EQUIVARIANT ALGEBRAIC TOPOLOGY AND THE EQUIVARIANT BROWN REPRESENTABILITY THEOREM

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Equivariant Algebraic Topology and the Equivariant Brown Representability Theorem

by

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A thesis submitted to the School of Graduate
Studies in partial fulfillment of the
requirements for the degree of
Master of Science

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Memorial University of Newfoundland
1992

St. John's Newfoundland

ISBN 0-315-82661-4

Abstract

The main purpose of this thesis is to give a complete proof of the Equivariant Brown Representability Theorem, in the process developing the equivariant algebraic topology needed in the final proof. The proof of the theorem in the category of path-connected G-spaces is given in Chapter 4 and follows the proof of the non-equivariant case given in Spanier ([Sp], pp. 406-411). There is another account of the proof given in Switzer ([Sw], pp. 152-157), which is closer to the original account given by Brown [Br1]. Equivariant versions of the theorem are announced in [LMS] and [V], for example, but no details of the proofs are given.

In Chapter 1, the basic theory of G-spaces and G-maps is presented. G-final and G-initial structures on a set are defined and sufficient conditions are given which allow such G-space structures to be constructed.

The equivariant homotopy groups are defined in Chapter 2 and the isomorphisms $\pi_n^H(X) \cong \pi_n(X^H)$ and $\pi_n^H(X, A) \cong \pi_n(X^H, A^H)$ are established. These two results are then used to prove the results about equivariant homotopy groups needed in Chapter 4.

In Chapter 3, G-CW-complexes are defined and all the necessary homotopic properties of G-CW-complexes are developed, culminating in the proof of the Equivariant Whitehead Theorem.

Finally, in Chapter 5, we prove an equivariant version of the statement that if a functor satisfies the Wedge Axiom and the Mayer-Victoris Axiom given in Brown's original version of his theorem, then it also satisfies the Equalizer Axiom given in Spanier's version of the theorem. This immediately gives a Switzer style version of the main result.

Acknowledgements

I would like to thank the School of Graduate Studies and the Department of Mathematics and Statistics for their financial support during the writing of this thesis. I would especially like to thank Dr. Peter Booth for his suggestions, encouragement, and valuable advice given during our numerous discussions concerning the topics and results presented in the thesis.

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Chapter 1 Equivariant Initial and Final G-space Structures

Let G be a topological group and let X be a topological space. Then X is called a G-space if there is a map (continuous function)

$$\phi : G \times X \to X$$

such that the following conditions are satisfied:

(i) $\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$ for $g_1, g_2 \epsilon G, x \epsilon X$

(ii) $\phi(e, x) = x$ for all $x \epsilon X$, where e is the identity in G.

Usually $\phi(g, x)$ will be denoted by gx, so that (i) and (ii) become:

(i) $g_1(g_2 x) = (g_1 g_2)x$ for $g_1, g_2 \epsilon G,\ x \epsilon X$

(ii) $ex = x$ for all $x \epsilon X$

Such a ϕ is called an action of G on X.

For each $x \epsilon X$, $Gx = \{gx | g \epsilon G\}$ is called the orbit of x. If $S \subseteq X$ then $GS = \bigcup_{x \epsilon S} Gx$. A subset $A \subseteq X$ is fixed by G if $GA = A$.

If X and Y are G-spaces then a map $f : X \to Y$ is said to be equivariant or a G-map if for all $x \epsilon X$, $g \epsilon G$ we have $f(gx) = gf(x)$. Hence f is a G-map iff the following diagram commutes

$$\begin{array}{ccc} G \times X & \xrightarrow{\phi_X} & X \\ {\scriptstyle 1_G \times f} \downarrow & & \downarrow {\scriptstyle f} \\ G \times Y & \xrightarrow[\phi_Y]{} & Y \end{array}$$

where ϕ_X and ϕ_Y are the actions of G on X and Y respectively.

1.1 Proposition. The composition of two G-maps is a G-map.

Proof: Assume $f : X \to Y$ and $h : Y \to Z$ are G-maps. Then

$$\begin{aligned}(h\circ f)(gx) &= h(f(gx))\\ &= h(gf(x))\\ &= gh(f(x))\\ &= g(h\circ f)(x) \qquad \blacksquare\end{aligned}$$

Examples of G-spaces

1. Trivial G-spaces

Any topological space X can be given the trivial action of G, defined by

$$\begin{aligned}\phi : G\times X &\rightarrow X\\ \phi(g,x) &= x.\end{aligned}$$

In particular the spaces $E^n = \{x\epsilon\mathbb{R}^n |\ |x| \leq 1\}$, $S^n = \{x\epsilon\mathbb{R}^{n+1}|\ |x| = 1\}$, and $I^n = \{(x_1, x_2, \ldots, x_n)\epsilon\mathbb{R}^n | 0 \leq x_i \leq 1,\ 1 \leq i \leq n\}$ will always be given the trivial action of G.

2. Coset or homogeneous G-spaces

If H is a closed subgroup of G and G is locally compact Hausdorff then the set of left cosets $\frac{G}{H}$, with the quotient topology, is a G-space with action defined by

$$\begin{aligned}\phi : G\times \frac{G}{H} &\rightarrow \frac{G}{H}\\ \phi(g', gH) &= g'gH\end{aligned}$$

To show that ϕ is continuous consider the composite

$$G\times G \xrightarrow{1_G\times q} G\times\frac{G}{H}\xrightarrow{\phi}\frac{G}{H}$$

where q is the quotient map. Since G is locally compact Hausdorff and $q : G \rightarrow \frac{G}{H}$ is an identification, it follows that $1_G \times q$ is also an identification (see for example

[Bro], 4.3.2, p. 101). So ϕ is continuous if and only if $\phi \circ (1_G \times q)$ is continuous. But $\phi \circ (1_G \times q)$ is continuous since it is equal to the composite

$$G \times G \xrightarrow{m} G \xrightarrow{q} \frac{G}{H}$$

where $m(g', g) = g'g$ is the group multiplication.

For the remainder of the thesis we assume G is compact Hausdorff. It is known (see for example [D], Proposition 3.3, p. 23) that $\frac{G}{H}$ is Hausdorff if and only if H is closed in G. So it will also be convenient to henceforth assume all subgroups H of G referred to are closed.

3. Function Spaces

Let $\mathrm{Map}_G(X, Y)$ denote the space of G-maps from the G-space X to the G-space Y, with the compact open topology. This space can be given a G-action defined by

$$\begin{aligned} G \times \mathrm{Map}_G(X,Y) &\rightarrow \mathrm{Map}_G(X,Y) \\ (g, f) &\longmapsto (x \longmapsto gf(g^{-1}x)) \end{aligned}$$

If Y is locally compact Hausdorff, then we have the following exponential law for G-spaces and G-maps. There is a bijection

$$\alpha : \mathrm{Map}_G(X \times Y, Z) \rightarrow \mathrm{Map}_G(X, \ \mathrm{Map}_G(Y, Z))$$

defined by

$$f \longmapsto \hat{f}$$

where $\hat{f}(x)(y) = f(x, y)$ (see [D], p. 35).

G-initial structures

If X is a set, then a G-space structure on X is a topology on X along with a continuous action of G on X.

Let J be an indexing set and let $\{f_j : X \to X_j\}_{j\epsilon J}$ be a family of functions from the set X to the topological spaces $\{X_j\}_{j\epsilon J}$. Then a topology on X is said to be <u>initial</u> with respect to $\{f_j\}_{j\epsilon J}$ if it satisfies the following universal property: for any topological space Z, a function $h : Z \to X$ is continuous if and only if each composite $f_j \circ h$ is continuous. Such a topology exists; it is the topology with subbasis consisting of all sets of the form $\{f_j^{-1}(U) | U \text{ open in } X,\ j\epsilon J\}$. (For more details see [Bro], p. 153).

Now let $\{f_j : X \to X_j\}_{j\epsilon J}$, as above, be a family of G-maps. Then the G-space structure on X is said to be <u>G-initial</u> with respect to the functions $\{f_j\}_{j\epsilon J}$ if it satisfies the following universal property: for any G-space Z, a function $p : Z \to X$ is a G-map if and only if each composite $f_j \circ p$ is a G-map.

We also note that if $p : Z \to X$ is a G-map it follows from Proposition 1.1 that the composite $f_j \circ p$ is a G-map for each $j\epsilon J$. Hence, to establish that X has the G-initial structure with respect to the family of G-maps $\{f_j : X \to X_j\}_{j\epsilon J}$, we only need to show that the continuity of p follows from the continuity of the compositions $f_j \circ p$, for all $j\epsilon J$.

<u>Example: Product of G-spaces</u>

For any family of G-spaces $\{X_j\}_{j\epsilon J}$ the product $\prod_{j\epsilon J} X_j$ can be given a G-initial structure with respect to the projections $p_j : \prod_{j\epsilon J} X_j \to X_j$, where $\prod_{j\epsilon J} X_j$ has the initial topology. The action of G on $\prod_{j\epsilon J} X_j$ is the <u>diagonal</u> action defined by $\phi : G \times \left(\prod_{j\epsilon J} X_j\right) \to \prod_{j\epsilon J} X_j$ where $\phi(g, (x_j | j\epsilon J)) = (gx_j | j\epsilon J)$. To show that ϕ is continuous, consider the following diagram

$$\begin{array}{ccc} G \times \prod_{j\epsilon J} X_j & \xrightarrow{\phi} & \prod_{j\epsilon J} X_j \\ {\scriptstyle 1_G \times p_j}\downarrow & & \downarrow{\scriptstyle p_j} \\ G \times X_j & \xrightarrow[\phi_j]{} & X_j \end{array}$$

where ϕ_j is the action of G on X_j. Then each composite $p_j \circ \phi$ is equal to the composite $\phi_j \circ (1_G \times p_j)$ which is clearly continous. By the universal property for initial topologies, ϕ is continuous. We notice that the projections $p_j : \prod_{j \epsilon J} X_j \rightarrow X_j$ are G-maps.

Let $h : Z \rightarrow \prod_{j \epsilon J} X_j$ be a function from the G-space Z to $\prod_{j \epsilon J} X_j$ such that $p_j \circ h$ is a G-map for each $j \epsilon J$. Then for all $z \epsilon Z$, $g \epsilon G$ we have

$$\begin{aligned} gh(z) &= (gp_j(h(z))|j \epsilon J) \\ &= (g(p_j \circ h)(z)|j \epsilon J) \\ &= ((p_j \circ h)(gz)|j \epsilon J) \\ &= (p_j(h(gz))|j \epsilon J) \\ &= h(gz) \end{aligned}$$

Thus h is a G-map and the G-space structure on $\prod_{j \epsilon J} X_j$ is G-initial with respect to $\{p_j\}_{j \epsilon J}$.

Sufficient conditions for the general case are given in the following result.

1.2 Proposition. Let $\{f_j : X \rightarrow X_j\}_{j \epsilon J}$ be a family of functions from the set X to the G-spaces $\{X_j\}_{j \epsilon J}$. Then X can be given a G-initial structure with respect to $\{f_j\}_{j \epsilon J}$ if the following conditions are satisfied:

(i) the function $h : X \rightarrow \prod_{j \epsilon J} X_j$ defined by $h(x) = (f_j(x)|j \epsilon J)$, is injective, and

(ii) $h(X)$ is fixed by G, i.e. if $g \epsilon G$ and $y \epsilon h(X)$, then $gy \ \epsilon h(X)$.

Proof: Let X be given the initial topology with respect to $\{f_j\}_{j \epsilon J}$. We have to define an action of G on X such that the universal property holds.

Using condition (ii) we see that, for each $g \epsilon G$, $x \epsilon X$, there exists an $\bar{x} \epsilon X$ with $h(\bar{x}) = gh(x)$, and this $\bar{x}$ is unique by condition (i). Define $\phi : G \times X \rightarrow X$ by $\phi(g, x) = \bar{x}$. We have to show that ϕ is continuous.

It follows by the transitive rule for initial topologies (see [Bro], 5.6.8, p. 154) that X has the initial topology relative to the injection h, and so $h : X \to h(X)$ is a homeomorphism.

Let θ be the diagonal action on $h(X)$. Then $\phi = h^{-1} \circ (1_G \times h) \circ h$, i.e. we have the following commutative diagram.

$$\begin{array}{ccc} G \times h(X) & \xrightarrow{\theta} & h(X) \\ {\scriptstyle 1_G \times h}\uparrow & & \downarrow{\scriptstyle h^{-1}} \\ G \times X & \xrightarrow[\phi]{} & X \end{array}$$

So ϕ is continuous since it is the composite of continuous functions.

Let $p : Z \to X$ be a function from the G-space Z to X. Assume $f_j \circ p$ is a G-map for each $j \epsilon J$. Let ϕ_X, ϕ_Z be the actions of G on X and Z, respectively. Then p is a G-map if $p \circ \phi_Z = \phi_X \circ (1_G \times p)$. But from the argument above $\phi_X = h^{-1} \circ \theta \circ (1_G \times h)$. So we have to show that the outer perimeter of the following diagram commutes:

$$\begin{array}{ccc} G \times Z & \xrightarrow{\phi_Z} & Z \\ {\scriptstyle 1_G \times p}\downarrow & & \downarrow{\scriptstyle p} \\ G \times X & \xrightarrow[\phi_X]{} & X \\ {\scriptstyle 1_G \times h}\downarrow & & \uparrow{\scriptstyle h^{-1}} \\ G \times h(X) & \xrightarrow{\theta} & h(X) \end{array}$$

Let $(g, z) \epsilon G \times Z$. Then

$$\begin{aligned}
& \theta(1_G \times h)(1_G \times p)(g,z) \\
= & \ \theta(1_G \times h)(g, p(z)) \\
= & \ \theta(g, (f_j(p(z))|j\epsilon J)) \\
= & \ (gf_j(p(z))|j\epsilon J) \\
= & \ (g(f_j \circ p)(z)|j\epsilon J) \\
= & \ ((f_j \circ p)(gz)|j\epsilon J) \\
= & \ (f_j(p(gz))|j\epsilon J) \\
= & \ h(p(gz)) \\
= & \ hp\phi_Z(g,z)
\end{aligned}$$

So $p(gz) = gp(z)$ and p is a G-map. ■

Examples of G-initial structures and Proposition 1.2

1. Product G-spaces.

   The product topology is a special case of the above proposition with h the identity. Special examples of product G-spaces are $\frac{G}{H} \times *$, $\frac{G}{H} \times S^n$, $\frac{G}{H} \times E^n$, and $\frac{G}{H} \times I$, where G acts trivially on $*$, S^n, E^n, and I.

2. G-subspaces.

   If A is a subset of X that is fixed by G, then the inclusion $i : A \to X$ satisfies the conditions of Proposition 1.2 where $h = i$. So if $\phi : G \times X \to X$ is the action of G on X then $\phi|G \times A \to A$ is the action of G on A which gives A the G-initial structure on A, where A has the subspace topology. A is then called a G-subspace of X.

If $(X, *)$ is a based space with basepoint $*$, then $(X, *)$ will be called a based G-space if the action of G on X is such that $*$ is fixed by G.

For G-subspaces we have the following:

1.3 Proposition. Let $f : X \to Y$ be a G-map. Then

(i) If A is a G-subspace of X, then $f(A)$ is a G-subspace of Y.

(ii) If B is a G-subspace of Y, then $f^{-1}(B)$ is a G-subspace of X.

Proof:

(i) We have to show that $Gf(A) = f(A)$. Let $a \epsilon A, g \epsilon G$. Then $gf(a) = f(ga) \epsilon f(A)$ since $ga \epsilon A$. Hence $Gf(A) \subseteq f(A)$ and since $ef(A) = f(A)$ we have $Gf(A) = f(A)$.

(ii) Let $x \epsilon f^{-1}(B)$ and $g \epsilon G$. Then $f(gx) = gf(x) \epsilon B$ since $f(x) \epsilon B$ and $GB = B$. Hence $gx \epsilon f^{-1}(B)$, $Gf^{-1}(B) \subseteq f^{-1}(B)$ and $Gf^{-1}(B) = f^{-1}(B)$. ■

G-final structures

Let J be an indexing set and let $\{f_j : X_j \to X\}_{j \epsilon J}$ be a family of functions from the spaces $\{X_j\}_{j \epsilon J}$ to the set X. Then a topology on X is said to be final with respect to the functions $\{f_j\}_{j \epsilon J}$ if the following universal property is satisfied: for any topological space Z, a function $h : X \to Z$ is continuous if and only if each composite $h \circ f_j$ is continuous. The final topology exists and consists of all sets $U \subseteq X$ such that $f_j^{-1}(U)$ is open in X_j for all $j \epsilon J$.

Now let $\{f_j : X_j \to X\}_{j \epsilon J}$, as above, be a family of G-maps. Then a G-space structure on X is said to G-final with respect to $\{f_j\}_{j \epsilon J}$ if it satisfies the following universal property: for any G-space Z, a function $h : X \to Z$ is a G-map if and only if each composite $h \circ f_j$ is a G-map. Again we note that from Proposition 1.1 it follows that if $h : X \to Z$ is a G-map, then the composite $h \circ f_j$ is also a G-map for each $j \epsilon J$. Hence, to establish that X has the G-final structure with respect to the family of G-maps $\{f_j : X_j \to X\}_{j \epsilon J}$, we only need to show that the continuity of h follows from the continuity of the composites $h \circ f_j$, for all $j \epsilon J$.

Example: Sum of G-spaces

For any family of G-space $\{X_j\}_{j \epsilon J}$ the topological sum $\coprod_{j \epsilon J} X_j$, i.e. the disjoint union of the spaces X_j with the usual final topology, can be given a G-final structure with respect to the inclusions $i_j : X_j \hookrightarrow \coprod_{j \epsilon J} X_j$. The action of G on $\coprod_{j \epsilon J} X_j$ is defined by:

$$\begin{aligned} \phi : G \times \coprod_{j \epsilon J} X_j &\rightarrow \coprod_{j \epsilon J} X_j \\ (g, x_j) &\longmapsto gx_j \quad \text{for } g \epsilon G,\ x_j \epsilon X_j, \text{ and } j \epsilon J \end{aligned}$$

To show that ϕ is continuous we note that since G is compact Hausdorff, $G \times \coprod_{j \epsilon J} X_j$ has the final topology with respect to $1_G \times i_j : G \times X_j \rightarrow G \times \coprod_{j \epsilon J} X_j$ (see [Bro], 4.3.2, p. 101). Then ϕ is continuous since each composite $\phi \circ (1_G \times i_j) = i_j \phi_j$, where ϕ_j is the action of G on X_j. We notice that the inclusions $i_j : X_j \rightarrow \coprod_{j = \epsilon J} X_j$ are G-maps for all $j \epsilon J$.

Next let $h : \coprod_{j \epsilon J} X_j \rightarrow Z$ be a function from $\coprod_{j \epsilon J} X_j$ to the G-space Z with $h \circ i_j$ a G-map for all $j \epsilon J$. Then we have for $x \epsilon X_j$, $g \epsilon G$:

$$\begin{aligned} h(gx) &= hi_j(gx) \\ &= (h \circ i_j)(gx) \\ &= g(h \circ i_j)(x) \\ &= gh(i_j(x)) \\ &= gh(x) \end{aligned}$$

Thus the universal property is satisfied.

Sufficient conditions for the existence of G-final structures are given in the following result.

1.4 Proposition. Let $\{f_j : X_j \rightarrow X\}_{j \epsilon J}$ be a family of functions from the G-spaces $\{X_j\}_{j \epsilon J}$ to the set X. Then X can be given a G-final structure with respect to $\{f_j\}_{j \epsilon J}$ if the following conditions are satisfied:

(i) the function $p : \coprod_{j\epsilon J} X_j \to X$ defined by $p(x_i) = f_i(x_i)$ for $x_i \epsilon X_i$, is surjective, and

(ii) If $p(x_i) = p(x_j)$ for $x_i \epsilon X_i$, $x_j \epsilon X_j$, with $i, j \epsilon J$, then $p(gx_i) = p(gx_j)$ for all $g \epsilon G$.

Proof: From (i) it follows that for each $x \epsilon X$, there is $\bar{x} \epsilon \coprod_{j\epsilon J} X_j$ such that $p(\bar{x}) = x$. Define the action of G on X by:

$$\phi : G \times X \to X$$

$$\phi(g, x) = p(g\bar{x})$$

This action is well-defined by condition (ii)

If X is given the final topology with respect to the functions $\{f_j\}_{j\epsilon J}$, then $p : \coprod_{j\epsilon J} X_j \to X$ is an identification. Since G is locally compact Hausdorff

$$1_G \times p : G \times \coprod_{j\epsilon J} X_j \to G \times X$$

is also an identification. Hence it follows from the following commutative diagram.

$$\begin{array}{ccc} G \times \coprod_{j\epsilon J} X_j & \xrightarrow{1_G \times p} & G \times X \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \phi} \\ \coprod_{j\epsilon J} X_j & \xrightarrow[p]{} & X \end{array}$$

that ϕ is continuous.

Let $h : X \to Z$ be a function from X to the G-space Z. Assume $h \circ f_j$ is a G-map for each $j \epsilon J$. Let $x \epsilon X$. Then we have to show $h(gx) = gh(x)$ for any $g \epsilon G$.

$$\begin{aligned} h(gx) &= h(p(g\bar{x})) \text{ where } \bar{x}\epsilon X_j \text{ for some } j\epsilon J \\ &= h(f_j(g\bar{x})) \\ &= (h \circ f_j)(g\bar{x}) \\ &= g(h \circ f_j)(\bar{x}) \quad \text{since each} \quad h \circ f_j \quad \text{is a } G\text{-map} \\ &= gh(f_j(\bar{x})) \\ &= gh(x) \end{aligned}$$

So h is a G-map. ■

Examples of G-final structures and Proposition 1.4

1. Topological sums A topological sum $\coprod_{j\epsilon J} X_j$ of G-spaces is covered by the above proposition with p the identity on $\coprod_{j\epsilon J} X_j$.

2. Quotient G-space

   Let $\sim$ be an equivalence relation on the G-space X such that if $x \sim y$ then $gx \sim gy$ for all $x, y\epsilon X$, $g\epsilon G$. Then by Proposition 1.4, the quotient space $\frac{X}{\sim}$ can be given the G-final structure with the action on $\frac{X}{\sim}$ defined by

$$\phi : G \times \frac{X}{\sim} \to \frac{X}{\sim}$$
$$\phi(g, [x]) = [gx]$$

   The p of Proposition 1.5 is the obvious identification map $X \to \frac{X}{\sim}$.

   In particular if A is a G-subspace of X then $\frac{X}{A}$, the quotient space obtained by shrinking A to a point, can be given the G-final structure with respect to the quotient map $p : X \to \frac{X}{A}$.

   Given a based set $(Z, *)$, the rule $f \longmapsto fp$, where f is a based function $Y/A \to Z$, determines a bijective correspondence between based functions $Y/A \to Z$ and functions $Y \to Z$ which take A to $*$. Hence it follows from Proposition 1.4 that p satisfies the following universal property:

If Z is a based G-space, then the rule $f \longmapsto fp$, where $f : (Y/A, \{A\}) \to (Y, *)$ is a based G-map, determines a bijective correspondence between

(i) based G-maps $(Y/A, \{A\}) \to (Z, *)$, and

(ii) G-maps of pairs $(Y, A) \to (Z, *)$.

If $\{(X_j, *)\}_{j \in J}$ is a family of based G-spaces, then the wedge $\bigvee_{j \in J} X_j$ is the quotient G-space formed from the topological sum $\coprod_{j \in J} X_j$ by identifying all the basepoints, and using the identified point as basepoint for the wedge.

3. <u>G-adjunction squares and G-adjunction spaces</u>

A diagram of G-spaces and G-maps

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ X & \xrightarrow[\bar{f}]{} & Z \end{array}$$

will be called a <u>G-adjunction square</u> if the following universal property is satisfied:

if Z' is any G-space and $j : Y \to Z'$, $k : X \to Z'$ are G-maps such that $j \circ f = k \circ i$, then there exists a unique G-map $h : Z \to Z'$ such that $h \circ \bar{i} = j$ and $h \circ \bar{f} = k$.

For the given maps f and i the G-space Z above is unique up to G-homeomorphism in the following sense. If Z' is any other G-space such that

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}'} \\ X & \xrightarrow[\bar{f}']{} & Z' \end{array}$$

is also a G-adjunction square, then there are G-maps $h : Z \to Z'$ and h' :

$Z' \to Z$ determined by the universal property for the G-adjunction squares involving Z and Z', respectively, which are homeomorphic inverses of each other. Thus, $h\bar{i} = \bar{i}'$, $h\bar{f} = \bar{f}'$, $h'\bar{i}' = \bar{i}$, and $h'\bar{f}' = \bar{f}$. The uniqueness of the universal property, for G-adjunction squares ensures that $h'h = 1_Z$ and $hh' = 1_{Z'}$.

If A is a G-subspace of X, then given the G-map $f : A \to Y$ we can form the G-adjunction square

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ X & \xrightarrow[\bar{f}]{} & X \cup_f Y \end{array}$$

Here, $X \cup_f Y$ is the quotient space of $X \coprod Y$ formed by identifying a with $f(a)$ for all $a \epsilon A$, and giving $X \cup_f Y$ the G-final structure with respect to the inclusion $\bar{i} : Y \to Y \cup_f Y$ and the function $\bar{f} : X \to Y \cup_f Y$ defined by $\bar{f}(x) = [x]$ for $x \epsilon X$. The action of G on $X \cup_f Y$ is given by

$$\phi : G \times (X \cup_f Y) \to X \cup_f Y$$
$$\phi(g, [x]) = [gx] \text{ for } g \epsilon G,\ x \epsilon X$$
$$\phi(g, [y]) = [gy] \text{ for } g \epsilon G,\ y \epsilon Y.$$

We see that ϕ is continuous as in the proof of Proposition 1.4. The G-space $X \cup_f Y$ will be called the <u>G-adjunction space</u> determined by f. The corresponding function $p : X \coprod Y \to X \cup_f Y$ is defined in the obvious way.

1.5 <u>Proposition</u>. The square used in constructing any G-adjunction space is a G-adjunction square.

<u>Proof</u>. If Z is a G-space and $j : Y \to Z$, $k : X \to Z$ are G-maps such that $j \circ f = k \circ i$ then we can define $h : X \cup_f Y \to Z$ by $h([\bar{f}(x)] = k(x)$ and $h([y]) = j(y)$ for all $x \epsilon X$, $y \epsilon Y$. It is easily seen that h is unique. ■

4. Properties of G-adjunction squares

The following results will be utilized in Chapter 3 on G-CW-complexes.

1.6 Proposition. If A, B and C are G-spaces, with A a G-subspace of B and B a G-subspace of C, then the following diagram is a G-adjunction square:

$$\begin{array}{ccc} B & \xrightarrow{p} & B/A \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle i'} \\ C & \xrightarrow[p']{} & C/A \end{array}$$

where i and i' are the inclusions and p and p' are the corresponding quotient G-maps.

Proof: If Z is a G-space and $j : B/A \to Z$, $k : C \to Z$ are G-maps with $j \circ p = k \circ i$ then we can define a unique G-map $h : C/A \to Z$ by $h([c]) = k(c)$ if $c \notin A$ and $h([a]) = j(A)$ if $a \in A$. Then $h \circ i' = j$ and $h \circ p' = k$. The surjectivity of p' implies that h is unique. Finally, h is a G-map since C/A has the G-final structure with respect to i' and p'. ■

1.7 Corollary. If J is an indexing set and for each $j \in J$, (X_j, A_j) is a pair of based G-spaces with A_j a G-subspace of X_j, then the following diagram is a G-adjunction square:

$$\begin{array}{ccc} \coprod_{j\in J} A_j & \xrightarrow{p_A} & \bigvee_{j\in J} A_j \\ {\scriptstyle \coprod_{j\in J} i_j}\big\downarrow & & {\scriptstyle \bigvee_{j\in J} i_j}\big\downarrow \\ \coprod_{j\in J} X_j & \xrightarrow[p_X]{} & \bigvee_{j\in J} X_j \end{array}$$

where p_A and p_X are the obvious quotient G-maps and $i_j : A_j \to X_j$ is the inclusion for each $j \in J$, and $\coprod_{j\in J} i_j$ and $\bigvee_{j\in J} i_j$ are the obvious G-maps.

1.8 Proposition. Let $(X_j, A_j, *)$ be a pair of based G-spaces for each $j \epsilon J$ and let $\{f_j : A_j \to Y_j\}_{j \epsilon J}$ be a family of based G-maps. Then the following is a G-adjunction square:

$$\begin{array}{ccc} \bigvee_{j\epsilon J} A_j & \xrightarrow{\bigvee_{j\epsilon J} f_j} & \bigvee_{j\epsilon J} Y_j \\ \Big\downarrow{\scriptstyle \bigvee_{j\epsilon J} i_j} & & \Big\downarrow{\scriptstyle \bigvee_{j\epsilon J} \bar{i}_j} \\ \bigvee_{j\epsilon J} X_j & \xrightarrow[\bigvee_{j\epsilon J} \bar{f}_j]{} & \bigvee_{j\epsilon J}(X_j \cup_{f_j} Y_j) \end{array}$$

Proof: Given based G-maps $\ell : \bigvee_{j\epsilon J} Y_j \to Z$ and $k : \bigvee_{j\epsilon J} X_j \to Z$ such that $\ell\left(\bigvee_{j\epsilon J} f_j\right) = k\left(\bigvee_{j\epsilon J} i_j\right)$, then for each restriction $\ell_j : Y_j \to Z$ and $k_j : X_j \to Z$ there exists a G-map $h_j : X_j \cup_{f_j} Y_j \to Z$ such that $h_j \circ \bar{i}_j = \ell_j$ and $h_j \circ \bar{f}_j = k_j$. Then if we define $h : \bigvee_{j\epsilon J}(X_j \cup_{f_j} Y_j) \to Z$ by $h|(X_j \cup_{f_j} Y_j) = h_j$ for each $j \epsilon J$, we have $h \circ \bigvee_{j\epsilon J} \bar{i}_j = \ell$, $h \circ \bigvee_{j\epsilon J} \bar{f}_j = k$. ■

1.9 Proposition. Let X, Y and Z be G-spaces, with A a G-subspace of X. Then if $f : A \to Y$, and $g : Y \to Z$ are G-maps, the outer perimeter of the following diagram is a G-adjunction square for $i : A \to X$ and the G-map gf:

$$\begin{array}{ccccc} A & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ i\downarrow & & \downarrow \bar{i} & & \downarrow \bar{\bar{i}} \\ X & \overrightarrow{\bar{f}} & X \cup_f Y & \overrightarrow{\bar{g}} & (X \cup_f Y) \cup_g Z \end{array}$$

Proof. Let $\ell : Z \to W$ and $k : X \to W$ be G-maps such that $\ell \circ (gf) = k \circ i$. Then since the left-hand square is a G-adjunction square, there exists a G-map $h : X \cup_f Y \to W$ such that $h \circ \bar{i} = \ell \circ g$ and $h \circ \bar{f} = k$. But the right-hand square is also a G-adjunction square so there exists a G-map $h' : (X \cup_f Y) \cup_g Z \to W$ such that $h' \circ \bar{\bar{i}} = \ell$ and $h' \circ \bar{g} = h$. Hence $h' \circ \bar{g} \circ \bar{f} = h \circ \bar{f} = k$ and so the outer perimeter is a G-adjunction square. ■

5. <u>*G*-mapping cylinder</u>

The <u>*G*-mapping cylinder</u> for the G-map $f : X \to Y$ is defined as the G-adjunction space formed via the diagram

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\ X \times I & \xrightarrow[\bar{f}]{} & (X \times I) \cup_f Y \end{array}$$

We will denote $(X \times I) \cup_f Y$ by M_f and call it the G-mapping cylinder for f. The action of G on $X \times I$ is the diagonal action with G acting trivially on I. If $(X, *)$ and $(Y, *)$ are based G-spaces and $f : X \to Y$ is a based G-map then the <u>reduced</u> G-mapping cylinder, denoted by $\tilde{M}_f$, is the quotient G-space $\tilde{M}_f = \frac{M_f}{* \times I}$, with $\{* \times I\}$ as basepoint.

Let $f : X \to Y$ be a based G-map and $\tilde{M}_f$ be the reduced G-mapping cylinder. Then we have the following commutative diagram:

$$X \xrightarrow{f} Y$$

$$\tilde{M}_f$$

where i and r are the based G-maps defined by

$$\begin{array}{rcl} i : (X, *) & \hookrightarrow & (\tilde{M}_f, *) \\ x & \longmapsto & [x, 0], \quad \text{and} \\ r : (\tilde{M}_f, *) & \to & (Y, *) \\ [y] & \longmapsto & y \\ [x, t] & \longmapsto & f(x), \end{array}$$

where $t \epsilon I$, $x \epsilon X$ and $y \epsilon Y$.

We also have the based G-map $j : (Y,*) \to (\tilde{M}_f,*)$ defined by $y \longmapsto [y]$. By an argument analogous to that in non-equivariant case, r is a G-homotopy equivalence with inverse j.

6. Expanding sequence of G-spaces

Let $X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots$ be an expanding sequence of G-spaces,i.e. X_n is a G-subspace of X_{n+1} for all $n \geq 0$, and the set $X = \bigcup_{n\geq 0} X_n$. Let $p : \coprod_{n\geq 0} X_n \to X$ be defined by $p(x) = i_n(x)$ where $x \epsilon X_n$ and $i_n : X_n \hookrightarrow X$ be the inclusion. So, by Proposition 1.4, X can be given the G-final structure with respect to the inclusions $\{i_n : X_n \to X\}_{n\geq 0}$. From the universal property, it follows that $f : X \to Z$ is a G-map if and only if $f|X^n$ is a G-map for each $n \geq 0$.

1.10 Proposition. If $X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots \subset X$ is an expanding sequence of G-spaces where X has the G-final structure with respect to the inclusions $\{i_n : X_n \hookrightarrow X\}_{n\geq 0}$, and A is locally compact Hausdorff, then

$$X_0 \times A \subset X_1 \times A \subset \ldots X_n \times A \subset \ldots \subset X \times A$$

is also an expanding sequence of G-spaces with $X \times A$ having the G-final structure with respect to the inclusions $\{i_n \times 1_A : X_n \times A \hookrightarrow X \times A\}_{n\geq 0}$.

Proof: Let $h : X \times A \to Z$ be a function from $X \times A$ to the G-space Z such that $h|X_n \times A$ is a G-map for all $n \geq 0$. By the exponential law for G-spaces and G-maps, for each $h|X_n \times A$ there exists a G-map $h' : X_n \to \mathrm{Map}_G(A, Z)$ and a function $\tilde{h} : X \to \mathrm{Map}_G(A, Z)$ such that $h' = \tilde{h}|X_n$. Since X has the G-final structure with respect to the inclusions, it follows that $\tilde{h}$ is a G-map and hence h is also a G-map. Thus $X \times A$ has the G-final structure with respect to the inclusions $\{i_n \times A : X_n \times A \hookrightarrow X \times A\}_{n\geq 0}$. ■

7. Cones

Let X be a based G-space. Then since $X \times \{0\} \cup * \times I$ is a G-subspace of

$X\times I$, $C(X)=\frac{X\times I}{X\times\{0\}\cup *\times I}$ is a quotient G-space. It will be called the cone of X.

Define $S^n_H=\dfrac{\frac{G}{H}\times S^n}{\frac{G}{H}\times *}$ and $E^n_H=\dfrac{\frac{G}{H}\times E^n}{\frac{G}{H}\times *}$, where $\frac{G}{H}\times S^n$ and $\frac{G}{H}\times E^n$ have the diagonal action with G acting trivially on S^n and E^n. The following fact is needed in the proof of the Brown Representability Theorem:

$$C(S^n_H)=C\left(\frac{\frac{G}{H}\times S^n}{\frac{G}{H}\times *}\right)\cong\frac{\frac{G}{H}\times C(S^n)}{\frac{G}{H}\times *}\cong\frac{\frac{G}{H}\times E^{n+1}}{\frac{G}{H}\times *}=E^{n+1}_H$$

That the left hand $\cong$ above is a homeomorphism follows from the following.

1.11 Proposition. Let X and Y be based G-spaces, where the underlying space of X is locally compact Hausdorff. Then there is a homeomorphism $\psi:C\left(\dfrac{X\times Y}{X\times *}\right)\to\dfrac{X\times CY}{X\times *}$ defined by the rule that the equivalence classes of (x,y,t), in the two spaces, correspond, where $x\epsilon X$, $y\epsilon Y$, $t\epsilon I$.

Proof. Recalling that if A is a subspace of B then $C\left(\dfrac{B}{A}\right)\cong\dfrac{B\times I}{(B\times\{0\})\cup(A\times I)}$, we notice that $C\left(\dfrac{X\times Y}{X\times *}\right)\cong\dfrac{X\times Y\times I}{(X\times Y\times\{0\})\cup(X\times\{*\}\times I}$.

We also note that $X\times\left(\dfrac{B}{A}\right)$ is homeomorphic to the quotient space of $X\times B$ in which $\{x\}\times A$, for each $x\epsilon X$, is identified to a separate point. Hence $X\times CY$ is homeomorphic to the quotient space of $X\times Y\times I$ where for each $x\epsilon X$, $(\{x\}\times Y\times\{0\}\cup\{x\}\times\{*\}\times I)$ is identified to a separate point. Since $\dfrac{X\times CY}{X\times *}$ is a quotient space of $X\times CY$ and CY is a quotient space of $Y\times I$, then, using [Bro], 4.3.2, p. 101, $\frac{X\times CY}{X\times *}$ is (essentially) a quotient space of $X\times Y\times I$ under the equivalence relation we will denote by $\sim$. Then we have $(x,y,0)\sim(x,*,0)\sim(*,*,0)$ for $x\epsilon X$, $y\epsilon Y$, and $(x,*,t)\sim(x,*,0)\sim(*,*,0)$ for $x\epsilon X$, $t\epsilon I$. Hence $\dfrac{X\times CY}{X\times *}$ is homeomorphic to $\dfrac{X\times Y\times I}{(X\times Y\times\{0\})\cup(X\times\{*\}\times I)}$.

In both cases the actions of G agree with that on the quotient G-space $\dfrac{X\times Y\times I}{(X\times Y\times\{0\})\cup(X\times\{*\}\times I}$, that is itself induced by the diagonal action of G on $X\times Y\times I$.

Using p, q, r and s to denote the quotient maps:

$$p : X \times Y \to \frac{X \times Y}{X \times *}, \quad q : \left(\frac{X \times Y}{X \times *}\right) \times I \to C\left(\frac{X \times Y}{X \times *}\right),$$

$$r : Y \times I \to CY \quad \text{and} \quad s : X \times CY \to \frac{X \times CY}{X \times *},$$

we note that we have

$$q(p \times 1_I) : X \times Y \times I \to C\left(\frac{X \times Y}{X \times *}\right) \quad \text{and}$$

$$s(1_X \times r) : X \times Y \times I \to \frac{X \times CY}{X \times *}.$$

Since I and X are locally compact Hausdorff, $q(p \times 1_I)$ and $s(1_X \times r)$ are identifications, ψ is a bijection, and since $\psi q(p \times 1_I) = s(1_X \times r)$ it follows that ψ is a homeomorphism. ∎

8. Suspension

Let X be a G-space with basepoint $*$. Then $X \times \{0\} \cup X \times \{1\} \cup * \times I$ is a G-subspace of $X \times I$ and the suspension of X is the quotient G-space defined by

$$SX = \frac{X \times I}{X \times \{0\} \cup X \times \{1\} \cup * \times I}$$

9. Smashed product

Let X and Y be based G-spaces. Then $X \vee Y = (X \times \{*\}) \cup (\{*\} \times Y)$ is a G-subspace of $X \times Y$ and the G-smashed product is the quotient G-space

$$X \wedge Y = \frac{X \times Y}{X \vee Y}$$

Finally, let $(Y, *)$ be a based G-space and X is a G-space. We define the G-space X^+ by adjoining to X an additional point ∞ on which G acts trivially. Then we have $X^+ \wedge Y \cong \frac{X \times Y}{X \times *}$.

If H is a closed subgroup of G and X is a G-space, we define $X^H = \{x \epsilon X | hx = x \text{ for all } h\}$ with the subspace topology. Then, if X is Hausdorff, X^H is closed in X

(see [D], 3.9, p. 25). If $f : X \to Y$ is a G-map then $f^H : X^H \to Y^H$ denotes the restriction of f to X^H.

1.12 Proposition. If X and Y are Hausdorff G-spaces, then the adjunction space $X^H \cup_{f^H} Y^H$ is a closed subspace of $X \cup_f Y$.

Proof. Since $A^H \subset X^H$, X^H is closed in X and Y^H is closed in Y, it follows from a result of general topology (see [Du], VI, 6.4, p. 128) that $X^H \cup_{f^H} Y^H$ will be a closed subspace in $X \cup_f Y$. ■

From this it follows that since $X^H \cup_{f^H} Y^H$ and $(X \cup_f Y)^H$ have the same underlying sets, and $(X \cup_f Y)^H$ is a subspace of $X \cup_f Y$, they must be homeomorphic, i.e. $X^H \cup_{f^H} Y^H = (X \cup_f Y)^H$.

In order to show that the G-CW-complexes defined in Chapter 3 are Hausdorff we need the following result about adjunction spaces.

1.13 Proposition. If X and Y are normal spaces and A is a closed subset of X then for any map $f : A \to Y$, $X \cup_f Y$ is also normal.

Proof. See [FP], Proposition A.4.8(iv), p. 260. ■

1.14 Proposition. (i) If X and Y are G-spaces then $(X \times Y)^H = X^H \times Y^H$.

(ii) If $\{X_j | j \in J\}$ is an indexed family of based G-spaces, then $\left(\bigvee_{j \in J} X_j\right)^H = \bigvee_{j \in J}(X_j^H)$.

Proof. Both (i) and (ii) follow quite easily from the definition of the actions of G on the spaces $X \times Y$ and $\bigvee_{j \in J} X_j$, respectively. ■

Chapter 2 Equivariant Homotopy Groups

Two G-maps $f,g : X \to Y$ are said to be G-homotopic if there exists a G-map $F : X \times I \to Y$, with G acting trivially on I, such that for all $x \epsilon X$, $F(x,0) = f(x)$ and $F(x,1) = g(x)$. Note that each map $F_t : X \to Y$ defined by $F_t(x) = F(x,t)$ is itself a G-map since for $x \epsilon X$, $g \epsilon G$ we have $F_t(gx) = F(gx,t) = gF(x,t) = gF_t(x)$. The set of G-homotopy classes of G-maps from X to Y will be denoted by $[X : Y]_G$.

If A is a G-subspace of X and B is a G-subspace of Y then two G-maps $f,g : (X,A) \to (Y,B)$ are said to be G-homotopic relative to A if there is a G-map $F : (X \times I, A \times I) \to (Y,B)$ such that for $x \epsilon X$, $F(x,0) = f(x)$ and $F(x,1) = g(x)$ and for $a \epsilon A$, $F(a,t) = f(a)$ for $t \epsilon I$. The set of relative G-homotopy classes of G-maps from (X,A) to (Y,B) is denoted by $[(X,A) : (Y,B)]_G$.

If X and Y are based G-spaces, then the based G-maps $f,g : X \to Y$ are based G-homotopic if there exists a G-homotopy $F : X \times I \to Y$ between f and g with $F(*,t) = *$ for $t \epsilon I$. We write $f \simeq_G g$ to denote the based G-homotopy. The set of based G-homotopy classes of based G-maps from X to Y will be denoted by $[X : Y]_G^0$ or $[(X,*) : (Y,*)]_G^0$.

Let $(X,*)$ be a based G-space. Then for each closed subgroup H of G, the underlying set of the nth equivariant homotopy group of type H is defined by

$$\begin{aligned} \pi_n^H(X,*) &= [S_H^n : X]_G^0 \\ &\cong \left[\left(\frac{G}{H} \times S^n, \frac{G}{H} \times *\right) : (X,*)\right]_G^0 \end{aligned}$$

the canonical bijection $\cong$ being defined using the universal property of quotient G-spaces, where $S_H^n = \dfrac{\frac{G}{H} \times S^n}{\frac{G}{H} \times *} \cong \left(\frac{G}{H}\right)^+ \wedge S^n$. Using the fact that $A \wedge SB \cong S(A \wedge B)$ we have

$$S_H^n \cong \left(\frac{G}{H}\right)^+ \wedge S^n \cong \left(\frac{G}{H}\right)^+ \wedge S(S^{n-1}) \cong S\left(\left(\frac{G}{H}\right)^+ \wedge S^{n-1}\right)$$

Since S^n_H is a suspension for $n \geq 1$, we can define a group operation on $\pi^H_n(X,*)$ in a way analogous to the non-equivariant case. If $[f], [g] \epsilon \pi^H_n(X,*)$ are two elements, then the product of $[f]$ and $[g]$, denoted $[f]\cdot[g]$, is the G-homotopy class of the composite G-map

$$S^n_H \xrightarrow{\nu_H} S^n_H \vee S^n_H \xrightarrow{f \vee g} X \vee X \xrightarrow{\nabla} X$$

where ∇ is the folding map defined by $\nabla(x,*) = x$, $\nabla(*,x) = x$ for all $x \epsilon X$, and ν_H is the comultiplication G-map given by:

$$\begin{aligned} \nu_H : S^n_H &\rightarrow S^n_H \vee S^n_H \\ [gH,[z,t]] &\longmapsto \begin{cases} ([gH,[z,2t]],*) & \text{if } 0 \leq t \leq \frac{1}{2}, \quad z \epsilon S^{n-1} \\ (*,[gH,[z,2t-1]]) & \text{if } \frac{1}{2} \leq t \leq 1, \quad z \epsilon S^{n-1} \end{cases} \end{aligned}$$

This uses the fact that $S^n_H = \left(\frac{G}{H}\right)^+ \wedge S^n = \left(\frac{G}{H}\right)^+ \wedge S(S^{n-1})$ so that an element of S^n_H is the equivalence class $[gH,[z,t]]$ where $g \epsilon G$, $z \epsilon S^{n-1}$ and $t \epsilon I$. Then ν_H is easily seen to be a G-map.

2.1 Proposition. Let H be a closed subgroup and $(X,*)$ be a based G-space. There exists a natural isomorphism

$$\pi^H_n(X,*) \cong \pi_n(X^H,*).$$

To prove this proposition we need the following lemmas. First let $\text{Map}_G\,(X,Y)$ denote the set of G-maps from X to Y and $\text{Map}\,(X,Y)$ denote the set of all maps from X to Y.

2.2 Lemma. If X and Y are G-spaces, with G acting trivially on the locally compact Hausdorff space X, then for each closed subgroup H of G there is a bijection

$$\text{Map}_G\left(\frac{G}{H} \times X, Y\right) \cong \text{Map}\,(X, Y^H).$$

Proof: Define $\theta : \text{Map}_G\left(\frac{G}{H} \times X, Y\right) \rightarrow \text{Map}\,(X, Y^H)$ by $\theta(F)(x) = F(H, x)$ for $F \epsilon\, \text{Map}_G\left(\frac{G}{H} \times X, Y\right)$ and $x \epsilon X$, and define $\phi : \text{Map}\,(X, Y^H) \rightarrow \text{Map}_G\left(\frac{G}{H} \times X, Y\right)$ by $\phi(f)(gH, x) = gf(x)$ for $f \epsilon\, \text{Map}\,(X, Y^H)$. We first show θ and ϕ are well-defined and continuous.

Since $h\theta(F)(x) = hF(H, x) = F(hH, x) = F(H, x) = \theta(F)(x)$, it follows that the image of $\theta(F)$ is a subset of Y^H. Defining $i : X \rightarrow \frac{G}{H} \times X$ by $i(x) = (H, x)$, where $x \epsilon X$, we see that $\theta(F) = Fi$, so $\theta(F)$ is continuous and θ is well-defined.

We notice that if $g_1, g_2 \epsilon G$ and $g_2^{-1} g_1 \epsilon H$, i.e. $g_1 = g_2 h$ for some $h \epsilon H$, then $g_1 f(x) = g_2 h f(x) = g_2 f(x)$ for all $x \epsilon X$. Hence $\phi(f)$ is well-defined. The function $m : G \times X \rightarrow X$, defined by $m(g, x) = gf(x)$ where $g \epsilon G$ and $x \epsilon X$, is clearly continuous since $q \times 1_X : G \times X \rightarrow \frac{G}{H} \times X$ is an identification. Now $\phi(f)(q \times 1_X) = m$ so $\phi(f)$ is continuous. Also, $g'\phi(f)(gH, x) = g'gf(x) = \phi(f)(g'gH, x) = \phi(f)(g'(gH, x))$ so that $\phi(f)$ is a G-map and ϕ is a well-defined function. To establish the bijection we have to show that $\phi \circ \theta$ and $\theta \circ \phi$ equal the respective identity functions.

Let $F \epsilon\, \text{Map}_G\left(\frac{G}{H} \times X, Y\right)$. Then

$$\begin{aligned}
(\phi \circ \theta)(F)(gH, x) &= \phi(\theta(F))(gH, x) \\
&= g\theta(F)(x) \\
&= gF(H, x) \\
&= F(gH, x)
\end{aligned}$$

So $(\phi \circ \theta)(F) = F$. Now let $f \epsilon\, \text{Map}\,(X, Y^H)$. Then

$$\begin{aligned}
(\theta \circ \phi)(f)(x) &= \theta(\phi(f))(x) \\
&= \phi(f)(H, x) \\
&= f(x)
\end{aligned}$$

Hence $(\theta \circ \phi)(f) = f$. ■

2.3 Lemma. If X and Y are based G-spaces with G acting trivially on the locally compact Hausdorff space X, then for each closed subgroup H of G there is a bijection:

$$\mathrm{Map}_G\left(\left(\frac{G}{H}\times X,\ \frac{G}{H}\times *\right):(Y,*)\right)\cong\ \mathrm{Map}\left((X,*):(Y^H,*)\right)$$

Proof: We define θ and ϕ as in Lemma 2.2, noting that $\theta(F)(*) = F(H,*) = *$ and $\phi(f)(gH,*) = gf(*) = g* = *$.

So θ and ϕ take based maps to based maps, which allows us to establish the bijection between the two sets. ■

2.4 Lemma. If X and Y are G-spaces with G acting trivially on the locally compact Hausdorff space X, A is any G-subspace of X and Z is any G-subspace of Y, with $*\epsilon A\subset X$ and $*\epsilon Z\subset Y$ then for each closed subgroup H of G there is a bijection

$$\mathrm{Map}_G\left(\left(\frac{G}{H}\times X,\ \frac{G}{H}\times A,\ \frac{G}{H}\times *\right):(Y,Z,*)\right)\cong\ \mathrm{Map}\left((X,A,*):(Y^H,Z^H,*)\right)$$

Proof: Define θ and ϕ as in Lemma 2.2. Let $F\epsilon\ \mathrm{Map}_G\left(\frac{G}{H}\times X,\ \frac{G}{H}\times A,\ \frac{G}{H}\times *\right)$ and $f\epsilon\ \mathrm{Map}((X,A,*),\ (Y^H,Z^H,*))$. If $a\epsilon A$ then $\theta(F)(a) = F(H,a)\epsilon Z^H$ since $hF(H,a) = F(hH,a) = F(H,a)$ for all $h\epsilon H$, and $\phi(f)(gH,a) = gf(a)\epsilon Z$ since $f(a)\epsilon Z$ and $GZ = Z$. Since θ and ϕ are inverses of each other, this allows us to establish the bijection. ■

If X is replaced by $X\times I$ and A by $*\times I$ in Lemma 2.4, then we have the bijection $\mathrm{Map}_G\left(\left(\frac{G}{H}\times X\times I,\ \frac{G}{H}\times *\times I\right):(Y,*)\right)\cong\ \mathrm{Map}\left((X\times I,\ *\times I):(Y^H,*)\right)$.
So there is a bijection between the based G-homotopies of $\dfrac{\frac{G}{H}\times X}{\frac{G}{H}\times *}$ to Y and based homotopies of X to Y^H.

Proof of Proposition 2.1

Let $\alpha:\ \mathrm{Map}^0_G\ (S^n_H, X)\rightarrow\ \mathrm{Map}^0\ (S^n, X^H)$ be defined by $\alpha(f)(z) = f(H,z)$ for $z\epsilon S^n$. Then by Lemma 2.3 there is a bijection

$$\alpha_* : [S^n_H : X]^0_G \to [S^n : X^H]^0$$

where $\alpha_*([f]) = [\alpha(f)]$. We have to show that α_* is a homomorphism. Let $f_0, f_1 : S^n_H \to X$. Then α_* is a homomorphism if $\alpha_*([f_0]\cdot[f_1]) = \alpha_*([f_0])\cdot\alpha_*([f_1])$. Let $\alpha(f_0) = g_0$ and $\alpha(f_1) = g_1$. Then $[f_0]\cdot[f_1]$ is the G-homotopy class of the composite

$$S^n_H \xrightarrow{\nu_H} S^n_H \xrightarrow{f_0 \vee f_1} X \vee X \xrightarrow{\nabla} X.$$

Similarly $[g_0]\cdot[g_1]$ is the G-homotopy class of the composite

$$S^n \xrightarrow{\nu} S^n \vee S^n \xrightarrow{g_0 \vee g_1} X^H \vee X^H \xrightarrow{\nabla^H} X^H.$$

Hence α_* will be a homomorphism if we have

$$\alpha_*([\nabla \circ (f_0 \vee f_1) \circ \nu_H]) = [\nabla^H \circ (g_0 \vee g_1) \circ \nu].$$

The map $\nabla^H \circ (g_0 \vee g_1) \circ \nu$ is given by:

$$S^n \xrightarrow{\nu} S^n \vee S^n \xrightarrow{g_0 \vee g_1} X^H \vee X^H \xrightarrow{\nabla^H} X$$

$$[z,t] \longmapsto \begin{cases} ([z,2t],*) \longmapsto g_0([z,2t]) & 0 \le t \le \frac{1}{2},\ z \epsilon S^{n-1} \\ (*,[z,2t-1]) \longmapsto g_1([z,2t-1]) & \frac{1}{2} \le t \le 1,\ z \epsilon S^{n-1}. \end{cases}$$

The map $\nabla \circ (f_0 \vee f_1) \circ \nu_H$ is given by:

$$S^n_H \xrightarrow{\nu_H} S^n_H \vee S^n_H \xrightarrow{f_0 \vee f_1} X \vee X \xrightarrow{\nabla} X$$

$$[gH,[z,t]] \longmapsto \begin{cases} ([gH,[z,2t]],*) \longmapsto f_0([gH,[z,2t]]) & 0 \le t \le \frac{1}{2},\ z \epsilon S^{n-1} \\ (*,[gH,[z,2t-1]]) \longmapsto f_1([gH,z,2t-1]]) & \frac{1}{2} \le t \le 1,\ z \epsilon S^{n-1}. \end{cases}$$

If $f : S^n_H \to X$ then $\alpha_*([f])$ is the G-homotopy class of the composite

$$S^n \xrightarrow{i_H} S^n_H \xrightarrow{f} X$$
$$[z,t] \longmapsto [H,[z,t]] \longmapsto f([H,[z,t]]).$$

Hence $\alpha_*([\nabla \circ (f_0 \vee f_1) \circ \nu_H])$ is the G-homotopy class of

$$S^n \longrightarrow S^n_H \vee S^n_H \longrightarrow X$$

$$[z,t] \longmapsto \begin{cases} ([H,[z,2t]],*) & \longmapsto f_0([H,[z,2t]]) \quad 0 \le t \le \frac{1}{2} \\ (*,[H,[z,2t-1]]) & \longmapsto f_1([H,[z,2t-1]]) \quad \frac{1}{2} \le t \le 1. \end{cases}$$

But $f_0([H,[z,2t]]) = g_0([z,2t])$ and $f_1([H,[z,2t-1]]) = g_1([z,2t-1])$, which implies that $\alpha_*([\nabla \circ (f_0 \vee f_1) \circ \nu_H]) = [\nabla^H \circ (g_0 \vee g_1) \circ \nu]$. Thus α_* is a homomorphism and consequently an isomorphism. ■

Let $(X, A, *)$ be a pair of G-spaces with basepoint $*$. Then for each closed subgroup H of G the nth relative equivariant homotopy group of type H is defined by:

$$\begin{aligned} \pi^H_n(X,A,*) &= [(E^n_H, S^{n-1}_H, *) : (X,A,*)]^0_G \\ &\cong \left[\left(\frac{G}{H} \times E^n, \frac{G}{H} \times S^n, \frac{G}{H} \times *\right) : (X,A,*)\right]^0_G. \end{aligned}$$

2.5 Proposition. There is a natural ismorphism

$$\pi^H_n(X,A,*) \cong \pi_n(X^H, A^H, *).$$

Proof: This follows from Lemma 2.4, with the proof being similar to the proof of Proposition 2.1. ■

2.6 Theorem. Given a G-map $\alpha : (E^n_H, S^{n-1}_H, *) \to (X, A, *)$, then $[\alpha] = 0$ in $\pi^H_n(X,A,*)$ if and only if α is G-homotopic relative to S^{n-1}_H to a G-map $E^n_H \to A$.

Proof: From Proposition 2.5 it follows that $[\alpha] = 0$ in $\pi_n^H(X, A, *)$ if and only if for the corresponding map $\alpha' : (E^n, S^{n-1}, *) \to (X^H, A^H, *)$, we have $[\alpha'] = 0$ in $\pi_n(X^H, A^H, *)$. Then α' is homotopic relative to S^{n-1} to a map $E^n \to A^H$ by a homotopy

$$F' : (E^n, S^{n-1}, *) \times I \to (X^H, A^H, *)$$

(see for example [Sp], Thm. 1, p. 372.)

Also a G-homotopy from α to a G-map $E^n_H \to A$ is given by

$$F : \left(\frac{G}{H} \times E^n,\ \frac{G}{H} \times S^{n-1},\ \frac{G}{H} \times *\right) \times I \to (X, A, *)$$

$$\text{where } F(gH, z, t) = gF'(z, t) \quad \text{for} \quad g \epsilon G,\ z \epsilon E^n \text{ and } t \epsilon I.$$

This G-homotopy is relative to S^{n-1}_H since for $z_0 \epsilon S^{n-1}$ we have for all $t \epsilon I$ and $g \epsilon G$:

$$\begin{aligned} F_t(gH, z_0) &= gF'_t(z_0) \\ &= g\alpha'(z_0) \\ &= g\alpha(H, z_0) \\ &= \alpha(gH, z_0). \end{aligned}$$

Conversely, assume that there is a G-homotopy $F : \left(\frac{G}{H} \times E^n,\ \frac{G}{H} \times S^{n-1},\ \frac{G}{H} \times *\right) \times I \to (X, A, *)$ relative to S^{n-1}_H with $F_0 = \alpha$ and $F_1(E^n) \subseteq A$. Then $[\alpha] = [F_1]$ in $\pi_n^H(X, A, *)$ and a G-homotopy K from F_1 to the constant map $\frac{G}{H} \times E^n \to * \epsilon X$ is given by

$$K : \left(\frac{G}{H} \times E^n,\ \frac{G}{H} \times S^{n-1},\ \frac{G}{H} \times *\right) \times I \to (X, A, *)$$

$$\text{where } K(gH, z, t) = F_1(gH, (1-t)z + t*). \quad \blacksquare$$

Let $n \geq 0$. A pair $(X, A, *)$ of based G spaces is said to be G-n-connected if every path component of X intersects A and, for $1 \leq k \leq n$, we have $\pi_k^H(X, A, *) = 0$ for each closed subgroup H of G. As a direct consequence of Theorem 2.6 we have the following corollary.

2.7 Corollary: A pair $(X, A, *)$ is G-n-connected for $n \geq 0$ if and only if for $0 \leq k \leq n$ every G-map $\alpha : (E_H^k,\ S_H^{k-1}, *) \to (X, A, *)$ is G-homotopic relative to S_H^{k-1} to some G-map $E_H^k \to A$.

Finally we note that from the naturality of the isomorphisms $\pi_n^H(X, *) \cong \pi_n(X^H, *)$ and $\pi_n^H(X, A, *) \cong \pi_n(X^H, A^H, *)$ we have that for each closed subgroup H of G the following diagram commutes

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \pi_{n+1}^H(X, A, *) & \xrightarrow{\partial} & \pi_n^H(A, *) & \xrightarrow{i_*} & \pi_n^H(X, *) & \xrightarrow{j_*} & \pi_n^H(X, A, *) & \xrightarrow{\partial} \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & \pi_n(X^H, A^H, *) & \xrightarrow{\partial} & \pi_n(A^H, *) & \xrightarrow{i_*} & \pi_n(X^H, *) & \xrightarrow{j_*} & \pi_n(X^H, A^H, *) & \xrightarrow{\partial} \cdots \end{array}$$

Since the bottom row is exact and the vertical homomorphisms are in fact isomorphisms, it follows that the top row is also exact.

Chapter 3 G-CW-Complexes and the Equivariant Whitehead Theorem

Let n be a fixed positive integer. The G-space X is said to be obtained from the G-space A by attaching equivariant n-cells if for a family of closed subgroups $\{H_j | j \epsilon J\}$ of G we have for each $j \epsilon J$ an indexing set L_j and for each $\lambda \epsilon L_j$ there is a copy $(E^n_\lambda, S^{n-1}_\lambda)$ of the pair of trivial G-spaces (E^n, S^{n-1}), and we have the following G-adjunction space:

$$\begin{array}{ccc} \coprod_{\substack{j\epsilon J \\ \lambda\epsilon L_j}} \frac{G}{H_j} \times S^{n-1}_\lambda & \xrightarrow{\phi} & A \\ \downarrow & & \downarrow \\ \coprod_{\substack{j\epsilon J \\ \lambda\epsilon L_j}} \frac{G}{H_j} \times E^n_\lambda & \xrightarrow{\bar{\phi}} & X = A \cup_\phi \left(\coprod_{\substack{j\epsilon J \\ \lambda\epsilon L_j}} \frac{G}{H_j} \times E^n_\lambda \right) \end{array}$$

We denote $\phi|\frac{G}{H_j} \times S^{n-1}_\lambda$ by ϕ^j_λ and $\bar{\phi}|\frac{G}{H_j} \times E^n_\lambda$ by $\bar{\phi}^j_\lambda$. We call ϕ^j_λ the attaching map for the equivariant n-cell $\frac{G}{H_j} \times E^n_\lambda$ and we call $\bar{\phi}^j_\lambda$ the characteristic map. We say that the pair (X, A) is a G-adjunction of equivariant n-cells.

3.1 Lemma. If, in the situation just described, A is normal and T_1, then X is normal and T_1.

Proof. Since H_j is closed in G for each $j \epsilon J$, it follows (see p. 3) that $\frac{G}{H_j}$ is Hausdorff for each $j \epsilon J$. Since G is compact $\frac{G}{H_j}$ is compact, and hence $\frac{G}{H_j} \times E^n_\lambda$ is a compact Hausdorff G-space for each $j \epsilon J$, $\lambda \epsilon L_j$. Thus $\frac{G}{H_j} \times E^n_\lambda$ is normal since every compact Hausdorff space is normal (see [Hu], Proposition 2.7, p. 63). Hence the topological sum $\coprod_{\substack{j\epsilon J \\ \lambda\epsilon L_j}} \frac{G}{H_j} \times E^n_\lambda$ is normal and by Proposition 1.13, X is normal. Also X is T_1 since both A and $\coprod_{\substack{j\epsilon J \\ \lambda\epsilon L_j}} \frac{G}{H_j} \times E^n_\lambda$ are T_1 (see [Hu], Proposition 3.7, p. 125). ■

Let $(A, *)$ be a based G-space, and let $\{H_j | j \epsilon J\}$ be a family of closed subgroups of G. If for each $j \epsilon J$ there is an indexing set L_j, and for each $\lambda \epsilon L_j$ there is a copy $(E^n_{H_j,\lambda}, S^{n-1}_{H_j,\lambda})$ of $(E^n_{H_j}, S^{n-1}_{H_j})$, and $\phi : \bigvee_{\substack{j \epsilon J \\ \lambda \epsilon L_j}} S^{n-1}_{H_j,\lambda} \to A$ is a based G-map, then there is a based G-adjunction square:

$$\begin{array}{ccc} \bigvee_{\substack{j \epsilon J \\ \lambda \epsilon L_j}} S^{n-1}_{H_j,\lambda} & \xrightarrow{\phi} & A \\ \downarrow & & \downarrow \\ \bigvee_{\substack{j \epsilon J \\ \lambda \epsilon L_j}} E^n_{H_j,\lambda} & \xrightarrow{\bar{\phi}} & X = A \cup_\phi \left(\bigvee_{\substack{j \epsilon J \\ \lambda \epsilon L_j}} E^n_{H_j,\lambda} \right) \end{array}$$

We say that the pair $(X, A, *)$ is a based G-adjunction of based equivariant n-cells, or that X is obtained from A by attaching based equivariant n-cells.

A G-CW-complex or simply G-complex is a G-space X such that there is an expanding sequence of G-spaces

$$X^0 \subset X^1 \subset \ldots \subset X^n \subset \ldots$$

with the following properties

(i) there is a family of closed subgroups H_j of G indexed by J, i.e. $\{H_j | j \epsilon J\}$, with the property that for each $j \epsilon J$ there is an associated indexing set L_j, and $X_0 = \coprod_{\substack{j \epsilon J \\ \lambda \epsilon L_j}} \left(\frac{G}{H_j}\right) \times \{\lambda\}$,

(ii) (X^n, X^{n-1}) is a G-adjunction of equivariant n-cells, and

(iii) X has underlying set $\cup_{n \geq 0} X^n$ and carries the G-final structure with respect to the inclusions $\{i_n : X^n \hookrightarrow X\}_{n \geq 0}$.

The dimension of X is the largest n such that X contains an equivariant n-cell. If no such n exists then the dimension of X is said to be infinite.

More generally, a relative G-CW-complex is defined as follows. If A is any G-space, then (X, A) is a relative G-CW-complex if there exists an expanding sequence of G-spaces (not pairs of G-spaces):

$$A = (X,A)^{-1} \subset (X,A)^0 \subset (X,A)^1 \subset \ldots \subset (X,A)^n \subset \ldots$$

with the following properties:

(i) there is an indexed family $\{H_j | j \epsilon J\}$ of closed subgroups of G, such that for each $j \epsilon J$ there is an associated indexing set L_j, and $(X,A)^0 = A \amalg \left(\coprod_{\substack{j \epsilon J \\ \lambda \epsilon L_j}} \left(\frac{G}{H_j}\right) \times \{\lambda\} \right)$.

(ii) $((X,A)^n, (X,A)^{n-1})$ is a G-adjunction of equivariant n-cells for $n \geq 0$.

(iii) X has underlying set $\cup_{n \geq 0}(X,A)^n$ and carries the G-final structure with respect to the inclusions $\{i_n : (X,A)^n \hookrightarrow (X,A)\}_{n \geq -1}$.

The relative dimension of (X, A), written $\dim(X - A)$, is the largest value of n such that the construction of X from A includes an equivariant n-cell. If no such n exists, the relative dimension is infinite.

A based G-CW-complex can be defined as a based G-space X by taking the definition of G-CW-complex, requiring X^0 to be a 1-point space *, replacing G-spaces by based G-spaces, maps by based G-maps, disjoint unions by wedges and G-adjunctions by based G-adjunctions.

Similarly, a based relative G-CW-complex can be defined as a based G-space X by taking the definition of relative G-CW-complex and replacing G-space by based G-space, G-map by based G-map, disjoint union by wedge, and G-adjunction by based G-adjunction.

3.2 Proposition. A G-CW-complex is normal and T_1 (and hence Hausdorff).

Proof. Since X^0 is discrete, it is normal and T_1. By applying Lemma 3.1, we see that each X^n is normal and T_1 for each $n \geq 0$. Since $X = \cup_{n \geq 0} X^n$, it is a normal G-space. If x is a point of X, then $x \epsilon X^n$ for some $n \geq 0$. But X^n is T_1, so that $\{x\}$ is closed in X^n. Since X^n is closed in X, it follows that $\{x\}$ is also closed in X. Hence, X is T_1. ■

Since any connected G-CW-complex is G-homotopy equivalent to a based G-CW-complex (see [FP], Cor. 2.6.10, p. 82 for the non-equivariant version), it suffices to consider only based G-CW-complexes in our discussion.

In fact, each based G-CW-complex is also an ordinary G-CW-complex if we neglect basepoints. This easily follows from the following result.

3.3 Lemma. Let $(A,*)$ be a based G-space and $(X,A,*)$ be a based G-adjunction of based equivariant n-cells. Then, disregarding the basepoint, (X,A) is a G-adjunction of equivariant n-cells.

Proof. Consider the following diagram:

$$\begin{array}{ccccccc} \coprod\limits_{\substack{j\in J\\ \lambda\in L_j}} \frac{G}{H_j}\times S^{n-1}_\lambda & \longrightarrow & \bigvee\limits_{\substack{j\in J\\ \lambda\in L_j}} \frac{G}{H_j}\times S^{n-1}_\lambda & \longrightarrow & \bigvee\limits_{\substack{j\in J\\ \lambda\in L_j}} S^{n-1}_{H,\lambda} & \xrightarrow{\phi} & A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \coprod\limits_{\substack{j\in J\\ \lambda\in L_j}} \frac{G}{H_j}\times E^n_\lambda & \longrightarrow & \bigvee\limits_{\substack{j\in J\\ \lambda\in L_j}} \frac{G}{H_j}\times E^n_\lambda & \longrightarrow & \bigvee\limits_{\substack{j\in J\\ \lambda\in L_j}} E^n_{H,\lambda} & \xrightarrow{\bar{\phi}} & X \end{array}$$

The right-hand square is a based G-adjunction square since $(X,A,*)$ is a based G-adjunction of based equivariant n-cells. The middle square is a G-adjunction square by Propositions 1.6 and 1.8. The left-hand square is a G-adjunction square by Corollary 1.7. Finally, by Proposition 1.9, the outer perimeter of the above diagram is a G-adjunction square. The result follows. ■

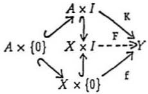
3.4 Theorem. If $(X,*)$ is a based G-CW-complex then X is a G-CW-complex.

Proof. This follows easily from Lemma 3.3 and the relevant definitions. ■

From this we can conclude that results given below for ordinary G-CW-complexes (e.g. Propositions 3.5, 3.6 and 3.9) also hold for based G-CW-complexes.

The G-map $i : A \to X$ is called a G-cofibration if for all G-homotopies $K : A\times I \to Y$ and G-maps $f : X \to Y$ such that $fi = K_0$, there exists a G-homotopy

$F : X \times I \to Y$ such that $F_0 = f$ and $F_t i = K_t$ for $0 \leq t \leq 1$. This is equivalent to saying that the following commutative diagram can be completed by means of a G-map $F : X \times I \to Y$.



If A is a G-subspace of X, then A is called a <u>G-retract</u> of X if there exists a G-map $r : X \to A$ such that $r|A$ is the identity. The G-map r is called a <u>G-retraction</u> of X onto A.

3.5 <u>Proposition.</u> If X is obtained from the G-space A by attaching equivariant n-cells, then $X \times \{0\} \cup A \times I$ is a G-retract of $X \times I$.

<u>Proof</u>: We know that there exists a non-equivariant retraction $\rho : E^n \times I \to E^n \times \{0\} \cup S^{n-1} \times I$ (see for example [Sp], 3.2.4, p. 117). Then if H is a closed subgroup of G an equivariant retraction is given by:

$$1 \times \rho : \frac{G}{H} \times E^n \times I \quad \to \quad \frac{G}{H} \times E^n \times \{0\} \cup \frac{G}{H} \times S^{n-1} \times I$$

$$(gH, z, t) \quad \longmapsto \quad (gH, \rho(z,t)).$$

Let $\bar{\phi}^j_\lambda : \frac{G}{H_j} \times E^n_\lambda \to X$ be the characteristic maps for the equivariant n-cells $\frac{G}{H_j} \times E^n_\lambda$, $j \epsilon J$, $\lambda \epsilon L$ of X. Then a G-retraction can be defined by:

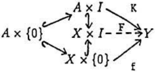
$$R : X \times I \longrightarrow X \times \{0\} \cup A \times I$$

$$(z, t) \longmapsto (z, t) \ , \ (\bar{\phi}^j_\lambda(gH_j, z), t) \longmapsto \phi^j_\lambda((1 \times \rho)(gH_j, z, t))$$

where $x \epsilon A$, $t \epsilon I$, $gH_j \epsilon \dfrac{G}{H_j}$ and $z \epsilon X$.■

3.6 Corollary. If X is obtained from the G-space A by attaching equivariant n-cells, then the inclusion $i : A \hookrightarrow X$ is a G-cofibration.

Proof: Let $R : X \times I \to X \times \{0\} \cup A \times I$ be the G-retraction given by the above proposition. We have to show that the following diagram can be completed:



where Y is any G space, K and f are any G-maps. This can be done by letting F be the composite

$$X \times I \xrightarrow{R} X \times 0 \cup A \times I \xrightarrow{f \cup K} Y.$$

The diagram will be commutative since $R|A \times I = A \times I$ and $R|X \times 0 = X \times 0$. ■

We can extend this result to the case where (X, A) is a relative G-CW-complex.

3.7 Proposition. Let (X, A) be a relative G-CW-complex. Then the inclusion $i : A \hookrightarrow X$ is a G-cofibration.

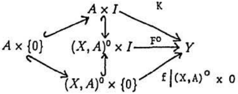
Proof: Let $K : A \times I \to Y$ be a G-homotopy and $f : X \times \{0\} \to Y$ be a G-map with $f|A \times \{0\} = K_0$. We will construct a G-homotopy $F : X \times I \to Y$ inductively on the n-skeletons of (X, A).

We have

$$A = (X, A)^{-1} \subset (X, A)^0 \subset (X, A)^1 \subset \ldots \subset (X, A)^n \subset \ldots \subset X$$

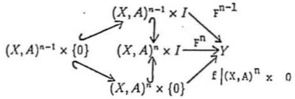
where $(X, A)^k$ is obtained from $(X, A)^{k-1}$ by attaching equivariant k-cells.

Consider the following diagram:



The G-homotopy F^0 exists since $(X,A)^0$ is obtained from A by attaching 0-cells, so by the previous corollary $A \hookrightarrow (X,A)^0$ is a G-cofibration.

In general we have:



Hence we can construct a sequence of G-homotopies $\{F^n : n \geq -1\}$ with $F^{-1} = K$ such that:

(i) $F_0^n = f|(X,A)^n$

(ii) $F^n|((X,A)^{n-1} \times I) = F^{n-1}$

Define $F : X \times I \to Y$ by $F(x,t) = F^n(x,t)$ for $x \epsilon (X,A)^n$, $t \epsilon I$. F is well-defined since $F^n|((X,A)^{n-1} \times I) = F^{n-1}$ so that if $x \epsilon (X,A)^0$, $F^n(x,t) = F^{n-1}(x,t) = \ldots = F^{-1}(x,t) = K(x,t)$. We also know that F is a G-map since $F|((X,A)^n \times I) = F^n$ and $(X,A) \times I$ has the G-final structure with respect to $\{(X,A)^n \times I\}_{n \geq -1}$ (by Lemma 3.3).

Finally $F_0 = f$, since by (i) if $x \epsilon (X,A)^n$,

$$F_0(x) = F_0^n(x) = f(x),$$

and $F|A \times I = K$, since if $x \epsilon A$ then

$$F(x,t) = F^{-1}(x,t) = K(x,t),$$

which implies that $i : A \hookrightarrow X$ is a G-cofibration. ■

3.8 Lemma. For any family $\{X_j\}_{j \epsilon J}$ of topological spaces, $\pi_k\left(\bigvee_{j \epsilon J}(X_j \times E^{n+1}), \bigvee_{j \epsilon J}(X_j \times S^n)\right) = 0$ for $0 < k \leq n$.

Proof: We use the fact that $\bigvee_{j\in J}(X_j \times E^{n+1}) \cong \left(\bigvee_{j\in J} X_j\right) \times E^{n+1}$ and $\bigvee_{j\in J}(X_j \times S^n) \cong \left(\bigvee_{j\in J} X_j\right) \times S^n$. Let $X = \bigvee_{j\in J} X_j$ and let $p : X \times S^n \to X$ and $q : X \times E^{n+1} \to X$ be the projection maps. Then since p and q are fibrations with fibres S^n and E^{n+1}, respectively, we have the following diagram:

$$
\begin{array}{ccccccccc}
\cdots \to & \pi_k(S^n) & \to & \pi_k(X \times S^n) & \xrightarrow{p_*} & \pi_k(X) & \xrightarrow{\partial} & \pi_{k-1}(S^n) & \to \cdots \\
 & & & \Big\downarrow i_* & \searrow^{p_*} & & & & \\
\cdots \to & \pi_k(E^{n+1}) & \to & \pi_k(X \times E^{n+1}) & \xrightarrow{q_*} & \pi_k(X) & \xrightarrow{\partial} & \pi_{k-1}(E^{n+1}) & \to \cdots
\end{array}
$$

with the two horizontal rows exact. Since $qi = p$ we have $q_*i_* = p_*$. Also $\pi_k(S^n) = 0$ for $0 < k < n$ and $\pi_k(E^{n+1}) = 0$ for $0 < k \le n+1$. By exactness, p_* and q_* are both isomorphisms for $0 < k < n$, and hence, i_* is an isomorphism for $0 < k < n$. For $k = n$, q_* is an isomorphism and p_* is an epimorphism. Hence i_* must be an epimorphism for $k = n$.

Now consider the long exact sequence

$$\cdots \to \pi_k(X \times S^n) \xrightarrow{i_*} \pi_k(X \times E^{n+1}) \to \pi_k(X \times E^{n+1}, X \times S^n) \xrightarrow{\partial} \pi_{k-1}(X \times S^n) \to \cdots$$

Since i_* is an isomorphism for $0 < k < n$ and an epimorphism for $k = n$ it follows that $\pi_k(X \times E^{n+1}, X \times S^n) = 0$ for $0 < k \le n$. ■

3.9 Proposition. If X is obtained from the based Hausdorff G-space A by attaching based equivariant $n+1$-cells, then (X, A) is G-n-connected.

Proof: Let $(X, A, *)$ be a based G-adjunction of based equivariant n-cells:

$$\begin{array}{ccc} \bigvee\limits_{\substack{j\in J\\ \lambda\in L_j}} \frac{G}{H_j} \times S^n_\lambda & \xrightarrow{\phi} & A \\ \downarrow & & \downarrow \\ \bigvee\limits_{\substack{j\in J\\ \lambda\in L_j}} \frac{G}{H_j} \times E^{n+1}_\lambda & \xrightarrow{\bar{\phi}} & X \end{array}$$

where $\{H_j | j\epsilon J\}$ is an indexed family of closed subgroups of G and $\{L_j | j\epsilon J\}$ is an indexed family of sets.

Taking H-fixed point sets we see from Propositions 1.12 and 1.14 that the following is an adjunction space:

$$\begin{array}{ccc} \bigvee\limits_{\substack{j\in J\\ \lambda\in L_j}} \left(\frac{G}{H_j}\right)^H \times S^n_\lambda & \xrightarrow{\phi^H} & A^H \\ \downarrow & & \downarrow \\ \bigvee\limits_{\substack{j\in J\\ \lambda\in L_j}} \left(\frac{G}{H_j}\right)^H \times E^{n+1}_\lambda & \xrightarrow{\bar{\phi}^H} & X^H \end{array}$$

Since $\left(\frac{G}{H_j}\right)^H \times S^n_\lambda$ is a deformation retract of a neighbourhood of $\left(\frac{G}{H_j}\right)^H \times E^{n+1}_\lambda$, and $\pi_k\left(\bigvee\left(\frac{G}{H_j}\right)^H \times E^{n+1}_\lambda, \bigvee\left(\frac{G}{H_j}\right)^H \times S^n\right) = 0$ for $0 < k \leq n$ (Lemma 3.8) it follows as a special case of the Blakers-Massey excision theorem, (see [DKP], p. 211), that (X^H, A^H) is n-connected. Hence $\pi_k(X^H, A^H) = 0$ for $1 \leq k \leq n$ and so, by Proposition 2.5, $\pi^H_k(X, A) = 0$ for $1 \leq k \leq n$. So (X, A) is G-n-connected. ■

3.10 Lemma. If X is obtained from the based G-space A by attaching based equivariant n-cells, and $(Y, B, *)$ is a based pair of G-spaces such that $\pi^H_n(Y, B, *) = 0$ for all closed subgroups H of G, then any G-map $f : (X, A, *) \to (Y, B, *)$ is based G-homotopic relative to A to a G-map $k : (X, *) \to (B, *)$.

Proof: Let $\bar{\phi}_j : \left(\frac{G}{H_j} \times E^n, \frac{G}{H_j} \times S^{n-1}, \frac{G}{H_j} \times *\right) \to (X, A, *)$ correspond to the attaching maps of the equivariant n-cells. Then the composite

$$\left(\frac{G}{H_j} \times E^n, \frac{G}{H_j} \times S^{n-1}, \frac{G}{H_j} \times *\right) \xrightarrow{\bar{\phi}_j} (X, A, *) \xrightarrow{f} (Y, B, *)$$

represents a homotopy class in $\pi_n^{H_j}(Y, B, *)$. By Theorem 2.6, there exists a based G-homotopy

$$F_j : \frac{G}{H_j} \times E^n \times I \to Y$$

such that

(i) $F_j(gH_j, z, t) = f(\phi_j(gH_j, z))$ for $(gH_j, z) \epsilon \frac{G}{H_j} \times S^{n-1}$

(ii) $F_j(gH_j, z, 0) = f(\phi_j(gH_j, z))$

(iii) $F_j(gH_j, z, 1) \epsilon B$.

We can thus define a based G-homotopy

$$K : X \times I \to Y$$

$$\text{by} \qquad K(x, t) = f(x) \quad \text{for} \quad x \epsilon A,\ t \epsilon I$$

$$\text{and} \quad K(\phi_j(gH_j, z), t) = F_j(gH_j, z, t) \quad \text{for} \quad (gH_j, z) \epsilon \frac{G}{H_j} \times E^n,\ t \epsilon I.$$

Clearly the G-homotopy is relative to A, and

$$K(\phi_j(gH_j, z), 1) = F_j(gH_j, z, 1) \epsilon B \quad \text{for} \quad (gH_j, z) \epsilon \frac{G}{H_j} \times E^n.$$

Then $k(x) = K(x, 1)$ is the required G-map. ■

3.11 Proposition. If $(X, A, *)$ is a based relative G-CW-complex of relative dimension $\leq n$, and $(Y, B, *)$ is G-n-connected, then any G-map $f : (X, A, *) \to (Y, B, *)$ is based G-homotopic, relative to A, to a G-map from $(X, *) \to (B, *)$.

Proof: We have the following sequence of G-spaces: $A = (X,A)^0 \subset \ldots \subset (X,A)^n \subset \ldots \subset (X,A)$ where $((X,A)^k,(X,A)^{k-1})$ is a based G-adjunction of equivariant k-cells. Since $(X,A)^1$ is obtained from $(X,A)^0 = A$ by attaching based equivariant 1-cells, by Lemma 3.10, $f|(X,A)^1$ is based G-homotopic relative to A to a G-map $(X,A)^1 \to B$ via a based G-homotopy $K^1 : (X,A)^1 \times I \to (Y,B)$. Since $(X,A)^1 \hookrightarrow (X,A)$ is a G-cofibration we can extend K^1 to a based G-homotopy $F^1 : (X,A) \times I \to (Y,B)$ such that $F^1|(X,A)^1 \times I = K^1$.

Now assume that $F^{m-1} : (X,A) \times I \to (Y,B)$ with $m \geq 2$, has been constructed such that $F_1^{m-1}|(X,A)^m : (X,A)^m \to (Y,B)$ is a based G-map with $F_1^{m-1}((X,A)^{m-1}) \subset B$. By Lemma 3.10 there is a based G-homotopy

$$K^m : (X,A)^m \times I \to (Y,B)$$

such that $K_1^m(X,A) \subset B$ and K^m is relative to $(X,A)^{m-1}$.

Since $(X,A)^m \hookrightarrow (X,A)$ is a G-cofibration we can extend K^m to a based G-homotopy

$$F^m : (X,A) \times I \to (Y,B)$$

such that $F_1^m((X,A)^m) \subset B$.

Hence we have a sequence of based G-homotopies $F^0, F^1, \ldots, F^n, \ldots$ such that:

(i) $F_1^{m-1} = F_0^m$

(ii) $F_1^m((X,A)^m) \subset B$

(iii) F^m is a G-homotopy relative to $(X,A)^{m-1}$.

These can be combined consecutively to give a based G-homotopy

$$F : (X,A) \times I \to (Y,B)$$

such that F is relative to A and $F_1(X,A) \subset B$. ∎

Let X and Y be path-connected based G-spaces. Then a based G-map $f : (X,*) \to (Y,*)$ is a based G-n-equivalence if $f^H : (X^H,*) \to (Y^H,*)$ is an n-equivalence for all closed subgroups H of G (by an n-equivalence we mean a map $f : (X,*) \to (Y,*)$ such that $f_* : \pi_q(X,*) \to \pi_q(Y,*)$ is an isomorphism for $0 < q < n$ and an epimorphism for $q = n$). If $f : X \to Y$ is a based G-n-equivalence for all $n \geq 1$, then f is said to be a based G-weak homotopy equivalence.

If $\tilde{M}_f$ is the reduced G-mapping cylinder for f with the G-maps $i : X \to \tilde{M}_f$ and $r : \tilde{M}_f \to Y$ defined in Chapter 1, then since r is a G-homotopy equivalence, f is a G-n-equivalence if and only if i is.

3.12 Proposition. Let $f : (X,*) \to (Y,*)$ be a based G-map. Then $(\tilde{M}_f, X, *)$ is G-n-connected if and only if $i : X \to \tilde{M}_f$ is a based G-n-equivalence.

Proof: Consider the long exact sequence

$$\cdots \longrightarrow \pi_k^H(X) \xrightarrow{i_*} \pi_k^H(\tilde{M}_f) \xrightarrow{j_*} \pi_k^H(\tilde{M}_f, X) \xrightarrow{\partial} \pi_{k-1}^H(X) \xrightarrow{i_*} \pi_{k-1}^H(\tilde{M}_f) \longrightarrow \cdots$$

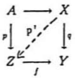
where $k \leq n$. If i is a based G-n-equivalence then $i_* : \pi_{k-1}^H(X) \to \pi_{k-1}^H(\tilde{M}_f)$ is an isomorphism. This implies that $\ker i_* = \operatorname{im} \partial = 0$. Since $i_* : \pi_k^H(X) \to \pi_k^H(\tilde{M}_f)$ is an epimorphism we have $\ker j_* = \pi_k^H(\tilde{M}_f)$ and hence $\operatorname{im} j_* = 0$. By exactness $0 = \operatorname{im} j_* = \ker \partial = \pi_k^H(\tilde{M}_f, X)$. So $(\tilde{M}_f, X)$ is G-n-connected.

Conversely, if $(\tilde{M}_f, X)$ is G-n-connected then $\pi_k^H(\tilde{M}_f, X) = 0$ for $0 \leq k \leq n$ and by a similar argument to the above, using exactness of the sequence, it can be easily shown that $i : X \to \tilde{M}_f$ is a based G-n-equivalence. ∎

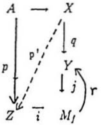
The key to proving the equivariant version of the Whitehead Theorem is the following lemma.

3.13 Lemma. Let $f : (Z,*) \to (Y,*)$ be a based G-n-equivalence (n finite or infinite) and let $(X, A, *)$ be a based relative G-CW-complex with $\dim(X - A) \leq n$. Then for any based G-maps $p : A \to Z$ and $q : X \to Y$ such that $q|A = f \circ p$, there exists a based G-map $p' : X \to Z$ such that $p'|A = p$ and $f \circ p' \simeq_G q$ relative to A.

Proof: Let $\tilde{M}_f$ be the reduced G-mapping cylinder of f, with inclusion maps $i : Z \hookrightarrow \tilde{M}_f$ and $j : Y \hookrightarrow \tilde{M}_f$ and the G-retraction $r : M_f \to Y$. Proving the lemma amounts to essentially completing the following diagram where $p'|A = p$ and $f \circ p' \simeq_G q$



We replace this diagram with the following



We then have $j \circ q|A = j \circ f \circ p = j \circ r \circ i \circ p \simeq_G i \circ p$ since $j \circ r \simeq_G 1_{(M_f)}$. Let $K : A \times I \to \tilde{M}_f$ be the based G-homotopy from $j \circ q|A$ to $i \circ p$. Since $A \hookrightarrow X$ is a G-cofibration, (Proposition 3.7) there is a based G-homotopy $F : X \times I \to \tilde{M}_f$ with $F_0 = j \circ q$ and $F_t|A = K_t$. Let $q' = F_1$. Then $q'|A = i \circ p$ and $r \circ q' \simeq_G r \circ j \circ q$ relative to A. Since $q' : (X, A) \to (\tilde{M}_f, Z)$ and $(\tilde{M}_f, Z)$ is G-n-connected and $\dim(X - A) \leq n$, by Proposition 3.9, q' is based G-homotopic relative to A to some map $p' : X \to Z$. Hence $p'|A = p$ and

$$f \circ p' = r \circ i \circ p' \simeq_G r \circ q' \simeq_G r \circ j \circ q = q.$$

So $f \circ p' \simeq_G q$ relative to A. ■

3.14 Corollary. Let $f : X \to Y$ be a based G-n-equivalence and let P be a based G-CW-complex of dimension $\leq n$. Then the induced function $f_* : [P, X]_G \to [P : Y]_G$, where $f_*([p]) = [f \circ p]$, is surjective. If $\dim P \leq n - 1$, then f_* is injective. In particular, if f is a based G-weak homotopy equivalence then f_* is bijective.

Proof: To prove the first part, apply the previous lemma to the relative G-CW-complex (P, ϕ)

$$\begin{array}{ccc} \phi & \longrightarrow & P \\ {\scriptstyle p}\downarrow & \swarrow {\scriptstyle p'} & \downarrow{\scriptstyle q} \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

For any based G-map $q : P \to Y$ there exists a based G-map $p' : P \to X$ such that

$$f \circ p' \simeq_G q$$

which implies that $f_*([p']) = [f \circ p'] = [q']$ and f_* is surjective.

To show that f_* is injective for $\dim P \leq n - 1$ apply the lemma to the relative G-CW-complex $(P \times I,\ P \times \dot{I})$.

Suppose $p_0, p_1 : P \to X$ are based G-maps such that $f \circ p_0 \simeq_G f \circ p_1$. Let $p : P \times \dot{I} \to X$ be defined by $p(z,0) = p_0(z)$ and $p(z,1) = p_1(z)$. Then there is a based G-homotopy $F : P \times I \to Y$ such that $F_0 = f \circ p_0$ and $F_1 = f \circ p_1$. Since $\dim(P \times I) \leq n$, we can "complete" the following diagram.

$$\begin{array}{ccc} P \times \dot{I} & \longrightarrow & P \times I \\ {\scriptstyle p}\downarrow & \swarrow {\scriptstyle p'} & \downarrow{\scriptstyle F} \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

We have $p'|P \times \dot{I} = p$ which implies that $p'|P \times 0 = p_0$ and $p'|P \times 1 = p_1$ so p' is a G-homotopy from p_0 to p_1 and $[p_0] = [p_1]$. Hence f_* is injective. ■

3.15 Theorem. (Equivariant Whitehead) A based G-map between based G-CW-complexes is a based G-weak homotopy equivalence if and only if it is a based G-homotopy equivalence.

Proof: Clearly a based G-map which is a based G-homotopy equivalence is also a based G-weak homotopy equivalence. Assume $f : X \to Y$ is a based G-weak homotopy equivalence. By the corollary, $f_* : [Y : X]_G \to [Y : Y]_G$ and $f_* : [X :$

$X]_G \to [X : Y]_G$ are bijections. If $p : Y \to X$ is a based G-map with $f_*[p] = [1_Y]$ then $f \circ p \simeq_G 1_Y$. Also we have

$$f_*([p \circ f]) = [f \circ p \circ f] = [1_Y \circ f] = [f \circ 1_X] = f_*([1_X]).$$

Now f_* is injective so $[p \circ f] = [1_X]$, which means that $p \circ f \simeq_G 1_X$. So we have $f \circ p \simeq_G 1_Y$ and $p \circ f \simeq_G 1_X$, and f is a based G-homotopy equivalence. ■

Even though Theorem 3.15 is not needed in the proof of the Brown representability theorem, we have included it since it is a direct consequence of the results of this chapter.

Chapter 4 Equivariant Brown Representability Theorem

Let $\mathcal{C}_0$ be the category of path-connected based G-spaces with the inclusion $i : * \hookrightarrow X$ a G-cofibration for each $X \epsilon \mathcal{C}_0$ and let the morphisms be the based G-homotopy classes of based G-maps. Unless otherwise stated all G-maps are assumed to be based G-maps and all G-homotopies are assumed to be based G-homotopies.

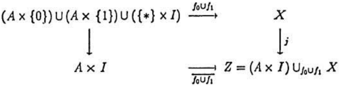
Given two G-maps $f_0, f_1 : A \to X$ then the G-map $j : X \to Z$ is said to be an equalizer of $[f_0]$ and $[f_1]$ if the following hold:

(i) $j \circ f_0 \simeq_G j \circ f_1$

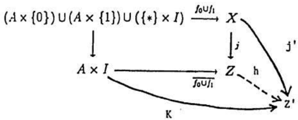
(ii) If $j' : X \to Z'$ is a G-map such that $j' \circ f_0 \simeq_G j' \circ f_1$ then there exists a G-map $g : Z \to Z'$ such that $g \circ j \simeq_G j'$.

4.1 Lemma Any pair of G-maps in $\mathcal{C}_0$ has an equalizer.

Proof: Let $f_0, f_1 : A \to X$ be two G-maps in $\mathcal{C}_0$. Consider the G-adjunction space Z defined via the following diagram:



Then $F = \overline{f_0 \cup f_1} : A \times I \to Z$ is a G-homotopy such that $F_0 = F|A \times 0 = j \circ f_0$ and $F_1 = F|A \times I = j \circ f_1$. Furthermore, if $j' : X \to Z'$ is a G-map such that there is a G-homotopy $K : A \times I \to Z'$ from $j' \circ f_0$ to $j' \circ f_1$ then we have the following commutative diagram:



By the universal property of G-adjunction spaces there is a G-map h such that $h \circ j = j'$. Hence $j : X \to Z$ is an equalizer of f_0 and f_1. ■

4.2 Example. If f_0 is the constant map, then $Z = CA \cup_{f_1} X$.

A G-homotopy functor is a contravariant functor $F : \mathcal{C}_0 \to \mathcal{S}_0$ from $\mathcal{C}_0$ to the category $\mathcal{S}_0$ of pointed sets such that the following two properties hold:

Equalizer Axiom: If $j : X \to Z$ is an equalizer of $f_0, f_1 : A \to X$ and $u \epsilon F(X)$ is such that $F(f_0)u = F(f_1)u$ there exists $v \epsilon F(Z)$ such that $F(j)v = u$.

Wedge Axiom: If $\{X_\lambda\}_{\lambda \epsilon L}$ is a family of G-spaces in $\mathcal{C}_0$ then there is a bijection of sets

$$\{F[i_\lambda]\}_\lambda : F(\vee_\lambda X_\lambda) \to \prod_{\lambda \epsilon L} F(X_\lambda)$$

where $\{F[i_\lambda]\}_\lambda$ is the morphism induced by the inclusions $i_\lambda : X_\lambda \hookrightarrow \bigvee_{\lambda \epsilon L} X_\lambda$.

Note that if $L = \{1,2\}$ and $X_1 = * = X_2$, then

$$\{F[i_1], F[i_2]\} : F(X_1 \vee X_2) \to F(X_1) \times F(X_2)$$

is a bijection. But $X_1 \vee X_2 = *$, so that we have a bijection $\{F[i_1], F[i_2]\} : F(*) \to F(*) \times F(*)$. This can happen only if $F(*)$ is a single point.

4.3 Proposition. If Y is any pointed G-space in $\mathcal{C}_0$ then $\pi^Y = [\ : Y]_G^0$ is a G-homotopy functor.

Proof: Let $j : X \to Z$ be an equalizer of $f_0, f_1 : A \to X$ and let $u \epsilon \pi^Y(X)$ be such that $\pi^Y(f_0)u = \pi^Y(f_1)u$. Thus $u \circ f_0 \simeq_G u \circ f_1$ and since j is an equalizer, there is a G-map $v : Z \to Y$ such that $v \circ j \simeq_G u$, which means $\pi^Y(j)v = u$. So $\pi^Y = [\ : Y]_G^0$ satisfies the Equalizer Axiom.

For the Wedge Axiom, let $\{[f_\lambda]\}_{\lambda \epsilon L}$ be an element of $\prod_{\lambda \epsilon L} \pi^Y(X_\lambda)$, where $f_\lambda : X_\lambda \to Y$ is a G-map. Define $f : \vee_\lambda X_\lambda \to Y$ by $f(x) = f_\lambda(x)$ for $x \epsilon X_\lambda$. Then we have $f \circ i_\lambda = f_\lambda$ for all $\lambda \epsilon L$, which means that $\{\pi^Y[i_\lambda]\}_\lambda[f] = \{[f_\lambda]\}_{\lambda \epsilon L}$. So $\{\pi^Y[i_\lambda]\}_\lambda$ is surjective.

Now assume that $[f]$ and $[g]$ are two elements of $[\vee_\lambda X_\lambda : Y]_G$ such that $\{\pi^Y[i_\lambda]\}_\lambda[f] = \{\pi^Y[i_\lambda]\}[g]$. For each $\lambda \epsilon L$ there is a G-homotopy $K^\lambda : X_\lambda \times I \to Y$ such that $K^\lambda(*, t) = *$ for $t \epsilon I$, $K_0^\lambda = f \circ i_\lambda$, and $K_1^\lambda = g \circ i_\lambda$. Using the fact that $\vee_\lambda(X_\lambda \times I) \cong (\vee_\lambda X_\lambda) \times I$ we get a G-homotopy $K : (\vee_\lambda X_\lambda) \times I \to Y$ with $K \circ (i_\lambda \times 1) = K^\lambda$ and $K_0 = f$, $K_1 = g$. Hence $[f] = [g]$ and $\{\pi^Y[i_\lambda]\}_\lambda$ is an injection and thus a bijection. ■

If Y is an object of C_0, then for any element $u \epsilon F(Y)$ we have the natural transformation $T_u : \pi^Y \to F$ defined by $T_u([f]) = F([f])u$. To show that this is a natural transformation let $f : X \to Z$ be a based G-map, where X and Z are G-spaces in C_0, and consider the following diagram:

$$\begin{array}{ccc} [X:Y]_G & \xrightarrow{T_u} & F(X) \\ {\scriptstyle \pi^Y(f)} \uparrow & & \uparrow {\scriptstyle F(f)} \\ [Z:Y]_G & \xrightarrow{T_u} & F(Z) \end{array}$$

If $\alpha : Z \to Y$, then $\pi^Y(f)[\alpha] = [\alpha \circ f]$ and $T_u([\alpha \circ f]) = F(\alpha \circ f)u = F(f)F(\alpha)u$. Hence the diagram commutes and T_u is a natural transformation.

For any suspension SX, $F(SX)$ is a group (for the non-equivariant case see [Sp], Lemma 6, p. 407, the equivariant case follows by the analogous argument). Let $\nu_H : SX \to SX \vee SX$ be the comultiplication map. Then the group multiplication on $F(SX)$ is given by the composite:

$$F(SX) \times F(SX) \xrightarrow{(F[i_1], F[i_2])^{-1}} F(SX \vee SX) \xrightarrow{F(\nu_H)} F(SX).$$

4.4 Lemma. Let H be a closed subgroup of G and let F be a G-homotopy functor. If $u \epsilon F(Y)$ then $T_u : \pi^Y(S^q_H) \to F(S^q_H)$ is a group homomorphism.

Proof. Let $[f], [g] \epsilon \pi^Y(S^q_H)$. Then

$$\begin{aligned}
T_u(\nabla(f \vee g)\nu_H) &= F(\nabla(f \vee g)\nu_H)(u) \\
&= F(\nu_H)\{F[i_1], F[i_2]\}^{-1}\{F[i_1], F[i_2]\}F(\nabla(f \vee g))(u) \\
&= F(\nu_H)\{F[i_1], F[i_2]\}^{-1}\{F[(f \vee g)i_2], F[(f - \vee g)i_2]\}(u) \\
&= F(\nu_H)\{F[i_1], F[i_2]\}^{-1}\{F[f], F[g]\}(u) \\
&= F(\nu_H)\{F[i_1], F[i_2]\}^{-1}(F[f](u), F[g](u))
\end{aligned}$$

and the last line is the product of $F(f)(u)$ and $F(g)(u)$ in $F(S^q_H)$, as required. ∎

The element $u \epsilon F(Y)$ is said to be G-n-universal for $n \geq 1$, if for all closed subgroups H of G, $T_u : \pi^Y(S^q_H) \to F(S^q_H)$ is an isomorphism for $1 \leq q < n$ and an epimorphism for $q = n$. If u is G-n-universal for all n, then u is called a G-universal element and Y is called a classifying space for F.

4.5 Lemma. Let F be a G-homotopy functor with universal elements $u \epsilon F(Y)$ and $u' \epsilon F(Y')$ and assume $f : Y \to Y'$ is a based G-map such that $F(f)u' = u$. Then f is a based G-weak homotopy equivalence.

Proof: For $q \geq 1$ and H any closed subgroup of G, we have the following commutative diagram:

$$\begin{array}{ccc}
[S^q_H : Y]^0_G & \xrightarrow{f_*} & [S^q_H : Y']^0_G \\
& {\scriptstyle T_u} \searrow \quad \swarrow {\scriptstyle T_{u'}} & \\
& F(S^q_H) &
\end{array}$$

where $f_*([\alpha]) = [f \circ \alpha]$. The commutativity follows from the fact that if $\alpha : S^q_H \to Y$ is a based G-map then

$$\begin{aligned} T_{u'}(f_*([\alpha])) &= T_{u'}([f \circ \alpha]) \\ &= F(f \circ \alpha)u' \\ &= F(\alpha)F(f)u' \\ &= F(\alpha)u \\ &= T_u([\alpha]) \end{aligned}$$

Since T_u and $T_{u'}$ are bijective, f_* must also be a bijection. ■

Let Y be a G-subspace of Z in $\mathcal{C}_0$ and $u \epsilon F(Z)$. We will write $u|Y$ for $F(i)u \epsilon F(Y)$ where $i : Y \to Z$ denotes the inclusion.

The key to proving the Brown Representability Theorem is the construction of a classifying space Y and a G-universal element $u \epsilon F(Y)$. This is done by constructing a sequence of G-spaces $Y_0 \subset Y_1 \subset \ldots \subset Y_n \subset \ldots$ such that for each $n \geq 0$ Y_{n+1} is obtained from Y_n by attaching equivariant $(n+1)$-cells, and such that there exists a G-n-universal element $u_n \epsilon F(Y_n)$ with $u_{n+1}|Y_n = u_n$. The classifying space for F will be $Y = \cup_{n \geq 0} Y_n$.

The sequence of G-spaces is defined in the following two lemmas.

4.6 Lemma. Let F be a G-homotopy functor with Y_0 a G-space in $\mathcal{C}_0$ and $u_0 \epsilon F(Y_0)$. Then there exists a G-space Y_1 obtained from Y_0 by attaching equivariant 1-cells, and a G-1-universal element $u_1 \epsilon F(Y_0)$ such that $u_1|Y_0 = u_0$

Proof: For each closed subgroup H of G and for each $\lambda \epsilon F(S^1_H)$ let $S^1_{H,\lambda}$ be an equivariant 1-sphere and let $\vee_{H,\lambda} S^1_{H,\lambda}$ be the wedge of these 1-spheres. Define $Y_1 = Y_0 \vee (\vee_{H,\lambda} S^1_{H,\lambda})$. Let $g_{H,\lambda}$ be the composite $S^1_H \overrightarrow{\cong} S^1_{H,\lambda} \hookrightarrow Y_1$. By the Wedge Axiom we know that there is an element $u_1 \epsilon F(Y_1)$ such that $u_1|Y_0 = u_0$ and $F(g_{H,\lambda})u_1 = \lambda$ for all $\lambda \epsilon F(S^1_H)$. Since $T_u([g_{H,\lambda}]) = F([g_{H,\lambda}])u_1 = \lambda$ it follows that $T_{u_1} : \pi^{Y_1}(S^1_H) \to F(S^1_H)$ is surjective and u_1 is G-1-universal. ■

4.7 Lemma. Let Y_n be a Hausdorff G-space in $\mathcal{C}_0$, F be a G-homotopy functor, and $u_n \epsilon F(Y_n)$ be a G-n-universal element for F, with $n \geq 1$. Then there exists an object Y_{n+1} in C_0 obtained from Y_n by attaching equivariant $(n+1)$-cells and a G-$(n+1)$-universal element $u_{n+1}\epsilon F(Y_{n+1})$ with $u_{n+1}|Y_n = u_n$.

Proof: For each closed subgroup H of G and for each $\lambda\epsilon F(S_H^{n+1})$, let $S_{H,\lambda}^{n+1}$ be an equivariant $(n+1)$-sphere and form the G-space $Y_n \vee (\vee_{H,\lambda} S_{H,\lambda}^{n+1})$. For each based G-map $\alpha : S_H^n \to Y_n$ such that $T_{u_n}(\alpha) = F(\alpha)u_n = 0$ attach an equivariant $(n+1)$-cell to Y_n by α to obtain the based G-space Y_{n+1}. Let $g_{H,\lambda}$ be the composite based G-map $S_H^{n+1} \overrightarrow{\cong} S_{H,\lambda}^{n+1} \hookrightarrow Y_n \vee (\vee_{H,\lambda} S_{H,\lambda}^{n+1})$. By the Wedge Axiom there is an element $\bar{u}\epsilon F(Y_n \vee (\vee_{H,\lambda} S_{H,\lambda}^{n+1}))$ such that $\bar{u}|Y_n = u_n$ and $F(g_{H,\lambda})\bar{u} = \lambda$ for $\lambda\epsilon F(S_H^{n+1})$.

For each based G-map $\alpha : S_H^n \to Y$ such that $F(\alpha)u_n = 0$, let $S_{H,\alpha}^n$ be an equivariant n-sphere and let $f_0 : \vee_\alpha S_{H,\alpha}^n \to Y_n \vee (\vee_{H,\lambda} S_{H,\lambda}^{n+1})$ be the constant map. Define $f_1 : \vee_\alpha S_{H,\alpha}^n \to Y_n \vee (\vee_{H,\lambda} S_{H,\lambda}^{n+1})$ by $f_1|S_{H,\alpha}^n = \alpha$. Then according to Example 4.2 we have the following G-adjunction space:

$$\begin{array}{ccc} (\vee_\alpha S_{H,\lambda}^n \times 0) \cup (\vee_\alpha S_{H,\alpha}^n \times 1) & \xrightarrow{f_0 \cup f_1} & Y_n \vee (\vee_{H,\lambda} S_{H,\lambda}^{n+1}) \\ \downarrow & & \downarrow j \\ \vee_\alpha S_{H,\alpha}^n \times I & \xrightarrow[f_0 \cup f_1]{} & Z = C(\vee_\alpha S_{H,\alpha}^n) \cup_{f_1} Y_n \vee (\vee_{H,\lambda} S_{H,\lambda}^{n+1}) \end{array}$$

Using the fact that $C(S_H^n) = E_H^{n+1}$, it follows that $C(\vee_\alpha S_{H,\alpha}^n) = \vee_\alpha C(S_{H,\alpha}^n) = \vee_\alpha E_{H,\alpha}^{n+1}$. Hence Z is homeomorphic to Y_{n+1} and $j : Y_n \vee (\vee_{H,\lambda} S_{H,\lambda}^{n+1}) \hookrightarrow Y_{n+1}$ is an equalizer of f_0 and f_1.

Since f_0 is the constant map $F(f_0)\bar{u} = 0$. We can also show $F(f_1)\bar{u} = 0$.
Since $f_1|S_{H,\alpha}^n = \alpha$ and $F(\alpha)u_n = 0$ for each α, then from the bijection given in the Wedge Axiom, we have

$$F(f_1)\bar{u} = \{F(\alpha)u_n\}_\alpha = 0.$$

So $F(f_0)\bar{u} = 0 = F(f_1)\bar{u}$. By the Equalizer Axiom there exists an element $u_{n+1}\epsilon F(Y_{n+1})$ such that $F(j)u_{n+1} = \bar{u}$ which implies $u_{n+1}|Y_n = u_n$. We have to show that u_{n+1} is G-$(n+1)$-universal.

For each closed subgroup H of G we have the following commutative diagram:

$$\pi^H_{q+1}(Y_{n+1},Y_n) \xrightarrow{\partial} \pi^H_q(Y_n) \xrightarrow{i_*} \pi^H_q(Y_{n+1}) \xrightarrow{j_*} \pi^H_q(Y_{n+1},Y_n)$$

$$T_{u_n} : \pi^H_q(Y_n) \to F(S^q_H), \qquad T_{u_{n+1}} : \pi^H_q(Y_{n+1}) \to F(S^q_H)$$

Since Y_{n+1} is obtained from Y_n by attaching equivariant $(n+1)$-cells we know from Proposition 3.9 that $\pi^H_q(Y_{n+1},Y_n) = 0$ for $1 \leq q \leq n$ and for any closed subgroup H of G. Hence i_* is an isomorphism for $q < n$ and an epimorphism for $q = n$. Since T_{u_n} is an isomorphism for $q < n$ and an epimorphism for $q = n$, $T_{u_{n+1}}$ must also be an isomorphism for $q < n$ and an epimorphism for $q = n$. We thus only have to show that $T_{u_{n+1}}$ is injective for $q = n$ and surjective for $q = n+1$.

Let $\beta \epsilon \pi^H_n(Y_{n+1})$ with $T_{u_{n+1}}(\beta) = 0$. Since i_* is surjective for $q = n$, there exists $\alpha \epsilon \pi^H_n(Y_n)$ such that $i_*(\alpha) = \beta$. Hence $T_{u_n}(\alpha) = T_{u_{n+1}}(i_*(\alpha)) = T_{u_{n+1}}(\beta) = 0$. By definition of Y_{n+1} there is an equivariant $(n+1)$-cell attached to $Y_n \vee (\vee_{H,\lambda} S^{n+1}_{H,\lambda})$ via α. This implies that $i_*(\alpha) = 0 = \beta$. So $\ker T_{u_{n+1}} = 0$ and $T_{u_{n+1}}$ is injective for $q = n$.

To prove that $T_{u_{n+1}}$ is surjective for $q = n+1$, let $\lambda \epsilon F(S^{n+1}_H)$ and note that the G-map $j \circ g_{H,\lambda} : S^{n+1}_H \to Y_{n+1}$ is such that

$$\begin{aligned} T_{u_{n+1}}([j \circ g_{H,\lambda}]) &= F(j \circ g_{H,\lambda})u_{n+1} \\ &= F(g_{H,\lambda})F(j)u_{n+1} \\ &= F(g_{H,\lambda})\bar{u} \\ &= \lambda. \end{aligned}$$

Hence u_{n+1} is an isomorphism for $q < n+1$ and and epimorphism for $q = n+1$. Therefore u_{n+1} is a G-$(n+1)$-universal element. ■

4.8 Lemma. Let $\{Y_n\}_{n \geq 0}$ be a sequence of based G-spaces in $\mathcal{C}_0$ such that the inclusion $i_n : Y_n \to Y_{n+1}$ is a G-cofibration for all $n \geq 0$ and let $Y = \cup_{n \geq 0} Y_n$ where Y has the G-final structure with respect to $\{Y_n\}_{n \geq 0}$. Let $1_n : Y_n \to Y_n$ be the identity and

$j_n : Y_n \hookrightarrow Y$ be the inclusion. If $i : \vee Y_n \to \vee Y_n$, $1 : \vee Y_n \to \vee Y_n$, and $j : \vee Y_n \to Y$ are the based G-maps induced by the i_n's, 1_n's, and j_n's respectively, then j is an equalizer of i and 1.

Proof: It is clear that $j_{n+1} \circ i_n = j_n \circ 1_n$. Hence $j \circ i = j \circ 1$. Now assume $j' : \vee Y_n \to Y'$ is such that $j' \circ i \simeq_G j \circ 1$. Let $j'_n : Y_n \to Y'$ be defined by $j'_n = j'|Y_n$ so that $j'_{n+1} \circ i_n \simeq_G j'_n$. Since i_n is a G-cofibration we can define a sequence of G-maps $g_n : Y_n \to Y'$ such that $j_n' \simeq_G g_n$ and $g_{n+1} \circ i_n = g_n$. Using the universal property for expanding sequences of G-spaces define $g : Y \to Y'$ by $g|Y_n = g_n$. Then $g \circ j_n = g_n \simeq_G j_n'$ so that $g \circ j \simeq_G j'$ and j is an equalizer of i and 1. ■

4.9 Theorem. Let F be a G-homotopy functor, Y_0 a G-space of C_0 whose underlying space is normal and T_1, and $u_o \epsilon F(Y_0)$. Then there exists a classifying space Y for F such that (Y, Y_0) is a based relative G-CW-complex and a G-universal element $u \epsilon F(Y)$ such that $u|Y_0 = u_0$.

Proof: By using Lemmas [4.6] and [4.7] we can construct a sequence of G-spaces $Y_0 \subset Y_1 \subset \ldots \subset Y_n \subset \ldots$ with Y_{n+1} obtained from Y_n by attaching equivariant $(n+1)$-cells, and G-universal elements $u_n \epsilon F(Y_n)$ with $u_{n+1}|Y_n = u_n$. Lemma 3.1 ensures that each G-space Y_n is normal and T_1, and hence Hausdorff. Thus Lemma 4.7 can be applied to each Y_n. Let $Y = \cup_{n \geq 0} Y_n$ have the G-final structure with respect to $\{Y_n\}_{n \geq 0}$. From Lemma [4.8] we know that $j : \vee Y_n \to Y$ is an equalizer of $i : \vee Y_n \to \vee Y_n$ and $1 : \vee Y_n \to \vee Y_n$. By the Wedge Axiom there exists an element $\bar{u} \epsilon F(\vee Y_n)$ such that $\bar{u}|Y_n = u_n$. Since $\bar{u}|Y_{n+1} = u_{n+1}$ and $u_{n+1}|Y_n = u_n$ we have that $F(i_n)\bar{u} = F(1_n)\bar{u}$ for all $n \geq 0$ and hence

$$F(i)\bar{u} = F(1)\bar{u}.$$

We deduce from the Equalizer Axiom that there exists an element $u \epsilon F(Y)$ such that $F(j)u = \bar{u}$. Since $\bar{u}|Y_n = u_n$, $u|Y_n = u_n$ for $n \geq 0$. In particular $u|Y_0 = u_0$.

For $1 \leq q < n$ and H a closed subgroup of G, we have the following commutative diagram:

For $1 \leq q < n$ and H a closed subgroup of G, we have the following commutative diagram:

$$\begin{array}{ccc} \pi_q^H(Y_n) & \xrightarrow{(j_n)_*} & \pi_q^H(Y) \\ & {\scriptstyle T_{u_n}} \searrow \quad \swarrow {\scriptstyle T_u} & \\ & F(S^q_H) & \end{array}$$

Since Y is obtained from Y_n by attaching equivariant cells of dimension $\geq n+1$ and $q < n$, (j_n) is an isomorphism. Also T_{u_n} is an isomorphism since u_n is G-n-universal. Thus T_u is an isomorphism for $q < n$. The above argument holds for any n and thus T_u is G-universal. ■

The completion of the proof of the Equivariant Brown Representability Theorem is very similar to the proof of the Whitehead Theorem in Chapter 3.

4.10 Corollary. Let Y be a G-space whose underlying space is normal and T_1, and let $u \epsilon F(Y)$ be a G-universal element for a G-homotopy functor F. Let $(X, A, *)$ be a based relative G-CW-complex, where A and X are G-spaces in C_0. Given a G-map $g : A \to Y$ and an element $v \epsilon F(X)$ such that $v|A = F(g)u$, there exists a G-map $g' : X \to Y$ such that $g'|A = g$ and $F(g')u = v$.

Proof: We are given that $F(f)v = F(g)u$ and have to find a g' such that $F(g')u = v$. The proof is similar to that of Lemma 3.13 in that we must complete the diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & \swarrow {\scriptstyle g'} & \downarrow{\scriptstyle \bar{g}} \\ Y & \xrightarrow[h]{} & Y' \end{array}$$

where h is a G-weak homotopy equivalence and $\bar{g}|A = h \circ g$. So we first have to define Y' and the G-maps h and $\bar{g}$.

Let $j : X \vee Y \to Z$ be the equalizer of the composite maps:

$$A \xrightarrow{f} X \xrightarrow{i_X} X \vee Y$$
$$A \xrightarrow{g} Y \xrightarrow{i_Y} X \vee Y$$

By the Wedge Axiom there is an element $\bar{v}\epsilon F(X \vee Y)$ such that $\bar{v}|X = v$ and $\bar{v}|Y = u$. Since $F(f)v = F(g)u$ we have $F(i_X \circ f)v = F(i_Y \circ g)u$ and by the Equalizer Axiom there is an element $\bar{u}\epsilon F(Z)$ such that $F(j)\bar{u} = \bar{v}$. By Theorem [4.9] we can construct a G-space Y' containing Z with $u'\epsilon F(Y')$ a G-universal element and $u'|Z = \bar{u}$. Define h to be the composite

$$\underbrace{Y \overset{i_Y}{\hookrightarrow} X \vee Y \overset{j}{\to} Z \overset{i_Z}{\hookrightarrow} Y'}_{h}$$

Then $F(h)u' = u$ and by Lemma 4.4 h must be a G-weak homotopy equivalence. Since j is an equalizer of $i_X \circ f$ and $i_Y \circ g$, the composite

$$A \overset{f}{\hookrightarrow} X \overset{i_X}{\hookrightarrow} X \vee Y \overset{j}{\to} Z \overset{i_Z}{\hookrightarrow} Y'$$

is G-homotopic to the composite

$$A \overset{g}{\to} Y \overset{i_Y}{\hookrightarrow} X \vee Y \overset{j}{\to} Z \overset{i_Z}{\hookrightarrow} Y' .$$

Let k be the composite $i_Z \circ j \circ i_X$. Then $h \circ g \simeq_G k \circ f$. Since f is a G-cofibration, there is a G-map $\bar{g} : X \to Y'$ such that $\bar{g} \simeq_G k$ and $\bar{g}|A = h \circ g$. By Lemma 3.13, there is a G-map $g' : X \to Y$ such that $g'|A = g$ and $h \circ g' \simeq_G \bar{g}$. Since $h \circ g' \simeq_G \bar{g}$ and $\bar{g} \simeq_G k$, we have

$$F(g')u = F(g')F(h)u' = F(\bar{g})u' = F(k)u' = F(i_X)F(j)F(i_Z)u' .$$

But $F(j)F(i_Z)u' = \bar{v}$ so $F(g')u = \bar{v}|X = v$, and g is the required G-map. ■

4.11 Theorem (Equivariant Brown Representibility Theorem). Let G be a compact Hausdorff group. If $F : \mathcal{C}_0 \to \mathcal{S}_0$ is a G-homotopy functor, then a classifying space

Y for F exists with a G-universal element $u \epsilon F(Y)$. Further, for any based G-CW-complex X in $\mathcal{C}_0$, $T_u : \pi^Y(X) \to F(X)$ is a natural equivalence.

Proof: The existence of Y and u follows from Theorem 4.9. Taking Y_0 to be a 1-point G-space, Y will be a G-CW-complex, and hence is normal and T_1 by Proposition 3.2.
(i) T_u is surjective. Taking $v \epsilon F(X)$ we will apply Corollary [4.10] with $A = *$ and g the constant map. Since $F(*)$ is a single point, $F(f)v = F(g)u$, so there exists a G-map $g' : X \to Y$ such that $F(g')u = v$.
(ii) T_u is injective. Let $g_0, g_1 : X \to Y$ be two G-maps such that $T_u([g_0]) = T_u([g_1])$. Let $X' = \frac{X \times I}{* \times I}$, i.e. the based G-CW-complex with n-skeleton $\frac{(X^n \times I) \cup (X^{n-1} \times I)}{* \times I}$ for $n \geq 0$, and and define $h : X' \to X$ by $h([x,t]) = x$. Let $v \epsilon F(X')$ be defined by $v = F(h)F(g_0)u$. Identifying $A = \frac{X \times I}{* \times I}$ with $X \vee X$ we will define $g : A \to Y$ by $g([x,0]) = g_0(x)$ and $g([x,1]) = g_1(x)$.

We need to show that $F(g)u = v|A$. Let $\nabla : Y \vee Y \to Y$ be the folding map, $i : A = X \vee X \to X'$ be the inclusion, and $\{F[i_1], F[i_2]\} : F(X \vee X) \to F(X) \times F(X)$ the known bijection. Then

$$
\begin{aligned}
F(g)u &= F(\nabla(g_0 \vee g_1))u \\
&= \{F[i_1], F[i_2]\}^{-1}\{F[i_1], F[i_2]\}F(\nabla(g_0 \vee g_1))u \\
&= \{F[i_1], F[i_2]\}^{-1}(F(g_0)u, F(g_1)u) \\
&= \{F[i_1], F[i_2]\}^{-1}(F(g_0)u, F(g_0)u) \\
&= F(\nabla(g_0 \vee g_1))u \\
&= F(g_0 h i)u \\
&= F(i)F(h)F(g_0)u \\
&= F(i)v \\
&= v|A
\end{aligned}
$$

By Corollary 4.10 there is a map $g' : X' \to Y$ such that $g'|A = g$. The composite

By Corollary 4.10 there is a map $g' : X' \to Y$ such that $g'|A = g$. The composite

$$X \times I \xrightarrow{p} \frac{X \times I}{* \times I} \xrightarrow{g'} Y$$

is such that $g' \circ p|X \times \{0\} = g_0$ and $g' \circ p|X \times \{1\} = g_1$. Thus $g' \circ p$ is a G-homotopy from g_0 to g_1, which implies $[g_0] = [g_1]$ and so T_u is injective. ■

Chapter 5 The Mayer-Vietoris Condition and the Equalizer Axiom

Following Brown's original account, [Brl], we could define a homotopy functor $F : \mathcal{C}_0 \to \mathcal{S}_0$ as one which satisfies the Wedge Axiom and the following:

Mayer-Vietoris Axiom. Let $(X : A_1, A_2)$ be a triad of G-spaces, i.e. A_1 and A_2 are G-subspaces of X with $X = A_1 \cup A_2$. Then if there are $x_1 \epsilon F(A_1)$, $x_2 \epsilon F(A_2)$ such that $x_1|A_1 \cap A_2 = x_2|A_1 \cap A_2$, then there exists $v \epsilon F(X)$ such that $v|A_1 = x_1$, $v|A_2 = x_2$. Such functors F will be called MV-homotopy functors.

Let A, X_1, X_2 and Z be based G-spaces and let $f_1 : A \to X_1$, $f_2 : A \to X_2$, $g_1 : X_1 \to Z$ and $g_2 : X_2 \to Z$ be based G-maps such that $g_1 f_1 \simeq_G g_2 f_2$. Then (f_1, g_1, f_2, g_2) will be called a weak pushout if, for every based G-space Z' and pair of based G-maps $g_1' : X_1 \to Z'$ and $g_2' : X_2 \to Z'$ such that $g_1' f_1 \simeq_G g_2' f_2$, there exists a based G-map $g : Z \to Z'$ such that $g g_1 \simeq_G g_1'$ and $g g_2 \simeq_G g_2'$.

5.1 Proposition. Given that $j : X \to Z$ is an equalizer of $f_1 : A \to X$ and $f_2 : A \to X$, then $(f_1 \vee f_2, j, \nabla, j f_1)$ is a weak pushout. Here $f_1 \vee f_2 : A \vee A \to X$ is the obvious G-map and $\nabla : A \vee A \to A$ is the folding G- map.

Proof: We first notice that $j(f_1 \vee f_2) = j f_1 \vee j f_2 \simeq_G j f_1 \vee j f_1 = j f_1 \nabla$. Assuming that Z' is a based G-space, and $g_1' : X \to Z'$ and $g_2' : A \to Z'$ are based G-maps such that $g_1'(f_1 \vee f_2) \simeq_G g_2' \nabla$, we have

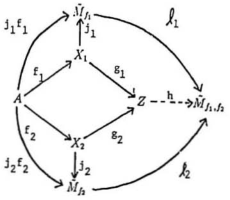
$$g_1' f_1 \vee g_1' f_2 = g_1'(f_1 \vee f_2) \simeq_G g_2' \nabla = g_2' \vee g_2',$$

so $g_1' f_1 \simeq_G g_2' \simeq_G g_1' f_2$. Now j is an equalizer so there is a based G-map $g : Z \to Z'$ such that $g j \simeq_G g_1'$. Further $g_2' \simeq_G g_1' f_1 \simeq_G g j f_1$. ■

5.2 Proposition. Let $F : CW_0 \to \mathcal{S}_0$ be an MV-homotopy functor and let $f_1 : A \to X_1$, $f_2 : A \to X_2$, $g_1 : X_1 \to Z$, $g_2 : X_2 \to Z$ be a weak pushout in $\mathcal{C}_0$. Then if there are $x_1 \epsilon F(X_1)$, $x_2 \epsilon F(X_2)$ such that $F(f_1)x_1 = F(f_2)x_2$ then there exists $v \epsilon F(Z)$ such that $F(g_1)v = x_1$ and $F(g_2)v = x_2$.

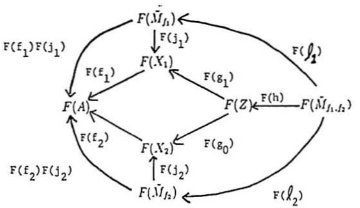
Proof: Let $\tilde{M}_{f_1}$ and $\tilde{M}_{f_2}$ be the G-mapping cylinders of f_1 and f_2. Let $\tilde{M}_{f_1, f_2}$ be the quotient G-space of $(A \times [0,2]) \coprod (X_1 \coprod X_2)$ formed by identifying $(a, 0)$ with

$f_1(a)$, $(a,2)$ with $f_2(a)$ for all $a \epsilon A$, and by shrinking $\{*\} \times [0,2]$ to a single point. Let $\ell_1 : \bar{M}_{f_1} \to \bar{M}_{f_1,f_2}$ be the inclusion and let $\ell_2 : \bar{M}_{f_2} \to \bar{M}_{f_1,f_2}$ be the G-map and homeomorphism into defined by $\ell_2([a,t]) = [a, 2-t]$ where $a \epsilon A$, $t \epsilon I$, and $\ell_2([x]) = [x]$, where $x \epsilon X_2$. Then there are G-maps $j_1 : X_1 \hookrightarrow \bar{M}_{f_1}$, $r_1 : \bar{M}_{f_1} \to X_1$, $j_2 : X_2 \hookrightarrow \bar{M}_{f_2}$, and $r_2 : \bar{M}_{f_2} \to X_2$ such that j_1 and r_1, j_2 and r_2 are homotopy equivalences and homotopy inverses respectively. Then we have the following commutative diagram:



Since $\ell_1 j_1 f_1 \simeq_G \ell_1 i_1 = \ell_2 i_2 \simeq_G \ell_2 j_2 f_2$, (where i_1 and i_2 are the inclusions of A into $\bar{M}_{f_1}$ and $\bar{M}_{f_2}$, respectively) then, as we are working with a weak pushout, there exists a G-map $h : Z \to \bar{M}_{f_1,f_2}$ which makes the above diagram G-homotopy commutative.

Applying the functor F to the above diagram we have the following commutative diagram in S_0:



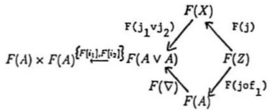
Now j_1 and j_2 are based G-homotopy equivalences, so $F(j_1)$ and $F(j_2)$ are bijections.

We notice that $(\tilde{M}_{f_1,f_2}, \tilde{M}_{f_1}, \tilde{M}_{f_2})$ is (essentially) a triad of G-spaces (to make this literally true we have to replace [0,1] in the definition of $\tilde{M}_{f_2}$ by [1,2]).

Now let $u_1 \epsilon F(\tilde{M}_{f_1})$ and $u_2 \epsilon F(\tilde{M}_{f_2})$ be such that $F(j_1)u_1 = x_1$ and $F(j_2)u_2 = x_2$. Then we have $F(f_1)F(j_1)u_1 = F(f_2)F(j_2)u_2$, so by the Mayer-Victoris Axiom there exists $\bar{v} \epsilon F(\tilde{M}_{f_1,f_2})$ such that $F(\ell_1)\bar{v} = u_1$ and $F(\ell_2)\bar{v} = u_2$. Then let $F(h)\bar{v} = v \epsilon F(Z)$. By the commutativity of the above diagram we have $F(g_1)v = F(g_1)F(h)\bar{v} = F(j_1)F(\ell_1)\bar{v} = F(j_1)u_1 = x_1$, and $F(g_2)v = F(g_2)F(h)\bar{v} = F(j_2)F(\ell_2)\bar{v} = F(j_2)u_2 = x_2$. So $v \epsilon F(Z)$ is the desired element. ■

5.3 Proposition. If $F : CW_0 \to S_0$ is a MV-homotopy functor then it satisfies the Equalizer Axiom.

Proof: Let $j : X \to Z$ be an equalizer of $f_1, f_2 : A \to X$. Considering the weak pushout described in Proposition 5.1 we have the corresponding commutative diagram in S_0:



where $\{F[i_1], F[i_2]\}$ is the bijection given in the Wedge Axiom. Assume that $F(f_1)u = F(f_2)u = w$ for $u \epsilon F(X)$. Then $\{F[i_1], F[i_2]\}F(f_1 \vee f_2)u = (w, w) = \{F[i_1], F[i_2]\}F(\nabla)F(f_1)u$. Since $\{F[i_1], F[i_2]\}$ is a bijection we have:

$$F(f_1 \vee f_2)u = F(\nabla)F(f_1)u$$

By Proposition 5.2 there exists $v \epsilon F(Z)$ such that $F(j)v = u$ and $F(j \circ f_1)v = F(f_1)u$. In particular F satisfies the Equalizer Axiom. ∎

5.4 Theorem (alternate form of the equivariant Brown-Representability Theorem). Let G be a compact Hausdorff group. If $F : C_0 \rightarrow S_0$ is an MV-homotopy functor, then there exists a classifying space Y for F and a G-universal element $u \epsilon F(Y)$. Further, for based G-CW-complexes X in C_0, $T_u : \pi^Y(X) \rightarrow F(X)$ is a natural equivalence.

Proof. This follows from Theorem 4.10 and Proposition 5.3. ∎

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