of closures (in βX) of zero-sets of X.*

Proof - Let $Z \in \mathbf{Z}[\beta X]$. Then $Z = \mathbf{Z}(f^\beta)$, for some $f \in C^*(X)$. Note that $\mathbf{Z}(f^\beta) = \bigcap_{n=1}^{\infty} \{p \in \beta X : |f^\beta(p)| \leq \frac{1}{n}\}$. Since X is dense in βX, and f^β is continuous, it follows that $\mathbf{Z}(f^\beta) = \bigcap_{n=1}^{\infty} \mathrm{cl}_{\beta X}\{x \in X : |f(x)| \leq \frac{1}{n}\}$. Since each set $\{x \in X : |f(x)| \leq \frac{1}{n}\}$ is a zero-set of X, the result follows. □.

A maximal ideal of $C(X)$ is said to possess the *Countable Intersection Property* (abbreviated, C.I.P), if every countable subfamily of zero-sets of it's members has nonempty intersection.

Theorem 1.5.3 - *The real maximal ideals in $C(X)$ are precisely those with C.I.P.*

Proof - If $p \in \beta X \sim \upsilon X$, then M_p is not a real maximal ideal and we show that it cannot have C.I.P. We can find an $f \in C(X)$, $f > 0$, $f^\beta(p) = \infty$. Define $g = \frac{1}{f}$, so $g \in C(X)$, and $p \in \mathbf{Z}(g^\beta) \subseteq \beta X \sim \upsilon X$. By lemma 1.5.2, $\mathbf{Z}(g^\beta) = \bigcap_{n=1}^{\infty} \mathrm{cl}_{\beta X} Z_n$, where each Z_n is a zero-set on X. Since this intersection is contained in $\beta X \sim \upsilon X$, $\bigcap_{n=1}^{\infty} Z_n = \emptyset$, and M_p does not have C.I.P.

Conversely, if the maximal ideal M_p does not have C.I.P, then there exists a sequence $\{f_n\}$, in M_p, such that $\bigcap_{n=1}^{\infty} \mathbf{Z}(f_n) = \emptyset$. Since $p \in \bigcap_{n=1}^{\infty} \mathrm{cl}_{\beta X}\mathbf{Z}(f_n)$, it follows that $p \in \bigcap_{n=1}^{\infty} \mathbf{Z}(f_n^{\beta})$. Let $\bigcap_{n=1}^{\infty} \mathbf{Z}(f_n^{\beta}) = \mathbf{Z}(h)$, where $h \in C(\beta X)$. Since $\mathbf{Z}(h|_X) = \emptyset$, $g = (\frac{1}{h})|_X \in C(X)$, and its continuous extension g^{β} is such that, $g^{\beta}(p) = \infty$, proving that $p \in \beta X \sim \upsilon X$. □

Theorem 1.5.4 - *The topological space X is realcompact if and only if every real maximal ideal is fixed.*

Proof Let $X = \upsilon X$. Then all real maximal ideals are of the form $M_p = \{ f \in C(X) : p \in \mathrm{cl}_{\beta X} \mathbf{Z}(f)\}$, for $p \in X$. Now for $Z \in \mathbf{Z}[X]$, $p \in \mathrm{cl}_{\beta X} Z$ if and only if $p \in Z$. This follows, since if $p \in \mathrm{cl}_{\beta X}Z \sim Z$, then there exists a neighborhood, N (in βX), such that $N \cap X$ is a neighborhood (in X) of p disjoint from Z (by regularity of X). This contradicts the fact that $p \in \mathrm{cl}_{\beta X} Z$. Therefore, $M_p = \{ f \in C(X) : p \in \mathbf{Z}(f) \}$, and $\bigcap_{f \in M_p} \mathbf{Z}(f) \neq \emptyset$. Conversely, assume all real maximal ideals are fixed. Let $p \in \upsilon X$, so M_p is a real maximal ideal and so $\bigcap \mathbf{Z}[M_p] \neq \emptyset$. If $q \in \bigcap \mathbf{Z}[M_p]$, and $q \neq p$, then there exists a zero-set neighborhood Z of p in βX not containing q. But, $p \in \mathrm{cl}_{\beta X}(Z \cap X) = \mathrm{cl}_{\beta X}Z \cap \mathrm{cl}_{\beta X}X = Z$, so $Z \cap X \in \mathbf{Z}[M_p]$, a contradiction. Therefore, $\{p\} = \bigcap \mathbf{Z}[M_p]$, $\subseteq X$, so $p \in X$, and $X = \upsilon X$, i.e., X is realcompact. □

Theorem 1.5.5 - *Every Lindelof topological space X is realcompact.*

Proof - Since every family of closed sets with C.I.P., has nonempty intersection, the proof follows from Theorems 1.5.3, and 1.5.4. □

Example 1.5.6 - **R, Q,** and **N** are realcompact.

All these spaces are Lindelof. □

The next result is a Banach-Stone theorem for realcompact spaces.

Theorem 1.5.7 - *If the topological spaces X and Y are realcompact, then X is homeomorphic to Y if and only if C(X) is isomorphic to C(Y).*

Proof - Since being a real maximal ideal is an algebraic invariant, [19, Chapter 5], if we assume that C(X) is isomorphic to C(Y), we obtain that υX is homeomorphic to υY, and the result follows, since X and Y are realcompact. □

Comparing the above theorem to Corollary 1.4.6, it is clear that, just as $C^*(X)$ distinguishes among compact spaces, C(X) distinguishes among realcompact spaces.

Finally, it is worth pointing out a purely topological connection between compactness and realcompactness. Recall, that if $C^*(X) = C(X)$, then X is pseudocompact.

Theorem 1.5.8 - *A topological space is compact if and only if it is realcompact and pseudocompact.*

Proof - One direction is clear. Assume that X is realcompact and pseudocompact. Then, every point in βX is real, since there are no unbounded functions in C(X). Therefore, every maximal ideal in C(X) is real, and by Theorem 1.5.4, is fixed. The result then follows from Theorem I.2.5. □

In the next section, we outline an embedding for realcompact spaces that guarantees a rich supply of such spaces. The following is an example of a non-realcompact space.

Example 1.5.9 - Let Ω be the first uncountable ordinal, and consider the ordinal space $[0, \Omega)$. Since it is not closed in the Hausdorff space $[0, \Omega]$, it is not compact. In view of the above theorem, it cannot be realcompact if it is pseudocompact. In section 0.1, we saw that all countably compact spaces are pseudocompact, so it suffices to show that $[0, \Omega)$ is countably compact.

If $\{x_n\}$ is a sequence in $[0, \Omega)$, we can rearrange it to be increasing, and therefore have it's supremum, x as a limit. Clearly, x is a cluster point of the original sequence, $\{x_n\}$. That $[0, \Omega)$ is countably compact, now follows from the fact that every sequence has a cluster point. □

1.6 Alternative Descriptions of βX and υX

In this section we outline some alternate methods for the construction of βX and υX.

(a) Via z – ultrafilters on X -

In section 1.4, βX was constructed using the maximal ideals in $C(X)$. In the light of Theorem 1.1.3, we would expect that a similar construction could be carried out using the z – ultrafilters on X. In fact, βX can be defined as the collection of all z – ultrafilters on X, with a topology defined by the basic open sets $\mathbf{B} = \{\{F \in \beta X : Z \notin F\} : Z \in \mathbf{Z}[X]\}$. To embed X into a dense subspace of βX, each point $x \in X$ is associated with the z-ultrafilter $e(x) = \{Z \in \mathbf{Z}[X] : x \in Z\}$.

Similar to the definition of real maximal ideal given in section 1.5, we define a z-ultrafilter $F \in \beta X$ to be *real* if it has C.I.P., i.e., every countable family of zero-sets in F has nonempty intersection. Clearly, then, by Theorem 1.5.3, $\mathbf{Z}^{\leftarrow}[F]$ is a real maximal ideal. We then define υX to be the subspace of βX consisting of all the real z – ultrafilters on X.

This is the approach adopted in [10].

(b) Via embedding in products of real lines -

Historically, the Stone-Čech compactification of X was first constructed by embedding X into a product of a suitable number of copies of the space [0,1]. We conclude immediately, that X is compact if and only if it is homeomorphic to a closed subspace of a product of a number of copies of the unit interval [0,1]. We now extend this idea, and show that υX can be constructed from an analogous embedding.

Consider the set $\mathbf{R}^{C^*(X)}$, endowed with the product topology and a map $\sigma_* : X \rightarrow \mathbf{R}^{C^*(X)}$, defined as follows : for each $x \in X$, $\sigma_*(x)(f) = f(x)$, for every $f \in C^*(X)$. In an analogous manner, let $\mathbf{R}^{C(X)}$ be given the product topology and define a map $\sigma : X \rightarrow \mathbf{R}^{C(X)}$, where for each $x \in X$, $\sigma(x)(f) = f(x)$ for every $f \in C(X)$. It turns out that σ_* (respectively σ) is a homeomorphism of X into $\mathbf{R}^{C^*(X)}$ (respectively $\mathbf{R}^{C(X)}$), and $\sigma_*[X]$ (respectively $\sigma[X]$) is C^*-embedded (respectively C-embedded) in $\mathbf{R}^{C^*(X)}$ (respectively $\mathbf{R}^{C(X)}$). We then define βX to be $cl\sigma_*[X]$ and υX to be $cl\sigma[X]$, where in each case, the closure is taken with respect to the appropriate product topology. We refer the reader to [19, chapter 11] for details.

From this construction of υX, it follows that X is realcompact if and only if it is homeomorphic to a closed subspace of a product of a number of copies of the real line. As a consequence, closed subspaces, intersections, and products of realcompact spaces are realcompact. This then, greatly increases the number of examples of realcompact spaces we have available to us.

(c) Via Uniformities -

Since X is completely regular, the topology on X coincides with the topology generated by $C^*(X)$ (Theorem 0.6.5). Therefore, this topology can be induced by the weak uniformity generated by $C^*(X)$, which we refer to as U^*, [45, section 11.4]. To see how such a uniformity is formed, we first look at the following subset of $X \times X$. For $f \in C^*(X)$, and $\epsilon > 0$, let $U_{f,\epsilon} = \{(x,y) \in X \times X : |f(x) - f(y)| < \epsilon\}$. The family, $\{U_{f,\epsilon} : f \in C^*(X), \epsilon > 0\}$ is a subbase for $\mathbf{U}^*$. This means that all finite intersections of members of this family form a base for $\mathbf{U}^*$, and so $\mathbf{U}^*$ consists of all those subsets of $X \times X$ which contain a member of this base. We now define βX to be the completion of the uniform space $(X,\mathbf{U}^*)$, [45, section 11.5, example 1].

Since the topology on X also coincides with the weak topology generated by $C(X)$, (Theorem 0.6.5), it can be induced by the weak uniformity generated by $C(X)$, which we refer to as $\mathbf{U}$. Then, υX is defined to be the completion of the uniform space $(X, \mathbf{U})$. This enables us to conclude that the topological space X is compact (respectively, realcompact) if and only if the uniform space $(X,\mathbf{U}^*)$, (respectively, $(X,\mathbf{U})$) is complete.

(d) Via measure theory -

We close this section with a final construction involving measure theory. Define $\mathbf{A}$ to be the algebra of sets generated by the family of zero-sets, $\mathbf{Z}[X]$, and m to be a measure defined on $\mathbf{A}$, for which the following regularity condition holds : for each $A \in \mathbf{A}$, $m(A) = \sup\{m(Z) : Z \in \mathbf{Z}[X], Z \subseteq A\}$. Note that m is determined by its values on $\mathbf{Z}[X]$. If the range of m is $\{0,1\}$ then it is called a *0-1 measure*. We say that m is *finitely additive* if $m(\cup E_n) = \sum m(E_n)$, whenever $\{E_n\}$ is a finite collection of disjoint subsets of $\mathbf{A}$. In a similar fashion m is *countably additive* if $m(\cup E_n) = \sum m(E_n)$, whenever $\{E_n\}$ is a countable collection of pairwise disjoint subsets of **A** whose union is in **A**. We use the symbol M_0 to denote the collection of all finitely additive 0-1 measures on **A**, which have the above mentioned regularity condition.

A topology can be specified for M_0, called the *vague topology*. A typical subbasic neighborhood of $m_0 \in M_0$, is given by sets of the form : $\{m \in M_0 : |\int f\, dm - \int f\, dm_0| < \epsilon\}$, where $f \in C^*(X)$, and $\epsilon > 0$.

For each $x \in X$, the 0-1 measure, m_x, *concentrated at x*, is defined by : $m_x(Z) = 1$, if $x \in Z$, and $m_x(Z) = 0$, if $x \notin Z$, $(Z \in \mathbf{A})$. Clearly, for each $x \in X$, m_x is countably additive, and belongs to M_0. If M_0 is given the vague topology, then the map $\psi : X \rightarrow M_0$, defined by $\psi(x) = m_x$, for each $x \in X$, is a homeomorphism of X into M_0.

The Stone-Cech compactification of X, is then defined to be the space M_0, under the vague topology, and the realcompactification of X is defined as the subspace of M_0, consisting of all the countably additive measures of M_0. We refer the reader to [7, section 1.7], for further details.

CHAPTER 2

The Banach Space C(X)

This chapter deals with the Banach space structure of C(X). We take a compact Hausdorff space X, and equip the linear space C(X) with the uniform norm. Several interesting properties of this Banach space are then studied. Some theorems which relate the Banach space structure of C(X) with the topology of X are given. For example, C(X) is separable if and only if X is metrizable. We prove the Riesz Representation Theorem, which characterizes the dual of this Banach space. The question of when a given Banach space is isometric to the Banach space C(X), for some compact Hausdorff space X, is then answered. We obtain a version of the Banach-Stone Theorem which asserts that the Banach spaces C(X) and C(Y) are isometric when and only when the spaces X and Y are homeomorphic. Finally, for a compact Hausdorff extremally disconnected topological space X, C(X) characterizes the class of Banach spaces which enjoy the Hahn-Banach extension property.

2.1 The Banach Space C(X)

Throughout this chapter, we assume that X is a compact Hausdorff topological space. The ring C(X), which is also an **R**-vector space, becomes a real normed linear space under the norm $\| \cdot \|$, defined by $\| f \| = \sup\{|f(x)| : x \in X\}$ for $f \in C(X)$. This norm is referred to as the *uniform norm*, or *supremum norm*. It is easy to see that C(X) is complete under this norm. For the remainder of this chapter, C(X) will refer to this Banach space.

Note - If $c \in \mathbf{R}$, and $f \in C(X)$, the function $c \cdot f$ will usually be written, simply as cf.

For $f, g \in C(X)$, the set $[f;g] = \{cf + (1-c)g : 0 \leq c \leq 1\}$ is called the *line segment* joining f and g. We recall that a subset A of C(X) is *convex*, if whenever $f, g \in A$, $[f;g] \subseteq A$. The *convex hull* of a subset B of C(X), written conv(B), is the smallest convex set containing B, and consists of all finite linear combinations $\sum_{i=1}^{n} c_i f_i$, $f_i \in B$, $c_i \geq 0$ and $\sum_{i=1}^{n} c_i = 1$.

An *extreme point* of the convex subset A is an element $f \in C(X)$ that is not an internal point of any line segment whose endpoints belong to A; in other words, f is an extreme point of A if and only if whenever $f = cg + (1-c)h$, for $g, h \in A$ and $0 < c < 1$, we have $f = g = h$. It suffices to take $c = \frac{1}{2}$ in the above definition. For, if $f = cg + (1-c)h$, with $0 < c < \frac{1}{2}$, then $f = \frac{1}{2}h + \frac{1}{2}(2f - h)$ and $2f - h \in A$. If $\frac{1}{2} < c < 1$, then $f = \frac{1}{2}g + \frac{1}{2}(2f - g)$

and $2f - g \in A$.

In $\mathbf{R}^2$, with the Euclidean topology, the region consisting of the boundary and interior of a triangle is a convex set whose only extreme points are the three vertices. In contrast, for a closed disc in this space, every point on the circumference is an extreme point. It is easy to see that an open convex set has no extreme points. However, not every closed bounded convex set need posses an extreme point, as the following example illustrates.

Example 2.1.1 - As usual, c_0 denotes the Banach space (under the supremum norm) of all real sequences which converge to 0. Let A be the closed unit sphere in c_0. If $x = \{x_n\} \in A$, we first consider the case where there exists $r \in \mathbf{N}$ such that $x_n = 0$ for $n > r$. Set

$$a_n = x_n,\ (n \leq r),\quad \frac{1}{n}\ (n > r)$$
$$b_n = x_n,\ (n \leq r),\quad -\frac{1}{n}\ (n > r)$$

Then, $x = \frac{a+b}{2}$, where $a = \{a_n\}$, $b = \{b_n\}$, with $a \neq b$, and $a, b \in A$. Thus, x is not an extreme point of A. On the other hand, if no such r exists, then choose k such that $|x_n| \leq \frac{1}{2}$ for $n > k$. Define

$$a_n = x_n,\ (n \leq k),\quad 2x_n\ (n \text{ even},\ n > k),\quad 0\ (n \text{ odd},\ n > k)$$
$$b_n = x_n,\ (n \leq k),\quad 0\ (n \text{ even},\ n > k)\quad 2x_n\ (n \text{ odd},\ n > k)$$

Here again, $x = \frac{a+b}{2}$, where $a = \{a_n\}$ and $b = \{b_n\}$ $a \neq b$, $a, b \in A$, so x is not an extreme point of A. Thus, A has no extreme points. □

Let E be a linear space, and K, a convex subset of E. Then, a nonempty subset A of K, is a *face* of K, if it is convex, and each line segment contained in K that has an internal point in A is contained in A, i.e., if $x, y \in K$ and $cx + (1 - c)y \in A$, for a scalar c, such that $0 < c < 1$, then $x, y \in A$. We recall the following elementary facts about faces of a convex set in a linear space [27, section 15].

If A is a face of a convex set B, and B is a face of a convex set C, then, A is a face of C. Extreme points of a convex set are precisely those faces which are singleton sets. If f is a linear map between vector spaces, carrying a convex set K into a convex set M, and if L is a face of M, then $f^{-1}(L) \cap K$ is either empty or a face of K. Also, the set of points at which a continuous linear functional assumes a maximum on a convex set K is a face of K, and hence contains an extreme point.

The classical theorem on the existence of extreme points for a compact convex set is the following, [36, Theorem 26, page 179].

Theorem 2.1.2 (Krein-Millman) - *Every nonempty, compact, convex subset of a Banach space is the closed convex hull of the set of all its extreme points.*

The set $B = \{f \in C(X): \| f \| \leq 1\}$ is the closed unit sphere in C(X). Note that B is always convex. The following theorem characterizes all the extreme points of B.

Theorem 2.1.3 - *The extreme points of the closed unit sphere B in $C(X)$ are precisely the functions $f \in C(X)$ such that $|f(x)| = 1$ for all $x \in X$.*

Proof - If f satisfies the condition, then f is an extreme point of B. For, if $f = \frac{1}{2}g + \frac{1}{2}h$, where g and h are in B, then $|g(x) + h(x)| = 2$, for every $x \in X$. This is possible only if g(x) and h(x) are simultaneously +1 or -1 at each point x, and hence $f = g = h$.

On the other hand, if $f \in B$ and $x_0 \in X$ is such that $|f(x_0)| < 1$, we may write $f = \frac{1}{2}\, 1 + \frac{1}{2}\, h$, where $|h(x)| \leq 1$ for every $x \in X$. Clearly, $h \in B$ and $1(x_0) \neq h(x_0)$, so f is not an extreme point. □

This result tells us that the extreme points on the closed unit sphere of C(X), are precisely the square roots of **1**. Observing that a compact space can have, at most, a finite number of components, Theorem 0.2.1 leads to the following interesting result.

Theorem 2.1.4 - *For $m < \aleph_0$, the compact space X has m components if and only if the closed unit sphere in $C(X)$ has exactly 2^m extreme points.*

As a consequence, X is connected if and only if the closed unit sphere B in C(X) has exactly two extreme points, namely the functions **1** and **- 1**. We also point out that if X is connected, then B cannot be compact. For, if x,y $\in X$, $x \neq y$, by complete regularity of X, there exists an Urysohn function f for $(\{x\}, \{y\})$, and $\| f \| \leq 1$. It is easy to check that f does not belong to the closure of $\mathrm{conv}(\{-1, 1\})$. So by the Krein-Millman theorem, B cannot be compact.

Next, we provide another result which relates the Banach space structure of C(X) with the topology of X. Explicitly, we give conditions on X under which C(X) is a separable Banach space. We need a few preliminary results from general topology.

Lemma 2.1.5 - *If F is a compact subset of X, and $x \in X \setminus F$, then there exist disjoint open sets U, V, containing x and F, respectively.*

Lemma 2.1.6 - *For every open set U of X and $x \in U$, x has a compact neighborhood contained in U.*

Lemma 2.1.7 **(Urysohn's lemma for compact spaces)** - *For $K \subseteq U$, where K is compact and U open, there exists $f \in C^*(X)$, $0 \leq f \leq 1$, such that $f[K] = \{1\}$ and f vanishes outside a compact subset of U.*

We refer to [18, Propositions 4.23 and 4.30, and Lemma 4.32] for the straightforward proofs of Lemmas 2.1.5, 2.1.6, and 2.1.7.

For $f \in C(X)$, we define the *support* of f, written *supp(f)*, to be the closure of the subset $\{x \in X : f(x) \neq 0\}$ of X. Since X is compact, supp(f) is always a compact set. For an open subset U of X and $f \in C(X)$, we say that f is *subordinate* to U, if $0 \leq f \leq 1$, and $\mathrm{supp}(f) \subseteq U$. In this terminology, the function f in Lemma 2.1.7 is subordinate to the open set U.

Lemma 2.1.8 - *If* $\{U_j\}$, $(j = 1,2, \ldots, n)$ *is a finite open cover of a compact subset* K *of* X, *then there exists* $h_j \in C(X)$, $(j = 1, 2, \ldots, n)$ *such that* h_j *is subordinate to* U_j, *and* $\sum_1^n h_j = 1$ *on* K.

Proof - By lemma 2.1.6, each $x \in K$ has a compact neighborhood N_x such that $N_x \subseteq U_j$ for some j, $(1 \leq j \leq n)$. If $\mathrm{int}(N_x)$ denotes the interior of N_x, then $\{\mathrm{int}(N_x)\}$, as x ranges over K, is an open cover of K, and so reduces to a finite subcover, say, $\{\mathrm{int}(N_{x_k})\}$, $k = 1,2, \ldots, n$. Define $F_j = \bigcup_{N_{x_k} \subseteq U_j} N_{x_k}$. Note that F_j is compact. By Lemma 2.1.7, for each $j = 1, 2, \ldots, n$, there exist $g_j \in C(X)$, such that $g_j = 1$ on F_j, and $\mathrm{supp}(g_j) \subseteq U_j$. Therefore, $\sum_{k=1}^{n} g_k \geq 1$ on K, and so again by lemma 2.1.7, there exists $f \in C(X)$ with $f[K] = 1$ and $\mathrm{supp}(f) \subseteq \{ x \in X : \sum_{k=1}^{n} g_k(x) > 0 \}$. Defining $g_{n+1} = 1 - f$, it follows that $\sum_{k=1}^{n+1} g_k > 0$ everywhere on X. Let $h_j = \frac{g_j}{\sum_{k=1}^{n+1} g_k}$, $(j = 1, 2, \ldots, n)$. Then, $\mathrm{supp}(h_j) = \mathrm{supp}(g_j) \subseteq U_j$ and $\sum_{j=1}^{n} h_j = 1$ on K. □

Note : This collection $\{h_j\}$ is called a *partition of unity* on K subordinate to the collection $\{U_j\}$.

A family A of functions in C(X) is said to be *separating* if for $x,y \in X$, $x \neq y$, there exists $f \in A$ such that $f(x) \neq f(y)$.

Theorem 2.1.9 - *The Banach space C(X) is separable if and only if X is metrizable.*

Proof. - Let $X = (X, \tau)$ be such that C(X) is separable and let D be a countable dense subset of C(X). If τ' denotes the weakest topology on X which makes each function in D continuous, then $\tau' \subseteq \tau$. Clearly, τ' is semimetrizable, since D is countable and **R** is metrizable. To show that τ' is metrizable, we need only check that D is separating. If $x,y \in X$ and $f(x) = f(y)$ for every $f \in D$, then $f(x) = f(y)$ for every $f \in C(X)$ since D is dense in C(X). However, X is completely regular, so $x = y$. This proves that D is separating. Finally, since τ is compact, and τ' is Hausdorff (since D is separating), the identity map from (X, τ) to (X, τ') is a homeomorphism. Therefore, $\tau = \tau'$, and so X is metrizable.

Assume X is metrizable with metric d. Let $f \in C(X)$, and $\epsilon > 0$ be given. Since X is compact, f is uniformly continuous and therefore there exists, $n \in \mathbf{N}$ such that for every $x,y \in X$ with $d(x,y) < \frac{1}{n}$, $|f(x) - f(y)| < \epsilon$. Since the collection $\{U_n\}$ of open spheres of diameter $\frac{1}{n}$ is an open covering of

X, there exists a finite subcollection $\{U_{n_i}\}$, $(i = 1$ to $m)$, covering X. By Lemma 2.1.8, we can always find a finite collection of members of $C(X)$; $\{f_{n_i}\}$, $i = 1, 2, \ldots, m$, with range contained in $[0,1]$, such that f_{n_i} is subornate to U_{n_i} and $\sum_{i=1}^{m} f_{n_i} = 1$. For each $i = 1, 2, \ldots, m$, choose $x_{n_i} \in U_{n_i}$. Clearly, $|f(x) - f(x_{n_i})| < \epsilon$ for $x \in U_{n_i}$ and for each i. Therefore,

$$\begin{aligned} |f(x) - \sum_{i=1}^{m} f(x_{n_i}) \cdot f_{n_i}(x)| &= |\sum_{i=1}^{m} f(x) \cdot f_{n_i}(x) - \sum_{i=1}^{m} f(x_{n_i}) \cdot f_{n_i}(x)| \\ &\leq \sum_{i=1}^{m} |f(x) - f(x_{n_i})| \cdot f_{n_i}(x) \\ &\leq \epsilon \sum_{i=1}^{m} f_{n_i}(x) = \epsilon, \end{aligned}$$

where the last inequality follows since f_{n_i} is subordinate to U_{n_i}, so $f_{n_i} \geq 0$ and $f_{n_i}(x) = 0$ for $x \notin U_{n_i}$. Since f is arbitrary, the set of finite linear combinations (with real coefficients) of the functions $\{f_{n_i}\}$, $n \in \mathbf{N}$, is dense in $C(X)$. However, since the rationals are dense in $\mathbf{R}$, the (countable) collection of finite linear combinations of these functions with rational coefficients is dense in $C(X)$. Therefore, $C(X)$ is separable. □

2.2 The Riesz Representation Theorem

In this section, we describe the continuous linear functionals on the Banach space $C(X)$. This normed linear space is called the *dual* of $C(X)$, and we use the symbol, $[C(X)]^*$ to represent it. We start by giving some preliminaries from measure theory. We shall use [18] for our reference.

A σ-algebra on X is a nonempty family of subsets of X that is closed under countable unions and complements. The *Borel sets* in X are the members of the smallest σ-algebra which contains all the open (equivalently, closed) subsets of X. A measure defined on the σ-algebra B_X of Borel sets is called a *Borel measure.* If μ is a Borel measure and $E \in B_X$, then μ is *outer regular* on E if $\mu(E) = \inf \{\mu(U) : U \supseteq E, U \text{ open}\}$, and *inner regular* on E if $\mu(E) = \sup \{\mu(K) : K \subseteq E, K \text{ is compact}\}$. A *Radon measure* is a Borel measure that is finite on all compact sets, outer regular on all Borel sets and inner regular on all open sets. Note that the word "compact" can be replaced by "closed" in the above definitions, since X is compact and Hausdorff.

A linear functional I on C(X) is said to be *positive* if $I(f) \geq 0$, whenever $f \geq 0$. Clearly this implies that if $f \geq g$, if and only if $I(f) \geq I(g)$. It is worth pointing out that the positivity implies a rather strong continuity property, namely, for each compact $K \subseteq X$, there exists a constant M_K such that $|I(f)| \leq M_K \| f \|$, for $f \in C(X)$ with $\text{supp}(f) \subseteq K$, [18, Proposition 7.1]

Let I be a fixed positive linear functional on C(X). For each open set U of X, define $\mu(U) = \sup \{I(f) : f \in C(X), f \text{ is subordinate to } U\}$. Clearly, 0 is subordinate to each open set, so the set function μ is well-defined. If $\{U_i : i \in N\}$ is a sequence of open subsets of X whose union is U, then, $\mu(U) \leq \sum_{i=1}^{\infty} \mu(U_i)$. To see this, let $f \in C(X)$, f subordinate to U. If supp(f)

$= K$, by compactness, a finite union $\bigcup_{i=1}^{n} U_i$ covers K. By Lemma 2.1.8, there exist $g_1, g_2, \cdots, g_n$, such that for each $i = 1, 2, \ldots, n$, g_i is subordinate to U_i and $\sum_{i=1}^{n} g_i = 1$ on K. Then, $f = \sum_{i=1}^{n} f \cdot g_i$, and $f \cdot g_i$ is subordinate to U_i. Now, $I(f) = \sum_{i=1}^{n} I(f \cdot g_i) \leq \sum_{i=1}^{n} \mu(U_i) \leq \sum_{i=1}^{\infty} \mu(U_i)$. It now follows that $\mu(U) \leq \sum_{i=1}^{\infty} \mu(U_i)$.

For an arbitrary subset E of X, we define $\mu^*(E) = \inf\{\mu(U) : U \supseteq E, U \text{ open}\}$. Note that if U and V are open subsets of X such that $U \subseteq V$, then $\mu(U) \leq \mu(V)$, and it follows that μ and μ^* coincide on open sets. From the previous paragraph, it is clear that for any $E \subseteq X$,

$$\mu^*(E) = \inf\left\{\sum_{i=1}^{\infty} \mu(U_i) : U_i \text{ open, } E \subseteq \bigcup_{i=1}^{\infty} U_i\right\}. \qquad (*)$$

Theorem 2.2.1 - *The set function μ^* defines an outer measure on X.*

Proof - Since **0** is subordinate to the empty set $\emptyset$, it is straightforward to see that $\mu^*(\emptyset) = 0$. By (*) above, if $E \subseteq F$, $\mu^*(E) \leq \mu^*(F)$. To complete the proof, let $E = \bigcup_{i=1}^{\infty} E_i$ be an arbitrary subset of X. Given $\epsilon > 0$, for each i, there exists a sequence $\{U_{i,k}\}$ $(k = 1, 2, \ldots)$ of open sets such that $E_i \subseteq \bigcup_{k=1}^{\infty} U_{i,k}$ and by (*), $\sum_{k=1}^{\infty} \mu(U_{i,k}) \leq \mu^*(E_i) + \frac{\epsilon}{2^i}$. Since $E \subseteq \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{\infty} U_{i,k}$, and since $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \mu(U_{i,k}) \leq \sum_{i=1}^{\infty} \mu^*(E_i) + \epsilon$, we have $\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i) + \epsilon$.

However ϵ is arbitrary, so $\mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$. □

Theorem 2.2.2 - *Every open subset of X is μ^*-measurable.*

Proof - Let U be an open subset of X. We must show that for each $E \subseteq X$, $\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \sim U)$. We assume that $\mu^*(E) < \infty$; otherwise the result is trivial. We consider the following two cases.

Let E be open in X. Since $E \cap U$ is open, for a given $\epsilon > 0$, there exists $f \in C(X)$ such that f is subordinate to $E \cap U$ and $I(f) > \mu(E \cap U) - \epsilon$. Also, $V = E \sim \{supp(f)\}$ is open and therefore there exists $g \in C(X)$ subordinate to V such that $I(g) > \mu(V) - \epsilon$. However, $f + g$ is subordinate to E, so $\mu^*(E) = \mu(E) \geq I(f) + I(g) \geq \mu(E \cap U) + \mu(V) - 2\epsilon \geq \mu^*(E \cap U) + \mu^*(E \sim U) - 2\epsilon$.

Now, if E is arbitrary, then, for $\epsilon > 0$, there exists an open set $W \supseteq E$ such that $\mu^*(E) + \epsilon > \mu(W)$. Therefore $\mu^*(E) + \epsilon > \mu^*(U \cap W) + \mu^*(W \sim U) \geq \mu^*(E \cap U) + \mu^*(E \sim U)$.

In both cases, since ϵ is arbitrary, we have $\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \sim U)$. □

The outer measure μ^* restricted to the Borel sets is clearly a Borel measure, and will be denoted by $\bar{\mu}$. By the very definition, it is outer regular on all μ^*-measurable sets, hence on all Borel sets. We will show that it is also inner regular on all open sets.

Recall that the *characteristic function*, χ_A of a subset $A \subseteq X$, is the function that assumes the value 1 on A and 0 outside A.

Lemma 2.2.3 - *For each compact subset* K *of* X,

$$\bar{\mu}(K) = \inf\left\{ I(f) : f \in C(X),\ f \geq \chi_K \right\} \qquad (**)$$

Proof - Let $f \in C(X)$, $f \geq \chi_K$. For $\epsilon > 0$, define the open set $U = \{x \in X : f(x) > 1 - \epsilon\}$, and choose g subordinate to U. On U, $f(x) > 1 - \epsilon$, so $g(x) \leq 1 \leq \frac{f(x)}{1-\epsilon}$. Outside U, $g = 0$, and since f is never negative, it follows that $g \leq \frac{f}{1-\epsilon}$. This implies that $I(g) \leq \frac{I(f)}{1-\epsilon}$. Therefore, $\bar{\mu}(K) \leq \bar{\mu}(U) \leq \frac{I(f)}{1-\epsilon}$. Since ϵ is arbitrary, $\bar{\mu}(K) \leq I(f)$. For any open set $V \subseteq K$, there exists $f \in C(X)$ such that f is subordinate to V, and $f \geq \chi_K$ (Lemma 2.1.7). Therefore, $I(f) \leq \bar{\mu}(V)$, and since $\bar{\mu}(K)$ is the infimum of $\mu(V)$ for such V, the result follows. □

Theorem 2.2.4 - *The Borel measure* $\bar{\mu}$ *is inner regular on all open sets.*

Proof - Let U be an open subset of X. Note that $\bar{\mu}(U) = \mu(U) = \sup\{I(f): f$ is subordinate to $U\}$. If $K = \text{supp}(f)$ and $g \in C(X)$, $g \geq \chi_K$, then $g \geq f$, and so $I(g) \geq I(f)$. By (**), it follows that $\bar{\mu}(K) \geq I(f)$. Hence, $\bar{\mu}(U) = \sup\{\bar{\mu}(K) : K \subseteq U, K \text{ compact}\}$. □

In order to establish that $\bar{\mu}$ is a Radon measure, we need only show that it is finite for all compact sets. This follows from (**), since $1 \geq \chi_K$ for any compact set K, so $I(1) \geq \mu(K)$.

Theorem 2.2.5 - *For every positive linear functional I on $C(X)$, there corresponds a unique Radon measure $\bar{\mu}$ on X such that $I(f) = \int f\, d\bar{\mu}$, for every $f \in C(X)$.*

Proof - We have already seen that we can associate a Radon measure $\bar{\mu}$ (constructed from the set function μ defined earlier), with the given positive linear functional I on C(X). It suffices to check the relation $I(f) = \int f\, d\bar{\mu}$ for those functions f whose range is a subset of [0, 1]. For, if the range of f is contained in [a, b], $a < b$, then there exists $g \in C(X)$, with its range contained in [0, 1], and $f = \frac{1}{b-a} g - \frac{a}{b-a}$.

For $n \in \mathbf{N}$, and $1 \leq j \leq n$, define $K_j = \{x \in X: f(x) \geq \frac{j}{n}\}$, $K_0 =$ supp(f), and $f_j = \min\{\max\{f(x) - \frac{j-1}{n}, 0\}, \frac{1}{n}\}$, $(x \in X)$. Note that $f_j \in C(X)$ for $j = 1, 2, \dots, n$. We claim that $\frac{\chi_{K_j}}{n} \leq f_j \leq \frac{\chi_{K_{j-1}}}{n}$, $j = 1, 2, \dots, n$. Since $K_j \subseteq K_{j-1}$, we consider the three possible cases. If $x \notin K_{j-1}$, then $f_j(x) = 0$ since $f(x) < \frac{j-1}{n}$. If $x \in K_{j-1} \sim K_j$, then $\frac{j-1}{n} \leq f(x) \leq \frac{j}{n}$, so $f_j(x) < \frac{1}{n}$. Finally, if $x \in K_j$, then $f(x) \geq \frac{j-1}{n} > \frac{1}{n}$, so $f_j(x) = \frac{1}{n}$. Integrating the inequality in the above claim, we obtain

$$\frac{\bar{\mu}(K_j)}{n} \leq \int f_j d\bar{\mu} \leq \frac{\bar{\mu}(K_{j-1})}{n}$$

If U is any open set containing K_{j-1}, then nf_j is subordinate to U, and so $I(f_j) \leq \frac{1}{n}\,\bar{\mu}(U)$. Also, since $n f_j \geq \chi_{K_j}$, by (**), it follows that $\frac{\bar{\mu}(K_j)}{n} \leq I(f_j)$.

We next show that $f(x) = \sum_{i=1}^{n} f_j(x)$, $(x \in X)$. If $x \notin K_0$, then $f_j(x) = 0$ for $j = 1, 2, \ldots, n$. Next, if $x \in K_{j-1} \sim K_j$, then $f(x) \in \left[\frac{j-1}{n}, \frac{j}{n}\right]$, whence $f_i(x) = 0$ for $i > j$, $f_i(x) = \frac{1}{n}$ for $i < j$, and $f_j(x) = f(x) - \frac{j-1}{n}$. If $x \in K_n$, $f_j(x) = \frac{1}{n}$ for $j = 1, 2, \ldots, n$.

By the above result, and the two previous inequalities, we obtain

$$\frac{1}{n}\sum_{j=1}^{n} \bar{\mu}(K_j) \leq \int f\, d\bar{\mu} \leq \frac{1}{n}\sum_{j=0}^{n-1} \bar{\mu}(K_j) \tag{1}$$

and

$$\frac{1}{n}\sum_{j=1}^{n} \bar{\mu}(K_j) \leq I(f) \leq \frac{1}{n}\sum_{j=0}^{n-1} \bar{\mu}(K_j) \tag{2}$$

From (1) and (2), it follows that

$$\left| I(f) - \int f d\bar{\mu} \right| \leq \frac{1}{n}\left[\bar{\mu}(K_0) - \bar{\mu}(K_n)\right] \leq \frac{1}{n}\bar{\mu}(K_0).$$

Since $\overline{\mu}$ is finite on compact sets, and n is arbitrary, we have $I(f) = \int f d\overline{\mu}$.

To prove uniqueness, let ν be a Radon measure such that $I(f) = \int f d\nu$, for every $f \in C(X)$. We show that for any open subset U of X, $\nu(U) = \sup\{I(f) : f \in C(X), f \text{ subordinate to } U\}$. For $f \in C(X)$, f subordinate to U, $f \leq \chi_U$, so $I(f) = \int f d\nu \leq \int \chi_U d\nu = \nu(U)$. If K is a compact subset of X, by Lemma 2.1.7, there exists $g \in C(X)$, g subordinate to U, and $\chi_K \leq g$. Therefore, $\nu(K) = \int \chi_K d\nu \leq \int g d\nu = I(g)$. By inner regularity of ν on open sets, $\nu(U) = \sup\{I(f) : f \in C(X), f \text{ subordinate to } U\}$. However, since $\overline{\mu}$ and ν agree on open sets, they agree on Borel sets by outer regularity. □

The above result characterizes positive linear functions on C(X). We now extend the result to all bounded (equivalently, continuous) linear functionals on C(X). We recall the following result [36, p255, Proposition 23], which expresses every bounded linear functional on C(X) as the difference of two positive linear functionals.

Lemma 2.2.6 - *For each bounded linear functional F on C(X), there exist positive linear functionals F_+, and F_-, such that $F = F_+ - F_-$, and $\| F \| = F_+(1) + F_-(1)$.*

Theorem 2.2.7 (The Riesz Representation Theorem) - *For each bounded linear functional F on C(X), there exists a unique signed Radon measure μ on X such that $F(f) = \int f d\mu$, for every $f \in C(X)$. Moreover,*

$\| F \| = |\mu|(X)$.

Proof - Let $F = F_+ - F_-$. Then by theorem 2.2.5, there exist Radon measures μ_1, and μ_2, such that $F_+ = \int f\, d\mu_1$, and $F_- = \int f\, d\mu_2$. If we set $\mu = \mu_1 - \mu_2$, then μ is a signed Radon measure, and $F(f) = \int f\, d\mu$, for every $f \in C(X)$.

To show uniqueness of μ, assume that μ, ν are both signed Radon measures such that $\int f d\mu = F(f) = \int f d\nu$ for every $f \in C(X)$. Then the signed Radon measure $\eta = \mu - \nu$ has the property that $\int f d\eta = 0$, for every $f \in C(X)$. If $\eta = \eta^+ - \eta^-$ is the Jordan decomposition of η, then integration with respect to η^+ or η^-, will yield the same positive linear functional. However, by uniqueness of such measures (Theorem 2.2.5), this implies that $\eta^+ = \eta^-$. Therefore, $\eta = 0$, i.e., $\mu = \nu$.

Finally, if $|\mu| = \mu^+ + \mu^-$ denotes the total variation of μ, then, for any $f \in C(X)$, $|F(f)| = |\int f d\mu| \leq \int |f|\, d|\mu| \leq \| f \| \, |\mu|(X)$. This implies that $\| F \| \leq |\mu|(X)$. However, $|\mu|(X) \leq \mu_1(X) + \mu_2(X) = \int 1 d\mu_1 + \int 1 d\mu_2 = F_+(1) + F_-(1) = \| F \|$. Thus $\| F \| = |\mu|(X)$. □

Corollary 2.2.8 - *The dual* $[C(X)]^*$ *of the Banach space* $C(X)$ *is isometrically isomorphic to the space* $\mathbf{M}$ *of signed Radon measures on* X *with norm given by* $\| \mu \| = |\mu|(X)$ *for* $\mu \in \mathbf{M}$.

2.3 Arens-Kelley Characterization of the Banach Space C(X)

In this section, we address the following question : under what conditions is a given Banach space E isometric to C(X) for a suitable topological space X. It is well known that every Banach space E is isometric to a (closed) subspace of C(X) for some compact Hausdorff space X. In fact, the space X is the closed unit sphere of the dual E^*, of E (endowed with the relativized weak* topology). We now provide necessary and sufficient conditions under which E is isometric to "all" of C(X). As a preliminary, we describe the extreme points on the closed unit sphere of the dual space $[C(X)]^*$ of C(X). We denote the closed unit sphere of $[C(X)]^*$ by the symbol B^*.

For each $x \in X$, the *Dirac measure*, δ_x, [concentrated at x] is a Radon measure such that for every Borel set E, $\delta_x(E) = 1$, if $x \in E$, and $\delta_x(E) = 0$ if $x \notin E$. For $x \in X$, we define the linear functional $F_x \in [C(X)]^*$ by $F_x(f) = f(x)$, for every $f \in C(X)$. For any $y \in X$, $\{y\}$ and $X \sim \{y\}$ are Borel sets in X, and $\delta_y(X \sim \{y\}) = 0$, so for $f \in C(X)$, $f(x) = f(y)\chi_{\{y\}}$ almost everywhere with respect to δ_y. Therefore, $\int f\delta_y = f(y)\int \chi_{\{y\}} d\delta_y = f(y) = F_y(f)$. By Theorem 2.2.7, for each $x \in X$ there is a one-to-one correspondence between δ_x and F_x. We say that the linear functional F_x *corresponds to* the Dirac measure δ_x.

Theorem 2.3.1 - *The set of extreme points on the closed unit sphere* B^* *of the dual space* $[C(X)]^*$ *(equivalently, the space* $\mathbf{M}$ *of Corollary 2.2.8) consists* *precisely of the linear functionals* F_x, *and* $-F_x$, *(equivalently, the Dirac measures* $\delta_x, -\delta_x$*), as* x *ranges over the points of* X.

Proof - Let $x_0 \in X$ be given. We show that F_{x_0} is an extreme point of B^*. Assume the contrary, then there exist $F_1, F_2 \in B^*, F_1 \neq F_2$ and $F_{x_0} = \frac{1}{2} F_1 + \frac{1}{2} F_2$. By the Riesz representation Theorem, there exist two unique Radon measures μ_1, μ_2 such that for $f \in C(X)$, $F_i(f) = \int f(x) d\mu_i$, and $\| F_i \| = |\mu_i|(X)$, $(i = 1, 2)$. There exists an (open) neighborhood U of x_0 such that $|\mu_i|(U) < 1$, $(i = 1, 2)$. By complete regularity of X, there exists $f \in C(X)$, $\| f \| = 1$, $f(x_0) = 1$, $f(x) = 0$, for $x \notin U$. Therefore, $F_i(f) = \int_U f(x) d\mu_i \leq |\mu_i|(U) < 1$, $(i = 1,2)$. However, $F_{x_0}(f) = 1$, which contradicts the fact that $F_{x_0} = \frac{1}{2} F_1 + \frac{1}{2} F_2$. Hence, F_{x_0} is an extreme point of B^*. Similarly, $-F_{x_0}$ is an extreme point of B^*.

To prove the converse, let F be an extreme point of B^*. Therefore, $\| F \| = 1$, and there exists a Radon measure μ associated with F.. We must now show that there exists $x \in X$ such that $F(f) = f(x)$ for every $f \in C(X)$, or $F(f) = -f(x)$ for every $f \in C(X)$. Assume the contrary. Since X is compact, we can find two disjoint open sets, U_1, U_2, such that $0 < |\mu|(U_i) < 1$, $(i = 1,2)$. Define $G \in [C(X)]^*$ by $G(f) = |\mu|(U_1) \int_{U_2} f(x) d\mu - |\mu|(U_2) \int_{U_1} f(x) d\mu$, where $f \in C(X)$ and $x \in X$. We now show that $F + G$ belongs to B^*. Let $f \in C(X)$, $\| f \| \leq 1$. Then,

$$(F+G)(f) = \int_X f d\mu + |\mu|(U_1) \int_{U_2} f\, d\mu - |\mu|(U_2) \int_{U_1} f\, d\mu$$

$$= \int_{(X\sim U_1)\sim U_2} f\, d\mu + (1+|\mu|(U_1)) \int_{U_2} f\, d\mu + (1-|\mu|(U_2)) \int_{U_1} f\, d\mu$$

Therefore,

$$|(F+G)(f)| \leq \left| \int_{(X\sim U_1)\sim U_2} f d\mu \right| + 1 + |\mu|(U_1) \left| \int_{U_2} f d\mu \right| + (1-|\mu|(U_2)) \left| \int_{U_1} f d\mu \right|$$

$$\leq \int_{(X\sim U_1)\sim U_2} f\, d|\mu| + (1+|\mu|(U_1)) \int_{U_2} f\, d|\mu| + (1-|\mu|(U_2)) \int_{U_1} f\, d|\mu|$$

$$\leq 1 - |\mu|(U_1) - |\mu|(U_2) + (1+|\mu|(U_1))\, |\mu|(U_2) + (1-|\mu|(U_2))\, |\mu|(U_1) = 1.$$

Therefore, $\| F + G \| \leq 1$ and $F + G \in B^*$, implying also that $F - G \in B^*$.

□

Define $X^* = \{F_x : x \in X\}$, and $-X^* = \{-F_x : x \in X\}$. Since X is completely regular, there is a bijection between X and X^*, and between X and $-X^*$. If $[C(X)]^*$ is given the *weak*-topology*, it follows that the above bijections become homeomorphisms, where X^*, and $-X^*$ are given subspace topologies. To see this let $\{x_\alpha\}$ be a net in X, converging to x. Then by definition of the weak*-topology, F_{x_α} converges to F_x. On the other hand, if the net $\{F_{x_\lambda}\}$ converges to F_x in X^*, it follows that $f(x_\lambda)$ converges to $f(x)$ for every $f \in C(X)$, so $\{x_\lambda\}$ converges to x in the weak topology generated by $C(X)$. However X is completely regular, so this topology coincides with the given topology on X. (see Theorem 0.6.5).

A maximal (proper) linear subspace M of $[C(X)]^*$ is called a *hyperspace*, and for any $F \in [C(X)]^*$, the set $M + F = \{ G + F : G \in M \}$, is called a *hyperplane.* It is clear, by definition of the weak* topology, that any closed hyperplane, $\mathbf{H}$ in $[C(X)]^*$ is of the form $\{F \in [C(X)]^* : F(f) = c\}$, where $f \in C(X)$ and $c \in \mathbf{R}$. Furthermore, we say that a hyperplane $\mathbf{H}$ *supports* a convex set $A \subseteq [C(X)]^*$ if $\mathbf{H} \cap A \neq \emptyset$, and A is contained in the set $\{F \in [C(X)]^* : F(f) \leq c\}$, or A is contained in the set $\{F \in [C(X)]^* : F(f) \geq c\}$.

For any $g \in C(X)$ such that $\| g \| = 1$, consider the hyperplane $\mathbf{H} = \{F \in [C(X)]^* : F(g) = 1\}$. Since there exists $x_0 \in X$ such that $g(x_0) = 1$, $F_{x_0} \in \mathbf{H} \cap B^*$, and clearly $\mathbf{H}$ supports B^*. Therefore, any hyperplane supporting B^*, can be written in this form. This discussion and Theorem 2.3.1 leads to the following theorem.

Theorem 2.3.2 - *The spaces* X^*, *and* $-X^*$ *(both with the relativized weak* topology) consisting of the extreme points of the closed unit sphere of* $[C(X)]^*$ *(under the norm topology) are both homeomorphic to the topological space* X. *Furthermore,* X^*, *and* $-X^*$ *have no common limit points.*

Proof - We need only prove the last statement. Since X^* lies on the hyperplane $\{F \in [C(X)]^* : F(\mathbf{1}) = 1\}$, and $-X^*$ lies on the hyperplane $\{F \in [C(X)]^* : F(\mathbf{1}) = -1\}$, and these hyperplanes do not intersect, the result follows. □

As a consequence of this theorem, we prove the following version of the Banach-Stone theorem for the Banach space $C(X)$.

Theorem 2.3.3 (Banach-Stone) - *The compact Hausdorff spaces X and Y are homeomorphic if and only if the Banach spaces $C(X)$ and $C(Y)$ are isometric.*

Proof - The if part is clear. Assume $C(X)$ and $C(Y)$ are isometric under a map ϕ. Then the adjoint map $\phi^* : [C(Y)]^* \rightarrow [C(X)]^*$, defined $\phi^*(F)(f) = F(\phi(f))$, for every $F \in [C(Y)]^*$ and every $f \in C(X)$, is an isometry and in particular, $[C(Y)]^*$ and $[C(X)]^*$ are weak* homeomorphic. In addition, ϕ^* carries $Y^* \cup -Y^*$ onto $X^* \cup -X^*$.

Define $\phi^*(Y^*) \cap X^* = X_1$, and $\phi^*(Y^*) \cap -X^* = X_2$. If $\phi^*(F_y) = -F_x$, for any $F_y \in Y^*$, then $\phi^{*-1}(F_x) = -F_y$. Therefore we can define a bijection, $\alpha : X_1 \cup X_2 \rightarrow X^*$, as follows : if $F \in X_1$, then $\alpha(F) = F$, and if $F \in X_2$, then $\alpha(F) = -F$. By Theorem 2.3.2, X^* and $-X^*$ have no limit points in common, so the subspace $X_1 \cup X_2$ consists of two components, namely X_1 and X_2. Since for $i = 1,2$ the map α restricted to X_i is continuous and closed, α itself is continuous and closed. Therefore Y^* is homeomorphic to X^*, and by Theorem 2.3.2 we conclude that Y is homeomorphic to X. □

Before giving the main theorem of this section, which characterizes the conditions under which a given Banach space is isometric to $C(X)$ for some suitable compact Hausdorff space X, we prove the following lemma.

Lemma 2.3.4 - *Let Y be a dense subset of a topological space X, and E a complete linear subspace of $C(X)$. If for every pair of subsets K and L of*

Y, whose closures are disjoint, there exists $f \in E$, such that $0 \leq f(x) \leq 1$, for all $x \in X$, and $f(x) = 1$ for $x \in K$, $f(x) = 0$, for $x \in L$, then $E = C(X)$.

Proof - Since E is complete, it is closed, and therefore, the proof follows if it can be shown that E is dense in C(X). If $g \in C(X)$, then for arbitrary $\epsilon > 0$, there exists $a \in \mathbf{R}$ and $n \in \mathbf{N}$, such that $a \leq g(x) \leq n\epsilon + a$, for all $x \in X$. For each m, $0 \leq m \leq n$, we define $L_m = \{x \in X : g(x) \leq (m-1)\epsilon + a\}$, and $K_m = \{x \in X : g(x) \geq m\epsilon + a\}$. Therefore, for each m, K_m is disjoint from L_m (and both are closed) so by hypothesis, there exists $f_m \in E$ such that $0 \leq f_m \leq 1$, for all $x \in X$, and $f_m(x) = 1$ for $x \in K_m$, and $f_m(x) = 0$ for $x \in L_m$. Define $f = af_0 + \epsilon f_1 + \cdots + \epsilon f_n$. Clearly, $f \in E$ and for any $x \in X$, there is an m, such that $a + (m-1)\epsilon \leq g(x) \leq a + m\epsilon$. Since $K_{m-1} \supseteq K_{m-2} \supseteq \dots$, and $L_{m+1} \subseteq L_{m+2} \dots$, for $1 \leq p \leq m-1$, $f_p(x) = 1$, and for $p \geq m+1$, $f_p(x) = 0$. Therefore $f(x) = a + \epsilon(m-1) + \epsilon f_m(x)$, implying that $a + (m-1)\epsilon \leq f(x) \leq a + m\epsilon$. Thus $|g(x) - f(x)| \leq \epsilon$, so E is dense in C(X). □

For any element x of a linear space E, we refer to the element −x as the *antipodal point* to x.

Theorem 2.3.5 *(Arens, Kelley [2]) - A Banach space E is isometric to C(X) (under the usual sup norm), for some compact Hausdorff space X if and only if the closed unit sphere B^* of the dual space E^* under the weak* topology is as follows : (1) there are two supporting hyperplanes of B^* which together* *contain all the extreme points of* B^*, *and (2) any set of extreme points of* B^* *whose closure contains no antipodal points, lies entirely in some hyperplane supporting* B^*.

Proof - Assume E is isometric to C(X). In Theorem 2.3.2 and the discussion following it, we showed that all the extreme points of B^* lie in it's two supporting hyperplanes $\{F \in [C(X)]^* : F(1) = 1\}$, and $\{F \in [C(X)]^* : F(1) = -1\}$. Thus, condition (1) of the theorem is met. To establish condition (2), let Φ be a set of extreme points of B^*, whose closure contains no pairs of antipodal points. Then $\Phi = \Phi^+ \cup \Phi^-$, where Φ^+ lies in the hyperplane $\{F \in [C(X)]^* : F(1) = 1\}$, and Φ^- lies in the hyperplane $\{F \in [C(X)]^* : F(1) = -1\}$. By Theorem 2.3.2 Φ^+ and Φ^- can be mapped onto two subsets A and B of X, respectively, whose closures are disjoint. By normality of X, there exists $g \in C(X)$, such that $0 \leq g(x) \leq 1$, for all $x \in X$, and $g(x) = 1$ for $x \in A$ and $g(x) = 0$ for $x \in B$. By definition of g, the hyperplane $\{F \in [C(X)]^* : F(g) = 1\}$ contains Φ and since $\| g \| = 1$, it supports B^*.

To prove the converse, let E be an arbitrary Banach space satisfying conditions (1) and (2). Consider B^*, the closed unit sphere of the dual space E^*. By condition (1), there is a set of extreme points X' of B^* lying in one of the two hyperplanes of support of B^*, and this set of extreme points has no limit points in common with the remaining extreme points of B^* lying in the other hyperplane supporting B^*. Let X be the closure (in the weak* topology) of X'. Clearly, X is compact Hausdorff since B^* is compact Hausdorff (Alaoglu

Theorem). We can consider each element $f \in E$ to be a continuous real-valued function over X by assigning to each $F \in X$, the real number $F(f)$; i.e. $f(F) = F(f)$. Note that continuity is clear from the definition of weak* convergence. To show that E is normed as a function space over X, we recall the Krein-Millman theorem. Since B^* is the closed convex hull of its extreme points, $\sup\{|f(F)| : F \in B^*\} = \sup\{|f(F)| : F \in X\} = \| f \|$. To complete the proof, we have to show that E is isometric to "all" of $C(X)$. Recall Lemma 2.3.4. Let K and L be subsets of X' (which is obviously dense in X), with disjoint closures. Then, K and $-L = \{-F : F \in L\}$ have no antipodal limit points, so condition (2) can be applied. Let $\{F \in E^* : F(g) = 1\}$ be the hyperplane containing $K \cup -L$ and supporting B^*. Therefore $g \in E$, $-1 \leq g(F) \leq 1$, for all $F \in X$ and $g(F) = 1$ for $F \in K$ and $g(F) = -1$ for $F \in L$. Since $\| g \| = 1$, by Lemma 2.3.4, E is isometric to $C(X)$. □

2.4 C(X) as a Hahn-Banach Space

A topological space X is called *extremally disconnected* if the closure of every open set is open. For example, every discrete topological space is clearly extremally disconnected and the Stone-Cech compactification of such spaces is extremally disconnected as well, [43, Proposition 10.47]. Extremally disconnected spaces have interesting properties, as is illustrated by the following results. Every open (or dense) subspace of an extremally disconnected space is C^*-embedded [19, Problem 1H.6] and [43, Proposition 10.47]. If Y is dense in X, where X is extremally disconnected, then every continuous map from Y into a compact space, extends continuously to all of X [43, Exercise 2J.4]. It turns out that for X extremally disconnected, compact, and Hausdorff, C(X) characterizes a class of Banach spaces.

A normed linear space E is called a *Hahn-Banach* space, if for every continuous linear map $f : K \to E$, where K is a subspace of a normed linear space H, there exists a map $F : H \to E$, such that $F|_K = f$ and $\| F \| = \| f \|$. The map F is said to be a *norm-preserving extension* of f. The name is well motivated since the classical Hahn-Banach Theorem asserts that, **R** and **C** are Hahn-Banach spaces.

Since any continuous linear map between normed linear spaces has a unique norm-preserving extension defined between the completions of these spaces, we may assume, without loss of generality, that the space H mentioned above is complete. In addition, if E is a Hahn-Banach space then it must necessarily be a Banach space. Let $i : E \to E$, be the identity map on the Hahn-Banach space E, and $\hat{i}$, its extension to $\hat{E}$, the completion of E. Since $\hat{i}$ and i agree on the dense subspace E, $\hat{i}$ is the identity on $\hat{E}$. Clearly, this means that $E = \hat{E}$, so E is complete.

We may ask, if every Banach space is a Hahn-Banach space. This was shown by Banach, and Mazur [5] (and many others), to be not true.

Problem : How can we characterize the class of Hahn-Banach spaces?

If **B** is a collection of sets, we say that it has *binary intersection property* (abbreviated B.I.P.), if for every subcollection **A**, such that any two members of **A** intersect, $\bigcap \mathbf{A} \neq \emptyset$. Using this notion L. Nachbin [30] obtained the following condition for a Banach space to be a Hahn-Banach space.

Theorem 2.4.1 *(Nachbin [30]) - A (real) normed linear space E is a Hahn-Banach space if its closed spheres have B.I.P.*

Proof- Let the collection of spheres in E have B.I.P and $f : K \to E$ be a continuous linear map, where K is a subspace of a normed linear space H. By an *extension* of (f, K), we shall mean a pair (g, T), where T is a linear subspace of H such that, $K \subseteq T$, $g : T \to E$ is continuous linear, g agrees with f on K and $\| g \| = \| f \|$. We partial order the family of all extensions of (f, K), by setting $(g_1, T_1) \geq (g_2, T_2)$ if and only if $T_2 \subseteq T_1$ and g_2 agrees with g_1 on T_2. By Zorn's lemma, we obtain a maximal extension, say (ϕ, B). We claim that $B = H$. If not, let $z \in H \sim B$. For every $h \in \phi(B)$, define $\rho(h) = \| f \| \inf \{ \| x - z \|, x \in \phi^{-1}(h)\}$. If S_h is the sphere centered at h, with radius $\rho(h)$, then it is easy to verify that the family $\{S_h : h \in \phi(B)\}$ has B.I.P. Therefore, let $\xi \in \bigcap_{h \in \phi(B)} S_h$, i.e., $\| \phi(x) - \xi \| \leq \| f \| \, \| x - z \|$, for every $x \in B$. Let M be the vector space spanned by B and z, i.e., if $x \in M$, then $x = b + \lambda z$, where $b \in B$ and $\lambda \in \mathbf{R}$. Next for each such $x \in M$, define a map $\psi : M \to E$ by $\psi(x) = \phi(b) + \lambda\xi$. Clearly (ψ, M) is an extension of (f, K), and since $(\psi, M) \geq (\phi, B)$, we get a contradiction to the maximality of (ϕ, B). So $B = H$, and the proof is complete. □

Next, we show that C(X), for X compact Hausdorff and extremally disconnected, is a Hahn-Banach space, by verifying that its closed spheres have B.I.P.

It is clear from section 0.1, that C(X) is an *archimedian ordered vector lattice*, under the partial order $\leq$. We use this lattice structure on C(X) to describe it's closed spheres.

For $f,g \in C(X)$, the set $[f,g] = \{h \in C(X) : f \leq h \leq g\}$ is called an *order interval*. Consider the closed sphere centered at f, of radius ϵ, namely $S(f,\epsilon) = \{g \in C(X) : \| f - g \| \leq \epsilon\} = \{g \in C(X) : g - \epsilon \leq f \leq g + \epsilon\} = [f - \epsilon, f + \epsilon]$. Therefore, all closed spheres in C(X) can be regarded as order intervals.

We say that C(X) is *order complete* if every nonempty subset of C(X) which is bounded above has a supremum; equivalently, that every nonempty subset of C(X) which is bounded below has an infimum.

Theorem 2.4.2 - *For the (compact Hausdorff) extremally disconnected topological space X, C(X) under the partial order $\leq$, is order complete.*

Proof - Let $A = \{f_\alpha : \alpha \in \Lambda\}$ be a nonempty subset of C(X), which is bounded below by g. Therefore, $A - g = \{f_\alpha - g : \alpha \in \Lambda\}$ is bounded below by **0**, and if $A - g$ has an infimum h, then A will have an infimum $g + h$. So, without loss of generality, we assume that all the elements in A are positive.

Let $G_{\alpha\epsilon} = \{x \in X : f_\alpha(x) < \epsilon\}$, where $\alpha \in \Lambda$, $\epsilon > 0$. The set $G_\epsilon = \bigcup_{\alpha \in \Lambda} G_{\alpha\epsilon}$ is open and $X = \bigcup_{\epsilon > 0} G_\epsilon$. Since X is extremally disconnected, clG_ϵ is open and we define $l(x) = \inf\{\epsilon > 0 : x \in clG_\epsilon\}$, for each $x \in X$. To show that $l \in C(X)$, let $0 \leq a < b$, and consider the set $l^{-1}((a,b)) = l^{-1}([0,b)) \sim l^{-1}([0,a])$. Since $l^{-1}([0,b)) = \bigcup_{\epsilon < b} clG_\epsilon$, and $l^{-1}([0,a]) = \bigcup_{\epsilon \leq a} clG_\epsilon = clG_a$, $l^{-1}([0,b))$ is open and $l^{-1}([0,a])$ is closed, so $l^{-1}((a,b))$ is open and l is therefore continuous.

We now show that l is a lower bound for A. For $x \in X$, if $l(x) = 0$, then since all the members of A are positive, $l(x) \leq f_\alpha(x)$, for every $\alpha \in \Lambda$. Therefore, assume that there exists $x \in X$, such that $l(x) \neq 0$. Then there exists $\epsilon > 0$, such that $0 < \epsilon < l(x)$. Clearly $x \notin clG_\epsilon$, so $f_\alpha(x) \geq \epsilon$ for each $\alpha \in \Lambda$. Therefore $f_\alpha(x) \geq l(x)$, and l is a lower bound for A.

Let k be any lower bound for A. If $x \in G_\epsilon$, then there exists $f_\alpha \in A$, $f_\alpha(x) < \epsilon$, so $k(x) < \epsilon$. Therefore by continuity of k, if $x \in clG_\epsilon$, then $k(x) \leq \epsilon$; i.e. $k(x)$ is a lower bound for the set $\{\epsilon > 0 : x \in clG_\epsilon\}$. As $l(x)$ is the infimum of this set, $k(x) \leq l(x)$, so $k \leq l$, since $X = \bigcup_{\epsilon > 0} G_\epsilon$. Therefore, C(X) is order complete. □

Combining the previous two theorems, we obtain Theorem 2.4.3.

Theorem 2.4.3 - *For the (compact Hausdorff) extremally disconnected topological space X, C(X) is a Hahn-Banach space.*

Proof - Since all the closed spheres in $C(X)$ are order intervals, the proof follows from Theorem 2.4.1 if we show that the family of all order intervals in $C(X)$ has B.I.P. Let $\mathbf{C} = \{[f_\alpha, g_\alpha] : \alpha \in \Lambda\}$ be a mutually intersecting family of order intervals. Then, $g_\alpha \geq f_\beta$ for any $\alpha, \beta \in \Lambda$. Therefore, $\{f_\alpha : \alpha \in \Lambda\}$ is bounded above, and $\{g_\beta : \beta \in \Lambda\}$ is bounded below. By the completeness of $C(X)$, there exists $f = \sup\{f_\alpha : \alpha \in \Lambda\}$, and $g = \inf\{g_\beta : \beta \in \Lambda\}$. Since $f \lesssim g$, $[f,g]$ is a nonempty order interval, and $[f,g] \subseteq \cap \mathbf{C}$. Therefore, the family of order intervals in $C(X)$ has B.I.P. ⊡

Independently, Nachbin [30] and Goodner [21] characterized those Hahn-Banach spaces whose unit sphere contains an extreme point, as those $C(X)$ for which X is a suitable extremally disconnected compact Hausdorff space. Later, Kelley [25] showed that the assumption of the extreme point was not necessary. The next theorem characterizes all Hahn-Banach spaces as isometric to $C(X)$, where X is compact Hausdorff and extremally disconnected.

Theorem 2.4.4 *(Kelley [25]) - A real Banach space E is a Hahn-Banach space if and only if it is isometric to $C(X)$, where X is a compact Hausdorff extremally disconnected topological space.*

Proof - One way is clear from Theorem 2.4.3.

Given a Hahn-Banach space E, let P denote the weak*-closure of the set of all the extreme points on the closed unit sphere B^* of the dual space E^*. By Alaoglu's theorem, it is clear that P is weak*-compact. Using Zorn's Lemma, we obtain an open set W^* of P, maximal for the property that $(-W) \cap W = \emptyset$. Then, it is not hard to see that $(-W) \cup W$ is dense in P. We claim that $\overline{W}$, the weak*-closure of W in P, is the required candidate X in the statement of the theorem.

We make several observations. In what follows, the topology referred to will always mean the weak*-topology relativized from P, and the adjoint of a map f will be denoted by f^*.

If U and V are disjoint open subsets of P such that $-(U \cup V)$ and $(U \cup V)$ are disjoint, and their union is dense in P, we shall call the pair (U,V), *a tearing* of P. For such a tearing, let $T = (\{0\} \times \overline{U}) \cup (\{1\} \times \overline{V})$, so that T consists of "disjoint copies" of $\overline{U}$ and $\overline{V}$. We topologize T by taking as an open base, sets of the form $\{0\} \times G$ and $\{1\} \times H$ where G is open in $\overline{U}$ and H is open in $\overline{V}$. Note that $\{0\} \times \overline{U}$ and $\{1\} \times \overline{V}$ are disjoint, clopen (i.e., both open and closed) subsets of T and are respectively homeomorphic to the compact sets $\overline{U}$ and $\overline{V}$. Hence, it follows that T is compact and Hausdorff.

Next, define a map $H : E \to C(T)$ by $H(x)(0,u) = u(x)$ and $H(x)(1,v) = v(x)$, for $x \in E$, $u \in \overline{U}$ and $v \in \overline{V}$. The fact that $H(x) \in C(T)$, follows from weak*-convergence.

We first claim that H is a linear isometry. Clearly, $\overline{U} \cup \overline{V} \subseteq B^*$, so that $\| H(x) \| \leq \| x \|$. To see the reverse inequality, for each $x \in E$, let $x^{**} \in E^{**}$ be defined by $x^{**}(z) = z(x)$ for $z \in E^*$. By Hahn-Banach theorem, for each x

$\in E$, there exists $z \in E^*$ such that $\|z\| = 1$ and $z(x) = \|x\|$. Clearly, by linearity of z, x^{**} maps the unit sphere B^* onto the closed interval $[-\|x\|, \|x\|]$. Now, the set of values in E^* at which x^{**} assumes $\|x\|$ is a weak*-closed face of B^*, so is weak*-compact, and hence by the Krein-Millman Theorem, contains an extreme point t, which is also an extreme point of B^*. Therefore $t \in P$, so either t or $-t$ is in $\overline{U} \cup \overline{V}$. Assuming $t \in \overline{U} \cup \overline{V}$, $\|H(x)\| \geq |t(x)| = \|x\|$.

We have already seen (Theorem 2.3.1) that the extreme points on the unit sphere of $[C(T)]^*$ are precisely the linear functionals $F_t, -F_t$, $t \in T$, where $F_t(f) = f(t)$ for $f \in C(T)$. Consider the adjoint map $H^*: C(T)^* \to E^*$. Now, for $u \in \overline{U}$, $H^*(F_{(0,u)}(x)) = F_{(0,u)}H(x) = H(x)(0,u) = u(x)$ for each $x \in E$, so that $H^*(F_{(0,u)}) = u$. Similarly, for $v \in \overline{V}$, $H^*(F_{(1,v)}) = v$.

Let u be an extreme point of B^*, $u \in U$ and S^* the closed unit sphere in $[C(T)]^*$. The set $H^{*-1}(u) \cap S^*$ is a (compact) face of S^* and therefore contains an extreme point of S^*. Since $u \notin \overline{V}$, the only extreme point H^* maps onto u must be $F_{(0,u)}$. Therefore $H^{*-1}(u) \cap S^* = \{F_{(0,u)}\}$. Similarly, for an extreme point v in B^*, $v \in V$, we have $H^{*-1}(v) \cap S^* = \{F_{(1,v)}\}$.

Since E is a Hahn-Banach space, for the linear isometry $H: E \to C(T)$, H^{-1} is a linear isometry from $H[E]$ to E, so has an extension G from $C(T)$ onto E of norm 1 such that $G \circ H$ is the identity on E. Then G^* carries B^* into S^*, and $(G \circ H)^* = H^* \circ G^*$ is the identity on E^*. This means that for extreme points u, v of B^*, $u \in U$ and $v \in V$, $G^*(u) = F_{(0,u)}$ and

$G^*(v) = F_{(1,v)}$. Recalling Theorem 2.3.2, let $T^+ = \{F_t : t \in T\}$ and $T^- = \{-F_t : t \in T\}$. Such points u and v are dense in U, V and $(-U \cup V) \cup (U \cup V)$ is dense in P, so G^* carries a dense subset of P onto a dense subset of $T^+ \cup T^-$. Therefore, since P and $T^+ \cup T^-$ are compact, it follows that G^* carries P onto $T^+ \cup T^-$.

Next, for extreme points u, v of B^*, $u \in U$ and $v \in V$, $G^* \circ H^*(F_{(0,u)}) = G^*(u) = F_{(0,u)}$, and $G^* \circ H^*(F_{(1,v)}) = G^*(v) = F_{(1,v)}$, so $G^* \circ H^*$ is the identity on a dense subset of $T^+ \cup T^-$. Thus, G^* is a homeomorphism on P, and H^* is an inverse of this homeomorphism on $T^+ \cup T^-$.

For the tearing (U, V), we have the following properties:

Property 1: $\bar{U} \cap \bar{V} = \emptyset$

Property 2: $-(\bar{U} \cup \bar{V}) \cap (\bar{U} \cup \bar{V}) = \emptyset$

Since G^* is a homeomorphism on P, Property 1 is trivial. Since H^* maps T^+ onto $\bar{U} \cup \bar{V}$, Property 2 follows since $T^+ \cap T^- = \emptyset$.

By the above arguments, every extreme point of S^* belongs to $G^*(B^*)$ and since $G^*(B^*)$ is compact and convex, it follows from the Krein-Millman theorem that $S^* \subseteq G^*(B^*)$. However G^* has norm 1, so $G^*(B^*) = S^*$. Finally, $H^* \circ G^*$ is the identity on E^* and since G^* maps B^* onto $[C(T)]^*$, it follows that H^* is one-to-one. We use this fact to show that H is onto. If H[E] is a proper (closed) subspace of C(T), then there exists a nonzero linear functional F on C(T) which vanishes on H[E]. Therefore, $H^*(F)(x) = F(H(x))$

$= 0$ for all $x \in E$, and consequently, $H^*(F)$ is zero on E^*, contradicting the fact that H^* is one-to-one..

To complete the proof, we take the tearing $(W, \emptyset)$ of P. Therefore, T is homeomorphic to $\overline{W}$, and we take $\overline{W}$ to be our candidate for X in the statement of the theorem. Clearly, X is compact Hausdorff, and E is isometric to C(X). We need only prove that X is extremally disconnected. By Property 2, $X \cap -X = \emptyset$, so X is both open and closed in P. Let U be an open subset of X and $V = X \sim \overline{U}$. Then (U, V) is a tearing of P, and by Property 1, $\overline{U} \cap \overline{V} = \emptyset$, so $\overline{U}$ is open proving that X is extremally disconnected. □

Chapter 3

The Topological Vector Space C(X)

When endowed with the compact-open topology, C(X) becomes a (real) locally convex Hausdorff topological vector space. In general, this topology is not metrizable. In the first section, we give conditions on X, necessary and sufficient, for C(X) to be metrizable, under the compact-open topology. In the subsequent two sections, we study the necessary and sufficient conditions on X, which make C(X) a barreled, respectively a bornological space.

3.1 C(X) as a Locally Convex Hausdorff Topological Vector Space

For each subset K of the completely regular topological space X, and each open subset U of $\mathbf{R}$, we define $[K,U] = \{f \in C(X) : f[K] \subseteq U\}$. If Φ is a nonempty family of subsets of X, then the collection $\mathbf{C} = \{[K,U] : K \in \Phi, U$ open in $\mathbf{R}\}$ is a subbase for a topology on C(X). This topology is known as the *Φ-open topology* and is said to be *generated* by $\mathbf{C}$. The family Φ is usually taken to be the family of all compact subsets of X, and in this case, the resulting Φ-open topology on C(X), is called the *compact-open topology*. A net $\{f_\delta\}$, of members of $C(X)$, converges to $f \in C(X)$ with respect to the compact-open topology, if and only if $\{f_\delta\}$ converges to f uniformly on each compact subset of X. It is for this reason that the compact-open topology is often called the *topology of uniform convergence on compact sets.*

It is easily seen that the compact-open topology on $C(X)$ is compatible with the vector space structure on $C(X)$, thus rendering it a (real) topological vector space.

It is useful to describe the compact-open topology on $C(X)$ by the use of semi-norms. For each compact subset K of X, the function $p_K : C(X) \rightarrow \mathbf{R}$, defined by $p_K(f) = \sup\{|f(x)| : x \in K\}$, where $f \in C(X)$, is a seminorm on $C(X)$, and the family $\{p_K : K \text{ a compact subset of } X\}$ generates the compact-open topology. To describe this process, define the set $V_{p_K} = \{f \in C(X) : p_K(f) < 1\}$. The family $\mathbf{B} = \{\bigcap_{\text{finite}} \alpha_i V_{p_{K_i}} : \alpha_i > 0, K_i$ is a compact subset of $X\}$, is a base for the neighborhoods of $\mathbf{0}$ for the compact-open topology. Thus, a neighborhood of $f \in C(X)$ will be of the form, $f + N = \{f + g : g \in N\}$, where $N \subseteq C(X)$, and N contains a member of $\mathbf{B}$. It is worth noting that the compact-open topology on $C(X)$ makes each seminorm p_K (K a compact subset of X) continuous, and is the weakest topology rendering all such seminorms continuous.

A *vector topology* on E is said to be *locally convex* if every neighborhood of the zero element in E contains a convex neighborhood of the zero element. Clearly, $C(X)$, under the compact-open topology, is a locally convex Hausdorff topological vector space.

We shall use the abbreviations T.V.S., (L.C.H.T.V.S.) respectively, for topological vector space (locally convex Hausdorff topological vector space).

Note : *Unless otherwise stated, $C(X)$ will carry the compact-open topology.*

Clearly, if X is compact (Hausdorff), the compact-open topology coincides with the familiar uniform norm topology. Therefore, under the strong condition of compactness of X, $C(X)$ is metrizable (in fact, normable). We now ask if a weaker condition on X will guarantee the metrizability of $C(X)$. The next theorem answers this question.

A topological space X is called *hemicompact*, if there exists a countable family $\{K_n, n \in \mathbf{N}\}$ of compact subsets of X, such that, $X = \bigcup_{n=1}^{\infty} K_n$, and each compact subset of X is contained in some K_n. The collection $\{K_n : n \in \mathbf{N}\}$, is referred to as a *fundamental system of compact subsets* of X. For example, since $\{[-n,n] : n \in \mathbf{N}\}$ is a fundamental system of compact subsets of $\mathbf{R}$, we conclude that $\mathbf{R}$ is hemicompact.

It is well known, that a (Hausdorff) T.V.S. is metrizable if and only if it has a countable base for the neighborhood system at $\mathbf{0}$, [23, corollary 14(b), p134]. We use this fact to derive a metrization theorem for $C(X)$.

Theorem 3.1.1 - *$C(X)$ is metrizable if and only if X is hemicompact.*

Proof - Assume X is hemicompact. Then there exists a fundamental system $\Phi = \{K_n : n \in \mathbf{N}\}$, of compact subsets of X. Let T′ be the Φ-open topology on C(X). Since each $f \in C(X)$ is bounded on each $K_n \in \Phi$, T′ is a vector topology. In addition, T′ is metrizable since the countable collection $\{[K_n,(-n,n)] : n \in \mathbf{N}\}$ forms a base for the neighborhood system at **0**. If T denotes the compact-open topology on C(X), then $T \supseteq T'$. Therefore, the metrizability of T will follow if we can show that $T' \supseteq T$. Since both T and T′ are vector topologies, we need only consider their neighborhoods of **0**. For K, a compact subset of X and $\epsilon > 0$, $[K,(-\epsilon,\epsilon)]$ is a basic neighborhood of **0** for T. Since X is hemicompact, there exists $K_n \in \Phi, K_n \supseteq K$, so $[K,(-\epsilon,\epsilon)] \supseteq [K_n,(-\epsilon,\epsilon)]$, implying that $T' \supseteq T$.

To prove the converse, let C(X) be metrizable. Therefore, C(X) is first countable, so there exists a countable collection $\{K_n : n \in \mathbf{N}\}$ of compact subsets of X such that, for a countable collection $\{U_n : n \in \mathbf{N}\}$ of open neighborhoods of 0 in $\mathbf{R}$, $\{[K_n,U_n] : n \in \mathbf{N}\}$ is a countable base of neighborhoods of **0** in C(X). Let K be an arbitrary compact subset of X, and $0 < \epsilon < 1$. Then there exists some $[K_n,U_n]$, such that $[K_n,U_n] \subseteq [K,(-\epsilon,\epsilon)]$. This implies that $U_n \subseteq (-\epsilon,\epsilon)$ and $K \subseteq K_n$, since if not, there would exist $x \in K \sim K_n$. The complete regularity of X, guarantees the existence of $f \in C(X)$, $f[K_n] = \{0\}$, and $f(x) \notin (-\epsilon,\epsilon)$, which yields a contradiction, since this would mean that $f \in [K_n,U_n]$, but $f \notin [K,(-\epsilon,\epsilon)]$.

Finally we show that $X = \bigcup_{n=1}^{\infty} K_n$. Let $x \in X$. Since $\{x\}$ is compact, the above argument can be applied to the set $[\{x\},(-\epsilon,\epsilon)]$, where $\epsilon > 0$. This tells us that there exists a K_n, such that $x \in K_n$. Therefore, X is hemicompact. □

3.2 The Barreledness of C(X)

Bourbaki [Espaces vectoriels topologiques, *Hermann, Paris,* (1953, 1955)] introduced the notion of barreled spaces. Nachbin [31] and Shirota [40] independently obtained necessary and sufficient conditions for C(X), to be barreled.

Recall, that a family F of linear maps from a T.V.S. E to a T.V.S. H is called ***equicontinuous*** if and only if, for every neighborhood V of zero in H, there exists a neighborhood U of zero in E such that $f[U] \subseteq V$, for every $f \in F$. If for every $x \in E$, $\{f(x) : f \in F\}$ is bounded in H, then F is said to be *point-wise bounded.*

The importance of the notion of barreled spaces, is seen from the fact that they form the most general class of T.V.S. for which the uniform boundedness principle holds, namely if E is barreled, then any nonempty family of point-wise bounded continuous linear maps from E to a locally convex topological vector space H, is equicontinuous [27, p104, Theorem 12.3].

We recall the following definitions from the theory of topological vector spaces.

A nonempty subset A of a (real) topological vector space E, is *absorbent,* if for any $x \in E$, there exists $\epsilon > 0$, such that $x \in bA$ for $|b| \geq \epsilon$. The set

A is *balanced* if $bA \subseteq A$, whenever $|b| \leq 1$. If A is both convex and balanced, then A is *absolutely convex*. A closed absolutely convex absorbent subset of E is called a *barrel*. In any locally convex topological vector space, there exists a base for the neighborhoods of the zero element consisting entirely of barrels. For example, in C(X), we can define sets of the following form : $V_{p_K} = \{f \in C(X) : p_K(f) \leq 1\}$, where K is a compact subset of X, and as before, $p_K(f) = \sup\{|f(x)| : x \in K\}$. Finite intersections of these sets are certainly barrels, and can be employed (as a base) to generate all neighborhoods of 0, [42, p107]. However, in general, not every barrel is a neighborhood of the zero element, (see example 3.2.2). A locally convex topological vector space is *barreled* if every barrel is a neighborhood of the zero element.

Recall, that a subset A of a topological space Y is *nowhere dense* if the interior of its closure is empty, and is of *first category* (in Y) if it is the union of a countable collection of nowhere dense subsets of Y. If A is not of first category in Y, it is of *second category* in Y. Finally, the topological space Y is called a *Baire* space if every nonempty open subset of Y is of second category in Y. We know from Baire's Category Theorem that every complete metrizable topological space (e.g., every Banach space), is a Baire space.

Lemma 3.2.1 - *Every L.C.H.T.V.S. E, which is also a Baire space, is barreled.*

Proof - Let B be a barrel in E. Then $E = \bigcup_{n \in N} nB$, and since E is Baire, there exists $m \in \mathbf{N}$, such that mB has an interior point. Therefore, B itself has an interior point, b. But $-b \in B$, so if θ is the zero element of E, then $\theta = \frac{1}{2}b + \frac{1}{2}(-b)$ is an interior point of B, so B is a neighborhood of θ. Therefore, E is barreled. □

Thus every Banach space is barreled, so if we are to find a non-barreled normed space we will have to look for them in the class of incomplete normed spaces.

Example 3.2.2 - Consider the vector space $C([0,1])$, endowed with the $\mathbf{L_1}$ norm, namely, if $f \in C([0,1])$, then $\| f \| = \int_0^1 |f(x)|\, dx$. Let $B = \{f \in C([0,1]) : \sup\{|f(x)| : x \in [0,1]\} \leq 1\}$. To show that B is a barrel, we need only show that it is closed. Assume $f \in C([0,1])$ is adherent to B and $\{f_n\}$ is a sequence of functions in B converging to f. This means that $\int_0^1 |f_n(x) - f(x)|\, dx$ converges to 0. Therefore, a subsequence of $\{f_n\}$ will converge (point-wise) to f, almost everywhere. By continuity of f, $\sup\{|f(x)| : x \in [0,1]\} \leq 1$, so $f \in B$, proving that B is closed.

However, B cannot be a neighborhood of **0**, since it cannot contain any sphere $S(\mathbf{0},\epsilon) = \{f \in C([0,1]) : \int_0^1 |f(x)|\, dx \leq \epsilon\}$, where $\epsilon > 0$. This is evident, since we can always produce $g \in C([0,1])$, such that $\int_0^1 |g(x)|\, dx \leq \epsilon$, yet

$\sup\{\,|g(x)| : x \in [0,1]\} > 1$. Therefore $C([0,1])$, under this norm, is not barreled.

Before proceeding to the main theorem of this section, we need some preliminary definitions and results.

Lemma 3.2.3 - *For any continuous linear functional F defined on $C(X)$, there exists a compact subset of X, called the support of F, written $supp(F)$, with the following two properties :*

(1) *whenever $f \in C(X)$ vanishes on $supp(F)$, $F(f) = 0$.*

(2) *if T is a closed subset of X with property (1), then $T \supseteq supp(F)$.*

We refer to [7, p92, 2.4-8], for a straight-forward proof that such a subset always exists.

We recall that if E is a T.V.S., then E', the *dual* of E, is the set of all continuous linear functionals defined on E. Similarly, the *bidual* of E is $(E')' = E''$.

A useful tool, when dealing with the dual spaces, is the notion of a polar. For $V \subseteq E$, the *polar* of V, written V°, is the set $V^\circ = \{x' \in E' ; \sup\{|x'(x)| ; x \in V\} \leq 1\}$, and for $A \subseteq E'$, the polar of A is the set $A^\circ = \{x \in E : \sup\{|x'(x)| ; x' \in A\} \leq 1\}$. Some useful elementary properties of polars can be found in [32, section 9.3]. The Bipolar Theorem asserts that, $V^{\circ\circ}$ is the (weak) closure of the absolutely convex hull of V [32, section 9.3]. Therefore, if $V \subseteq$

E, and V is absolutely convex and closed, then $V^{oo} = V$.

If S is *any* subset of X, we define, $p_S(f) = \sup\{|f(x)| : x \in S\}$, $V_{p_S} = \{f \in C(X) : p_S(f) < 1\}$, $\hat{V}_{p_S} = \{f \in C(X) : p_S(f) \leq 1\}$, where $f \in C(X)$. Note that, if S is not compact, $p_S(f)$ may be infinite.

Theorem 3.2.4 *(Nachbin [31], Shirota [40]) - The topological vector space C(X) is barreled if and only if for each closed non-compact subset S of X, there is some function f ∈ C(X), and f is unbounded on S.*

Proof - Assume that there exists a closed non-compact subset S of X, and no member of C(X) is unbounded on S. Then $\hat{V}_{p_S} =$ $\{f \in C(X) : \sup\{|f(x)| : x \in S\} \leq 1\}$ is absorbent, and since it is already closed and absolutely convex, it is therefore a barrel. Consequently, we show that C(X) is not barreled, by proving that $\hat{V}_{p_S}$ is not a neighborhood of **0**.

If $\hat{V}_{p_S}$ is a neighborhood of **0**, then there exists a compact set $K \subseteq X$ and $\epsilon > 0$, such that $\epsilon V_{p_K} \subseteq \hat{V}_{p_S}$. This implies that $S \subseteq K$: for otherwise there exists $x \in S \sim K$. Then by complete regularity of X, there exists $f \in C(X)$ and $f(x) = 2$, and $f[K] = \{0\}$. This implies that $f \in \epsilon V_{p_K}$, but $f \notin \hat{V}_{p_S}$, which is a contradiction. However, if $S \subseteq K$, S is compact, which is again a contradiction. Therefore, $\hat{V}_{p_S}$ cannot be a neighborhood of **0**, and so C(X) cannot be barreled.

To prove the converse, let V be an arbitrary barrel in C(X).

Under the uniform norm, $\| \cdot \|$, $C^*(X)$ is a Banach space, and hence is barreled. For any $K \subseteq X$, K compact, $\{f \in C^*(X) : \| f \| < 1\} \subseteq V_{p_K} \cap C^*(X)$. Therefore, this norm topology on $C^*(X)$ is finer that the subspace topology (inherented from $C(X)$), so $V \cap C^*(X)$ is closed in $C^*(X)$, and therefore, clearly a barrel in $C^*(X)$. Then, there exists $d > 0$, such that $dV_{p_X} \subseteq V \cap C^*(X) \subseteq V$.

We now prove the following technical result. For any $S \subseteq X$, if all the elements of $C(X)$ that vanish on S, belong to V, then $aV_{p_S} \subseteq V$, for some $a > 0$. We assume the hypothesis, and prove the result, where $a = \frac{d}{2}$. If for any $f \in C(X)$, $f \in \frac{d}{2}V_{p_S}$, then $p_S(f) \leq \frac{d}{2}$. Define $g = \max(f,\frac{d}{2}) + \min(f,-\frac{d}{2})$. Since $2g$ vanishes on S, by hypothesis $2g \in V$. Also, $p_X(2(f-g)) \leq d$, so $2(f-g) \in V$. Finally, by convexity of V, $f = \frac{1}{2}(2g) + \frac{1}{2}(2(f-g)) \in V$. In light of this technical result, the proof will be complete if we can produce a compact set $K \subseteq X$, with the property that each element of $C(X)$ which vanishes on K, belongs to V.

Since V is absolutely convex and closed, $V = V^{\circ\circ}$. Therefore, if $f \in V$, then $f \in V^{\circ\circ}$, so $\sup\{|F(f)| : F \in V^{\circ}\} \leq 1$. Therefore, for f to be a member of V, it suffices that $F(f) = 0$ for every $F \in V^{\circ}$. This means, if f vanishes on $\bigcup_{F \in V^{\circ}} \text{Supp}(F)$, then $f \in V$. Our choice for K then, is $cl_X(\bigcup_{F \in V^{\circ}} \text{Supp}(F))$.

We now show that K is compact.

Our approach will be to assume that K is not compact, and get a contradiction to the fact that V is absorbent. We will find a function $f \in C(X)$, which V cannot absorb, by showing that there can be no $a > 0$, such that $f \in aV = (\frac{1}{a}V^{\circ})^{\circ}$. This means, that there exists no $a > 0$, such that, $\sup\{|F(f)| : F \in V^{\circ}\} \leq a$. This will be accomplished by constructing a sequence $\{F_m : m \in N\}$, where for each $m \in N$, $F_m \subseteq V^{\circ}$ and $F_m(f) = m$.

At this point, we use the hypothesis. Since K is closed but not compact, there exists $g \in C(X)$, and g is unbounded on K. Therefore, $\{U_n : n \in N\} = \{(|g|)^{\leftarrow}(n,\infty) : n \in N\}$ is a decreasing sequence of open sets, such that $U_n \cap K \neq \emptyset$, for every n. Let's choose $F_n \in V^{\circ}$, such that $U_n \cap \text{supp}(F_n) \neq \emptyset$. Since X is completely regular there exist functions in $C(X)$ which vanish on $X \sim U_n$, and we claim that for at least one of these, say f_n, $F_n(f_n) \neq 0$. If this was not the case, then $X \sim U_n$ would be a closed subset of X, with the property that, for each function h in $C(X)$, which vanishes on it, $F_n(h) = 0$. However, $U_n \cap \text{supp}(F_n) \neq \emptyset$, so $X \sim U_n$ does not contain $\text{supp}(F_n)$, which is a contradiction to Lemma 3.2.3. Without loss of generality, we assume that $F_n(f_n) = 1$.

For each m, there exists $f_m \in C(X)$, such that, $f_m[X \sim cl(U_m)] = \{0\}$, where the closure is in X. Since $\{cl(U_m) : m \in N\}$ is decreasing, $\{X \sim cl(U_m) : m \in N\}$ is increasing, so $X \sim cl(U_m) \supseteq X \sim cl(U_n)$, for $m \geq n$ (n fixed). Therefore, $f_m[X \sim cl(U_n)] = \{0\}$, for $m \geq n$, so for any sequence $\{c_n : n \in N\}$, of real numbers, $\sum_{n \in N} c_n f_n$ reduces to a finite sum on each $X \sim$

$cl(U_n)$, if each f_n is constructed as above. Since $\bigcap_{n \in N} cl(U_n) = \emptyset$, $\bigcup_{n \in N}(X \sim cl(U_n)) = X$. Therefore, $\sum_{n \in N} c_n f_n \in C(X)$.

For every $m \in \mathbf{N}$ (m fixed), since $supp(F_m)$ is compact, and $\bigcap_{n \in N} cl(U_n) = \emptyset$, $\{cl(U_n) \cap supp(F_m) : n \in \mathbf{N}\}$ cannot have F.I.P. Therefore $cl(U_n) \cap supp(F_m) = \emptyset$, for all but a finite number of n's, since $\{clU_n : n \in \mathbf{N}\}$ is decreasing. By going to a subsequence if necessary, we may assume $cl(U_n) \cap supp(F_m) = \emptyset$, for $n > m$ (m fixed), so for $n > m$, $X \sim cl(U_n) \supseteq supp(F_m)$, and since f_n vanishes on $X \sim cl(U_n)$, it will vanish on $supp(F_m)$, for $n > m$. This implies that $F_m(f_n) = 0$, for $n > m$.

Recall that for every $m \in \mathbf{N}$, we obtained $F_m \in V^\circ$, such that $F_m(f_m) = 1$. Therefore, for each $m \in \mathbf{N}$, a real number c_m may be chosen such that $F_m(\sum_{n \in N} c_n f_n) = \sum_{n \in N} c_n F_m(f_n) = c_m + \sum_{n=1}^{m-1} c_n F_m(f_n) = m$. Clearly, $f = \sum_{n \in N} c_n f_n$ is the desired member of $C(X)$ which cannot be absorbed by V. ▶

Therefore, K is compact and $\frac{d}{2} V_{p_K} \subseteq V$, so V is a neighborhood of **0**, and $C(X)$ is barreled. □

3.3 The Bornological space $C(X)$

The idea of a bornological space was first introduced by Mackey [28] in 1946. It was an attempt to obtain a class of spaces for which a bounded linear map between these spaces was necessarily continuous. In 1954, both Nachbin [31] and

Shirota [40] gave necessary and sufficient conditions on X, for C(X) to be bornological. We devote this final section to prove this elegant characterization.

A subset B of a T.V.S. E is *bounded*, if for each neighborhood U, of zero in E, there exists $\epsilon > 0$, such that, $B \subseteq \alpha U$, where $|\alpha| \geq \epsilon$. A set $V \subseteq E$ is called a *bornivore*, if for any bounded set $B \subseteq E$, there exists $\epsilon > 0$, such that, for any scalar a, where $|a| > \epsilon$, $B \subseteq aV$. Obviously, such a set V, *absorbs* all bounded sets in E.

A L.C.T.V.S. E is *bornological*, if for any L.C.T.V.S. H, and linear map $F : E \rightarrow H$, if F maps bounded sets in E to bounded sets in H, then F is continuous. Equivalently, E is bornological, if each absolutely convex bornivore is a neighborhood of the zero element in E, [32, section 13.2].

A linear functional F defined on C(X) is *multiplicative*, if $F(f \cdot g) = F(f)F(g)$, for any $f,g \in C(X)$. For the result to follow, C(X), under the compact-open topology, is considered to be a locally convex Hausdorff commutative topological algebra.

Lemma 3.3.1 - *Every nonzero continuous multiplicative linear functional on C(X), is of the form,* $F_x(f) = f(x), f \in C(X)$, *where* $x \in X$.

We refer to [23, p309, Corollary 14] for a proof.

Recall from sections 1.4 and 1.5, that the symbols βX and υX stand for the Stone-Cech and realcompactifications of X, respectively.

Lemma 3.3.2 - *A point $p \notin \upsilon X$ if and only if there exists a sequence of neighborhoods in βX of p, whose intersection misses X.*

The straightforward proof can be found in [7, p22, Theorem 1.5-1].

Recall from section 2.4, that for $f,g \in C(X)$, the set $[f,g] = \{h \in C(X) : f \leq h \leq g\}$, is an order interval. If K is any compact subset of X, then $\max\{p_K(f), p_K(g)\} = \epsilon > 0$, and $p_K(h) \leq \epsilon$, for any $h \in [f,g]$. Therefore, $[f,g] \subseteq \alpha V_{p_K}$, for any scalar α, where $|\alpha| > \epsilon$. This shows that all order intervals are bounded.

Let V be an absolutely convex subset of $C(X)$, which absorbs all order intervals in $C(X)$. If K is a closed set in βX, such that, for each $f \in C(X)$, whenever f^{β} vanishes on K, $f \in V$, then K is called a *support set* of V. Since βX, itself, is a support set of V, the definition is not vacuous. Intersection of all the support sets of V is called the *support* of V, written $\text{supp}(V)$.

Lemma 3.3.3 - *For an absolutely convex subset V of $C(X)$ which absorbs all order intervals, the support of V is a support set of V.*

We refer to [7, p101] for the proof. We are now ready to prove the main result.

Theorem 3.3.4 *(Nachbin [31], Shirota [40])* - *The topological vector space $C(X)$ is bornological if and only if X is realcompact.*

Proof - Assume X is not realcompact. Then there exists $p \in \upsilon X \sim X$. Since $f^{\beta}(p)$ is finite, for every $f \in C(X)$, we can define a linear functional F_p on $C(X)$ as follows : for every $f \in C(X)$, $F_p(f) = f^{\beta}(p)$. By Lemma 3.3.1, F_p is not continuous.

Let B, a bounded subset of $C(X)$, contain the sequence $\{f_n : n \in \mathbf{N}\}$, and assume that F_p maps this sequence into an unbounded set in $\mathbf{R}$. For each $n \in \mathbf{N}$, define $U_n = \{q \in \beta X : |f_n^{\beta}(q)| > |f_n^{\beta}(p)| - 1\}$. Each U_n is a neighborhood (in βX) of p, and by Lemma 3.3.2, since $p \in \upsilon X$, the intersection of these neighborhoods will meet X. Therefore, there exists $x_0 \in \bigcap_{n \in N} U_n \cap X$, which means that $\{f_n(x_0) : n \in \mathbf{N}\}$ is unbounded, since we assumed that $\{f_n^{\beta}(p) : n \in \mathbf{N}\}$ is unbounded. However, this is a contradiction to the boundedness of B. Therefore, the discontinuous functional F_p maps bounded sets to bounded sets, so $C(X)$ is not bornological.

To prove the converse, we assume that X is realcompact and show that every absolutely convex set $V \subseteq C(X)$, which absorbs each order interval, is a neighborhood of $\mathbf{0}$. Since all order intervals are bounded, this will prove that $C(X)$ is bornological. We first prove that $\mathrm{supp}(V) \subseteq X$. Let $p \in \beta X \sim X$. By realcompactness of X, $p \notin \upsilon X$, so by Lemma 3.3.2., there exists a decreasing sequence $\{W_n\}$ of closed neighborhoods (in βX) of p, and $\bigcap_{n \in N} W_n \cap X = \emptyset$. For each $n \in \mathbf{N}$, if $\beta X \sim \mathrm{int}(W_n)$ is not a support set for V, then there exists $f_n \in C(X)$, such that $f_n^{\beta}[\beta X \sim \mathrm{int}(W_n)] = \{0\}$, and $f_n \notin V$, where

$\text{int}(W_n)$ denotes the (topological) interior of W_n. Define $f = \sup\{n|f_n| : n \in \mathbf{N}\}$. For $m \in \mathbf{N}$, (m fixed), if $n > m$, then $X \sim W_m \subseteq \beta X \sim \text{int}W_n$, so f_n vanishes on $X \sim W_m$, and $f = \max\{|f_1|, 2|f_2|, \dots, m|f_m|\}$, on $X \sim W_m$. Therefore, for each $m \in \mathbf{N}$, f is continuous on $X \sim W_m$. Since $\{X \sim W_m\}$ is an increasing sequence of open sets whose union is X, $f \in C(X)$. Since V absorbs the order interval $[-f,f]$, there exists a positive scalar a, such that $[-f,f] \subseteq aV$. By definition of f, for each $n \in \mathbf{N}$, $nf_n \in [-f,f]$, and $f_n \in V$, whenever $n \geq a$, since V is balanced. This is a contradiction, so at least one of the sets $\beta X \sim \text{int}W_n$ is a support set of V, and since $p \in \text{int}W_n$, $p \notin \text{supp}(V)$. Therefore, $\text{supp}(V) \subseteq X$.

Again, since V absorbs all order intervals, there exists a positive scalar a, such that $[-1,1] \subseteq aV$. If we choose $d = \frac{1}{a}$, then $dV_{p_X} \subseteq V$. In the proof of Theorem 3.2.4, we already showed that for any set $S \subseteq X$, if all the elements of $C(X)$ that vanish on S, belong to V, then $\frac{d}{2}V_{p_S} \subseteq V$. By Lemma 3.3.3, $\text{supp}(V)$ has this property, and since it is a closed subset of βX, and contained in X, it is compact in X. Therefore, $\frac{d}{2}V_{p_{\text{supp}(V)}} \subseteq V$, and V is a neighborhood of $\mathbf{0}$, proving that $C(X)$ is bornological. □

We conclude with the remark that the characterization Theorem 3.2.4 and Theorem 3.3.4 together, yield, provide an affirmative solution to a question posed by Dieudonne whether there exist barreled spaces that are not bornological, and bornological spaces that are not barreled.

REFERENCES

[1] Alo, R.A., and Shapiro, H.L., Normal topological spaces, *Cambridge University Press, Cambridge*, (1974).

[2] Arens, R.F., and Kelley, J. L., Characterizations of the space of continuous functions over a compact Hausdorff space, *Trans. Amer. Math. Soc.* **62**, *No. 3*, (1947), 499-508.

[3] Atiyah, M.F., Macdonald, I.G., Introduction to commutative algebra, *Addison-Wesley, Massachusetts*, (1969).

[4] Aull, C.E. (editor), Rings of continuous functions, *Lecture notes in pure and applied mathematics* **95**, *Marcel Decker, New York*, (1985).

[5] Banach, ~~S.~~ and Mazur, S., Zur theorie der linearen dimension, *Studia Math* **4**, (1933), 100 - 112.

[6] Beauzamy, B., Introduction to Banach spaces and their geometry, *North-Holland Mathematics Studies* **68**, *North-Holland, Amsterdam*, (1982).

[7] Beckenstein, E., Narici, L., and Suffel, C., Topological algebras, *North Holland Mathematics Studies* **24**, *North-Holland, Amsterdam*, (1977).

[8] Berberian, S.K., Measure and intergration, *Chelsea, New York,* (1970).

[9]. Chandler, R.E., Hausdorff compactifications, *Lecture notes in pure and applied mathematics* **23**, Marcel Decker, New York, (1976).

[10] Comfort, W.W., and Negrepontis, S., The theory of ultrafilters, *Springer-Verlag, New York,* (1974).

[11] Cronin-Scanlon, J., Advanced calculus : A start in analysis (revised edition), *D.C. Heath, Lexington, Massachusetts,* (1969).

[12] Day, M.M., Normed linear spaces, *Academic Press, New York,* (1962).

[13] Dean, R.A., Elements of abstract algebra, *John Wiley and Sons, New York,* (1966).

[14] Dugundji, J., Topology, *Allyn and Bacon, Boston,* (1966).

[15] Dunford, N., Schwartz, J., Linear operators, part I : General theory, *Interscience, New York,* (1958).

[16] Edwards, R.E., Functional analysis : Theory and applications, *Holt, Rinehart and Winston, New York.,* (1965).

[17] Fichtenholtz, G. and Kantorovitch, L., Sur les operations lineaires dans l'espace des fonctions bornees, *Studia Math.* **5**, (1934), 69 - 98.

[18] Folland, G.B., Real analysis : Modern techniques and their applications, *John Wiley and Sons, New York*, (1984)

[19] Gillman, L., and Jerison, M., Rings of continuous functions, *Graduate texts in mathematics* **43**, *Springer-Verlag, New York*, (1976).

[20] Goldberg, R.R., Methods of real analysis, *Xerox College Publishing, Waltham Massachusetts*, (1964).

[21] Goodner, D.B., Projections in normed linear spaces, *Trans. Amer. Math. Soc.* **69**, (1950), 89 - 108.

[22] Halmos, P., Measure theory, *D. Van. Nostrand, Princeton, New Jersey*, (1950).

[23] Husain, T., Topology and maps, *Mathematical concepts and methods in science and engineering* **5**, *Plenum Press, New York*, (1977).

[24] Jarchow, H., Locally convex spaces, *B.G. Teubner, Stuttgart, West Germany*, (1981).

[25] Kelley, J.L. , Banach spaces with extension property, *Trans. Amer. Math. Soc.* **72** (1952), 323 - 326.

[26] Kelley, J.L., General topology, *D. Van Nostrand., Princeton, New Jersey*,

(1955).

[27] Kelley, J.L., Namioka, I., et al., Linear topological spaces, *D. Van Nostrand, Princeton, New Jersey,* (1963).

[28] Mackey, G., On convex topological linear spaces, *Trans. Amer. Math. Soc.* **60**, (1946), 519 - 537.

[29] McCoy, N.H., Introduction to modern algebra (third edition), *Allyn and Bacon, Boston,* (1975).

[30] Nachbin, L., A theorem of Hahn-Banach type for linear transformations, Trans. Amer. Math. Soc. **68**, (1950), 28 - 46.

[31] Nachbin, L., Topological vector spaces of continuous functions, Proc. Nat. Acad. Sci. USA **40**, (1954), 471 - 474.

[32] Narici, L., Beckenstein, E., Topological vector spaces, *Monographs and textbooks in pure and applied Mathematics, Marcel Decker* **95**, *New York,* (1985).

[33] Pryce, J.D., Basic methods of linear functional analysis, *Hutchinson University Library, London,* (1973).

[34] Robertson, A., and Robertson, W., Topological vector spaces (second edi-

tion), *Cambridge tracts* **53**, *Cambridge University Press, Cambridge*, (1973).

[35] Ross, K.A., Elementary analysis : The theory of calculus, *Undergraduate texts in mathematics*, *Springer-Verlag, New York*, (1980).

[36] Royden, H.L., Real analysis, *Macmillan, New York*, (1964).

[37] Schaefer, H.H., Topological vector spaces, *Graduate texts in mathematics*, *vol.3*, *Springer-Verlag, New York*, (1971).

[38] Semadeni, Z., Banach spaces of continuous functions (vol.1), *Polish Scientific Publishers, Warszawa*, (1971).

[39] Shapiro, L.W., Introduction to abstract algebra, *McGraw-Hill international series in pure and applied mathematics*, *McGraw-Hill, New York*, (1975).

[40] Shirota, T., On locally convex vector spaces of continuous functions, *Proc. Japan Acad.* **30**, (1954), 294 - 298.

[41] Steen, L.A. and Seebach, J.A.,Jr., Counterexamples in topology (second edition), *Springer-Verlag, New York*, (1978).

[42] Taylor, A.E. and Lay, D.C., Introduction to functional analysis (second edition), *John Wiley, New York*, (1980).

[43] Walker, R.C., The Stone-Cech compactification, *Springer-Verlag, New York*, (1974).

[44] Weir, M., Hewitt-Nachbin spaces *North-Holland mathematics studies* **17**, *North-Holland, Amsterdam*, (1975).

[45] Wilansky, A., Topology for analysists, *Xerox College Publishers, Waltham, Massachusetts*, (1970).

[46] Willard, S., General topology, *Addison-Wesley, Massachusetts*, (1970)