F-FIBRATIONS AND GROUPS OF GAUGE TRANSFORMATIONS

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by

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ABSTRACT

A study of the relationships between various notions of "Universal fibration" which arise in the literature, has been done by P. Booth, P. Heath and R. Piccinini, within the context of an admissible category of fibrations. This general category, of which the usual categories of fibrations that arise in practise are particular examples, is defined within the general framework introduced by J. P. May for the notion of F- fibration, by specifying certain axioms. Using a generalized version of the exponential law we show that the category of F- fibrations is directly related to this notion of an admissible category of fibrations. The result is that the axioms defining admissibility can be simplified.

The appropriate notion of equivalence in an admissible category A, is an extension of the notion of fibre homotopy equivalence, called F - homotopy equivalence. If $p: E \to B$ is an A- fibration (object of A), we denote by $F(p)$, the space of all F - homotopy equivalences $p \to p$ over B and by $F^1(p)$, the space of all F- homotopy equivalences $p \to p$ over B which extend $1: F \to F$ on a distinguished fibre F. We show that if the category A admits an "Aspherical Universal" A- fibration $p_\infty: E_\infty \to B_\infty$ (this is the situation in the usual categories of fibrations that arise in practise) and $k: B \to B_\infty$ is the classifying map for p, then $F(p)$ (resp. $F^1(p)$) has the same weak homotopy type as $\Omega L(B,B_\infty;k)$ (resp. $\Omega L_*(B, B_\infty;k)$). Here, $L(B,B_\infty;k)$ denotes the path component of the function space $L(B,B_\infty)$ which contains k and $L_*(B,B_\infty;k)$ is the based version. In particular, we show that if B_∞ is an H-group

then $F(p)$ (resp. $F^1(p)$) has the same weak homotopy type as $L(B,\Omega B_\infty)$ (resp. $L_*(B,\Omega B_\infty)$); if B is an H-cogroup, then $F^1(p)$ has the same weak homotopy type as $L_*(B,\Omega B_\infty)$. With a connectivity condition on B_∞ it is also possible to obtain some computations of the homotopy groups of $F(p)$ and $F^1(p)$ within the stable range. In the case where the admissible category A is the category of principal G-bundles over smooth manifolds, with G a compact Lie group, the spaces $F(p)$ and $F^1(p)$ are the groups of gauge transformations that arise in theoretical Physics. The results obtained in the general situation are valid here up to homotopy equivalence.

Introduction

This thesis is divided into three chapters, the first of which contains a description of the terminology, notation and most of the machinery needed in the actual research part contained in the next two chapters. There, we actually deal with two distinct problems. The first problem treated in Chapter II, is concerned with admissible categories of fibrations. This notion of an <u>admissible category</u> was introduced in [4] to be a general framework in which the concept of "Universal fibration" could be studied. As known, "Universal fibrations" appear in the literature in various forms; thus the need to set all these concepts into a unified context and study their relationships.

In [4], an admissible category of fibrations was defined as a non-empty, full subcategory A of the category of all F-spaces over CW-complexes and F-maps (previously defined by J. P. May in [14]) satisfying four axioms. The first three axioms can be summarized as the requirement that all F-spaces induced from objects of A and all F-spaces homeomorphic to objects of A, are themselves objects of A. The fourth axiom indirectly describes the objects of A; it is stated as follows:

A4 - If $q:Y \to A$ and $r:Z \to B$ are objects of A, then $q*r : Y*Z \to A \times B$ is a Serre fibration.

Here, $q*r$ is the functional bundle construction given in [4].

Now, it is not clear from this definition what conditions are imposed on the objects of A in order that axiom A4 be satisfied. Indeed, it is shown in [4] that the usual categories of fibrations considered in practise, for example, the category of Hurewicz fibrations and the category of numerable principal G-bundles, are examples of admissible categories; therefore, one would expect that in a general admissible category the objects are some generalized notion of a fibration. In fact, in section 2, we see that such is the case, the generalized notion of fibration being that of a weak F-fibration. This concept is a generalization of the notion of a Serre fibration, just as the concept of F-fibration in [14] is a generalization of the notion of a Hurewicz fibration. Because the objects of the categories of fibrations we will consider are all Hurewicz fibrations, we choose to deal with the concept of F-fibration. In this situation axiom A4 would require that all functional bundles $q*r$ are Hurewicz fibrations. The definition of an admissible category of fibrations can now be redefined in terms of F-fibrations without the use of axiom A4. Although this new concept of admissibility is not equivalent to the original definition, it does imply admissibility in the sense of [4]. Hence, the general theory of A-fibrations (objects of A), and, in particular, the theory of Universal A-fibrations developed in [4], remains unchanged in this new setting. This fact is used to our advantage in section 3, where we introduce several notions of n-Universality, n finite or infinite; the case $n = \infty$ is considered in [4]. As in [4], we distinguish four types: an A-fibration $p_n : E_n \to B_n$, n finite or infinite, is said to be (i) Free n-Universal if the appropriately defined equivalence classes of A-fibrations over a CW-complex B of dimension $\leq n$ are classified by the free homotopy classes $[B, B_n]$; (ii) Grounded n-Universal if the appropriately defined equivalence classes of grounded A-fibrations over a CW-complex B of dimension $\leq n$ are classified by the based homotopy classes $[B, B_n]_*$; (iii) Aspherical n-Universal if the total space of the associated principal fibration is n-aspherical (that is, all homotopy groups vanish in dimensions zero through to n); and (iv) Extension n-Universal if for every relative CW-complex (B,L) of dimension $\leq n$ and every A-fibration $p: E \to B$, every partial map $p|L \to p_n$ can be extended to a map $p \to p_n$.

Now, it is shown in [4] that the four notions of ∞-Universality are not, in general, equivalent but there do exist certain relationships between them. Indeed, these relationships also hold for the more restrictive notion of n-Universality, n finite, and we give the corresponding results. Of course, in discussing the equivalence of these four notions in a given admissible category one must first be sure that they all exist. In fact, for the notions of ∞-Universality, we show that the problem of existence and equivalence of the four notions can be reduced to just the problem of the existence of an Aspherical ∞-Universal A-fibration. This result is used in section 4, where we examine the problem of the existence and equivalence of the four notions of ∞-Universality in the category of numerable fibre bundles with fibre F and structure group G. There, our technique is to use the close relationship between fibre bundles and their associated principal G-bundles, as well as the Aspherical ∞-Universal G-bundle given by the Milnor construction [16], to give a necessary and sufficient condition for the existence of an Aspherical ∞-Universal fibre bundle and hence, for the equivalence of the four notions of ∞-Universality.

Our second problem is discussed at length in Chapter III; it is a description of the homotopy properties of the spaces of F-homotopy equivalences for A-fibrations. In the case in which the admissible category A is the category of principal G-bundles over smooth manifolds, with G a compact Lie group, these are the groups of <u>gauge transformations</u>, as described by Atiyah, Singer and others. This concept of a gauge transformation did not originate in Mathematics but rather, in theoretical Physics. The notion of a local gauge transformation was first given in 1918 by Hermann Weyl and extended in 1929 [21] by the same author to the theory of electromagnetic fields in interaction with charged particle fields. The basic ideas, translated into more modern terms, are relatively simple. Consider the trivial U(1)-bundle $(M^4 \times U(1), pr_1, M^4)$, where M^4 is Minkowski space ($\mathbb{R}^4$ with Minkowski's metric). It is known that the electromagnetic field is characterized by a differential 2-form F, the Faraday tensor, or rather, by its norm (field action). Now, Maxwell's equations show that $dF = 0$ and so, F is a 2-cycle in the deRham Cohomology of M^4. But, M^4 being contractible, $H_D^2(M^4) \cong H^2(M^4,\mathbb{R}) \cong 0$ and so, there is a differential 1-form A on M^4 such that $F = dA$. This 1-form A is called the vector potential; one shows that it gives rise to a Ehresmann connection of $M^4 \times U(1)$. The problem then is to study the transformations of this connection which do not alter the field action. This procedure can be generalized to more general physical situations, which was done by Yang and Mills in 1954 [15]. As pointed out by Atiyah, Singer and others, the study of Gauge Theories in these physical situations can be done Mathematically in the right framework; namely, the Theory of Fibre Bundles. Roughly speaking the Mathematical setup is as follows: we have a principal G-bundle $p:E \to B$, where B is a smooth manifold and G is a compact, connected Lie group. The connections of E correspond to the vector potentials and the gauge transformations correspond to the G-automorphisms of the bundle. The action of the gauge transformations on the vector potentials then correspond to the action of the group $\mathcal{G}(p)$ of all G-automorphisms of p on the space $\mathcal{U}$ of all connections on E. Because this action need not be effective, the projection map $\pi: \mathcal{U} \to \mathcal{U}/\mathcal{G}(p)$ is not a principal $\mathcal{G}(p)$-bundle. Recently I. M. Singer studied this map in [17] and determined that if $B = S^3$ or S^4 and $G = SU(n)$, $n > 1$, then π does not have a section (in physics terms a continuous gauge). The proof of this result entailed a study of the homotopy properties of the group $\mathcal{G}(p)$ and of a based version $\mathcal{G}^1(p)$. This is a motivation for our study of the groups $\mathcal{G}(p)$ and $\mathcal{G}^1(p)$ in section 3. Actually, the results we obtain are general and not just restricted to principal G-bundles over a smooth manifold, where the group G is a compact Lie group. Indeed, for any principal G-bundle p over a CW-complex B, we show that the fibration $\omega_p:\mathcal{G}(p) \to G$ with fibre $\mathcal{G}^1(p)$ is homotopy equivalent to a well known loop fibration. More specifically, if B_G is the classifying space for the group G, $k:B \to B_G$ is the classifying map for p and $L(B,B_G;k)$ is the component of the function space $L(B,B_G)$ containing the map k, then ω_p is homotopy equivalent to the loop fibration $\Omega L(B,B_G;k) \to \Omega B_G$; hence, $\mathcal{G}(p) \simeq \Omega L(B,B_G;k)$ and $\mathcal{G}^1(p) \simeq \Omega L_*(B,B_G;k)$ (here, $L_*(B,B_G;k)$ is the based version of $L(B,B_G;k)$). A better characterization of the groups $\mathcal{G}(p)$ and $\mathcal{G}^1(p)$ can be obtained with suitable restrictions on B or B_G; specifically, if B is an H-cogroup, then $\mathcal{G}^1(p) \simeq L_*(B,G)$ (see [17; Theorem 5]) and if B_G is an H-group, then $\mathcal{G}(p) \simeq L(B,G)$ and $\mathcal{G}^1(p) \simeq L_*(B,G)$. We also show that if the group G satisfies a connectivity condition and the dimension of the base space is suitably restricted, then the homotopy groups of $\mathcal{G}(p)$ (resp. $\mathcal{G}^1(p)$) are isomorphic to the homotopy groups of $L(B,G)$ (resp. $L_*(B,G)$) in low dimensions. This result is particularly useful for computations because it can be applied to a wide range of G-bundles.

Now, all these results on the groups $\mathcal{G}(p)$ and $\mathcal{G}^1(p)$ are actually specializations to the category of principal G-bundles of corresponding results in a general admissible category A. These results on the spaces of F-homotopy equivalences of A-fibrations are discussed in section 2 with greater generality.

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Chris Morgan

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I. PRELIMINARIES

§1. The category of k-spaces

To avoid the usual restrictions required for the existence of an exponential law in the category of topological spaces Top, we work in the convenient category K of k-spaces[(*)] [1],[6]; that is, spaces endowed with the final topology with respect to all maps from all compact Hausdorff spaces. Any topological space can be retopologized as such, that is, k-ified, and is called a k-space. As is usual for a convenient category, K is large enough to contain many of the spaces arising in practise, for example, CW-complexes, and is closed under standard operations such as, the formation of subspaces, product spaces $X \times Y$, pullback spaces $X \sqcap Y$ and function spaces $L(X,Y)$. These basic constructions in K are obtained as the k-ifications of the appropriate constructions in Top. The appropriate topology for the function space $M(X,Y)$ in Top is, of course, the compact-open topology; then, $L(X,Y) = k(M(X,Y))$.

Theorem 1.1.0. (exponential law of k-spaces; [1; Theorem 2.12]). If X, Y and Z are k-spaces, then the function that assigns to each map $f:X\times Y \to Z$ the map $g:X \to L(Y,Z)$, defined by $g(x)(y) = f(x,y)$, $x \in X$, $y \in Y$, is a natural homeomorphism

$$\theta:L(X\times Y,Z) \to L(X,L(Y,Z))$$

(*)- The original concept of a k-space was introduced by Kelley in [12].

§2. F-spaces and F-maps

The terminology and notation in this section are due to J. P. May [14].

Let F be a category, with distinguished object F, together with a faithful underlying space functor $F \to K$. Thus each object of F is a k-space and the set $F(P,Q)$ of morphisms from P to Q in F is a subspace of $L(P,Q)$. For each object P in F we agree to identify the spaces $P\times *$ and $*\times P$ with P, where $*$ denotes any one-point space.

<u>Definition</u> 1.2.1. An <u>F-space</u> is a morphism $p:X \to A$ of K, such that, for each $a \in A$, the fibre $p^{-1}(a)$, which we shall denote by X_a, is an object of F. Observe that, for any object P of F, the projection map $pr_1:A\times P \to A$ is an F-space and, in particular, the constant map $P \to *$ is an F-space.

By an <u>F-map</u> $(f_1,f_0):p \to r$ we mean a commutative diagram of K

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Z \\ \downarrow p & & \downarrow r \\ A & \xrightarrow{f_0} & B \end{array}$$

such that, for each $a \in A$, the restriction $f_1|X_a:X_a \to Z_{f_0(a)}$ is a morphism of F. Observe that once the map f_1, in the diagram above, is specified, f_0 is completely determined; hence, we denote by $F(p,r)$ the space of all F-maps $p \to r$, topologized as a subspace of $L(X,Z)$. We say that f_1 <u>covers</u> f_0 and denote by $F_{f_0}(p,r)$ the space of all F-maps $p \to r$ which cover $f_0\colon A \to B$, topologized as a subspace of $L(X,Z)$. If $B = A$ and $f_0 = 1\colon A \to A$, then f_1 is said to be an <u>F-map over A</u>.

A pair of F-maps (f_1,f_0) and (g_1,g_0) from p to r are said to be <u>F-homotopic</u>, denoted $(f_1,f_0) \simeq_F (g_1,g_0)$, if there exists an F-map (H,h) of the form

$$\begin{array}{ccc} X\times I & \xrightarrow{\quad H \quad} & Z \\ \downarrow{\scriptstyle p\times 1_I} & & \downarrow{\scriptstyle r} \\ A\times I & \xrightarrow[\quad h \quad]{} & B \end{array},$$

where I denotes the unit interval $[0,1]$, such that $(H(-,0), h(-,0)) = (f_1,f_0)$ and $(H(-,1),h(-,1)) = (g_1,g_0)$. Thus, for each $t \in I$, (H_t,h_t) is an F-map, where $H_t(x) = H(x,t)$ and $h_t(a) = h(a,t)$, $x \in X$, $a \in A$. We call the pair (H,h) an <u>F-homotopy</u> from (f_1,f_0) to (g_1,g_0). In the case where $B = A$ and $h\colon A\times I \to A$ is the projection map, we have the notion of an <u>F-homotopy over A</u>.

An F-map $g\colon X \to Z$ over A is said to be an <u>F-homotopy equivalence</u> over A if there exists an F-map $g'\colon Z \to X$ over A such that the composites $g\cdot g'$ and $g'\cdot g$ are F-homotopic over A to their respective identity maps. We call g' an <u>F-homotopy universe</u> for g and we say that p is <u>F-homotopy equivalent</u> to r over A. If, in particular, p is F-homotopy equivalent to $pr_1\colon A\times F \to A$ over A, then p is said to be F-homotopy trivial.

An F-map $g:X \to Z$ over A is said to be an F-homeomorphism over A if there exists an F-map $g':Z \to X$ over A such that $g \cdot g' = 1_Z$ and $g' \cdot g = 1_X$. We call g' an F-inverse for g and we say that p is F-homeomorphic to r over A. If, in particular, p is F-homeomorphic to $pr_1:A \times F \to A$ over A, then p is said to be a trivial F-space or equivalently, F-trivial.

By restriction to one-point base spaces, the previous definitions specialize to give the notions of F-homotopies, F-homotopy equivalences and F-homeomorphisms between spaces in F.

Henceforth we shall always assume that our category F satisfies the following two conditions:

(1.2.2)
(i) for each object P in F, $F(F,P) \neq \emptyset$.
(ii) every morphism in F is an F-homotopy equivalence over a point.

We shall call such a category F, a category of fibres and denote it by the pair (F,F). Oberve that conditions (i) and (ii) of (1.2.2) immediately imply that $F(P,F) \neq \emptyset$, for every object P in F, and hence, $F(P,Q) \neq \emptyset$, for all objects P, Q in F. Moreover, every morphism in F has an F-homotopy inverse in F.

Proposition 1.2.3. Let $r:Z \to B$ be an F-space and let $f:A \to B$ be a map. Then the induced map $r_f:A \sqcap Z \to A$ is an F-space and $(\bar{f},f):r_f \to r$ is an F-map. Moreover, if $p:X \to Z$ is an F-space and $(g,f):p \to r$ is an F-map, then the unique map $h:X \to A \sqcap Z$, defined by $h(x) = (p(x), g(x))$, is an F-map over A.

Proof: For each $a \in A$, the fibre $(A \sqcap Z)_a = a \times Z_{f(a)}$ is identified with $Z_{f(a)}$. But, for each $a \in A$, $Z_{f(a)}$ belongs to F because r is an F-space; hence, $r_f: A \sqcap Z \to A$ is an F-space. Now, for each pair $(a,z) \in A \sqcap Z$, $\bar{f}(a,z) = z$ and so $\bar{f}|(A \sqcap Z)_a : Z_{f(a)} \to Z_{f(a)}$ can be identified with $1: Z_{f(a)} \to Z_{f(a)}$ which belongs to F; thus, $(\bar{f}, f)$ is an F-map. To see that $h: p \to r_f$ is an F-map over A, observe that, for each $a \in A$, $h|X_a : X_a \to (A \sqcap Z)_a = a \times Z_{f(a)}$ is defined by $h|X_a(x) = (a, g(x))$ and so can be identified with the map $g|X_a : X_a \to Z_{f(a)}$ which belongs to F. //

Recall that an open covering $A = \{U_\alpha\}_{\alpha\in\Lambda}$ of a space A is said to be numerable if it admits a refinement by a locally finite partition of unity; that is to say, there exists a family of maps $\{\lambda_\gamma : A \to I\}_{\gamma\in\Gamma}$ such that, for each $x \in X$, $\sum_{\gamma\in\Gamma} \lambda_\gamma(x) = 1$, all but a finite number of the λ_γ's vanish outside some neighbourhood of each point of X and the collection $\{\lambda_\gamma^{-1}(0,1]\}_{\gamma\in\Gamma}$ refines A (every set $\lambda_\gamma^{-1}(0,1]$ is contained in some U_α).

Definition 1.2.4. Let $p: X \to A$ be an F-space and let $A = \{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of A. For each $\alpha\in\Lambda$, let $X|U_\alpha$ denote the subspace $p^{-1}(U_\alpha)$ of X. Then p is said to be locally F-trivial if, for each $\alpha\in\Lambda$, $p|(X|U_\alpha): X|U_\alpha \to U_\alpha$ is F-homeomorphic to $pr_1: U_\alpha \times F \to U_\alpha$ over U_α. If, in addition, A is a numerable cover of A, then p is said to be a numerable F-space.

Notice that, in view of [7;Theorem 4.8], every numerable F-space

$p:X \to A$ has the covering homotopy property (CHP) with respect to all topological spaces and hence, with respect to all $\underline{k}$-spaces W; in other words, for every map $g:W \to X$ and every homotopy $H:W\times I \to A$ of $p\cdot g$, there exists a homotopy $\overline{H}:W\times I \to X$ such that $p\cdot\overline{H} = H$ and $\overline{H}|W\times 0 = g$. Hence, numerable F-spaces are Hurewicz fibrations(*).

Proposition 1.2.5. Let $r:Z \to B$ be a numerable F-space and let $f:A \to B$ be a map. Then the induced map $r_f:A\sqcap Z \to A$ is a numerable F-space.

Proof: First observe that, by (1.2.3), $r_f:A\sqcap Z \to A$ is an F-space.

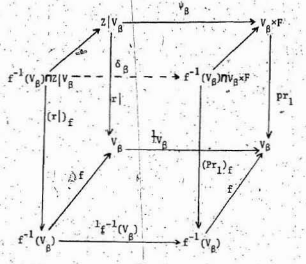
Let $\mathcal{B} = \{V_\beta\}_{\beta\in\Lambda}$ be a numerable cover of B over which r is F-trivial. Then there exists a refinement of $\mathcal{B}$ by a locally finite partition of unity $\{\lambda_\gamma:B \to I\}_{\gamma\in\Gamma}$; furthermore, for each $\beta \in\Lambda$, there exists an F-homeomorphism $\psi_\beta:Z|V_\beta \to V_\beta\times F$ over V_β. Now, let $\mathcal{A} = \{f^{-1}(V_\beta)\}_{\beta\in\Lambda}$ and, for each $\gamma \in \Gamma$, define

$$\pi_\gamma:A \to I$$

to be the composite $\lambda_\gamma\cdot f$. Then $\{\pi_\gamma\}_{\gamma\in\Gamma}$ is a locally finite partition of unity which defines a refinement of $\mathcal{A}$; hence, $\mathcal{A}$ is a numerable cover of A. It remains to show that $r_f:A\sqcap Z \to A$ is F-trivial over each set $f^{-1}(V_\beta)$ belonging to $\mathcal{A}$.

To this end, consider the commutative diagram

(*) Throughout this work, the notion of Hurewicz fibration is always relative to K.



and observe that, for each $\beta \in \Lambda$, ψ_β induces a map

$$\delta_\beta : f^{-1}(V_\beta) \sqcap Z|V_\beta \to f^{-1}(V_\beta) \sqcap V_\beta \times F,$$

defined by $\delta_\beta(a,z) = (a, \psi_\beta(z))$, which completes the above diagram. Now, it is clear from the definition that δ_β is an F-map, in fact, one can easily see that, since ψ_β is an F-homeomorphism over V_β, δ_β is also an F-homeomorphism over $f^{-1}(V_\beta)$ with F-inverse,

$$\delta_\beta^{-1} : f^{-1}(V_\beta) \sqcap V_\beta \times F \to f^{-1}(V_\beta) \sqcap Z|V_\beta,$$

defined by $\delta_\beta^{-1}(a,(b,y)) = (a, \psi_\beta^{-1}(b,y))$. Now observe that

$(r|)_f: f^{-1}(V_\beta) \sqcap Z|V_\beta \to f^{-1}(V_\beta)$ can be identified with $r_f|: A \sqcap Z|f^{-1}(V_\beta) \to f^{-1}(V_\beta)$ and furthermore, $(pr_1)_f: f^{-1}(V_\beta) \sqcap V_\beta \times F \to f^{-1}(V_\beta)$ can be identified with the trivial F-space $pr_1: f^{-1}(V_\beta) \times F \to f^{-1}(V_\beta)$. //

§3. G-spaces and the functional exponential law

Given a category of fibres $(F,\mathbf{F})$, we construct from $\mathbf{F}$ a new category $\mathbf{G}$ as follows: the objects of $\mathbf{G}$ are the function spaces $F(P,Q)$, where P, Q are objects in $\mathbf{F}$; the morphisms between objects $F(P,Q)$ and $F(P',Q')$ are the induced maps $F(\phi,\psi): F(P,Q) \to F(P',Q')$ defined by composition; that is, for all $f \in F(P,Q)$, $F(\phi,\psi)(f) = \psi \cdot f \cdot \phi$, where $\phi \in F(P',P)$ and $\psi \in F(Q,Q')$. Notice that $F(P',P)$ and $F(Q,Q')$ are not empty because of (1.2.2). The distinguished object of $\mathbf{G}$ is the object $G = F(F,F)$.

<u>Proposition</u> 1.3.1. $(G,\mathbf{G})$ is a category of fibres.

<u>Proof:</u> It is clear from the construction that $\mathbf{G}$ is a category. We must show that $\mathbf{G}$ satisfies conditions (i) and (ii) of (1.2.2.).

To this end, let $F(P,Q)$ be an object of $\mathbf{G}$. Since $(F,\mathbf{F})$ is a category of fibres, $F(P,F) \neq \emptyset$ and $F(F,Q) \neq \emptyset$; so, there exists $\eta \in F(P,F)$ and $\psi \in F(F,Q)$ such that $F(\eta,\psi): F(F,F) \to F(P,Q)$. Thus $G(G,F(P,Q)) \neq \emptyset$ and condition (i) of (1.2.2) is satisfied.

To see that (ii) of (1.2.2.) is satisfied, let

$F(\phi,\psi):F(P,Q) \to F(P',Q')$ be a morphism of G, where $\phi \in F(P',P)$ and $\psi \in F(Q,Q')$. Since (F,F) is a category of fibres, ϕ and ψ are F-homotopy equivalences over a point and so, there exist maps $\phi^{-1} \in F(P,P')$ and $\psi^{-1} \in F(Q',Q)$ such that $\phi\cdot\phi^{-1} =_F 1_P$, $\phi^{-1}\cdot\phi =_F 1_{P'}$, $\psi^{-1}\cdot\psi =_F 1_Q$ and $\psi\cdot\psi^{-1} =_F 1_{Q'}$. Now let $h:P\times I \to P$ be an F-homotopy from $\phi\cdot\phi^{-1}$ to 1_P; let $h':P'\times I \to P'$ be an F-homotopy from $\phi^{-1}\cdot\phi$ to $1_{P'}$; let $k:Q\times I \to Q$ be an F-homotopy from $\psi^{-1}\cdot\psi$ to 1_Q; and let $k':Q'\times I \to Q'$ be an F-homotopy from $\psi\cdot\psi^{-1}$ to $1_{Q'}$. Then define

$$H: F(P,Q)\times I \to F(P,Q) \text{ and } H': F(P',Q')\times I \to F(P',Q')$$

by $H(f,t) = k_t\cdot f\cdot h_t$ and $H'(g,t) = k'_t\cdot g\cdot h'_t$. Since the adjoints of H and H' can be identified with the composites

$$F(P,Q)\times P\times I \xrightarrow{1_{F(P,Q)}\times 1_P\times\Delta} F(P,Q)\times P\times I\times I \xrightarrow{1_{F(P,Q)}\times h\times 1_I} F(P,Q)\times P\times I \xrightarrow{e\times 1_I} Q\times I \xrightarrow{k} Q$$

and $F(P',Q')\times P'\times I \xrightarrow{1_{F(P',Q')}\times 1_{P'}\times\Delta} F(P',Q')\times P'\times I\times I \xrightarrow{1_{F(P',Q')}\times k\times 1_I}$ $F(P',Q')\times P'\times I \xrightarrow{e'\times 1_I} Q'\times I \xrightarrow{k'} Q'$, respectively (here $\Delta:I \to I\times I$ is the diagonal map and e and e' are evaluation maps), H and H' are continuous; furthermore, because $H_t = F(h_t,k_t)$ and $H'_t = F(h'_t,k'_t)$, for each $t \in I$, H and H' are G-homotopies over a point. Now observe that $H(-,0) = F(\phi\cdot\phi^{-1},\psi^{-1}\cdot\psi) = F(\phi^{-1},\psi^{-1})\cdot F(\phi,\psi)$, $H(-,1) = F(1_P,1_Q) = 1_{F(P,Q)}$ and $H'(-,0) = F(\phi^{-1}\cdot\phi,\psi\cdot\psi^{-1}) = F(\phi,\psi)\cdot F(\phi^{-1},\psi^{-1})$, $H'(-,1) = F(1_{P'},1_{Q'}) = 1_{F(P',Q')}$; hence, $F(\phi,\psi)$ and $F(\phi^{-1},\psi^{-1})$ are inverse G-homotopy equivalences. //

Observe that if we restrict the category G above to the full subcategory, with objects $F(F,P)$, for P an object of F, then we again get a category of fibres, denoted (G^*,G), which we call the <u>associated principal category of fibres</u>.

We now construct an important type of G-space, as given in [4], and examine some of its properties.

Let Z be a space and define Z^+ to be the set $Z \cup \{\infty\}$ topologized as the <u>k</u>-ification of the following topology: $C \subset Z^+$ is closed if, and only if, $C = Z^+$ or C is closed in Z. Then, any map $f:A \to Z$ defined on a closed subspace A of Y can be identified with a map $\tilde{f}:Y \to Z^+$ defined by

$$\tilde{f}(y) = \begin{cases} f(y), & \text{if } y \in A \\ \infty, & \text{otherwise.} \end{cases}$$

Let $q:Y \to A$ and $r:Z \to B$ be F-spaces, where A is a T_1-space (every singleton set is closed). Form the set

$$Y*Z = \bigcup_{a\in A,\, b\in B} F(Y_a, Z_b)$$

and define a function $j:Y*Z \to L(Y,Z^+)$ by $j(f)(y) = f(y)$, if $y \in Y_a$, $f:Y_a \to Z_b$, and $j(f)(y) = \infty$ otherwise. The condition that A is T_1 ensures, that each $f \in Y*Z$ has a closed domain when considered as a partial map from Y to Z. We topologize $Y*Z$ with the <u>k</u>-ification of the initial topology with respect to j and the function $q*r:Y*Z \to A\times B$, defined by $q*r(f:Y_a \to Z_b) = (a,b)$.

Notice that, because $L(Y,Z^+)$ and $A\times B$ are k-spaces, the k-ified initial topology on $Y*Z$ is just the initial topology in K with respect to j and $q*r$. Furthermore, if for each $a \in A$ and $b \in B$ we consider the constant F-spaces $q|Y_a:Y_a \to a$ and $r|Z_b:Z_b \to b$, then the fibre of $q*r$ over the point $(a,b) \in A\times B$ is the space Y_a*Z_b of all F-homotopy equivalences $Y_a \to Z_b$ over a point, topologized with the initial topology(*) with respect to the function $j:Y_a*Z_b \to L(Y,Z^+)$.

Proposition 1.3.2. $q*r:Y*Z \to A\times B$ is a G-space.

Proof: For each point $(a,b) \in A \times B$ we must show that Y_a*Z_b is homeomorphic to $F(Y_a,Z_b)$. Now, the identification of the underlying sets is clear. To see that the topologies are the same, observe that, for any space W, a function $g:W \to Y_a*Z_b$ is continuous if, and only if, the same function $g:W \to F(Y_a,Z_b)$ is continuous. //

We call $q*r:Y*Z \to A\times B$ a functional G-space and denote by $q*_1r$ and $q*_2r$ the composites $Y*Z \xrightarrow{q*r} A \times B \xrightarrow{pr_1} A$ and $Y*Z \xrightarrow{q*r} A \times B \xrightarrow{pr_2} B$, respectively.

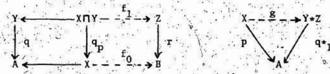
Theorem 1.3.3. (Functional Exponential Law; [4; Lemma 1.2]). Let $p:X \to A$, $q:Y \to A$ and $r:Z \to B$ be F-spaces, where A is Hausdorff. The function that assigns to each F-map $(f_1,f_0):q_p \to r$, the fibre

(*) Throughout this work, the notion of initial topology will always be relative to K.

preserving map $g:p \to q*_1 r$ over A, defined by $g(x)(y) = f_1(x,y)$, is a homeomorphism

$$\theta: F(q_p, r) \to L_1(p, q*_1 r)$$

Here $L_1(p, q*_1 r)$ denotes the space of all fibre preserving maps $p \to q*_1 r$ that cover $1:A \to A$, topologized as a subspace of $L(X, Y*Z)$. The Theorem is illustrated by the following diagram:



Observe that each $g:X \to Y*Z$ over A determines the corresponding $f_0:X \to B$ as the composite $q*_2 r \cdot g$.

<u>Proof of 1.3.3.</u>: Let $(f_1, f_0):q_p \to r$ be an F-map. Then $f_1:X\sqcap Y \to Z$ is such that, for each $x \in X$, $f_1|(X\sqcap Y)_x:Y_{p(x)} \to Z_{f_0(x)}$ belongs to $F(Y_{p(x)}, Z_{f_0(x)})$. Thus, for each $x \in X$, $\theta(f_1)(x):Y_{p(x)} \to Z_{f_0(x)}$ belongs to $Y*Z$ and so $\theta(f_1):X \to Y*Z$ is a function over A. Because $Y*Z$ has the initial topology with respect to $q*r$ and j, the continuity of $\theta(f_1)$ is equivalent to the continuity of the composites $q*r\cdot\theta(f_1)$ and $j\cdot\theta(f_1)$. But, since for each $x \in X$, $\theta(f_1)(x):Y_{p(x)} \to Z_{f_0(x)}$, $q*r\cdot\theta(f_1)(x) = (p(x), f_0(x)) = (p, f_0)(x)$ and hence, $q*r\cdot\theta(f_1)$ is continuous. To see that $j\cdot\theta(f_1)$ is continuous, observe that the

condition that A is Hausdorff ensures that $X \sqcap Y$ is closed in $X \times Y$ and hence, $f_1 : X \sqcap Y \to Z$ can be identified with the map $\tilde{f}_1 : X \times Y \to Z^+$, defined by

$$\tilde{f}_1(x,y) = \begin{cases} f_1(x,y), & \text{if } (x,y) \in X \sqcap Y \\ \infty & \text{, otherwise.} \end{cases}$$

But, $\tilde{f}_1$ corresponds, by (1.1.1.), to a map $\theta(\tilde{f}_1) : X \to L(Y, Z^+)$, defined by

$$\theta(\tilde{f}_1)(x)(y) = \begin{cases} f_1(x,y), & \text{if } (x,y) \in X \sqcap Y \\ \infty & \text{, otherwise.} \end{cases}$$

Now observe that $\theta(\tilde{f}_1) = j \cdot \theta(f_1)$.

The injectivity of θ is immediate. To see that θ is surjective, let $g : X \to Y * Z$ be a map over A. Then, the composite $j \cdot g : X \to L(Y, Z^+)$ is continuous and corresponds, by (1.1.1.), to a map $\theta^{-1}(j \cdot g) : X \times Y \to Z^+$, defined by

$$\theta^{-1}(j \cdot g)(x,y) = \begin{cases} g(x)(y), & \text{if } (x,y) \in X \sqcap Y \\ \infty & \text{, otherwise} \end{cases}$$

But, $\theta^{-1}(j \cdot g)$ can be identified to a map $f_1 : X \sqcap Y \to Z$, defined by $f_1(x,y) = g(x)(y)$. Now observe that $(f_1, q *_2 r \cdot g) : q_p \to r$ is an F-map with $\theta(f_1) = g$.

The fact that θ is a homeomorphism follows from the commutative diagram

$$\begin{array}{ccc} F(q_p,r) & \xrightarrow{\theta} & L_1(p,q*_1r) \\ \downarrow \lambda & & \downarrow j_* \\ L(X\times Y,Z^+) & \xrightarrow{\theta} & L(X,L(Y,Z^+)) \end{array},$$

where $\lambda: F(q_p,r) \to L(X\times Y,Z^+)$ is defined by $\lambda(f_1) = \tilde{f}_1$ and $j_*: L_1(p,q*_1r) \to L(X,L(Y,Z^+))$ is defined by $j_*(g) = j\cdot g$, and the observation that $F(q_p,r)$ has the initial topology with respect to λ and $L_1(p,q*_1r)$ has the initial topology with respect to j_*. //

Let $p:X \to A$ be an F-space. By a <u>section</u> to p we mean a map $s:A \to X$ such that $p\cdot s = 1_A$. We denote by $\sec p$, the space of all sections to p, topologized as a subspace of $L(A,X)$. Given two sections s and s' to p, we say that s is <u>vertically homotopic</u> to s' if there exists a homotopy $H:A\times I \to X$ such that $H(-,0) = s$, $H(-,1) = s'$ and $H(-,t) \in \sec p$, for all $t \in I$.

Taking p to be the identity on A in (1.3.3.), we obtain

<u>Corollary</u> 1.3.4. The function that assigns to each F-map $(f_1,f_0):q \to r$, the section s to $q*_1r$, defined by $s(a)(y) = f_1(y)$, $y \in Y_a$, is a homeomorphism

$$\theta: F(q,r) \to \sec q*_1r$$

Taking p to be the projection map $pr_1:A\times I \to A$ in (1.3.3.) we obtain

Corollary 1.3.5. A pair of F-maps (f_1,f_0) and (g_1,g_0) from q to r are F-homotopic if, and only if, their corresponding sections to $q*_1r$ are vertically homotopic.

Corollary 1.3.6. Given $a \in A$, the fibre $Y_a * Z$ of $q*_1r$ over a is a subspace of $L(Y_a, Z)$. More precisely, it is the subspace $F(c,r)$, where $c:Y_a \to a$.

Proof: Observe that $Y_a*Z \cong \sec c*_1r$, where $c*_1r:Y_a*Z \to a$. But by (1.3.4.), $\sec c*_1r \cong F(c,r) \subset L(Y_a,Z)$. //

Let $r:Z \to B$ be an F-space and let $f:A \to B$ be a map. By a lift of f over r we mean a map $g:A \to Z$ such that $r \cdot g = f$. We denote by $\mathrm{Lift}(f,r)$, the space of all lifts of f over r, topologized as a subspace of $L(A,Z)$. Given two lifts g and g' of f over r, we say that g is vertically homotopic to g' if there exists a homotopy $H:A\times I \to Z$ such that $H(-,0) = g$, $H(-,1) = g'$ and $H(-,t) \in \mathrm{Lift}(f,r)$, for all $t \in I$.

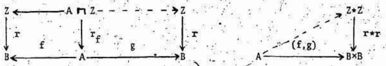
Proposition 1.3.7. Let $r:Z \to B$ be an F-space and let $f, g:A \to B$ be maps. Then there is a homeomorphism between (i) the space of all lifts of $(f,g):A \to B\times B$ over $r*r:Z*Z \to B\times B$ and (ii) the space of all F-maps from $r_f:A \sqcap Z \to A$ to $r_g:A \sqcap Z \to A$ over A.

Here $(f,g):A \to B\times B$ is defined by $(f,g)(a) = (f(a),g(a))$.

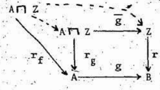
Proof: By setting $q = r:Z \to B$, $p = f:A \to B$ and $f_0 = g:A \to B$ in (1.3.3.), we obtain a homeomorphism

$$\theta: F_g(r_f, r) \to \mathrm{Lift}((f,g), r*r),$$

as illustrated by the following diagram:



Now, consider the following pullback diagram,



and observe that, by the universal property of pullbacks (see (1.2.3.)), there exists a homeomorphism $\phi: F_g(r_f, r) \to F_1(r_f, r_g)$. The required homeomorphism is now given by the composite $\phi \cdot \theta^{-1}$. //

As an immediate consequence of (1.3.7.) we have the following result:

Corollary 1.3.8. Any two F-maps $r_f \to r_g$ over A are F-homotopic if, and only if, their corresponding lifts of (f,g) over $r*r$ are vertically homotopic.

In the remainder of this section we shall discuss some results, which will be needed in the sequel, about functional G-spaces and, in particular, about induced functional G-spaces.

Proposition 1.3.9. Let $q:Y \to A$ and $r:Z \to B$ be F-spaces and let $f:A' \to A$ and $g:B' \to B$ be maps. Form the induced F-spaces $q_f:A' \sqcap Y \to A'$ and $r_g:B' \sqcap Z \to B'$. Then the functional G-space $q_f * r_g:(A' \sqcap Y)*(B' \sqcap Z) \to A' \times B'$ is G-homeomorphic over $A' \times B'$ to the G-space induced from $q*r:Y*Z \to A \times B$ by the map $f \times g:A' \times B' \to A \times B$.

Proof: Recall that $(A' \sqcap Y)*(B' \sqcap Z) = \bigcup_{a' \in A', b' \in B'} F(a' \times Y_{f(a')}, b' \times Z_{g(b')}) = \bigcup_{a' \in A', b' \in B'} (a', b') \times F(Y_{f(a')}, Z_{g(b')})$. But,

$(A' \times B') \sqcap Y*Z = \{((a',b'),h) \mid q*r(h) = (f(a'), g(b'))\} = \bigcup_{a' \in A', b' \in B'} (a', b') \times F(Y_{f(a')}, Z_{g(b')})$; hence, as sets $(A' \sqcap Y)*(B' \sqcap Z)$ and $(A' \times B') \sqcap Y*Z$ coincide. To see that the topologies are the same, observe that it is sufficient to show that, for any space W, a function $W \to (A' \sqcap Y)*(B' \sqcap Z)$ is continuous if, and only if, the same function $W \to (A' \times B') \sqcap Y*Z$ is continuous. So, let $\psi:W \to (A' \sqcap Y)*(B' \sqcap Z)$ be a function and assume that ψ is continuous. Then, by (1.3.3.) and (1.2.3.), we obtain the following commutative diagram of F-maps

$$\begin{array}{ccccccccc}
Y & \xleftarrow{\bar{f}} & A' \sqcap Y & \xleftarrow{\bar{k}_1} & W \sqcap (A' \sqcap Y) & \xrightarrow{\psi'} & B' \sqcap Z & \xrightarrow{\bar{g}} & Z \\
\downarrow q & & \downarrow q_f & & \downarrow (q_f)_{k_1} & & \downarrow r_g & & \downarrow r \\
A & \xleftarrow{f} & A' & \xleftarrow{k_1} & W & \xrightarrow{k_2} & B' & \xrightarrow{g} & B
\end{array},$$

where $k_1 = (q_f{}^*{}_1 r_g)\cdot\psi$, $k_2 = (q_f{}^*{}_2 r_g)\cdot\psi$ and $\psi'(w,(a',y)) = \psi(w)(a',y)$. Applying (1.3.3.) to the F-map $(\bar{g}\cdot\psi', g\cdot k_2):(q_f)_{k_1} \to r$, we now obtain a map $\phi: W \to Y{\star}Z$ which factors as the composite $W \xrightarrow{\psi} (A'\times B')\sqcap Y{\star}Z \xrightarrow{\overline{f\times g}} Y{\star}Z$. But, since $(q{\star}r)_{f\times g}\cdot\psi = (k_1,k_2)$, the following diagram commutes

$$\begin{array}{ccccc} W & \xrightarrow{\psi} & (A'\times B')\sqcap Y{\star}Z & \xrightarrow{\overline{f\times g}} & Y{\star}Z \\ & \searrow^{(k_1,k_2)} & \downarrow{\scriptstyle (q\star r)_{f\times g}} & & \downarrow{\scriptstyle q\star r} \\ & & A'\times B' & \xrightarrow{f\times g} & A\times B \end{array}$$

and hence, by the universal property of pullbacks, $\psi: W \to (A'\times B')\sqcap Y{\star}Z$ is continuous. This argument is clearly reversible. //

Proposition 1.3.10. Let $q:Y \to A$, $q':Y' \to A$, $r:Z \to B$ and $r':Z' \to B$ be F-spaces such that q is F-homotopy equivalent to q' over A and r is F-homotopy equivalent to r' over B. Then $q{\star}r: Y{\star}Z \to A\times B$ is G-homotopy equivalent to $q'{\star}r': Y'{\star}Z' \to A\times B$ over $A\times B$.

Proof: Let $f:Y \to Y'$ and $g:Z \to Z'$ be F-homotopy equivalences over A and B, respectively. Then there exist F-maps $f^{-1}:Y' \to Y$ and $g^{-1}:Z' \to Z$ over A and B, respectively, such that $f^{-1}\cdot f \simeq_F 1_Y$ over A, $f\cdot f^{-1} \simeq_F 1_{Y'}$ over A, $g^{-1}\cdot g \simeq_F 1_Z$ over B and $g\cdot g^{-1} \simeq_F 1_{Z'}$ over B. So, let $(h,pr_1):q\times 1_I \to q$ be an F-homotopy from $f^{-1}\cdot f$ to 1_Y; let $(h',pr_1):q'\times 1_I \to q'$ be an

F-homotopy from $f\cdot f^{-1}$ to 1_Y; let $(k,pr_1):r\times 1_I \to r$ be an F-homotopy from $g^{-1}\cdot g$ to 1_Z; and let $(k',pr_1):r'\times 1_I \to r'$ be an F-homotopy from $g\cdot g^{-1}$ to $1_{Z'}$. Now, define a function $F(f^{-1},g):Y\star Z \to Y'\star Z'$ over $A\times B$ by the rule $F(f^{-1},g)(\psi:Y_a \to Z_b) = g|Z_b\cdot\psi\cdot f^{-1}|Y'_a$. Then, for each pair $(a,b) \in A\times B$, $F(f^{-1},g)|F(Y_a,Z_b) = F(f^{-1}|Y'_a, g|Z_b)$ which belongs to $G(F(Y_a,Z_b),F(Y'_a,Z'_b))$; hence, $F(f^{-1},g)$ is a G-function over $A\times B$. Since $Y'\star Z'$ has the initial topology with respect to $q'\star r':Y'\star Z' \to A\times B$ and $j':Y'\star Z' \to L(Y,(Z')^+)$, $F(f^{-1},g)$ is continuous if, and only if, the composites $(q'\star r')\cdot F(f^{-1},g)$ and $j'\cdot F(f^{-1},g)$ are continuous. But, $(q'\star r')\cdot F(f^{-1}.g) = q\star r$ which is continuous. To see that $j'\cdot F(f^{-1},g)$ is continuous, consider the following commutative diagram

$$\begin{array}{ccc} Y\star Z & \xrightarrow{F(f^{-1},g)} & Y'\star Z' \\ \downarrow j & & \downarrow j' \\ L(Y,Z^+) & \xrightarrow{L(f^{-1},g^+)} & L(Y',(Z')^+) \end{array}$$

where $g^+:Z^+ \to (Z')^+$ is the map induced by $g:Z \to Z'$ and $L(f^{-1},g^+)$ is defined by composition of functions, and observe that the composite $L(f^{-1},g^+)\cdot j$ is continuous; hence, $F(f^{-1},g)$ is a G-map over $A\times B$. In a similar manner, we show that $F(f,g^{-1})$: $Y'\star Z' \to Y\star Z$ is a G-map over $A\times B$. Now define

$$H:Y\star Z\times I \to Y\star Z \quad \text{and} \quad H':Y'\star Z'\times I \to Y'\star Z'$$

by $H(\psi,t) = F(h_t,k_t)(\psi)$ and $H'(\psi',t) = F(h'_t,k'_t)(\psi')$. Since, for each $t \in I$, h_t and h'_t are F-maps over A and k_t and k'_t are F-maps over B, $F(h_t,k_t):Y*Z \to Y*Z$ and $F(h'_t,k'_t):Y'*Z' \to Y'*Z'$ are G-maps over $A\times B$; hence, the following diagrams commute

$$\begin{array}{ccc} Y*Z\times I & \xrightarrow{H} & Y*Z \\ \downarrow{\scriptstyle q*r\times 1_I} & & \downarrow{\scriptstyle q*r} \\ (A\times B)\times I & \xrightarrow{pr_1} & A\times B \end{array} \qquad \begin{array}{ccc} Y'*Z'\times I & \xrightarrow{H'} & Y'*Z' \\ \downarrow{\scriptstyle q'*r'\times 1_I} & & \downarrow{\scriptstyle q'*r'} \\ (A\times B)\times I & \xrightarrow{pr_1} & A\times B \end{array}$$

and so, $(H,pr_1):q*r\times 1_I \to q*r$ and $(H',pr_1):q'*r'\times 1_I \to q'*r'$ are G-functions. Now, because of the topology on $Y*Z$, H is continuous if, and only if, $q*r\cdot H$ and $j\cdot H$ are continuous. But, $q*r\cdot H = pr_1\cdot(q*r\times 1_I)$ which is continuous; furthermore, since the adjoint of $j\cdot H$ can be identified with the composite

$$Y*Z\times Y\times I \xrightarrow{j\times 1_Y\times\Delta} L(Y,Z^+)\times Y\times I\times I \xrightarrow{1_{L(Y,Z^+)}\times h\times 1_I} L(Y,Z^+)\times Y\times I \xrightarrow{e\times 1_I} Z^+\times I \xrightarrow{k^+} Z^+$$

(here, $\Delta:I \to I\times I$ is the diagonal map, e is the evaluation map and k^+ is the map induced by $k:Z\times I \to Z$), $j\cdot H$ is continuous and hence, (H,pr_1) is a G-homotopy. In a similar manner, we show that H' is continuous and hence, (H',pr_1) is also a G-homotopy. Now observe that $H(-,0) = F(f^{-1}\cdot f,g^{-1}\cdot g) = F(f,g^{-1})\cdot F(f^{-1},g)$, $H(-,1) = F(1_Y,1_Z) = 1_{Y*Z}$ and $H'(-,0) = F(f\cdot f^{-1},g\cdot g^{-1}) = F(f^{-1},g)\cdot F(f,g^{-1})$, $H'(-,1) = F(1_{Y'},1_{Z'}) = 1_{Y'*Z'}$; hence $F(f^{-1},g)$ and $F(f,g^{-1})$ are inverse G-homotopy equivalences over $A\times B$. //

By adapting the proof of (1.3.10.) to the notion of F-homeomorphism we have the following result.

Proposition 1.3.11. Let $q:Y \to A$, $q':Y' \to A$, $r:Z \to B$ and $r':Z' \to B$ be F-spaces such that q is F-homeomorphic to q' over A and r is F-homeomorphic to r' over B. Then $q*r:Y*Z \to A\times B$ is G-homeomorphic to $q'*r':Y'*Z' \to A\times B$ over $A\times B$.

Proposition 1.3.12. If $q:Y \to A$ and $r:Z \to B$ are trivial F-spaces, then $q*r:Y*Z \to A\times B$ is a trivial G-space.

Proof: *Since* q *is F-homeomorphic to* $pr_1:A\times F \to A$ *over* A and r is F-homeomorphic to $pr_1:B\times F \to B$ over B, by (1.3.11), $q*r$ is G-homeomorphic to pr_1*pr_1 over $A\times B$. The result now follows from (1.3.9.) and the observation that both $pr_1:A\times F \to A$ and $pr_1:B\times F \to B$ are induced from $F \to *$ by constant maps. //

Proposition 1.3.13. Let $q:Y \to A$ and $r:Z \to B$ be numerable F-spaces. Then $q*r:Y*Z \to A\times B$ is a numerable G-space.

Proof: Let $\mathcal{A} = \{U_\alpha\}_{\alpha\in\Lambda}$ be a numerable cover of A over which q is F-trivial and let $\mathcal{B} = \{V_\beta\}_{\beta\in\Lambda'}$ be a numerable cover of B over which r is F-trivial. Then there exist refinements of $\mathcal{A}$ and $\mathcal{B}$ by locally finite partitions of unity $\{\lambda_\gamma:A \to I\}_{\gamma\in\Gamma}$ and $\{\eta_\delta:B \to I\}_{\delta\in\Gamma'}$, respectively. Now, let $\mathcal{C} = \{U_\alpha\times V_\beta\}_{(\alpha,\beta)\in\Lambda\times\Lambda'}$ and, for each pair $(\gamma,\delta)\in\Gamma\times\Gamma'$, define

$$\pi_{(\gamma,\delta)}:A\times B \to I$$

to be the composite $A\times B \xrightarrow{\lambda_\gamma \times \eta_\delta} I\times I \xrightarrow{\nu} I$, where ν denotes multiplication of real numbers. Then $\{\pi_{(\gamma,\delta)}\}_{(\gamma,\delta)\in\Gamma\times\Gamma'}$ is a locally finite partition of unity which defines a refinement of C; hence, C is a numerable cover of $A\times B$.

To see that $q*r$ is G-trivial over each set $U_\alpha\times V_\beta$ in C, observe that since $q|(Y|U_\alpha):Y|U_\alpha \to U_\alpha$ is F-homeomorphic to $pr_1:U_\alpha\times F \to U_\alpha$ over U_α and $r|(Z|V_\beta):Z|V_\beta \to V_\beta$ is F-homeomorphic to $pr_1:V_\beta\times F \to V_\beta$ over V_β, by (1.3.12.), $q|(Y|U_\alpha)*r|(Z|V_\beta):Y|U_\alpha*Z|V_\beta \to U_\alpha\times V_\beta$ is a trivial G-space and hence, G-homeomorphic to $pr_1:(U_\alpha\times V_\beta)\times F*F \to U_\alpha\times V_\beta$ over $U_\alpha\times V_\beta$. Now notice that $Y|U_\alpha*Z|V_\beta = Y*Z|U_\alpha\times V_\beta$ and $q|(Y|U_\alpha)*r(Z|V_\beta) = q*r|(Y*Z|U_\alpha\times V_\beta)$. //

Notice that, if $\phi_\alpha:U_\alpha\times F \to Y|U_\alpha$ is an F-homeomorphism over U_α and $\psi_\beta:V_\beta\times F \to Z|V_\beta$ is an F-homeomorphism over V_β, then a G-homeomorphism $\theta_{(\alpha,\beta)}:(U_\alpha\times V_\beta)\times F*F \to Y*Z|U_\alpha\times V_\beta$ can be explicitly given by the formula

$$\theta_{(\alpha,\beta)}((a,b),f) = \psi_{\beta,b}\cdot f\cdot\phi^{-1}_{\alpha,a}\ ;$$

where $\phi^{-1}_{\alpha,a}$ denotes the restriction $\phi^{-1}_\alpha|Y_a:Y_a \to F$ and $\psi_{\beta,b}$ denotes the restriction $\psi_\beta|F:F \to Z_b$. Hence, as a consequence of (1.3.13.), we have the following result.

Corollary 1.3.14. If $q:Y \to A$ and $r:Z \to B$ are numerable F-spaces, then the initial topology on $Y*Z$ with respect to the functions $j:Y*Z \to L(Y,Z^+)$ and $q*r:Y*Z \to A\times B$ coincides with the final topology on $Y*Z$ with respect to the injections $\theta_{(\alpha,\beta)}:(U_\alpha\times V_\beta)\times F*F \to Y*Z$,

$(\alpha,\beta)\epsilon\Lambda\times\Lambda'$. Here, the notion of final topology is relative to K.

By considering F-spaces with the same base space we can construct a very special type of G-space whose properties are closely related to those of functional G-spaces. More precisely, if $q:Y\to B$ and $r:Z\to B$ are F-spaces, we define

$$(qr)_F:(YZ)_F\to B$$

to be the G-space induced from $q\star r:Y\star Z\to B\times B$ by the diagonal map $\Delta:B\to B\times B$; thus, $(YZ)_F=\bigcup_{b\epsilon B}F(Y_b,Z_b)$, topologized with the initial topology with respect to $j:(YZ)_F\to L(Y,Z^+)$ and the projection map $(qr)_F:(YZ)_F\to B$. Notice that $(YZ)_F$ is a subspace of the fibred mapping space (YZ) defined in [2]. We shall refer to $(qr)_F:(YZ)_F\to B$ as a functional G-space.

If we set $A=B$ and $f_0=p:X\to B$ in (1.3.3.) we get the following particular version of the Functional Exponential Law.

Theorem 1.3.15. Let $p:X\to B$, $q:Y\to B$ and $r:Z\to B$ be F-spaces, where B is Hausdorff. The function that assigns to each F-map $(f,p):q_p\to r$, the fibre preserving map $g:p\to(qr)_F$ over B, defined by $g(x)(y)=f(x,y)$, is a homeomorphism

$$\theta:F_p(q_p,r)\to L_1(p,(qr)_F).$$

Setting $p=1:B\to B$ in (1.3.15.), we obtain

Corollary 1.3.16. The function that assigns to each F-map $f:q\to r$ over B, the section s to $(qr)_F$, defined by

$s(b)(y) = f(y)$, $y \in Y_b$, is a homeomorphism

$$\theta : F_1(q,r) \to \sec(qr)_F$$

Setting $p = pr_1 : B\times I \to B$ in (1.3.15.), we obtain

Corollary 1.3.17. A pair of F-maps $q \to r$ over B are F-homotopic if, and only if, their corresponding sections to $(qr)_F$ are vertically homotopic.

The following result is a consequence of (1.3.9).

Proposition 1.3.18. Let $q:Y \to B$ and $r:Z \to B$ be F-spaces and let $f:A \to B$ be a map. Then the functional G-space $(q_f r_f)_F : (A\sqcap Y\ A\sqcap Z)_F \to A$ is G-homeomorphic over A to the G-space induced from $(qr)_F$ by the map $f:A \to B$.

Proof: Consider the following commutative diagram

$$\begin{array}{ccccc}
(A\sqcap Y\ A\sqcap Z)_F & \longrightarrow & (A\sqcap Y)*(A\sqcap Z) & \longrightarrow & Y*Z \\
\downarrow {\scriptstyle (q_f r_f)_F} & & \downarrow {\scriptstyle q_f * r_f} & & \downarrow {\scriptstyle q*r} \\
A & \xrightarrow{\ \Delta\ } & A\times A & \xrightarrow{\ f\times f\ } & B\times B
\end{array}$$

and observe that, by (1.3.9.), the right square is a pullback diagram and by definition, so is the left square; hence, the composite square is a pullback diagram. But, observe that the following commutative diagram

$$\begin{array}{ccccc} A\sqcap(YZ)_F & \longrightarrow & (YZ)_F & \longrightarrow & Y*Z \\ \downarrow{\scriptstyle ((qr)_F)_f} & & \downarrow{\scriptstyle (qr)_F} & & \downarrow{\scriptstyle q*r} \\ A & \xrightarrow{\ f\ } & B & \xrightarrow{\ \Delta\ } & B\times B \end{array}$$

is also a composition of pullback diagrams and $\Delta\cdot f = (f\times f)\cdot\Delta$; hence, by uniqueness of induced G-spaces, $(q_f r_f)_F$ is G-homeomorphic to $((qr)_F)_f$ over A. //

The next result is proved in the same manner as (1.3.10.).

Proposition 1.3.19. Let $q:Y \to B$, $q':Y' \to B$, $r:Z \to B$ and $r':Z' \to B$ be F-spaces such that q is F-homotopy equivalent to q' over B and r is F-homotopy equivalent to r' over B. Then $(qr)_F:(YZ)_F \to B$ is G-homotopy equivalent to $(q'r')_F:(Y'Z')_F \to B$ over B.

The following result is a consequence of (1.3.12) and the definition of $(qr)_F$ as the pullback of $q*r$ over the diagonal map Δ.

Proposition 1.3.20. If $q:Y \to B$ and $r:Z \to B$ are trivial F-spaces, then $(qr)_F:(YZ)_F \to B$ is a trivial G-space.

From (1.3.13.) and (1.2.5.) we obtain the following result.

Proposition 1.3.21. If $q:Y \to B$ and $r:Z \to B$ are numerable F-spaces, then $(qr)_F:(YZ)_F \to B$ is a numerable G-space.

Hence, as an immediate consequence, we have

Corollary 1.3.22. If $q:Y \to B$ and $r:Z \to B$ are numerable F-spaces, then the initial topology on $(YZ)_F$ with respect to the functions $j:(YZ)_F \to L(Y,Z^{\dagger})$ and $(qr)_F:(YZ)_F \to B$ coincides with the final topology on $(YZ)_F$ with respect to the injections $\theta_\beta:\mathring{U}_\beta \times F*F \to (YZ)_F$, $\beta\in\Lambda'$.

Note: Many of the results presented in this chapter have already been published elsewhere, but for completeness we have included all proofs. Theorem 1.3.3. and its subsequent Corollaries 1.3.4., 1.3.5. and 1.3.6. can be found in [4] along with Proposition 1.3.7. Analogous statements of Corollary 1.3.14., Theorem 1.3.15. and Corollary 1.3.16. for more general functional bundle constructions can be found in [2] and [3].

II. ADMISSIBLE CATEGORIES OF FIBRATIONS

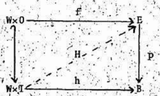
§1. F-fibrations

We remind the reader that all spaces and maps considered are objects and morphisms of the convenient category K.

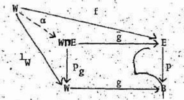
Definition 2.1.1. An F-space $p:E \to B$ is said to have the F-covering homotopy property (abbreviated F-CHP) with respect to an F-space $q:Y \to A$ if, for every F-map $(f_1, f_0):q \to p$ and every homotopy $h:A\times I \to B$ of f_0, there exists a homotopy $H:Y\times I \to E$ of f_1 such that (H,h) is an F-homotopy. If $p:E \to B$ has the F-covering homotopy property with respect to all F-spaces, then p is called an F-fibration [14].

Proposition 2.1.2. Every F-fibration is a Hurewicz fibration.

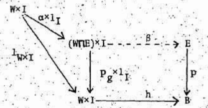
Proof: Let $p:E \to B$ be an F-fibration and let W be any space. Given a map $f:W \to E$ and a homotopy $h:W\times I \to B$ of $g = p\cdot f$ we must show that there exists a homotopy $H:W\times I \to E$ such that the resulting triangles commute:



To this end, consider the following pullback diagram



and observe that, by (1.2.3), p_g:WΠE → W is an F-space and $(\bar{g},g):p_g \to p$ is an F-map. Now, 1_W:W → W is not, in general, an F-space but nonetheless, by the universal property of pullbacks in K, there does exist a unique map α:W → WΠE such that $p_g \cdot \alpha = 1_W$ and $\bar{g}\cdot\alpha = f$. This gives rise to the following diagram



which can be completed, because p is an F-fibration, by an F-map β:(WΠE)×I → E such that β(-,o) = $\bar{g}$. Now set H = β· $(\alpha\times 1_I)$ and observe that H has the required properties. //

Although the notion of an F-fibration is, by (2.1.2.), a generalization of the covering homotopy property in the category of k-spaces, it is difficult, in general, to recognize which F-spaces possess this property. The following theorem will prove very useful in this regard.

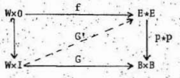
<u>Theorem</u> 2.1.3. The following statements are equivalent:

(*i*) $p:E \to B$ is an F-fibration.

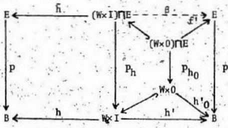
(*ii*) p has the F-covering homotopy property with respect to all F-spaces induced from p.

(*iii*) $p*p:E*E \to B\times B$ is a Hurewicz fibration.

<u>Proof</u>: (*i*)$\Rightarrow$(*iii*): Let W be any space. Given a map $f:W \to E*E$ and a homotopy $G:W\times I \to B\times B$ of $p*p\cdot f$, we must show that there exists a homotopy $G':W\times I \to E*E$ such that the following triangles commute:



To this end, let $G = (h,h')$, where h is the composite $W\times I \overset{G}{\to} B\times B \overset{pr_1}{\to} B$ and h' is the composite $W\times I \overset{G}{\to} B\times B \overset{pr_2}{\to} B$. By (1.3.3.), the completion of the above diagram is equivalent to the completion of the following diagram



where $h_0 = h(-,0) = p*_1 p \cdot f$, $h'_0 = h'(-,0) = p*_2 p \cdot f$ and $f'(w,0,y) = f(w,0)(y)$, $w \in W$, $y \in E$. Now, since p is an F-fibration, the homotopy $h' : W \times I \to B$ of h'_0 can be lifted to a homotopy $H' : (W \sqcap E) \times I \to E$ of f' such that $(H', h') : p_{h_0} \times 1_I \to p$ is an F-homotopy. We shall show that there exists an F-map $\alpha : (W \times I) \sqcap E \to (W \sqcap E) \times I$ over $W \times I$ such that $\alpha | (W \times 0) \sqcap E = 1_{(W \times 0) \sqcap E}$. Then set $\beta = H' \cdot \alpha$ and observe that β has the required properties.

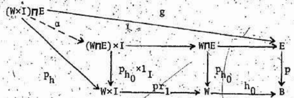
We construct the F-map α in the following way. Consider the pullback diagram

$$\begin{array}{ccc} (W \times I) \sqcap E & \xrightarrow{\bar{h}} & E \\ \downarrow p_h & & \downarrow p \\ W \times I & \xrightarrow{h} & B \end{array}$$

and define a homotopy $h^* : W \times I \times I \to B$ of h as follows:

$$h^*(w,t,s) = \begin{cases} h(w,t-s), & t-s \geq 0 \\ h(w,0), & t-s \leq 0 \end{cases}$$

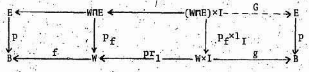
Since p is an F-fibration the homotopy h^* of h can be lifted to a homotopy $\bar{H} : (W \times I) \sqcap E \times I \to E$ of $\bar{h}$ such that $(\bar{H}, h^*) : p_h \times 1_I \to p$ is an F-homotopy. Now, define an F-map $g : (W \times I) \sqcap E \to E$ by the rule $g(w,t,y) = \bar{H}(w,t,y,t)$, where $p(y) = h(w,t)$. Then $p \cdot g(w,t,y) = p \cdot \bar{H}(w,t,y,t) = h^* \cdot (p_h \times 1_I)(w,t,y,t) = h^*(w,t,t) = h(w,0)$ and so the following diagram commutes:



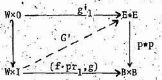
But the composite square is a pullback diagram of F-spaces and hence, by (1.2.3.), there exists a unique F-map $\alpha:(W\times I)\Pi E \to (W\Pi E)\times I$ over $W\times I$ such that $\alpha = (p_h, g)$. Now observe that $\alpha(w,0,y) = (p_h(w,0,y),\ g(w,0,y)) = (w,0,\bar{H}(w,0,y,0)) = (w,0,\bar{h}(w,0,y)) = (w,0,y)$.

(iii) $\Longrightarrow$ *(ii)*: Given a space W and a map $f:W \to B$, let $(g_1, g_0):p_f \to p$ be an F-map, and let $g:W\times I \to B$ be a homotopy of g_0. We must show that there exists a homotopy $G:(W\Pi E)\times I \to E$ of g_1 such that (G,g) is an F-homotopy.

To this end, consider the following commutative diagram



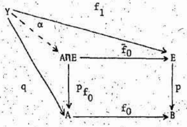
and observe that the existence of the required homotopy G completing the above diagram is equivalent, by (1.3.3.), to the existence of a homotopy $G':W\times I \to E{*}E$ completing the following diagram:



Here, $g'_1(w,0)(y) = g_1(w,y)$. But G' exists from the assumption on $p*p$.

(ii) ⇒ *(i)*: Let $q:Y \to A$ be an F-space, $(f_1,f_0):q \to p$ an F-map and $h:A\times I \to B$ a homotopy of f_0. We must show that there exists a homotopy $H:Y\times I \to E$ of f_1 such that (H,h) is an F-homotopy.

To this end, consider the following pullback diagram



and observe that, by (1.2.3.), there exists an F-map $\alpha:Y \to A\Pi E$ such that $p_{f_0}\cdot\alpha = q$ and $\bar{f}_0\cdot\alpha = f_1$. Now, by assumption, p has F-CHP with respect to all induced F-spaces and so, the homotopy $h:A\times I \to B$ of f_0 can be lifted to a homotopy $\bar{h}:(A\Pi E)\times I \to E$ of $\bar{f}_0$ such that $(\bar{h},h)$ is an F-homotopy. Set $H = \bar{h}\cdot(\alpha\times 1_I)$ and observe that H has the required properties. //

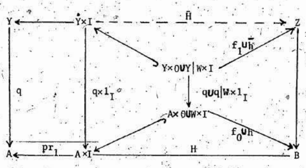
Notice that the proof of $(i) \Rightarrow (iii)$ in (2.1.3.) can be adapted to the more general situation of two different F-fibrations, yielding

<u>Proposition</u> 2.1.4. If $q:Y \to A$ and $r:Z \to B$ are F-fibrations, then $q * r : Y * Z \to A \times B$ is a Hurewicz fibration.

As a consequence of this result we have the following generalized version of the relative covering homotopy property in the category of F-fibrations.

<u>Corollary</u> 2.1.5. Let $q:Y \to A$ and $r:Z \to B$ be F-fibrations and let $(f_1, f_0):q \to r$ be an F-map. Let W be a closed subspace of A such that the inclusion $W \to A$ is a cofibration and let $(\bar{h},h)$: $q|W \times 1_I \to r$ be an F-homotopy such that $\bar{h}_0 = f_1|(Y|W)$ and $h_0 = f_0|W$. Then, for every homotopy $H:A \times I \to B$ of f_0 with $H|W \times I = h$, there is a homotopy $\bar{H}:Y \times I \to Z$ of f_1 such that $(\bar{H},H)$ is an F-homotopy and $\bar{H}|(Y|W \times I) = \bar{h}$.

The result is illustrated by the following diagram:



Proof: The completion of the above diagram is equivalent, by (1.3.3.), to the completion of the following diagram

$$\begin{array}{ccc} A\times 0\cup W\times I & \xrightarrow{f\cup g} & Y*Z \\ \downarrow & \nearrow & \downarrow{\scriptstyle q*r} \\ A\times I & \xrightarrow{(pr_I,H)} & A\times B \end{array} ,$$

where $f(a,0)(y) = f_1(y,0)$, $y\varepsilon Y_a$ and $g(w,t)(y) = \bar{h}(y,t)$, $y\varepsilon Y_w$. That the latter diagram can be completed follows from (2.1.4.) and [20; Theorem 4]. //

Proposition 2.1.6. Let $p:E \to B$ be an F-fibration and let $f:A \to B$ be a map. Then $p_f:A\Pi E \to A$ is an F-fibration.

Proof: By (2.1.3.) it is sufficient to show that $p_f*p_f:(A\Pi E)*(A\Pi E) \to A\times A$ is a Hurewicz fibration.

Now, since p is, by assumption, an F-fibration, $p*p:E*E \to B\times B$ is a Hurewicz fibration. But, by (1.3.9.), p_f*p_f is G-homeomorphic over $A\times A$ to the G-space induced from $p*p$ by the map $f\times f:A\times A \to B\times B$; hence, p_f*p_f is a Hurewicz fibration. //

The next result is a generalization of Dold's fibre homotopy equivalence theorem ([7; Theorem 6.3]) to the category of F-fibrations. We refer the reader to [14; Theorem 2.6] for the proof; recall that every morphism of F is already an F-homotopy equivalence over a point.

Theorem 2.1.7. Let $p:E \to B$ and $p':E' \to B$ be F-fibrations and let $g:E \to E'$ be an F-map over B. Assume that B admits a numerable cover $\mathcal{B}$ such that the inclusion map $V \to B$ is null-homotopic for each $V \in \mathcal{B}$. Then g is an F-homotopy equivalence over B.

Notice that the assumption on B is invariant under homotopy equivalence and is satisfied by such spaces as CW-complexes and more generally, by spaces which are paracompact and locally contractible.

Now recall that, if p and p' are F-spaces over B, then $F_1(p,p')$ is the space of all F-maps $p \to p'$ over B. If B is a CW-complex and both p and p' are F-fibrations, then, by (2.1.7.), the space $F_1(p,p')$ coincides with the space of all F-homotopy equivalences $p \to p'$ over B; hence, as a consequence of (1.3.7.), we have the following result.

Proposition 2.1.8. Let $p:E \to B$ be an F-fibration and let $f,g:A \to B$ be maps, where A is a CW-complex. Then there is a homeomorphism between (*i*) the space of all lifts of $(f,g):A \to B\times B$ over $p*p$ and (*ii*) the space of all F-homotopy equivalences $p_f \to p_g$ over A.

The next result follows immediately from (1.3.16) and the previous observation.

Proposition 2.1.9. Let $p:E \to B$ and $p':E' \to B$ be F-fibrations, where B is a CW-complex. Then there is a homeomorphism between

(i) the space of all sections to $(pp')_F:(EE')_F \to B$ and (ii) the space of all F-homotopy equivalences $p \to p'$ over B.

Proposition 2.1.10. Let $p:E \to B$ be an F-fibration and let $f,g:A \to B$ be homotopic maps, where A is a CW-complex. Then $p_f:A\sqcap E \to A$ and $p_g: A\sqcap E \to A$ are F-homotopy equivalent over A.

Proof: Let $H:A\times I \to B\times B$ be a homotopy from (f,f) to (f,g). By (2.1.8.), the F-homotopy equivalence $1:p_f \to p_f$ corresponds to a lifting $\theta:A \to E{\star}E$ of (f,f) over $p{\star}p$. But, since p is an F-fibration, by (2.1.3.), $p{\star}p:E{\star}E \to B\times B$ is a Hurewicz fibration; hence, there exists a homotopy $\bar{H}:A\times I \to E{\star}E$ such that $p{\star}p\cdot\bar{H} = H$ and $\bar{H}|A\times 0 = \theta$. The restriction of $\bar{H}$ to $A\times 1$ now corresponds, by (2.1.8.), to an F-homotopy equivalence $p_f \to p_g$ over A. //

§2. Admissible categories

The notion of an admissible category of fibrations was introduced in [4] as a general framework in which various notions of "Universal fibration" could be discussed. The specific problem of existence of Free Universal fibrations in a general theory is taken up in [14] within the context of F-fibrations. In this section we give a simplified reformulation of admissibility in terms of F-fibrations which is closely related to the original concept. The general theory of A-fibrations now reduces to the general theory of F-fibrations.

Definition 2.2.1. An admissible category of fibrations is a non-empty, full subcategory A of the category of all F-fibrations over CW-complexes and F-maps, satisfying the following axioms:

A1 - If $r:Z \to B$ belongs to $\mathcal{A}$, A is a CW-complex and $f:A \to B$ is a map, then the induced F-space $r_f:A\sqcap Z \to A$ belongs to $\mathcal{A}$.

A2 - If $r:Z \to B$ belongs to $\mathcal{A}$ and $q:Y \to B$ is an F-space such that q is F-homeomorphic to r over B, then q belongs to $\mathcal{A}$.

We call the objects of $\mathcal{A}$, <u>$\mathcal{A}$-fibrations</u>. Notice that, because of (2.1.3.) and (2.1.4.), an admissible category of fibrations can equivalently be defined as a non-empty, full subcategory $\mathcal{A}$ of the category of all F-spaces over CW-complexes and F-maps, satisfying axioms A1, A2 and an additional axiom:

A3 - If $q:Y \to A$ and $r:Z \to B$ are $\mathcal{A}$-fibrations, then $q*r:Y*Z \to A\times B$ is a Hurewicz fibration.

This is essentially the definition of admissibility in [4], the slight difference being, that in [4], the functional G-spaces $q*r:Y*Z \to A\times B$ are required to have the covering homotopy property with respect to all CW-complexes; that is to say, $q*r$ is a Serre fibration. In fact, observe that if the notion of F-fibration is generalized to one of <u>weak F-fibration</u>, that is to say, the F-covering homotopy property with respect to all F-spaces over CW-complexes, then (2.2.1.) with this new concept is actually equivalent to the definition of admissibility in [4]. Indeed, one can easily verify that the statements of (2.1.3.) and (2.1.4.) remain valid if the notion of F-fibration is replaced by weak F-fibration and the notion of Hurewicz fibration by Serre fibration.

In any event, it is clear that (2.2.1.) implies admissibility in the sense of [4].

In the remainder of this section we present some examples, both general and specific, of admissible categories of fibrations. The specific examples we consider are the usual categories of fibrations that arise in practise.

B. General Examples

1. F-fibrations - Let A_F denote the category of all F-fibrations over CW-complexes and F-maps. A_F is clearly the largest admissible category of fibrations.

2. Trivial F-spaces - Let A_T denote the category of all trivial F-spaces over CW-complexes and F-maps. A_T clearly satisfies axioms A1 and A2, and by (1.3.12.) and (2.1.3.), every trivial F-space is an F-fibration; hence, A_T is admissible.

3. Numerable F-spaces - Let A_N denote the category of all numerable F-spaces over CW-complexes and F-maps. If $p:E \to B$ is an object of A_N, then, by (1.3.13.), $p*p:E*E \to B\times B$ is a numerable G-space and consequently, by [7; Theorem 4.8], a Hurewicz fibration. But, then, p is an F-fibration, by (2.1.3.). Now axiom A2 is clearly satisfied and the verification of A1 follows from (1.2.5.); hence, A_N is admissible.

4. The Category A_p - Given an F-fibration $p:E \to B$, where B is a CW-complex, let A_p denote the category consisting of all the F-spaces over CW-complexes which are induced from p and all the F-spaces over CW-complexes which are F-homeomorphic to an F-space induced from p. Then A_p is clearly an admissible category of fibrations with distinguished object p. Notice that if p is a trivial F-space, then A_p is the smallest admissible category of fibrations.

C. Specific Examples

1. Hurewicz Fibrations - Let (F,F) be the category whose objects are all spaces of the homotopy type of a fixed space F and whose morphisms are all homotopy equivalences. Let A_F denote the category of all Hurewicz fibrations over CW-complexes with fibres in the category (F,F). If $p:E \to B$ is an object of A_F, then, by [3: Corollary 7], $p*p:E*E \to B\times B$ is a Hurewicz fibration and so by (2.1.3.), p is a F-fibration. Now A_F clearly satisfies axioms A1 and A2; hence, A_F is admissible.

2. Numerable Fibre Bundles [7] - Let G be a topological group and let F be a left G-space on which G acts effectively; that is to say, there exists a map $\mu:G\times F \to F$ such that:

(*i*) $\mu(g'\cdot g,y) = \mu(g',\mu(g,y))$, for all $g,g'\varepsilon G$ and all $y\varepsilon F$.

(*ii*) $\mu(e,y) = y$, for e = identity of G and all $y\varepsilon F$

(*iii*) $\mu(g,y) = \mu(g',y)$ if, and only if, $g = g'$, for all $g,g'\varepsilon G$ and all $y\varepsilon F$.

Conditions (i) and (ii) are just the statement that G acts on F; condition (iii) is the statement that G is an effective topological group.

We shall denote the image $\mu(g,y)$, $g\in G, y\in F$, by the multiplicative notation $g\cdot y$. Then, by (1.1.1.), there exists a map

$$\mu' : G \to L(F,F)$$

defined by $\mu'(g) = \bar{g}: F \to F$, where $\bar{g}(y) = g\cdot y$, for $g\in G, y\in F$. Denote the image of μ' in $L(F,F)$ by $F\star F$ and observe that $F\star F$ is a group of homeomorphisms of F, the group structure being given by composition of functions. Now, it is easily verified that the map μ' is a group homomorphism and moreover, because the action of G on F is effective, μ' is injective; hence, as groups, G and $F\star F$ are isomorphic. For this reason, G is frequently referred to as an effective topological transformation group of F.

Now observe that, because the topology on G need not coincide with the k-ified compact-open topology on $F\star F$, G and $F\star F$ need not be homeomorphic. However, in practise, the two topologies usually do coincide; for example, in case G is compact and F is Hausdorff (for then $F\star F$ is Hausdorff [2; Corollary 5.2]). Observe, furthermore, that with the k-ified compact-open topology, $F\star F$ is an H-group.

Now recall that a numerable fibre bundle consists of a collection $(E,p,B;F,G)$ such that:

(i) $p:E \to B$ is a map of E onto B.

(ii) for each $b\in B$, E_b is homeomorphic to F.

(iii) G is an effective topological transformation group of F, called the structure group of the bundle.

(iv) there exists a numerable covering $\{V_j : j \in J\}$ of B, the open sets V_j being called co-ordinate neighbourhoods, together with a family of homeomorphisms,

$$\phi_j : V_j \times F \to E|V_j, \quad j \in J$$

called co-ordinate functions, satisfying:

(a) $p \cdot \phi_j(x,y) = x$, for all $x \in V_j$, $y \in F$

(b) if the map $\phi_{j,x} : F \to E_x$ is defined by setting

$$\phi_{j,x}(y) = \phi_j(x,y)$$

then, for each pair i,j in J, and each $x \in V_i \cap V_j$, the homeomorphism

$$\phi_{j,x}^{-1} \cdot \phi_{i,x} : F \to F$$

coincides with the operation of an element of G (it is unique since G is effective)

(c) for each pair i,j in J, the function

$$g_{ji} : V_i \cap V_j \to G$$

defined by, $\mu'(g_{ji}(x)) = \phi_{j,x}^{-1} \cdot \phi_{i,x}$, is continuous; the collection $\{g_{ji} : i,j \in J\}$ is called a system of co-ordinate transformations of the bundle.

Hence, define a category F as follows (see [14; 6.11]). Let F have for objects all pairs (P,x) such that P is a left G-space and $x : F \to P$ is a homeomorphism of left G-spaces. Let the set of morphisms from (P,x) to (P',x') be the collection $\{x' \cdot g \cdot x^{-1} \mid g \in F*F\}$. Then F is clearly a category of fibres with distinguished object (F,1).

Let $A_{(F,G)}$ denote the category of all numerable fibre bundles over CW-complexes with fibres in the category (F,(F,1)). Then $A_{(F,G)}$ is admissible by example B3.

3. Numerable Principal G-bundles. - Let $\xi = (E,p,B;F,G)$ be a numerable fibre bundle where $F = G$ and G operates on itself by left translations. Then ξ is called a numerable principal G-bundle. Notice that the space $G*G$ of all left translations in G is precisely the space of all right G-homeomorphisms $G \to G$ and moreover, the map $e: G*G \to G$, which evaluates at the identity of G, is a homeomorphism. Hence, in this situation, the category of fibres $(F,(G,1))$ has for objects all pairs (P,x) such that P is a right G-space and $x: G \to P$, defined by $x(g) = x \cdot g$, $x \in P$, is a homeomorphism of right G-spaces. The set of morphisms from (P,x) to (P',x') is then the set of all right G-homeomorphisms $\{x' \cdot g \cdot x^{-1} | g \in G\}$.

Let A_G denote the category of all numerable principal G-bundles over CW-complexes with fibres in the category $(F,(G,1))$. Then A_G is admissible by example C2.

4. Vector Bundles - Let $V^k_{\mathbb{F}}$ be a k-dimensional topological vector space; real for $\mathbb{F} = \mathbb{R}$, complex for $\mathbb{F} = \mathbb{C}$ and quaternionic for $\mathbb{F} = \mathbb{H}$. Then a k-dimensional vector bundle is a collection $(E,p,B;V^k_{\mathbb{F}})$ such that:

(*i*) $p: E \to B$ is a map of E onto B.

(*ii*) for each $b \in B$, E_b is homeomorphic and isomorphic as a vector space to $V^k_{\mathbb{F}}$.

(*iii*) for each $b \in B$, there exists an open neighbourhood U_b of b and a homeomorphism

$$\phi(U_b):U_b\times V^k_{\mathbb{F}} \to E|U_b$$

such that:

(a) $p\cdot\phi(U_b)(x,y) = x$, for all $x\varepsilon U_b, y\varepsilon V^k_{\mathbb{F}}$

(b) for all $x\varepsilon U_b$, $\phi(U_b)|\{x\}\times V^k_{\mathbb{F}}:V^k_{\mathbb{F}} \to E_x$ is an isomorphism of vector spaces.

Let $U_{\mathbb{F}}(k)$ denote the orthogonal group $O(k)$, for $\mathbb{F} = \mathbb{R}$, the unitary group $U(k)$, for $\mathbb{F} = \mathbb{C}$ and the symplectic group $Sp(k)$ for $\mathbb{F} = \mathbb{H}$. Then, any k-dimensional vector bundle with fibre $V^k_{\mathbb{F}}$ over a paracompact base space can be viewed as a numerable fibre bundle with structure group $U_{\mathbb{F}}(k)$([11;5.7.4.]), the action of $U_{\mathbb{F}}(k)$ on $V^k_{\mathbb{F}}$ being given by multiplication of matrices. Notice that, if $\mu':U_{\mathbb{F}}(k) \to V^k_{\mathbb{F}} * V^k_{\mathbb{F}}$ denotes the adjoint of this action, then, for each $g\varepsilon U_{\mathbb{F}}(k)$, the homeomorphism $\mu'(g):V^k_{\mathbb{F}} \to V^k_{\mathbb{F}}$ is also a vector space isomorphism. Furthermore, because $V^k_{\mathbb{F}}$ is Hausdorff and $U_{\mathbb{F}}(k)$ is compact, μ' is not only a group isomorphism but also a homeomorphism of spaces.

Define a category of fibres F as in C2 but with the following modifications: the spaces $P^k_{\mathbb{F}}$ are required to be k-dimensional topological vector spaces such that the distinguished homeomorphism $x:V^k_{\mathbb{F}} \to P^k_{\mathbb{F}}$ is also a vector space isomorphism. The distinguished object of F is, of course, the pair $(V^k_{\mathbb{F}},1)$.

Let $A_{V^k_{\mathbb{F}}}$ denote the category of all k-dimensional vector bundles over CW-complexes with fibres in the category $(F,(V^k_{\mathbb{F}},1))$. Then $A_{V^k_{\mathbb{F}}}$ is admissible by example C2.

§3. Universality in admissible categories

For a given CW-complex B let $\mathcal{E}A(B)$ be the collection (assumed to be a set) of all equivalence classes of A-fibrations over B under the equivalence relation: $p \sim p'$ if, and only if, p is F-homotopy equivalent to p' over B. Notice that, in view of (2.1.7.), this equivalence relation can be restated as: $p \sim p'$ if, and only if, there exists an F-map $p \to p'$ over B.

Let HCW denote the homotopy category of CW-complexes. Then, as a consequence of (2.1.10.), $\mathcal{E}A$ is a contravariant functor from HCW to Set, the category of sets and functions. Furthermore, each A-fibration $p: E \to B$ defines a natural transformation

$$\zeta : [\ ,B] \to \mathcal{E}A(\ )$$

by the rule: for each CW-complex A, $\zeta_A : [A,B] \to \mathcal{E}A(A)$, where [A,B] denotes the set of all free homotopy classes of maps $A \to B$, is the function which assigns to each homotopy class [f], the equivalence class of the induced A-fibration $p_f : A \sqcap E \to A$. This relationship suggests the following definition.

Definition 2.3.1. An A-fibration $p_n : E_n \to B_n$ is said to be Free n-Universal in A, n finite or infinite, if, for each CW-complex A of dimension $\leq n$, $\zeta_A : [A,B_n] \to \mathcal{E}A(A)$ is a bijection.

Notice that, if $p_n : E_n \to B_n$ is a Free n-Universal A-fibration, n finite or infinite, then B_n is path-connected. This follows from the observation that, if $b, b' \in B_n$, the inclusions

$$b, b' : * \to B_n$$

induce A-fibrations which are F-homotopy equivalent to $F \to *$.

Definition 2.3.2. *Let B be a CW-complex with base point* b_0.
A grounded A-fibration (p,k) is a sequence

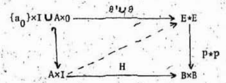
$$F \xrightarrow{k} E_{b_0} \hookrightarrow E \xrightarrow{p} B$$

such that $k \in F{*}E_{b_0}$ and $p:E \to B$ is an A-fibration.

A grounded F-map between grounded A-fibrations (p,k) and (p',k') is an F-map $(f_1,f_0):p \to p'$ such that f_0 is a based map and $(f_1|E_{b_0})\cdot k \simeq_F k'$. Setting $B' = B$ and $f_0 = 1_B$, we have the notion of a grounded F-map over B. But, by (2.1.7.), every grounded F-map over B is an F-homotopy equivalence; we call such a morphism a grounded F-homotopy equivalence over B. One can easily verify that this notion of grounded F-homotopy equivalence is an equivalence relation on the subcategory of all grounded A-fibrations over B.

Proposition 2.3.3. Let (p,k) be a grounded A-fibration, (A,a_0) a based CW-complex and $f,g:A \to B$ based homotopic maps. Then (p_f,k) and (p_g,k) are grounded F-homotopy equivalent over A.

Proof: Let $H:A{\times}I \to B{\times}B$ be a based homotopy from (f,f) to (f,g). By (2.1.8.), the grounded F-homotopy equivalence $1:(p_f,k) \to (p_f,k)$ corresponds to a lifting $\theta:A \to E{*}E$ of (f,f) over $p{*}p$ such that $\theta(a_0) = 1_{E_{b_0}}$. Now, consider the commutative diagram



where $\theta'(a_0,t) = \theta(a_0)$. Since p is an F-fibration, $p*p$ is a Hurewicz fibration; hence, by [20; Theorem 4], there exists a homotopy $K:A\times I \to E*E$ completing the above diagram. The restriction of K to $A\times 1$ now corresponds, by (2.1.8.), to a grounded F-homotopy equivalence $(p_f,k) \to (p_g,k)$ over A. //

Let $EA^F(B)$ denote the collection (assumed to be a set) of all equivalence classes of grounded A-fibrations over B and let $\underline{HCW}_*$ denote the homotopy category of based CW-complexes. Then, as a consequence of (2.3.3.), $EA^F(\ )$ is a contravariant functor from $\underline{HCW}_*$ to $\underline{Set}_*$, the category of based sets and based functions. Furthermore, each grounded A-fibration (p,k) defines a natural transformation

$$\zeta:[\ ,B]_* \to EA^F(\ )$$

in the obvious manner, where, for each based CW-complex (A,a_0), $[A,B]_*$ denotes the set of all based homotopy classes of based maps $A \to B$. This relationship suggests the following definition.

Definition 2.3.4. A grounded A-fibration (p_n,k) is said to be Grounded n-Universal in A, n finite or infinite, if, for each based CW-complex (A,a_0) of dimension $\le n$, $\zeta_A:[A,B_n]_* \to EA^F(A)$ is a bijection. If, for all choices of base point $b_n \varepsilon B_n$ and all F-homotopy equivalences $k:F \to (E_n)_{b_n}$, the pair (p_n,k) is Grounded n-Universal, we say that p_n is Grounded n-Universal in A.

Given an A-fibration $p:E \to B$ and the A-fibration $c:F \to *$, form the functional G^*-space $c*_2p:F*E \to B$. Now observe that, by (2.1.4.), $c*_2p$ is a Hurewicz fibration; we call $c*_2p$ <u>the associated principal fibration</u>. This terminology originated with an analogous construction in the theory of Hurewicz fibrations.

<u>Definition</u> 2.3.5. An A-fibration $p_n:E_n \to B_n$ is said to be <u>Aspherical n-Universal</u> in A, n finite or infinite, if, for all choices of base point, $\pi_i(F*E_n) = 0$, $0 \leq i \leq n$.

<u>Definition</u> 2.3.6. An A-fibration $p_n:E_n \to B_n$ is said to be <u>Extension n-Universal</u> in A, n finite or infinite, if, for every relative CW-pair (B,L) with dim $B \leq n$, and every A-fibration $p:E \to B$, each F-map $(f_{1L}, f_{0L}):p|L \to p_n$ can be extended to an F-map $(f_1, f_0):p \to p_n$.

Notice that (2.3.6.) is a generalization of the notion of n-Universality in [19] for the category of numerable principal G-bundles. Theorems 19.3 and 19.4 in [19], relating the notions of Free n-Universal, Aspherical n-Universal and Extension n-Universal principal G-bundles, can be generalized to their corresponding statements in A. Because (2.2.1.) implies admissibility in the sense of [4], the proofs of these generalizations can be readily obtained by adapting the proofs of Theorems 3.1 and 3.2 in [4], for the case $n = \infty$, to the case of finite n. Furthermore, one can easily verify that the relationship in [4] of Grounded ∞-Universality to the other notions of ∞-Universality also holds for the more restrictive notion of n-Universality, n finite. Hence, we have

<u>Theorem</u> 2.3.7. ([4; Theorem 3.1]). An $\mathcal{A}$-fibration is Aspherical n-Universal if, and only if, it is Extension (n+1) -Universal.

<u>Theorem</u> 2.3.8. ([4; Theorem 3.2]). Every Aspherical n-Universal $\mathcal{A}$-fibration is Free n-Universal.

<u>Theorem</u> 2.3.9. ([4; Theorem 3.3]). Every Aspherical n-Universal $\mathcal{A}$-fibration is Grounded n-Universal.

<u>Theorem</u> 2.3.10. ([4; §3]). Every Grounded n-Universal $\mathcal{A}$-fibration is Free n-Universal.

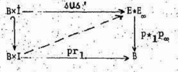
Notice that, in order to show that the four notions of n-Universality coincide in a given admissible catagory $\mathcal{A}$, it is sufficient, from (2.3.7.), (2.3.9.) and (2.3.10.) to show:

(2.3.11.) - Every Free n-Universal $\mathcal{A}$-fibration is Aspherical n-Universal (or equivalently, Extension (n+1) -Universal).

That (2.3.11.) does not hold in general can be seen by the following counterexample: consider the admissible category $\mathcal{A}$ of all trivial $\mathbb{Z}_2$-bundles, where $\mathbb{Z}_2$ is the discrete group of order 2. The trivial $\mathbb{Z}_2$-bundle $c:\mathbb{Z}_2 \to *$ is clearly Free n-Universal in $\mathcal{A}$ for any n, finite or infinite, but is not Aspherical n-Universal, for any n, since the space $\mathbb{Z}_2 * \mathbb{Z}_2$ of all $\mathbb{Z}_2$-automorphisms of $\mathbb{Z}_2$ is homeomorphic to $\mathbb{Z}_2$ and $\mathbb{Z}_2$ is not even path-connected.

Theorem (2.3.12.) An Aspherical ∞-Universal A-fibration is a terminal object in the homotopy category of A-fibrations.

Proof: Let $p_\infty : E_\infty \to B_\infty$ be an Aspherical ∞-Universal A-fibration and let $p: E \to B$ be any A-fibration. We must show that if (f_1, f_0) and (g_1, g_0) are any two F-maps $p \to p_\infty$, then $(f_1, f_0) \simeq_F (g_1, g_0)$. By (1.3.5.), it is sufficient to show that, if $\delta, \delta' : B \to E * E_\infty$ are sections which correspond, by (1.3.4.), to (f_1, f_0) and (g_1, g_0), respectively, then δ is vertically homotopic to δ'. Notice that the existence of a vertical homotopy from δ to δ' is equivalent to the completion of the following diagram:



Now observe that, since $p *_1 p_\infty$ has fibre $F * E_\infty$ and p_∞ is Aspherical ∞-Universal, $p *_1 p_\infty$ is a weak homotopy equivalence.(*) Thus, by [18; 7.6.22.], there exists a map $H: B \times I \to E * E_\infty$ such that $H|B \times \dot{I} = \delta \cup \delta'$ and $p *_1 p_\infty \circ H \simeq pr_1$ relative to $B \times \dot{I}$. But, $p *_1 p_\infty$ is a Hurewicz fibration; hence, we can replace the map H by a map $\bar{H}$ such that $\bar{H}|B \times \dot{I} = \delta \cup \delta'$ and $p *_1 p_\infty \circ \bar{H} = pr_1$. //

The following result reduces the problem of the existence of the four notions of ∞-Universality in a given admissible

(*) also called an ∞-equivalence

category A and of the equivalence of these four notions to just the existence of Aspherical ∞-Universal A-fibrations.

<u>Theorem</u> 2.3.13. Let A be an admissible category of fibrations in which there exists an Aspherical ∞-Universal A-fibration. Then, there exist Free ∞-Universal, Grounded ∞-Universal and Extension ∞-Universal A-fibrations and moreover, the four notions of ∞-Universality coincide.

<u>Proof</u>: The first part is an immediate consequence of (2.3.7.), (2.3.8.) and (2.3.9.), for the case $n = \infty$.

To see that the four notions of ∞-Universality coincide, let $p'_\infty : E'_\infty \to B'_\infty$ be an Aspherical ∞-Universal A-fibration and let $p_\infty : E_\infty \to B_\infty$ be any Free ∞-Universal A-fibration. By (2.3.11.), it is sufficient to show that p'_∞ is Aspherical ∞-Universal.

Now, since p_∞ is Free ∞-Universal, there exists an F-map $k = (k_1, k_0) : p'_\infty \to p_\infty$. But, by (2.3.8.), p'_∞ is itself Free ∞-Universal and hence, there exists an F-map $k' : (k'_1, k'_0) : p_\infty \to p'_\infty$. The composites $k \cdot k'$ and $k' \cdot k$ now define F-maps $p_\infty \to p_\infty$ and $p'_\infty \to p'_\infty$, respectively. However, p_∞ is classified by $1 : B_\infty \to B_\infty$ and p'_∞ is classified by $1 : B'_\infty \to B'_\infty$; hence, $k'_0 \cdot k_0 \simeq 1_{B'_\infty}$ and $k_0 \cdot k'_0 \simeq 1_{B_\infty}$ and so, $k_0 : B'_\infty \to B_\infty$ is a homotopy equivalence.

Now, the F-map $k : p'_\infty \to p_\infty$ induces a G^*-map $\bar{k} = (\bar{k}_1, k_0)$: $c\ast_2 p'_\infty \to c\ast_2 p_\infty$, where $\bar{k}_1 : F \ast E'_\infty \to F \ast E_\infty$ is defined by, $\bar{k}_1(\theta : F \to (E'_\infty)_{b'_\infty}) = k_1 | (E'_\infty)_{b'_\infty} \cdot \theta$. In particular, $\bar{k}$ can be viewed as a fibre preserving map of Hurewicz fibrations, which restricts to a homotopy equivalence on the fibres and which is also a homotopy equivalence on the bases. An application of the Five Lemma to the commutative diagram arising from the exact homotopy sequence of both fibrations now shows that $\bar{k}_1 : F*E'_\infty \to F*E_\infty$ is a weak homotopy equivalence, from which we deduce that p_∞ is Aspherical ∞-Universal. //

§4. Universality in the category of fibre bundles

The problem of equivalence of the four notions of ∞-Universality, as defined in section 3, is discussed in [4] for specific admissible categories of fibrations. In this section we shall discuss this problem in the admissible category $A_{(F,G)}$ of numerable fibre bundles over CW-complexes with fibre F and structure group G. We first show that these four notions are equivalent in the admissible category A_G of numerable principal G-bundles over CW-complexes (this is also shown in [4]), and then, using this fact and the close relationship between fibre bundles and their associated principal G-bundles, we give a necessary and sufficient condition for the existance of Aspherical ∞-Universal fibre bundles and hence, by (2.3.13.), for the equivalence of the four notions of ∞-Universality in $A_{(F,G)}$.

For a given numerable principal G-bundle $p:E \to B$, observe that the function $G*E \to E$ which evaluates at the identity of G is a homeomorphism, its inverse being adjoint to the right action $E \times G \to E$; hence, to say that p is Aspherical ∞-Universal means that all the homotopy groups of E vanish. Now, recall that Milnor constructed in [16], for a topological group G, a numerable principal G-bundle

$$p_G : E_G \to B_G$$

with the property that p_G is Free ∞-Universal and E_G is a contractible space. But, by the above observation, p_G is also Aspherical ∞-Universal; hence, A_G possesses an Aspherical ∞-Universal G-bundle and, by (2.3.13.), we have

Theorem 2.4.1. Let $p_\infty : E_\infty \to B_\infty$ be a numerable principal G-bundle over a CW-complex B_∞. Then p_∞ is ∞-Universal in all the four senses described in section 3 if it is ∞-Universal in any one of these senses.

Now, the existence in $A_{(F,G)}$ of a Free ∞-Universal fibre bundle is a direct consequence of the existence of the Milnor bundle $p_G : E_G \to B_G$. This can be seen as follows:

Let $\xi = (E,p,B;F,G)$ be a numerable fibre bundle with co-ordinate neighbourhoods $\{V_j : j \in J\}$ and co-ordinate functions $\{\phi_j : j \in J\}$. Let $T = \bigcup_{j \in J} V_j \times G \times j$, topologized with the final topology with respect to the inclusions $V_j \times G \times j \to T$, and define on T the following equivalence relation:

$(x,g,i) \sim (x',g',j)$ if, and only if, $x = x'$ and $g_{ji}(x) \cdot g = g'$.

Form the quotient $\tilde{E} = T/\sim$, topologized with the quotient topology, and define $\tilde{p}:\tilde{E} \to B$ by $\tilde{p}([x,g,i]) = x$. Then $\tilde{\xi} = (\tilde{E},\tilde{p},B,G)$ is a numerable principal G-bundle ([19; Theorem 3.2]), called the associated principal G-bundle of p. Notice that the co-ordinate transformations of $\tilde{\xi}$ are precisely the same as those for ξ and furthermore, the above construction works in reverse; that is to say, if ξ is a principal G-bundle, then, by replacing G by F in the above construction, the resulting bundle $\tilde{\xi}$ is a numerable fibre bundle with fibre F and structure group G.

Now, Steenrod has shown ([19; Theorem 3.3]) that there is a 1 - 1 correspondance between equivalence classes of bundles and equivalence classes of systems of co-ordinate transformations. Hence, the rule which assigns to each fibre bundle its associated principal G-bundle, sets up a 1 - 1 correspondance between equivalence classes of fibre bundles with fibre F and structure group G and equivalence classes of principal G-bundles. Therefore, since $p_G:E_G \to B_G$ is Free ∞-Universal, replacing G by F in the construction described yields a numerable fibre bundle which is Free ∞-Universal in $A_{(F,G)}$. Conversely, given a Free ∞-Universal numerable fibre bundle, the same construction yields a Free ∞-Universal principal G-bundle.

Although the existence of a Free ∞-Universal principal G-bundle guarantees the existence of a Free ∞-Universal fibre bundle and conversely, we shall see that the same cannot be said for the notion of Aspherical ∞-Universality. This is because, unlike the former situation where the classification of fibre bundles and principal G-bundles depends only on the base space B and the topological group G, the existence of Aspherical ∞-Universal fibre bundles depends directly on the fibre F and its relation to the group G. We now examine this latter situation.

Recall that, if $\xi = (E,p,B;F,G)$ is a numerable fibre bundle, then the action $\mu : G \times F \to F$ (assumed to be effective) gives rise to a continuous bijective homomorphism $\mu' : G \to F*F$. Now, consider the associated principal G-bundle $\tilde{\xi} = (\tilde{E},\tilde{p},B;G)$ and the associated principal fibration $\bar{\xi} = (F*E, \alpha *_2 p, B; F*F)$ and define

$$\bar{\mu}' : \tilde{\xi} \to \bar{\xi}$$

over B, by $\bar{\mu}'([x,g,i]) = \phi_{i,x} \cdot \mu'(g)$.

<u>Lemma</u> 2.4.2. $\bar{\mu}' : \tilde{E} \to F*E$ is a continuous bijection.

<u>Proof</u>: (*i*) <u>$\bar{\mu}'$ is well-defined:</u>

Suppose $(x,g,j) \sim (x',g',k)$; then $x = x'$ and $g_{kj}(x) \cdot g = g'$.

Now,
$$\begin{aligned} \phi_{k,x} \cdot \mu'(g') &= \phi_{k,x} \cdot \mu'(g_{kj}(x) \cdot g) \\ &= \phi_{k,x} \cdot \mu'(g_{kj}(x)) \cdot \mu'(g) \\ &= \phi_{k,x} \cdot \phi_{k,x}^{-1} \cdot \phi_{j,x} \cdot \mu'(g) \\ &= \phi_{j,x} \cdot \mu'(g) \ ; \end{aligned}$$

hence, $\bar{\mu}'([x,g,j]) = \bar{\mu}'([x',g',k])$.

(*ii*) <u>$\bar{\mu}'$ is injective:</u>

Suppose $\bar{\mu}'([x,g,j]) = \bar{\mu}'([x',g',k])$; then $x = x'$ and $\phi_{j,x} \cdot \mu'(g) = \phi_{k,x} \cdot \mu'(g')$. Now,

$$\begin{aligned} & \phi_{j,x} \cdot \mu'(g) = \phi_{k,x} \cdot \mu'(g') \\ \Longleftrightarrow \quad & \phi_{k,x}^{-1} \cdot \phi_{j,x} \cdot \mu'(g) = \mu'(g') \end{aligned}$$

$$\Longleftrightarrow \quad \mu'(g_{k,j}(x))\cdot\mu'(g) = \mu'(g')$$

$$\Longleftrightarrow \quad \mu'(g_{k,j}(x)\cdot g) = \mu'(g')$$

But, μ' is injective; hence, $g_{k,j}(x)\cdot g = g'$ and so, $(x,g,j) \sim (x',g',k)$.

(*iii*) $\bar{\mu}'$ is surjective:

Let $f:F \to E_b$ be an element of $F*E$. Then, for some $g\varepsilon G$, $f = \phi_{k,b}\cdot g = \phi_{k,b}\cdot\mu'(g)$; hence, $\bar{\mu}'([b,g,k]) = \phi_{k,b}\cdot\mu'(g) = f$.

(*iv*) $\bar{\mu}'$ is continuous:

Because $\tilde{E}$ has the final topology with respect to the inclusions $\lambda_i:V_i\times G\times i \to \tilde{E}$, $\bar{\mu}'$ is continuous if, and only if, for each $i\,\varepsilon\,J$, the composite

$$V_i\times G\times i \xrightarrow{\lambda_i} \tilde{E} \xrightarrow{\bar{\mu}'} F*E$$

is continuous. But, $\bar{\mu}'\cdot\lambda_i$ can be written as the following composition,

$$V_i\times G\times i \xrightarrow{\phi_i' \times \mu'} F*E\times F*F \xrightarrow{\gamma} F*E$$

where $\phi_i':V_i \to F*E$ is the adjoint of the co-ordinate function $\phi_i:V_i\times F \to E|V_i$ and γ is the action, by composition, of $F*F$ on $F*E$; hence, $\bar{\mu}'$ is continuous. //

The following result now follows quite easily from (2.4.2.) and the topology on $F*E$.

Theorem 2.4.3. Let $\xi = (E,p,B;F,G)$ be a numerable fibre bundle. If $\mu':G \to F*F$ is a homeomorphism, then $\tilde{\xi}$ and $\bar{\xi}$ are equivalent principal G-bundles. Moreover, the bundle structure on $\bar{\xi}$ arising from the action of $F*F$ on $F*E$ by composition coincides with the bundle structure induced from $\tilde{\xi}$ by $\bar{\mu}'$.

Proof: It is sufficient, by (2.4.2.), to show that $(\bar{\mu}')^{-1}$ is continuous. Then the bundle structure on $\tilde{\xi}$ induces, via the map $\bar{\mu}'$, an equivalent bundle structure on $\bar{\xi}$.

To see that $(\bar{\mu}')^{-1}$ is continuous, observe that, by (1.3.14.), $F{*}E$ has the final topology with respect to the co-ordinate functions $\phi_i:V_i\times F{*}F \to F{*}E$, $i\varepsilon J$, and hence, $(\bar{\mu}')^{-1}$ is continuous if, and only if, the composites $(\bar{\mu}')^{-1}\circ\phi_i$ are continuous, for each $i\varepsilon J$. But, for each $i\varepsilon J$, $(\bar{\mu}')^{-1}\circ\phi_i = \lambda_i\circ(1_{V_i}\times(\mu')^{-1})$ and $(\mu')^{-1}$ is continuous by assumption; hence, $(\bar{\mu}')^{-1}$ is continuous.

The proof of the second part follows from the observation that the following diagram is commutative:

$$\begin{array}{ccc} \tilde{E}\times G & \xrightarrow{\ \eta\ } & \tilde{E} \\ {\scriptstyle \bar{\mu}'\times\mu'}\downarrow & & \downarrow{\scriptstyle \bar{\mu}'} \\ F{*}E\times F{*}F & \xrightarrow{\ \gamma\ } & F{*}E \end{array}$$

Here, η denotes the action of G on $\tilde{E}$, defined by $\eta([b,g,i],a) = [(b,g\cdot a,i)]$. //

Although $\bar{\xi}$ is not in general a principal G-bundle, notice that both $\tilde{\xi}$ and $\bar{\xi}$ are Hurewicz fibrations and moreover, the action of G on F determines a fibre preserving map of Hurewicz fibrations. This leads us to the following result on the existence of Aspherical ∞-Universal fibre bundles.

Theorem 2.4.4. There exists an Aspherical ∞-Universal fibre bundle in $A_{(F,G)}$ if, and only if, $\mu':G \to F{*}F$ is a weak homotopy equivalence.

Proof: ($\Longrightarrow$): Let $\xi = (E,p,B;F,G)$ be an object of $A_{(F,G)}$ which is Aspherical ∞-Universal. Then, by (2.3.8.), ξ is also Free ∞-Universal; hence, the associated principal G-bundle $\tilde{\xi} = (\tilde{E},\tilde{p},B;G)$ is Free ∞-Universal in A_G and thus, by (2.4.1.), Aspherical ∞-Universal.

Now the action of G on F determines, by (2.4.2.), a continuous bijection $\bar{\mu}':\tilde{E} \to F{*}E$ over B which is, because all the homotopy groups of $\tilde{E}$ and $F{*}E$ vanish, a weak homotopy equivalence. Since the restriction of $\bar{\mu}'$ to the fibre over a point $b \in B$ can be identified with the map $\mu':G \to F{*}F$, an application of the Five Lemma shows that μ' is also a weak homotopy equivalence.

($\Longleftarrow$): Let $\tilde{\xi} = (\tilde{E}_G,\tilde{p}_G,B_G;F,G)$ be the numerable fibre bundle associated to the Milnor bundle $\xi = (E_G,p_G,B_G;G)$. We show that $\tilde{\xi}$ is Aspherical ∞-Universal.

Let $\eta = (F{*}\tilde{E}_G,\alpha{*}_2\tilde{p}_G,B_G;F{*}F)$ be the associated principal fibration of $\tilde{\xi}$. Then the action of G on F determines, by (2.4.2.), a continuous bijection $\bar{\mu}':E_G \to F{*}\tilde{E}_G$ over B_G whose restriction to the fibre over a point $b_\infty \in B_G$ can be identified with $\mu':G \to F{*}F$. But, by assumption, μ' is a weak homotopy equivalence; hence, by the Five Lemma, $\bar{\mu}'$ is also a weak homotopy equivalence. Since all the homotopy groups of E_G vanish, all the homotopy groups of $F{*}\tilde{E}_G$ vanish and so, $\tilde{\xi}$ is Aspherical ∞-Universal.//

As an immediate consequence of (2.4.4.) and (2.3.13.) we have the following result:

Theorem 2.4.5. The four notions of ∞-Universality are equivalent in $A_{(F,G)}$ if, and only if, $\mu':G \to F_*F$ is a weak homotopy equivalence.

Now recall from example C4 that any k-dimensional vector bundle with fibre $V^k_{\mathbb{F}}$ and base space a CW-complex has structure group $U_{\mathbb{F}}(k)$; furthermore, since $U_{\mathbb{F}}(k)$ is compact and $V^k_{\mathbb{F}}$ is Hausdorff, the adjoint $\mu':U_{\mathbb{F}}(k) \to V^k_{\mathbb{F}*}V^k_{\mathbb{F}}$ of the action of $U_{\mathbb{F}}(k)$ on $V^k_{\mathbb{F}}$ is a homeomorphism. Hence, as a consequence of (2.4.5.) we obtain:

Corollary 2.4.6. Let $\xi = (E,p,B;V^k_{\mathbb{F}},U_{\mathbb{F}}(k))$ be an object of $A_{V^k_{\mathbb{F}}}$. Then ξ is ∞-Universal in all four senses if it is ∞-Universal in any one of these senses.

III. THE HOMOTOPY OF SPACES OF F-HOMOTOPY EQUIVALENCES

§1. Some technical results

In this section we shall discuss some results, which will be needed in the sequel, about Hurewicz fibrations and the inducing of H-group structures on function spaces. Once more, we recall that all spaces and maps considered are objects and morphisms of the convenient category K.

Definition 3.1.1. Let $p:E \to B$ and $p':E' \to B'$ be maps and let $g = (g_1, g_0): p \to p'$ be a fibre preserving map such that g_1 and g_0 are n-equivalences (resp. homotopy equivalences). Then g is called an n-equivalence (resp. homotopy equivalence) and we say that p is n-equivalent (resp. homotopy equivalent) to p'.

Lemma 3.1.2. Let (P,Q) be a relative CW-complex, with inclusion map $i:Q \to P$, such that dim. $(P\backslash Q) \le n$. Let $g = (g_1, g_0): p \to p'$ be a map of Hurewicz fibrations, where g_1 is an n-equivalence and g_0 is an (n+1)-equivalence, n finite or infinite, and let $f:i \to p$ be a map. Then, given a lifting $\alpha:P \to E'$ of the map $g \cdot f$, there exists a lifting $\beta:P \to E$ of f such that $g_1 \cdot \beta$ is homotopic to α relative to $g \cdot f$.

This result is a trivial generalization of [18;7.8.11.]; its proof involves two applications of [18;7.6.22.] and then two applications of [20; Theorem 4].

Lemma 3.1.3. Let $g = (g_1, g_0):p \to p'$ be a map of Hurewicz fibrations, where g_1 is an n-equivalence and g_0 is an (n+1)-equivalence, n finite or infinite, and let $k:A \to B$ and $k':A' \to B'$

be maps. If there is a map $f_0: A \to A'$ such that $k' \cdot f_0 = g_0 \cdot k$ and f_0 is a Hurewicz fibration and an n-equivalence, then the induced map $f_1: A \sqcap E \to A' \sqcap E'$, defined by $f_1(a,x) = (f_0(a), g_1(x))$, is an n-equivalence; hence, $f = (f_1, f_0): p_k \to p'_{k'}$ is an n-equivalence of Hurewicz fibrations.

<u>Proof</u>: Let (a_0, x_0) be the base point of $A \sqcap E$, where a_0 and x_0 are base points of A and E, respectively. For each $j \geq 0$, consider the induced homomorphism (function for $j = 0$)

$$(f_1)_*: \pi_j(A \sqcap E, (a_0, x_0)) \to \pi_j(A' \sqcap E', (f_0(a_0), g_1(x_0))),$$

defined by $(f_1)_*([h]) = [f_1 \cdot h]$. We show that $(f_1)_*$ is bijective for $0 \leq j \leq n-1$ and surjective for $j = n$.

(i) <u>$(f_1)_*$ is injective for $0 \leq j \leq n-1$:</u>

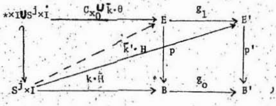
Let $[h], [h'] \in \pi_j(A \sqcap E, (a_0, x_0))$ be such that $f_1 \cdot h$ is based homotopic to $f_1 \cdot h'$ and let $*$ denote the base point of S^j. Then there exists a homotopy $H: S^j \times I \to A' \sqcap E'$ such that $H(-,0) = f_1 \cdot h$, $H(-,1) = f_1 \cdot h'$ and $H(*,t) = (f_0(a_0), g_1(x_0))$, for all $t \in I$.

Consider the following commutative diagram

$$\begin{array}{ccc} * \times I \cup S^j \times I & \xrightarrow{C_{a_0} \cup p_k \cdot \theta} & A \\ \downarrow & \nearrow & \downarrow f_0 \\ S^j \times I & \xrightarrow{p'_{k'} \cdot H} & A' \end{array},$$

where $C_{a_0}(*,t) = a_0$, $t \in I$, and $\theta(-,0) = h$, $\theta(-,1) = h'$.

Since f_0 is an n-equivalence and dim $(S^j \times I \setminus * \times I \cup S^j \times \dot{I}) \leq n$ (j is at most n-1), there exists, by [18;7.6.22.], a map $H': S^j \times I \to A$ such that $H' | * \times I \cup S^j \times \dot{I} = C_{a_0} \cup p_k \cdot \theta$ and $f_0 \cdot H'$ is homotopic to $p'_{k'} \cdot H$ relative to $* \times I \cup S^j \times \dot{I}$. But, because f_0 is a fibration, we can replace H' by a map $\bar{H}: S^j \times I \to A$ such that the above triangles commute. Now, consider the following commutative diagram .

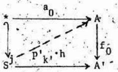


where $\bar{k}: A \sqcap E \to E$ and $\bar{k}': A' \sqcap E' \to E'$ are the projections, and observe that, if γ denotes the map pair $(C_{x_0} \cup \bar{k} \cdot \theta, k \cdot \bar{H})$, then $\bar{k}' \cdot H$ is a lift of $g \cdot \gamma$. Applying (3.1.2) to the above diagram, we obtain a lifting $G: S^j \times I \to E$ of γ such that $g_1 \cdot G$ and $\bar{k}' \cdot H$ are homotopic relative to $g \cdot \gamma$. G and $\bar{H}$ now determine, by the universal property of pullbacks, a unique map $\beta: S^j \times I \to A \sqcap E$ which is a based homotopy between h and h'.

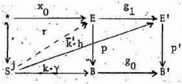
(ii) $(f_1)_*$ is surjective for $0 \leq j \leq n$:

Let $[h] \in \pi_j(A' \sqcap E', (f_0(a_0), g_1(x_0)))$. We show that there exists a based map $\alpha: S^j \to A \sqcap E$ such that $f_1 \cdot \alpha$ is based homotopic to h.

To this end, consider the following commutative diagram



and observe that, because f_0 is an n-equivalence and dim $(S^j \setminus *) \leq n$ (j is at most n), there exists, by [18;7.6.22.], a map $\delta: S^j \to A$ such that δ is a based map and $f_0 \cdot \delta$ is based homotopic to $p'_{k'} \cdot h$. But, because f_0 is a fibration, we can replace δ by a map $\gamma: S^j \to A$ such that the above triangles commute. Now, form the following commutative diagram



and observe that, if ϕ denotes the map pair $(x_0, k \cdot \gamma)$, then $\bar{k}' \cdot h$ is a lifting of $g \cdot \phi$. Hence, by (3.1.2), there exists a lifting $\beta: S^j \to E$ of ϕ such that $g_1 \cdot \beta$ is homotopic to $\bar{k}' \cdot h$ relative to $g \cdot \phi$. The maps β and γ now determine, by the universal property of pullbacks, a unique map $\alpha: S^j \to A \sqcap E$ by the rule, $\alpha(x) = (\gamma(x), \beta(x)), x \in S^j$. In particular, $\alpha(*) = (\gamma(*), \beta(*)) = (a_0, x_0)$ and so α is a based map. It remains to show that $f_1 \cdot \alpha$ is based homotopic to h.

To this end, let $K: S^j \times I \to A'$ be defined by, $K(x,t) = p'_{k'} \cdot h(x)$, $x \in S^j$, $t \in I$, and let $G: S^j \times I \to E'$ denote a homotopy from $g_1 \cdot \beta$ to $\bar{k}' \cdot h$ relative to $g \cdot \phi$. Then,

$p' \cdot G = g_0 k \cdot \gamma = p' \cdot \bar{k}' \cdot h = k' \cdot p'_{k'} \cdot h = k' \cdot K$ and so there exists a unique map

$J:S^j \times I \to A'\sqcap E'$, defined by $J(x,t) = (K(x,t), G(x,t))$. Now, observe that, because $p'_{k'} \cdot f_1 \cdot \alpha = f_0 \cdot \gamma = p'_{k'} \cdot h$ and $\bar{k}' \cdot f_1 \cdot \alpha = g_1 \cdot \bar{k} \cdot \alpha = g_1 \cdot \beta$, J is a based homotopy from $f_1 \cdot \alpha$ to h.//

The following two results are trivial consequences of (3.1.3.).

Corollary 3.1.4. Let $g = (g_1, g_0)$: $p \to p'$ be a map of Hurewicz fibrations, where g_1 is an n-equivalence and g_0 is an (n+1)-equivalence, n finite or infinte, and let $F = p^{-1}(b_0)$ and $F' = (p')^{-1}(g_0(b_0))$. Then $g_1|F:F \to F'$ is an n-equivalence.

Notice that, if the spaces involved are path-connected, then (3.1.4.) is a trivial consequence of the Five Lemma.

Corollary 3.1.5. Let $g = (g_1, g_0): p \to p'$ be a map of Hurewicz fibrations, where g_1 is an n-equivalence and g_0 is an (n+1)-equivalence, n finite or infinite, and let $k:A \to B$ and $k':A \to B'$ be maps such that $g_0 \cdot k = k'$. Then the induced map $\bar{g}_1 : A\sqcap E \to A\sqcap E'$ is an n-equivalence over A.

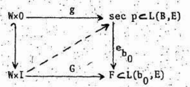
Lemma 3.1.6. Let $p:E \to B$ be a Hurewicz fibration over a CW-complex B and, for a given base point $b_0 \in B$, let $F = p^{-1}(b_0)$. Then, if sec $p \neq \emptyset$, the map

$$e_{b_0} : \sec p \to F,$$

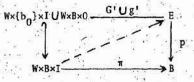
defined by $e_{b_0}(s) = s(b_0)$, $s \in \sec p$, is a Hurewicz fibration with fibre $\sec_* p$.

Notice that, if F is not path-connected, then e_{b_0} need not be surjective.

Proof: For a given space W, let $g:W \to \sec p$ be a map and let $G:W\times I \to F$ be a homotopy of $e_{b_0}\circ g$. We must show that there exists a homotopy $H:W\times I \to \sec p$ such that the following triangles commute:



But, by (1.1.1), the completion of the above diagram is equivalent to the completion of the following diagram



where $G'(w,b_0,t) = G(w,t)(b_0)$, $g'(w,b,0) = g(w,0)(b)$ and $\pi(w,b,t) = b$. Now observe that the latter diagram can be completed by [20; Theorem 4].//

Lemma 3.1.7. Let $p:E \to B$ and $p':E' \to B$ be Hurewicz fibrations over a CW-complex B and let $f:p \to p'$ be an n-equivalence over B, n finite or infinite. For a given base point $b_0 \in B$, let $F = p^{-1}(b_0)$ and $F' = (p')^{-1}(b_0)$. Then, if both sec p and sec p' are non-empty and $\dim B = m \leq n$, f induces a map

$$\bar{f}: \sec p \to \sec p',$$

defined by $\bar{f}(s) = f \cdot s$, $s \in \sec p$, such that the resulting map $\alpha = (\bar{f}, f|F): e_{b_0} \to e'_{b_0}$ is an (n-m)-equivalence. In the case $n = \infty$, α is an ∞-equivalence and there is no condition on the dimension of B.

Notice that, because of (3.1.4), $\bar{f}|: \sec_* p \to \sec_* p'$ is also an (n-m) -equivalence when n is finite and $m \neq 0$ and an ∞-equivalence when n is infinite.

Proof: The commutativity of the diagram

$$\begin{array}{ccc} \sec p & \xrightarrow{\bar{f}} & \sec p' \\ \downarrow{\scriptstyle e_{b_0}} & & \downarrow{\scriptstyle e'_{b_0}} \\ F & \xrightarrow{f|} & F' \end{array}$$

is clear; furthermore, by (3.1.4), $f|: F \to F'$ is an n-equivalence and hence, an (n-m) -equivalence. Therefore, it is sufficient to show that $\bar{f}: \sec p \to \sec p'$ is an (n-m) -equivalence.

Let $s: B \to E$ be the base point of sec p. For each $0 \leq j \leq n-m$, consider the induced homomorphism (function for $j = 0$)

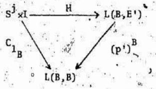
$$\bar{f}_*: \pi_j(\sec p, s) \to \pi_j(\sec p', f \cdot s),$$

defined by $\bar{f}_*([h]) = [\bar{f} \cdot h]$. We show that $\bar{f}_*$ is bijective for $0 \leq j \leq n-m-1$ and surjective for $j = n-m$.

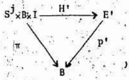
(i) $\bar{f}_*$ is injective for $0 \leq j \leq n-m-1$:

Let $[h], [h'] \in \pi_j(\sec p, s)$ be such that $\bar{f} \cdot h$ is based homotopic to $\bar{f} \cdot h'$ and let $*$ denote the base point of S^j.

Then there exists a homotopy $H: S^j \times I \to \sec p'$ such that $H(-,0) = \bar{f} \cdot h$, $H(-,1) = \bar{f} \cdot h'$ and $H(*,t) = f \cdot s$, for all $t \in I$. Now, observe that H can be viewed as a map $S^j \times I \to L(B,E')$ and the condition that the image of H lies in sec p' is equivalent to the fact that the following diagram commutes

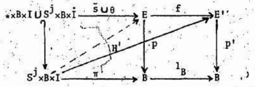


where $(p')^B(g) = p' \cdot g$ and C_{1_B} is the constant map to 1_B. Applying (1.1.1.) to the above diagram, we obtain the following commutative diagram

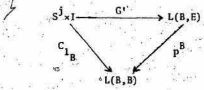


where $H'(x,b,t) = H(x,t)(b)$ and $\pi(x,b,t) = b$.

Now, consider the commutative diagram



where $\bar{s}(*,b,t) = s(b)$ and $\theta(x,b,0) = h(x)(b)$, $\theta(x,b,1) = h'(x)(b)$, and observe that H' is a lifting of $f \cdot \gamma$, where γ denotes the map pair $(\bar{s} \cup \theta, \pi)$. Because $j < n-m$, $\dim.(S^j \times B \times I \setminus * \times B \times I \cup S^j \times B \times \dot{I}) \leq (n-m-1)+m+1 = n$ and hence, by (3.1.2), there exists a lifting $G: S^j \times B \times I \to E$ of γ such that $f \cdot G$ is homotopic to H' relative to $f \cdot \gamma$. Applying (1.1.1.) to the homotopy G we obtain a homotopy $G': S^j \times I \to L(B,E)$, defined by $G'(x,t)(b) = G(x,b,t)$, such that the following diagram commutes:

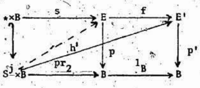


Now observe that the image of G' lies in sec p and moreover, G' is a based homotopy from h to h'.

(ii) $\bar{f}_*$ is surjective for $0 \leq j \leq n-m$:

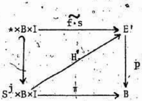
Let $[h] \in \pi_j(\sec p', f \cdot s)$. We show that there exists a based map $\alpha: S^j \to \sec p$ such that $\bar{f} \cdot \alpha$ is based homotopic to h.

To this end, consider the commutative diagram



where $h'(x,b) = h(x)(b)$, and observe that, because for each $x \in S^j$, $h(x) \in \sec p'$, h' is a lift of $f \cdot \gamma$, where γ denotes the map pair (s, pr_2). Since $j \leq n-m$, $\dim.(S^j \times B \setminus * \times B) \leq n-m+m = n$ and so, by (3.1.2.), there exists a lifting $\alpha': S^j \times B \to E$ such that $f \cdot \alpha'$ is homotopic to h' relative to $f \cdot \gamma$. Applying (1.1.1.) to the map α', we obtain the based map $\alpha:S^j \to \sec p$ as required.

It remains to show that $\bar{f}\cdot\alpha$ is based homotopic to h. But, recall that $f.\alpha'$ is homotopic to h' relative to $f\cdot\gamma$. Hence, there exists a homotopy $H':S^j\times B\times I \to E'$ from $f\cdot\alpha'$ to h' such that the following diagram commutes:



Here, $\widetilde{f\cdot s}(*,b,t) = f\cdot s(b)$ and $\pi(x,b,t) = b$. Applying (1.1.1.) to the map H', we now obtain a based homotopy $H:S^j\times I \to \sec p'$ from $\bar{f}\cdot\alpha$ to h.//

If we now consider $p:E \to B$ and $p':E' \to B$ as objects of the admissible category A_F of Hurewicz fibrations with fibres of the homotopy type of a fixed space F, we have the following analogue of (3.1.7.) for the notion of fibre homotopy equivalence.

<u>Lemma</u> 3.1.8. Let $p:E \to B$ and $p':E' \to B$ be objects of the category A_F and let $f:p \to p'$ be a fibre homotopy equivalence over B. Then, if both $\sec p$ and $\sec p'$ are non-empty, the induced map $\alpha = (\bar{f},f|): e_{b_0} \to e'_{b_0}$ is a homotopy equivalence. Furthermore, the restriction $\bar{f}|:\sec_* p \to \sec_* p'$ is also a homotopy equivalence.

<u>Proof:</u> Observe that $f|$ is a homotopy equivalence by [7; Theorem 6.3]; furthermore, if $\bar{f}:\sec p \to \sec p'$ is a homotopy equivalence, then $\bar{f}|:\sec_* p \to \sec_* p'$ is a homotopy equivalence by [5; Corollary 1.5]. Hence, it is sufficient to show that $\bar{f}:\sec p \to \sec p'$ is a homotopy

equivalence.

Now, let $f^{-1}:p' \to p$ denote a fibre homotopy inverse of f over B and define

$$\bar{f}^{-1}:\sec p' \to \sec p$$

by $\bar{f}^{-1}(s') = f^{-1}\cdot s'$. Let $(h,pr_1):p\times 1_I \to p$ be a fibre homotopy such that $h_0 = h(-,0) = f^{-1}\cdot f$ and $h_1 = h(-,1) = 1_E$, and let $(h',pr_1):p'\times 1_I \to p'$ be a fibre homotopy such that $h'_0 = h'(-,0) = f\cdot f^{-1}$ and $h'_1 = h'(-,1) = 1_{E'}$. Then define

$$H:\sec p\times I \to \sec p \quad \text{and} \quad H':\sec p'\times I \to \sec p'$$

by $H(s,t) = h_t\cdot s$ and $H'(s',t) = h'_t\cdot s'$. Since $p\cdot h_t = p$ and $p'\cdot h'_t = p'$, $p\cdot H(s,t) = p\cdot h_t\cdot s = p\cdot s = 1_B$ and $p'\cdot H'(s',t) = p'\cdot h'_t\cdot s' = p'\cdot s' = 1_B$; hence, for all $(s,t) \in \sec p\times I$ and all $(s',t) \in \sec p'\times I$, $H(s,t) \in \sec p$ and $H'(s',t) \in \sec p'$. Furthermore, since the adjoints of H and H' can be identified with the composites $\sec p\times B\times I \xrightarrow{e\times 1_I} E\times I \xrightarrow{h} E$ and $\sec p'\times B\times I \xrightarrow{e'\times 1_I} E'\times I \xrightarrow{h'} E'$, respectively ($e$ and e' are evaluation maps), H and H' are continuous. Now observe that $H(-,0) = \bar{f}^{-1}\cdot\bar{f}$, $H(-,1) = 1_{\sec p}$ and $H'(-,0) = \bar{f}\cdot\bar{f}^{-1}$, $H'(-,1) = 1_{\sec p'}$; hence, $\bar{f}$ and $\bar{f}^{-1}$ are inverse homotopy equivalences.//

Given spaces A and B, we shall denote by $L(A,B;k)$, the space of all maps $A \to B$ which are homotopic to a given map $k:A \to B$. Equivalently, $L(A,B;k)$ is the path component of the function space $L(A,B)$ containing the map k. In case we deal with based spaces and based maps, this component will be denoted by $L^*(A,B;k)$ and the function space of based maps by $L^*(A,B)$.

Now recall that an H-group(*) consists of a based space (B, b_0), together with a continuous multiplication

$$\mu : B\times B \to B$$

that respects base points, satisfying:

(i) if $C_{b_0} : B \to B$ is the constant map to the base point b_0,

$$B \xrightarrow{(1_B, C_{b_0})} B\times B \xrightarrow{\mu} B \quad\text{and}\quad B \xrightarrow{(C_{b_0}, 1_B)} B\times B \xrightarrow{\mu} B$$

are based homotopic to 1_B;

(ii) μ is homotopy associative; that is to say, the square

$$\begin{array}{ccc} B\times B\times B & \xrightarrow{\mu\times 1_B} & B\times B \\ \downarrow{\scriptstyle 1_B\times\mu} & & \downarrow{\scriptstyle \mu} \\ B\times B & \xrightarrow{\mu} & B \end{array}$$

is homotopy commutative (with respect to base points);

(iii) there exists a based map $\phi : B \to B$, called a homotopy inverse for B, such that

$$B \xrightarrow{(1_B, \phi)} B\times B \xrightarrow{\mu} B \quad\text{and}\quad B \xrightarrow{(\phi, 1_B)} B\times B \xrightarrow{\mu} B$$

are based homotopic to C_{b_0}.

Recall, also, that an H-cogroup (**) consists of a based space (A, a_0), together with a continuous comultiplication $\nu : A \to A\vee A$

(*)-For an alternate description see [8;4.10.]; more generally, see [8],[9] and [10].

(**)-For an alternate description see the dual of Theorem 4.10 in [8]; more generally, see [8],[9] and [10].

(here, $A \vee A$ is the one-point union of A with itself) that respects base points, satisfying:

(i) if $C_{a_0}: A \to A$ is the constant map to the base point a_0,

$$A \xrightarrow{\nu} A \vee A \xrightarrow{(1_A, C_{a_0})} A \quad \text{and} \quad A \xrightarrow{\nu} A \vee A \xrightarrow{(C_{a_0}, 1_A)} A$$

are based homotopic to 1_A;

(ii) ν is <u>homotopy associative</u>; that is to say, the square

$$\begin{array}{ccc} A & \xrightarrow{\nu} & A \vee A \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle 1_A \vee \nu} \\ A \vee A & \xrightarrow{\nu \vee 1_A} & A \vee A \vee A \end{array}$$

is homotopy commutative (with respect to base points);

(iii) there exists a based map $\psi: A \to A$, called a homotopy <u>coinverse</u> for A, such that

$$A \xrightarrow{\nu} A \vee A \xrightarrow{(1_A, \psi)} A \quad \text{and} \quad A \xrightarrow{\nu} A \vee A \xrightarrow{(\psi, 1_A)} A$$

are based homotopic to C_{a_0}.

<u>Lemma</u> 3.1.9. If A is an H-cogroup, $L^*(A,B)$ is an H-group, for every space B; if B is an H-group and A is any space, both $L(A,B)$ and $L^*(A,B)$ are H-groups. Finally, if $L(A,B)$ is an H-group, its path components have the same homotopy type. A similar result holds for $L^*(A,B)$.

<u>Proof</u>: We first show that, if A is an H-cogroup, $L^*(A,B)$ is an H-group.

Let $a_0 \varepsilon A$ and $b_0 \varepsilon B$ be base points and define

$$\bar{\nu}: L*(A,B) \times L*(A,B) \longrightarrow L*(A,B)$$

as follows: for any two maps $f,g \varepsilon L*(A,B)$, let $\bar{\nu}(f,g)$ be the composite $\nabla_B \cdot f\vee g \cdot \nu$, where $\nabla_B: B\vee B \to B$ is the folding map. $\bar{\nu}$ is clearly well-defined and continuous. Let us take as base point of $L*(A,B)$, the constant map $C^A_{b_0}$ to b_0 and let $C(L*(A,B))$ denote the constant map of $L*(A,B)$ to base point $C^A_{b_0}$. We want to show that $\bar{\nu}\cdot(1_{L*(A,B)}, C(L*(A,B)))$ is based homotopic to $1_{L*(A,B)}$.

Since A is an H-cogroup, $(1_A, C_{a_0})\cdot\nu$ is based homotopic to 1_A; hence, let $h: A\times I \to A$ denote a based homotopy from $(1_A, C_{a_0})\cdot\nu$ to 1_A and define

$$H: L*(A,B)\times I \to L*(A,B)$$

by the rule, $H(f,t) = f\cdot h_t$. Since the adjoint of H can be identified with the composite $L*(A,B)\times A\times I \xrightarrow{1_{L*(A,B)}\times h} L*(A,B)\times A \xrightarrow{e} B$ (e is the evaluation map), H is continuous; furthermore, because the composition of $C^A_{b_0}$ with any map is $C^A_{b_0}$, H is a based homotopy.

Now observe that $H(f,0) = f\cdot h_0 = f\cdot(1_A, C_{a_0})\cdot\nu = \nabla_B\cdot f\vee C^A_{b_0}\cdot\nu = \bar{\nu}(1_{L*(A,B)}, C(L*(a,B)))(f)$ and $H(f,1) = f\cdot h_1 = f\cdot 1_A = 1_{L*(A,B)}(f)$; hence, $\bar{\nu}(1_{L*(A,B)}, C(L*(A,B)))$ is based homotopic to $1_{L*(A,B)}$. In a similar fashion we show that $\bar{\nu}(C(L*(A,B)), 1_{L*(A,B)})$ is based homotopic to $1_{L*(A,B)}$.

For a homotopy coinverse ψ of A, define

$$\bar{\psi}:L^*(A,B) \to L^*(A,B)$$

by $\bar{\psi}(f) = f\cdot\psi$, for every $f \in L^*(A,B)$. We are going to show that $\bar{\psi}$ is a homotopy inverse for $L^*(A,B)$. In fact, consider the composition

$$L^*(A,B) \xrightarrow{(1_{L^*(A,B)},\bar{\psi})} L^*(A,B)\times L^*(A,B) \xrightarrow{\bar{\nu}} L^*(A,B)$$

and observe that, if $f \in L^*(A,B)$, then $\bar{\nu}\cdot(1_{L^*(A,B)},\bar{\psi})(f) = \nabla_B\cdot(f\vee f\cdot\psi)\cdot\nu = f\cdot(1_A,\psi)\cdot\nu$. But, since ψ is a homotopy coinverse for A, $(1_A,\psi)\cdot\nu$ is based homotopic to C_{a_0}; hence, let $h:A\times I \to A$ denote a based homotopy from $(1_A,\psi)\cdot\nu$ to C_{a_0} and define

$$H:L^*(A,B)\times I \to L^*(A,B)$$

by the rule, $H(f,t) = f\cdot h_t$. One can easily check that H defines a based homotopy from $\bar{\nu}\cdot(1_{L^*(A,B)},\bar{\psi})$ to $C(L^*(A,B))$. In a similar fashion we show that $\bar{\nu}\cdot(\bar{\psi},1_{L^*(A,B)})$ is based homotopic to $C(L^*(A,B))$. The construction of a based homotopy between $\bar{\nu}\cdot(\bar{\nu}\times 1_{L^*(A,B)})$ and $\bar{\nu}\cdot(1_{L^*(A,B)}\times\bar{\nu})$ from a based homotopy between $(\nu\vee 1_A)\cdot\nu$ and $(1_A\vee\nu)\cdot\nu$ is also straightforward.

The proof that $L(A,B)$ is an H-group if B is an H-group is analogous to that above. In this situation the multiplication

$$\bar{\mu}:L(A,B)\times L(A,B) \to L(A,B)$$

is defined as follows: for any two maps $f,g \in L(A,B)$, $\bar{\mu}(f,g)$ is the composite $\mu\cdot f\times g\cdot\Delta_A$, where $\Delta_A:A \to A\times A$ is the diagonal map. A homotopy inverse for $L(A,B)$,

$$\bar{\phi}:L(A,B) \to L(A,B),$$

is defined by $\bar{\phi}(f) = \phi \cdot f$. The multiplication and homotopy inverses for $L^*(A,B)$, where B is an H-group, are defined in the same fashion.

We now prove that the H-group structure of $L^*(A,B)$ implies that its components have the same homotopy type. Indeed, we shall prove that, for every $k \in L^*(A,B)$, $L^*(A,B;k)$ has the same homotopy type as $L^*(A,B,C^A_{b_0})$.

If $f \in L^*(A,B;k)$, $\bar{\nu}(\bar{\psi}(k),f) \simeq \bar{\nu}(\bar{\psi}(k),k) \simeq C^A_{b_0}$;
on the other hand, $\bar{\nu}(k,C^A_{b_0}) = \nabla_B \cdot k \vee C^A_{b_0} \cdot \nu = k \cdot (1_A, C_A) \cdot \nu \simeq k$
and so, if $f \simeq C^A_{b_0}$, $\bar{\nu}(k,f) \simeq \bar{\nu}(k,C^A_{b_0}) \simeq k$. Hence, define

$$\theta : L^*(A,B;k) \longrightarrow L^*(A,B;C^A_{b_0})$$

by $\theta(f) = \bar{\nu}(\bar{\psi}(k),f)$, and

$$\theta' : L^*(A,B;C^A_{b_0}) \longrightarrow L^*(A,B;k)$$

by $\theta'(f) = \bar{\nu}(k,f)$.

Then,

$$\theta'\theta = \theta' \cdot \bar{\nu}(\bar{\psi}(k),-) = \bar{\nu}(k,\bar{\nu}(\bar{\psi}(k),-)) \simeq \bar{\nu}(\bar{\nu}(k,\bar{\psi}(k)),-) \simeq \bar{\nu}(C^A_{b_0},-) \simeq 1_{L^*(A,B;k)}$$

and

$$\theta \cdot \theta' = \theta \cdot \bar{\nu}(k,-) = \bar{\nu}(\bar{\psi}(k),\bar{\nu}(k,-)) \simeq \bar{\nu}(\bar{\nu}(\bar{\psi}(k),k),-) \simeq \bar{\nu}(C^A_{b_0},-) \simeq 1_{L^*(A,B;C^A_{b_0})};$$

hence, θ and θ' are inverse homotopy equivalences.

The proof of the corresponding result for $L(A,B)$ is given in a similar fashion.//

§2. Spaces of $\mathcal{F}$-homotopy Equivalences

Throughout this section we shall always assume that A is an admissible category of fibrations which admits Aspherical n-Universal A-fibrations, n finite or infinite. Notice that, by (2.3.7.), (2.3.8.) and (2.3.9.), any such Aspherical n-Universal A-fibration is Free n-Universal, Grounded n-Universal and equivalently Extension (n+1)-Universal. Because the notion of an n-equivalence of Hurewicz fibrations appears extensively in this section, we wish to remind the reader that if the map on the base spaces is also an (n+1)-equivalence, then the restriction to the fibres is always an n-equivalence (see (3.1.4.)). This fact should be borne in mind whenever the situation arises.

Let $p:E \to B$ be an A-fibration and, for a given base point $b_0 \in B$, let $F = p^{-1}(b_0)$. We denote by $\mathcal{F}(p)$, the space of all $\mathcal{F}$-homotopy equivalences $p \to p$ over B, and by $\mathcal{F}^1(p)$, the space of all $\mathcal{F}$-homotopy equivalences $p \to p$ over B which extend $1:F \to F$. Notice that, both $\mathcal{F}(p)$ and $\mathcal{F}^1(p)$ are non-empty, since $1:p \to p$ belongs to both and furthermore, by (2.1.9.), $\mathcal{F}(p)$ is homeomorphic to $\sec(pp)_{\mathcal{F}}$, the space of all sections to $(pp)_{\mathcal{F}}$: $(EE)_{\mathcal{F}} \to B$, and $\mathcal{F}^1(p)$ is homeomorphic to $\sec_*(pp)_{\mathcal{F}}$, the space of all based sections to $(pp)_{\mathcal{F}}$; that is to say, of all sections to $(pp)_{\mathcal{F}}$ which map $b_0 \in B$ to $1_F \in F*F$. One can easily see that the function $\omega_p:\mathcal{F}(p) \to F*F$, defined by $\omega_p(f) = f|F$, coincides with the map $e_{b_0}:\sec(pp)_{\mathcal{F}} \to F*F$, defined by $e_{b_0}(s) = s(b_0)$, and consequently, by (3.1.6.),

<u>Proposition</u> 3.2.1. $\omega_p : F(p) \to F{*}F$ is a Hurewicz fibration with fibre $F^1(p)$. ∎

We shall show that, if $p: E \to B$ is an A- fibration with dim $B = m$ and p is induced from an Aspherical n-Universal A- fibration $p_n : E_n \to B_n$, n finite or infinite, then $\omega_p : F(p) \to F{*}F$ can be approximated up to $(n-m)$-equivalence by a certain loop fibration whose homotopy, in special situations, is computable. In the case $n = \infty$, the dimension of B can be finite or infinite and the approximation is up to ∞- equivalence.

We proceed as follows: let $p_n : E_n \to B_n$ be an Aspherical n-Universal A- fibration, n finite or infinite, and let $e_0, e_1 : L(I,B_n) \to B_n$ denote the evaluation fibrations at 0 and 1, respectively. Consider the pullback diagram

$$\begin{array}{ccc} E_n & \xleftarrow{\ \pi\ } & L(I,B_n) \sqcap E_n \\ \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle (p_n)_{e_0}} \\ B_n & \xleftarrow{\ e_0\ } & L(I,B_n) \end{array}$$

where π denotes the projection map. Now, define a homotopy $f: L(I,B_n) \times I \to B_n$ from e_0 to e_1, by $f(\ell,t) = \ell(t)$, $\ell \in L(I,B_n)$, and observe that, since p_n is an F- fibration, the following diagram can be completed by a homotopy F :

$$\begin{array}{ccc} (L(I,B_n) \sqcap E_n) \times I & \overset{F}{\dashrightarrow} & E_n \\ \downarrow{\scriptstyle (p_n)_{e_0} \times 1_I} & & \downarrow{\scriptstyle p_n} \\ L(I,B_n) \times I & \xrightarrow{\ f\ } & B_n \end{array}$$

such that (F,f) is an F-homotopy with $F(-,0) = \pi$. Applying (I.3.3.) to the F- map $(F(-,1),e_1)$, we now obtain a map

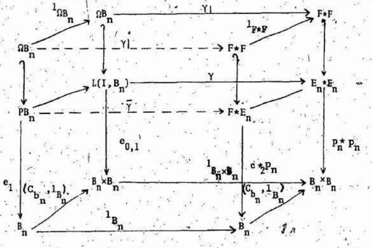
$$\gamma\colon e_{0,1} \to p_n * p_n$$

over $B_n \times B_n$, where $e_{0,1}\colon L(I,B_n) \to B_n \times B_n$ is defined by $e_{0,1}(\ell) = (\ell(0),\ell(1))$ and $\gamma(\ell)(x) = F((\ell,x),1)$.

<u>Proposition</u> 3.2.2. $\gamma\colon e_{0,1} \to p_n * p_n$ is an n-equivalence over $B_n \times B_n$.

<u>Proof</u>: First, observe that $e_{0,1}$ is a Hurewicz fibration; it can be identified with the well-known fibration $L(I,B_n) \to L(\dot{I},B_n)$ which maps an element $f \in L(I,B_n)$ to its restriction $f|\colon \dot{I} \to B_n$.

Now, let b_n be the base point of B_n and let $F = p_n^{-1}(b_n)$. Pulling back the fibrations $e_{0,1}$ and $p_n * p_n$ over the map $(C_{b_n}, 1_{B_n})\colon B_n \to B_n \times B_n$, we obtain the following commutative diagram of fibrations



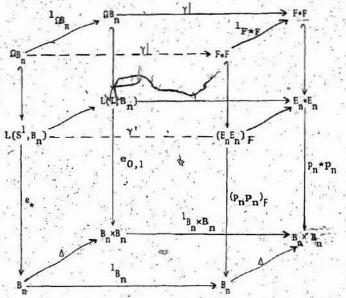
where the induced map $\bar{\gamma}\colon PB_n \to F*E_n$ is just the restriction of γ to PB_n, the space of all paths in B_n originating at the point b_n. The identification of $c*_2p_n$ with the G-space induced from $p_n * p_n$ by the map $(C_{b_n}, 1_{B_n})$ follows from (1.3.9.).

Now, it is well-known that PB_n is a contractible space and, for $0 \leq j \leq n$ and for all choices of base point, $\pi_j(F*E_n) = 0$, since p_n is Aspherical n-Universal; hence, $\bar{\gamma}$ induces isomorphisms in homotopy, for $0 \leq j \leq n$. Applying the Five Lemma to the commutative diagram arising from the exact homotopy sequence of both fibrations and using the fact that B_n is path-connected (p_n is Free n-Universal), we obtain that $\gamma|: \Omega B_n \to F*F$ is an n-equivalence. Applying the Five Lemma once again for the fibrations $e_{0,1}$ and $p_n * p_n$, we obtain that $\gamma: L(I,B_n) \to E_n * E_n$ is an n-equivalence; the bijection $\gamma_*: \pi_0(L(I,B_n),\ell) \to \pi_0(E_n*E_n,\gamma(\ell))$ is immediate, since $\pi_0(L(I,B_n),\ell) \equiv \pi_0(B_n,\ell(0)) \equiv 0$ and $\pi_0(E_n*E_n,\gamma(\ell)) \equiv 0$ from the fibration $F*E_n \to E_n*E_n \xrightarrow{p_n*1p_n} B_n$.//

Recall that the functional fibration $(p_np_n)_F: (E_nE_n)_F \to B_n$ was defined as the pullback of $p_n * p_n$ over the diagonal map $\Delta: B_n \to B_n \times B_n$. If we now pullback $e_{0,1}$ over Δ, we can easily check that we obtain the evaluation fibration $e_*: L(S^1,B_n) \to B_n$, where e_* evaluates at the base point $* \in S^1$. The map $\gamma: e_{0,1} \to p_n * p_n$ over $B_n \times B_n$ then induces a map

$$\gamma': L(S^1,B_n) \to (E_nE_n)_F$$

over B_n such that the following diagram commutes:



Notice that $\gamma': L(S^1,B_n) \to (E_nE_n)_F$ is just the restriction of γ to $L(S^1,B_n)$.

The following result now follows immediately from (3.2.2.) and (3.1.5.).

Proposition 3.2.3. $\gamma': L(S^1,B_n) \to (E_nE_n)_F$ is an n-equivalence over B_n.

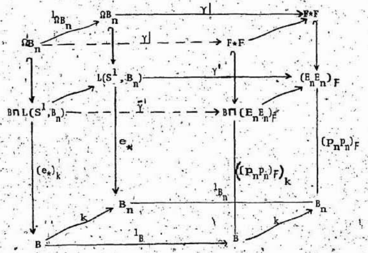
Recall that if A and B are spaces and $k: A \to B$ is a map, then $L(A,B;k)$ denotes the component of $L(A,B)$ containing k. In the case we deal with based spaces and based maps, this component is denoted by $L^*(A,B;k)$. Now, let $a_0 \in A$ be the base point and let $e_{a_0}: L(A,B) \to B$ denote evaluation at the point a_0. It is well-known that e_{a_0} is a Hurewicz fibration and moreover, its restriction to the component $L(A,B;k)$ is also a Hurewicz fibration. If B is path-connected, then e_{a_0} maps $L(A,B;k)$ onto B. Applying the loop functor Ω to e_{a_0}, we obtain a map

$$\Omega e_{a_0} : \Omega L(A,B;k) \to \Omega B$$

defined by, $\Omega e_{a_0}(\ell) = e_{a_0} \cdot \ell$, $\ell \in \Omega L(A,B;k)$; in fact, Ωe_{a_0} is a Hurewicz fibration with fibre $\Omega L^*(A,B;k)$.

Theorem 3.2.4. Let $p_n: E_n \to B_n$ be an Aspherical n-Universal A- fibration, n finite or infinite, and let $p:E \to B$ be an A- fibration with $\dim B = m \leq n$. Let b_0 be the base point of B and let $k:B \to B_n$ be the classifying map for p. Then there exists a fibre preserving map $\alpha = (\alpha_1, \alpha_0)$ from $\Omega e_{b_0}: \Omega L(B,B_n;k) \to \Omega B_n$ to $\omega_p: F(p) \to F*F$ such that α is an (n-m)-equivalence. In case $n = \infty$, α is an ∞-equivalence and there is no condition on the dimension of B.

Proof: Consider the commutative diagram of fibrations



where $\bar{\gamma}': B \sqcap L(S^1, B_n) \to B \sqcap (E_n E_n)_F$ is the map induced by γ'. Since γ' is an n-equivalence (see(3.2.3.)), $\bar{\gamma}'$ is an n-equivalence, by (3.1.5).

Now observe the following facts:

(i) By (1.3.18.), $((p_n)_k(p_n)_k)_F\colon (B\sqcap E_n\ B\sqcap E_n)_F \to B$ can be identified with the induced G-space $((p_np_n)_F)_k\colon B\sqcap(E_nE_n)_F \to B$; furthermore, the evaluation fibration $e_{b_0}\colon \sec\ ((p_n)_k(p_n)_k)_F \to F{*}F$ can be identified with the fibration $\omega_{(p_n)_k}\colon F((p_n)_k) \to F{*}F$.

(ii) Since sec e_* is non-empty, sec $(e_*)_k$ is non-empty; furthermore, because $(e_*)_k$ is the pullback of e_* over k, sec $(e_*)_k$ is homeomorphic to Lift (e_*,k). But, by (1.1.1.), we can identify Lift (e_*,k) with $\Omega L(B,B_n;k)$; hence, the evaluation fibration $e_{b_0}\colon \sec\ (e_*)_k \to \Omega B_n$ coincides with the loop fibration $\Omega e_{b_0}\colon \Omega L(B,B_n;k) \to \Omega B_n$.

If we now apply (3.1.7.) to the n-equivalence γ and make the appropriate identifications with the evaluation fibrations, we obtain an $(n-m)$ - equivalence $\phi = (\beta,\gamma|)$ from $\Omega e_{b_0}\colon \Omega L(B,B_n;k) \to \Omega B_n$ to $\omega_{(p_n)_k}\colon F((p_n)_k) \to F{*}F$.

Since k is the classifying map for p_n, p is F-homotopy equivalent to $(p_n)_k$ over B. But then, by (1.3.19.), $(pp)_F$ is G-homotopy equivalent to $((p_n)_k(p_n)_k)_F$ over B. Let $\eta\colon ((p_n)_k(p_n)_k)_F \to (pp)_F$ denote such a G-homotopy equivalence. If we now apply (3.1.8.) to η and make the appropriate identifications with the evaluation fibrations, we obtain a homotopy equivalence $\psi = (\bar{\eta},\eta|)$ from $\omega_{(p_n)_k}$ to ω_p. The composite $\psi\cdot\phi\colon \Omega e_{b_0} \to \omega_p$ now has the required properties.//

The previous Theorem shows that the problem of computing the homotopy groups of $F(p)$ and $F^1(p)$ for an A-fibration $p\colon E \to B$ is equivalent, in certain dimensions, to computing the homotopy groups of $\Omega L(B,B_n;k)$ and $\Omega L*(B,B_n;k)$, respectively; however, the computation of the homotopy groups of these loop spaces is, in general, a difficult problem. Nevertheless, in some cases we can obtain good answers. For example, if $k = C_{b_n}$, the constant map to a point $b_n \in B_n$, then $p = pr_1: B\times F \to B$ and the loop fibration $\Omega e_{b_0}: \Omega L(B,B_n;C_{b_n}) \to \Omega B_n$ can be identified, via (1.1.1.), to the evaluation fibration $e_{b_0}: L(B,\Omega B_n) \to \Omega B_n$ with fibre $L*(B,\Omega B_n)$; in fact, for a trivial A-fibration $pr_1: B\times F \to B$ we can see directly that $F(pr_1) \equiv sec(pr_1 pr_1)_F \equiv L(B,F*F)$, since $(pr_1 pr_1)_F$ is the trivial fibration $pr_1: B\times F*F \to B$ (see(1.3.20)), and $F^1(p) \equiv sec_*(pr_1 pr_1)_F \equiv L*(B,F*F)$. Hence, in this situation, certain computations are possible. Furthermore, we shall see that, under suitable conditions, the problem of computing the homotopy of $F(p)$ and $F^1(p)$ for an A-fibration $p:E \to B$ with classifying map k not homotopic to a constant map, can be reduced, in certain dimensions, to the simplified problem of computing the homotopy of $F(pr_1)$ and $F^1(pr_1)$.(*)

Theorem 3.2.5. Let $p_n: E_n \to B_n$ be an Aspherical n-Universal A-fibration, n finite or infinite, over an H-group B_n, and let $p: E \to B$ be an A-fibration with $\dim B = m \le n$. Let b_0 be the base point of B and let $k:B \to B_n$ be the classifying map for p. Then there exists a fibre preserving map $\beta = (\beta_1,\beta_0)$ from $e_{b_0}: L(B,\Omega B_n) \to \Omega B_n$ to $\omega_p: F(p) \to F*F$ such that β is an (n-m) - equivalence. In the case $n = \infty$, β is an ∞-equivalence and there is no condition on the dimension of B.

Proof: First observe that, by (3.2.4.), there exists an (n-m) - equivalence

(*)- See (3.2.5.), (3.2.6.) and (3.2.7.).

$\alpha = (\alpha_1, \alpha_0)$ from $\Omega e_{b_0} : \Omega L(B, B_n; k) \to \Omega B_n$ to $\omega_p : F(p) \to F * F$.

Now, let $C_{b_n} : B \to B_n$ denote the constant map to the base point $b_n \in B_n$. Since B_n is an H-group, by (3.1.9), both $L(B, B_n)$ and $L*(B, B_n)$ are H-groups; furthermore, the map

$$\phi : L(B, B_n; C_{b_n}) \to L(B, B_n; k)$$

defined by $\phi(f) = \bar{\mu}(k, f)$, where $\bar{\mu}$ is the multiplication in $L(B, B_n)$, is a homotopy equivalence and so is its restriction

$$\phi| : L*(B, B_n; C_{b_n}) \to L*(B, B_n; k).$$

Now, ϕ induces a map $\phi_k : B_n \to B_n$, defined by $\phi_k(x) = \mu(k, x)$, where μ is the multiplication in B_n, such that the following diagram commutes:

$$\begin{array}{ccc} L(B, B_n; C_{b_n}) & \xrightarrow{\phi} & L(B, B_n; k) \\ \downarrow e_{b_0} & & \downarrow e_{b_0} \\ B_n & \xrightarrow{\phi_k} & B_n \end{array}$$

Notice that ϕ_k is a homotopy equivalence and hence, if we apply the loop functor Ω to the above diagram, we obtain the commutative diagram

$$\begin{array}{ccc} \Omega L(B, B_n; C_{b_n}) & \xrightarrow{\Omega\phi} & \Omega L(B, B_n; k) \\ \downarrow \Omega e_{b_0} & & \downarrow \Omega e_{b_0} \\ \Omega B_n & \xrightarrow{\Omega\phi_k} & \Omega B_n \end{array}$$

where $\Omega\phi_k$ and $\Omega\phi$ are homotopy equivalences. Let us denote the map pair $(\Omega\phi, \Omega\phi_k)$ by η. Now, observe that the loop fibration $\Omega e_{b_0} : \Omega L(B, B_n; C_{b_n}) \to \Omega B_n$ can be identified, via (1.1.1.), with the evaluation fibration $e_{b_0} : L(B, \Omega B_n) \to \Omega B_n$ with fibre $L*(B, \Omega B_n)$ and the map $\alpha \cdot \eta : e_{b_0} \to \omega_p$ has the required properties.//

Theorem 3.2.6. Let $p_n : E_n \to B_n$ be an Aspherical n-Universal A-fibration, n finite or infinite, and let $p: E \to B$ be an A-fibration where B is an H-cogroup and $\dim \cdot B = m \leq n$. Then there exists an (n-m)- equivalence $\delta: L*(B, \Omega B_n) \to F^1(p)$. In the case $n = \infty$, δ is an ∞ - equivalence and there is no condition on the dimension of B.

Proof: Let $k: B \to B_n$ be the classifying map for p. Then, by (3.2.4.) there exists an (n-m) - equivalence $\alpha_1 : \Omega L*(B, B_n; k) \to F^1(p)$. Now, let $C_{b_n} : B \to B_n$ denote the constant map to the base point $b_n \in B_n$. Since B is an H-cogroup, by (3.1.9.), $L*(B, B_n)$ is an H-group and furthermore, the map

$$\theta : L*(B, B_n; C_{b_n}) \to L*(B, B_n; k)$$

defined by $\theta(f) = \bar{\nu}(k, f)$, where $\bar{\nu}$ is the multiplication in $L*(B, B_n)$, is a homotopy equivalence; hence,

$$\Omega\theta : \Omega L*(B, B_n; C_{b_n}) \to \Omega L*(B, B_n; k)$$

is a homotopy equivalence.

Now, form the composite $\alpha_1 \cdot \Omega\theta$ and observe that $\Omega L*(B, B_n; C_{b_n})$ can be identified, via (1.1.1.), with the space $L*(B, \Omega B_n)$.//

If we now assume that the classifying space B_∞ is n-connected,

$n > 0$, then $F*F$ is $(n-1)$ - connected and from the fibration $\omega_p: F(p) \to F*F$ we obtain that

for $0 \leq j \leq n-1$, $\pi_j(F^1(p)) \cong \pi_j(F(p))$.

This observation can actually be extended to the following stable range result.

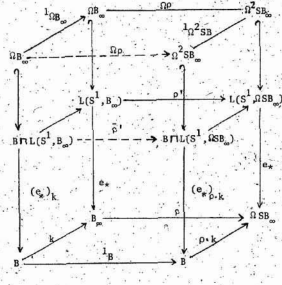
Theorem 3.2.7. Let $p_\infty: E_\infty \to B_\infty$ be an Aspherical ∞ - Universal A-fibration and let $p: E \to B$ be an A-fibration with classifying map k. If B_∞ is n-connected, $n > 0$, and $\dim B = m < 2n$, then, for $0 \leq j \leq 2n-m-1$, $\pi_j(F(p)) \cong \pi_j(L(B,F*F))$ and $\pi_j(F^1(p)) \cong \pi_j(L*(B,F*F))$.

Proof: Let SB_∞ denote the suspension of B_∞ and let $\rho: B_\infty \to \Omega SB_\infty$ denote the adjoint of the identity map on SB_∞. Since B_∞ is n-connected, by [18;8.5.10 and 7.5.9], ρ is a $(2n+1)$ - equivalence. Now, observe that the following diagram commutes

$$\begin{array}{ccc} L(S^1,B_\infty) & \xrightarrow{\rho'} & L(S^1,\Omega SB_\infty) \\ \Big\downarrow e_* & & \Big\downarrow e_* \\ B_\infty & \xrightarrow{\rho} & \Omega SB_\infty \end{array}$$

where ρ' is the obvious map induced by ρ; furthermore, the restriction of ρ' to the fibre ΩB_∞ is precisely the map $\Omega\rho: \Omega B_\infty \to \Omega^2 SB_\infty$. Since ρ is a $(2n+1)$ - equivalence, $\Omega\rho$ is a $2n$ - equivalence; hence, by the Five Lemma, ρ' is a $2n$ - equivalence.

Next, consider the commutative diagram



and observe that, by (3.1.5.), the induced map $\bar{\rho}'$ is a 2n - equivalence. By (3.1.7.), $\bar{\rho}'$ induces a (2n-m) - equivalence $\phi = (B, \Omega\rho)$ from $e_{b_0} : \sec (e_*)_k \to \Omega B_\infty$ to $e_{b_0} : \sec (e_*)_{\rho \cdot k} \to \Omega^2 SB_\infty$. However, recall that $e_{b_0} : \sec (e_*)_k \to \Omega B_\infty$ can be identified with $\Omega e_{b_0} : \Omega L(B, B_\infty; k) \to \Omega B_\infty$ and furthermore, via (1.1.1), $e_{b_0} : \sec (e_*)_{\rho \cdot k} \to \Omega^2 SB_\infty$ can be identified with $\Omega e_{b_0} : \Omega L(B, \Omega SB_\infty; \rho \cdot k) \to \Omega^2 SB_\infty$; hence, for $0 \leq j \leq 2n-m-1$,

$$\pi_j(\Omega L(B, B_\infty; k)) \cong \pi_j(\Omega L(B, \Omega SB_\infty; \rho \cdot k)) \text{ and } \pi_j(\Omega L*(B, B_\infty; k)) \cong \pi_j(\Omega L*(B, \Omega SB_\infty; \rho \cdot k))$$

But, by (3.2.4.), there exists an ∞ - equivalence $\alpha = (\alpha_1, \alpha_0)$ from $\Omega e_{b_0} : \Omega L(B, B_\infty; k) \to \Omega B_\infty$ to $\omega_p : F(p) \to F*F$; hence, for $0 \leq j \leq 2n-m-1$,

$$\pi_j(F(p)) \cong \pi_j(\Omega L(B,B_\infty;k)) \cong \pi_j(\Omega L(B,\Omega SB_\infty;\rho\cdot k))$$

and

$$\pi_j(F^1(p)) \cong \pi_j(\Omega L*(B,B_\infty;k)) \cong \pi_j(\Omega L*(B,\Omega SB_\infty;\rho\cdot k))$$

Finally, observe that since ΩSB_∞ is an H-group, (3.1.9.) shows that we can replace $\Omega L(B,\Omega SB_\infty;\rho\cdot k)$ by $\Omega L(B,\Omega SB_\infty;c)$ and $\Omega L*(B,\Omega SB_\infty;\rho\cdot k)$ by $\Omega L*(B,\Omega SB_\infty;c)$ in the computations above (here $c:B \to \Omega SB_\infty$ denotes any constant map). Moreover, via (1.1.1.), we can identify $\Omega L(B,\Omega SB_\infty;c)$ with $L(B,\Omega^2 SB_\infty)$ and $\Omega L*(B,\Omega SB_\infty;c)$ with $L_*(B,\Omega^2 SB_\infty)$; hence, for $0 \leq j \leq 2n-m-1$,

$$\pi_j(F(p)) \cong \pi_j(L(B,\Omega^2 SB_\infty)) \cong \pi_j(L(B,F*F))$$

and

$$\pi_j(F^1(p)) \cong \pi_j(L*(B,\Omega^2 SB_\infty)) \cong \pi_j(L*(B,F*F)). //$$

We conclude this section with the following observation:

Let $p: E \to B$ and $p':E' \to B$ be F-homotopy equivalent A-fibrations over B and for a given base point $b_0 \in B$, let $F = p^{-1}(b_0)$ and $F = (p')^{-1}(b_0)$. Then, by (2.1.7.), the space $F_1(p,p')$ of all F-maps $p \to p'$ is just the space of all F-homotopy equivalences $p \to p'$ over B and furthermore, $F_1(p,p')$ is homeomorphic to sec $(pp')_F$, the space of all sections to $(pp')_F$: $(EE')_F \to B$. If $f: p \to p'$ is an F-homotopy equivalence over B, then $F_1^f(p,p')$ will denote the space of all F-homotopy equivalences $p \to p'$ over B which extend $f|: F \to F$. By (2.1.9.), $F_1^f(p,p')$ is homeomorphic to $\sec_*(pp')_F$, the space of all based sections to $(pp')_F$; that is to say, it is the subspace of sec $(pp')_F$ which maps $b_0 \in B$ to $f| \in F*F$. Notice that because $e_{b_0}: \sec (pp')_F \to F*F$ is not necessarily onto, $e_{b_0}^{-1}(1:F \to F)$ may be empty. Now, by (1.3.19.), the functional fibration $(pp')_F: (EE')_F \to B$ is G-homotopy equivalent to $(pp)_F: (EE)_F \to B$ over B. Let $\eta: (pp)_F \to (pp')_F$ denote such a G-homotopy equivalence over B. Applying (3.1.8.) to η and making the appropriate identifications with the spaces of sections we obtain a homotopy equivalence $\phi = (\bar{\eta},\eta|):\omega_p \to \omega_{(p,p')}$, where $\omega_{(p,p')}$ is the fibration $F_1(p,p') \to F*F$. Hence, the conclusions of (3.2.4.), (3.2.5.), (3.2.6.) and (3.2.7.) remain valid if we replace the fibration ω_p by the more general fibration $\omega_{(p,p')}$, for F-homotopy equivalent A-fibrations p and p'.

§3. Some Results on Groups of Gauge Transformations

We now apply the general theory of section 2 to the admissible category A_G of numerable principal G-bundles to obtain some specific calculations of the homotopy of spaces of bundle equivalences. Recall that, for any n finite or infinite, there exists an Aspherical n-Universal principal G-bundle; namely, the (Aspherical) ∞- Universal G-bundle

$$p_G: E_G \to B_G$$

obtained by the Milnor construction [16]. Because the general results of section 2, for $p_\infty: E_\infty \to B_\infty$ an Aspherical ∞-Universal A-fibration, actually have a stronger formulation in A_G, we give the corresponding results for $p_G: E_G \to B_G$.(*) The results of section 2 in A_G, for $p_n: E_n \to B_n$ an Aspherical n-Universal principal G-bundle, n finite, remain the same as in a general admissible category.

(*)- See (3.3.3.), (3.3.4.), (3.3.5.) and (3.3.6.).

Let $p:E \to B$ be a numerable principal G-bundle and for a given base point $b_0 \in B$, let $G = p^{-1}(b_0)$. We shall denote by $\mathcal{G}(p)$, the space of all G-automorphisms of p, and by $\mathcal{G}^1(p)$, the space of all G-automorphisms of p which extend $1:G \to G$. Because the only G-map $G \to G$ which has a fixed point is 1_G, $\mathcal{G}^1(p)$ can also be viewed as the space of all based G-automorphisms of p. Notice that the spaces $\mathcal{G}(p)$ and $\mathcal{G}^1(p)$ are actually topological groups, the group operation being given by composition of functions, and moreover, by (2.1.9.), $\mathcal{G}(p)$ is homeomorphic to $\sec(pp)_G$ and $\mathcal{G}^1(p)$ is homeomorphic to $\sec_*(pp)_G$. It should also be noted that the group $\mathcal{G}(p)$ appears in theoretical physics, where it is called the group of all gauge transformations of p. In the physics context G is a compact, connected Lie group and the principal G-bundle p is actually a smooth bundle over a smooth manifold B. Since the results we obtain are of a more general nature, we elect to stay in the general admissible category A_G and still call $\mathcal{G}(p)$ the group of gauge transformations of p.

Now consider the ∞-Universal G-bundle $p_G:E_G \to B_G$ and observe that by (3.2.2.), p_G has the property that the map $\gamma:e_{0,1} \to p_G * p_G$ is an ∞-equivalence over $B_G \times B_G$. But, recall that, for any principal G-bundle and, in particular, for the ∞-Universal G-bundle $p_G:E_G \to B_G$, the associated principal fibration $c*_2p_G:G*E_G \to B_G$ coincides with p_G and the total space E_G is contractible; hence, the induced map $\tilde{\gamma}:PB_G \to G*E_G \cong E_G$ over B_G is not only an ∞-equivalence (see proof of (3.2.2.)) but also a homotopy equivalence and, by [7; Theorem 6.1], a fibre homotopy equivalence over B_G. Applying [7; Theorem 6.3] to $\tilde{\gamma}$, we obtain the stronger result:

Proposition 3.3.1. $\gamma: L(I,B_G) \to E_G * E_G$ is a fibre homotopy equivalence over $B_G \times B_G$.

Since the restriction of the induced map $\gamma': L(S^1,B_G) \to (E_G E_G)_G$, in (3.2.3.), to ΩB_G is precisely the map $\gamma|: \Omega B_G \to G$, by [7;Theorem 6.3] and (3.3.1.) above, we also have:

Proposition 3.3.2. $\gamma': L(S^1,B_G) \to (E_G E_G)_G$ is a fibre homotopy equivalence over B_G.

This now gives us the following formulation of (3.2.4.) in A_G for for the case $n = \infty$.

Theorem 3.3.3. Let $p_G:E_G \to B_G$ be the ∞-Universal G-bundle obtained by the Milnor construction and let $p:E \to B$ be any numerable principal G-bundle. Let b_0 be the base point of B and let $k:B \to B_G$ be the classifying map for p. Then there exists a fibre preserving map $\alpha = (\alpha_1,\alpha_0)$ from $\Omega e_{b_0}: \Omega L(B,B_G;k) \to \Omega B_G$ to $\omega_p: \mathcal{G}(p) \to G$ such that α_0,α_1 and $\alpha_1|: \Omega L*(B,B_G;k) \to \mathcal{G}^1(p)$ are homotopy equivalences.

Proof: By (3.3.2), $\gamma': L(S^1,B_G) \to (E_G E_G)_G$ is a fibre homotopy equivalence over B_G; hence, by [7; Theorem 6.3], the induced map $\bar{\gamma}': B\sqcap L(S^1,B_G) \to B\sqcap(E_G E_G)_G$ is also a fibre homotopy equivalence over B. If we now apply (3.1.8), instead of (3.1.7), to the map $\bar{\gamma}'$, in the proof of (3.2.4.), the resulting map $\phi = (\beta,\gamma|): \Omega e_{b_0} \to \omega_{(P_G)_k}$ has the property that $\gamma|$, β and $\beta|$ are homotopy equivalences. Setting $\alpha = \psi\cdot\phi$, as in the proof of (3.2.4), gives the required result.//

(3.2.5.) now has the following formulation in A_G for the case $n = \infty$.

Theorem 3.3.4. Let $p_G:E_G \to B_G$ be the Milnor bundle, where B_G is an H-group, and let $p:E \to B$ be any numerable principal G-bundle. Then, for a given base point $b_0 \in B$, the evaluation fibration $e_{b_0}: L(B,G) \to G$ is fibre homotopically equivalent to $\omega_p: \mathcal{G}(p) \to G$ over G.

Proof: Let $k:B \to B_G$ be the classifying map for p. Then, by (3.3.3.), there exists a homotopy equivalence $\alpha = (\alpha_1,\alpha_0)$ from $\Omega e_{b_0}: \Omega L(B,B_G;k) \to \Omega B_G$ to $\omega_p: \mathcal{G}(p) \to G$. Now, construct the homotopy equivalence $\eta = (\eta_1,\eta_0)$ from $\Omega e_{b_0}: \Omega L(B,B_G;C_{b_\infty}) \to \Omega B_G$ to $\Omega e_{b_0}: \Omega L(B,B_G;k) \to \Omega B_G$ as in the proof of (3.2.5.). Identifying $\Omega e_{b_0}: \Omega L(B,B_G;C_{b_\infty}) \to \Omega B_G$ with the evaluation fibration $e_{b_0}: L(B,\Omega B_G) \to \Omega B_G$, we obtain a homotopy equivalence $\alpha\cdot\eta = (\alpha_1\cdot\eta_1,\alpha_0\cdot\eta_0): e_{b_0} \to \omega_p$.

Let $g = \alpha_0\cdot\eta_0: \Omega B_G \to G$ and let $g^{-1}:G \to \Omega B_G$ be a homotopy inverse of g. Then g^{-1} induces a map $\bar{g}^{-1}: L(B,G) \to L(B,\Omega B_G)$, defined in the obvious manner, such that the following diagram commutes:

$$\begin{array}{ccc} L(B,G) & \xrightarrow{\ \bar{g}^{-1}\ } & L(B,\Omega B_G) \\ \downarrow{\scriptstyle e_{b_0}} & & \downarrow{\scriptstyle e_{b_0}} \\ G & \xrightarrow{\ g^{-1}\ } & \Omega B_G \end{array}$$

Now, it is easily verified that $\bar{g}^{-1}$ is a homotopy equivalence; hence, by [5; Corollary 1.5], $\bar{g}^{-1}|: L*(B,G) \to L*(B,\Omega B_G)$ is a homotopy equivalence. Let us denote the homotopy equivalence $(\bar{g}^{-1},g^{-1})$ by θ. Then $\alpha\cdot\eta\cdot\theta = (\alpha_1\cdot\eta_1\cdot\bar{g}^{-1},g\cdot g^{-1})$ defines a homotopy equivalence from

$e_{b_0}: L(B,G) \to G$ to $\omega_p: \mathcal{G}(p) \to G$. But, since $g \cdot g^{-1}$ is homotopic to 1_G we can replace the homotopy equivalence $\alpha_1 \cdot \eta_1 \cdot \bar{g}^{-1}$ by a map $\beta : L(B,G) \to \mathcal{G}(p)$ such that β is fibre homotopic to $\alpha_1 \cdot \eta_1 \cdot \bar{g}^{-1}$ and β covers $1: G \to G$.//

We now study $\mathcal{G}(p)$ and $\mathcal{G}^1(p)$ for p a numerable principal U-bundle over a sphere $S^n (n \geq 1)$, where U is the infinite unitary group.

Recall that the classifying space BU of U is of the same homotopy type as ΩSU and hence, is an H-group; moreover, from the Bott periodicity theorem we have that $\Omega U \simeq BU \times \mathbb{Z}$ and so, for $i \geq 1$,

$$\pi_{i-1}(U) \cong \pi_i(BU) \cong \begin{cases} \mathbb{Z}, & i \text{ even} \\ 0, & i \text{ odd} \end{cases}$$

Now consider the fibration $\omega_p: \mathcal{G}(p) \to U$. By (3.3.4.), ω_p is fibre homotopically equivalent to the evaluation fibration $e_*: L(S^n,U) \to U$ over U, $* \in S^n$. Thus, for $i \geq 0$, $\pi_i(\mathcal{G}^1(p)) \cong \pi_i(L*(S^n,U))$ and $\pi_i(\mathcal{G}(p)) \cong \pi_i(L(S^n,U))$. But, for $i \geq 0$, $\pi_i(L*(S^n,U)) \cong \pi_{i+n}(U)$ (see [22;2.10]) ; hence,

if n is even, $\pi_i(\mathcal{G}^1(p)) \cong \begin{cases} 0, & i \text{ even} \\ \mathbb{Z}, & i \text{ odd} \end{cases}$

and, if n is odd, $\pi_i(\mathcal{G}^1(p)) \cong \begin{cases} \mathbb{Z}, & i \text{ even} \\ 0, & i \text{ odd} \end{cases}$.

Now, in view of [13;Theorem 2.2]

$L(S^n,U) \cong U \times L*(S^n,U)$;

thus, for $i \geq 0$,

$$\pi_i(\mathcal{G}(p)) \cong \pi_i(U) \oplus \pi_i(L*(S^n,U)) \cong \pi_i(U) \oplus \pi_i(\mathcal{G}^1(p))$$

and hence;

if n is even, $\pi_i(\mathcal{G}(p)) \cong \begin{cases} 0, & i \text{ even} \\ \mathbb{Z} \oplus \mathbb{Z}, & i \text{ odd} \end{cases}$

and, if n is odd, $\pi_i(\mathcal{G}(p)) \cong \begin{cases} \mathbb{Z}, & i \text{ even} \\ \mathbb{Z}, & i \text{ odd} \end{cases}$

The next result is a stronger version of (3.2.6.) in A_G for the case $n = \infty$. Its proof follows easily from (3.3.3) and (3.1.9.)

Theorem 3.3.5. Let $p_G: E_G \to B_G$ be the Milnor bundle and let $p:E \to B$ be any numerable principal G-bundle. If B is an H-cogroup, then $\mathcal{G}^1(p)$ has the same homotopy type as $L*(B,G)$.

Notice that, as a consequence of this result, for a numerable principal G-bundle p over a sphere $S^n (n \geq 1)$, the homotopy of the group $\mathcal{G}^1(p)$ is completely determined by the homotopy of G; more precisely, for $i \geq 0$, $\pi_i(\mathcal{G}^1(p)) \cong \pi_i(L*(S^n,G)) \cong \pi_{i+n}(G)$.

Since the group $G*G$ of all G-automorphisms of G can be identified with G, we have the following formulation of (3.2.7) in A_G.

Theorem 3.3.6. Let $p:E \to B$ be a numerable principal G-bundle. If G is $(n-1)$ - connected, $n > 0$, and $\dim B = m < 2n$, then, for $0 \leq i \leq 2n-m-1$, $\pi_i(\mathcal{G}^1(p)) \cong \pi_i(L*(B,G))$ and $\pi_i(\mathcal{G}(p)) \cong \pi_i(L(B,G))$.

If p is any numerable principal G-bundle over a sphere $S^j (j \geq 1)$, *where G is $(n-1)$ - connected, $n > 0$, and $j < 2n$, then combining* (3.3.5.) and (3.3.6.) we have that

$$\text{for } i \geq 0, \quad \pi_i(\mathcal{G}^1(p)) \cong \pi_{i+j}(G)$$

and, for $0 \leq i \leq 2n-j-1$, $\pi_i(\mathcal{G}(p)) \cong \pi_i(L(S^j,G)) \cong \pi_i(G) \oplus \pi_{i+j}(G)$

Now, further computations of the homotopy of $\mathcal{G}(p)$ and $\mathcal{G}^1(p)$ can also be obtained for numerable principal G-bundles p in which the fibration $\omega_p : \mathcal{G}(p) \to G$ has a section. This is a consequence of the following result.

Theorem 3.3.7. Let $p: E \to B$ be a numerable principal G-bundle and, for a given base point $b_0 \in B$, let $G = p^{-1}(b_0)$. Then the following statements are equivalent:

(i) ω_p has a section

(ii) ω_p is fibre homotopically equivalent to $e_{b_0} : L(B,G) \to G$ over G

(iii) there exists a fibre homotopy equivalence $\alpha: B \times G \to (EE)_G$ over B which extends $1: G \to G$

(iv) the G-morphism $(\nu, pr_1): p|G \times 1_G \to p$, defined by $\nu(a,g) = g\cdot a$, can be extended to a G-morphism $(\mu, pr_1): p \times 1_G \to p$.

Proof: The equivalence of (iii) and (iv) is a consequence of (1.3.3.) and the equivalence of (i) and (iii) is a consequence of (1.1.1.)

(ii) $\Longrightarrow$(i): Let $f: e_{b_0} \to \omega_p$ be a fibre homotopy equivalence over G. Now, e_{b_0} has a section, namely, $s: G \to L(B,G)$ defined by $s(g) = C_g$, the constant map to $g \in G$; hence, $f\cdot s: G \to \mathcal{G}(p)$ defines a section to ω_p.

(iii) $\Longrightarrow$(ii): Let $g: pr_1 \to (pp)_G$ be a fibre homotopy equivalence over B which extends $1: G \to G$. Then, by (3.1.8.), g induces a homotopy equivalence $\bar{g}: \sec pr_1 \to \sec(pp)_G$ over G. Identifying $e_{b_0} : \sec pr_1 \to G$ with $e_{b_0} : L(B,G) \to G$ and $e_{b_0} : \sec(pp)_G \to G$ with $\omega_p : \mathcal{G}(p) \to G$ and applying [7;Theorem 6.1], we have that $\bar{g}: e_{b_0} \to \omega_p$ is a fibre homotopy equivalence over G.//

Generally speaking it is difficult to determine if sec $\omega_p \neq \emptyset$ for a given numerable principal G-bundle p; however, using (3.3.7.) one can obtain some examples. The most obvious example of a bundle p in which ω_p has a section is, of course, the trivial G-bundle $pr_1: B \times G \to B$. Another example is given by a numerable principal G-bundle p in which the classifying space B_G for the group G is an H-group; this is a consequence of (3.3.4.) and (ii) of (3.3.7.). Using (iv) of (3.3.7.), we can produce two more examples; namely,

(1) If $p:E \to B$ is a numerable principal G-bundle, where G is any abelian group, then, because of commutativity in G, the right action $\mu: E \times G \to E$ defines a right G-map; in fact, one can easily check that $(\mu, pr_1): p \times 1_G \to p$ defines a G-morphism which extends the G-morphism (ν, pr_1) given in (iv) of (3.3.7.). Hence, ω_p has a section. Notice that, because μ actually gives rise, by (1.3.3.), to a homeomorphism $\mu': B \times G \to (EE)_G$ over B, ω_p can be identified with $e_{b_0}: L(B,G) \to G$.

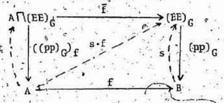
(2) Let G be a closed, normal subgroup of a topological group H and let $p:H \to H/G$ denote the projection. Let $\mu: H \times G \to H$ be defined by $\mu(h,g) = g \cdot h$, where $\cdot$ denotes the multiplication in H. Then, because G is normal in H, the following diagram commutes:

$$\begin{array}{ccc} H \times G & \xrightarrow{\mu} & H \\ \downarrow{\scriptstyle p \times 1_G} & & \downarrow{\scriptstyle p} \\ H/G \times G & \xrightarrow{pr_1} & H/G \end{array}$$

Since multiplication in H is associative, μ is a right G-map and furthermore, (μ, pr_1) extends the G-morphism (ν, pr_1) given in (iv) of (3.3.7.). Hence, ω_p has a section. Notice that, because μ actually gives rise to a homeomorphism $\mu': H/G \times G \to (HH)_G$ over H/G (see(1.3.3.)), ω_p can be identified with the evaluation fibration $e_{b_0}: L(H/G, G) \to G$, $b_0 \in H/G$.

<u>Theorem</u> 3.3.8. Let $p: E \to B$ be a numerable principal G-bundle and let $f: A \to B$ be a map. If $\omega_p: \mathcal{G}(p) \to G$ has a section, then $\omega_{p_f}: \mathcal{G}(p_f) \to G$ has a section.

<u>Proof</u>: Consider the following pullback diagram



and observe that if $s \in \sec(pp)_G$, $s \cdot f \in \mathrm{Lift}((pp)_G, f)$ and hence $t: A \to A \sqcap (EE)_G$, defined by $t(a) = (a, s\cdot f(a))$, is a section to $((pp)_G)_f$. Therefore, define $\phi: \sec(pp)_G \to \sec((pp)_G)_f$ by $\phi(s) = t$. The continuity of ϕ is a straight forward application of (1.1.1.) and the universal property of pullbacks. If $a_0 \in A$ is the base point, $e_{a_0} \cdot \phi = e_{f(a_0)}$; hence, ϕ defines a fibre preserving map over G.

Now, observe that, since p_f is induced from p, by (1.3.18.), $(p_f p_f)_G: (A \sqcap E, A \sqcap E)_G \to A$ can be identified with $((pp)_G)_f: A \sqcap (EE)_G \to A$. Thus, if we identify $e_{f(a_0)}: \sec(pp)_G \to G$ with $\omega_p: \mathcal{G}(p) \to G$, $e_{a_0}: \sec(p_f p_f)_G \to G$ with $\omega_{p_f}: \mathcal{G}(p_f) \to G$ and let $\nu: G \to \mathcal{G}(p)$ be a section, then $\phi \cdot \nu$ defines a section to ω_{p_f}.//

The existence of at least one non-trivial, numerable principal G-bundle for which sec $\omega_p \neq \emptyset$ can thus be used to generate many examples of such G-bundles. In fact, if the ∞- Universal G-bundle p_G is such that ω_{p_G} has a section, then every numerable principal G-bundle has this property. This is the situation when the classifying space is an H-group or when the group G is abelian.

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