TOPOLOGY OF FIBRATIONS

FRANCESCO CASERTA

TOPOLOGY OF FIBRATIONS

by

© Francesco Caserta

A Thesis submitted in partial fulfillment
of the requirements for the degree of
Master of Science

Department of Mathematics and Statistics
Memorial University of Newfoundland

July 1984

St. John's Newfoundland Canada

ABSTRACT

This thesis contains a systematic exposition of the topology of fibrations, including Hurewicz, Dold and Serre fibrations and quasifibrations. The fundamental properties and the classical results due to Hurewicz and Dold are discussed in a detailed way. Many examples illustrate the theory; some of them are used to describe properties peculiar of each class of fibrations. The thesis concludes with a discussion of some recent developments. These are: the functional space studied by P. Booth, P. Heath, C. Morgan and R. Piccinini and its application to fibred exponential laws; the theory of F-spaces and F-fibrations introduced by P. May; a categorical interpretation of a fibration as an algebra over the monad which sends each map to its associated fibration.

ACKNOWLEDGEMENTS

I wish to express my hearty thanks to my supervisor Dr. C. Morgan for the care and precision with which he supervised my thesis and my gratitude to Dr. R. Piccinini for his invaluable guide and assistance during my Master's program.

I thank also the Italian Council of Research which supported me by a fellowship and the Department of Mathematics of University of Naples which granted me a leave from my position there.

Lastly, I would thank Ms. Levinia Vatcher for her helpful typing work.

Francesco Caserta

TABLE OF CONTENTS

INTRODUCTION

Fibrations form an important class of maps in geometric and algebraic topology. In geometric topology each geometric object (a differentiable manifold, a p.l. manifold, a topological manifold, or a Poincaré duality space) carries its own specific fibration (the differential tangent bundle, the p.l. tangent bundle, the topological tangent bundle, the Spivak spherical fibration, respectively) containing relevant information on the geometry of that object. One is then interested in classifying such fibrations and in computing algebraic invariants of the classifying space. In algebraic topology the exact homotopy sequence and the Serre spectral sequence give powerful tools for computing algebraic invariants of the total space, the base or the fibre of a fibration when two of them are already known.

This thesis deals with the topology of fibrations, Hurewicz, Dold and Serre fibrations and quasifibrations. The material is organized into two chapters: the first chapter is devoted to classical results and the second to some recent developments. Each chapter is further divided in three sections.

In section I.1 we discuss preliminary notions and results. We start by defining the categories we will deal with, that is, the category of maps and map pairs and the category of maps with a fixed target space and fibre maps over that space; we then define in these categories an appropriate notion of homotopy. We introduce the standard procedure for factorizing any map as a homotopy equivalence followed by a fibration, and the modification of this construction when we deal with fibre maps. The section continues with a short discussion on shrinkable maps, a class of maps introduced by Dold to tackle local-to-global problems and with a discussion on properties of cozero sets, which will be used in the proof of the Hurewicz uniformization theorem in section I.2 (theorem 32). The section ends with a result by Dold which says that for fibre maps over "nice" base spaces the property of being a fibre homotopy equivalence is a local concept (theorem 14).

Section I.2 is devoted to Hurewicz fibrations and is the core of the thesis. It is ideally divided into three parts. In the first part we deduce some immediate consequences of the definition, give the main examples of fibrations and discuss how lifting functions can characterize intrinsically fibrations. In the second part we deal with the basic properties held by fibrations; for example, any fibration gives rise to a functor from the fundamental groupoid of the base to the homotopy category of topological spaces; any fibre map gives rise to a natural transformation between these two functors; any map can be factorized in a standard way as a homotopy equivalence followed by a fibration and this fibration has the same fibre exact homotopy type as the original map, if that map is already a fibration; given a fibration on a cylinder, the restrictions over the bottom and top bases have the same fibre homotopy type. The third part is devoted to classical results of Hurewicz and Dold. The section ends with a brief introduction of a new concept, that of a π-fibration, where π is a partition of the base. A generalization of a Dold's theorem (theorems 42 and 45) to π-fibrations is given. The introduction of this notion of a π-fibration is motivated by its association to any fibre map, in analogy with the fibration associated to any map.

In section I.3 three other classes of maps related to the covering homotopy property are introduced, namely, Dold fibrations, Serre fibrations and quasifibrations, and their main properties discussed. Each of these classes generalizes in a different direction Hurewicz fibrations. The class of Dold fibrations is, in a certain meaning, the closure of the class of Hurewicz fibrations; indeed, it is closed under fibre homotopy equivalence, unlike Hurewicz fibrations, and maps of the same fibre homotopy type as a Hurewicz fibration are Dold fibrations. Serre fibrations keep that important relation between the homotopy groups of the total space, base space and fibre, given by the so-called exact homotopy sequence, but in contrast to quasifibrations, which are defined just as those maps for which the homotopy exact sequence holds, it is easier to check if a map is a Serre fibration, since they are defined as those maps for which the covering homotopy property with respect to all cubes I^n, $n \geq 0$, holds. Examples of maps characteristic for each class are also presented.

In section II.1 we present a construction, originally due to P. Booth and then developed jointly with P. Heath, C. Morgan and R. Piccinini, which associates to a pair of maps a map whose domain is the set, appropriately topologized, of all maps between fibres. This construction generalizes the usual mapping space of two spaces, topologized with the compact-open topology. It allows us to state fibred exponential laws, generalizing the classical one, when spaces are replaced by maps and maps between spaces with map pairs. This functional construction has turned out to be useful in unifying problems in homotopy theory (cfr. [7]) and for studying universal fibrations.

In section II.2 F-spaces and F-fibrations are discussed. They were first introduced by P. May [33] to construct classifying spaces for fibrations where fibres are not finite CW-complexes, arising for example in Sullivan's proof of the Adams conjecture, and to classify spherical fibrations oriented with respect to an extraordinary cohomology. Roughly speaking, an F-space is a map whose fibres are constrained to lie in a fixed category F of spaces and F-fibre maps are fibre maps whose restriction on each fibre is in F. The notion of a fibration is then appropriately adapted to this context and all main properties held by fibrations remain true for F-fibrations. We show, also, that the functional construction, as given in section II.1, adapted to this context gives analogous F-fibred exponential laws. We conclude this section with two results due to C. Morgan. The former claims that the functional construction applied to F-fibrations gives a Hurewicz fibration (theorem 13) and the latter claims that the converse is also true when the maps considered coincide (theorem 16). This last result gives a bridge between the theory of F-fibrations and the classical theory of fibrations. Using this bridge and the F-fibred exponential law, it can be proved that, under mild conditions on the spaces involved, the main results on F-fibrations are quickly deducible from the analogue classical ones. Furthermore, P. Booth, P. Heath, C. Morgan and R. Piccinini have found useful the theory of F-spaces and F-fibrations to analyze the relationships between different notions of universality for fibrations.

In section II.3 it is shown that the standard procedure for factorizing any map as a homotopy equivalence followed by a fibration gives rise to a monad on the category of maps over a fixed space, and that fibrations are essentially the algebras over this monad. Expressive examples of the general notion of a monad on a category are presented and, also, a necessary discussion on Moore paths is given.

Convention In the text, "proposition N" means the N-th proposition in the section where that quotation appears; "proposition M.N" means the N-th proposition in the M-th section of the chapter where that quotation appears; "proposition L.M.N" means the N-th proposition in the M-th section of the chapter L. Similar considerations apply for theorems, lemmas, corollaries. As usual, "[N]" refers to the N-th item in the bibliography. To simplify notation the symbol "∘", normally used to denote the composition of functions, will be omitted, except in cases where ambiguity may arise. Furthermore, the unitary path constant at a point b of a topological space B will be denoted by $\tilde{b}$.

Chapter I

TOPOLOGY OF FIBRATIONS: CLASSICAL RESULTS

I. PRELIMINARY NOTIONS AND RESULTS

Given topological spaces A and B, we denote by $M(A,B)$ the <u>set</u> of all continuous functions (i.e. maps) $f:A \to B$. The set $M(A,B)$ topologized with the compact-open topology will be denoted by $M(A,B)$ or B^A. We recall that the compact-open topology on $M(A,B)$ has as a subbasis the sets of the kind $\langle K,U\rangle = \{f:A \to B:\ f(K) \subseteq U\}$, where $K \subseteq A$ is compact and $U \subseteq B$ is open; hence a generic open set of $M(A,B)$ is of the form $\bigcup_{j \in J} \langle K_1^j, A_1^j\rangle \cap \ldots \cap \langle K_{m_j}^j, A_{m_j}^j\rangle$. We list here the main properties of the compact-open topology; their proofs can be found in [18; chap. XII].

<u>Proposition 1</u> The following properties hold:

(i) B^A is Hausdorff if and only if B is Hausdorff.

(ii) Given maps $f:A \to B$ and $g:B \to C$, the functions $f^*: k \in C^B \to kf \in C^A$ and $g_*: h \in B^A \to gh \in C^A$ are continuous; more generally, if B is Hausdorff and locally compact the function $T:(f,g) \in B^A \times C^B \to gf \in C^A$ is continuous.

(iii) For any spaces A and B and $a \in A$ the function $\omega_a: h \in B^A \to h(a) \in B$ is continuous; ω_a is called <u>the evaluation map at a</u>. More generally, if B is Hausdorff and locally compact the function $\omega: (h,a) \in B^A \times A \to h(a) \in B$ is continuous and is called the <u>evaluation map</u>.

(iv) If $f:A\times B \to C$ is any map then the function $\hat{f}:a\varepsilon A \to f_a \varepsilon C^B$, where $f_a: b\varepsilon B \to f(a,b)\varepsilon C$, is continuous and is called the adjoint of f. If B is Hausdorff and locally compact, then for any map $h:A \to C^B$ the function $\hat{h}:(a,b)\varepsilon A\times B \to h(a)(b)\varepsilon C$ is continuous and is called the adjoint of h. Hence, when B is Hausdorff and locally compact there is a one-to-one correspondence between $M(A\times B,C)$ and $M(A,C^B)$, called the exponential correspondence.

(v) The subspace of B^A consisting of the constant maps $c_b: a\varepsilon A \to b\varepsilon B$, $b\varepsilon B$, is canonically homeomorphic to B, and for every $a\varepsilon A$ the map $h\varepsilon B^A \to c_{h(a)}\varepsilon B^A$ is a retraction onto this subspace.

We denote by Top the category whose objects are topological spaces and whose morphisms are continuous functions. The law of composition is given by the usual composition of functions. Top is called the category of topological spaces and maps.

Given a homotopy $H:A\times I \to B$, we can define for each $t\varepsilon I$ the map $H_t: a\varepsilon A \to H(a,t)\varepsilon B$ and for every $a\varepsilon A$ the path $H_a: t\varepsilon I \to H(a,t)\varepsilon B$. By proposition 1 the function $a\varepsilon A \to H_a\varepsilon B^I$ is continuous and, since I is Hausdorff and locally compact, the exponential correspondence gives a one-to-one correspondence between homotopies $A\times I \to B$ and maps $A \to B^I$. If $f:A \to B$ is a map such that $H_0=f$, then H is called a <u>homotopy of</u> f; if $g:A \to B$ is another map such that $H_1=g$, then H is called a <u>homotopy from</u> f <u>to</u> g. If $S\subseteq A$ is any subset, we say that H is <u>stationary on</u> S if $H(a,t)=H(a,0)$ for every $a\in S$ and $t\in I$; in particular, we say that H is a <u>stationary homotopy</u> if it is stationary on A and we say that H is <u>stationary at</u> $a\in A$ if it is stationary on $\{a\}$. If $H_0=f$ and H is stationary, we also say that H is <u>stationary at</u> f. H is called <u>semi-stationary</u> if $H(a,t)=H(a,0)$ for every $a\in A$ and $0\leq t\leq 1/2$. Given homotopies $H,K:A\times I \to B$ such that $H_1=K_0$, their <u>product</u> $H.K:A\times I \to B$ is defined by $H.K(a,t)=H(a,2t)$, if $0\leq t\leq 1/2$, and $H.K(a,t)=K(a,2t-1)$, if $1/2\leq t\leq 1$. The <u>inverse</u> $H^{-1}:A\times I \to B$ of H is defined by $H^{-1}(a,t)=H(a,1-t)$.

Given a partition π of B, that is a collection of non-empty subsets of B which cover B and which are pairwise disjoint, let $[b]$ denote, for every $b\in B$, the unique element of π containing b. We say that a homotopy $H:A\times I \to B$ is <u>π-stationary</u> if $[H(a,t)]=[H(a,0)]$ for every $a\in A$ and $t\in I$. If π <u>refines</u> the partition π' (in symbols $\pi\leq\pi'$), that is $[b]_\pi\subseteq[b]_{\pi'}$ for every $b\in B$, then a π-stationary homotopy is also π'-stationary. We observe that if π is the coarsest partition of B (i.e. $\pi=\{B\}$) then a π-stationary homotopy is just an ordinary homotopy. If π is the finest (or discrete) partition of B (i.e. $\pi=\{\{b\}\mid b\in B\}$) then a π-stationary homotopy is a stationary homotopy, as defined earlier.

Associated to the category Top is the category HTop defined as follows: the objects are topological spaces, the morphisms are homotopy classes of maps between topological spaces and the composition of morphisms is given by the homotopy class of the composite of the representatives. HTop is called the <u>homotopy category of topological spaces</u> and from the categorical point of view it can be regarded as the quotient category of Top with respect to the congruence given by the homotopy relation on maps [31;p.52].

Given a map $p:E \to B$ and a point $b \in B$ we call $F_b = p^{-1}(b)$ the <u>fibre of</u> p <u>over</u> b (possibly empty). If U is any subset of B we define the <u>restriction of</u> p <u>over</u> U to be the map $p_U : e \in E_U = p^{-1}(U) \to p(e) \in U$. If $f:A \to B$ is a map we define the <u>pullback of</u> p <u>along</u> f to be the map $p_f : E_f \to A$ defined by $E_f = \{(a,e) \in A \times E : f(a) = p(e)\}$ and $p_f(a,e) = a$. The pullback $p_f = E_f \to A$ is characterized (up to isomorphism) by the commutativity of the square

$$\begin{tikzcd} E_f \arrow[r, "pr_2"] \arrow[d, "p_f"'] & E \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

and the following universal property: given any space X and maps $g_1 : X \to A$ and $g_2 : X \to E$ such that $fg_1 = pg_2$, the map $g : x \in X \to (g_1(x), g_2(x)) \in E_f$ is the only map such that $g_1 = p_f g$ and $g_2 = pr_2 g$. A <u>lifting</u> of the map $f:A \to B$ over $p:E \to B$ is any map $\tilde{f}:A \to E$ such that $f = p\tilde{f}$; a lifting of the identity map

$1_B : B \to B$ is called a *section of* p. The set of all liftings of f over p will be denoted by $L(f,p)$ and the set of all sections of p by Sec(p). Two liftings $\tilde{f}$ and $\tilde{\tilde{f}}$ are said to be *vertically homotopic* if there is a homotopy $H: A\times I \to E$ from $\tilde{f}$ to $\tilde{\tilde{f}}$ such that pH is the homotopy stationary at f, that is, $pH(a,t)=f(a)$ for every $a\in A$ and $t\in I$. There is a one-to-one correspondence between liftings of f and sections of p_f given by associating to the lifting $\tilde{f}$ the section $a\in A \to (a,\tilde{f}(a))\in E_f$; under this correspondence two liftings are vertically homotopic if and only if their corresponding sections are vertically homotopic.

Given maps $p:E \to B$ and $p':E' \to B'$, a *map pair from p to* p' is a couple (f,g) of maps $f:E \to E'$ and $g:B \to B'$ such that $p'f=gp$, that is, such that the diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "g"] & B' \end{tikzcd}$$

commutes. This is equivalent to the requirement that $f(F_b)\subseteq F'_{g(b)}$ for every $b\in \mathrm{Im}\,p$; indeed, from the definition of a map pair we deduce that if $e\in F_b$ then $p'f(e)=gp(e)=g(b)$, that is $f(e)\in F'_{g(b)}$; on the other hand, from the relation $f(F_b)\subseteq F'_{g(b)}$ for every $b\in \mathrm{Im}\,p$, we deduce that $p'f(e)=gp(e)$ for every $e \in E$. We will write $(f,g):p \to p'$ to mean that (f,g) is a map pair from p to p'. The <u>composition</u> of the map pair $(f,g):p \to p'$ with the map pair $(f',g'):p' \to p''$ is the map pair $(f'f,g'g):p \to p''$. Two map pairs (f,g) and (f',g') from p to p' are said to be <u>homotopic</u> if there is a homotopy $H:E\times I \to E'$ from f to f' and a homotopy $K:B\times I \to B'$ from g to g' with $p'H=K(p\times 1_I)$, that is, such that the following diagram commutes

$$\begin{tikzcd}
E \arrow[rrrr, "f"] \arrow[dr, "i_0"] \arrow[ddd, "p"'] & & & & E' \arrow[ddd, "p'"] \\
& E\times I \arrow[urrr, "H"] \arrow[d, "p\times 1_I"] & E \arrow[l, "i_1"'] \arrow[urr, "f'"'] \arrow[d, "p"] & & \\
& B\times I \arrow[drrr, "K"] & B \arrow[l, "i_1"'] \arrow[drr, "g'"] & & \\
B \arrow[ur, "i_0"] \arrow[rrrr, "g"'] & & & & B'
\end{tikzcd}$$

We say that (H,K) is a <u>homotopy pair</u> from (f,g) to (f',g'). The map pair (f,g) is said to be a <u>homotopy equivalence from</u> p <u>to</u> p' if there exists a map pair (f',g') from p' to p such that $(f'f,g'g)$ is homotopic to $(1_E,1_B)$ and (ff',gg') is homotopic to $(1_{E'},1_{B'})$, in which case p and p' are said to have the <u>same homotopy type</u>.

Given maps $p:E \to B$, $p':E' \to B'$ and $g:B \to B'$, we call a map $f:E \to E'$ a <u>fibre (preserving) map from</u> p <u>to</u> p'

<u>over</u> g if $p'f=gp$. We denote by $M_g(p,p')$ the set of all fibre maps from p to p' over g. Two fibre maps from p to p' over g, $f,f':E \to E'$, are said to be <u>fibre homotopic over</u> g, written $f \simeq_g f'$, if the map pairs (f,g) and (f',g) from p to p' are homotopic by a homotopy pair (H,K) with K stationary at g; in other words, f and f' are fibre homotopic over g if there exists a homotopy $H:E \times I \to E'$ from f to f' with $p'H(e,t)=gp(e)$, that is, vertical with respect to p'. If $B=B'$ and $g=1_B$ we will speak of fibre maps <u>over B</u>, of fibre homotopies <u>over B</u> and write $M_B(p,p')$ for $M_{1_B}(p,p')$ and $f \simeq_B f'$ for $f \simeq_{1_B} f'$. If $f:E \to E'$ is a fibre map from p to p' over B and $g:E' \to E$ is a fibre map from p' to p over B, we say that g is a <u>left (right) fibre homotopy inverse</u> for f (over B) if gf (fg) is fibre homotopic over B to 1_E $(1_{E'})$. g is said to be a <u>fibre homotopy inverse</u> for f (over B) if it is both a left and a right fibre homotopy inverse, in which case f is called a <u>fibre homotopy equivalence</u> over B and we say that p and p' have the same <u>fibre homotopy type</u> (over B). If B is a one-point space the above definitions reduce to the usual notions of homotopy theory.

We denote by M the category whose objects are maps between topological spaces and whose morphisms are map pairs as defined earlier. The law of composition is given by the composition of map pairs. M is called the <u>category of maps</u>

and map pairs and from the categorical point of view it can be regarded as the category of the morphisms of Top. For a fixed topological space B, the category Top_B has as objects maps with target space B, as morphisms fibre maps over B as defined earlier and composition of morphisms given by the ordinary composition of maps; Top_B is called the category of maps over B and it can be regarded as a (not full) subcategory of M.

Proposition 2 Given maps $p:E \to B$ and $p':E' \to B$, let $f:E \to E'$ and $g, g':E' \to E$ be fibre maps over B. If g is a left fibre homotopy inverse for f and g' is a right fibre homotopy inverse for f, then g and g' are fibre homotopic over B and moreover f is a fibre homotopy equivalence over B with fibre homotopy inverses g and g'.

Proof From $gf \simeq_B 1_E$ and $fg' \simeq_B 1_{E'}$ we deduce that $g = g1_{E'} \simeq_B g(fg') = (gf)g' \simeq_B 1_E g' = g'$. Hence, $fg \simeq_B fg' \simeq_B 1_{E'}$ and $g'f \simeq_B gf \simeq_B 1_E$. So f is a fibre homotopy equivalence over B with fibre homotopy inverses g and g'.

Remark 3 In proposition 2 the case in which B is a one-point space is of particular interest and will be used in the proof of theorems 2.42 and 2.45.

Proposition 4 Let $p:D \to B$ and $q:E \to B$ be maps of the same fibre homotopy type over B. If $\lambda:A \to B$ is any map, then the pullbacks of p and q along ℓ have the same fibre homotopy type over A.

Proof Let $p':D' \to A$ and $q':E' \to A$ denote the pullbacks of p and q along ℓ, respectively. Let $f:D \to E$ be a fibre homotopy equivalence over B between p and q with fibre homotopy inverse $g:E \to D$ and let $H:D\times I \to D$ and $K:E\times I \to E$ be vertical homotopies from gf to 1_D and from fg to 1_E, respectively. Define $f':(a,d)\in D' \to (a,f(d))\in E'$, $g':(a,e)\in E' \to (a,g(e))\in D'$, $H':(a,d,t)\in D'\times I \to (a,H(d,t))\in D'$ and $K':(a,e,t)\in E'\times I \to (a,K(e,t))\in E'$. Then f' and g' are fibre maps over A and H' and K' are vertical homotopies from g'f' to $1_{D'}$ and from f'g' to $1_{E'}$, respectively. This proves that p' and q' have the same fibre homotopy type over A.

Remark 5 The pullbacks of a map $p:E \to B$ along two homotopic maps $f',f'':A \to B$ may not have the same fibre homotopy type over A. Simple examples of this kind can be obtained by taking A={a}, a one-point space, and f' and f" such that f'(a) and f"(a) can be joined by a path in B, but their anti-images by p have different homotopy types. We will show in the next section that this cannot happen when p is a fibration.

Let $p:E \to B$ and $p':E' \to B$ be maps, $f:E \to E'$ a fibre map over B, π a partition of B and $q_\pi:B \to B/\pi$ the quotient map determined by π. We say that f is a <u>π-fibre homotopy equivalence</u> if f, regarded as a fibre map over B/π from $q_\pi p$ to $q_\pi p'$, is a fibre homotopy equivalence.

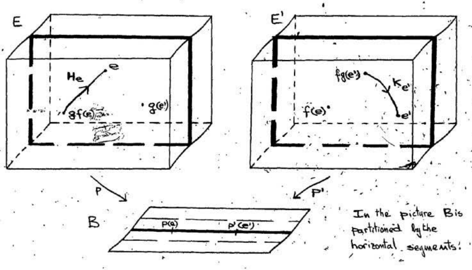
$$
\begin{tikzcd}
E \arrow[rr, "f"] \arrow[dr, "p"] \arrow[ddr, bend right, "q_\pi p"'] & & E' \arrow[dl, "p'"'] \arrow[ddl, bend left, "q_\pi p'"] \\
& B \arrow[d, "q_\pi"] & \\
& B/\pi &
\end{tikzcd}
$$

In other words, if we denote by $[b]$ the unique element of π containing $b \in B$, then $f: E \to E'$ is a π-fibre homotopy equivalence if there exist maps $g: E' \to E$, $H: E \times I \to E$ and $K: E' \times I \to E'$ such that

(i) $[pg(e')] = [p'(e')]$ for every $e' \in E'$;

(ii) $H_0 = gf$, $H_1 = 1_E$ and $[pH(e,t)] = [p(e)]$ for every $e \in E$ and $t \in I$;

(iii) $K_0 = fg$, $K_1 = 1_{E'}$, and $[p'K(e',t)] = [p'(e')]$ for every $e' \in E'$ and $t \in I$.



We observe that if π is the coarsest partition of B then a π-fibre homotopy equivalence is just a homotopy equivalence. For π the finest (or discrete) partition of B (i.e. $\pi=\{\{b\} \mid b \in B\}$) we have the notion of a fibre homotopy equivalence. Given any map $f:E \to B$ and a partition π of B, we say that f is a <u>π-homotopy equivalence</u> if f, regarded as a fibre map over B from f to 1_B, is a π-fibre homotopy equivalence.

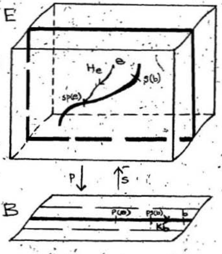
$$\begin{tikzcd}
E \arrow[rr, "f"] \arrow[dr, "p"] \arrow[ddr, bend right, "q_\pi p"'] & & B \arrow[dl, "1_B"'] \arrow[ddl, bend left, "q_\pi"] \\
& B \arrow[d, "q_\pi"] & \\
& B/\pi &
\end{tikzcd}$$

In other words a map $f: E \to B$ is a π-homotopy equivalence if there exist maps $g: B \to E$, $H: E\times I \to E$ and $K: B\times I \to B$ with the following properties:

(i) $[fg(b)]=[b]$ for every $b\in B$;

(ii) $H_0=gf$, $H_1=1_E$ and $[fH(e,t)]=[f(e)]$ for every $e\in E$ and $t\in I$;

(iii) $K_0=fg$, $K_1=1_B$ and $[K(b,t)]=[b]$ for every $b\in B$ and $t\in I$.



In the picture B is partitioned by the horizontal segments

We observe that if π is the coarsest partition of B the notion of π-homotopy equivalence coincides with the usual notion of homotopy equivalence.

Associated to a map $f: E \to E'$ is an important space $\Lambda_f = \{(e,\alpha) \varepsilon E \times E'^I : \alpha(0) = f(e)\}$, first introduced by Hurewicz in [25]. There are projection maps $pr_1 : (e,\alpha) \varepsilon \Lambda_f \to e \varepsilon E$ and $pr_2 : (e,\alpha) \varepsilon \Lambda_f \to \alpha \varepsilon E'^I$, a decomposition map $\rho : \alpha \varepsilon E^I \to (\alpha(0), f\alpha) \varepsilon \Lambda_f$, an "inclusion" map $\iota : e \varepsilon E \to (e, \widetilde{f(e)}) \varepsilon \Lambda_f$ and a map $\bar{f} : (e,\alpha) \varepsilon \Lambda_f \to \alpha(1) \varepsilon E'$. If $\omega_0 : E'^I \to E'$ denotes the evaluation map at $0 \varepsilon I$, then $pr_2 : \Lambda_f \to E'^I$ is the pullback of f along ω_0.

$$\begin{tikzcd}
E^I \arrow[dr, "\rho"] \arrow[drr, bend left, "\omega_0"] \arrow[ddr, bend right, "f_*"'] & & \\
& \Lambda_f \arrow[r, "pr_1"] \arrow[d, "pr_2"] & E \arrow[d, "f"] \\
& E'^I \arrow[r, "\omega_0"] & E'
\end{tikzcd}$$

Λ_f is sometimes called the <u>cocylinder</u> of f (cfr. [46, p.43]). This is due to a sort of duality with the <u>cylinder</u> of f, Z_f, which is the space obtained from $E \times I \cup E'$ by identifying $(e,0)$ with $f(e)$ and topologizing with the quotient topology. Indeed, the cylinder of f can be regarded as the pushout of f along $i_0 : e \varepsilon E \to (e,0) \varepsilon E \times I$

$$\begin{tikzcd}
E \arrow[r, "i_0"] \arrow[d, "f"'] & E\times I \arrow[d] \arrow[dd, bend left, "f\times 1_I"] \\
E' \arrow[r] \arrow[dr, "i_0"'] & Z_f \arrow[d] \\
& E'\times I
\end{tikzcd}$$

and the cylinder functor $-\times I$ is left adjoint to the path space functor $(-)^I$.

In the case $f:E \to E'$ is a fibre map over B from $p:E \to B$ to $p':E' \to B$ there is a modification of the above construction, which yields Λ_f when B is a one-point space. This is the space $R_f = \{(e,\alpha) \in E\times E'^I : \alpha(0)=f(e)$ and α vertical$\} \subseteq \Lambda_f$, introduced by Dold in [13]. Associated with R_f are the maps $\bar{f}:(e,\alpha) \in R_f \to \alpha(1) \in E'$ and $\iota: e \in E \to (e, \widetilde{p(e)}) \in R_f$. As a first example of the usefulness of R_f, there is the following result, which will be improved by proposition 8.

Proposition 6 A fibre map $f:E \to E'$ over B admits a right fibre homotopy inverse if and only if $\bar{f}:R_f \to E'$ admits a section.

Proof Let $g:E' \to E$ be a right fibre homotopy inverse for f and let $K:E'\times I \to E'$ be a vertical homotopy from fg to $1_{E'}$.

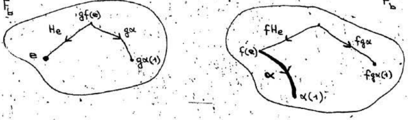
Define $s:E' \to R_f$ by $s(e')=(g(e'),K_{e'})$; s is well defined since $K_{e'}(0)=K(e',0)=fg(e')$ and the path $K_{e'}$ is vertical; furthermore, s is a section of $\bar{f}$ because $\bar{f}s(e')=\bar{f}(g(e'),K_{e'})=K_{e'}(1)=K(e',1)=e'$. Conversely, let $s:E' \to R_f$ be a section of $\bar{f}$ and define $g:E' \to E$ and $K:E'\times I \to E'$ by $g(e')=pr_1 s(e')$ and $K(e',t)=[pr_2 s(e')](t)$. Then g is a fibre map over B, since $pg(e')=p'fg(e')=p'(f(pr_1 s(e')))=p'([pr_2 s(e')](0))=p'([pr_2 s(e')](1))=p'(e')$, and K is a vertical homotopy from fg to 1_E.

Proposition 7 Let $f:E \to E'$ be a fibre map over B from $p:E \to B$ to $p':E' \to B$ and let π be the partition of E' given by the fibres of p', that is $\pi=\{F'_b | b\varepsilon \mathrm{Im}\, p'\}$. Then f is a fibre homotopy equivalence over B if and only if $\bar{f}:R_f \to E'$ is a π-homotopy equivalence.

Proof Suppose f is a fibre homotopy equivalence over B and let $g:E' \to E$ be a fibre homotopy inverse for f with $H:E\times I \to E$ and $K:E'\times I \to E'$ vertical homotopies from gf to 1_E and from fg to $1_{E'}$, respectively. We start to define for each path α in E' and $t\varepsilon I$ the path $\alpha_t: s\varepsilon I \to \alpha((1-t)s)\varepsilon E'$; the function $(\alpha,t)\varepsilon E'^I\times I \to \alpha_t \varepsilon E'^I$ so defined is continuous because its adjoint is the map $(\alpha,t,s)\varepsilon E'^I\times I\times I \to \alpha((1-t)s)\varepsilon E'$. The path α_t follows the path α from $\alpha(0)$ to $\alpha(1-t)$; in particular we have that $\alpha_0=\alpha$ and $\alpha_1=\widetilde{\alpha(0)}$. Now define maps

$\bar{g}:E' \to R_f$ and $J:R_f \times I \to R_f$ by $\bar{g}(e')=(g(e'),\overline{fg(e')})$ and

$$J(e,\alpha,t)=\begin{cases}(e,\alpha_{3t}) & \text{if } 0\le t\le 1/3\\ (H(e,2-3t),\overline{fH(e,2-3t)}) & \text{if } 1/3\le t\le 2/3\\ (g\alpha(3t-2),\overline{fg\alpha(3t-2)}) & \text{if } 2/3\le t\le 1\end{cases}$$



We have that:

(i) $\bar{f}\bar{g}(e')=fg(e')$ and so $\bar{f}\bar{g}(e')$ lies in the same fibre of e', since f and g are fibre maps over B;

(ii) $J_0=1_{R_f}$, $J_1(e,\alpha)=(g\alpha(1),\overline{fg\alpha(1)})=\bar{g}\bar{f}(e,\alpha)$ and

$$\bar{f}J(e,\alpha,t) = \begin{cases}\alpha(1-3t) & \text{if } 0\le t\le 1/3\\ fH(e,2-3t) & \text{if } 1/3\le t\le 2/3\\ fg\alpha(3t-2) & \text{if } 2/3\le t\le 1\end{cases}$$

so J is a homotopy from 1_{R_f} to $\bar{g}\bar{f}$ and $\bar{f}J(e,\alpha,t)$ lies in the fibre of $f(e)$;

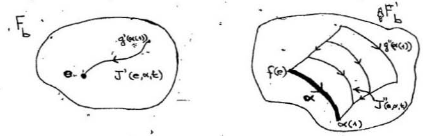
(iii) the map $K:E'\times I \to E'$ is a homotopy from $\bar{f}\bar{g}=fg$ to $1_{E'}$ and $K(e',t)$ lies in the same fibre of e'.

Hence, $\bar{f}$ is a π-homotopy equivalence.

Suppose now that $\bar{f}:R_f \to E'$ is a π-homotopy equivalence, so there exist maps $g=(g',g''):E' \to R_f$, $J=(J',J''):R_f \times I \to R_f$ and $L:E' \times I \to E'$ such that:

(i) $g''(e')(1)$ ($=\bar{f}g(e')$) and e' belong to the same fibre of p';

(ii) $J'(e,\alpha,0)=g'(\alpha(1))$, $J'(e,\alpha,1)=e$, $J''(e,\alpha,0)=g''(\alpha(1))$, $J''(e,\alpha,1)=\alpha$ and all points $J''(e,\alpha,t)(1)$ lie in the same fibre of $\alpha(1)$;



(iii) $L(e',0)=g''(e')(1)$, $L(e',1)=e'$ and the path $L_{e'}$ lies in the same fibre of e'.

It follows from these properties and the definition of R_f that the path $g''(e')$ lies in the same fibre of e', all the paths $J''(e,\alpha,t)$ lie in the same fibre of $\alpha(1)$ and $J'(e,\alpha,t)$ lies in the same fibre of e. The map $g':E' \to E$ is a fibre map over B, since $pg'(e')=p'fg'(e')=p'(g''(e')(0))=p'(e')$. If we define $H:(e,t)\in E\times I \to J'(e,\widetilde{f(e)},t)\in E$, we have that H is continuous, $H(e,0)=J'(e,\widetilde{f(e)},0)=g'f(e)$, $H(e,1)=J'(e,\widetilde{f(e)},1)=e$ and H is vertical. Moreover the function $K:E'\times I \to E'$ defined by $K(e',t)=g''(e')(2t)$, if $0\leq t\leq 1/2$, and $K(e',t)=L(e',2t-1)$, if $1/2\leq t\leq 1$, is continuous, $K(e',0)=g''(e')=fg'(e')$, $K(e',1)=L(e',1)=e'$ and K is vertical. Hence f is a fibre homotopy equivalence over B with fibre homotopy inverse g'.

We now introduce a class of maps which was shown by Dold in [13] to play an important role in local-to-global problems. A map $p:E \to B$ is called <u>shrinkable</u> if it admits a section $s:B \to E$ such that $sp \simeq_B 1_E$. Equivalently, p is shrinkable if p, regarded as a fibre map over B from p to 1_B, is a fibre homotopy equivalence over B.

$$\begin{tikzcd} E \arrow[rr, shift left, "p"] \arrow[dr, "p"'] & & B \arrow[ll, shift left, dashed, "s"] \arrow[dl, "1_B"] \\ & B & \end{tikzcd}$$

If π is the discrete partition of B, then the π-homotopy equivalences $p:E \to B$ as defined earlier, are just the shrinkable maps.

<u>Example</u> Let F be a contractible space and let $K:F\times I \to F$ be a fixed deformation of F to $\bar{e}\in F$. Then, for every space B the projection map $pr_1:B\times F \to B$ is shrinkable; indeed, $s:b\in B \to (b,\bar{e})\in B\times F$ is a section and $1_B\times K:B\times F\times I \to B\times F$ defines a vertical homotopy from the identity of $B\times F$ to $s\circ pr_1$.

It follows from the definition that a shrinkable map is onto and that all of its fibres are contractible spaces. But there are maps where each fibre is contractible, but yet the map is not shrinkable. For example, let B be the subset (the so-called "polish circle") of the plane R^2 given by $B=A_1\cup A_2\cup A_3\cup A_4$, where $A_1=\{(x,\sin(1/x))|0<x<1/\pi\}$, $A_2=\{(1/\pi,y)|-2<y<0\}$, $A_3=\{(x,-2)|0<x<1/\pi\}$, $A_4=\{(0,y)|-2<y<1\}$ and let $b_0=(0,1)$, $P(B,b_0)=\{\alpha\in B^I: \alpha(0)=b_0\}$ and $p:P(B,b_0) \to B$ defined by $p(\alpha)=\alpha(1)$; then it can be shown that all fibres of p are contractible spaces, but p is not shrinkable.

Shrinkable maps can be regarded as "contractible" objects in the category Top_B. By a "contractible" object in Top we mean any space E which has the same homotopy type as a one-point space $\{*\}$, which is, of course, a terminal object in Top (i.e. $M(E,\{*\})$ has exactly one element). Generalizing this notion to Top_B, we have that a "contractible" object in Top_B is any map $p:E \to B$ which has the same fibre homotopy type over B as the identity map $1_B: B \to B$, which is, of course, a terminal object in Top_B.

A nice and useful relationship between fibre homotopy equivalences and shrinkable maps is provided by the following proposition, due to Dold [13; lemma 3.4]. This result will be applied in the proof of theorem 14, the main result of this section.

Proposition 8 Given maps $p: E \to B$ and $p': E' \to B$, a fibre map $f: E \to E'$ over B is a fibre homotopy equivalence over B if and only if $\overline{f}: R_f \to E'$ is shrinkable.

Proof Dold gave a very complicated proof of the shrinkability of $\overline{f}$ when f is a fibre homotopy equivalence. Our proof of the above equivalence is an immediate application of proposition 7 and corollary 2.47. Since corollary 2.47 involves the notion of a π-fibration, it appears in section 2 because there it finds its natural setting; of course the proof of corollary 2.47 is independent of the result we are proving.

We now discuss some notions and results which will be mainly used in section 2, particularly in the proof of the Hurewicz uniformization theorem. Before, we prove the following result.

Lemma 9 If $f_1, \ldots, f_n: X \to R$ are real-valued maps on a topological space X, then:

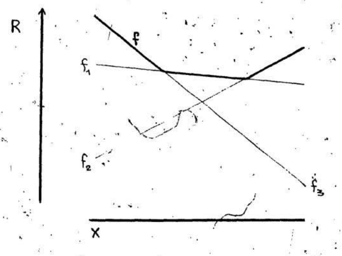
(i) the function $f: X \to R$ defined by $f(x) = \sup\{f_1(x), \ldots, f_n(x)\}$

is continuous;

(ii) the function $g:X \to R$ defined by $g(x)=\inf\{f_1(x),\dots,f_n(x)\}$ is continuous.

<u>Proof</u> First of all, we observe that since we are dealing with finite sets of numbers $f(x)=\sup\{f_1(x),\dots,f_n(x)\} \iff f_i(x) \le f(x)$ for every $i=1,\dots,n$ and there exists $\bar{i}\in\{1,\dots,n\}$ such that $f(x)=f_{\bar{i}}(x)$; similarly $g(x)=\inf\{f_1(x),\dots,f_n(x)\} \iff g(x) \le f_i(x)$ for every $i=1,\dots,n$ and there exists $\bar{i}\in\{1,\dots,n\}$ such that $g(x)=f_{\bar{i}}(x)$.

(i) It is enough to prove that for every $s\in R$ the sets $f^{-1}(]s,+\infty[)$ and $f^{-1}(]-\infty,s[)$ are open. For the former we have that $f^{-1}(]s,+\infty[)=\bigcup_{i=1}^{n} f_i^{-1}(]s,+\infty[)$ because $s<f(x) \iff$ there exists some $i\in\{1,\dots,n\}$ such that $s<f_i(x)$. For the latter we have that $f^{-1}(]-\infty,s[)=\bigcap_{i=1}^{n} f_i^{-1}]-\infty,s[)$ because $f(x)<s \iff f_i(x)<s$ for every $i=1,\dots,n$.

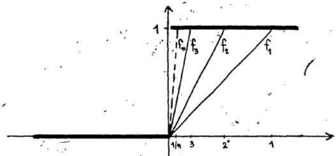


(ii) can be proved either independently from (i) following a similar argument or using (i) and the observation that $\inf\{f_1(x),\ldots,f_n(x)\}=-\sup\{-f_1(x),\ldots,-f_n(x)\}$. Indeed, $g(x)=\inf\{f_1(x),\ldots,f_n(x)\} \Leftrightarrow g(x)<f_i(x)$ for every $i=1,\ldots,n$ and there exists $\bar{i}\in\{1,\ldots,n\}$ such that $g(x)=f_{\bar{i}}(x) \Leftrightarrow -f_i(x)<-g(x)$ for every $i=1,\ldots,n$ and there exists $\bar{i}\in\{1,\ldots,n\}$ such that $-g(x)=-f_{\bar{i}}(x) \Leftrightarrow$ $-g(x)=\sup\{-f_1(x),\ldots,-f_n(x)\}$.

<u>Remark 10</u> In the more general case of any family $\{f_j \mid j\in J\}$ of real-valued maps bounded above, we have only that $f: x\in X \to \sup_{j\in J}\{f_j(x)\}\in R$ is <u>lower semicontinuous</u>, that is, $f^{-1}(]s,+\infty[)$ is open for every $s\in R$. Indeed we have the following counterexample: the sequence of maps $\{f_n \mid n\in N\}$ defined by

$$f_n: x\in R \to \begin{cases} 0 & \text{if } x<0 \\ nx & \text{if } 0<x<1/n \\ 1 & \text{if } x>1/n \end{cases}$$

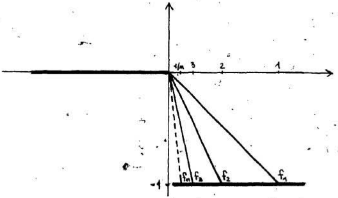
has as its supremum the function 0 on $x<0$ and 1 on $x>0$.



Similarly if $\{f_j | j \varepsilon J\}$ is any family of real-valued maps bounded below we have only that $g: x \varepsilon X \to \inf_{j \varepsilon J}\{f_j(x)\} \varepsilon R$ is <u>upper semicontinuous</u>, that is $g^{-1}(]-\infty, s[)$ is open for every $s \varepsilon R$. Indeed we have the following counterexample: the sequence of maps $\{f_n | n \varepsilon N\}$ defined by

$$f_n : x \varepsilon R \to \begin{cases} 0 & \text{if } x<0 \\ -nx & \text{if } 0<x<1/n \\ -1 & \text{if } x>1/n \end{cases}$$

has as its infimum the function 0 on $x<0$ and -1 on $x>0$.



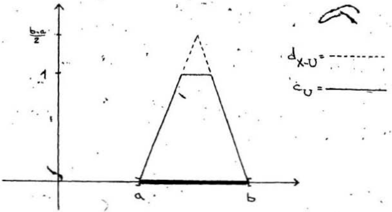
In a topological space X an open subset U is called a <u>cozero set</u> if there exists a continuous function

$c:X \to [0,1]$ such that $U=c^{-1}(]0,1])$; c is said to be a numeration of U. Not any open set can be a cozero set. For example a cozero set U must be an F_σ-set, that is the union of at most countably many closed sets; in fact $U=\bigcup_{n\in N} c^{-1}[1/n,1]$. If X is a normal space then Urysohn's theorem implies that U is a cozero set if and only if it is an F_σ-set. In fact, let $U=\bigcup_{n\in N} C_n$, with C_n closed and $C_1\subseteq C_2\subseteq\ldots$, and let $f_n:X \to [0,1]$ be a continuous function such that $f_n(x)=0$ for every $x\in X-U$ and $f_n(x)=1$ for every $x\in C_n$; then $f(x)=\sum_{n=1}^{\infty} f_n(x)/2^n$ is well defined and continuous. Now $U\supseteq f^{-1}(]0,1])$, because from $f(x)>0$ it follows that $f_n(x)>0$ for some n and hence $x\in U$. $U\subseteq f^{-1}(]0,1])$, because if $x\in U$ it follows that $x\in C_n$ for some n and hence $f_n(x)=1$ and so $f(x)\geq 1/2^n$. From this observation we have that in a perfectly normal space, which is by definition a normal space where each open set is an F_σ-set, any open set is a cozero set.

Metric spaces are perfectly normal and in this case the metric gives a canonical numeration for each open set. In fact let (X,d) be a metric space. We recall that for any subset $A\subseteq X$ the function $d_A: x\in X \to d(x,A)=\inf\{d(x,a)\,|\,a\in A\}\in R^+$ is continuous and satisfies $d_A(x)=0$ if and only if $x\in\bar{A}$ [18; p.185]. Hence if U is an open subset of X the map $d_{X-U}:X \to R^+$ is such that $d_{X-U}(x)=0$ if and only if $x\in\overline{X-U}=X-U$; by lemma 9, $c_U: x\in X \to \inf\{1,d_{X-U}(x)\}\in[0,1]$ is continuous and so a numeration of U.

Example: X=R and U=]a,b[. In this case

$$d_{X-U}(x)=\begin{cases} 0 & \text{if } x<a \text{ or } x>b \\ x-a & \text{if } a<x<(a+b)/2 \\ b-x & \text{if } (a+b)/2<x<b \end{cases}$$



The next proposition states that cozero sets behave well under Boolean operations and that they transfer their property to the subbasic sets determined by them in the path space.

Proposition 11 (i) The intersection of finitely many cozero sets is a cozero set;

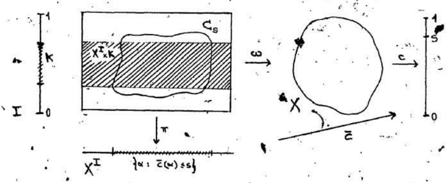
(ii) the union of any locally finite family of cozero sets is cozero set;

(iii) if U is a cozero set in X and $K\subseteq I$ a compact, then the subbasic open set $\langle K,U\rangle\subseteq X^I$ is a cozero set.

Proof (i) If $U_1,\dots,U_n$ are cozero sets with numerations $c_1,\dots,c_n$, respectively, then $c=c_1\cdot\ldots\cdot c_n:X\to I$ is such that $c^{-1}(]0,1])=\bigcap_{i=1}^{n}c_i^{-1}(]0,1])$ and hence c is a numeration of $U_1\cap\dots\cap U_n$.

(ii) Let $\mathcal{U}=\{U_j\,|\,j\in J\}$ be a locally finite family of cozero sets in X and let $c_j:X\to I$ be a fixed numeration of U_j, $j\in J$. Define $c:x\in X\to\sup_j\{c_j(x)\}\in I$. c is well defined because each c_j is bounded in I; moreover it is continuous because if $\bar{x}\in X$ is a fixed point and V a neighbourhood of $\bar{x}$ meeting only an empty or finite set of elements of $\mathcal{U}$, say $\{U_{j_1},\dots,U_{j_n}\}$, then, for every $x\in V$, $c(x)=\sup_j\{c_j(x)\}=\sup\{0,c_{j_1}(x),\dots,c_{j_n}(x)\}$ and hence continuous on V by lemma 9.

(iii) We can exclude the case $K=\emptyset$ which gives $\langle K,U\rangle=X^I$. Let $c:X\to I$ be a numeration of U; we must construct a continuous function $\bar{c}:X^I\to I$ such that $\bar{c}^{-1}(]0,1])=\langle K,U\rangle$. For any $\alpha\in X^I$ we define $\bar{c}(\alpha)=\inf\{c\alpha(t)\in I\,|\,t\in K\}$. $\bar{c}$ is well defined since for any α the set $\{c\alpha(t)\in I\,|\,t\in K\}$ is of course bounded below; moreover, since K is compact, for any α the set $\{c\alpha(t)\in I\,|\,t\in K\}$ is compact and so, in particular, closed and hence for any α there exists some $t_0\in K$ such that $\bar{c}(\alpha)=c\alpha(t_0)$. It is easy to see that $\bar{c}(\alpha)>0$ if and only if $\alpha\in\langle K,U\rangle$; in fact, if $\bar{c}(\alpha)>0$ it follows that $c\alpha(t)>0$ for every $t\in K$ and so $\alpha(t)\in U$ for every $t\in K$; conversely, if $\alpha(t)\in U$ for every $t\in K$ then $c\alpha(t)>0$ for every $t\in K$ and hence $\bar{c}(\alpha)>0$ because $\bar{c}(\alpha)=c\alpha(t_0)$ for some $t_0\in K$. Now it remains to prove that $\bar{c}$ is continuous and to this end it will be enough to show that for any $s\in I$ the set $\{\alpha\in X^I:\ \bar{c}(\alpha)<s\}$ is open and that the set $\{\alpha\in X^I:\ \bar{c}(\alpha)\le s\}$ is closed. For any $t\in I$ let $\omega_t:X^I\to X$ be the evaluation map at t. We have that $\bar{c}(\alpha)<s$ if and only if there exists some $t\in K$ such that $c\alpha(t)<s$; observing that $c\alpha(t)=c\omega_t(\alpha)$ we get that $\{\alpha\in X^I:\ \bar{c}(\alpha)<s\}=\bigcup_{t\in K}\{\alpha\in X^I:\ c\omega_t(\alpha)<s\}$, which is open. To prove that $\{\alpha\in X^I:\ \bar{c}(\alpha)\le s\}$ is closed, consider the evaluation map $\omega:X^I\times I\to X$ and the closed subspace of $X^I\times I$ $C_s=\{(\alpha,t)\in X^I\times I:\ c\omega(\alpha,t)\le s\}$; it is easy to see that $\{\alpha\in X^I:\ \bar{c}(\alpha)\le s\}=\pi(C_s\cap X^I\times K)$, where π is the projection on X^I.

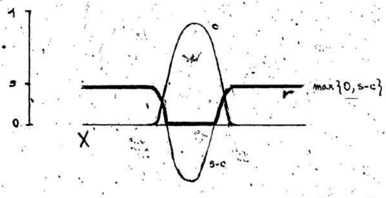


In fact $\bar{c}(\alpha)<s$ <=> there exists some $t\epsilon K$ such that $c\omega(\alpha,t)=c\alpha(t)<s$ <=> there exists some $t\epsilon K$ such that $(\alpha,t)\epsilon C_s$ <=> $\alpha\epsilon\pi(C_s\cap X^I\times K)$. To conclude we have only to observe that $C_s\cap X^I\times K$ is closed and that π is a closed map since I is compact.

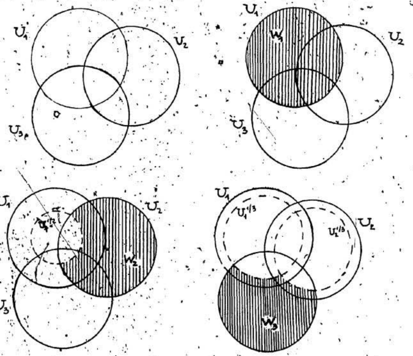
<u>Proposition 12</u> (i) Let $\mathcal{U}=\{U_n|n\epsilon N\}$ be a covering of X by cozero sets; then there is a locally finite covering $\mathcal{W}=\{W_n|n\epsilon N\}$ of X by cozero sets which refines $\mathcal{U}$.

(ii) Let $\{\mathcal{U}_n|n\epsilon N\}$ be a sequence of locally finite families $\mathcal{U}_n=\{U_j|j\epsilon J_n\}$ of cozero sets of X such that $\mathcal{U}=\{U_j|j\epsilon J=\bigcup_{n\epsilon N}J_n\}$ covers X. (For convenience we are assuming that the sets J_n are pairwise disjont). Then there exists a locally finite refinement $\mathcal{U}'=\{U'_j|j\epsilon J\}$ of $\mathcal{U}$ by cozero sets.

<u>Proof</u> (i) Let $c_n:X\to I$ be a numeration of U_n. For any positive $s\epsilon I$ define $U_n^s=\{x\epsilon X:\ c_n(x)<s\}$; U_n^s is a cozero set because the continuous function $x\epsilon X\to\max(0,s-c_n(x))\epsilon I$ is a numeration of it.



Define $W_n = U_n \cap \bigcap_{i=1}^{n-1} U_i^{1/n}$; W_n is a cozero set because it is a finite intersection of cozero sets. Note that $W_n = \{x \in U_n : c_i(x) < 1/n$ for every $i=1,\dots,n-1\}$ and that $W_n \subseteq U_n$. The sequence $W_1, W_2, \dots$ covers X: in fact given $x \in X$ let U_i be the first element where x appears; then $x \in W_i$. To show local finiteness take $x \in X$ and let U_i be the first element where x appears; let $n_0 \in N$ be the least integer such that $1/n_0 < c_i(x)$ and let $V = \{y \in U_i : c_i(y) > 1/n_0\}$. Then V is a neighbourhood of x and $V \cap W_n = \emptyset$ for every $n > \max\{i, n_0\}$: in fact if $y \in V \cap W_n$ it would follow that $c_i(y) < 1/n$, since $n > i$, contradicting the fact that $c_i(y) > 1/n_0 > 1/n$.



(ii) For any $n\in N$ let $S_n=\bigcup_{j\in J_n} U_j$. Since each U_j is a cozero set and $\mathcal{U}_n$ is locally finite, we have that each S_n is a cozero set. Furthermore, since $\mathcal{U}$ covers X, we have that the sequence $S=\{S_n|n\in N\}$ covers X. We can therefore apply (i) to to obtain a locally finite cover $\{W_n|n\in N\}$ of X by cozero sets with $W_n\subseteq S_n$, for each n. For each $j\in J$ let $U_j'=U_j\cap W_n$, where n is the unique integer such that $j\in J_n$, and let $\mathcal{U}'=\{U_j'|j\in J\}$. Each element of $\mathcal{U}'$ is a cozero set and $\mathcal{U}'$ refines $\mathcal{U}$. Furthermore $\mathcal{U}'$ covers X; indeed, if $x\in X$ there is some integer n such that $x\in W_n\subseteq\bigcup_{j\in j} U_j$ and so there is some $j\in J_n\subseteq J$ such that $x\in U_j\cap W_n=U_j'$. It remains to prove that $\mathcal{U}'$ is locally finite. For a fixed $x\in X$, let V be a neighbourhood of x such that $\{n\in N: V\cap W_n\neq\emptyset\}$ is finite, say $\{n_1,\dots,n_p\}$. For every $k=1,\dots,p$ let $V_k\subseteq V$ be a neighbourhood of x such that $\{j\in J_{n_k}: V_k\cap U_j\neq\emptyset\}$ is empty or finite. Define $V'=V_1\cap\dots\cap V_p\subseteq V$ and observe that $\{n\in N: V'\cap W_n\neq\emptyset\}\subseteq\{n\in N: V\cap W_n\neq\emptyset\}$ and that $\{j\in J_n: V'\cap U_j'\neq\emptyset\}\subseteq\{j\in J_n: V_k\cap U_j\neq\emptyset\}$. Then $\{j\in J: V'\cap U_j'\neq\emptyset\}=\bigcup_{n\in N}\{j\in J_n: V'\cap U_j'\neq\emptyset\}=\bigcup_{k=1}^{p}\{j\in J_{n_k}: V'\cap U_j'\neq\emptyset\}\subseteq\bigcup_{k=1}^{p}\{j\in J_{n_k}: V_k\cap U_j\neq\emptyset\}$ and so the set $\{j\in J: V'\cap U_j'\neq\emptyset\}$ is finite.

An open cover of a space X is called <u>numerable</u> if it is locally finite and each element of the cover is a cozero set.

The proof of the following result can be found in [13; cor.3.2] and will be omitted. To give it here, it would first require a discussion of the section extension property introduced by Dold in that same paper. Although this property has some technical advantages it will not be needed in this text.

Proposition 13 Let $p:E \to B$ be a map. If B admits a numerable cover $\mathcal{U}=\{U\}$ such that $p_U:E_U \to U$ is shrinkable for every $U \in \mathcal{U}$, then p is shrinkable.

Theorem 14 Let $p:E \to B$ and $p':E \to B$ be maps and let $f:E \to E'$ be a fibre map over B. If B admits a numerable cover $\mathcal{U}=\{U\}$ such that $f_U:E_U \to E'_U$ is a fibre homotopy equivalence over U for every $U \in \mathcal{U}$, then f is a fibre homotopy equivalence (over B).

Proof Consider $\bar{f}:R_f \to E'$, we have that for every $U \in \mathcal{U}$ $\bar{f}^{-1}(E'_U)=R_{f_U}$ and so the restriction of $\bar{f}$ over E'_U is $\overline{f_U}:R_{f_U} \to E'_U$. From the hypothesis and proposition 8 we deduce that for every $U \in \mathcal{U}$ the map $\overline{f_U}$ is shrinkable. On the other hand, the open cover $\{E'_U \mid U \in \mathcal{U}\}$ of E' is numerable because $\mathcal{U}$ is numerable. Hence, by proposition 13, $\bar{f}$ is shrinkable and so, by proposition 8, f is a fibre homotopy equivalence over B.

2. HUREWICZ FIBRATIONS

A map $p:E \to B$ is said to have the *covering homotopy property* (*CHP*) *with respect to the space* X if for every map $f:X \to E$ and every homotopy $H:X\times I \to B$ of pf, there exists a homotopy $\tilde{H}:X\times I \to E$ of f lifting (i.e. covering) H, that is $H=p\tilde{H}$. In other words, given any commutative diagram (ignore the dotted arrow)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X\times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B \end{tikzcd}$$

we can fill the dotted arrow by a homotopy making the enlarged diagram commutative. p is said to have the *covering homotopy property with respect to the couple* (X,A) if for every map $f:X \to E$ and every homotopy $H:X\times I \to B$ and $\tilde{H}_A:A\times I \to E$ such that $H_0=pf$, $(\tilde{H}_A)_0=f|A$ and $p\tilde{H}_A=H|A\times I$, there exists a lifting of H, say $\tilde{H}:X\times I \to E$, such that $\tilde{H}_0=f$ and $\tilde{H}|A\times I=\tilde{H}_A$. In other words, given any commutative diagram (ignore the dotted arrow)

$$\begin{tikzcd} X\cup A\times I \arrow[r, "f\cup\tilde{H}_A"] \arrow[d, "i_0\cup i_{A\times I}"'] & E \arrow[d, "p"] \\ X\times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B \end{tikzcd}$$

we can fill the dotted arrow by a homotopy making the enlarged diagram commutative.

Now let $q:D \to X$ be any map. Then p is said to have the <u>covering homotopy property with respect to the map</u> q if for any map pair (f,g) from q to p and homotopy $H:X\times I \to B$ of g there exists a homotopy $\tilde{H}:D\times I \to E$ of f such that $p\tilde{H}=H(q\times 1_I)$. In other words, given any commutative diagram (ignore the dotted arrow)

$$\begin{tikzcd}
D \arrow[rr, "f"] \arrow[dr, "i_0"] \arrow[ddd, "q"'] & & E \arrow[ddd, "p"] \\
& D\times I \arrow[ur, dashed, "\tilde{H}"'] \arrow[d, dash, "q\times 1_I"] & \\
& X\times I \arrow[dr, "H"] & \\
X \arrow[ur, "i_0"] \arrow[rr, "g"'] & & B
\end{tikzcd}$$

we can fill the dotted arrow by a homotopy making the enlarged diagram commutative. Now let A be a subset of X and let $q_A:D_A \to A$ denote the restriction of q over A. p is said to have the <u>covering homotopy property with respect to</u> (q,q_A) if for every map pair (f,g) from q to p and every homotopy $H:X\times I \to B$ and $\tilde{H}_A:D_A\times I \to E$ such that $H_0=g$, $p\tilde{H}_A=(H|A\times I)\circ(q_A\times 1_I)$ and $(\tilde{H}_A)_0=f|A$, there exists a homotopy $\tilde{H}:D\times I \to E$ of f such that $p\tilde{H}=H(q\times 1_I)$ and $\tilde{H}|D_A\times I=\tilde{H}_A$. In other words, given any commutative diagram (ignore the dotted arrow)

$$
\begin{tikzcd}
D \arrow[rrr, "f"] \arrow[dr, "i_0"] \arrow[ddd, "q"'] & & & E \arrow[ddd, "p"] \\
& D\times I \arrow[urr, dashed, "\tilde{H}"] \arrow[d, "q\times 1_I"'] & D_A\times I \arrow[l, "i_{D_A}\times I"'] \arrow[ur, "\tilde{H}_A"'] \arrow[d, "q_A\times 1_I"] & \\
& X\times I \arrow[drr, "H"] & A\times I \arrow[l, "i_{A\times I}"'] \arrow[dr, "H|A\times I"] & \\
X \arrow[ur, "i_0"] \arrow[rrr, "g"] & & & B
\end{tikzcd}
$$

we can fill the dotted arrow by a homotopy making the enlarged diagram commutative.

Proposition 1 For any map $p:E \to B$ the following properties are equivalent:

(i). p has the CHP with respect to all spaces;

(ii) p has the CHP with respect to all maps;

(iii) p has the CHP with respect to all pullbacks $p_f:E_f \to X$ of p along any map $f:X \to B$.

Proof (i) $\Rightarrow$ (ii). Let $q:D \to A$ be any map and suppose given the following commutative diagram (the filled arrows are data and the dotted arrow unknown)

$$
\begin{tikzcd}
D \arrow[rr, "f"] \arrow[dr, "i_0"] \arrow[dd, "q"'] & & E \arrow[dd, "p"] \\
& D\times I \arrow[ur, dashed, "\tilde{H}"] \arrow[d, "q\times 1_I"] & \\
A \arrow[r, "i_0"] \arrow[rr, bend right, "g"'] & A\times I \arrow[r, "H"] & B
\end{tikzcd}
$$

From it we can extract the commutative diagram

$$\begin{tikzcd}
D \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
D\times I \arrow[r, "H(q\times 1_I)"] \arrow[ur, dashed, "\tilde{H}"] & B
\end{tikzcd}$$

The existence of $\tilde{H}$ now follows from the hypothesis.

(ii) $\Rightarrow$ (iii). If p has the CHP with respect to all maps, in particular it has the CHP with respect to all pullbacks $p_f : E_f \to X$, with $f : X \to B$ any map.

(iii) => (i). Suppose given the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
X\times I \arrow[r, "H"] \arrow[ur, dashed, "\tilde{H}"] & B
\end{tikzcd}$$

and from it consider the commutative diagram

$$\begin{tikzcd}
X \arrow[dr, "(1_X{,}f)"] \arrow[rrrd, "f"] \arrow[ddddr, "1_X"'] & & & \\
& E_{H_0} \arrow[rr, "pr"] \arrow[dr, "i_0"] \arrow[ddd, "p_{H_0}"'] & & E \arrow[ddd, "p"] \\
& & E_{H_0}\times I \arrow[ur, "\bar{H}"'] \arrow[d, "p_{H_0}\times 1_I"] & \\
& & X\times I \arrow[dr, "H"] & \\
& X \arrow[ur, "i_0"] \arrow[rr, "H_0"'] & & B
\end{tikzcd}$$

where the existence of $\bar{H}$ follows from the hypothesis. Then $\tilde{H}=\bar{H}\circ((1_X,f)\times 1_I):X\times I\to E$ solves our initial problem; indeed, $p\tilde{H}=p\circ\bar{H}\circ((1_X,f)\times 1_I)=H\circ(p_{H_0}\times 1_I)\circ((1_X,f)\times 1_I)=H\circ(1_X\times 1_I)=H$ and $\tilde{H}\circ i_0=\bar{H}\circ((1_X,f)\times 1_I)\circ i_0=\bar{H}\circ i_0\circ(1_X,f)=f$.

A map $p:E\to B$ is called a Hurewicz fibration, or simply a fibration, if it has the CHP with respect to all spaces. The importance of this concept lies in the fact that if $p:E\to B$ is a fibration then the liftability up to homotopy of a map $f:X\to B$, that is, the existence of a map an $\tilde{f}:X\to E$ with $f\simeq p\tilde{f}$, is equivalent to the strict liftability of f, that is, the existence of an $\tilde{f}:X\to E$ with $f=p\tilde{f}$. Indeed, if p is a fibration and $\tilde{f}$ is a lifting of f up to homotopy and $H:X\times I\to B$ a homotopy from $p\tilde{f}$ to f, then H can be lifted to $\tilde{H}:X\times I\to E$ and $\tilde{H}_1$ gives a strict lifting of f. Hence the liftability of f is not just a property of f but of its homotopy class; so the lifting problem over a fibration can be tackled with the tools of algebraic topology, which are generally homotopy invariant.

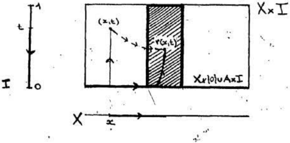
A map $p:E\to B$ is called a regular fibration if for any map $f:X\to E$ and any homotopy $H:X\times I\to B$ of pf there exists a homotopy $\tilde{H}:X\times I\to E$ of f lifting H such that $\tilde{H}$ is stationary at every point at which H is stationary. Not all fibrations are regular. The first correct example of a fibration which is not regular was found by P. Tulley in [43]; in that paper are also given sufficient conditions on the space B so that all fibrations with base space B are regular.

<u>Proposition 2</u> If p is a fibration the following properties hold:

(i) p has the covering homotopy property with respect to all closed cofibered pairs (X,A);

(ii) p has the covering homotopy property with respect to all pairs (q,q_A), where $q:D \to X$ is a fibration and (X,A) is a closed cofibered pair.

<u>Proof</u> (i) If (X,A) is a cofibered pair, $X\times\{0\}\cup A\times I$ is not only a retract of $X\times I$, but a strong deformation retract. In fact, let $r:X\times I \to X\times\{0\}\cup A\times I$, be a retraction. For every $(x,t)\in X\times I$, the map $s\in I \to r(x,ts)\in X\times\{0\}\cup A\times I$ is a path from $r(x,0)=(x,0)$ to $r(x,t)$ and hence the map $s\in I \to pr_1 r(x,ts)\in X$ is a path in X from x to $pr_1 r(x,t)$. Furthermore, for every $(x,t)\in X\times I$, the map $s\in I \to (pr_2 r(x,t)-t)s+t\in I$ is a path in I from t to $pr_2 r(x,t)$. Then the map $D:(x,t,s)\in X\times I\times I \to (pr_1 r(x,ts), (pr_2 r(x,t)-t)s+t)\in X\times I$ is a strong deformation retraction of $X\times I$ onto $X\times\{0\}\cup A\times I$ since $D(x,t,0)=(x,t)$, $D(x,t,1)=(pr_1 r(x,t), pr_2 r(x,t))=r(x,t)$, $D(x,0,s)=(x,0)$ and $D(a,t,s)=(a,t)$.



We then observe that, since (X,A) is a closed cofibered pair, there exists a map $\phi:X\to I$ such that $A=\phi^{-1}(0)$. Indeed, define $\phi(x)=\max_{t\in I}|t-pr_2r(x,t)|$; we have that $\phi(a)=\max_{t\in I}|t-pr_2r(a,t)|=\max_{t\in I}|t-pr_2(a,t)|=\max_{t\in I}|t-t|=0$ for every $a\in A$. Furthermore if $\bar{x}\in X-A$, since $X-A$ is open and r is continuous, there exists a neighbourhood V of $0\in I$ such that $r(\bar{x},t)\in X-A$ for every $t\in V$; so $\phi(\bar{x})\neq 0$. Using the above map $\phi:X\to I$ with $\phi^{-1}(0)=A$, we can construct a map $\psi:X\times I\to I$ with that $\psi^{-1}(0)=X\times\{0\}\cup A\times I$ putting $\psi(x,t)=t\phi(x)$. We now can apply theorem 3 of [41] which claims that a fibration has the relative lifting property with respect to all couples such that the subspace is a strong deformation retract of the ambient space and such that there exists a function on the ambient space into I whose zero set is the subspace. Indeed, our previous considerations say that $(X\times I, X\times\{0\}\cup A\times I)$ has these properties; so every homotopy $H:X\times I\to B$ can be lifted to $\tilde{H}:X\times I\to E$ with prescribed restriction on $X\times\{0\}\cup A\times I$.

(ii) Given a map pair (f,g) from q to p and homotopies $H:X\times I\to B$ and $\tilde{H}_A:D_A\times I\to E$ such that $p\tilde{H}_A=(H|A\times I)\,(q_A\times 1_I)$, consider $H(q\times 1_I):D\times I\to B$. We have that $H\circ(q\times 1_I)\circ i_0=H\circ i_0\circ q=gq=pf$ and $p\tilde{H}_A=(H|A\times I)\circ(q_A\times 1_I)=H\circ i_{A\times I}\circ q_A\times 1_I=H\circ(q\times 1_I)\circ i_{D_A\times I}=H\circ(q\times 1_I)|D_A\times I$. For theorem 12 in [42], since (X,A) is a cofibered pair with A closed and $q:D\to X$ is a fibration, (D,D_A) is a cofibered pair with D_A of course closed.

$$\begin{tikzcd}
D \arrow[rrr, "f"] \arrow[dr, "i_0"] \arrow[ddddd, "q"'] & & & E \arrow[ddddd, "p"] \\
& D\times I \arrow[urr, dashed, "\tilde{H}"] \arrow[ddd, "q\times 1_I"'] & & \\
& & D_A\times I \arrow[ul] \arrow[uur, "\tilde{H}_A"'] \arrow[d, "q_A\times 1_I"] & \\
& & A\times I \arrow[dl] \arrow[ddr, "H|A\times I"] & \\
& X\times I \arrow[drr, "H"] & & \\
X \arrow[ur, "i_0"] \arrow[rrr, "g"] & & & B
\end{tikzcd}$$

So we can apply the previous (i) to the pair (D,D_A) and data $f:D\to E$, $H\circ(q\times 1_I):D\times I\to B$ and $\tilde{H}_A:D_A\times I\to E$ obtaining a homotopy $\tilde{H}:D\times I\to E$ such that $p\tilde{H}=H\circ(q\times 1_I)$, $\tilde{H}_0\, i=f$ and $\tilde{H}|A\times I=\tilde{H}_A$, as required.

We derive some immediate consequences of the definition of a fibration.

Proposition 3 If $p:E\to B$ and $q:B\to A$ are fibrations, then their composition $qp:E\to A$ is a fibration.

Proof Let X be any space, $f:X\to E$ a map and $H:X\times I\to A$ a homotopy of $(qp)f$. Since H is homotopy of $q(pf)$ and q is a fibration, there exists a homotopy $H':X\times I\to B$ of pf with $qH'=H$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "pf"] \arrow[dd, "i_0"'] & E \arrow[d, "p"] \\
& B \arrow[d, "q"] \\
X\times I \arrow[uur, dashed, "\tilde{H}"] \arrow[ur, dashed, "H'"] \arrow[r, "H"] & A
\end{tikzcd}$$

Since H' is a homotopy of pf and p is a fibration, there exists a homotopy $\widetilde{H}:X\times I \to E$ of f with $p\widetilde{H}=H'$. Now $(qp)\widetilde{H}=q(p\widetilde{H})=qH'=H$ and so $\widetilde{H}$ is a homotopy of f lifting H, as required.

We recall that in a category C a commutative square

$$\begin{tikzcd} D \arrow[r, "g'"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ A \arrow[r, "g"] & B \end{tikzcd}$$

is called <u>cartesian</u> if for every pair of morphisms $f_1:X \to A$ and $f_2:X \to E$ such that $gf_1=pf_2$, there exists a unique morphism $f:X \to D$ such that $f_1=p'f$ and $f_2=g'f$.

$$\begin{tikzcd} X \arrow[dr, dashed] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\ & D \arrow[r, "g'"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ & A \arrow[r, "g"] & B \end{tikzcd}$$

<u>Proposition 4</u> If the following commutative square in Top

$$\begin{tikzcd}
D \arrow[r, "g'"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\
A \arrow[r, "g"] & B
\end{tikzcd}$$

is cartesian and p (respectively g) is a fibration, then p' (respectively g') is a fibration.

<u>Proof</u> Let $f: X \to D$ be any map and $H: X\times I \to A$ a homotopy of $p'f$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_0"'] & D \arrow[r, "g'"] \arrow[d, "p'"] & E \arrow[d, "p"] \\
X\times I \arrow[r, "H"] & A \arrow[r, "g"] & B
\end{tikzcd}$$

Since p is a fibration and gH is a homotopy of $p(g'f)$, there exists a homotopy $\bar{H}: X\times I \to E$ of $g'f$ with $p\bar{H}=gH$. The square on the right is cartesian, so there exists a (unique) homotopy $\tilde{H}: X\times I \to D$ with $p'\tilde{H}=H$ and $g'\tilde{H}=\bar{H}$.

$$\begin{tikzcd}
X\times I \arrow[drr, "\tilde{H}"'] \arrow[ddrr, bend right, "H"'] \arrow[drrr, bend left, "\bar{H}"] & X \arrow[l, "i_0"'] \arrow[dr, "f"] & & \\
& & D \arrow[r, "g'"] \arrow[d, "p'"] & E \arrow[d, "p"] \\
& & A \arrow[r, "g"] & B
\end{tikzcd}$$

It remains to prove that $\tilde{H}$ is a homotopy of f, that is, $\tilde{H}i_0=f$. Since the above square is cartesian, it is enough to show that $p'(\tilde{H}i_0)=p'f$ and $g'(\tilde{H}i_0)=g'f$. Indeed, $p'(\tilde{H}i_0)=(p'\tilde{H})i_0= Hi_0=f$ and $g'(\tilde{H}i_0)=(g'\tilde{H})i_0=\bar{H}i_0=g'f$.

<u>Corollary 5</u> If $p:E \to B$ is a fibration and A is a subspace of B, then the restriction of p over A, $p_A:E_A \to A$, is a fibration.

<u>Proof</u> Indeed the commutative square

$$\begin{tikzcd} E_A \arrow[r, "i'"] \arrow[d, "p_A"'] & E \arrow[d, "p"] \\ A \arrow[r, "i"] & B \end{tikzcd}$$

is cartesian.

<u>Corollary 6</u> If $p':D \to A$ and $p:E \to B$ are topologically equivalent, that is, there exist homeomorphisms $g':D \to E$ and $g:A \to B$ such that $pg'=gp'$, then p' is a fibration if and only if p is a fibration.

<u>Proof</u> Indeed the commutative square

$$\begin{tikzcd} D \arrow[r, "g'"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ A \arrow[r, "g"] & B \end{tikzcd}$$

is cartesian.

Corollary 7 If $p:E \to B$ is a fibration and $g:A \to B$ any map, then its pullback along g, $p_g:E_g \to A$, is a fibration.

Proof We know that the commutative square

$$\begin{tikzcd} E_g \arrow[r, "pr_2"] \arrow[d, "p_g"'] & E \arrow[d, "p"] \\ A \arrow[r, "g"] & B \end{tikzcd}$$

is cartesian.

Proposition 8 $p:E \to B$ and $q:L \to C$ are fibrations if and only if their cartesian product $p\times q:E\times L \to B\times C$ is a fibration.

Proof Suppose p ans q are fibrations. Let X be any space, $f:X \to E\times L$ a map and $H:X\times I \to B\times C$ a homotopy of $(p\times q)f$. Let $f'=pr_1 f$, $f''=pr_2 f$ and $H'=pr_1 H$, $H''=pr_2 H$ be the components of f and H, respectively, that is, the compositions of f and H with the projections onto the factors. Since $(p\times q)f=(pf')\times(qf'')$, we have that H' is a homotopy of pf' and H'' is a homotopy of qf''. Now p and q are fibrations, so there exist homotopies $\tilde{H}':X\times I \to E$ and $\tilde{H}'':X\times I \to L$ of f' and f'', respectively, such that $p\tilde{H}'=H'$ and $q\tilde{H}''=H''$. Then $\tilde{H}=(\tilde{H}',\tilde{H}'')$ is a homotopy of f lifting H, since $(p\times q)\tilde{H}=(p\tilde{H}',q\tilde{H}'')=(H',H'')=H$.

Conversely, suppose that $p\times q$ is a fibration. Choose an element $c\epsilon C$ and consider the restriction of $p\times q$ over $B\times\{c\}$, $(p\times q)_{B\times\{c\}}:(p\times q)^{-1}(B\times\{c\})\to B\times\{c\}$. By corollary 5 it is a fibration and furthermore we have that $(p\times q)^{-1}(B\times\{c\})=E\times L_c$. It now follows that that map $p':(e,\ell)\epsilon E\times L_c\to p(e)\epsilon B$ is a fibration. Let $f:X\to E$ be any map and $H:X\times I\to B$ a homotopy of pf. Pick an element $\ell\epsilon L_c$ and consider the map $f':x\epsilon X\to(f(x),\ell)\epsilon E\times L_c$. Then H is a homotopy of $p'f'$ and so there exists a homotopy $\tilde{H}':X\times I\to E\times L_c$ of f' with $p'\tilde{H}'=H$. The composition of $\tilde{H}'$ with the projection onto E gives the required homotopy $\tilde{H}$. A similar argument shows that $q:L\to C$ is a fibration.

<u>Proposition 9</u> If $p:E\to B$ is a fibration, then $\operatorname{Im}p$ is a union of path components of B; in particular, if B is path connected then p is onto.

<u>Proof</u> Let $\pi_0(B)$ be the set of all path components of B and let $\pi'=\{P\epsilon\pi_0(B):\ P\cap\operatorname{Im}p\neq\emptyset\}$. We claim that $\operatorname{Im}p=\bigcup_{P\epsilon\pi'}P$. Indeed, if $b\epsilon\operatorname{Im}p$ and P is the path component containing b then $P\epsilon\pi'$ and hence $b\epsilon\bigcup_{P\epsilon\pi'}P$. Conversely, suppose $b\epsilon\bigcup_{P\epsilon\pi'}P$, that is $b\epsilon P$ for some $P\epsilon\pi'$. Let $b'\epsilon P\cap\operatorname{Im}p$ and let α be a path joining b' to b. Since $p:E\to B$ is a fibration, there exists a lifting $\tilde{\alpha}$ of α. Hence $b=\alpha(1)=p\tilde{\alpha}(1)$ and so $b\epsilon\operatorname{Im}p$.

<u>Proposition 10</u> Let $p:E\to B$ be a fibration and let E' be a

union of path components of E. Then $p'=p|E':E' \to B$ is a fibration.

<u>Proof</u> Let $f:X \to E'$ be a map and let $H:X\times I \to B$ be a homotopy of pf. Since p is a fibration, there exists a lifting of H $\tilde{H}:X\times I \to E$ with $\tilde{H}_0=f$. Now, for every $x\varepsilon X$, $\tilde{H}_x(0)=f(x)\varepsilon E'$. Since E' is a union of path components of E, the path $\tilde{H}_x$ must lie in E' and hence $Im\tilde{H}\subseteq E'$. So $\tilde{H}$ is a lifting of H over p'.

<u>Proposition 11</u> Let $p:E \to B$ be a fibration and let $f:X \to E$ be a map. Then f is homotopic to a map whose image is all contained in the fibre over $b\varepsilon B$ if and only if pf is homotopic to the map constant at $b\varepsilon B$.

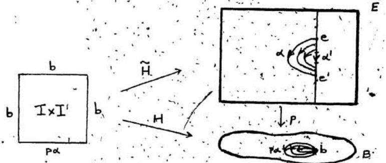
<u>Proof</u> Let $H:X\times I \to B$ be a homotopy of pf such that $H(x,1)=b$ for every $x\varepsilon X$. Since p is a Hurewicz fibration, there is a homotopy of f, $\tilde{H}:X\times I \to E$, lifting H; in particular, $p\tilde{H}(x,1)=H(x,1)=b$ and hence the image of $\tilde{H}_1$ is contained in the fibre over b. Conversely, if $H:X\times I \to E$ is a homotopy of f with ImH_1 contained in the fibre over b, then $pH:X\times I \to B$ is a homotopy from pf to the map constant at $b\varepsilon B$.

<u>Proposition 12</u> Let $p:E \to B$ be a fibration. Then the following properties hold:

(i) if $e,e'\varepsilon E$ are points lying in the same fibre over a point $b\varepsilon B$ which can be joined by a path in E whose projection is homotopic rel.İ to the constant loop at b, then e and e' can be joined by a path which is contained in the fibre;

(ii) if e, e' and e'' are points of E such that there exists paths in E joining e, e' and e, e'', with the same projection on B (so in particular e' and e'' lie in the same fibre), then e' and e'' can be joined by a path which is contained in the fibre.

<u>Proof</u> (i) Let α be a path in E joining e to e' such that there exists a homotopy rel. $\dot{I}$, $H: I \times I \to B$, from the loop $p\alpha$ to the loop constant at $p(e) = b$.



Since p is a fibration, there exists a homotopy $\tilde{H}: I \times I \to E$ of α lifting H. Now the image of the restriction of $\tilde{H}$ to $\dot{I} \times I \cup I \times \{1\}$ is contained in the fibre over b of e and e'. So the path $\alpha': I \to E$ defined by

$$\alpha'(t) = \begin{cases} \tilde{H}(0, 3t) & \text{if } 0 \leq t \leq 1/3 \\ \tilde{H}(3t-1, 1) & \text{if } 1/3 \leq t \leq 2/3 \\ \tilde{H}(1, -3t+3) & \text{if } 2/3 \leq t \leq 1 \end{cases}$$

joins e to e' and its image is contained in the fibre over b.

(ii) Let α,α' be paths joining e to e' and e to e", respectively, such that $p\alpha=p\alpha'$. Then $\alpha^{-1}.\alpha'$ is a path joining e' to e" whose projection on B is $p(\alpha^{-1}.\alpha')=(p\alpha^{-1}).(p\alpha)=(p\alpha)^{-1}.(p\alpha)$ and hence a loop homotopic rel. $\dot{I}$ to the constant loop at $p(e')=p(e")$. Applying (i), we deduce the existence of a path joining e' to e" which is contained in the fibre.

<u>Remark 13</u> In the proof of (i) we could have used the fact that $(I,\dot{I})$ is a closed cofibered pair and proposition 2 to deduce that the map $h:\dot{I}\times I\cup I\times\{0\}\to E$ defined by $h(t,0)=\alpha(t)$, $h(0,s)=e$ and $h(1,s)=e'$ admits an extension $\tilde{H}$ to $I\times I$ lifting H. In this case the path α' is simply defined by $\alpha'(t)=\tilde{H}(t,1)$.

We now present some examples of fibrations:

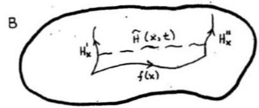
(i) For any pair of spaces B and F, the projection map $pr_1:B\times F\to B$ is a fibration. Indeed, let $f:X\to B\times F$ be any map and $H:X\times I\to B$ a homotopy of pr_1f. If $f':X\to B$ and $f":X\to F$ denote the components of f, then $pr_1f=f'$ and so H is simply a homotopy of f'. Hence, if we define $\tilde{H}:(x,t)\in X\times I\to(H(x,t),f"(x))\in B\times F$, then $\tilde{H}$ is a homotopy of f lifting H.

(ii) We recall that a map $p:E\to B$ is a <u>fibre bundle</u> if there exist a space F, an open covering $\{U\}$ of B and homeomorphisms $h_U:U\times F \to p^{-1}(U)=E_U$ such that the diagram

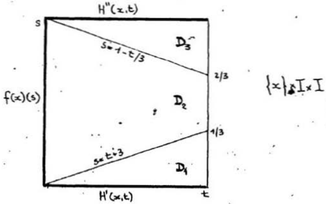
$$\begin{tikzcd} U\times F \arrow[rr, "h_U", "\approx"'] \arrow[dr, "pr_1"'] & & p^{-1}(U) \arrow[dl, "p_u"] \\ & U & \end{tikzcd}$$

commutes. It follows from example (i) and corollary 6 that each map $p_U:E_U \to U$ is a fibration. If B is a paracompact space, it then follows from the Hurewicz Uniformization Theorem (theorem 32) that p is itself a fibration. There are examples of fibre bundles over non-paracompact spaces which are not fibrations.

(iii) For any space B the map $\pi:B^I \to B\times B$ defined by $\pi(\alpha)=(\alpha(0),\alpha(1))$ is a fibration. Given any map $f:X \to B^I$ and a homotopy $H:X\times I \to B\times B$ of πf, we must find a homotopy of f, $\tilde{H}:X\times I \to B^I$, which is a lifting of H. If $H',H'':X\times I \to B$ denote the components of H, the fact that H is a homotopy of πf means that $f(x)(0)=H'(x,0)$ and $f(x)(1)=H''(x,0)$, that is, $f(x)$ is a path in B with the same origin as H'_x and with end point equal to the origin of H''_x.



The conditions on $\widetilde{H}$ mean that, for any $x\varepsilon X$ and $t\varepsilon I$, $\widetilde{H}(x,t)$ must be a path in B, equal to f(x) when t=0, and with initial point equal to H'(x,t) and end point equal to H"(x,t) when x and t are generic. If we denote by $H:X\times I\times I \to B$ the adjoint of $\widetilde{H}$, the above conditions mean that H must satisfy the relations $H(x,0,s)=f(x)(s)$, $H(x,t,0)=H'(x,t)$ and $H(x,t,1)=H''(x,t)$.



At this point we could invoke the fact that $X\times(I\times\dot{I}\cup\{0\}\times I)$ is a retract of $X\times I\times I$ (since $I\times\dot{I}\cup\{0\}\times I$ is a retract of $I\times I$) to deduce the existence of such a map H and so of $\widetilde{H}$, but we prefer to construct H explicitly. To this end, let $D_1=\{(t,s)\in I\times I:\ 0<s<t/3\}$, $D_2=\{(t,s)\in I\times I:\ t/3<s<-(t/3)+1\}$ and $D_3=\{(t,s)\in I\times I:\ (-t/3)+1<s<1\}$ and define $H_1:(x,t,s)\in X\times D_1 \to H'(x,-3s+t)\in B$, $H_2:(x,t,s)\in X\times D_2 \to f(x)((3s-t)/(3-2t))\in B$ and $H_3:(x,t,s)\in X\times D_3 \to H''(x,3s+t-3)\in B$; H_1, H_2 and H_3 coincide on the intersection of their domains of definition because $H_1(x,t,t/3)=H'(x,0)=f(x)(0)=H_2(x,t,t/3)$ and $H_2(x,t,1-t/3)=f(x)(1)=H''(x,0)=H_3(x,t,1-t/3)$, giving rise to a map H which satisfies the required properties. Roughly speaking, the path $\tilde{H}(x,t)$ ($\tilde{H}$=adjoint of H) has been obtained covering first the tract of H'_x from $H'_x(t)$ to $H'_x(0)$ (reversing the original versus), second covering all $f(x)$ and then covering the tract of H''_x from $H''_x(0)$ to $H''_x(t)$.

As a consequence of the fact that the maps $\pi:B^I \to B\times B$, $pr_1:B\times B \to B$ and $pr_2:B\times B \to B$ are fibrations and by proposition 3, we get that the maps $\pi_0:\alpha\in B^I \to \alpha(0)\in B$ and $\pi_1:\alpha\in B^I \to \alpha(1)\in B$ are fibrations. It is easy to see that the function $f:\alpha\in B^I \to \alpha^{-1}\in B^I$ is a homeomorphism over B, that is, the following diagram commutes

$$\begin{tikzcd}
B^I \arrow[rr, "f", "\cong"'] \arrow[dr, "\pi_0"'] & & B^I \arrow[dl, "\pi_1"] \\
& B &
\end{tikzcd}$$

As other examples along this line, we have that for any fixed $\bar{b}\epsilon B$ the maps $p:\alpha\epsilon P(B,\bar{b})=\{\alpha\epsilon B^I;\ \alpha(0)=\bar{b}\}\to\alpha(1)\epsilon B$ and $p':\alpha\epsilon P'(B,\bar{b})=\{\alpha\epsilon B^I;\ \alpha(1)=\bar{b}\}\to\alpha(0)\epsilon B$ are fibrations. Indeed, if we consider the "inclusions" $i:b\epsilon B\to(\bar{b},b)\epsilon B\times B$ and $i':b\epsilon B\to(b,\bar{b})\epsilon B\times B$ and take the pullbacks of $\pi:B^I\to B\times B$ along i and i', respectively

$$\begin{tikzcd} P(B,\bar{b}) \arrow[r, leftrightarrow] \arrow[dr, "p"'] & (B^I)_i \arrow[r] \arrow[d, "\pi_i"] & B^I \arrow[d, "\pi"] \\ & B \arrow[r, "i"] & B\times B \end{tikzcd} \qquad \begin{tikzcd} P'(B,\bar{b}) \arrow[r, leftrightarrow] \arrow[dr, "p'"'] & (B^I)_{i'} \arrow[r] \arrow[d, "\pi_{i'}"] & B^I \arrow[d, "\pi"] \\ & B \arrow[r, "i'"] & B\times B \end{tikzcd}$$

then the maps π_i and $\pi_{i'}$ are fibrations and their total spaces $(B^I)_i=\{(b,\alpha)\epsilon B\times B^I:\ \alpha(0)=\bar{b}$ and $\alpha(1)=b\}$ and $(B^I)_{i'}=\{(b,\alpha)\epsilon B\times B^I:\ \alpha(0)=b$ and $\alpha(1)=\bar{b}\}$ are homeomorphic over B to $P(B,\bar{b})$ and $P'(B,\bar{b})$, respectively, by the identifications $\alpha\epsilon P(B,\bar{b})<\!\!-\!\!>(\alpha(1),\alpha)\epsilon(B^I)_i$ and $\alpha\epsilon P'(B,\bar{b})<\!\!-\!\!>(\alpha(0),\alpha)\epsilon(B^I)_{i'}$. The above maps $p:P(B,\bar{b})\to B$ and $p':P'(B,\bar{b})\to B$ are called the <u>path fibrations</u> associated to the pointed space $(B,\bar{b})$. It is easy to see that the function $f:\alpha\epsilon P(B,\bar{b})\to\alpha^{-1}\epsilon P'(B,\bar{b})$ is a homeomorphism over B.

(iv) Let (Z,A) be a closed cofibred pair with Z locally compact, Hausdorff and let B be any space. We want to show that the map $j^*:B^Z\to B^A$ induced by the inclusion map $j:A\to Z$

is a fibration. Let $f:X \to B^Z$ be any map and let $H:X\times I \to B^A$ be a homotopy of j^*f.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_0"'] & B^Z \arrow[d, "j^*"] \\
X\times I \arrow[r, "H"] \arrow[ur, dashed] & B^A
\end{tikzcd}$$

Since Z is locally compact, Hausdorff and A is a closed subspace of Z, A is also locally compact, Hausdorff. Hence the adjoints of f and H, $f:X\times Z \to B$ and $H:X\times A\times I \to B$, are continuous and make the following diagram commutative (ignore the dotted arrow)

$$\begin{tikzcd}
X\times A \arrow[r, "1_X\times j"] \arrow[d, "i_0"'] & X\times Z \arrow[d, "i_0"] \arrow[ddr, "f"] & \\
X\times A\times I \arrow[r, "1_X\times j\times 1_I"] \arrow[drr, "H"'] & X\times Z\times I \arrow[dr, dashed] & \\
& & B
\end{tikzcd}$$

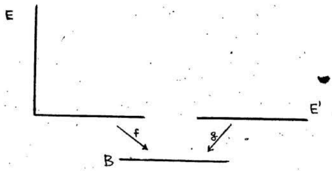
Since (Z,A) is a cofibred pair, $(X\times Z, X\times A)$ is also a cofibred pair; indeed, if $r: Z\times I \to Z\times\{0\}\cup A\times I$ is a retraction then $1_X\times r: X\times Z\times I \to X\times(Z\times\{0\}\cup A\times I)=X\times Z\times\{0\}\cup X\times A\times I$ is also a retraction. We can then fill the dotted arrow in the above diagram by a homotopy $K: X\times Z\times I \to B$ making the enlarged diagram commutative. The adjoint of K, $K: X\times I \to B^Z$, is a homotopy of f lifting H, which shows that j^* is a fibration. The fibration $\pi: B^I \to B\times B$ of example (iii) can be seen as a particular case of this example, taking as a cofibred pair $(I,\dot{I})$ and identifying $B^{\dot{I}}$ with $B\times B$.

(v) Let $p: E \to B$ be a fibration and let A be a locally compact, Hausdorff space. Then the map $p_*: \ell\in M(A,E) \to p\ell\in M(A,B)$ is a fibration. Indeed, let $f: X \to M(A,E)$ be any map and let $H: X\times I \to M(A,B)$ be a homotopy of p_*f, that is, $H(x,0)=(p_*f)(x)= p_*(f(x))=p\circ(f(x))$. From this functional relation we deduce that $H(x,0)(a)=(p\circ(f(x)))(a)=p(f(x)(a))$ for every $a\in A$. So, if we consider the adjoint of f, $\hat{f}: (x,a)\in X\times A \to f(x)(a)\in E$, and the adjoint of H, $\hat{H}: (x,a,t)\in X\times A\times I \to H(x,t)(a)\in B$, which are continuous because A is locally compact, Hausdorff, we get that $\hat{H}(x,a,0)=H(x,0)(a)=p(f(x)(a))=p(\hat{f}(x,a))=(p\hat{f})(x,a)$, that is, the following diagram commutes

$$\begin{tikzcd}
X\times A \arrow[r, "\hat{f}"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
X\times A\times I \arrow[r, "\hat{H}"] & B
\end{tikzcd}$$

Since p is a fibration, there exists a homotopy of f, $\bar{H}:X\times A\times I \to E$, lifting H, that is, such that $\bar{H}(x,a,0)=f(x,a)$ and $p\bar{H}(x,a,t)=H(x,a,t)$. If we consider the adjoint of $\bar{H}$, say $\tilde{H}:X\times I \to M(A,E)$, the above relations yield $\tilde{H}(x,0)(a)=\bar{H}(x,a,0)=f(x,a)=f(x)(a)$ and $(p\circ(\tilde{H}(x,t)))(a)=p(\tilde{H}(x,t)(a))=p(\bar{H}(x,a,t))=H(x,a,t)=H(x,t)(a)$ for every $a\in A$, and hence $\tilde{H}(x,0)=f(x)$ and $(p_*\tilde{H})(x,t)=p_*(\tilde{H}(x,t))=p\circ(\tilde{H}(x,t))=H(x,t)$.

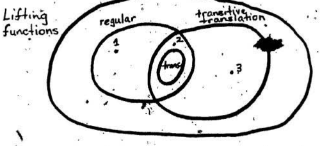
Unfortunately the property of being a fibration is not preserved under fiber homotopy equivalence, as the following simple example shows. Let $E=I\times\{0\}\cup\{0\}\times I\subseteq I\times I$, $E'=B=I$ and let $p:E \to B$ be the projection map on the first factor and $p':E' \to B$ the identity map.



Define fibre maps over B $f:E \to E'$ and $g:E' \to E$ by $f(x,y)=x$ and $g(x)=(x,0)$. Then the maps $H:E\times I \to E$ and $K:E'\times I \to E'$ given by $H((x,y),t)=(x,ty)$ and $K(x,t)=x$ are vertical homotopies from gf to 1_E and from fg to $1_{E'}$, respectively. Hence, p and p' have the same fibre homotopy type over B, but p' is a fibration and p is not a fibration, like it is easy to see.

According to the definition we have given of a fibration, we should check the covering homotopy property with respect to all spaces X; fortunately, it is possible to give an intrinisic characterization of a fibration.

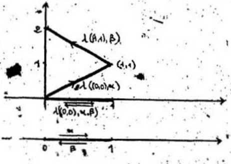
A (global) <u>lifting function</u> for p is a map $\lambda:\Lambda_p \to E^I$ such that $\rho\lambda=1_{\Lambda_p}$, that is, such that $\lambda(e,\alpha)(0)=e$ and $p\,\lambda(e,\alpha)=\alpha$; λ is said to be <u>regular</u> if $\lambda(e,\alpha)$ is stationary whenever α is stationary; λ is said to be be <u>transitive</u> if $\lambda(e,\alpha.\beta)=\lambda(e,\alpha).\lambda(\lambda(e,\alpha)(1),\beta)$. The <u>translation map</u> along to $\alpha\varepsilon B^I$ is the map $\lambda_\alpha:F_{\alpha(0)} \to F_{\alpha(1)}$ defined by $\lambda_\alpha(e)=\lambda(e,\alpha)(1)$; the translation maps are said to be <u>transitive</u> if $\lambda_{\alpha.\beta}=\lambda_\beta\lambda_\alpha$ for every pair of paths α and β such that $\alpha(1)=\beta(0)$. If U is any subspace of B^I and $\Lambda_p(U)$ denotes the subspace of Λ_p given by $\Lambda_p(U)=\{(e,\alpha)\varepsilon\Lambda_p:\ \alpha\varepsilon U\}$, then any section of $\rho:E^I \to \Lambda_p$ over $\Lambda_p(U)$, that is a map $\lambda:\Lambda_p(U) \to E^I$ such that $\rho\lambda=$ is called a lifting function <u>over</u> U. Now we will study relations among these concepts; the situation can be summarized in the following diagram



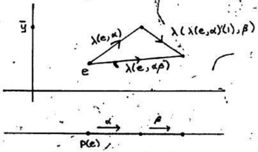
A transitive lifting function is regular and has transitive translation maps. Indeed, from the transitivity property we that for every $e\in E$ $\lambda(e,p(e))=\lambda(e,p(e))\cdot\lambda(\lambda(e,p(e))(1),p(e))$ and from this we deduce that $\lambda(e,p(e))$ must be constant. This follows from the general fact that if α and β are paths satisfying the relation $\alpha=\alpha\beta$ then α must be constant; in fact we have that $\alpha(t)=\alpha(2t)$, if $0<t<1/2$, and for every $\bar{t}\in[0,1]$ the sequence $\{\bar{t},\bar{t}/2,\bar{t}/4,\ldots\}$ converges to 0 and so the sequence $\{\alpha(\bar{t}),\alpha(\bar{t}/2),\alpha(\bar{t}/4),\ldots\}$ must converge to $\alpha(0)$; but $\alpha(\bar{t})=\alpha(\bar{t}/2)=\alpha(\bar{t}/4),\ldots$ and hence $\alpha(\bar{t})=\alpha(0)$. If λ is transitive then it has transitive translation maps since if α and β are paths with $\alpha(1)=\beta(0)$ then $\lambda_\beta\lambda_\alpha(e)=\lambda_\beta(\lambda(e,\alpha)(1))=\lambda(\lambda(e,\alpha)(1),\beta)(1)=[\lambda(e,\alpha)\cdot\lambda(\lambda(e,\alpha)(1),\beta)](1)=\lambda(e,\alpha\cdot\beta)(1)=\lambda_{\alpha\cdot\beta}(e)$.

We now give an example of a regular lifting function which has not transitive lifting maps. Take $E=R\times R$, $B=R$ and $p:E\to B$ defined by $p(x,y)=x$; using the continuous function $m:\alpha\in B^I \to |\alpha(1)-\alpha(0)|\in R$, we define a lifting function $\lambda:\Lambda_p \to E^I$ by $\lambda((x,y),\alpha)(t)=(\alpha(t),y+tm(\alpha))$; λ is of course regular, since $m(\tilde{b})=0$ for every $b\in B$, but it has not transitive translation maps.

We now give an example of a lifting functions which is regular and has transitive translation maps, but it is not transitive. Take $E=R\times R$, $B=R$ and $p(x,y)=x$; define the continuous function $r:B^I \to R$ by $r(\alpha)=\alpha(1)-\alpha(0)$ and observe that r is additive with respect to the product of paths, that is, $r(\alpha.\beta)=r(\alpha)+r(\beta)$ whenever $\alpha(1)=\beta(0)$, since $r(\alpha.\beta)=(\alpha.\beta)(1)-(\alpha.\beta)(0)=\beta(1)-\alpha(0)$ and $r(\alpha)+r(\beta)=\alpha(1)-\alpha(0)+\beta(1)-\beta(0)=\beta(1)-\alpha(0)$; then define $\lambda:\Lambda_p \to E^I$ by $\lambda((x,y),\alpha)(t)=(\alpha(t),y+tr(\alpha))$. λ is regular, since $\lambda((x,y),\tilde{x})(t)=(x,y)$ and has transitive translation maps, since $\lambda_{\alpha.\beta}(x,y)=(\alpha.\beta(1),y+r(\alpha.\beta))=(\beta(1),y+r(\alpha.\beta))$ and $\lambda_\beta\lambda_\alpha(x,y)=\lambda_\beta(\alpha(1),y+r(\alpha))=(\beta(1),y+r(\alpha)+r(\beta))=(\beta(1),y+r(\alpha.\beta))$, but is not transitive, taking for example $(x,y)=(0,0)$, $\alpha(t)=t$ and $\beta(t)=1-t$. Since in this case $\lambda((0,0),\alpha.\beta)(t)=((\alpha.\beta)(t),0)$ and $\lambda((0,0,\alpha)(t)=(\alpha(t),t)$ and $\lambda((1,1),\beta)(t)=(\beta(t),1-t)$



To complete the picture we now give an example of a lifting function which has transitive translation maps but is not regular. Take again $E=R\times R$, $B=R$ and $p(x,y)=x$ and fix some $\bar{y}\varepsilon R$; define $\lambda((x,y),\alpha)(t)=(\alpha(t),(1-t)y+t\bar{y})$. Then λ has transitive translation maps, but it is not regular.



Conditions on a fibration to have a lifting function with transitive translation maps were studied by Schlesinger in [37] and used by E. Brown in [11].

Proposition 14 (M. L. Curtis; W. Hurewicz) A map $p:E\to B$ is a (regular) fibration if and only if it admits a (regular) lifting function.

Proof Suppose p is a fibration. Consider the projection map $pr_1:\Lambda_p\to E$ and the homotopy $H:\Lambda_p\times I\to B$ given by $H((e,\alpha),t)=\alpha(t)$. Since $p(e)=\alpha(0)$ for each $(e,\alpha)\varepsilon\Lambda_p$, we have that $p\circ pr_1=H_0$. Then there exists a homotopy of pr_1, $\tilde{H}:\Lambda_p\times I\to E$, lifting H; its adjoint, $\lambda:\Lambda_p\to E^I$, is a lifting function for p, which is regular when p is a regular fibration.

$$
\begin{tikzcd}
E^I \arrow[drr, "p_0"] \arrow[dr, shift left, "\rho"] \arrow[ddr, "p_*"'] & & & \\
& \Lambda_p \arrow[r, "pr_1"] \arrow[d, "pr_2"] \arrow[ul, dashed, shift left, "\lambda"] & E \arrow[d, "p"] & \Lambda_p \arrow[l, "pr_1"'] \arrow[d, "i_0"] \\
& B^I \arrow[r, "p_0"] & B & \Lambda_p \times I \arrow[l, "H"'] \arrow[ul, dashed, "\tilde{H}"']
\end{tikzcd}
$$

Suppose now that p has a lifting function $\lambda : \Lambda_p \to E^I$. Let $f : X \to E$ be any map and $H : X \times I \to B$ a homotopy of pf. If $H : X \to B^I$ denotes the adjoint of H, then taking the composition $X \xrightarrow{(f,H)} \Lambda_p \xrightarrow{\lambda} E^I$ and its adjoint $\tilde{H} : X \times I \to E$, we get a homotopy of f lifting H. $\tilde{H}$ will be stationary at every point where H is stationary, if λ is regular.

$$
\begin{tikzcd}
E^I \arrow[drrr, "p_0"] \arrow[dr, shift left, "\rho"] \arrow[dddr, "p_*"'] & & & & \\
& \Lambda_p \arrow[rr, "pr_1"] \arrow[dd] \arrow[ul, shift left, "\lambda"] & & E \arrow[dd, "p"] & X \arrow[l, "f"'] \arrow[dd, "i_0"] \\
& & X \arrow[ul, "{(f,H)}"'] \arrow[ur, "f"'] \arrow[dl, "H"] & & \\
& B^I \arrow[rr, "p_0"] & & B & X \times I \arrow[l, "H"'] \arrow[uul, dashed, "\tilde{H}"']
\end{tikzcd}
$$

An unexpected by-product of proposition 14 is the following result.

Corollary 15 If $p:E \to B$ has the CHP with respect to all metric spaces and E and B are also metric, then p has the CHP with respect to all spaces.

Proof If B is metric, so is B^I and hence $\Lambda_p \subseteq E \times B^I$ is also metric. Since p has the CHP with respect to any metric space and, in particular, for Λ_p, p has a lifting function.

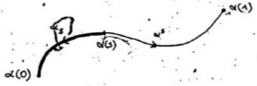
A more general kind of lifting function will be useful when we prove the fundamental Hurewicz Uniformization Theorem 32, which asserts that the property of being a fibration is a "local" property. Let $\bar{\Lambda}_p = \{(e,\alpha,s) \in E \times B^I \times I : \alpha(s) = p(e)\}$, that is, $\bar{\Lambda}_p$ is the following pullback

$$\begin{tikzcd}
E^I \times I \arrow[drr, "\omega"] \arrow[dr, shift right, "\rho"'] \arrow[ddr, "p_* \times 1_I"'] & & \\
& \bar{\Lambda}_p \arrow[ul, dashed, shift right, "\sigma"'] \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
& B^I \times I \arrow[r, "\omega"] & B
\end{tikzcd}$$

where $\rho: E^I \times I \to \bar{\Lambda}_p$ is the map $\rho(\alpha, s) = (\alpha(s), p\alpha, s)$. An <u>extended lifting function</u> for p is the first component $\bar{\lambda}: \bar{\Lambda}_p \to E^I$ of some section $\sigma: \bar{\Lambda}_p \to E^I \times I$ of ρ; more explicitly, it is a map $\bar{\lambda}: \bar{\Lambda}_p \to E^I$ such that $\bar{\lambda}(e, \alpha, s)(s) = e$ and $p \circ \bar{\lambda}(e, \alpha, s) = \alpha$. If U is any subspace of B^I and $\bar{\Lambda}_p(U)$ denotes the subspace of $\bar{\Lambda}_p$ given by $\bar{\Lambda}_p(U) = \{(e, \alpha, s) \in \bar{\Lambda}_p : \alpha \in U\}$, then any map $\bar{\lambda}: \bar{\Lambda}_p(U) \to E^I$ such that $\bar{\lambda}(e, \alpha, s)(s) = e$ and $p \circ \bar{\lambda}(e, \alpha, s) = \alpha$ for every $(e, \alpha, s) \in \Lambda_p(U)$, is called an extended lifting function <u>over</u> U.

<u>Proposition 16</u> Every fibration $p: E \to B$ has an extended lifting function.

<u>Proof</u> For any path $\alpha: I \to B$ and $s \in I$ define new paths α_s and α^s by $\alpha_s(t) = \alpha(s-t)$, if $0 < t < s$, and $\alpha_s(t) = \alpha(0)$, if $s < t < 1$, $\alpha^s(t) = \alpha(s+t)$, if $0 < t \le 1-s$, and $\alpha^s(t) = \alpha(1)$, if $1-s < t \le 1$.



The functions $(\alpha, s) \in B^I \times I \to \alpha_s \in B^I$ and $(\alpha, s) \in B^I \times I \to \alpha^s \in B^I$ are continuous because their adjoints are continuous. In fact the adjoint of the first function is

$$(\alpha,s,t)\varepsilon B^I\times I\times I \to \begin{cases}\alpha(s-t) & \text{if } 0\leq t\leq s\\ \alpha(0) & \text{if } s\leq t\leq 1\end{cases} \varepsilon B$$

and if we consider the subspaces S_1 and S_2 of $B^I\times I\times I$ defined by $S_1=\{(\alpha,s,t)\varepsilon B^I\times I\times I:\ 0\leq t\leq s\}$ and $S_2=\{(\alpha,s,t)\varepsilon B^I\times I\times I:\ s\leq t\leq 1\}$, we have that they are closed, their union is $B^I\times I\times I$ and the above function restricted to S_1 is the composition $(\alpha,s,t)\varepsilon S_1 \to (\alpha,s,t)\varepsilon B^I\times I\times I \to (\alpha,|s-t|)\varepsilon B^I\times I \xrightarrow{\omega} \alpha(|s-t|)\varepsilon B$, which is continuous, and restricted to S_2 is the composition $(\alpha,s,t)\varepsilon S_2 \to (\alpha,s,t)\varepsilon B^I\times I\times I \to \alpha\varepsilon B^I \xrightarrow{\omega} \alpha(0)\varepsilon B$, which is continuous. In a similar manner is proved that the adjoint of $(\alpha,s)\varepsilon B^I\times I \to \alpha^s\varepsilon B^I$ is continuous.

Let $\lambda:\Lambda_p \to E$ be a lifting function for p. Since $\lambda(e,\alpha_s)(0)=\lambda(e,\alpha^s)(0)=e$, we can glue them together to get a function $\bar{\lambda}:\bar{\Lambda}_p \to E^I$ defined by $\bar{\lambda}(e,\alpha,s)(t)=\lambda(e,\alpha_s)(s-t)$, if $0\leq t\leq s$, and $\bar{\lambda}(e,\alpha,s)(t)=\lambda(e,\alpha^s)(t-s)$, if $s\leq t\leq 1$. We claim that $\bar{\lambda}$ is continuous; again we will prove this by taking the adjoint of $\bar{\lambda}$, which is

$$(e,\alpha,s,t)\varepsilon\bar{\Lambda}_p\times I \to \begin{cases}\lambda(e,\alpha_s)(s-t) & \text{if } 0\leq t\leq s\\ \lambda(e,\alpha^s)(t-s) & \text{if } s\leq t\leq 1\end{cases} \varepsilon E$$

Let S_1 and S_2 the subspaces of $\bar{\Lambda}_p\times I$ defined by $S_1=\{(e,\alpha,s,t)\varepsilon\bar{\Lambda}_p\times I:\ 0\leq t\leq s\}$ and $S_2=\{(e,\alpha,s,t)\varepsilon\bar{\Lambda}_p\times I:\ s\leq t\leq 1\}$;

they are closed and cover $\bar{\Lambda}_p \times I$. The restriction of the above function on S_1 is given by the following composition

$(e,\alpha,s,t)\varepsilon S_1 \to (e,\alpha,s,t)\varepsilon\bar{\Lambda}_p\times I \to (e,\alpha_s,|s-t|)\varepsilon\Lambda_p\times I \xrightarrow{\lambda\times 1_I}$

$(\lambda(e,\alpha_s),|s-t|)\varepsilon E^I\times I \xrightarrow{\omega} \lambda(e,\alpha_s)(|s-t|)\varepsilon E$, which is continuous; the restriction to S_2 is given by the following composition

$(e,\alpha,s,t)\varepsilon S_2 \to (e,\alpha,s,t)\varepsilon\bar{\Lambda}_p\times I \to (e,\alpha^s,|t-s|)\varepsilon\Lambda_p\times I \xrightarrow{\lambda\times 1_I}$

$(\lambda(e,\alpha^s),|t-s|)\varepsilon E^I\times I \xrightarrow{\omega} E$, which is also continuous.

Since an extended lifting function gives rise to a lifting function, by restriction, Proposition 14 shows that the converse of Proposition 16 is also true.

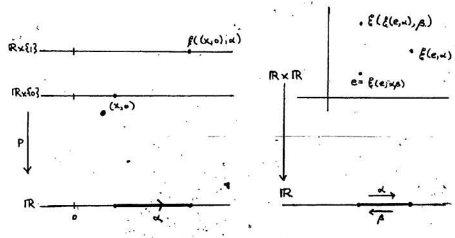
Beside the "global" lifting functions for p, there are what we can call the "end-point" lifting functions for p; both concepts are related to each other and sometimes the end-point liftings are a little more convenient since we have only to lift the end-point of the path. Given a map $p:E \to B$, an <u>end-point lifting function</u> for p is a map $\xi:\Lambda_p \to E$ such that $p\xi(e,\alpha)=\alpha(1)$. For each path α in B we can define the <u>translation map</u> along α $\xi_\alpha:F_{\alpha(0)} \to F_{\alpha(1)}$ by $\xi_\alpha(e)=\xi(e,\alpha)$. ξ is said to be <u>joinable</u> if there exists a homotopy $\Xi:\Lambda_p\times I \to E$ such that

$p\Xi((e,\alpha),t)=\alpha(t)$, $\Xi_0(e,\alpha)=e$ and $\Xi_1(e,\alpha)=\xi(e,\alpha)$; ξ is said <u>regular</u> if $\xi(e,p(e))=e$, that is, the translation maps along constant paths $\tilde{b}$, $\xi_{\tilde{b}}:F_b \to F_b$, are the identities; ξ is called transitive if for any $e\epsilon E$ and paths α and β in B such that $\alpha(0)=p(e)$ and $\alpha(1)=\beta(0)$ then $\xi(e,\alpha.\beta)=\xi(\xi(e,\alpha),\beta)$, that is, the translation maps along α,β and $\alpha.\beta$ are related by $\xi_{\alpha.\beta}=\xi_\beta\xi_\alpha$.

Regularity does imply joinability. In fact, if α_t $(0\leq t\leq 1)$ denotes the path obtained from α by putting $\alpha_t(s)=\alpha(ts)$, we can define a homotopy $\Xi:\Lambda_p\times I \to E$ by $\Xi(e,\alpha,t)=\xi(e,\alpha_t)$. Ξ is continuous, since it is the restriction to $\Lambda_p\times I$ of the following composition $(e,\alpha,t)\epsilon E\times B^I\times I \to (e,\alpha,f_t)\epsilon E\times B^I\times I^I \to (e,\alpha\circ f_t)\epsilon E\times B^I \to \xi(e,\alpha_t)\epsilon E$, where f_t is the map $s\epsilon I \to ts\epsilon I$. We have that $p\Xi(e,\alpha,t)= p\xi(e,\alpha_t)=\alpha_t(1)=\alpha(t)$, $\Xi_0(e,\alpha)=\Xi(e,\alpha,0)=\xi(e,\alpha_0)=\xi(e,p(e))=e$, since ξ is regular, and $\Xi_1(e,\alpha)=\Xi(e,\alpha,1)=\xi(e,\alpha_1)=\xi(e,\alpha)$.

Joinability does not imply regularity. For example take $E=R\times R$, $B=R$ and $p:E \to B$ given by $p(x,y)=x$; fix some $\bar{y}\epsilon R$ and define $\xi:\Lambda_p \to E$ by $\xi((x,y),\alpha)=(\alpha(1),\bar{y})$. Then ξ is joinable since there exists a homotopy $\Xi:\Lambda_p\times I \to E$ given by $\Xi((x,y),\alpha,t)=(\alpha,(t),(1-t)y+t\bar{y})$, but ξ is not regular.

Transitivity and joinability are independent concepts: in fact there are end-point lifting functions which are transitive and not joinable and viceversa. For example, take $E=R\times\{0,1\}$, $B=R$, $p(x,i)=x$ and $\xi((x,i),\alpha)=(\alpha(1),1)$ to illustrate the first situation and $E=R\times R$, $B=R$, $p(x,y)=x$ and $\xi((x,y),\alpha)=(\alpha(1),y+m(\alpha))$, with $m:\alpha\epsilon B^I \to |\alpha(1)-\alpha(0)|\epsilon R$, to illustrate the second situation.



Transitivity and regularity are also independent concepts. In fact the last example yields a regular but not transitive end-point lifting function and $E=R\times R$, $B=R$, $p(x,y)=x$ and $\xi((x,y),\alpha)=(\alpha(1),\bar{y})$, $\bar{y}\in R$ fixed, gives us a transitive but not regular end-point lifting function.

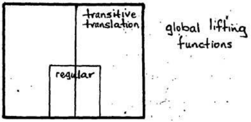
Now we will see how global and end-point lifting functions are related. Denote by G the set of all global lifting function for p, by T the set of all end-point lifting function for p and by G' and T' the subsets of G and T of those regular. There is a correspondence $f: G \to T$ sending

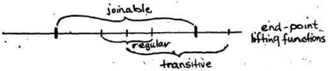
$\lambda \varepsilon G$ to $f(\lambda)$ defined by $f(\lambda)(e,\alpha)=\lambda(e,\alpha)(1)$ and a correspondence $g: T \to G$ sending $\xi \varepsilon T'(p)$ to $g(\xi)$ defined by $g(\xi)(e,\alpha)(t)=\xi(e,\alpha_t)$

Proposition 17 The following properties hold:

(i) $\xi \varepsilon \mathrm{Im} f$ if and only if ξ is joinable;

(ii) $f(\lambda)$ is regular if and only if $\lambda(e,p(e))(1)=e$; in particular $f(\lambda)$ is regular if λ is regular:





(iii) $f(\lambda)$ is transitive if and only if λ has transitive translation maps.

Proof (i) Let $\xi=f(\lambda)$ for some global lifting function $\lambda: \Lambda_p \to E^I$; then if $\Xi: \Lambda_p \times I \to E$ denotes the adjoint of λ, we have that $p\Xi(e,\alpha,t)=\alpha(t)$, $\Xi_0(e,\alpha)=e$ and $\Xi_1(e,\alpha)=\alpha(1)=\xi(e,\alpha)$ and hence ξ is joinable. Viceversa, if ξ is a joinable

end-point lifting function and $\Xi: \Lambda_p \times I \to E$ a homotopy such that $p\Xi(e,\alpha,t)=\alpha(t)$, $\Xi_0(e,\alpha)=e$ and $\Xi_1(e,\alpha)=\xi(e,\alpha)$ then the adjoint of $\Xi, \lambda: \Lambda_p \to E^I$, is such that $\xi=f(\lambda)$.
(ii) and (ii) are straightforward.

<u>Observation 18</u> If p admits a global lifting function λ such that $\lambda(e,p(e))(1)=e$ for every $e\varepsilon E$, then $gf(\lambda)$ is a regular global lifting function for p. Hence we can slightly weaken proposition 14 saying that p is a regular fibration if and only if it admits a (global) lifting function λ such that $\lambda(e,p(e))$ is a loop for every $e\varepsilon E$.

As a consequence of the proposition 17 and proposition 14 we have the following result.
<u>Proposition 19</u> $p:E \to B$ is a fibration if and only if it admits a joinable end-point lifting function and p is a regular fibration if and only if it admits a regular end-point lifting function.

There are other characterizations of a fibration, some of which are useful for generalizations in a categorical framework (cfr. [3],[25],[28],[29],[36] and [38]). Here we just mention one. Recall that a commutative square in any category C

$$\begin{tikzcd} A \arrow[r, "v_2"] \arrow[d, "u_1"'] & C \arrow[d, "v_2"] \\ C_1 \arrow[r, "v_1"] & B \end{tikzcd}$$

is said to be weak cartesian if for any pair of morphisms $f_1:X \to C_1$ and $f_2:X \to C_2$ such that $v_1f_1=v_2f_2$, there exists at least one morphism $f:X \to A$ such that $f_1=u_1f$ and $f_2=u_2f$; so, weak cartesian is cartesian without uniqueness.

Now, using the exponential correspondence, it is easy to see that a map $p:E \to B$ is a fibration if and only if the following commutative diagram in Top_*

$$\begin{tikzcd} E^I \arrow[r, "p_*"] \arrow[d, "\omega_0"'] & B^I \arrow[d, "\omega_0"] \\ E \arrow[r, "p"] & B \end{tikzcd}$$

is weak cartesian.

Although maps in general are not fibrations, there is a standard procedure for factorizing a given map $p:E \to B$ as a homotopy equivalence followed by a fibration. We factorize $p:E \to B$ as follows: let $\bar{p}:\Lambda_p \to B$ be defined by

$\bar{p}(e,\alpha)=\alpha(1)$ and let $\iota: E \to \Lambda_p$ be defined by $\iota(e)=(e,p(e))$, where $p(e)$ denotes the constant path at $p(e)$. Then $p=\bar{p}\iota$.

Proposition 20 For any given map $p: E \to B$, the map $\bar{p}: \Lambda_p \to B$ is a fibration and the "inclusion" map $\iota: E \to \Lambda_p$ is a homotopy equivalence.

Proof Consider the map $p\times 1_B: E\times B \to B\times B$ and let $\bar{\pi}: P \to E\times B$ be the pullback of the fibration $\pi: B^I \to B\times B$ along $p\times 1_B$; so $P=\{(e,b,\alpha)\varepsilon E\times B\times B^I: \alpha(0)=p(e)$ and $\alpha(1)=b\}$ and $\bar{\pi}: (e,b,\alpha)\varepsilon P \to (e,b)\varepsilon E\times B$. Then $\bar{\pi}$ is a fibration and hence the map $q=pr_2\circ\bar{\pi}: P \to B$ is also a fibration, being the composite of two fibrations.

$$\begin{tikzcd}
\Lambda_p \arrow[r, leftrightarrow] \arrow[d, "\bar{p}"'] & P \arrow[r] \arrow[d, "\bar{\pi}"] \arrow[dl, "q"'] & B^I \arrow[d, "\pi"] \\
B & E\times B \arrow[l, "pr_2"'] \arrow[r, "p\times 1_B"] & B\times B
\end{tikzcd}$$

But q is just the map $(e,b,\alpha)\varepsilon P \to \alpha(1)\,\varepsilon B$, since $b=\alpha(1)$, and so P is homeomorphic over B to Λ_p via the correspondence $(e,b,\alpha)\varepsilon P \leftrightarrow (e,\alpha)\varepsilon\Lambda_p$. This shows that $\bar{p}: \Lambda_p \to B$ is a fibration.

To prove that $\iota: E \to \Lambda_p$ is a homotopy equivalence, consider the map $\nu: (e,\alpha)\varepsilon\Lambda_p \to e\varepsilon E$. Then $\nu\iota=1_E$ and $\iota\nu\simeq 1_{\Lambda_p}$ via the homotopy $G: (e,\alpha,t)\varepsilon\Lambda_p\times I \to (e,\alpha_t)\varepsilon\Lambda_p$, where α_t is the path defined by $\alpha_t(s)=\alpha(s)$, if $0\leq s\leq t$, and $\alpha_t(s)=\alpha(t)$, if $t\leq s\leq 1$. G is continuous because the function $(\alpha,t)\in B^I\times I \to \alpha_t\in B^I$ is continuous; indeed, its adjoint is

$$(\alpha,t,s)\in B^I\times I\times I \to \begin{cases}\alpha(s) & \text{if } 0\leq s\leq t\\ \alpha(t) & \text{if } t\leq s\leq 1\end{cases}$$

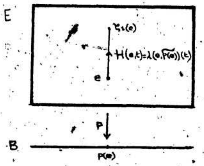
whose restriction on $B^I\times\{(t,s)\in I\times I:\ 0\leq s\leq t\}$ is the composition $(\alpha,t,s)\in B^I\times I\times I \to (\alpha,s)\in B^I\times I \xrightarrow{\omega} \alpha(s)\in B$ and whose restriction on $B^I\times\{(t,s)\in I\times I:\ t\leq s\leq 1\}$ is the composition $(\alpha,t,s)\in B^I\times I\times I \to (\alpha,t)\in B^I\times I \xrightarrow{\omega} \alpha(t)\in B$.

<u>Remark 21</u> Generally, the map ν is not a fibre map over B, the homotopy G is not vertical and p and $\bar{p}$ do not have the same fibre homotopy type over B. However, when p is a fibration it is possible to find a fibre homotopy inverse for ι. This is the statement of our next proposition. A generalization and improvement of this result will be given in the next section 3 (proposition 3.4).

<u>Proposition 22</u> If $p:E \to B$ is a fibration, the fibre map $\iota:E \to \Lambda_p$ is a fibre homotopy equivalence over B.

<u>Proof</u> Let $\lambda:\Lambda_p \to E^I$ be a lifting function for p and define $\zeta:\Lambda_p \to E$ as the composition $\Lambda_p \xrightarrow{\lambda} E^I \xrightarrow{\omega_1} E$, where ω_1 is the evaluation at 1; ζ is of course a fibre map over B because $p\zeta(e,\alpha)=p(\lambda(e,\alpha)(1))=\alpha(1)=\bar{p}(e,\alpha)$. We have that $\zeta\iota(e)=\lambda(e,\bar{p}(e))(1)$ and the map $H:E\times I \to E$ defined as

the adjoint of the composition $E \xrightarrow{\iota} \Lambda_p \xrightarrow{\lambda} E^I$, that is, $H(e,t)=\lambda(e,\widetilde{p(e)})(t)$, is a vertical homotopy from 1_E to ζ_ι.

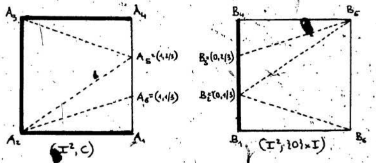


Looking at the composition $\iota\zeta$, we have that $\iota\zeta(e,\alpha)=(\lambda(e,\alpha)(1),p(\lambda(e,\alpha)(1)))$. A vertical homotopy $G:\Lambda_p\times I \to \Lambda_p$ from 1_{Λ_p} to $\iota\zeta$ can be constructed as follows. For any path $\alpha\varepsilon B^I$ and $t\varepsilon I$, let α_t be the path defined by $\alpha_t(s)=\alpha(s+t-st)$ (i.e. α_t is just the path α from $\alpha_t(0)=\alpha(t)$ to $\alpha_t(1)=\alpha(1)$) and define $G:\Lambda_p\times I \to \Lambda_p$ by the rule $G(e,\alpha,t)=(\lambda(e,\alpha)(t),\alpha_t)$. G is well defined, because $p(\lambda(e,\alpha)(t))=\alpha(t)=\alpha_t(0)$, continuous, because its components are continuous, satisfies the relations $G_0(e,\alpha)=G(e,\alpha,0)=(\lambda(e,\alpha)(0),\alpha_0)=(e,\alpha)$ and $G_1(e,\alpha)=G(e,\alpha,1)=(\lambda(e,\alpha)(1),\alpha_1)=(\lambda(e,\alpha)(1),p(\lambda(e,\alpha)(1)))=\iota\zeta(e,\alpha)$, and is vertical, because $\bar{p}(\lambda(e,\alpha)(t),\alpha_t)=\alpha_t(1)=\alpha(1)$.

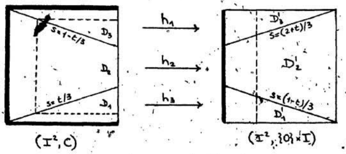
We now discuss a functor associated to any fibration and derive some important consequences. First we prove the so-called "C-lemma" (proposition 24). Let $C=I\times\dot{I}\cup\{0\}\times I$.

Lemma 23 There exists a relative homeomorphism $h:(I^2,C)\to(I^2,\{0\}\times I)$.

Proof Triangulate (I^2,C) and $(I^2,\{0\}\times I)$ as illustrated in the following pictures



Then the simplicial map $H:I^2\to I^2$ determined by $h(A_i)=B_i$ $i=1,\ldots,6$, is a relative homeomorphism between the pairs (I^2,C) and $(I^2,\{0\}\times I)$. More explicitly, let $D_1=\{(t,s)\in I^2:\ 0<s<t/3\}$, $D_2=\{(t,s)\in I^2:\ t/3<s<1-t/3\}$, $D_3=\{(t,s)\in I^2:\ 1-t/3<s<1\}$, $D_1'=\{(t,s)\in I^2:\ 0<s<(1-t)/3\}$, $D_2'=\{(t,s)\in I^2:\ (1-t)/3<s<(2+t)/3\}$ and $D_3'=\{(t,s)\in I^2:\ (2+t)/3<s<1\}$ and define homeomorphisms $h_1:D_1\to D_1'$, $h_2:D_2\to D_2'$ and $h_3:D_3\to D_3'$ by $h_1(t,s)=(3s,(1-t)/3)$, $h_2(t,s)=(t,[(1+2t)s+1-2t]/(3-2t))$ and $h_3(t,s)=(3-3s,(2+t)/3)$.



Then our previous h coincides with the map obtained by glueing together h_1, h_2 and h_3.

Proposition 24 ("C-lemma"). Let $p:E \to B$ be a fibration. Then for every map $J: X\times C \to E$ and for every homotopy $L: X\times I\times I \to B$ such that $pJ=L|X\times C$ there exists an extension of J $\bar{J}: X\times I\times I \to E$ lifting L.

Proof Let $h:(I^2,C) \to (I^2,\{0\}\times I)$ be a relative homeomorphism and consider the following commutative diagram (ignore the dotted arrow)

$$\begin{tikzcd}
X\times C \arrow[rr, "J"] \arrow[dr, "1_X\times h|C"] \arrow[ddd] & & E \arrow[ddd] \\
& X\times I \arrow[ur, "J'"] \arrow[d, "i_0"] & \\
& X\times I\times I \arrow[uur, dashed, "\tilde{L}"] \arrow[ddr, "L'"] & \\
X\times I\times I \arrow[ur, "1_X\times h"] \arrow[rr, "L"] & & B
\end{tikzcd}$$

where $J'=J\circ(1_X\times h|C)^{-1}$, $L'=L\circ(1_X\times h)^{-1}$ and $i_0(x,s)=(x,0,s)$. Since p' is a fibration, there exists a homotopy $\tilde{L}$ of J' lifting L'. The composition $\bar{J}=\tilde{L}\circ(1_X\times h)$ yields an extension of J lifting L.

We recall that the <u>fundamental groupoid</u> ΠB of a space B is the category whose objects are the points of B and whose morphisms from b to b' are the homotopy classes rel. $\dot{I}$ of paths having b as origin and b' as end. Composition of morphisms $[\alpha]:b\to b'$ and $[\beta]:b'\to b''$ is given by the rule $[\beta]\circ[\alpha]=[\alpha.\beta]$, where $\alpha.\beta$ denotes the usual product of unitary paths (i.e. first α and then β). ΠB is a groupoid in the usual categorical meaning, that is, every morphism is invertable, because $[\alpha^{-1}]\circ[\alpha]=[\alpha(0)]$.

Given a fibration $p:E\to B$ and an object $b\in|\Pi B|$, define $T_p(b)=F_b$. For a morphism $[\alpha]$ of ΠB we define $T_p([\alpha])$ as follows. Consider the commutative diagram

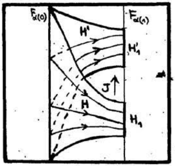
$$\begin{tikzcd}
F_{\alpha(0)} \arrow[rr, "i"] \arrow[d, "i_0"'] & & E \arrow[d, "p"] \\
F_{\alpha(0)}\times I \arrow[r, "pr_2"] & I \arrow[r, "\alpha"] & B
\end{tikzcd}$$

Since p is a fibration there exists a homotopy $H:F_{\alpha(0)}\times I \to E$ of the inclusion $i:F_{\alpha(0)} \to E$ lifting $\alpha\circ pr_2$. Restricting H to the top of the cylinder, we obtain a map $H_1:F_{\alpha(0)} \to F_{\alpha(1)}$. Define $T_p([\alpha])=[H_1]$.

<u>Proposition 25</u> For every fibration $p:E \to B$, $T_p:\Pi B \to \mathrm{HTop}$ defines a covariant functor from the fundamental groupoid of B to the homotopy category of topological spaces. Furthermore, if $p':E' \to B$ is a fibration and $f:E \to E'$ a fibre map over B, then f gives rise to a natural transformation $\phi_f:T_p \to T_{p'}$ defined by $\phi_f(b)=[f_b]$.

<u>Proof</u> We first show that if α and α' are paths in B homotopic rel.$\dot{I}$ and $H,H':F_{\alpha(0)}\times I \to E$ are homotopies of the inclusion map $F_{\alpha(0)} \to E$ lifting the composition $F_{\alpha(0)}\times I \xrightarrow{pr_2} I \xrightarrow{\alpha} B$ and $F_{\alpha(0)}\times I \xrightarrow{pr_2} I \xrightarrow{\alpha'} B$, respectively, then the maps $H_1,H_1':F_{\alpha(0)} \to F_{\alpha(1)}$ are homotopic. To this end, let $\Phi:I\times I \to B$ be a homotopy rel.$\dot{I}$ from α to α' and define the map $J:F_{\alpha(0)}\times C \to E$ by $J(e,t,0)=H(e,t)$, $J(e,t,1)=H'(e,t)$ and $J(e,0,s)=e$ and the map $L:F_{\alpha(0)}\times I\times I \to B$ by $L(e,t,s)=\Phi(t,s)$. Since $pJ=L|F_{\alpha(0)}\times C$, we can apply the C-lemma to obtain an extension of J $\bar{J}:F_{\alpha(0)}\times I\times I \to E$ lifting L. Now the map $K:F_{\alpha(0)}\times I \to F_{\alpha(1)}$ given by $K(e,s)=\bar{J}(e,1,s)$ is a homotopy from H_1 to H_1', as required.

E



B

It follows from the above observation that T_p is well defined; indeed, it shows that for every morphism [α] of ΠB the definition of $T_p([\alpha])$ is independent of the choice of the homotopy $H: F_{\alpha(0)}\times I \to E$ of the inclusion $F_{\alpha(0)} \to E$ lifting the composition $F_{\alpha(0)}\times I \xrightarrow{pr_2} I \xrightarrow{\alpha} B$ and of the choice of the representative in [α]. To show that T_p is a functor, that is, it preserves the identity morphisms and the composition law, we use again the above observation. Indeed, if $[\tilde{b}]: b \to b$ is the identity morphism of b, then in the definition of $T_p([\tilde{b}])$ we can take as representative of $[\tilde{b}]$ the constant path $\tilde{b}$ and as homotopy $H: F_b\times I \to E$ of the inclusion $F_b \to E$ lifting $F_b\times I \xrightarrow{pr_2} I \xrightarrow{\tilde{b}} B$ the composition

$F_b \times I \xrightarrow{pr_1} F_b \to E$, which gives $T_p([\tilde{b}])=[H_1]=[1_{F_b}]$, as required. For the composition law, let $[\alpha]:b \to b'$ and $[\beta]:b' \to b''$ be morphisms of ΠB, $H:F_{\alpha(0)} \times I \to E$ a homotopy of the inclusion $F_{\alpha(0)} \to E$ lifting the composition $F_{\alpha(0)} \times I \xrightarrow{pr_2} I \xrightarrow{\alpha} B$ and let $K:F_{\beta(0)} \times I \to E$ be a homotopy of the inclusion $F_{\beta(0)} \to E$ lifting the composition $F_{\beta(0)} \times I \xrightarrow{pr_2} I \xrightarrow{\beta} B$. Then define the map $G:F_{\alpha(0)} \times I \to E$ by

$$G(e,t)=\begin{cases} H(e,2t) & \text{if } 0<t<1/2 \\ K(H_1(e),2t-1) & \text{if } 1/2<t<1 \end{cases}$$

G is a homotopy of the inclusion $F_{\alpha(0)} \to E$ lifting the composition $F_{\alpha(0)} \times I \xrightarrow{pr_2} I \xrightarrow{\alpha.\beta} B$ and such that $G_1=K_1H_1$. Therefore $T_p([\beta]\circ[\alpha])=T_p([\alpha.\beta])=[G_1]=[K_1H_1]=[K_1][H_1]=T_p([\beta])T_p([\alpha])$.

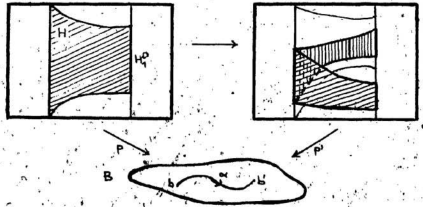
Now let $p':E' \to B$ be a fibration and let $f:E \to E'$ be a fibre map over B. We must prove that for every $b,b' \in B$ and morphism $[\alpha]:b \to B'$ in ΠB, the following diagram in HTop commutes

$$\begin{tikzcd}
F_b \arrow[r, "T_p([\alpha])"] \arrow[d, "\Phi_f(b)"'] & F_{b'} \arrow[d, "\Phi_f(b')"] \\
F'_b \arrow[r, "T_{p'}([\alpha])"] & F'_{b'}
\end{tikzcd}$$

By definition of T_p and Φ_f, this is equivalent to the following diagram in Top being homotopy commutative

$$\begin{tikzcd} F_b \arrow[r, "H_1"] \arrow[d, "f_b"'] & F_{b'} \arrow[d, "f_{b'}"] \\ F'_b \arrow[r, "H'_1"] & F'_{b'} \end{tikzcd}$$

where $H_1(e)=H(e,1)$ for some homotopy $H:F_b\times I \to E$ of the inclusion $F_b \to E$ lifting the composition $F_b\times I \xrightarrow{pr_2} I \xrightarrow{\alpha} B$ and $H'_1(e)=H'(e,1)$ for some homotopy $H':F'_b\times I \to E'$ of the inclusion $F'_b \to E'$ lifting the composition $F'_b\times I \xrightarrow{pr_2} I \xrightarrow{\alpha} B$. Define $J:F_b\times C \to E'$ by $J(e,t,0)=fH(e,t)$, $J(e,t,1)=H'(f(e),t)$ and $J(e,0,s)=f(e)$. Applying the C-lemma to J and $L:F_b\times I\times I \xrightarrow{pr_2} I \xrightarrow{\alpha} B$, we get an extension of J, $\bar{J}:F_b\times I\times I \to E'$, lifting L. The the map $K:F_b\times I \to F'_{b'}$ defined by $K(e,s)=\bar{J}(e,1,s)$ is a homotopy from $f_{b'}\circ H_1$ to $H'_1\circ f_b$, as required



<u>Corollary 26</u> Let $p:E \to B$ be a fibration. Then the fibres over points lying in the same path component of B have the same homotopy type.

<u>Proof</u> Since ΠB is a groupoid and any functor sends invertible morphisms to invertible morphisms, it follows that $T_p([\alpha])=[H_1]:F_b \to F_{b'}$ is an invertible morphism in HTop for any path α joining b to b'. This means that $H_1:F_b \to F_{b'}$ is a homotopy equivalence.

<u>Corollary 27</u> If $p:E \to B$ and $p':E' \to B$ are fibrations and $f:E \to E'$ is a fibre map over B such that $f_{\bar{b}}:F_{\bar{b}} \to F'_{\bar{b}}$ is a homotopy equivalence for some $\bar{b}\varepsilon B$, then $f_b:F_b \to F'_b$ is a homotopy equivalence for every b in the path component of B containing $\bar{b}$.

<u>Proof</u> Let α be a path joining b to $\bar{b}$. By proposition 25 we have the following commutative diagram in HTop

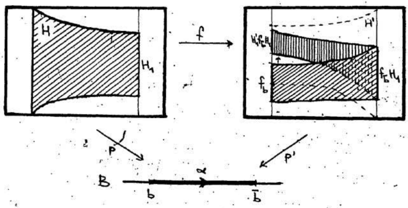
$$\begin{tikzcd}
F_b \arrow[r, "{[H_1]}"] \arrow[d, "{[f_b]}"'] & F_{\bar{b}} \arrow[d, "{[f_{\bar{b}}]}"] \\
F'_b \arrow[r, "{[H'_1]}"] & F'_{\bar{b}}
\end{tikzcd}$$

where, as usual, H_1 is defined from a homotopy $H:F_b\times I \to E$ of the inclusion $F_b \to E$ lifting the composition $F_b\times I \xrightarrow{pr_2} I \xrightarrow{\alpha} B$ and H'_1 is defined from a homotopy $H':F'_b\times I \to E'$ of the inclusion $F'_b \to E'$ lifting the composition $F'_b\times I \xrightarrow{pr_2} I \xrightarrow{\alpha} B$. Since $[H_1]$, $[H_1']$ and $[f_{\bar{b}}]$ are invertible morphisms of HTop, we deduce that $[f_b]$ must also be invertible, that is, f_b is a homotopy equivalence. Indeed, in any category a commutative diagram

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ Y \arrow[rr, "\psi"] & & Z \end{tikzcd}$$

with ϕ and ψ invertible, must have τ invertible, because $1_Y=\psi^{-1}\psi=\psi^{-1}1_Z\psi=\psi^{-1}\phi\phi^{-1}\psi=\psi^{-1}(\psi\tau)\phi^{-1}\psi=\tau(\phi^{-1}\psi)$ and $1_X=\phi^{-1}\phi=\phi^{-1}1_Z\phi=\phi^{-1}\psi\psi^{-1}\phi=\phi^{-1}\psi\psi^{-1}(\psi\tau)=(\phi^{-1}\psi)\tau$.

<u>Remark 28</u> Corollary 27 can be proved in an independent way as follows. Let $H:F_b\times I \to E$ be a homotopy of the inclusion $F_b \to E$ lifting the composition $F_b\times I \xrightarrow{pr_2} I \xrightarrow{\alpha} B$ and let $H':F'_{\bar{b}}\times I \to E'$ be a homotopy of the inclusion $F'_{\bar{b}} \to E'$ lifting the composition $F'_{\bar{b}}\times I \xrightarrow{pr_2} I \xrightarrow{\alpha^{-1}} B$. By hypothesis, the composition $F_b \xrightarrow{H_1} F_{\bar{b}} \xrightarrow{f_{\bar{b}}} F'_{\bar{b}} \xrightarrow{H_1'} F'_b$ is a homotopy equivalence.



Applying the C-lemma to the map $J:F_b\times C \to E'$ defined by $J(e,t,0)=fH(e,1-t)$, $J(e,t,1)=H'(f_{\bar{b}}H_1(e),t)$ and $J(e,0,s)=f_{\bar{b}}H_1(e)$ and to the map $L:F_b\times I\times I \to B$ defined by $L(e,t,s)=\alpha(1-t)$, we get an extension of J, $\bar{J}:F_b\times I\times I \to E'$, lifting L. Then the map $K:F_b\times I \to F'_b$ defined by $K(e,s)=\bar{J}(e,1,s)$ is a homotopy from f_b to $H'_1\circ f_{\bar{b}}\circ H_1$. Hence f_b is a homotopy equivalence.

The next proposition improves on proposition 14 and shows new examples of shrinkable maps. It is an observation contained in [12; p.166].

__Proposition 29__ If $p:E\to B$ is a fibration then the map $\rho:E^I \to \Lambda_p$ is shrinkable.

__Proof__ Let $\lambda:\Lambda_p \to E^I$ be a lifting function for p, that is, a section of ρ. Define $J:E^I\times C \to E$ by $J(\alpha,t,0)=\alpha(t)$, $J(\alpha,t,1)=\lambda(\alpha(0),p\alpha)(t)$ and $J(\alpha,0,s)=\alpha(0)$ and define $L:E^I\times I\times I \to B$ by

$L(\alpha,t,s)=p\alpha(t)$. Then the following diagram (ignore the dotted arrow)

$$\begin{tikzcd}
E^I\times G \arrow[r, "J"] \arrow[d] & E \arrow[d, "p"] \\
E^I\times I\times I \arrow[r, "L"] \arrow[ur, dashed, "\bar{J}"] & B
\end{tikzcd}$$

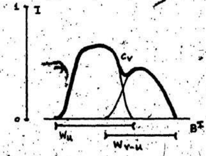
commutes and so, by proposition 24, there exists an extension of J, $\bar{J}:E^I\times I\times I \to E$, lifting L. The adjoint of $\bar{J}$ with respect to the variable t gives a vertical homotopy $K:E^I\times I \to E^I$ from the identity of E^I to $\lambda\rho$. Indeed, we have that $K(\alpha,0)(t)=\bar{J}(\alpha,t,0)=J(\alpha,t,0)=\alpha(t)$, $K(\alpha,1)(t)=\bar{J}(\alpha,t,1)=J(\alpha,t,1)=\lambda(\alpha(0),p\alpha)(t)=(\lambda\rho(\alpha))(t)$, and furthermore, using the relation $p\circ K(\alpha,s)=p\alpha$, obtained by observing that $(p\circ K(\alpha,s))(t)=p(K(\alpha,s)(t))=p\bar{J}(\alpha,t,s)=L(\alpha,t,s)=p\alpha(t)$, we have that $\rho(K(\alpha,s))=(K(\alpha,s)(0),p\circ K(\alpha,s))=(\bar{J}(\alpha,0,s),p\alpha)=(J(\alpha,0,s),p\alpha)=(\alpha(0),p\alpha)=\rho(\alpha)$, that is, K is vertical.

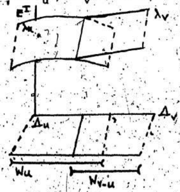
We now prove the Hurewicz Uniformization theorem stating that maps over "nice" base spaces which are locally fibrations are themselves fibrations. We first need the following results.

Lemma 30 Let $p:E \to B$ be a map. If there exists a numerable covering $\mathcal{W}=\{W_k \mid k\in K\}$ of B^I such that for each $k\in K$ there is an extended lifting function over W_k, then there is a lifting function for p.

<u>Proof</u> For any $k \varepsilon K$ let $\Lambda_k = \{(e,\alpha)\varepsilon\Lambda_p \mid \alpha\varepsilon W_k\}$ and $\bar{\Lambda}_k = \{(e,\alpha,s)\varepsilon\mathring{\bar{\Lambda}}_p \mid \alpha\varepsilon W_k\}$ and let $\bar{\lambda}_k : \bar{\Lambda}_k \to E^I$ be an extended lifting function over W_k. If $u \subseteq K$ is any subset, we define $W_u = \underset{k\varepsilon u}{\ } W_k$ (hence $W_{\emptyset} = \emptyset$); by proposition 1.11 (ii), W_u is a cozero set. For each $u \subseteq K$, we define a function $c_u : B^I \to R^+$ by $c_u(\alpha) = \sum_{k\varepsilon u} c_k(\alpha)$, if $u \neq \emptyset$, and $c_{\emptyset}(\alpha) = 0$; c_u is well defined, since is locally finite and continuous. Note that c_u is not in general a numeration of W_u, because for a given α it may happen that $c_u(\alpha) > 1$, but we still have that $c_u(\alpha) \neq 0$ if and only if $\alpha\varepsilon W_u$; moreover, c_u coincides with c_k if $u = \{k\}$. For any $u \subseteq K$ let $\Lambda_u = \{(e,\alpha)\varepsilon\Lambda_p ; \alpha\varepsilon W_u\}$.

Denote by L the set of all pairs (u,λ_u) with $u \subseteq K$ and $\lambda_u : \Lambda_u \to E^I$ a lifting function over W_u. L is not empty, because, for any $k\varepsilon K$, we have that $(\{k\},\lambda_k)\varepsilon L$, where λ_k denotes the lifting function over W_k induced by the extended lifting function $\bar{\lambda}_k$. Introduce an ordering on L by $(u,\lambda_u) < (v,\lambda_v)$ if and only if $u \subseteq v$ and $\lambda_u(e,\alpha) = \lambda_v(e,\alpha)$ for any $\alpha\varepsilon W_u$ with $c_u(\alpha) = c_v(\alpha)$. We observe that $c_u(\alpha) = c_v(\alpha)$ if and only if $\alpha\varepsilon W_u - W_{v-u}$.



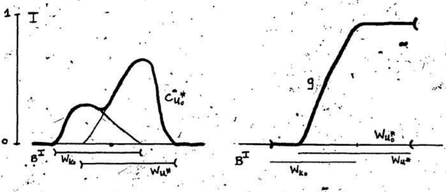


To check antisymmetry, suppose $(u,\lambda_u)\leq(v,\lambda_v)$ and $(v,\lambda_v)\leq(u,\lambda_u)$; it follows that $u=v$ and so $c_u(\alpha)=c_v(\alpha)$ for every $\alpha\in W_u$ and hence $\lambda_u(e,\alpha)=\lambda_v(e,\alpha)$. To check transitivity, let $(u,\lambda_u)\leq(v,\lambda_v)\leq(w,\lambda_w)$; observe that, since $u\subseteq v\subseteq w$, we have that $W_{v-u}\subseteq W_{w-u}$ and $W_{w-v}\subseteq W_{w-u}$ and hence $W_u-W_{w-u}\subseteq W_u-W_{v-u}$ and $W_u-W_{w-u}\subseteq W_v-W_{w-v}$. It follows that if $\alpha\in W_u-W_{w-u}$ then $\lambda_u(e,\alpha)=\lambda_v(e,\alpha)$ and $\lambda_v(e,\alpha)=\lambda_w(e,\alpha)$ and hence $\lambda_u(e,\alpha)=\lambda_w(e,\alpha)$; this proves that $(u,\lambda_u)\leq(w,\lambda_w)$.

We will prove now that any chain L_0 of L (i.e. any two elements of L_0 are related) has an upper bound $(\bar{u},\lambda_{\bar{u}})$ in L (i.e. $(u,\lambda_u)\leq(\bar{u},\lambda_{\bar{u}})$ for every (u,λ_u) in L_0). By abuse of notation we write $u\varepsilon L_0$ to mean that u is the first component of some element of L_0; we observe explicitly that if $u\varepsilon L_0$ then u is the first component of only one element of L_0, because if (u,λ_u) and (u,λ_u') belong to L_0, it follows that either $(u,\lambda_u)\leq(u,\lambda_u')$ or $(u,\lambda_u')\leq(u,\lambda_u)$ and in either case we have that $\lambda_u=\lambda_u'$. Define $\bar{u}=\bigcup_{u\varepsilon L_0} u$ and let $\alpha\varepsilon W_{\bar{u}}$. Since the family $\{W_k \mid k\varepsilon\bar{u}\}$ is in particular pointwise finite, the set $\{k\varepsilon\bar{u}\colon \alpha\varepsilon W_k\}$ is finite, say $\{k_1,\ldots,k_n\}$. It is easy to see that there is a $u\varepsilon L_0$ with $\{k_1,\ldots,k_n\}\subseteq u$; in fact, choose $u_1\varepsilon L_0$ with $k_1\varepsilon u_1,\ldots, u_n\varepsilon L_0$ with $k_n\varepsilon u_n$. Since L_0 is a chain, $\{u_1,\ldots,u_n\}$ has an upper bound in L_0; this will be our u. Now, define $\lambda_{\bar{u}}(e,\alpha)=\lambda_u(e,\alpha)$; $\lambda_{\bar{u}}$ is well defined, because if $u'\varepsilon L_0$ is another element containing $\{k_1,\ldots,k_n\}$, then either

$(u,\lambda_u)<(u',\lambda_{u'})$ or $(u',\lambda_{u'})<(u,\lambda_u)$ and in either case $\lambda_u(e,\alpha)=\lambda_{u'}(e,\alpha)$, since in the first case $\alpha\in W_u-W_{u'-u}$ and in the second case $\alpha\in W_{u'}-W_{u-u'}$. We want to prove that $\lambda_{\bar{u}}:\Lambda_{\bar{u}}\to E^I$ is continuous. Fix $\bar{\alpha}\in W_{\bar{u}}$ and let $V\subseteq W_{\bar{u}}$ be a neighbourhood of $\bar{u}$ such that $\{k\in\bar{u}: V\cap W_k\neq\emptyset\}$ is finite, say $\{k_1',\ldots,k_m'\}$. For any $\alpha\in V$ $\{k_1',\ldots,k_m'\}\subseteq\{k\in\bar{u}:\alpha\in W_k\}$ and so any $u\in L_0$ containing $\{k',\ldots,k_m'\}$ will contain $\{k\in\bar{u}:\alpha\in W_k\}$ for any $\alpha\in V$. Hence for any $\alpha\in V$ $\lambda_{\bar{u}}(e,\alpha)=\lambda_u(e,\alpha)$ and so continuity is proved. We want to show now that $(\bar{u},\lambda_{\bar{u}})$ is an upper bound for L_0, that is, $(u,\lambda_u)<(\bar{u},\lambda_{\bar{u}})$ for every $(u,\lambda_u)\in L_0$. Since $\bar{u}=\bigcup_{u\in L_0}u$, it follows that $u\supseteq\bar{u}$ for every $u\in L_0$; furthermore, if $\alpha\in W_u-W_{\bar{u}}$ it follows that $u\supseteq\{k\in\bar{u}:\alpha\in W_k\}$ and hence $\lambda_{\bar{u}}(e,\alpha)=\lambda_u(e,\alpha)$.

By Zorn's lemma, L has a maximal element (u^*,λ_{u^*}), that is, for every $(u,\lambda_u)\in L$, either $(u,\lambda_u)<(u^*,\lambda_{u^*})$ or they are not related. We claim that $u^*=K$. Suppose $k\in K-u^*$ and let $u_0^*=u^*\cup\{k_0\}$. Define a map $g:W_{u_0^*}\to I$ by $g(\alpha)=c_{u^*}(\alpha)/c_{u_0^*}(\alpha)$; then $\alpha\in W_{u^*}$ if and only if $g(\alpha)\neq 0$ and $\alpha\in W_{k_0}$ if and only if $g(\alpha)\neq 1$.



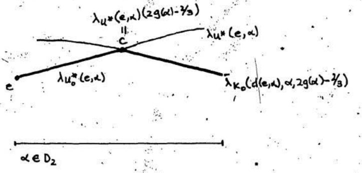
Let $D_1=\{\alpha\varepsilon W_{u_0^*}: 0\leq g(\alpha)\leq 1/3\}$, $D_2=\{\alpha\varepsilon W_{u_0^*}: 1/3\leq g(\alpha)\leq 2/3\}$ and $D_3=\{\alpha\varepsilon W_{u_0^*}: 2/3\leq g(\alpha)\leq 1\}$; D_1, D_2 and D_3 are closed subspaces of $W_{u_0^*}$ and cover it. Let $\Lambda_i=\{(e,\alpha)\varepsilon\Lambda_{u_0^*}:\alpha\varepsilon D_i\}$, $i=1,2,3$, so $\Lambda_1, \Lambda_2, \Lambda_3$ cover $\Lambda_{u_0^*}$ and define $d_1:\Lambda_1\to E$ by $d_1(e,\alpha)=e$, $d_2:\Lambda_2\to E$ by $d_2(e,\alpha)=\lambda_{u^*}(e,\alpha)(2g(\alpha)-2/3)$ (i.e. d_2 is the composition

$$(e,\alpha)\varepsilon\Lambda_2 \xrightarrow{(\lambda_{u^*},pr_2)} (\lambda_{u^*}(e,\alpha),\alpha)\varepsilon E^I\times D_2 \xrightarrow{1\times g}$$
$$(\lambda_{u^*}(e,\alpha),g(\alpha))\varepsilon E^I\times I \xrightarrow{1\times h} (\lambda_{u^*}(e,\alpha),h(g(\alpha)))=$$
$$(\lambda_{u^*}(e,\alpha),2g(\alpha)-2/3)\varepsilon E^I\times I \xrightarrow{\omega} \lambda_{u^*}(e,\alpha)(2g(\alpha)-2/3)\varepsilon E$$

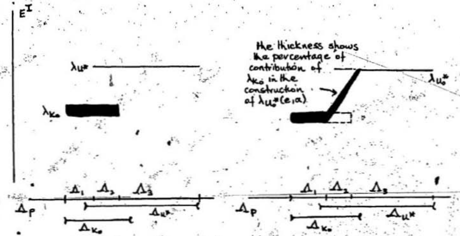
where $h(t)=(2t-2/3)$ and $d_3:\Lambda_3\to E$ by $d_3(e,\alpha)=\lambda_{u^*}(e,\alpha)(g(\alpha))$. d_1, d_2, d_3 give rise to a well defined and continuous function $d:W_{u_0^*}\to E$. Let $\bar{\lambda}_{k_0}:\bar{\Lambda}_{k_0}\to E^I$ be an extended lifting function for W_{k_0} and define $\lambda_{u_0^*}:\Lambda_{u_0}\to E^I$ by

$$\lambda_{u_0^*}(e,\alpha)(t)=\begin{cases}\lambda_{k_0}(e,\alpha,0)(t) & \text{if}(e,\alpha)\varepsilon\Lambda_1\\ \left.\begin{matrix}\lambda_{u^*}(e,\alpha)(t) & \text{if } 0<t<2g(\alpha)-\frac{2}{3}\\ \bar{\lambda}_{k_0}(d(e,\alpha),\alpha,2g(\alpha)-\frac{2}{3})(t) & \text{if } 2g(\alpha)-\frac{2}{3}<t<1\end{matrix}\right\} & \text{if}(e,\alpha)\varepsilon\Lambda_2\\ \left.\begin{matrix}\lambda_{u^*}(e,\alpha)(t) & \text{if } 0<t<g(\alpha)\\ \bar{\lambda}_{k_0}(d(e,\alpha),\alpha,g(\alpha))(t) & \text{if } g(\alpha)<t<1\end{matrix}\right\} & \text{if } (e,\alpha)\varepsilon\Lambda_3\end{cases}$$

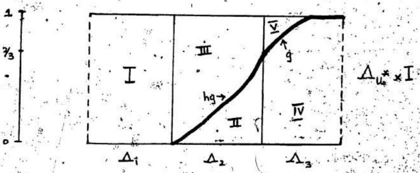
We can describe the construction of $\lambda_{u_0^*}$ by saying that if $(e,\alpha)\varepsilon\Lambda_1$ then its lifting is obtained by using the extended lifting function $\bar{\lambda}_{k_0}$ on W_{k_0}; if $(e,\alpha)\varepsilon\Lambda_2$ then $\alpha\varepsilon W_{u^*}\cap W_{k_0}$ and so we first lift α by λ_{u^*}, cut this lifting at the instant $t=2g(\alpha)-2/3$ and the glue it to that portion of the lifting of α by $\bar{\lambda}_{k_0}$ commencing at time $t=2g(\alpha)-2/3$.



If $(e,\alpha)\varepsilon\Lambda_3$ then $\alpha\varepsilon W_{u^*}$ and so we first lift α by λ_{u^*}, cut this lifting at the instant $t=g(\alpha)$ and then glue it to that portion of the lifting of α by $\bar{\lambda}_{k_0}$ commencing at time $t=g(\alpha)$. when $g(\alpha)=1$ we have that $\lambda_{u\gamma}(e,\alpha)=\lambda_{u*}(e,\alpha)$. So, using the extended lifting function $\bar{\lambda}_{k_0}$ over W_{k_0}, the lifting function λ_{u*} over W_{u*} and the numerical function g, we are able to connect the lifting functions λ_{k_0} on Λ_{k_0} and λ_{u*} on Λ_{u*} in such a way that this enlarged lifting function coincides with λ_{u*} on $\Lambda_{u\gamma}-\Lambda_{k_0}$ and λ_{k_0} on $\Lambda_{u\gamma}-\Lambda_{u*}$



The continuity of $\lambda_{u\gamma}$ is proved by checking the continuity of its adjoint $\Lambda_{u\gamma}\times I \to E$ and this is proved by subdividing $\Lambda_{u\gamma}\times I$ in the following five regions

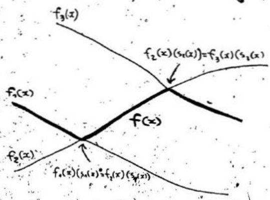


Here $h:I \to I$ is, as above, $h(t)=2t-2/3$. If $\alpha \in W_{u^\#}-W_{u^\#-u^*}=$ $W_{u^\#}-W_{k_0}$ then $g(\alpha)=1$ and so $\lambda_{u^\#}(e,\alpha)=\lambda_{u^*}(e,\alpha)$. Hence $(u^*,\lambda_{u^*})<(u^\#,\lambda_{u^\#})$, contradicting the maximality of (u^*,λ_{u^*}).

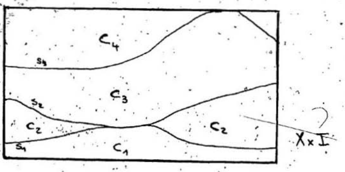
Lemma 31. Let $f_1,\dots,f_n:X \to E^I$ and $s_1,\dots,s_{n-1}:X \to I$ be maps such that $0<s_1(x)<\dots<s_{n-1}(x)<1$, for every $x \in X$, and $f_h(x)(s_h(x))=f_{h+1}(x)(s_h(x))$, for every $h=1,\dots,n-1$ and $x \in X$. Then the function $f:X \to E^I$ defined by

$$f(x)(t)=\begin{cases} f_1(x)(t) & \text{if } 0\le t\le s_1(x) \\ f_2(x)(t) & \text{if } s_1(x)\le t\le s_2(x) \\ \dots\dots & \dots\dots\dots\dots \\ f_n(x)(t) & \text{if } s_{n-1}(x)\le t\le 1 \end{cases}$$

is continuous.



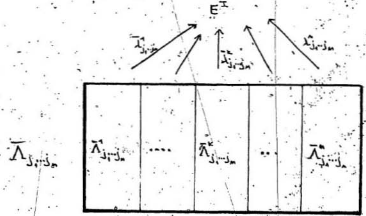
<u>Proof</u> Consider the adjoint of f and the n closed subsets of $X \times I$, $C_1=\{(x,t)\,\varepsilon\, X\times I:\ 0\leq t\leq s_1(x)\}$, $C_h=\{(x,t)\,\varepsilon\, X\times I:\ s_{h-1}(x)\leq t\leq s_h(x)\}$, if $h=2,\ldots,n-1$, and $C_n=\{(x,t)\,\varepsilon\, X\times I:\ s_{n-1}(x)\leq t\leq 1\}$. Now observe that the adjoint of f restricted to each C_h is equal to the restriction to C_h of the adjoint of f_h. Therefore, the adjoint of f is continuous and so f is continuous.



Hurewicz Uniformization Theorem 32 Let $p:E \to B$ be a map and assume that there is a numerable covering $\mathcal{U}=\{U_j \mid j \varepsilon J\}$ such that each map $p_{U_j}:E_{U_j} \to U_j$ is a (regular) fibration. Then p is a (regular) fibration.

Proof. The idea of the proof is to deduce from the given data the existence of a numerable covering of B^I with an extended lifting function over each element of the cover, and then to invoke Lemma 30 to deduce the existence of a global lifitng function for p.

For notational convenience let $E_j=E_{U_j}$, $p_j=p_{U_j}$, $\Lambda_j=\{(e,\alpha)\varepsilon\Lambda_p: \alpha(I)\subseteq U_j\}$ and $\bar{\Lambda}_j= \{(e,\alpha,s)\varepsilon\bar{\Lambda}_p: \alpha(I)\ U_j\}$. For each $j\varepsilon J$, let $\bar{\lambda}_j:\bar{\Lambda}_j \to E^I$ be an extended lifting function for p_j. Such functions do exist since, by hypothesis, each p_j is a fibration. Now, for any n-tuple $(j_1,\ldots,j_n)$ of indeces of J not necessarily distinct $(n=1,2\ldots)$, define $\bar{U}_{j_1\ldots j_n}=\{\alpha\varepsilon B^I: \alpha([(k-1)/n,k/n])\subseteq U_{j_k};\ k=1,\ldots,n\}$ and $\bar{\Lambda}_{j_1\ldots j_n}=\{(e,\alpha,s)\varepsilon\bar{\Lambda}_p:\alpha\varepsilon\bar{U}_{j_1\ldots j_n}\}$. $\bar{U}_{j_1\ldots j_n}$ is a cozero set of B^I because $\bar{U}_{j_1\ldots j_n}=\langle[0,1/n],U_{j_1}\rangle\cap\ldots\cap\langle[(n-1)/n,1],U_{j_n}\rangle$ and each subbasic set is a cozero set by proposition 1.11(iii). Using the extended lifting functions $\bar{\lambda}_{j_1},\ldots,\bar{\lambda}_{j_n}$ over $\bar{U}_{j_1},\ldots,\bar{U}_{j_n}$, respectively, we will construct an extended lifting function $\bar{\lambda}_{j_1\ldots j_n}:\bar{\Lambda}_{j_1\ldots j_n} \to E^I$ over $\bar{U}_{j_1\ldots j_n}$. This is done by subdividing $\bar{\Lambda}_{j_1\ldots j_n}$ into n closed (in $\bar{\Lambda}_{j_1\ldots j_n}$) subsets $\bar{\Lambda}^k_{j_1\ldots j_n}=\{(e,\alpha,s)\in\bar{\Lambda}_{j_1\ldots j_n}:(k-1)/n\leq s\leq k/n\}$ $k=1,\ldots,n$ and defining continuous functions $\bar{\lambda}^k_{j_1\ldots j_n}:\bar{\Lambda}^k_{j_1\ldots j_n}\to E^I$ such that $p(\bar{\lambda}^k_{j_1\ldots j_n}(e,\alpha,s)(t))=\alpha(t)$, $\bar{\lambda}^k_{j_1\ldots j_n}(e,\alpha,s)(s)=e$ and $\bar{\lambda}^k_{j_1\ldots j_n}(e,\alpha,k/n)=\bar{\lambda}^{k+1}_{j_1\ldots j_n}(e,\alpha,k/n)$. Setting $\bar{\lambda}_{j_1\ldots j_n}(e,\alpha,s)=\bar{\lambda}^k_{j_1\ldots j_n}(e,\alpha,s)$, where k is such that $s\in[(k-1)/n,k/n]$, we get the required extended lifting function



We start by defining for any path $\alpha\in B^I$ and natural numbers n and k, with $k<n$, the path

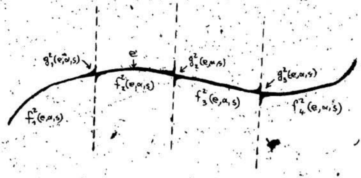
$$\alpha_{n,k}(t)=\begin{cases}\alpha((k-1)/n) & \text{if } 0<t<(k-1)/n\\ \alpha(t) & \text{if } (k-1)/n<t<k/n\\ \alpha(k/n) & \text{if } k/n<t<1\end{cases}$$

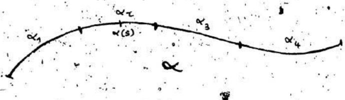
The function $\alpha\varepsilon B^I \to \alpha_{n,k}\varepsilon B^I$ is continuous. Indeed, its adjoint is continuous because restricted on $B^I\times[0,(k-1)/n]$ is equal to the composition $e\ B^I\times[0,(k-1)/n] \xrightarrow{pr_1} B^I \xrightarrow{\omega_{(k-1)/n}} I$, restricted on $B^I\times[(k-1)/n,k/n]$ is equal to the composition $B^I\times[(k-1)/n,k/n[\to B^I\times I \xrightarrow{\omega} I$ and restricted on $B^I\times[k/n,1]$ is equal to the composition $e\ B^I\times[k/n,1] \xrightarrow{pr_1} B^I \xrightarrow{\omega_{k/n}} I$. When it is clear from the context, we will drop n in $\alpha_{n,k}$ and simply write α_k. To each triple $(e,\alpha,s)\varepsilon\bar{\Lambda}^k_{j_1\ldots j_n}$ we will associate n paths in E denoted by $f^k_h(e,\alpha,s)$, $h=1,\ldots,n$. Cutting these paths at certain intersection points and glueing in a prescribed manner will give us the required path $\bar{\lambda}^k_{j_1\ldots j_n}(e,\alpha,s)$. For $h=k$ define $f^k_k(e,\alpha,s)=\bar{\lambda}_{j_k}(e,\alpha_k,s)$. This definition makes sense because $\alpha_k(I)\subseteq U_{j_k}$ and $p(e)=\alpha(s)=\alpha_k(s)$. We have that $p(f^k_k(e,\alpha,s)(t))=\alpha(t)$, if $t\varepsilon[(k-1)/n,k/n]$, and furthermore the function $f^k_k:\bar{\Lambda}^k_{j_1\ldots j_n} \to E^I$ is continuous because it is equal to the composition $(e,\alpha,s)\varepsilon\bar{\Lambda}^k_{j_1\ldots j_n} \to$

$(e,\alpha_k,s)\in\bar{\lambda}_{j_k} \xrightarrow{\bar{\lambda}_{j_k}} E^I$. For h=k-1, first define $g^k_{k-1}(e,\alpha,s)=$ $f^k_k(e,\alpha,s)((k-1)/n)$; the function $g^k_{k-1}:\bar{\lambda}^k_{j_1\ldots j_n}\to E$ is continuous and furthermore $pg^k_{k-1}(e,\alpha,s)=p(f^k_k(e,\alpha,s)((k-1)/n)=$ $\alpha((k-1)/n)=\alpha_{k-1}((k-1)/n)$. Now define $f^k_{k-1}(e,\alpha,s)=$ $\bar{\lambda}_{j_{k-1}}(g^k_{k-1}(e,\alpha,s),\alpha_{k-1},(k-1)/n)$. Then $p(f^k_{k-1}(e,\alpha,s)(t))=\alpha(t)$, if $t\in[(k-2)/n,(k-1)/n]$, and $f^k_{k-1}(e,\alpha,s)((k-1)/n)=g^k_{k-1}(e,\alpha,s)=$ $f^k_k(e,\alpha,s)((k-1)/n)$. The function $f^k_{k-1}:\bar{\lambda}^k_{j_1\ldots j_n}\to E^I$ is continuous because it is equal to the composition

$$(e,\alpha,s)\in\bar{\lambda}^k_{j_1\ldots j_n}\to(g^k_{k-1}(e,\alpha,s),\alpha_{k-1},(k-1)/n)\in\bar{\lambda}_{j_{k-1}}\xrightarrow{\bar{\lambda}_{j_{k-1}}}E^I.$$

Proceeding by induction on decreasing values of k, we define $f^k_h(e,\alpha,s)$ for all h<k-1. For h=k+1 we set $g^k_k(e,\alpha,s)=$ $f^k_k(e,\alpha,s)(k/n)$ and define $f^k_{k+1}(e,\alpha,s)=$ $\bar{\lambda}_{j_{k+1}}(g^k_k(e,\alpha,s),\alpha_{k+1},k/n)$. As above $f^k_{k+1}:\bar{\lambda}^k_{j_1\ldots j_n}\to E^I$ is continuous and we have that $p(f^k_{k+1}(e,\alpha,s)(t))=\alpha(t)$, if $t\in[k/n,(k+1)/n]$, and $f^k_k(e,\alpha,s)(k/n)=g^k_k(e,\alpha,s)=$ $f^k_{k+1}(e,\alpha,s)(k/n)$. Proceeding now by induction on increasing values of h, we define $f^k_h(e,\alpha,s)$ for all h>k+1. So we have constructed n continuous functions $f^k_1,\ldots,f^k_n:\bar{\lambda}^k_{j_1\ldots j_n}\to E^I$ such that $p(f^k_h(e,\alpha,s)(t))=\alpha(t)$, if $t\in[(h-1)/n,h/n]$, and $f^k_h(e,\alpha,s)(h/n)=f^k_{h+1}(e,\alpha,s)(h/n)$.





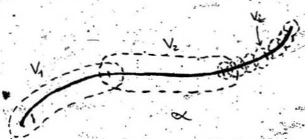
Applying lemma 30 to $f_1^k,\dots,f_n^k$ with $s_1(t)=1/n,\dots,s_{n-1}(t)=(n-1)/n$, we get a continuous function $\bar\lambda^k_{j_1\dots j_n}:\bar\Lambda^k_{j_1\dots j_n}\to E^I$ defined by

$$\bar\lambda^k_{j_1\dots j_n}(e,\alpha,s)(t)=\begin{cases} f_1^k(e,\alpha,s)(t) & \text{if } 0<t<1/n \\ \dots\dots\dots\dots\dots\dots & \\ f_h^k(e,\alpha,s)(t) & \text{if } (h-1)/n<t<h/n \\ \dots\dots\dots\dots\dots\dots & \\ f_n^k(e,\alpha,s)(t) & \text{if } (n-1)/n<t<n \end{cases}$$

It remains to show that $\bar\lambda^k_{j_1\dots j_n}(e,\alpha,k/n)=\bar\lambda^{k+1}_{j_1\dots j_n}(e,\alpha,k/n)$ when $k=1,\dots,n-1$. This will be proved by showing that, once k is fixed, $f_h^k(e,\alpha,k/n)=f_h^{k+1}(e,\alpha,k/n)$ for every $h=1,\dots,n$. For $h=k,k+1$ we have that $f_k^{k+1}(e,\alpha,k/n)=$ $\bar\lambda_{j_k}(g_k^{k+1}(e,\alpha,k/n),\alpha_k,k/n)=\bar\lambda_k(e,\alpha_k,k/n)=f_k^k(e,\alpha,k/n)$ and

$f^k_{k+1}(e,\alpha,k/n)=\bar{\lambda}_{j_{k+1}}(g^k_k(e,\alpha,k/n),\alpha_{k+1},k/n)=\bar{\lambda}_{j_{k+1}}(e,\alpha_{k+1},k/n)=$ $f^{k+1}_{k+1}(e,\alpha,k/n)$. Now assume by induction on decreasing values of h that $f^k_h(e,\alpha,k/n)=f^{k+1}_h(e,\alpha,k/n)$ for $h<k$; then $g^k_{h-1}(e,\alpha,k/n)=f^f_h(e,\alpha,k/n)((h-1)/n)=f^{k+1}_h(e,\alpha,k/n)((h-1)/n)=$ $g^{k+1}_{h-1}(e,\alpha,k/n)$ and hence $f^k_{h-1}(e,\alpha,k/n)=$ $\bar{\lambda}_{j_{h-1}}(g^k_{h-1}(e,\alpha,k/n),\alpha_{h-1},(h-1)/n)=$ $\bar{\lambda}_{j_{h-1}}(g^{k+1}_{h-1}(e,\alpha,k/n),\alpha_{h-1},(h-t)/n)=f^{k+1}_{h-1}(e,\alpha,k/n)$. In a similiar manner using induction on increasing values of h it can be shown that $f^k_h(e,\alpha,k/n)=f^{k+1}_h(e,\alpha,k/n)$ for all $h>k+1$.

We now study covering properties of the $\bar{U}_{j_1\ldots j_n}$'s. For any $\alpha\in B^I$ there exists some $n\in N$ and an n-tuple $(j_1,\ldots,j_n)\in J^n$ such that $\alpha\in\bar{U}_{j_1\ldots j_n}$; indeed, consider the open cover of I given by $\{\alpha^{-1}(U_j)|j\in J\}$, and let $\{\alpha^{-1}(U_{j_1}),\ldots,\alpha^{-1}(U_{j_m})\}$ be a finite subcovering with Lebesgue number $\varepsilon>0$. Then is for every $t\in I$ there exists some $k\in\{1,\ldots,m\}$ such that $]t-\varepsilon,t+\varepsilon[\cap I\subseteq\alpha^{-1}(U_{j_k})$. If $n\in N$ is such that $1/n<\varepsilon$, then for each $k=1,\ldots,n$ we have that $[(k-1)/n,k/n]\subseteq](2k-1)/2n)-\varepsilon,(2k-1)/2n)+\varepsilon]$ and hence for there is some element of $\{j_1,\ldots,j_m\}$, say j'_k, such that $[k-1)/n,k/n]\subseteq\alpha^{-1}(U_{j'_r})$, that is, $\alpha([(k-1)/n,k/n])\subseteq U_{j'_k}$. Therefore $\alpha\in\bar{U}_{j'_1\ldots j'_n}$. Let now $\bar{\mathcal{U}}_n=\{\bar{U}_{j_1\ldots j_n}|(j_1,\ldots,j_n)\in J_n\}$. $\bar{\mathcal{U}}_n$ is a locally finite family of cozero sets of B^I. This can be seen as follows. Let $\alpha\varepsilon B$ be a fixed path and for any $t\varepsilon[0,1]$ let V_t be an open neighbourhood of $\alpha(t)$ such that $\{j\varepsilon J\colon V_t \cap U_j \neq \emptyset\}$ is finite; such a neighbourhood does exist since $\mathcal{U}$ is locally finite. Consider the open covering of $\alpha([0,1/n])$ given by $\{V_t \mid 0<t<1/n\}$. Since $\alpha([0,1/n])$ is compact, we can find a finite subcovering, say $\{V_{t_1},\ldots,V_{t_n}\}$; let $V_1 = V_{t_1} \cap \ldots \cap V_{t_n}$ and observe that $V_1 \supseteq \alpha([0,1/n])$ and that $J_1 = \{j\varepsilon J\colon V_1 \cap U_j \neq \emptyset\} = \bigcup_{k=1}^{n} \{j\varepsilon J\colon V_{t_k} \cap U_j \neq \emptyset\}$ is finite. In a similar way we can find for every $k=2,\ldots,n$ an open set $V_k \supseteq \alpha([(k-1)/n,k/n])$ with $J_k = \{j\varepsilon J\colon V_k \cap U_j \neq \emptyset\}$ finite.



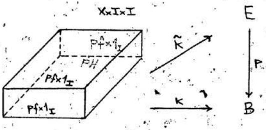
Define $V = \langle[0,1/n],V_1\rangle \cap \ldots \cap \langle[(n-1)/n,1],V_n\rangle$; V is a neighbourhood of α and we will prove that it meets at most a finite number of elements of $\bar{\mathcal{U}}_n$. Suppose that $\bar{U}_{j_1\ldots j_n}\cap V\neq\emptyset$; then there exists some $\beta\in B^I$ with $\beta([(k-1)/n,k/n])\subseteq U_{j_k}$ and $\beta([(k-1)/n,k/n])\subseteq V_k$ for every $k=1,\ldots,n$. Hence, for every $k=1,\ldots,n$, $U_{j_k}\cap V_k\neq\emptyset$ and so $j_k\in J_k$; this implies that $(j_1,\ldots,j_n)\in J_1\times\ldots\times J_n$ and hence the set $\{(j_1,\ldots,j_n)\in J^n: \bar{U}_{j_1\ldots j_n}\cap V\neq\emptyset\}\subseteq J_1\times\ldots\times J_n$ is finite.

Since $\{\bar{\mathcal{U}}_n|n\in\mathbb{N}\}$ is a sequence of locally finite families of cozero sets of B^I such that the family $\bar{\mathcal{U}}=\{\bar{U}_{j_1\ldots j_n}: (j_1,\ldots,j_n)\in J^\infty=\bigcup_{n\in\mathbb{N}} J^n\}$ covers B^I, we can use proposition 12(ii) to deduce the existence of a locally finite refinement $\mathcal{W}=\{W_{j_1\ldots j_n}|(j_1,\ldots,j_n)\in J^\infty\}$ of $\bar{\mathcal{U}}$ by cozero sets. Since each $W_{j_1\ldots j_n}$ is contained in $\bar{U}_{j_1\ldots j_n}$ and we have shown the existence of an extended lifting function over each $\bar{U}_{j_1\ldots j_n}$, it follows that there exists an extended lifting function over each $W_{j_1\ldots j_n}$. Applying lemma 30, we deduce the existence of a lifting function for p.

The following result generalizes proposition 12(i).

<u>Proposition 33</u> Let $p:E\to B$ be a fibration and let $f,f':X\to E$ be two maps such that $pf=pf'$ and such that there exists a homotpy H from f to f' whose projection on B is homotopic rel. $X\times\dot{I}$ to the homotopy stationary at pf. Then there exists a vertical homotopy from f to f'.

<u>Proof</u> Let $K: X\times I\times I \to B$ be a homotopy such that $K(x,t,0)=$ $K(x,t,0)=pH(x,t)$ and $K(x',t,1)=K(x,0,s)=K(x,1,s)=pf(x)$. Since p is a fibration there exists a lifting of K $\tilde{K}: X\times I\times I \to E$ with $\tilde{K}(x,t,0)=H(x,t)$



The restrictions of $\tilde{K}$ to the faces $X\times\{0\}\times I$, $X\times I\times\{1\}$ and $X\times\{1\}\times I$ give vertical homotopies because their projections on B are stationary at pf. Hence the homotopy $H': X\times I \to E$ defined by

$$H'(x,t) = \begin{cases} \tilde{K}(x,0,3t) & \text{if } 0\le t\le 1/3 \\ \tilde{K}(x,3t-1,1) & \text{if } 1/3\le t\le 2/3 \\ \tilde{K}(x,1,-3t+3) & \text{if } 2/3\le t\le 1 \end{cases}$$

is vertical with $H_0'=f$ and $h_1'=f'$.

As an application of lemma 33, we prove the following proposition which appears in a paper by James and Thomas [27].

<u>Proposition 34</u> Let $p:E \to B$ be a fibration. Then any two sections $s,s':B \to E$ of p which are homotopic are also

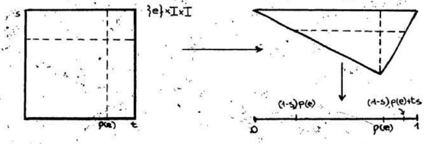
vertically homotopic.

Proof Let $H:B\times I \to E$ be a homotopy from s to s'. Then $spH:B\times I \to E$ is a homotopy from s to s and moreover $(spH)^{-1}.H$ is a homotopy from s to s', having as projection on B the homotopy $(pH)^{-1}.(pH)$. Therefore we can apply proposition 33 to deduce the existence of a vertical homotopy from s to s'.

Given a map $p:E \to B\times[0,1]$, let $E^i=p^{-1}(B\times\{i\})$ and let $p^i:E^i \to B$ be defined by $p^i(e)=pr_1(p(e))$.

Theorem 35 If $p:E \to B\times[0,1]$ is a fibration, then p^0 and p^1 have the same fibre homotopy type over B.

Proof Let $\pi:E \to B$ and $\rho:E \to [0,1]$ be the compositions of p with the projections maps, so that $p(e)=(\pi(e),\rho(e))$. Consider the map $L:E\times I\times I \to B\times I$ defined by $L(e,t,s)=(\pi(e),(1-s)\rho(e)+ts)$. On $\{e\}\times I\times\{0\}$ and $\{e\}\times\{\rho(e)\}\times I$ L is constant at $(\pi(e),\rho(e))$ and L maps $\{e\}\times I\times\{s\}$ linearly onto $\{\pi(e)\}\times[(1-s)\rho(e),(1-s)\rho(e)+s]$.

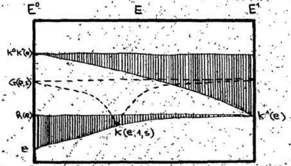


Now consider the following commutative diagram

$$\begin{tikzcd} E\times I \arrow[r, "pr_1"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ E\times I\times I \arrow[r, "L"] & B\times I \end{tikzcd}$$

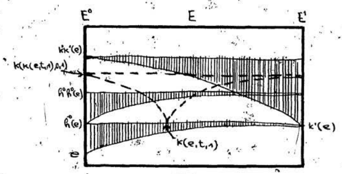
where $i_0(e,t)=(e,t,0)$. Since p is a fibration, there exists a map $K:E\times I\times I \to E$ lifting L and with $K(e,t,0)=e$. Since $pK(e,t,s)=L(e,t,s)=(\pi(e),(1-s)\rho(e)+ts)$, it follows that $\pi K(e,t,s)=\pi(e)$ and $\rho K(e,t,s)=(1-s)\rho(e)+ts$. Using K we define the following maps: $k^1:e\varepsilon E^0 \to K(e,1,1)\varepsilon E^1$, $k^0:e\varepsilon E^1 \to K(e,0,1)\varepsilon E^0$ and $H:(e,s)\varepsilon E\times I \to K(e,\rho(e),s)\varepsilon E$. Since $pK(e,1,1)=(\pi(e),(1-1)\rho(e)+1\cdot 1)=(\pi(e),1)$ and $pK(e,0,1)=(\pi(e),(1-1)\rho(e)+0\cdot 1)=(\pi(e),0)$, we have that k^1 and k^0 are well defined fibre maps over B. Furthermore, since $pH(e,s)=pK(e,\rho(e),s)=(\pi(e),(1-s)\rho(e)+\rho(e)s)=(\pi(e),\rho(e))=p(e)$ and $H(e,0)=K(e,\rho(e),0)=e$, we have that H is a vertical homotopy from 1_E to the fibre map $h:e\varepsilon E \to K(e,\rho(e),1)\varepsilon E$. We define set $H^i:(e,s)\varepsilon E^i\times I \to H(e,s)\varepsilon E^i$ and $h^i\ e\varepsilon E^i \to h(e)\varepsilon E^i$, $i=0,1$.

We claim that k^1 is a fibre homotopy equivalence over B with fibre homotopy inverse k^0. To this end, consider the homotopy $G:E^0\times I \to E^0$ defined by $G(e,s)=K(K(e,1,s),0,1)$.



G is well defined because $\rho K(K(e,1,s),0,1)=(1-1)\rho K(e,1,s)+0\cdot 1=0$; G is vertical because $\pi G(e,s)=\pi K(K(e,1,s),0,1)=\pi K(e,1,s)=\pi(e)$; furthermore, $G(e,0)=K(K(e,1,0),0,1)=K(e,0,1)=h^0(e)$ and $G(e,1)=K(K(e,1,1),0,1)=K(k^1(e),0,1)=k^0k^1(e)$ and so G is a vertical homotopy from h^0 to k^0k^1. Therefore the product of the homotopies H^0 and G is a vertical homotopy from 1_{E^0} to k^0k^1. A similar argument shows that 1_{E^1} is vertically homotopic to k^1k^0.

Remark 36 (a) The above proof is due to Dold [13;prop.6.6]. The map K and its derivatives k^0,k^1, H and h will also be used in the proof of the next theorem 39. We wish to point out that our homotopies from 1_{E^0} to k^0k^1 and from 1_{E^1} to k^1k^0 are slightly different and more natural than those constructed by Dold. Indeed, he first constructs a vertical homotopy from 1_{E^0} to h^0h^0 by the map $(e,t)\in E^0\times I\to H^0(H^0(e,t),t)\in E^0$ and then a vertical homotopy from h^0h^0 to k^0k^1 by the map $(e,t)\in E^0\times I\to K(K(e,t,1),0,1)\in E^0$.



(b) If we were not interested in the above maps K, k^0, k^1 and H and their uses, then the proof of theorem 35 could be shortened as follows (cfr.[46;p.39]). Let $i_0:E^0 \to E$ and $i_1:E^1 \to E$ be the inclusion maps and consider the homotopies $H:(e,t)\epsilon E^0\times I \to (\pi(e),t)\epsilon B\times I$ and $H':(e,t)\epsilon E^1\times I \to (\pi(e),1-t)\epsilon B\times I$. Since $H_0=pi_0$ and $H_0'=pi_1$ and p is a fibration, we can find homotopies $\tilde{H}:E^0\times I \to E$ and $\tilde{H}':E^1\times I \to E$ of i_0 and i_1, respectively, lifting H and H'. Now define $f:E^0 \to E^1$ and $g:E^1 \to E^0$ by $f(e)=\tilde{H}(e,1)$ and $g(e)=\tilde{H}'(e,1)$; it is clear that f and g are fibre maps over B. The maps $i_0:E^0 \to E$ and $i_0gf:E^0 \to E$ have the same projection on $B\times I$; furthermore, the map $G:E^0\times I \to E$ defined by $G(e,t)=\tilde{H}(e,2t)$, if $0\leq t\leq 1/2$, and $G(e,t)=\tilde{H}'(e,2t-1)$, if $1/2\leq t\leq 1$, is a homotopy from i_0 to i_0gf such that its projection on $B\times I$ is homotopic rel. $E^0\times\dot{I}$ to the homotopy stationary at pi_0, since $G(e,t)=G(e,1-t)$. Applying proposition 33 we deduce the existence of a vertical homotopy from 1_{E^0} to gf. A similar argument shows the existence of a vertical homotopy from 1_{E^1} to fg.

Corollary 37 Let $p:E \to B$ be a fibration. If $f',f'':A \to B$ are homotopic maps, then the pullbacks of p along f' and f" have the same fibre homotopy type over A.

Proof Let $H:A\times I \to B$ be a homotopy from f' to f" and let $p':E' \to A$, $p'':E'' \to A$ and $\bar{p}:\bar{E} \to A\times I$ denote the pullbacks of p along f', f" and H, respectively. Applying theorem 35 to the fibration $\bar{p}$, we get that $p'=\bar{p}^0$ and $p''=\bar{p}^1$ have the same fibre homotopy type over A.

We now introduce a relevant class $\mathcal{D}$ of spaces which includes as a sub-class the CW-complexes. As shown by Dold, such spaces play an important role in local-to-global considerations. A space A belongs to $\mathcal{D}$ if it admits a numerable cover $\mathcal{U}$ such that each element of $\mathcal{U}$ can be deformed in A to a point. Allaud in [1] calls such a space locally contractible in large, following E. Dyer and D.S. Kahn [19].

Proposition 38 The class $\mathcal{D}$ satisfies the following properties:

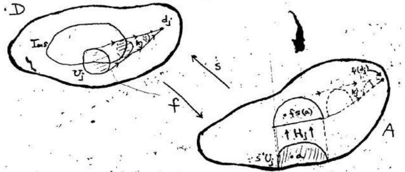
(i) if A is dominated by D and $D\varepsilon\mathcal{D}$, then $A\varepsilon$; in particular the class $\mathcal{D}$ is stable under homotopy equivalence;

(ii) for any space A, its suspension $SA\varepsilon\mathcal{D}$;

(iii) any CW-complex is in $\mathcal{D}$.

Proof (i) We recall that a space A is dominated by a space

D if there exist maps $f:D \to A$ and $s:A \to D$ such that $fs \simeq 1_A$, i.e. f admits a homotopy section. Let $\mathcal{U}=\{U_j \mid j\varepsilon J\}$ be a numerable covering of D with each element deformable in D to a point. We define an open cover of A by $\mathcal{U}'=\{s^{-1}U_j \mid j\varepsilon J\}$. $\mathcal{U}'$ is locally finite; indeed, since $\mathcal{U}$ is locally finite, for each $\bar{a}\varepsilon A$ there exists a neighbourhood V of $s(\bar{a})\varepsilon D$ such that $\{j\varepsilon J: U_j\cap V\neq\emptyset\}$ is finite; it follows that $s^{-1}V$ is a neighbourhood of $\bar{a}$ and $\{j\varepsilon J: s^{-1}U_j\cap s^{-1}V\neq\emptyset\}\subseteq\{j\varepsilon J: U_j\cap V\neq\emptyset\}$ since if $a\varepsilon s^{-1}U_j\cap s^{-1}V$, then $s(a)\varepsilon U_j\cap V$. Each $s^{-1}U_j$ is a cozero set; indeed, if $c_j:D \to I$ is a numeration of U_j then $c_js:A \to I$ is a numeration of $s^{-1}U_j$ because $c_js(a)\neq 0 \Leftrightarrow s(a)\varepsilon U_j \Leftrightarrow a\varepsilon s^{-1}U_j$. It remains to show that each $s^{-1}U_j$ can be deformed in A to a point. For each $j\varepsilon J$ let $K_j:U_j\times I \to D$ be a deformation of U_j to a point $d_j\varepsilon D$ (i.e. $K_j(d,0)=d$ and $K_j(d,1)=d_j$) and let $H:A\times I \to A$ a homotopy from 1_A to fs. Denote by H_j the restriction of H to $s^{-1}U_j\times I$ and consider the homotopy $K_j':s^{-1}U_j\times I \to A$ obtained as the composition $s^{-1}U_j\times I \xrightarrow{s\times 1_I} U_j\times I \xrightarrow{K_j} D \xrightarrow{f} A$, so $K_j'(a,0)=fs(a)$ and $K_j'(a,1)=f(d_j)$. Then the homotopy $H_j.K_j':s^{-1}U_j\times I \to A$, obtained by following for the first half time H_j and then K_j', is a deformation of $s^{-1}U_j$ to $f(d_j)$.



(ii) We recall that the suspension SA is obtained from the cylinder $A\times I$ identifying the bottom base $A\times\{0\}$ to a point, the top base $A\times\{1\}$ to another point and topologizing it with the quotient topology. A generic point of SA will be written by $[a,t]$, with $a\in A$ and $t\in[0,1]$; so $[a,0]=A\times\{0\}$, $[a,1]=A\times\{1\}$ and $[a,t]=\{(a,t)\}$, if $0<t<1$. We define $U^+=SA-\{[a,0]\}$ and $U^-=SA-\{[a,1]\}$. U^+ and U^- are open sets of SA because their anti-images by the identification map $q:A\times I\to SA$ are $A\times]0,1]$ and $A\times[0,1[$, respectively. Furthermore, U^+ and U^- are cozero sets; indeed, the functions $c^+:[a,t]\in SA\to t\in[0,1]$ and $c^-:[a,t]\in SA\to 1-t\in[0,1]$ are continuous because their compositions with the identification map q are the maps $(a,t)\in A\times I\to t\in[0,1]$ and $(a,t)\in A\times I\to 1-t\in[0,1]$, respectively, and the anti-images of $]0,1]$ by c^+ and c^- are U^+ and U^-, respectively. It remains to prove that U^+ and U^- are deformable in SA to a point. To this end, let $K^+:U^+\times I\to SA$ be the function defined by $K^+([a,t],s)=[a,(1-t)s+t]$. K^+ is well defined because when t=1 we have that $K^+([a,1],s)=[a,1]=[a',1]=K^+([a',1],s)$; furthermore, $K^+([a,t],0)=[a,t]$ and $K^+([a,t],1)=[a,1]$. Consider the following commutative diagram

$$\begin{tikzcd} A\times]0,1]\times I \arrow[r, "\bar{K}^+"] \arrow[d, "q^+\times 1_I"'] & A\times I \arrow[d, "q"] \\ U^+\times I \arrow[r, "K^+"] & SA \end{tikzcd}$$

where $\bar{K}^+(a,t,s)=(a,(1-t)s+t)$ and $q^+:A\times]0,1] \to U^+$ is the restriction of q. Since U^+ is open, q^+ is also an identification map [18;th.2.1,p.122] and so $q^+\times 1_I$ is an identification map [18;th.4.1,p.262]. Hence K^+ is a continuous deformation of U^+. Similarly one proves that $K^-:([a,t],s)\in U^-\times I \to [a,(1-s)t]\in SA$ is a continuous deformation of U^- to $[a,0]$.

(iii) It is well known that any CW-complex X is paracompact (Miyazaki's theorm) and locally contractible, i.e. each point admits a contractible open neighbourhood. Let $\mathcal{U}=\{U\}$ be an open cover of X such that each $U\in\mathcal{U}$ is contractible. Then there exists a partition of unity $\{\psi: X \to I\}$ subordinate to $\mathcal{U}$. Hence the open cover $\{\psi^{-1}(]0,1])\}$ is a numerable cover of X such that each of its elements is contractible in X. (A more elementary proof, i.e. not using the paracompactness of CW-complexes, is given in [13]).

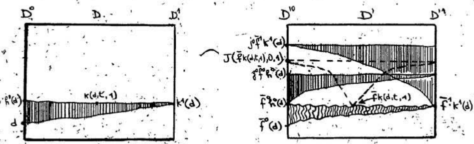
Theorem 39 Let $p:E \to B$ and $p':E' \to B$ be fibrations with $B\varepsilon\mathcal{D}$. If $f:E \to E'$ is a fibre map over B such that $f_b:F_b \to F_b'$ is a homotopy equivalence for every $b\varepsilon B$, then f is a fibre homotopy equivalence over B.

Proof Let $\mathcal{U}=\{U\}$ be a numerable cover of B such that each $U\varepsilon\mathcal{U}$ can be deformed in B to a point. We will show that for every $U\varepsilon$ the fibre map $f_U:E_U \to E_U'$ is a fibre homotopy equivalence over U between $p_U:E_U \to U$ and $p_U':E_U' \to U$. It will then follow from theorem 1.14 that f is a fibre homotopy equivalence over B.

Let $c:U\times I \to B$ be a deformation of U to a point $b_0\varepsilon B$, that is, $c(b,0)=b$ and $c(b,1)=b_0$ for every $b\varepsilon U$. Let $q:D \to U\times I$ and $q':D' \to U\times I$ denote the fibrations obtained by pulling back p and p', respectively, along c; so $D=\{(b,t,e)\varepsilon U\times I\times E:\ p(e)=c(b,t)\}$, $D'=\{(b,t,e')\varepsilon U\times I\times E':\ p'(e')=c(b,t)\}$, $q(b,t,e)=(b,t)$ and $q'(b,t,e')=(b,t)$. Keeping the same notations introduced in theorem 35, we have the fibrations $q^0:D^0 \to U$, $q^1:D^1 \to U$, $q'^0:D'^0 \to U$ and $q'^1:D'^1 \to U$. Now let $\bar{f}:D \to D'$ denote the fibre map over $U\times I$ induced by f and let $\bar{f}^0:D^0 \to D'^0$ and $\bar{f}^1:D^1 \to D'^1$ denote the restrictions of $\bar{f}$. Via the one-to-one correspondences $e\varepsilon E_U \leftrightarrow (b,0,e)\varepsilon D^0$, $e'\varepsilon E_U' \leftrightarrow (b,0,e')\varepsilon D'^0$, $(b,e)\varepsilon U\times F_{b_0} \leftrightarrow (b,1,e)\varepsilon D^1$ and $(b,e')\varepsilon U\times F_{b_0}' \leftrightarrow (b,1,e')\varepsilon D'^1$ we can identify q^0 with p_U, q'^0 with p_U', q^1 with $pr_1:U\times F_{b_0} \to U$ and q'^1 with $pr_1:U\times F_{b_0}' \to U$. Under these identifications $\bar{f}^0$ corresponds to f_U and $\bar{f}^1$ to

$1_U\times f_{b_0}$; so, in particular, $\bar f^1$ is a fibre homotopy equivalence over U. Let K, k^0, k^1 and h^0 denote the maps associated to $q:D\to U\times I$, as constructed in proof of theorem 35, and let J, j^0, j^1 and g^0 be the corresponding maps for $q':D'\to U\times I$.

We want to show that $\bar f^0$ is fibre homotopic over U to the composition $j^0\bar f^1k^1$. Since the latter is a fibre homotopy equivalence over U, each factor being a fibre homotopy equivalence over U, it will follow that $\bar f^0$, and hence f_U, is a fibre homotopy equivalence over U. We first observe that $\bar f^0=\bar f^0 1_{D^0}\simeq_U\bar f^0h^0=1_{D'^0}\bar f^0h^0\simeq_U g^0\bar f^0h^0$. Now consider the map $(d,t)\in D^0\times I\to J(\bar fK(d,t,1),0,1)\in D'^0$.



It is a vertical homotopy from $g^0\bar f^0h^0$ to $j^0\bar f^1k^1$; indeed, $J(\bar fK(d,0,1)0,1)=J(\bar fh^0(d),0,1)=g^0\bar fh^0(d)=g^0\bar f^0h^0(d)$ and $J(\bar fK(d,1,1),0,1)=J(\bar fk^1(d),0,1)=j^0\bar fk^1(d)=j^0\bar f^1k^1(d)$. So $\bar f^0$ is fibre homotopic over U to $j^0\bar f^1k^1$, as required.

Remark 40 Since it has been pointed out in corollary 27 that if the restriction f_b of a fibre map $f:E \to E'$ over a point $b \in B$ is a homotopy equivalence, then so is the restriction $f_{b'}$ over any point b' in the same path component containing b, we can weaken the statement of the above theorem requiring only f_b be a homotopy equivalence for one choice of b in each path component of B. In particular, if B is path connected it is enough to know that f_b is a homotopy equivalence for some $b \in B$.

<u>Proposition 41</u> Let $A, B \in \mathcal{D}$ be path-connected and let $f:A \to B$ be a map such that for some $a \in A$ the loop map $\Omega f:\Omega(A,a) \to \Omega(B,f(a))$ is a (free) homotopy equivalence. Then f is a homotopy equivalence.

<u>Proof</u> Factorize f as the homotopy equivalence $\iota:A \to \Lambda_f$ followed by the fibration $\bar{f}:\Lambda_f \to B$. Then f is a homotopy equivalence if and only if $\bar{f}$ is a homotopy equivalence. Now consider the commutative diagram

$$\begin{tikzcd} \Lambda_f \arrow[r, "\bar{f}"] \arrow[dr, "\bar{f}"'] & B \arrow[d, "1_B"] \\ & B \end{tikzcd}$$

and observe that if the fibre of $\bar{f}$ over some $b \in B$ is a contractible space then, by corollary 27 and theorem 39, it connected and belongs to , theorem 39 tells us that f_* is a fibre homotopy equivalence over A. Now $P(A,a)$ is a contractible space and hence $T_{f(a)}$ is a contractible space, as required.

The next result is also due to Dold [13].

<u>Theorem 42</u> If $p:E \to B$ and $p':E' \to B$ are fibrations and $f:E \to E'$ is a fibre map over B, then f is a fibre homotopy equivalence over B if and only if f is a homotopy equivalence.

<u>Proof</u> Suppose f is a homotopy equivalence and let $g:E' \to E$ be a homotopy inverse for f. We will use the fact that g is, in particular, a right homotopy inverse for f (i.e. $fg \simeq 1_{E'}$) to deduce the existence of a fibre map $g':E' \to E$ over B which is a right fibre homotopy inverse for f. Since g is also a left homotopy inverse for f (i.e. $gf \simeq 1_E$), it will follow from proposition 1.2 and remark 1.3 that $g' \simeq g$ and so g' is also a homotopy equivalence. Applying the same reasoning now to g', we will deduce the existence of a fibre map $f':E \to E'$ over B which is a right fibre homotopy inverse for g'. Applying proposition 1.2 once again, we will have that g' is a fibre homotopy equivalence over B with fibre homotopy inverse f, and hence, f is a fibre homotopy equivalence over B.

follows that $\bar{f}$ is a fibre homotopy equivalence (over B) and so, in particular, a homotopy equivalence. Hence, it is sufficient to show that the fibre of $\bar{f}$ over some $b\in B$ is a contractible space.

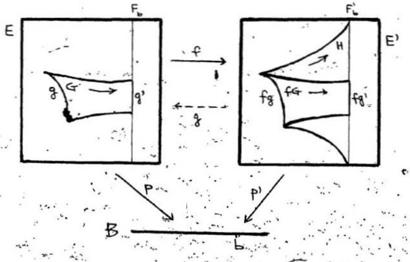
Let T_b denote the fibre of $\bar{f}$ over b (the so-called <u>homotopy fibre</u> of f) and define a map $q:T_b \to A$ by $q(a,\beta)=a$. Since q coincides with the pullback of the fibration $p':P'(B,b) \to B$ along f, it is a fibration and the fibre of q over $a\in A$ is $T_{b,a}=\{(a,\beta)\in A\times B^I;\ \beta(0)=f(a) \text{ and } \beta(1)=b\}$. If $b=f(a)$ we have that $T_{f(a),a}=\{a\}\times\Omega(B,f(a))$ and moreover there exists a fibre map $f_*:\alpha\in P'(A,a) \to (\alpha(0),f\alpha)\in T_{f(a)}$ over A, whose restriction to the fibre over a is the map $\Omega f:\Omega(A,a) \to \{a\}\times\Omega(B,f(a))$.

$$\begin{tikzcd}
\Omega(A,a) \arrow[r, "\Omega f"] \arrow[d] & \{a\}\times\Omega(B,f(a)) \arrow[d] & & \\
P'(A,a) \arrow[r, "f_*"] \arrow[ddr, "p'"'] & T_{f(a)} \arrow[dd, "q"] \arrow[dr] & & \\
& & \Lambda_f \arrow[dr, "\bar{f}"] & \\
& A \arrow[ur, "\iota"] \arrow[rr, "f"'] & & B
\end{tikzcd}$$

Since we know that Ωf is a homotopy equivalence and A is path

$$\begin{tikzcd}
E \arrow[rr, "f"] \arrow[rr, bend left=50, "f'"] \arrow[dr, "p"'] & & E' \arrow[ll, bend right=30, "g'"'] \arrow[ll, dashed, bend left=20, "\bar{g}"] \arrow[dl, "p'"] \\
& B &
\end{tikzcd}$$

Let $H:E'\times I \to E'$ be a homotopy from fg to $1_{E'}$. Then $p'H$ is a homotopy of pg because $p'H(x,0)=p'fg(x)=pg(x)$. Hence, using the fact that p is a fibration, we deduce the existence of a homotopy od f $G:E'\times I \to E$ lifting $p'H$. Define $g':E' \to E$ by $g'(x)=G(x,1)$ and observe that g' is a fibre map over B because $pg'(x)=pG(x,1)=p'H(x,1)=p'1_{E'}(x)=p'(x)$. Now the homotopies H and fG have the same projection on B because $p'fG=pG=p'H$. Hence the homotopy $(fG)^{-1}.H:E'\times I \to E'$, obtained by following for the first half time fG in the reverse direction and then H, is a homotopy from fg' to $1_{E'}$ whose projection on B is $(p'H)^{-1}.p'H$. Now $(p'H)^{-1}.p'H$ is a homotopy having the property that its value on (x,t) and $(x,1-t)$ coincide and so it can be deformed rel. $E\times\dot{I}$ to the homotopy stationary at p'. Therefore, by proposition 33, we deduce the existence of a vertical homotopy from fg' to $1_{E'}$.



With a similar argument, applied now to the homotopy equivalence g', we deduce the existence of a fibre map $f': E \to E'$ such that $g'f' \simeq_B 1_E$.

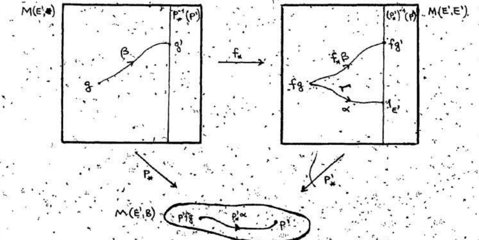
The reverse implication is obvious.

Remark 43 The above technique for proving that g' is a right fibre homotopy inverse for f cannot, in general, be used to show that g' is a left fibre homotopy inverse for f. Indeed, we cannot apply the above trick to the homotopies $J: E\times I \to E$ from gf to 1_E, and $K: E\times I \xrightarrow{f\times 1_I} E'\times I \xrightarrow{G} E$ from gf to $g'f$, because their projections pJ and pK are not related.

There is a more geometrical way to look at the proof of theorem 42, as suggested by L. Siebenmann in [39]; it gives an alternative proof when E and E' are locally compact, Hausdorff. Consider the mapping spaces $M(E',E)$, $M(E',E')$ and $M(E',B)$ and the maps $p_*:M(E',E) \to M(E',B)$, $p'_*:M(E',E') \to M(E',B)$ and $f_*:M(E',E) \to M(E',E')$ induced by p, p' and f, respectively. The diagram

$$\begin{tikzcd}
M(E',E) \arrow[rr, "f_*"] \arrow[dr, "p_*"'] & & M(E',E') \arrow[dl, "p'_*"] \\
& M(E',B) &
\end{tikzcd}$$

commutes, since $p'_*f_*(\ell)=p'f\ell=p\ell=p_*(\ell)$ for all $\ell\in M(E',E)$. Now p and p' are fibrations and E and E' are locally compact, Hausdorff, so p_* and p'_* are fibrations (example (v)). Let $g\in M(E',E)$ be any homotopy inverse of f; then fg is homotopic to $1_{E'}$ by some homotopy $E'\times I \to E'$ whose adjoint will be a path in $M(E',E')$, call it α.



We have that $p_*(g)=pg=p'fg=p'\circ\alpha(0)=p'_*\alpha(0)$ and using the fact that p_* is a fibration we can deduce the existence of a lifting of $p'_*\alpha$, call it β, with initial point at g. Let $g'=\beta(1)$; then g' is a fibre map over B because $pg'=p_*(g')=p_*(\beta(1))=p'_*\alpha(1)=p'_*(1_{E'})=p'1_{E'}=p'$. We claim that fg' is fibre homotopic over B to $1_{E'}$. Consider the path $f_*\beta$ in $M(E',E')$; its initial point is fg and its terminal point is fg' and furthermore its projection on $M(E',B)$ is $p'_*\alpha$ because $p'_*f_*\beta=p_*\beta=p'_*\alpha$. Hence $f_*\beta$ and α have the same projection and so applying proposition 12(ii) we deduce the existence of a path γ in $M(E',E')$ joining fg' to $1_{E'}$ and lying in the fibre over $p'\varepsilon M(E',B)$. Since $(p'_*)^{-1}(p')$ is the subspace of $M(E',E')$ consisting of all fibre maps $E'\to E'$ over B, the adjoint of γ defines a vertical homotopy from fg' to 1_{E}.

Corollary 44 Let $p:E \to B$ be a fibration. Then p is shrinkable if and only if p is a homotopy equivalence.

Proof We have already seen that a map $p:E \to B$ is shrinkable if and only if p, regarded as a fibre map over B from p to 1_B, is a fibre homotopy equivalence. On the other hand, since p and 1_B are fibrations, we have from theorem 42 that p is a fibre homotopy equivalence if and only if p is a homotopy equivalence.

Let $p:E \to B$ be a map and let π be a partition of B. We say that p is a π-fibration if for any map $f:X \to E$ and any π-stationary homotopy $H:X\times I \to B$ of pf, there exists a homotopy $\tilde{H}:X\times I \to E$ of f lifting H. We observe the following: if $\pi \leq \pi'$, then a π'-fibration is also a π-fibration, since a π-stationary homotopy is in particular π'-stationary; if π is the coarsest partition of B (i.e. $\pi=\{B\}$) then the notion of π-fibration coincides with that of Hurewicz fibration; if π is the discrete partition of B (i.e. $\pi=\{\{b\}|b\in B\}$) then any map is a π-fibration because in this case a π-stationary homotopy is just a stationary homotopy. Like Hurewicz fibrations, π-fibrations can be charaterized intrinsically. Indeed, if we consider the subspace B^I_π of B^I consisting of all paths α which are π-stationary, that is, $[\alpha(t)]=[\alpha(0)]$ for every $t\epsilon I$, and we consider the subspace Λ_p^π of Λ_p consisting of all couples in Λ_p with the second component in B_π^I, then considerations similar to those used in proposition 14 show that p is a π-fibration if and only if $\rho:E^I \to \Lambda_p$ admits a section over Λ_p^π. Our motivating example of a π-fibration is the map $\bar{f}:R_f \to E'$, for any fibre map $f:E \to E'$ over B from $p:E \to B$ to $p':E' \to B$. In this case the partition π of E' is given by the fibres of p', that is $\pi=\{F_b' | b\epsilon \mathrm{Im}p'\}$. To show that $\bar{f}:R_f \to E'$ is a π-fibration we will construct a section σ of $\rho:R_f^{\,I} \to \Lambda_{\bar{f}}$ over $\Lambda_{\bar{f}}^\pi$. To this end, we first define the map $L:E'^I \underline{\times} E'^I \times I \to E'^I$, where $E'^I \underline{\times} E'^I = \{(\alpha,\beta)\epsilon E'^I \times E'^I;\ \alpha(1)=\beta(0)\}$, by $L(\alpha,\beta,s)(t)=\alpha(2t/(2-s))$, if $0<t<1-s/2$, and $L(\alpha,\beta,s)(t)=\beta(2t+s-2)$, if $1-s/2<t<1$. L is continuous because its adjoint is the map

$$(\alpha,\beta,t,s)\epsilon E'^I \underline{\times} E'^I \times I \times I \to \begin{cases} \alpha(2t/(2-s)) & \text{if } 0<t<1-s/2 \\ \beta(2t+s-2) & \text{if } 1-s/2<t<1 \end{cases}$$

The path $L(\alpha,\beta,s)$ follows first the path α and then the path β up to $\beta(s)$, and so $L(\alpha,\beta,0)=\alpha$; moreover, if α and β lie in the same fibre of p', so does $L(\alpha,\beta,s)$ for every $s\epsilon I$. Now define $\sigma:\Lambda_{\bar{f}}^\pi \to R_f^{\,I}$ by $\sigma(e,\alpha,\beta)(s)=(e,L(\alpha,\beta,s))$; σ is well defined because $L(\alpha,\beta,s)(0)=\alpha(0)=f(e)$; it is continuous because its adjoint is the map $(e,\alpha,\beta,s)\epsilon\Lambda_{\bar{f}}^\pi \times I \to (e,L(\alpha,\beta,s))\epsilon R_f$, and furthermore $\sigma(e,\alpha,\beta)(0)=(e,L(\alpha,\beta,0))=$

(e,α) and $\bar{f}(\sigma(e,\alpha,\beta)(s))=\bar{f}(e,L(\alpha,\beta,s))= L(\alpha,\beta,s)(1)=\beta(s)$; so σ is a section of ρ over $\Lambda^{\pi}_{\bar{f}}$.

The next result generalizes theorem 42.

Theorem 45 If $p:E \to B$ and $p':E' \to B$ are π-fibrations and $f:E \to E'$ is a fibre map over B, then f is a fibre homotopy equivalence (over B) if and only if f is a π-fibre homotopy equivalence.

Proof The technique is the same as the proof of theorem 42. If f is a fibre homotopy equivalence over B, then there exist a fibre map $g:E' \to E$ over B, a vertical homotopy $H:E\times I \to E$ from gf to 1_E and a vertical homotopy $K:E'\times I \to E'$ from fg to $1_{E'}$. So, in particular, we have that:

(i) $[pg(e')]=[p'(e')]$ for every $e'\epsilon E'$;

(ii) $[pH(e,t)]=[p(e)]$ for every $e\epsilon E$ and $t\epsilon I$;

(iii) $[p'K(e',t)]=[p'(e')]$ for every $e'\epsilon E'$ and $t\epsilon I$.

This shows that f is a π-fibre homotopy equivalence.

Now suppose f is a π-fibre homotopy equivalence. This means that there exist a map $g:E' \to E$, a homotopy $H:E\times I \to E$ from gf to 1_E, and a homotopy $K:E'\times I \to E'$ from fg to $1_{E'}$ such that:

(i) $[pg(e')]=[p'(e')]$ for every $e'\epsilon E'$;

(ii) $[pH(e,t)]=[p(e)]$ for every $e\epsilon E$ and $t\epsilon I$;

(iii) $[p'K(e',t)]=[p'(e')]$ for every $e'\epsilon E'$ and $t\epsilon I$.

Because of property (iii), we have that $p'K$ is a π-stationary homotopy; furthermore, we have that $p'K(e',0)=p'fg(e')=$

$pg(e')$. Since p is a π-fibration, there is a homotopy of g $G:E'\times I \to E$ lifting $p'K$. The map $g':E' \to E$ defined by $g'(e')=G(e',1)$ is a fibre map over B, because $pg'(e')= pG(e',1)=p'K(e',1)=p'(e')$ Moreover, we have that $fg'\simeq 1_{E'}$ via the homotopy $(fG)^{-1}.K$. The projection of $(fG)^{-1}.K$ on B is the homotopy $(p'K)^{-1}.(p'K)$, which can be deformed rel. $E'\times \dot{I}$ to the homotopy stationary at p' by a π-stationary homotopy (with respect to the last variable) $L:E'\times I\times I \to B$. Since p' is a π-fibration, there is a lifting $\tilde{L}$ of L with $\tilde{L}_0=(fG)^{-1}.K$. Then the map $J:E'\times I \to E'$ given by

$$J(e',t)=\begin{cases}\tilde{L}(e',0,3t) & \text{if } 0\leq t\leq 1/3\\ \tilde{L}(e',3t-1,1) & \text{if } 1/3\leq t\leq 2/3\\ \tilde{L}(e',1,-3t+3) & \text{if } 2/3\leq t\leq 1\end{cases}$$

is a vertical homotopy from fg' to $1_{E'}$. We can now reapply the same argument to g'; indeed, g' is a π-fibre homotopy equivalence via the map $f:E \to E'$, the homotopy $E\times I \to E$ given by the product of the homotopies $E\times I \xrightarrow{f\times 1_I} E'\times I \xrightarrow{G^{-1}} E$ and H, and the homotopy $(fG)^{-1}.K:E'\times I \to E'$ (or J). We get in this way a fibre map $f':E \to E'$ over B with $g'f'\simeq_B 1_E$. Then, by proposition 1.2, g' is a fibre homotopy equivalence over B with fibre homotopy inverse f; hence f is a fibre homotopy equivalence.

Remark 46 If π is the coarsest partition of B (i.e. $\pi=\{B\}$) then the statement of theorem 45 is just Dold's theorem 42.

The following consequence of theorem 45 completes the proof of proposition 1.8.

Corollary 47 Let $p:E \to B$ be a π-fibration. Then p is shrinkable if and only if p is a π-homotopy equivalence.

Proof If p is shrinkable then p admits a section $s:B \to E$ and a vertical homotopy $H:E\times I \to E$ from 1_E to sp. So, in particular, the following properties hold:

(i) $[ps(b)]=[b]$ for every $b\in B$;

(ii) $[pH(e,t)]=[p(e)]$ for every $e\in E$;

(iii) ps is homotopic to 1_B via a π-stationary homotopy.

Hence p is a π-homotopy equivalence.

On the other hand, suppose p is a π-homotopy equivalence, that is, there is a map $q:B \to E$, a homotopy $H:E\times I \to E$ from qp to 1_E and a homotopy $K:B\times I \to B$ from pq to 1_B such that:

(i) $[pq(b)]=[b]$ for every $b\in B$;

(ii) $[pH(e,t)]=[p(e)]$ for every $e\in E$;

(iii) $[K(b,t)]=[b]$ for every $b\in B$ and $t\in I$.

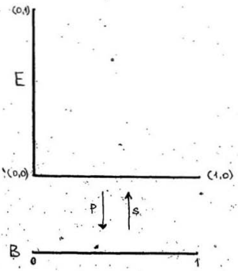
This means that p, regarded as a fibre map over B from p to 1_B, is a π-fibre homotopy equivalence. Since p and 1_B are π-fibrations, we can apply theorem 45 to deduce that p is a fibre homotopy equivalence (over B) from p to 1_B, that is, p is a shrinkable map.

3. DOLD AND SERRE FIBRATIONS. QUASIFIBRATIONS

We have seen in section 2 that maps of the same fibre homotopy type as a Hurewicz fibration may not be Hurewicz fibrations. But it was pointed out by Dold [13] and Weinzweig [45] that such maps do posses a weak covering homotopy property and that maps with this property exhibit many of the properties held by Hurewicz fibrations. More precisely a map $p:E \to B$ is said to have the weak covering homotopy property (WCHP) with respect to a space X if for every map $f:X \to E$ and semi-stationary homotopy $H:X\times I \to B$ of pf there exists a homotopy $\tilde{H}:X\times I \to E$ of f lifting H. p is called a Dold fibration if it has the WCHP with respect to all spaces.

Hurewicz fibrations are of course Dold fibrations. Shrinkable maps are also Dold fibrations. To prove it, let $p:E \to B$ be a shrinkable map, $s:B \to E$ a section of p and $K:E\times I \to E$ a vertical homotopy from 1_E to sp. Consider any map $f:X \to E$ and a semi-stationary homotopy $H:X\times I \to B$ of pf. Then H can be lifted by the homotopy $\tilde{H}:X\times I \to E$ of f given by $\tilde{H}(x,t)=K(f(x),2t)$, if $0\leq t\leq 1/2$, and $\tilde{H}(x,t)=sH(x,t)$, if $1/2\leq t\leq 1$.

An example of a Dold fibration which is not a Hurewicz fibration is obtained by considering $E=I\times\{0\}\cup\{0\}\times I\subseteq R^2$, $B=I$ and letting $p:E \to B$ be the projection on the first factor.



p is a shrinkable map, and hence a Dold fibration, since the maps $s:B \to E$ and $K:E\times I \to E$, given by $s(t)=(t,0)$ and $K((x,y),t)=(x,y-ty)$, are a section of p and a vertical homotopy from 1_E to sp, respectively. To show that p is not a Hurewicz fibration, consider a one-point space $P=\{*\}$ and the maps $f:P \to E$ and $H:P\times I \to B$ defined by $f(*)=(0,1)$ and $H(*,t)=t$. Then H is a homotopy of pf, but there is no lifting $\tilde{H}$ of H with $\tilde{H}_0=f$ because, otherwise, the inverse image by $\tilde{H}$ of the open set $\{0\}\times]0,1]\subseteq E$ would be the set $\{(*,0)\}\subseteq P\times I$, which is not open, contradicting the continuity of $\tilde{H}$.

The following result says that, unlike Hurewicz fibrations, being a Dold fibration is a property invariant under fibre homotopy equivalence.

Proposition 1 Let $p:E \to B$ and $p':E' \to B$ be maps having the same fibre homotopy type over B. Then if p is a Dold fibration, p' is also a Dold fibration.

Proof Let $f:E \to E'$ be a fibre homotopy equivalence over B with fibre homotopy inverse $g:E' \to E$. Furthermore, let $\ell:X \to E'$ be any map and let $H:X\times I \to B$ be a semi-stationary homotopy of $p'\ell$. Consider $g\ell:X \to E$; then H is a semi-stationary homotopy of $p(g\ell)$ because $p(g\ell)=(pg)\ell=p'\ell$. Since p is a Dold fibration, there is a homotopy $\bar{H}:X\times I \to E$ of $g\ell$ lifting H. Let $G:E'\times I \to E'$ be a vertical homotopy from fg to $1_{E'}$. Define $\tilde{H}:X\times I \to E'$ by

$$\tilde{H}(x,t) = \begin{cases} G(\ell(x),-4t+1) & \text{if } 0\le t\le 1/4 \\ f\bar{H}(x,2t-1/2) & \text{if } 1/4\le t\le 1/2 \\ f\bar{H}(x,t) & \text{if } 1/2\le t\le 1 \end{cases}$$

Then $\tilde{H}$ is a homotopy of ℓ lifting H; indeed, $\tilde{H}(x,0)=G(\ell(x),1)=\ell(x)$ and

$$p'\tilde{H}(x,t) = \begin{cases} p'G(\ell(x),-4t+1) & \text{if } 0\le t\le 1/4 \\ p'f\bar{H}(x,2t-1/2) & \text{if } 1/4\le t\le 1/2 \\ p'f\bar{H}(x,t) & \text{if } 1/2\le t\le 1 \end{cases}$$

$$= \begin{cases} p'\ell(x) & \text{if } 0\le t\le 1/4 \\ p'\ell(x) & \text{if } 1/4\le t\le 1/2 \\ p\bar{H}(x,t) & \text{if } 1/2\le t\le 1 \end{cases}$$

$$= H(x,t).$$

<u>Remark 2</u> The proof of the above proposition actually shows that a stronger result holds; namely, if $p':E' \to B$ is <u>dominated</u> by a Dold fibration $p:E \to B$ (i.e. there exist fibre maps $f:E \to E'$ and $g:E' \to E$ over B with $fg \simeq_B 1_{E'}$) then p' is a Dold fibration.

Dold fibrations, like Hurewicz fibrations, can be characterized intrinsically by lifting functions. Let $B_s^I = \{\alpha \in B^I : \alpha(t) = \alpha(0),\ 0 \leq t \leq 1/2\} \subseteq B^I$; an element of B_s^I is called a <u>semi-stationary path</u>. There is a natural map $\alpha \in B^I \to \bar{\alpha} \in B_s^I$, where $\bar{\alpha}$ is defined by $\bar{\alpha}(t) = \alpha(0)$, if $0 \leq t \leq 1/2$, and $\bar{\alpha}(t) = \alpha(2t-1)$, if $1/2 \leq t \leq 1$. Given a map $p:E \to B$, let $\Lambda_p^s = \{(e,\alpha) \in E \times B_s^I : \alpha(0) = p(e)\}$. Then a <u>Dold lifting function</u> for p is a map $\lambda : \Lambda_p^s \to E^I$ such that $\lambda(e,\alpha)(0) = e$ and $p \circ \lambda(e,\alpha) = \alpha$, that is, λ is a section of $\rho : E^I \to \Lambda_p$ over Λ_p^s. An argument similar to that used to prove proposition 2.14 gives the following result.

<u>Proposition 3</u> A map $p:E \to B$ is a Dold fibration if and only if p admits a Dold lifting function.

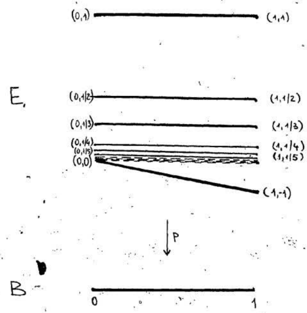
In section 2 we associated to any map $p:E \to B$ a Hurewicz fibration $\bar{p}:\Lambda_p \to B$ and an "inclusion" map $\iota:E \to \Lambda_p$ with $p = \bar{p}\iota$ and ι a homotopy equivalence; furthermore, we proved that if p is a fibration, then p and p' have the same fibre homotopy type (over B). We now generalize and improve that result.

<u>Proposition 4</u> A map $p:E \to B$ has the same fibre homotopy type over B as its associated fibration $\bar{p}:\Lambda_p \to B$ if and only if p is a Dold fibration.

<u>Proof</u> Suppose p and p' have the same fibre homotopy type over B. Since a Hurewicz fibration is also a Dold fibration, p has the same fibre homotopy type as a Dold fibration and hence, by proposition 1, p is itself a Dold fibration. Conversely, suppose p is a Dold fibration and let $\lambda:\Lambda_p \to E^I$ be a Dold lifting function for p. Define $\zeta:\Lambda_p \to E$ by $\zeta(e,\alpha)=\lambda(e,\bar{\alpha})(1)$. Then ζ is a fibre map over B such that $\zeta\iota\simeq_B 1_E$ and $\iota\zeta\simeq_B 1_{\Lambda_p}$. Indeed, the homotopy $H:(e,t)\varepsilon E\times I \to \lambda(e,p(e))(t)\varepsilon E$ is a vertical homotopy from 1_E to $\zeta\iota$ and the homotopy $K:\Lambda_p\times I \to \Lambda_p$ defined by $K(e,\alpha,t)=(\lambda(e,\bar{\alpha})(t),\alpha)$, if $0<t<1/2$, and $K(e,\alpha,t)=(\lambda(e,\bar{\alpha})(t),\alpha_{2t-1})$, if $1/2<t<1$, (here α_t denotes the path $\alpha_t(s)=\alpha(s+t-st)$, following α from $\alpha(t)$ to $\alpha(1)$) is a vertical homotopy from 1_{Λ_p} to $\iota\zeta$.

All main properties held by Hurewicz fibrations remain true for Dold fibrations. Actually, most of them (except the Hurewicz uniformization theorem) were for the first time proved by Dold in [13] in the context of Dold fibrations. We have preferred to state and prove them for Hurewicz fibrations because in that case technicalities simplify considerably, letting so the main ideas in the proofs appear in a clear way. We simply point out that, for example, the following results are also valid in the context of Dold fibrations: a Dold fibration $p:E \to B$ gives rise to a functor $T_p:\Pi B \to HTop$; if $p':E' \to B$ is another Dold fibration, then any fibre map $f:E \to E'$ over B gives rise to a natural transformation $\Phi_f:T_p \to T_{p'}$; the fibres of a Dold fibration over points lying in the same path component have the same homotopy type; the pullback of a Dold fibration is a Dold fibration and pullbacks along homotopic maps have the same fibre homotopy type; a map which is "locally" a Dold fibration is a Dold fibration; a fibre map $f:E \to E'$ over $B \varepsilon \mathcal{D}$ between Dold fibrations $p:E \to B$ and $p':E' \to B$, such that f_b is a homotopy equivalence for every $b \varepsilon B$, is a fibre homotopy equivalence; a fibre map $f:E \to E'$ over B between Dold fibrations is a fibre homotopy equivalence over B if and only if f is a homotopy equivalence.

We now introduce another class of maps which is related to the covering homotopy property. A map $p:E \to B$ is called a <u>Serre fibration</u> if it has the CHP with respect to all the cubes I^n, $n \geq 0$. The following map is an example of a "genuine" Serre fibration, that is, of a Serre fibration which fails to be a Dold fibration and so, a fortiori, to be a Hurewicz fibration. Let $E = \bigcup_{n \geq 1} I \times \{1/n\} \cup \{(t,-t) \mid t \varepsilon I\} \subseteq R^2$, $B = I$ and let $p:E \to B$ be the projection on the first factor.



The path components of E are the horizontal segments $I\times\{1/n\}$ and the slanting segment. To show that p is a Serre fibration, let $f: I^{n} \to E$ be any map and let $H: I^{n+1} \to B$ be a homotopy of pf; then Imf lies in some path component of E and so H can be canonically lifted to E. To show that p is not a Dold fibration, consider the space $X=\{1/n \mid n\in N\}\cup\{0\}\subseteq R$ with the subspace topology and let $f: X \to E$ be the map given by $f(x)=(0,x)$ and let $H: X\times I \to B$ be the homotopy defined by $H(x,t)=0$, if $0\leq t\leq 1/2$, and $H(x,t)=2t-1$, if $1/2\leq t\leq 1$; it is impossibe to find a lifting $\tilde{H}$ of H with $\tilde{H}_0=f$ because $\tilde{H}_1$ must then satisfy $\tilde{H}_1(x)=(1,x)$, if $x\neq 0$, and $\tilde{H}_1(0)=(1,-1)$ and so $\tilde{H}_1$ cannot be continuous at $x=0$.

On the other hand it is easy to see that the Dold fibration considered at beginning of this section is not a Serre fibration. Therefore, these two notions are independent of each other.

The following results state the main properties held by Serre fibrations; their proof can be found in [40 ;secc. 7.2 and 7.8].

Proposition 5 A Serre fibration $p:E \to B$ has the CHP with respect to any CW-complex.

Proposition 6 If $p:E \to B$ is a Serre fibration, then for every $e\in E$ and integer $n\geq 1$ the function $p_*:\pi_n(E,F_{p(e)},e) \to \pi_n(B,p(e))$ induced by p is bijective and the sequence of pointed sets $\pi_0(F_{p(e)},e) \xrightarrow{i_*} \pi_0(E,e) \xrightarrow{p_*} \pi_0(B,p(e))$ is exact.

It follows from proposition 6 that if we consider the exact homotopy sequence associated to the pointed pair $(E,F_{p(e)},e)$ (the horizontal line in the below diagram)

$$\begin{tikzcd}[column sep=small] \cdots \arrow[r] & \pi_n(F_{p(e)},e) \arrow[r, "i_*"] & \pi_n(E,e) \arrow[r, "j_*"] \arrow[dr, "p_* j_*"'] & \pi_n(E,F_{p(e)},e) \arrow[r, "\partial"] \arrow[d, "p_*"', "\cong"] & \pi_{n-1}(F_{p(e)},e) \arrow[r] & \cdots \\ & & & \pi_n(B,p(e)) \arrow[ur, "\bar{\partial}=\partial p_*^{-1}"'] & & \end{tikzcd}$$

and we define $\bar{\partial}=\partial p_*^{-1}:\pi_n(B,p(e)) \to \pi_{n-1}(F_{p(e)},e)$ $(n>1)$, then the sequence

$$\cdots \to \pi_n(F_{p(e)},e) \xrightarrow{i_*} \pi_n(E,e) \xrightarrow{p_*} \pi_n(B,p(e)) \xrightarrow{\bar{\partial}} \pi_{n-1}(F_{p(e)},e) \to \cdots$$

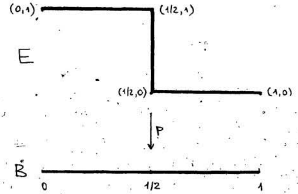
so defined is an exact sequence of groups and homomorphisms. This is called the exact homotopy sequence of the Serre fibration p.

Unlike Hurewicz and Dold fibrations, fibres of a Serre fibration over points lying in the same path component are not necessarily of the same homotopy type; an example is given by the Serre fibration previously constructed taking the fibres over 0 and 1. But it can be proved (cfr.[40; cor.7.8.4]) that fibres of a Serre fibration over points lying in the same path component have the same weak homotopy type, that is, there exists a map $f:F_b \to F_{b'}$ such that $f_*:\pi_n(F_b,e) \to \pi_n(F_{b'},f(e))$ is an isomorphism for every $e\in F_b$ and integer $n>0$.

A larger class of maps related to the covering homotopy property, which includes Hurewicz, Dold and Serre fibrations, was introduced by Dold and Thom in [16] and [17] in connection with the study of infinite symmetric products. This is the class of quasifibrations. A map $p:E \to B$ is a quasifibration if, for every $e \in E$ and integer $n \geq 1$, the function $p_*:\pi_n(E,F_{p(e)},e) \to \pi_n(B,p(e))$ is bijective and the sequence of pointed sets $\pi_0(F_{p(e)},e) \xrightarrow{i_*} \pi_0(E,e) \xrightarrow{p_*} \pi_0(B,p(e))$ is exact. Geometrically, the latter condition means that, for every $b \in \mathrm{Im}p$ and $e \in E$, $p(e)$ can be joined to b by a path if and only if e can be joined by a path in E to some point in F_b. We will call a subset U of B distinguished for the map $p:E \to B$ if the restriction of p to U, $p_U:E_U \to U$, is a quasifibration.

We have already observed that Serre fibrations are quasifibrations (proposition 6); therefore, Hurewicz fibrations, which are particular Serre fibrations, are also quasifibrations. With an argument similar to that used to prove proposition 6, it can be shown that Dold fibrations, too, are quasifibrations.

We now present a simple example of a "genuine" quasifibration, that is, a map which is a quasifibration but fails to be a Dold and a Serre fibration, and so, a fortiori, a Hurewicz fibration. Let $E=[0,1/2]\times\{1\}\cup\{1/2\}\times I\cup[1/2,1]\times\{0\}\subseteq R^2$, $B=I$ and let $p:E \to B$ be the projection on the first factor.



For every $e \in E$ the sequence of pointed sets

$\pi_0(F_{p(e)},e) \xrightarrow{i_*} \pi_0(E,e) \xrightarrow{p_*} \pi_0(B,p(e))$ is exact because E is path-connected. For the function

$p_*:\pi_n(E,F_{p(e)},e) \to \pi_n(B,p(e))$, $n>0$, we have that $\pi_n(B,p(e))=0$ because B is contractible and that $\pi_n(E,F_{p(e)},e)=0$ because the sequence $\pi_n(E,e) \xrightarrow{j_*} \pi_n(E,F_{p(e)},e) \xrightarrow{\partial} \pi_{n-1}(F_{p(e)},e)$ is exact (it is a part of the long exact sequence of the pointed pair $(E,F_{p(e)},e)$) and E and $F_{p(e)}$ are contractible; hence p_* is bijective. It is easily seen that p fails to be a Dold and a Serre fibration.

We recall that a <u>filtered space</u> is a pair $(B,\{S_n|n\geq 0\})$, where B is a topological space and $\{S_n|n\geq 0\}$ is an ascending chain $S_0 \subseteq S_1 \subseteq \ldots$ of closed subsets covering B such that B has the weak topology with respect to $\{S_n | n \geq 0\}$. The following proposition, resulting from propositions 2.2, 2.10 and 2.15 of [17], describes a general method for proving that a map is a quasifibration.

Proposition 7 Let $p: E \to B$ be a map onto a filtered space $(B, \{S_n | n \geq 0\})$. Then each S_n is distinguished for p and p is a quasifibration provided that:

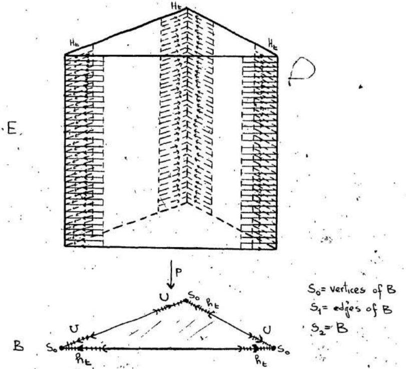
(i) S_0 and every open subset of $S_n - S_{n-1}$ $(n>0)$ is distinguished;

(ii) for each $n>0$ there is an open subset U of S_n containing S_{n-1} and homotopies $h: U \times I \to U$ and $H: E_U \times I \to E_U$ such that:

(a) $h_0 = 1_U$, $h_t(S_{n-1}) \subseteq S_{n-1}$ and $h_1(U) \subseteq S_{n-1}$;

(b) $H_0 = 1_{E_U}$ and H covers h, that is, $pH_t = h_t p$;

(c) $H_1: E_b \to E_{h_1(b)}$ is a weak homotopy equivalence for all $b \in U$.



Although quasifibrations do not satisfy the CHP, they do exibit a property of this kind.

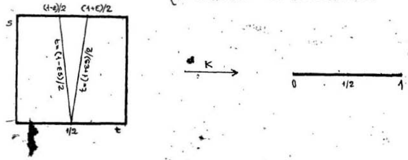
Proposition 8 Let $p:E \to B$ be a quasifibration. If P is a polyhedron, $f:P \to E$ a map and $H:P\times I \to B$ a homotopy of pf with $\mathrm{Im}H \subseteq \mathrm{Im}p$, then there exists a homotopy $H':P\times I \to B$ of pf, "arbitrarily near" to H and homotopic rel. $P\times\{0\}$ to H, such that H' can be lifted by $\widetilde{H}'$ with $\widetilde{H}'_0=f$.

The above property is called "rélévement des homotopies homotopes" in [16]. By a <u>polyhedron</u> we mean a topological space homeomorphic to the geometric realization of some simplicial complex (cfr.[40; p.113]). For a rigorous definition of the expression "arbitrarily near" we refer the reader to propositon 2.7 of [17], letting the next example give a feeling for its meaning in a concrete situation. Consider the "step" quasifibration $p:E \to B$ previously defined. Let $P=\{*\}$ be a one-point space, $f:P \to E$ the map defined by $f(*)=(0,1)$ and $H:P\times I \to B$ the homotopy of pf given by $H(*,t)=t$. Because of the "step", H cannot be lifted. For every $\varepsilon>0$ arbitrarily small define the homotopy $H':P\times I \to B$ by

$$H'(*,t) = \begin{cases} t/1-\varepsilon & \text{if } 0\leq t\leq (1-\varepsilon)/2 \\ 1/2 & \text{if } (1-\varepsilon)/2\leq t\leq (1+\varepsilon)/2 \\ (t-\varepsilon)/1-\varepsilon & \text{if } (1+\varepsilon)/2\leq t\leq 1 \end{cases}$$

H is homotopic rel. $P\times\{0\}$ to H' by the homotopy $K:P\times I\times I \to B$ defined as follows:

$$K(*,t,s) = \begin{cases} t/1-\varepsilon s & \text{if } 0\leq t\leq (1-\varepsilon s)/2 \\ 1/2 & \text{if } (1-\varepsilon s)/2\leq t\leq (1+\varepsilon s)/2 \\ (t-\varepsilon s)/1-\varepsilon s & \text{if } (1+\varepsilon s)/2\leq t\leq 1 \end{cases}$$

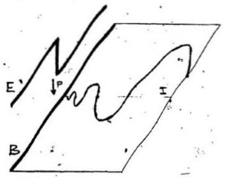


Now H'^{x} can be lifted by $\tilde{H}':P\times I \to E$, where

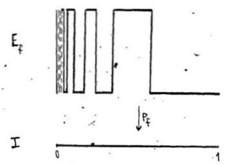
$$\tilde{H}'(*,t) = \begin{cases} (t/1-\varepsilon, 1) & \text{if } 0\le t\le (1-\varepsilon)/2 \\ (1/2, (1+\varepsilon-2t)/2\varepsilon) & \text{if } (1-\varepsilon)/2\le t\le (1+\varepsilon)/2 \\ ((t-\varepsilon)/1-\varepsilon, 0) & \text{if } (1+\varepsilon)/2\le t\le 1. \end{cases}$$

Intuitively we have modified H to stay for an ε-short while at $1/2$ and then we have used this pause to climb the step.

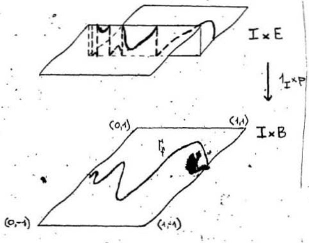
It is quite easy to see that being a quasifibration is a property invariant under fibre homotopy equivalence. However the pullback of a quasifibration need not be a quasifibration as shown by the following original example. Take again the "step" quasifibration $p:E \to B$, slightly modified for convenience by $E=[-1,0]\times\{1\}\cup\{0\}\times I\cup[0,1]\times\{0\}\subseteq R^2$, $B=[-1,1]$ and $p:E \to B$ the projection on the first factor. The map along which we pull back is $f:I \to B$ given by $f(t)=t\sin(1/t)$, if $t>0$, and $f(0)=0$.



The pullback $p_f : E_f \to I$ is illustrated in the following picture

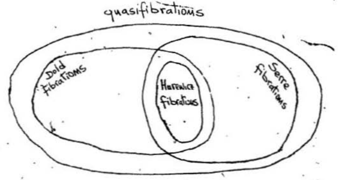


In fact E_f can be obtained by considering the map $1_I \times p : I\times E \to I\times B$ and then taking the anti-image $(1_I \times p)^{-1}(\Gamma_f)$, where $\Gamma_f \subseteq I\times B$ is the graph of f; $p_f : E_f \to I$ is then the restriction to E_f of the composition $pr_1(1_I \times p)$



The pullback $p_f : E_f \to I$ is not quasifibration because the sequence of pointed sets $\pi_0(F_{p_f(e)}, e) \xrightarrow{i_*} \pi_0(E_f, e) \xrightarrow{(p_f)_*} \pi_0(I, p_f(e))$ is not exact for every $e \in E_f$. Indeed, I and the fibres $F_{p_f(e)}$ are path connected but E_f has two path-components, the fibre over 0 and its complement.

We can summarize the relationship between the different kinds of fibrations we have introduced by the following diagram



Chapter II

RECENT DEVELOPMENTS

1. FUNCTIONAL SPACES AND FIBRED EXPONENTIAL LAWS

This section is devoted to generalizing the classical exponential correspondence (proposition I.1.1(iv)) in the case when:

(i) spaces are replaced by maps and maps between spaces are replaced by map pairs (1st Fibred Exponential Correspondence, theorem 3);

(ii) spaces are replaced by maps over a fixed base space B and maps between spaces are replaced by fibre maps over B (2nd Fibred Exponential Correspondence, theorem 9).

Convention Since we will be concerned with many maps (and their fibres) at the same time, we will henceforth denote the fibre of a map $p:E \to B$ over $b \in B$ by E_b and not by F_b, as before. Furthermore, maps of the type $\{*\} \times A \to B$, $A \to \{*\} \times B$ and $\{*\} \times A \to \{*'\} \times B$, where $\{*\}$ and $\{*'\}$ are one-point spaces, will be tacitly identified with the map $A \to B$ obtained by identifying in the canonical way the corresponding domains and codomains.

An important role in our arguments will be played by the following construction, first introduced in [9]. Although this construction is not absolutely indispensable, it does simplify proofs notably because it allows us to apply directly the classical exponential correspondence, rather than using partial (closed) maps and their exponential correspondence (cfr. [9; th.1.4]).

We associate to any space B a new space B^+ defined as follows: set theoretically, $B^+=B\cup\{\infty\}$, where $\infty\notin B$; the topology on B^+ has as open sets the empty set and all sets of the form $U\cup\{\infty\}$, with $U\subseteq B$ open. This topology is well defined; indeed, $B^+=B\cup\{\infty\}$ is open, the union of any family of open sets is an open set, since $\bigcup_{j\in J}(U_j\cup\{\infty\})=(\bigcup_{j\in J}U_j)\cup\{\infty\}$, and the intersection of two open sets is an open set, since $(U_1\cup\{\infty\})\cap(U_2\cup\{\infty\})=(U_1\cap U_2)\cup\{\infty\}$. The closed subsets of B^+ are B^+ and all the subsets of B which are closed in B. Furthermore, the topology induced on B by B^+ coincides with the original topology on B. If $f:B\to B'$ is a map, we define $f^+:B^+\to B'^+$ by $f^+(b)=f(b)$, if $b\in B$, and $f^+(\infty)=\infty'$. Since $(f^+)^{-1}(\emptyset)=\emptyset$ and $(f^+)^{-1}(U\cup\{\infty'\})=f^{-1}(U)\cup\{\infty\}$, we deduce that f^+ is continuous. From the above considerations and from the relation $(gf)^+=g^+f^+$, we notice that this construction gives rise to a covariant functor from Top to itself.

Let $A_0\subseteq A$ be a closed subspace and let $f:A_0\to B$ be a map. We define a map $\bar{f}:A\to B^+$ by $\bar{f}(a)=f(a)$, if $a\in A_0$, and $\bar{f}(a)=\infty$, otherwise. $\bar{f}$ is continuous because $\bar{f}^{-1}(\emptyset)=\emptyset$ and, for every open set $U'=U\cup\{\infty\}\subseteq B^+$, $\bar{f}^{-1}(U')=\bar{f}^{-1}(U\cup\{\infty\})=\bar{f}^{-1}(U)\cup\bar{f}^{-1}(\infty)=f^{-1}(U)\cup\bar{f}^{-1}(\infty)=V\cap A_0\cup(A-A_0)=V\cup(A-A_0)$, for some open set $V\subseteq A$. On the other hand, if $h:A\to B^+$ is a map, we define $A_h=h^{-1}(B)$ and $h^-:A_h\to B$ by $h^-(a)=h(a)$. Then A_h is a closed subspace of A, since B is closed in B^+, and h^- is of course continuous.

We are now ready to define our main construction, which generalizes the usual mapping space $M(D,E)$ with the compact-open topology. Given maps $q:D \to A$ and $p:E \to B$, with q having fibres closed in D, we define a map $q\cdot p:D\cdot E \to A\times B$ in the following way. The underlying set of $D\cdot E$ is $\coprod_{(a,b)\in A\times B} M(D_a,E_b) = \bigcup_{(a,b)\in A\times B} M(D_a,E_b)\times\{(a,b)\}$; $D\cdot E$ is topologized by the initial topology with respect to the functions $q\cdot p:D\cdot E \to A\times B$ and $j:D\cdot E \to M(D,E^+)$ given by $q\cdot p(f,a,b)=(a,b)$ and $j(f,a,b)=\bar{f}$. The space $D\cdot E$ is called the <u>functional space of</u> q <u>and</u> p.

<u>Remark 1</u> Our definition of the space $D\cdot E$ differs slightly from the definition presented in [10] because there the authors implicitly assumed p and q to be onto. In [10] the underlying set of $D\cdot E$ is defined to be $\bigcup_{(a,b)\in A\times B} M(D_a,E_b)$ and $q\cdot p$ is defined by $q\cdot p(f)=(a,b)$, if $f\in M(D_a,E_b)$. Now taking the union in this situation can lead to problems. For example, suppose that $A-\mathrm{Im}q$ contains at least two distinct points, a and a', and that B is not empty. Hence $D_a=D_{a'}=\emptyset$ and so, for every $b\in B$, we have that $M(D_a,E_b)=M(D_{a'},E_b)=\{\phi\}$, where ϕ denotes the "empty map" (cfr. [22, p. 33]). Therefore, when we take the union $\bigcup_{(a,b)\in A\times B} M(D_a,E_b)$, ϕ will appear only once and so $q\cdot p(\phi)$ is not well defined. Using the disjoint union avoids the problem.

<u>Proposition 2</u> Let $q: D \to A$ and $p: E \to B$ be maps with q having closed fibres. The following properties hold:

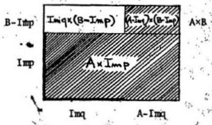
(i) $\operatorname{Im}(q\cdot p) = A \times \operatorname{Im}p \cup (A - \operatorname{Im}q) \times (B - \operatorname{Im}p)$ and $A \times B - \operatorname{Im}(q\cdot p) = \operatorname{Im}q \times (B - \operatorname{Im}p)$;

(ii) the fibre of $q\cdot p$ over $(a,b) \in A \times B$ is $\mathcal{M}(D_a, E_b) \times \{(a,b)\}$;

(iii) if $p': E' \to B'$ is a map and $(h,g): p \to p'$ a map pair, then the function $h_*: D\cdot E \to D\cdot E'$ given by $h_*(f,a,b) = (h_b f, a, g(b))$ is continuous and the following diagram commutes

$$\begin{tikzcd}
D\cdot E \arrow[r, "h_*"] \arrow[d, "q\cdot p"'] & D\cdot E' \arrow[d, "q\cdot p'"] \\
A\times B \arrow[r, "1_A \times g"] & A\times B'
\end{tikzcd}$$

<u>Proof</u> (i) We have that $(a,b) \in \operatorname{Im}(q\cdot p)$ if and only if $\mathcal{M}(D_a, E_b) \neq \emptyset$, that is, if and only if either $E_b \neq \emptyset$ or $D_a = E_b = \emptyset$; hence $\operatorname{Im}(q\cdot p) = A \times \operatorname{Im}p \cup (A - \operatorname{Im}q) \times (B - \operatorname{Im}p)$. On the other hand, we have that $(a,b) \notin \operatorname{Im}(q\cdot p)$ if and only if $\mathcal{M}(D_a, E_b) = \emptyset$, that is, if and only if $D_a \neq \emptyset$ and $E_b = \emptyset$; hence $A \times B - \operatorname{Im}(q\cdot p) = \operatorname{Im}q \times (B - \operatorname{Im}p)$.



(ii) Set theoretically the fiber of $q\cdot p$ over (a,b) is $M(D_a,E_b)\times\{(a,b)\}$. Since the subspace topology induced on it by $D\cdot E$ coincides with the initial topology with respect to the restrictions of $q\cdot p$ (which now is constant) and j, we have only to show that the compact-open topology k on $M(D_a,E_b)$ coincides with the initial topology, say τ, with respect to the function $j:M(D_a,E_b)\to M(D,E^+)$.

$\tau\subseteq k$: This is equivalent to showing that $j:M(D_a,E_b)\to M(D,E^+)$ is continuous. To this end it is enough to prove that the anti-image of a subbasic set $\langle K,U'\rangle$ of $M(D,E^+)$ is open in k. Let $U'=U\cup\{\infty\}$, with $U\subseteq E$ open, and define $K_a=K\cap D_a$ and $U_b=U\cap E_b$. Since D_a is closed in D, $K\cap D_a$ is closed in K and so K_a is compact. Now it is straightforward to see that $j^{-1}(\langle K,U'\rangle)=\langle K_a,U_b\rangle$.

$\tau\supseteq k$: We must prove that any open set of k is the anti-image by j of some open set of $M(D,E^+)$. Since the operation of taking the anti-image preserves intersections and unions, it is enough to check for a subbasic set $\langle K,U\rangle$ of k, $K\subseteq D_a$ compact and $U\subseteq E_b$ open. Now it is straightforward to see that $\langle K,U\rangle=j^{-1}(\langle K,V'\rangle)$, where $V'=V\cup\{\infty\}$ with V some open set of E such that $U=V\cap E_b$.

(iii) Since $D\cdot E'$ has the initial topology with respect to the functions $q\cdot p':D\cdot E'\to A\times B$ and $j':D\cdot E'\to M(D,E'^+)$, we have that $h_*:D\cdot E\to D\cdot E'$ is continuous if and only if the compositions $(q\cdot p')h_*$ and $j'h_*$ are continuous. Now the following diagram is commutative

$$\begin{tikzcd} M(D,E^+) \arrow[r, "(h^+)_*"] & M(D,E'^+) \\ D\cdot E \arrow[u, "j"] \arrow[r, "h_*"] \arrow[d, "q\cdot p"'] & D\cdot E' \arrow[u, "j'"'] \arrow[d, "q\cdot p'"] \\ A\times B \arrow[r, "1_A\times g"] & A\times B' \end{tikzcd}$$

Indeed, for the bottom square we have that $(q\cdot p')h_*(f,a,b)=q\cdot p'(h_b f,a,g(b))=(a,g(b))=(1_A\times g)(a,b)=[(1_A\times g)(q\cdot p)](f,a,b)$ and for the upper square we have that

$$[j'h_*(f,a,b)](d)=[j'(h_b f,a,g(b))](d)=\begin{cases} h_b f(d) & \text{if } d\in D_a \\ \infty' & \text{otherwise} \end{cases}$$

and

$$[(h^+)_* j(f,a,b)](d)=\begin{cases} h_b f(d) & \text{if } d\in D_a \\ \infty' & \text{otherwise} \end{cases}$$

Therefore, from the equalities $(q\cdot p')h_*=(1_A\times g)(q\cdot p)$ and $j'h_*=(h^+)_* j$, it follows that h_* is continuous.

<u>Theorem 3</u> (Fibred Exponential Correspondence I). Let $p:E\to B$, $q:D\to A$, $r:C\to A$ and $g:C\to B$ be maps with A Hausdorff and D locally compact, Hausdorff. Then there is a canonical one-to-one correspondence θ between fibre maps over g from the pullback q_r of q along r to p and liftings of

$(r,g):C \to A\times B$ over $q\cdot p:D\cdot E \to A\times B$. The correspondence $\theta:M_g(q_r,p) \to L((r,q),q\cdot p)$ is defined on $f:D_r \to E$ by $\theta(f)(c)=(f_c,r(c),g(c))$, $c\varepsilon C$.

The above result can be illustrated by the diagrams

$$\begin{tikzcd} D \arrow[d, "q"'] & D_r \arrow[l, "pr_2"'] \arrow[r, dashed, "f"] \arrow[d, "q_r"] & E \arrow[d, "p"] \\ A & C \arrow[l, "r"] \arrow[r, "g"'] & B \end{tikzcd} \qquad \begin{tikzcd} & D\cdot E \arrow[d, "q\cdot p"] \\ C \arrow[ur, dashed, "\theta(f)"] \arrow[r, "(r,g)"'] & A\times B \end{tikzcd}$$

<u>Proof</u> We must first check that θ is well defined, that is, $\theta(f):C \to D\cdot E$ is continuous for every $f\varepsilon M_g(q_r,p)$. For notational convenience let $\hat{f}=\theta(f)$. Since $D\cdot E$ has the initial topology with respect to the functions $q\cdot p:D\cdot E \to A\times B$ and $j:D\cdot E \to M(D,E^+)$, $\hat{f}$ is continuous if and only if the compositions $(q\cdot p)\hat{f}$ and $j\hat{f}$ are continuous. From $[(q\cdot p)\hat{f}](c)=q\cdot p(\hat{f}(c))=q\cdot p(f_c,r(c),g(c))=(r(c),g(c))$, we have that $(q\cdot p)\hat{f}=(r,g)$ and so $(q\cdot p)\hat{f}$ is continuous. The continuity of $j\hat{f}$ is proved in the following way. Since A is Hausdorff, we have that $D_r=\{(c,d)\varepsilon C\times D\colon r(c)=q(d)\}=(r\times q)^{-1}(\Delta)$ ($\Delta\subseteq A\times A$ is the diagonal) is closed in $C\times D$. Therefore, the function $\bar{f}:C\times D \to E^+$, given by $\bar{f}(c,d)=f(c,d)$, if $(c,d)\varepsilon D_r$, and $\bar{f}(c,d)=\infty$

otherwise, is continuous. Let $F:C \to M(D,E^+)$ be the adjoint of $\bar{f}$. Then F is continuous and we have that

$$[F(c)](d)=\bar{f}(c,d)=\begin{cases} f(c,d)=f_c(d) & \text{if } d\varepsilon D_{r(c)} \\ \infty & \text{otherwise} \end{cases}$$

and that

$$[j\hat{f}(c)](d)=[j(f_c,r(c),g(c)](d)=\overline{f_c}(d)=\begin{cases} f_c(d) & \text{if } d\varepsilon D_{r(c)} \\ \infty & \text{otherwise} \end{cases}$$

Hence $j\hat{f}=F$ and so $j\hat{f}$ is continuous.

It is straigthforward to see that θ is injective. Indeed, if $f,f':D_r \to E$ are distinct fibre maps over g, then there is some $(c,d)\varepsilon D_r$ with $f(c,d)\neq f'(c,d)$; it follows that $f_c\neq f'_c$ and so $\theta(f)(c)\neq\theta(f')(c)$; hence $\theta(f)\neq\theta(f')$. To prove that θ is surjective, let $k=(k^*,r,g):C \to D\cdot E$ be a lifting of $(r,g):C \to A\times B$; so, for every $c\varepsilon C$, $k^*(c)$ is a map from $D_{r(c)}$ to $E_{g(c)}$. Denote by $K:C\times D \to E^+$ the adjoint of jk; then K is continuous and we have that

$$K(c,d)=[jk(c)](d)=\overline{k^*(c)}(d)=\begin{cases} k^*(c)(d) & \text{if } d\varepsilon D_{r(c)} \\ \infty & \text{otherwise} \end{cases}$$

Let $k'=K|D_r:D_r \to E$; then k' is a fibre map over g from q_r to p, since $pk'(c,d)=pK(c,d)=p(k^*(c)(d))=g(c)=gq_r(c,d)$, and furthermore $k'_c=k^*(c)$, for every $c\varepsilon C$, since

$k'_c(d)=k'(c,d)=K(c,d)=k^*(c)(d)$. Therefore $\theta(k')=k$.

Corollary 4 Keeping the same notation and hypothesis as in theorem 3, we have that under the bijective correspondence $\theta: M_g(q_r,p) \to L((r,g),q\cdot p)$ two fibre maps over g are fibre homotopic over g if and only if their corresponding liftings are vertically homotopic.

Proof Let $f,f':D_r \to E$ be fibre maps over g. We must prove that f and f' are fibre homotopic over g if and only if their corresponding liftings of (r,g), $\hat{f}=\theta(f)$ and $\hat{f'}=\theta(f')$, are vertically homotopic. Suppose f and f' are fibre homotopic over g and let $H:D_r\times I \to E$ be a vertical homotopy from f to f'. If $R:C\times I \to A$ and $G:C\times I \to B$ denote the homotopies stationary at r and at g, respectively, and if we identify D_R, the domain of the pullback of q along R, with $D_r\times I$ via the correspondence $(c,t,d)\varepsilon D_R \leftrightarrow (c,d,t)\varepsilon D_r\times I$, then H can be regarded as a fibre map over G from the pullback q_R of q along R to p. By theorem 3 applied to the maps p, q, R and G, we have that $\hat{H}=\theta(H)$ is a lifting of $(R,G):C\times I \to A\times B$. Now $\hat{H}_0(c)=\hat{H}(c,0)=(H_{(c,0)},R(c,0),G(c,0))=(f_c,r(c),g(c))=\hat{f}(c)$ and $\hat{H}_1(c)=\hat{H}(c,1)=(H_{(c,1)},R(c,1),G(c,1))=(f'_c,r(c),g(c))=\hat{f'}(c)$; hence, since (R,G) is a stationary homotopy, we have that $\hat{H}$ is a vertical homotopy from $\hat{f}$ to $\hat{f'}$.

Conversely, suppose that the liftings $\hat{f},\hat{f'}:C \to D\cdot E$ are vertically homotopic, say by $K=(K^*,R,G):C\times I \to D\cdot E$. It follows that K is a lifting of (R,G), and, for every $t\varepsilon I$,

$K^*(c,t)$ is a map from $D_{r(c)}$ to $E_{g(c)}$; in particular, $K^*(c,0)=f_c$ and $K^*(c,1)=f'_c$. Now let $K'=\theta^{-1}(K):D_R \to E$. Identifying D_R with $D_r \times I$, we have that $K'_0(c,d)=K'(c,d,0)=K^*(c,0)(d)=f_c(d)=f(c,d)$, $K'_1(c,d)=K'(c,d,1)=K^*(c,1)(d)=f'_c(d)=f'(c,d)$ and that $pK'(c,d,t)=G(c,t)=g(c)$. Therefore K' is a vertical homotopy from f to f'.

<u>Corollary 5</u> Let $p:E \to B$, $q:D \to A$ and $r:C \to A$ be maps with A Hausdorff and D locally compact, Hausdorff. Then there is a canonical one-to-one correspondence ϕ between map pairs $(f,g):q_r \to p$ and liftings of $r:C \to A$ over $q_1 \cdot p:D \cdot E \to A$. The correspondence $\phi: M(q_r,p) \to L(r,q_1 \cdot p)$ is given by $\phi(f,g)(c)=(f_c,r(c),g(c))$, $c \in C$, and under this correspondence two map pairs are homotopic if and only if their corresponding liftings are vertically homotopic. In particular, map pairs from q to p are in one-to-one correspondence with sections of $q_1 \cdot p$ and two map pairs are homotopic if and only if their corresponding sections are vertically homotopic.

The above result can be illustrated by the diagrams

$$\begin{tikzcd}
D \arrow[d, "q"'] & D_r \arrow[l, "pr_2"'] \arrow[r, dashed, "f"] \arrow[d, "q_r"] & E \arrow[d, "p"] \\
A & C \arrow[l, "r"'] \arrow[r, dashed, "g"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & D \cdot E \arrow[d, "q_1 \cdot p"] \\
C \arrow[ur, dashed, "\psi(f,g)"] \arrow[r, "r"'] & A
\end{tikzcd}$$

Proof We start by observing that $M(q_r,p) = \coprod_{g\in M(C,B)} M_g(q_r,p)\times\{g\}$ and that $L(r,q_1\cdot p) = \coprod_{g\in M(C,B)} L((r,g),q\cdot p)$. Indeed, for the latter equality we have that if $h:C\to D\cdot E$ is a lifting of r over $q_1\cdot p = pr_1(q\cdot p)$, then h is a lifting of (r,g) over $q\cdot p$ for $g = pr_2(q\cdot p)h$; conversely, if h is a lifting of $(r,g):C\to A\times B$ over $q\cdot p$ for some $g\in M(C,B)$, then $(q_1\cdot p)h = pr_1(q\cdot p)h = pr_1(r,g) = r$ and so h is a lifting of r over $q_1\cdot p$. For each $g\in M(C,B)$, denote by $\theta_g: M_g(q_r,p)\to L((r,g),q\cdot p)$ the fibred exponential correspondence given by theorem 3. Define $\phi': M(q_r,p)\to L(r,q_1\cdot p)$ by $\phi'(f,g) = \theta_g(f)$. Then ϕ' is well defined; ϕ' is injective, since each θ_g is injective and the images of two distinct θ_g's are disjoint, and ϕ' is surjective, since the image of ϕ' is the union of the images of all θ_g's, which is $L(r,q_1\cdot p)$. Now, from $\phi(f,g) = (f_c, r(c), g(c)) = \theta_g(f) = \phi'(f,g)$, we get that our ϕ is bijective.

We now prove the second part of our statement. First observe that identifying the map $q_r\times 1_I: D_r\times I\to C\times I$ with $q_R: D_R\to C\times I$, where $R: C\times I\to A$ is the homotopy stationary at r, we can regard any homotopy pair (H,K) as a map pair from q_R to p. We can then apply what we have already proved to deduce a bijective correspondence between homotopy pairs (H,K) and liftings of $R: C\times I\to A$ over $q_1\cdot p$, these latter being vertical homotopies, since R is stationary. If follows that two map pairs from q_r to p are homotopic if and only if their corresponding liftings are vertically homotopic. The third part of our statement follows immediately from what we have already proved, taking $C=A$, $r=1_A$ and identifying $q_{1_A}:D_{1_A} \to C$ with $q:D \to A$.

<u>Corollary 6</u> Let $p:E \to B$ and $f,g:A \to B$ be maps with B Hausdorff and E locally compact, Hausdorff. Then there is a canonical one-to-one correspondence ϕ between fibre maps $h:E_f \to E_g$ over A and liftings of $(f,g):A \to B\times B$ over $p\cdot p:E\cdot E \to B\times B$. The correspondence $\phi: M_A(p_f,p_g) \to L((f,g),p\cdot p)$ is given by $\phi(h)(a)=(h_a,f(a),g(a))$ and under this correspondence two fibre maps are fibre homotopic over A if and only if their corresponding liftings are vertically homotopic.

<u>Proof</u> Consider the following commutative diagram (ignore the dotted arrows)

$$\begin{tikzcd}
E_f \arrow[dr, dashed, "h"] \arrow[drr, dashed, bend left, "h'"] \arrow[ddr, "p_f"'] & & \\
& E_g \arrow[r, "pr_2"] \arrow[d, "p_g"] & E \arrow[d, "p"] \\
& A \arrow[r, "g"] & B
\end{tikzcd}$$

Since p_g is a pullback, the set $M_A(p_f,p_g)$ of all fibres maps $h:E_f \to E_g$ over A is in one-to-one correspondence with the set of all maps $k:E_f \to E$ such that $pk=gp_f$, that is, the set $M_g(p_f,p)$. This correspondence, denoted by π, associates to h the fibre map over g given by $h'=pr_2h$. On the other hand, by theorem 3, there is a bijective correspondence θ between the set of all fibre maps $k:E_f \to E$ over g and the set of all liftings of $(f,g):A \to B\times B$ over $p\cdot p:E\cdot E \to B\times B$. Now, the following diagram commutes

$$\begin{tikzcd}
M_A(p_f,p_g) \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & L((f,g),p\cdot p) \\
& M_g(p_f,p) \arrow[ur, "\theta"'] &
\end{tikzcd}$$

Indeed, for every $h\varepsilon M_A(p_f,p_g)$ we have that $\theta\pi(h)(a)=\theta(h')(a)=(h'_a,f(a),g(a))$, $\phi(h)(a)=(h_a,f(a),g(a))$ and $h'_a:(a,e)\varepsilon E_{f,a}=\{a\}\times E_{f(a)} \to pr_2h(a,e)\varepsilon E_{g(a)}$ is equal to $h_a:(a,e)\varepsilon E_{f,a}=\{a\}\times E_{f(a)} \to h(a,e)\varepsilon E_{g,a}=\{a\}\times E_{g(a)}$ under the usual identification. This proves that ϕ is bijective. To prove the second part of our statement, observe that under the bijective correspondence π fibre homotopic maps over A correspond to fibre homotopic maps over g and that, by corollary 4, under the bijective correspondence θ, fibre homotopic maps over g correspond to vertically homotopic liftings.

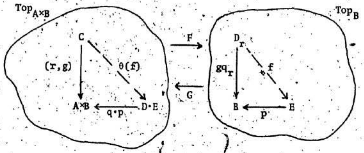
Since the ordinary topological exponential correspondence can be expressed in categorical language by the statement that for every locally compact, Hausdorff space B the functor $-\times B$: Top $\to$ Top is left adjoint to the functor $(-)^B$: Top $\to$ Top, it is natural to ask if our fibred exponential correspondence, which generalizes the classical one when A and B are one-point spaces, can be expressed by the adjointness of appropriate functors.

Suppose fixed a map $q: D \to A$ with closed fibres and a space B. Consider the categories $Top_{A\times B}$ and Top_B and define a functor $F: Top_{A\times B} \to Top_B$ by $F(r,g)=gq_r: D_r \to B$, on objects $(r,g): C \to A\times B$, and $F(m)=\bar{m}: D_r \to D_{r'}$, on morphisms $m:(r,g) \to (r',g')$, where $\bar{m}$ is the unique map making the following diagram commutative

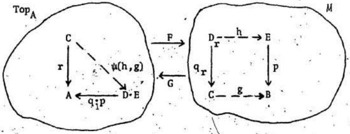
$$\begin{tikzcd}
D \arrow[d, "q"] & D_{r'} \arrow[l, "pr_2"'] \arrow[d, "q_{r'}"] & D_r \arrow[l, dashed, "\bar{m}"'] \arrow[ll, bend right, "pr_2"'] \arrow[d, "q_r"] \\
A & C' \arrow[l, "r'"] & C \arrow[l, "m"]
\end{tikzcd}$$

that is, $\bar{m}(c,d)=(m(c),d)$.

Now define a functor $G:Top_B \to Top_{A\times B}$ by $G(p)=q\cdot p:D\cdot E \to A\times B$, on objects $p:E \to B$, and $G(n)=n_*:(f,a,b)\in D\cdot E \to (n_b f,a,b)\in D\cdot E'$, on morphisms $n:p \to p'$. The continuity of n_* follows from proposition 2(iii) applied to the map pair $(n,1_B):p \to p'$. Then theorem 3 says that if A is Hausdorff and D is locally compact, Hausdorff, the functor F is left adjoint to the functor G with adjunction given by θ.



To express the fibred exponential correspondence of corollary 5 we have to change a little the categories involved. Suppose fixed a map $q:D \to A$ with closed fibres and consider the categories Top_A and $\mathcal{M}$, the latter being the category of maps and map pairs. Define a functor $F:Top_A \to \mathcal{M}$ by $F(r)=q_r:D_r \to C$, on objects $r:C \to A$, and $F(m)=(\bar{m},m):q_r \to q_{r'}$, on morphisms $m:r \to r'$ ($\bar{m}$ is the same as above). Now define a functor $G: M \to Top_A$ by $G(p)=q_1 \cdot p: D \cdot E \to A$, on objects $p: E \to B$, and $G(h,k)=h_*: (f,a,b) \varepsilon D \cdot E \to (h_b f, a, g(b)) \varepsilon D \cdot E'$, on morphisms $(h,k): p \to p'$. Then the fibred exponential correspondence of corollory 5 says that if A is Hausdorff and D locally compact, Hausdorff, the functor F is left adjoint to the functor G with adjunction ϕ.



We now discuss a modification of the map $q \cdot p$, in the case q and p have the same target space. This modification turns out to be more convenient when we are dealing with maps with the same target space B and with fibre maps over B and historically it came before the "$\cdot$"-construction.

Let $q: D \to B$ and $p: E \to B$ be maps, with q having closed fibres. Define a map $(qp): (DE) \to B$ as follows. As a set, let $(DE) = \coprod_{b \varepsilon B} M(D_b, E_b)$ and define $(qp): (f,b) \varepsilon (DE) \to b \varepsilon B$; then topologize (DE) with initial topology with respect to the functions (qp) and $j: (f,b) \varepsilon (DE) \to \bar{f} \varepsilon M(D, E^+)$. Since the fibres of q are closed, j is well defined.

Remark 7 As in the definition of $D\cdot E$, we have changed slightly the usual definition of (DE) as given, for example, in [4], replacing the union by the disjoint union. This circumvents the problem of the function (qp) not being well defined. This problem arises if and only if the set $(B-\mathrm{Im}q)\cap(B-\mathrm{Im}p)=B-(\mathrm{Im}q\cup\mathrm{Im}p)$ contains at least two distinct points. Indeed, (qp) is not well defined if and only if there exist distinct points $b,b'\in B$ with $M(E_b,E_b')\cap M(E_{b'},E_{b'}')\neq\emptyset$, or, equivalently, such that $E_b=E_{b'}$, $E_b'=E_{b'}'$, and $M(E_b,E_b')\neq\emptyset$; this happens if and only if b and b' are distinct points of $B-(\mathrm{Im}q\cup\mathrm{Im}p)$.

Proposition 8 Let $q:D\to B$ and $p:E\to B$ be maps with q having closed fibres. Then:

(i) $\mathrm{Im}(qp)=\mathrm{Im}p\cup(B-\mathrm{Im}q)$ and $B-\mathrm{Im}(qp)=\mathrm{Im}q\cap(B-\mathrm{Im}p)$;

(ii) the following square

$$\begin{tikzcd}
(DE) \arrow[r, "\overline{\Delta}"] \arrow[d, "(qp)"'] & D\cdot E \arrow[d, "q\cdot p"] \\
B \arrow[r, "\Delta"] & B\times B
\end{tikzcd}$$

where Δ is the diagonal map and $\overline{\Delta}:(f,b)\in(DE)\to(f,b,b)\in D\cdot E$, is cartesian; in particular, $(qp):(DE)\to B$ is in a canonical way fibre homeomorphic over B to the pullback of $q\cdot p: D\cdot E \to B\times B$ along Δ;

(iii) the fibre of (qp) over $b\in B$ is $M(D_b, E_b)\times\{b\}$.

Proof (i) We have that $b\in \mathrm{Im}(qp)$ if and only if $M(D_b,E_b)\neq\emptyset$ and this happens if and only if either $E_b\neq\emptyset$ or $D_b=E_b=\emptyset$. Therefore $\mathrm{Im}(qp)=\mathrm{Im}p\cup(B-\mathrm{Im}q)\cap(B-\mathrm{Im}p)=\mathrm{Im}p\cup(B-\mathrm{Im}q)$. On the other hand, $b\notin\mathrm{Im}(qp)$ if and only if $M(D_b,E_b)=\emptyset$ and this happens if and only if $D_b\neq\emptyset$ and $E_b=\emptyset$; hence $B-\mathrm{Im}(qp)=\mathrm{Im}q\cap(B-\mathrm{Im}p)$.

(ii) It is a straightforward consequence from the definitions.

(iii) It follows from proposition 2(ii).

Given maps $r:C\to B$ and $q:D\to B$, we define the fibred product of r and q to be the map $r\sqcap q:D_r\to B$, where D_r is the domain of the pullback of q along r and $r\sqcap q=rq_r$. The fibre of $r\sqcap q$ over $b\in B$ is $C_b\times D_b$. The fibred product makes Top_B a category with product.

Theorem 9 (Fibred Exponential Correspondence II). Let $r:C\to B$, $q:D\to B$ and $p:E\to B$ be maps with B Hausdorff and D locally compact, Hausdorff. Then there is a canonical one-to-one correspondence ϕ between fibre maps $f:D_r\to E$ over B from $r\sqcap q$ to p and fibre maps over B from r to (qp). The correspondence $\phi: M_B(r\sqcap q,p)\to M_B(r,(qp))$ is given by $\phi(f)(c)=(f_c,r(c))$ where $f_c:d\in D_{r(c)}\to f(c,d)\in E_{r(c)}$ (in other words we are regarding f as a fibre map over r from q_r to p). Furthermore, ϕ an ϕ^{-1} preserve the relation of fibre homotopy over B.

The above result can be illustrated by the diagrams

$$
\begin{tikzcd}
 & D_r \arrow[dl, "q_r"'] \arrow[ddd, "r \cap q"] \arrow[dr, "pr_2"] \arrow[rr, dashed, "f"] & & E \arrow[dddll, "p"] \\
C \arrow[ddr, "r"'] & & D \arrow[ddl, "q"] & \\
 & & & \\
 & B & &
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[rr, dashed, "\psi(f)"] \arrow[dr, "r"'] & & (DE) \arrow[dl, "(qp)"] \\
 & B &
\end{tikzcd}
$$

Proof Consider the following commutative diagrams (ignore the dotted arrows)

$$
\begin{tikzcd}
D \arrow[d, "q"'] & D_r \arrow[l, "pr_2"'] \arrow[r, dashed, "f"] \arrow[d, "q_r"] \arrow[dr, "r \cap q"] & E \arrow[d, "p"] \\
B & C \arrow[l, "r"] \arrow[r, "r"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[dr, dashed] \arrow[drr, dashed, "\psi(f)"] \arrow[ddr, "r"'] & & \\
 & (DE) \arrow[r, "\bar{\Delta}"] \arrow[d, "(qp)"] & D \cdot E \arrow[d, "q \cdot p"] \\
 & B \arrow[r, "\Delta"] & B \times B
\end{tikzcd}
$$

By the 1st fibred exponential correspondence (theorem 3), there is a canonical bijective correspondence $\theta: M_r(q_r, p) \to L((r,r), q\cdot p)$. Now, since $r \sqcap q = rq_r$, we have that $M_B(r\sqcap q, p) = M_r(q_r, p)$ and furthermore, since $\Delta r = (r,r)$, we have that $L(\Delta r, q\cdot p) = L((r,r), q\cdot p)$. By proposition 8(ii) the square in the right diagram is cartesian and hence there is a bijective correspondence $\pi: L(\Delta r, q\cdot p) \to M_B(r, (qp))$; explicitly, if $h \in L(\Delta r, q\cdot p)$ is given by $h(c) = (h^*(c), r(c), r(c))$, $c \in C$, with $h^*(c): D_{r(c)} \to E_{r(c)}$, then $\pi(h)(c) = (h^*(c), r(c))$. Now the bijectiveness of ϕ follows from the observation that $\phi = \pi\theta$, that is, $\phi: M_B(r\sqcap q, p) = M_r(q_r, p) \xrightarrow{\theta} L(\Delta r, q\cdot p) \xrightarrow{\pi} M_B(r, (qp))$. Indeed, $\phi(f)(c) = (f_c, r(c))$ and $\pi\theta(f)(c) = \pi(f_c, r(c), r(c)) = (f_c, r(c))$. The remaining part of the statement follows in the usual way from the invariance property held by θ and π.

<u>Corollary 10</u> Let $q: D \to B$ and $p: E \to B$ be maps with B Hausdorff and D locally compact, Hausdorff. Then there is a canonical one-to-one correspondence ϕ between fibre maps $f: D \to E$ over B and sections of $(qp): (DE) \to B$. The correspondence $\phi: M_B(q,p) \to \mathrm{Sec}(qp)$ is given by $\phi(f)(b) = (f_b, b)$. Furthermore, under this correspondence two fibre maps are fibre homotopic over B if and only if their corresponding sections are vertically homotopic.

Proof Apply theorem 9 to the maps $1_B:B \to B$, $q:D \to B$ and $p:E \to B$. Identifying $1_B \sqcap q:D_{1_B} \to B$ with $q:D \to B$, we have that the correspondence $\phi:M_B(1_B \sqcap q,p) \to M_B(1_B,(qp))$ of theorem 9 coincides with $\phi:M_B(q,p) \to \mathrm{Sec}(qp)$ and so ϕ is bijective and two fibre maps are fibre homotopic over B if and only if their corresponding sections are vertically homotopic.

2. $\mathcal{F}$-SPACES AND $\mathcal{F}$-FIBRATIONS

Let $\mathcal{F}$ denote a category with a faithful "underlying space" functor $\mathcal{F} \to \mathrm{Top}$. Thus each object of $\mathcal{F}$ is a space and the set $\mathcal{F}(F,F')$ of morphisms from F to F' in $\mathcal{F}$ is a subset of $M(F,F')$. We agree that $\mathcal{F}$ contains with each $F \in |\mathcal{F}|$ the spaces $F \times \{*\}$ and $\{*\} \times F$ and the evident homeomorphisms between these spaces and F.

Examples (i) Let G be a fixed topological group and define $\mathcal{F}$ to be the category of right (or left) G-spaces and G-maps.
(ii) Take as $\mathcal{F}$ the category of real (or complex) topological vector spaces and continuous linear transformations.
(iii) Let F be a fixed space and define $\mathcal{F}$ to be the category having as objects all spaces of the same homotopy type as F and as morphisms all homotopy equivalences between such spaces. A slight modification of this example is obtained by considering spaces of the same weak homotopy type as F and weak homotopy equivalences.

We say that a map $p:E \to B$ is an $\mathcal{F}$-space if the fibre E_b is an object of $\mathcal{F}$ for every $b \in B$. Given $\mathcal{F}$-spaces $q:D \to A$ and $p:E \to B$ and maps $f:D \to E$ and $g:A \to B$, we say that the couple (f,g) is an $\mathcal{F}$-map pair from q to p if (f,g) is a map pair from q to p and for every $a \in A$ the map $f_a:D_a \to E_{g(a)}$

is in F. We will denote by $M(q,p;F)$ the set of all F-map pairs from q to p. In particular we have the notion of an <u>F-homotopy pair</u> (H,K) as an F-map pair from $q\times 1_I$ to p. If $(H_0,K_0)=(f,g)$ we say that (H,K) is an <u>F-homotopy</u> of (f,g); if, furthermore, $(H_1,K_1)=(f',g')$ we say that (H,K) is an <u>F-homotopy from</u> (f,g) <u>to</u> (f',g').

Given F-spaces $q:D\to A$ and $p:E\to B$ and a map $g:A\to B$, we say that a map $f:D\to E$ is an <u>F-fibre map from q to</u> p <u>over</u> g if (f,g) is an F-map pair. We will denote by $M_g(q,p;F)$ the set of all such map pairs. If $f,f'\in M_g(q,p;F)$ we say that f and f' are <u>F-fibre homotopic over</u> g if there exists a homotopy $H:D\times I\to E$ such that (H,K) is an F-homotopy pair from (f,g) to (f'g), where K is the homotopy stationary at g. In other words, H must satisfy the relation $pH(d,t)=gq(d)$, for every $d\in D$ and $t\in I$, and the map $H_{a,t}:d\in D_a\to H(d,t)\in E_{g(a)}$ must be in F, for every $a\in A$ and $t\in I$. If A=B and $g=1_B$ we will speak of F-fibre maps <u>over B</u>, of F-fibre homotopies <u>over B</u> and we will write $M_B(q,p;F)$ for $M_{1_B}(q,p;F)$. If $p:E\to B$ and $p':E'\to B$ are F-spaces, we say that the F-fibre map $f:E\to E'$ over B is an <u>F-fibre homotopy equivalence over</u> B if there exists an F-fibre map $g:E'\to E$ over B such that gf is F-fibre homotopic over B to 1_E and fg is F-fibre homotopic over B to $1_{E'}$, in which case p and p' are said to have the same <u>F-fibre homotopy type</u> (over B).

If B is a one-point space, and so E and E' are objects of F and f is a map in F, the above definitions specialize to give the notions of F-homotopy, of F-homotopy equivalence and of F-homotopy type. We will denote by HF the category whose objects are the objects of F and whose morphisms are the F-homotopy classes $[f]_F$ of maps $f:E \to E'$ in F, viewed as morphisms over a point; composition of morphisms are given by the F-homotopy class of the composition of the represenatatives. HF is called the homotopy category of F.

The next result is the analogue of theorem I.1.14 in the context of F-spaces and F-fibre maps. Its proof can be found in [33; th.1.5].

Theorem 1 Let $p:E \to B$ and $p':E' \to B$ be F-spaces and let $f:E \to E'$ be an F-fibre map over B. Suppose there is a numerable cover $\mathcal{U}=\{U\}$ of B such that $f_U:E_U \to E'_U$ is an F-fibre homotopy equivalence over U, for every $U\in\mathcal{U}$. Then f is an F-fibre homotopy equivalence over B.

We now define the analogue of the notion of a Hurewicz fibration in the context of F-spaces. Let $p:E \to B$ be an F-space. We say that p is an F-fibration if given any F-space $q:D \to A$, an F-map pair $(f,g):q \to p$ and any homotopy $K:A\times I \to B$ of g, there exists a homotopy $H:D\times I \to E$ of f such that (H,K) is an F-homotopy pair. In other words, p is an F-fibration if given any commutative diagram of the kind (ignore the dotted arrow)

$$\begin{tikzcd}
D \arrow[rr, "f"] \arrow[dr, "i_0"] \arrow[ddd, "q"'] & & E \arrow[ddd, "p"] \\
& D\times I \arrow[ur, dashed, "H"] \arrow[d, "q\times 1_I"] & \\
& A\times I \arrow[dr, "K"] & \\
A \arrow[ur, "i_0"] \arrow[rr, "g"] & & B
\end{tikzcd}$$

with q an $\mathcal{F}$-space and $f_a: D_a \to E_{g(a)}$ in $\mathcal{F}$ for every $a\in A$, we can fill the dotted arrow by a homotopy $H: D\times I \to E$ making the enlarged diagram commutative and such that $H_{(a,t)}: D_a\times\{t\} \to E_{K(a,t)}$ is in $\mathcal{F}$, for every $(a,t)\in A\times I$.
If in the above definition we consider only semi-stationary homotopies $K: A\times I \to B$, we get the weaker notion of an <u>$\mathcal{F}$-Dold fibration</u>.

<u>Proposition 2</u> Every $\mathcal{F}$-fibration $p: E \to B$ is a Hurewicz fibration.

<u>Proof</u> By proposition I.2.1 p is a Hurewicz fibration if and only if p has the CHP with respect to all pullbacks of p, $p_f: E_f \to X$, along any map $f: X \to B$. Since the fibre of p_f over $x\in X$ is $\{x\}\times E_{f(x)}$, we have that p_f is an $\mathcal{F}$-space. Let $K: X\times I \to B$ be a homotopy of f. Since p is an $\mathcal{F}$-fibration, the dotted arrow in the following commutative diagram

$$
\begin{tikzcd}
E_f \arrow[rr, "pr_2"] \arrow[dr, "i_0"] \arrow[ddd, "p_f"'] & & E \arrow[ddd, "p"] \\
& E_f \times I \arrow[ur, dashed, "H"] \arrow[d, "p_f \times 1_I"] & \\
& X \times I \arrow[dr, "K"] & \\
X \arrow[ur, "i_0"] \arrow[rr, "f"] & & B
\end{tikzcd}
$$

can be filled by a homotopy $H: E_f \times I \to E$ making the enlarged diagram commutative, as required.

Remark 3 A similar result holds for F-Dold fibrations.

We now show that F-fibrations can be characterized intrisically. Let $p: E \to B$ be an F-space and consider the usual space $\Lambda_p = \{(e,\alpha) \in E \times B^I : \alpha(0) = p(e)\}$. We say that a map $\lambda: \Lambda_p \to E^I$ is an F-lifting function for p if $\lambda(e,\alpha)(0) = e$, $p \circ \lambda(e,\alpha) = \alpha$ and, for every $(\alpha,t) \in B^I \times I$, the map $\lambda_{\alpha,t}: e \in E_{\alpha(0)} \to \lambda(e,\alpha)(t) \in E_{\alpha(t)}$ is in F; $\lambda_{\alpha,t}$ is called the translation map along α at time t.

Proposition 4 An F-space $p: E \to B$ is an F-fibration if and only if it admits an F-lifting function.

Proof Suppose p is an F-fibration. The map $pr_2: (e,\alpha) \in \Lambda_p \to \alpha \in B^I$ is an F-space since the fibre over α is $E_{\alpha(0)} \times \{\alpha\}$. Consider the following commutative diagram (ignore the dotted arrow) -

$$\begin{tikzcd}
\Lambda_p \arrow[rr, "pr_1"] \arrow[dr, "i_0"] \arrow[ddd, "pr_2"'] & & E \arrow[ddd, "p"] \\
& \Lambda_p \times I \arrow[ur, dashed, "L"] \arrow[d, "pr_2 \times 1_I"] & \\
& B^I \times I \arrow[dr, "\omega"] & \\
B^I \arrow[ur, "i_0"] \arrow[rr, "\omega_0"] & & B
\end{tikzcd}$$

where ω_0 is the evaluation map at 0 and ω the evaluation map. Since p is an $\mathcal{F}$-fibration, we can fill the dotted arrow by a homotopy $L: \Lambda_p \times I \to E$ making the enlarged diagram commutative and such that, for every $(\alpha, t) \in B^I \times I$, the map $(e, \alpha, t) \in E_{\alpha(0)} \times \{(\alpha, t)\} \to L(e, \alpha, t) \in E_{\alpha(t)}$ is in $\mathcal{F}$. This means that the adjoint of L, $\lambda: \Lambda_p \to E^I$, is an $\mathcal{F}$-lifting function for p.

Suppose now p admits an $\mathcal{F}$-lifting function $\lambda: \Lambda_p \to E^I$. Let $q: D \to A$ be an $\mathcal{F}$-space, $(f, g): q \to p$ an $\mathcal{F}$-map pair and let $K: A \times I \to B$ be a homotopy such that the following diagram commutes (ignore the dotted arrow)

$$\begin{tikzcd}
D \arrow[rr, "f"] \arrow[dr, "i_0"] \arrow[ddd, "q"'] & & E \arrow[ddd, "p"] \\
& D \times I \arrow[ur, dashed, "H"] \arrow[d, "q \times 1_I"] & \\
& A \times I \arrow[dr, "K"] & \\
A \arrow[ur, "i_0"] \arrow[rr, "g"] & & B
\end{tikzcd}$$

Consider the composition $D \xrightarrow{(f,\hat{K}q)} \Lambda_p \xrightarrow{\lambda} E^I$, where $\hat{K}: A \to B^I$ is the adjoint of the homotopy K. Taking the adjoint of this composition we get a homotopy $H: D\times I \to E$ of f which makes the above diagram commutative. Since $H_{(a,t)} = \lambda_{K_a,t} \circ f_a$, we have that $(H,K): q\times 1_I \to p$ is an F-map pair, as required.

Most of the properties held by Hurewicz fibrations generalize to the context of F-spaces and F-fibrations without relevant changes in the proofs. For the sake of completeness we state here some of them without proof.

<u>Proposition 5</u>. ("C-lemma" for F-fibrations). Let $p: E \to B$ be an F-fibration and let $C = I\times \dot{I} \cup \{0\}\times I$. Let $q: D \to A$ be an F-space and let $L: A\times I\times I \to B$ and $J: D\times C \to E$ be maps making the following diagram commute (ignore the dotted arrow)

$$\begin{tikzcd}
D\times C \arrow[r, "J"] \arrow[d, "1_D\times i"'] & E \arrow[dd, "p"] \\
D\times I\times I \arrow[ur, dashed, "\overline{J}"'] \arrow[d, "q\times 1_{I\times I}"'] & \\
A\times I\times I \arrow[r, "L"] & B
\end{tikzcd}$$

and such that $J_{(a,t,s)}: D_a\times\{(t,s)\} \to E_{L(a,t,s)}$ is in F for every $(a,t,s)\in A\times C$. Then we can fill the dotted arrow by an extension $\overline{J}: D\times I\times I \to E$ of J such that $(\overline{J},L): q\times 1_{I\times I} \to p$ is an

$\mathcal{F}$-map pair.

Given an $\mathcal{F}$-fibration $p: E \to B$ and an object $b \in |\Pi B|$, the fundamental groupoid of B, define $T_p(b) = E_b$. For a morphism $[\alpha]$ of ΠB we define $T_p([\alpha])$ as follows. Consider the following commutative diagram (ignore the dotted arrow)

$$\begin{tikzcd}
E_{\alpha(0)} \arrow[r, "i"] \arrow[d, "i_0"'] & E \arrow[dd, "p"] \\
E_{\alpha(0)} \times I \arrow[ur, dashed, "H"] \arrow[d, "pr_2"'] & \\
I \arrow[r, "\alpha"] & B
\end{tikzcd}$$

Since p is an $\mathcal{F}$-fibration, we can fill the dotted arrow by a homotopy $H: E_{\alpha(0)} \times I \to E$ making the enlarged diagram commutative and such that $H_t: E_{\alpha(0)} \to E_{\alpha(t)}$ is in $\mathcal{F}$, for every $t \in I$. We define $T_p([\alpha]) = [H_1]_{\mathcal{F}}$, which is a morphism of $H\mathcal{F}$ from $E_{\alpha(0)}$ to $E_{\alpha(1)}$.

<u>Proposition 6</u> For any $\mathcal{F}$-fibration $p: E \to B$, $T_p: \Pi B \to H\mathcal{F}$ defines a covariant functor from the fundamental groupoid of B to the homotopy category of $\mathcal{F}$. Furthermore, if $p': E' \to B$ is an $\mathcal{F}$-fibration and $f: E \to E'$ is an $\mathcal{F}$-fibre map over B, then f gives rise to a natural transformation $\Phi_f: T_p \to T_{p'}$, defined by $\Phi_f(b) = [f_b]_{\mathcal{F}}$.

Theorem 7 Let $p:E \to B\times I$ be an $\mathcal{F}$-fibration and define $E^i = p^{-1}(B\times\{i\})$. Then the $\mathcal{F}$-spaces $p^0: e\in E^0 \to pr_1 p(e)\in B$ and $p^1: e\in E^1 \to pr_1 p(e)\in B$ have the same $\mathcal{F}$-fibre homotopy type over B.

Proposition 8 Let $p:E \to B$ be an $\mathcal{F}$-fibration. Then the pullback of p along any map $f:A \to B$ is an $\mathcal{F}$-fibration; furthermore, if $g:A \to B$ is homotopic to f, then p_f and p_g have the same $\mathcal{F}$-fibre homotopy type over A.

Theorem 9 Let $p:E \to B$ and $p':E' \to B$ be $\mathcal{F}$-fibrations, where B belongs to the class $\mathcal{D}$ (see section I.2). If $f:E \to E'$ is a $\mathcal{F}$-fibre map over B such that $f_b:E_b \to E'_b$ is an $\mathcal{F}$-homotopy equivalence for every $b\in B$, then f is an $\mathcal{F}$-fibre homotopy equivalence over B.

Theorem 10 Let $p:E \to B$ be an $\mathcal{F}$-space and assume there is a numerable covering $\mathcal{U}=\{U\}$ of B such that the restriction of p to U, $p_U:E_U \to U$, is an $\mathcal{F}$-fibration for each $U\in\mathcal{U}$. Then p is an $\mathcal{F}$-fibration.

In the context of $\mathcal{F}$-spaces and $\mathcal{F}$-maps there are analogues of the "$\cdot$"-construction and of the "round bracket"-construction, which we described in section 1. Namely, let $q:D \to A$ and $p:E \to B$ be $\mathcal{F}$-spaces, with q having closed fibres. We then define the F-functional space of q and p, denoted by $D_{\dot F}E$, to be the space $D_{\dot F}E = \coprod_{(a,b)\in A\times B} F(D_a, E_b)$ topologized with the subspace topology induced by $D\cdot E \supseteq D_{\dot F}E$; the map $q_{\dot F}p : D_{\dot F}E \to A\times B$ is defined taking the restriction of $q\cdot p$. Of course, since the fibre of $q_{\dot F}p$ over $(a,b)\in A\times B$ is $F(D_a, E_b)\times\{(a,b)\}$, $q_{\dot F}p$ is not generally an F-space. If, furthermore, a map $(r,g): C \to A\times B$ is given and A is Hausdorff and D locally compact, Hausdorff, it is easy to see that the fibred exponential correspondence $\theta: M_g(q_r, p) \to L((r,g), q\cdot p)$ restricts to a bijective correspondence $\theta_F: M_g(q_r, p; F) \to L((r,g), q_{\dot F}p)$ between F-fibre maps over g and liftings of (r,g) over $q_{\dot F}p$. Indeed, $f\in M_g(q_r, p)$ is an F-map if and only if $f_c: D_{r(c)} \to E_{g(c)}$ is in F, for every $c\in C$, that is, its adjoint $\theta(f)$ takes values in $D_{\dot F}E$. Hence we get the following result.

Theorem 11 (F-Fibred Exponential Correspondence I). Let $q: D \to A$ and $p: E \to B$ be F-spaces, $(r,g): C \to A\times B$ any map and let A be Hausdorff and D locally compact, Hausdorff. Then the function $\theta_F: M_g(q_r, p; F) \to L((r,g), q_{\dot F}p)$, defined by $\theta_F(f)(c) = (f_c, r(c), g(c))$, is bijective and two F-maps over g are F-fibre homotopic over g if and only if their corresponding liftings are vertically homotopic.

In the case we deal with F-spaces with the same target space B and F-fibre maps over B, the analogue of the "round bracket"-construction is defined in the following way. Let $q:D \to B$ and $p:E \to B$ be F-spaces, with q having closed fibres, and define $(DE) = \coprod_{b \in B} F(D_b, E_b)$, topologized with the subspace topology induced by $(DE) \supseteq (DE)_F$; the map $(qp)_F : (DE)_F \to B$ (generally not an F-space) is defined taking the restriction of (qp). With a same argument as above, we get the following result.

<u>Theorem 12</u> (F-Fibred Exponential Correspondence II). Let $q:D \to B$ and $p:E \to B$ be F-spaces, $r:C \to B$ any map, and let B be Hausdorff and D locally compact, Hausdorff. Then the function $\phi_F : M_r(q_r, p; F) \to M_B(r, (qp)_F)$, defined by $\phi_F(f)(c) = (f_c, r(c))$, is bijective and two F-fibre maps over r are F-fibre homotopic over r if and only if their corresponding fibre maps over B are fibre homotopic (over B).

$$\begin{tikzcd}
D \arrow[d, "q"'] & D_r \arrow[l, "pr_2"'] \arrow[r, dashed, "f"] \arrow[d, "q_r"] & E \arrow[d, "p"] \\
B & C \arrow[l, "r"'] \arrow[r, "r"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (DE)_F \arrow[d, "(qp)_F"] \\
C \arrow[ur, dashed, "\psi_F(f)"] \arrow[r, "r"] & B
\end{tikzcd}$$

The next result is due to C. Morgan [35].

<u>Theorem 13</u>. Let $q:D \to A$ and $p:E \to B$ be F-spaces, with A Hausdorff and D locally compact, Hausdorff. If q and p are F-fibrations, then $q \cdot p$ is a fibration.

<u>Proof</u> We must prove that for any commutative diagram (ignore the dotted arrow)

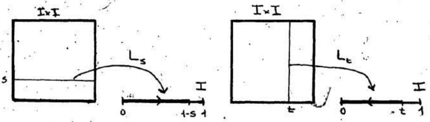
$$\begin{tikzcd}
X \arrow[r, "k"] \arrow[d, "i_0"'] & D \underset{F}{\cdot} E \arrow[d, "q \underset{F}{\cdot} p"] \\
X \times I \arrow[r, "{(R,G)}"] \arrow[ur, dashed] & A \times B
\end{tikzcd}$$

we can fill the dotted arrow by a homotopy $X \times I \to D \underset{F}{\cdot} E$ of k lifting (R,G). Let $r = R_0, g = G_0$; then $k(x) = (k^*(x), r(x), g(x))$, where the map $k^*(x): D_{r(x)} \to E_{g(x)}$ is in F. By theorem 11, k determines an F-fibre map $\hat{k}: D_r \to E$ over g, given by $\hat{k}(x,d) = k^*(x)(d)$, and the existence of the wanted homotopy is equivalent to the existence of an F-fibre map $K: D_R \to E$ over G such that $K(x,0,d) = \hat{k}(x,d)$.

$$\begin{tikzcd}
D \arrow[ddd, "q"'] & D_R \arrow[l, "pr"'] \arrow[rr, dashed, "K"] \arrow[ddd, "q_R"'] & & E \arrow[ddd, "p"] \\
& & D_r \arrow[ul, "i'_0"'] \arrow[ur, "\hat{k}"'] \arrow[d, "q_r"] & \\
& & X \arrow[dl, "i_0"'] \arrow[dr, "g"] & \\
A & X \times I \arrow[l, "R"'] \arrow[rr, "G"] & & B
\end{tikzcd}$$

The construction of K will be achieved after intermediate constructions.

Let $L: I \times I \to I$ be any map such that L restricted to $I \times \{0\}$ is the identity of I and L restricted to $\{0\} \times I \cup I \times \{1\}$ is constant at 0. A natural example of such a map is given by $L(t,s) = (1-s)t$; so $L_s: I \to I$ is the path going linearly from 0 to 1-s and $L_t: I \to I$ is the path going linearly from t to 0.

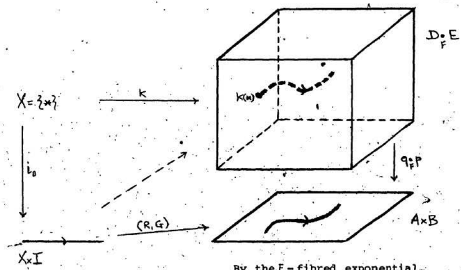


Using L define the homotopy $\bar{R}: X \times I \times I \to A$ by $\bar{R}(x,t,s) = R(x, L(t,s))$; in particular we have that $\bar{R}(x,t,0) = R(x,t)$, $\bar{R}(x,0,s) = r(x)$ and $\bar{R}(x,t,1) = r(x)$. Since q is an $\mathcal{F}$-fibration and $(X \times I, X \times \{0\})$ is a closed cofibred pair, we can apply proposition I.2.2, adapted to the context of $\mathcal{F}$-spaces, to deduce the existence of a homotopy $J: D_R \times I \to D$ of the projection of D_R on D, such that $(J, \bar{R})$ is an $\mathcal{F}$-map pair and $J(x,0,d,s) = d$. If we define $J_1: D_R \to D$ by $J_1(x,t,d) = J(x,t,d,1)$, then $J_1(x,0,d) = d$, $qJ_1(x,t,d) = \bar{R}(x,t,1) = r(x)$ and the maps $d \in D_{R(x,t)} \to$

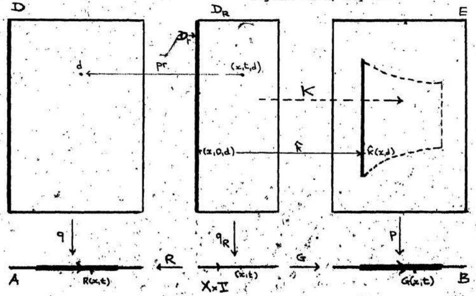
$J_1(x,t,d) \in D_{r(x)}$ are in $\mathcal{F}$. The map $J_1' : D_R \to D_r \times I$ defined by $J_1'(x,t,d) = (x, J_1(x,t,d), t)$ is well defined, since $qJ_1(x,t,d) = \bar{R}(x,t,1) = r(x)$, is an $\mathcal{F}$-fibre map over $X \times I$ and satisfies the relations $J_1'(x,0,d) = (x,d,0)$. Now, since p is an $\mathcal{F}$-fibration, there is a homotopy $\tilde{G} : D_r' \times I \to E$ of $\hat{k}$ such that $(\tilde{G}, G)$ is an $\mathcal{F}$-map pair:

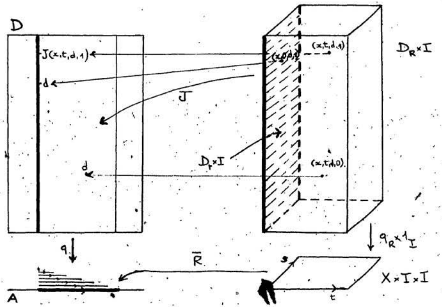
$$\begin{tikzcd}
D_R \arrow[r, "J_1'"] \arrow[dr, "q_R"'] & D_r \times I \arrow[r, "\tilde{G}"] \arrow[d, "q_r \times 1_I"] & E \arrow[d, "p"] \\
& X \times I \arrow[r, "G"] & B
\end{tikzcd}$$

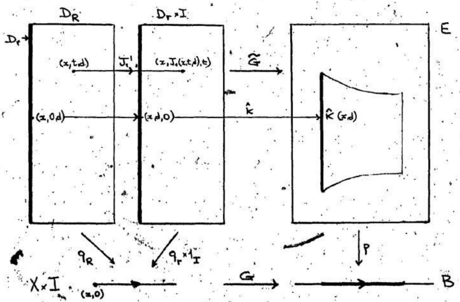
Define $K = \tilde{G} J_1'$. Then $K : D_R \to E$ is an $\mathcal{F}$-fibre map over G with $K(x,0,d) = \tilde{G} J_1'(x,0,d) = \tilde{G}(x,d,0) = \hat{k}(x,d)$, as required.



By the F-fibred exponential correspondence the existence of the above dotted arrow is equivalent to the existence of an F-fibre map over G $K : D_R \to E$ whose restriction to D_r is the adjoint of k, say $\hat{k}$.




D
J(x,t,d,1)
d
J
d
(x,t,d,1)
$D_R \times I$
$D_r \times I$
(x,t,d,0)
q
$q_R \times 1_I$
$\bar{R}$
s
t
A
$X \times I \times I$


D_R
D_r
$D_r \times I$
E
(x,t,d)
J_1'
$(x, J_1(x,t,d), t)$
$\tilde{G}$
(x,0,d)
(x,d,0)
$\hat{k}$
$\hat{k}(x,d)$
q_R
$q_r \times 1_I$
p
$X \times I$
(x,0)
G
B

<u>Corollary 14</u> Let $q:D \to B$ and $p:E \to B$ be F-spaces with B Hausdorff and D locally compact, Hausdorff. If q and p are F-fibrations, then $(qp)_F$ is a fibration.

<u>Proof</u> It follows from theorem 13 and from the fact that the commutative diagram

$$\begin{tikzcd}
(DE)_F \arrow[r, "\overline{\Delta}"] \arrow[d, "(qp)_F"'] & D \cdot_F E \arrow[d, "q \cdot_F p"] \\
B \arrow[r, "\Delta"] & B \times B
\end{tikzcd}$$

is cartesian.

<u>Remark 15</u> The converses of proposition 13 and corollary 14 are not true. Indeed, there are maps $q:D \to B$ and $p:E \to B$ such that $q \cdot p: D \cdot E \to B \times B$ is a fibration (and so $(qp):(DE) \to B$ is also a fibration), but q and p are not both fibrations. For example, let $D=[0,1/2]\times\{0\}\cup]1/2,1]\times\{1\}\subseteq R^2$, $B=E=[0,1]$ and let $q:D \to B$ be the projection map on the first factor and let $p:E \to B$ be the identity map. To simplify notation, define $\bar{t}=(t,0)\varepsilon D$, if $0<t<1/2$, and $\bar{t}=(t,1)\varepsilon D$, if $1/2<t<1$, and identify the underlying set of $D \cdot E$ with $D\times E$, identifying the triple $(f,t,s)\varepsilon D \cdot E$, where $(t,s)\varepsilon B\times B$ and $f:D_t=\{\bar{t}\} \to E_s=\{s\}$,

with the couple $(\bar{t},s)\in D\times E$. Furthermore, for every $(\bar{t}_0,s_0)\in D\times E$ define the map $[\bar{t}_0,s_0]:D\to E^+$ by $[\bar{t}_0,s_0](\bar{t})=s_0$, if $\bar{t}=\bar{t}_0$, and $[\bar{t}_0,s_0](\bar{t})=\infty$ otherwise. We want to prove that the initial topology τ on $D\times E$ with respect to $q\cdot p:(\bar{t},s)\in D\times E\to(t,s)\in B\times B$ and $j:(\bar{t},s)\in D\times E\to[\bar{t},s]\in M(D,E^+)$ coincides with the product topology π, and so $q\cdot p$ is a fibration in spite of the fact that q is not a fibration. Since π coincides with the initial topology with respect to $q\cdot p$ and since τ is the smallest topology making $q\cdot p$ and j continuous, it follows that it is equivalent to showing that π makes j continuous. To this end, let $\langle K,U'\rangle$ be any subbasic set of $M(D,E^+)$. If $U'=\emptyset$, then $\langle K,\emptyset\rangle=M(D,E^+)$, if $K=\emptyset$, and $\langle K,\emptyset\rangle=\emptyset$, if $K\neq\emptyset$; therefore the anti-image of $\langle K,\emptyset\rangle$ by j are $D\times E$ and $\emptyset$, respectively, which are open sets. If $U'=U\cup\{\infty\}$, with $U\subseteq E$ open, then $j^{-1}(\langle K,U'\rangle)=\{(\bar{t},s)\in D\times E:\ [\bar{t},s](K)\subseteq U'\}=\{(\bar{t},s)\in D\times E:\ \bar{t}\notin K\}\cup\{(\bar{t},s)\in D\times E:\ \bar{t}\in K \text{ and } s\in U\}$, which is open in the product topology because its complement is the closed set $K\times(E-U)$.

In spite of the above remark we have the following result, due also to C. Morgan [35], which, among other things, turns out to be a bridge between the theory of F-fibrations and the classical theory of Hurewicz fibrations.

Proposition 16 Let $p:E\to B$ be an F-space. Then p is an F-fibration if and only if $p_F p$ is a fibration.

Proof If p is an $\mathcal{F}$-fibration, then, by theorem 13, $p \sharp_{\mathcal{F}} p$ is a fibration. Conversely, suppose $p \sharp_{\mathcal{F}} p$ is a fibration and let us prove that p is an $\mathcal{F}$-fibration. By the universal property held by pullbacks, it is enough to show that given a map $g: X \to B$ and a homotopy $K: X \times I \to B$ of g, there exists a homotopy $H: E_g \times I \to E$ of the map $f: (x,e) \in E_g \to e \in E$ such that (H,K) is an $\mathcal{F}$-map pair.

$$
\begin{tikzcd}
E_g \arrow[rr, "f"] \arrow[dr, "i_0"] \arrow[ddd, "p_g"'] & & E \arrow[ddd, "p"] \\
& E_g \times I \arrow[ur, dashed, "H"] \arrow[d, "p_g \times 1_I"] & \\
& X \times I \arrow[dr, "K"] & \\
X \arrow[ur, "i_0"] \arrow[rr, "g"] & & B
\end{tikzcd}
$$

Let $G: X \times I \to B$ be the homotopy stationary at g and consider the following diagram (ignore the dotted arrow)

$$
\begin{tikzcd}
E \arrow[d, "p"'] & E_G \arrow[l, "pr"'] \arrow[r, dashed, "H"] \arrow[d, "p_G"] & E \arrow[d, "p"] \\
B & X \times I \arrow[l, "G"'] \arrow[r, "K"] & B
\end{tikzcd}
$$

Identifying $E_g \times I$ with E_G, we have that the properties on the required homotopy H are equivalent to the requirement that $H:E_G \to E$ is an F-fibre map over K with $H(x,0,e)=e$. Now, by theorem 11, the existence of such an H is equivalent to the existence of a homotopy $X\times I \to E_F E$ of $\hat{f}$, the adjoint of f, lifting (G,K). Since $p_F p$ is a fibration, such a homotopy does exist, which proves that p is an F-fibration.

3. ALGEBRAS OVER A MONAD AND FIBRATIONS

In this section we introduce a fundamental concept of category theory, that is, that of a monad (or dual standard construction or triple or triad) on a category and the related concept of an algebra over a monad. We show using Moore paths that the standard conversion of a map to a fibration, as presented in section I.2, gives rise to a monad and that fibrations are essentially the algebras for that monad. These observations are due to P. Malraison [34] and J.P. May [33; p.14]. Monads (or, to be precise, their duals "comonads") were first introduced by Godement [21; Appendix] in the special context of sheaf cohomology. Later Huber [25] found other applications of this concept, in particular to homotopy theory. More recently Eilenberg and Moore [20] and Kleisli [30] have studied their relationship with adjoint functors.

Let $\mathcal{C}$ be a category. A monad on $\mathcal{C}$ is a triple (T, ι, μ) where $T: \mathcal{C} \to \mathcal{C}$ is a (covariant) functor and $\iota: 1_{\mathcal{C}} \to T$ and $\mu: T^2 \to T$ are natural transformations such that for every $X \in |\mathcal{C}|$ the following diagrams commute

$$\begin{tikzcd}
T(X) \arrow[r, "\iota_{T(X)}"] \arrow[dr, equal] & T^2(X) \arrow[d, "\mu_X"] & T(X) \arrow[l, "T(\iota_X)"'] \arrow[dl, equal] \\
& T(X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] & T^2(X) \arrow[d, "\mu_X"] \\
T^2(X) \arrow[r, "\mu_X"] & T(X)
\end{tikzcd}$$

The term "monad" (suggested by S. Eilenberg) is due to the formal resemblance of its definition to that of a monoid, that is a semigroup with unit; in fact a monoid may be regarded as a set T with two functions $\iota:\{e\} \to T$ and $\mu: T\times T \to T$ such that the following diagrams commute

$$\begin{tikzcd}
T \arrow[r, "{(\bar{\iota},\, 1_T)}"] \arrow[dr, equal] & T\times T \arrow[d, "\mu"] & T \arrow[l, "{(1_T,\, \bar{\iota})}"'] \arrow[dl, equal] \\
 & T &
\end{tikzcd}
\qquad (\bar{\iota} = T \to \{e\} \xrightarrow{\iota} T)$$

$$\begin{tikzcd}
T\times T\times T \arrow[r, "\mu\times 1_T"] \arrow[d, "1_T\times\mu"'] & T\times T \arrow[d, "\mu"] \\
T\times T \arrow[r, "\mu"] & T
\end{tikzcd}$$

Thus, if we call ι the <u>unit</u> of the monad and μ the <u>multiplication</u>, then the above diagrams are to be interpreted as the left-unit law, the right-unit law and the associative law for multiplication. Actually the similarity between monads and monoids is not only formal: if we generalize the notion of an ordinary monoid (set with an associative multiplication and two-sided unit) to that of a monoid in a strict monoidal category [31;p.166], then the monads on C are just the monoids in the strict monoidal category C^C of the endofunctors of C where the product is given by the composition of endofunctors. P. Hilton in [24;p.77] credits this observation to Bénabou.

Given a monad $T=(T,\iota,\mu)$ on C, a <u>T-algebra</u> is a pair (X,ϕ) where X is an object of C and $\phi:T(X)\to X$ is a morphism of C such that the following diagrams commute

$$\begin{tikzcd} X \arrow[r, "\iota_X"] \arrow[dr, equal] & T(X) \arrow[d, "\phi"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} T^2(X) \arrow[r, "T(\phi)"] \arrow[d, "\mu_X"'] & T(X) \arrow[d, "\phi"] \\ T(X) \arrow[r, "\phi"] & X \end{tikzcd}$$

X is called the <u>underlying object</u> or the <u>carrier</u> of the T-algebra (X,ϕ).

A <u>morphism of T-algebras</u> $\lambda:(X,\phi)\to(X',\phi')$ is a morphism $\lambda:X\to X'$ in C such that the following diagram commutes

$$\begin{tikzcd} T(X) \arrow[r, "\phi"] \arrow[d, "T(\lambda)"'] & X \arrow[d, "\lambda"] \\ T(X') \arrow[r, "\phi'"] & X' \end{tikzcd}$$

It is easily seen that the identity morphism of the carrier of a T-algebra is a morphism of T-algebras and that the composition of morphisms of T-algebras is a morphism of T-algebras. So the T-algebras and their morphisms form a category C^T called the <u>Eilenberg-Moore category</u> corresponding to T.

Amongst T-algebras a distinguished role is played by the so-called <u>free T-algebras</u>, that is the pairs (TX, μ_X) with X any object in C; indeed the required commutativity of the diagrams

$$\begin{tikzcd} T(X) \arrow[r, "\iota_{T(X)}"] \arrow[dr, equal] & T^2(X) \arrow[d, "\mu_X"] \\ & T(X) \end{tikzcd} \qquad \begin{tikzcd} T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"] & T(X) \end{tikzcd}$$

follows from the definition of a monad. (TX, μ_X) is called the free T-algebra <u>on</u> X.

The name "free" is justified by the following fact: the "free" functor $U: \mathcal{C} \to \mathcal{C}^T$, sending each object of $\mathcal{C}$ to the free T-algebra on it and each morphism in $\mathcal{C}$ to its image by T, is left adjoint to the forgetful functor $F: \mathcal{C}^T \to \mathcal{C}$, sending each T-algebra to its underlying object and each morphism of T-algebras to itself viewed as a morphism in $\mathcal{C}$. This adjoint relationship has as unit $\iota: 1_{\mathcal{C}^T} \to T = FU$ and as a counit $\eta: UF \to 1_{\mathcal{C}^T}$; the natural transformation η is given by $\eta_{(X,\phi)} = \phi: UF(X,\phi) = (TX, \mu_X) \to (X, \phi)$. Here we can consider ϕ as a morphism of T-algebras because the required commutativity of the diagram

$$\begin{tikzcd}
T^2X \arrow[r, "\mu_{TX}"] \arrow[d, "T\phi"'] & TX \arrow[d, "\phi"] \\
TX \arrow[r, "\phi"] & X
\end{tikzcd}$$

is part of the definition of a T-algebra.

To prove that ι and η satisfy the conditions of adjointness [31;p.80], that is,

(i) for every $X \in |C|$ the composition

$U(X) \xrightarrow{U(\iota_X)} UFU(X) \xrightarrow{\eta_{U(X)}} U(X)$ is the identity of $U(X)$

(ii) for every $(X,\phi) \in |C^T|$ the composition

$F(X,\phi) \xrightarrow{\iota_{F(X,\phi)}} FUF(X,\phi) \xrightarrow{F(\eta_{(X,\phi)})} F(X,\phi)$ is the identity of $F(X,\phi)$

we have only to observe that (i) becomes

$(TX,\mu_X) \xrightarrow{T(\iota_X)} (T^2X,\mu_{TX}) \xrightarrow{\mu_X} (TX,\mu_X)$, which is the identity from the first axiom of a monad, and that (ii) becomes

$X \xrightarrow{\iota_X} T(X) \xrightarrow{\phi} X$, which is the identity from the first axiom of a T-algebra. We only mention that any pair of adjoint functors $F:C \to D$, $G:D \to C$ with $F \dashv G$ gives rise to a monad on C and that the monad associated to the above free and forgetful functors is our initial monad T [31;p.136].

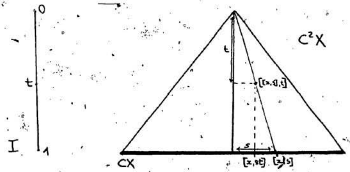
And now some simple and, we hope, expressive examples of monads:

(i) For a topological space X the <u>cone</u> of X, denoted by CX, is the space defined by $CX = X \times I / X \times \{0\}$, that is CX is obtained from $X \times I$ by identifying to one point the bottom base of the cylinder and topologizing with the quotient topology. For any $(x,t)\in X\times I$ we denote by $[x,t]$ the corresponding element of CX under the quotient map $\pi:X\times I \to X\times I/X\times\{0\}$; hence $[x,t] = \{(x,t)\}$, if $t>0$, and $[x,0]=X\times\{0\}$. If $f:X \to Y$ is a continuous function then $f\times 1_I:X\times I \to Y\times I$ is compatible with the identification process in $X\times I$ and $Y\times I$, giving rise to a map $C(f):C(X) \to C(Y)$. It is easy to check that given maps $f:X \to Y$ and $g:Y \to Z$, we have that $C(gf)=C(g)C(f)$ and $C(1_X)=1_{CX}$. So $C:\mathrm{Top} \to \mathrm{Top}$ defines a covariant functor (called the <u>cone functor</u>) from the category of topological spaces to itself. We define natural transformations $\iota:1_{\mathrm{Top}} \to C$ and $\mu:C^2 \to C$ by $\iota_X:x\in X \to [x,1]\in CX$ and $\mu_X:[[x,s],t]\in C^2X \to [x,st]\in CX$. μ_X is well defined; to check the continuity consider the following diagram (ignore the dotted arrows)

$$\begin{tikzcd}
X\times I\times I \arrow[r, "m"] \arrow[d, "\pi\times 1_I"'] & X\times I \arrow[d, "\pi"] \\
CX\times I \arrow[r, dashed, "m'"] \arrow[d, "\pi'"'] & CX \\
C^2X \arrow[ur, dashed] &
\end{tikzcd}$$

where $m:(x,s,t)\in X\times I\times I \to (x,st)\in X\times I$. The function $m':([x,s],t)\in CX\times I \to [x,st]\in CX$ is well defined because

$m'([x,0],t)=[x,0]=[x',0]=m'([x',0],t)$; moreover m' is continuous because $\pi\times 1_I$ is an identification map (I is Hausdorff and locally compact) and $m'(\pi\times 1_I)=\pi m$. We have that $\mu_X\pi'=m'$ because $\mu_X\pi'([x,s],t)=\mu_X[[x,s],t]=[x,st]=m'([x,s],t)$ and hence μ_X is continuous. Geometrically, $\mu_X: C^2X \to CX$ can be seen as the orthogonal projection of C^2X on its base CX



It is easy to check that the diagrams

$$\begin{tikzcd}
CX \arrow[r, "i_{CX}"] \arrow[dr, equal] & C^2X \arrow[d, "\mu_X"] & CX \arrow[l, "C(i_X)"'] \arrow[dl, equal] \\
& CX &
\end{tikzcd}
\qquad
\begin{tikzcd}
C^3X \arrow[r, "\mu_{CX}"] \arrow[d, "C(\mu_X)"'] & C^2X \arrow[d, "\mu_X"] \\
C^2X \arrow[r, "\mu_X"] & CX
\end{tikzcd}$$

commute. In fact for the former we have that $\mu_X \iota_{CX}[x,t]=\mu_X[[x,t],1]=[x,t]$ and $\mu_X C(\iota_X)[x,t]=\mu_X[[x,1],t]=[x,t]$. For the latter we have that $\mu_X C(\mu_X)[[[x,s],t],u]=\mu_X[[x,st],u]=[x,(st)u]$ and $\mu_X\mu_{CX}[[[x,s],t],u]=\mu_X[[x,s],tu]=[x,s(tu)]$; hence $\mu_X C(\mu_X)=\mu_X\mu_{CX}$ and so $C=(C,\iota,\mu)$ is a monad on Top. A C-algebra will be a pair (X,ϕ) with X a topological space and $\phi:CX\to X$ a map such that $\phi[x,1]=x$ for every $x\epsilon X$, and such that $\phi[x,st]=\phi[\phi[x,s],t]$ for every $x\epsilon X$ and $s,t\epsilon I$.

Considering the composition $X\times I \xrightarrow{\pi} CX \xrightarrow{\phi} X$ we see that the possible "multiplications" $\phi:CX\to X$ making X a C-algebra are in one-to-one correspondence with the <u>transitive contractions</u> $\Phi:X\times I\to X$, that is $\Phi_1=1_X$, Φ_0=constant at some point of X and satisfying the transitive rule $\Phi_{st}=\Phi_t\Phi_s$. This implies that the carrier of a C-algebra is a contractible space, but, because of the transitive rule, we view it as a special contractible space. As examples, all cones CX are spaces admitting a transitive contraction considering $\Phi:([x,s],t)\epsilon CX\times I\to[x,st]\epsilon CX$.

(ii) On the category Set consider the (covariant) functor $P:\mathrm{Set}\to\mathrm{Set}$ defined by $P(X)=2^X$=power set of X and with $P(f):P(X)\to P(Y)$ given by $P(f)(A)=f(A)$ for any function $f:X\to Y$ and $A\subseteq X$. We define natural transformations $\iota:1_{Set}\to P$ and $\mu:P^2\to P$ by $\iota_X:x\epsilon X\to\{x\}\epsilon P(X)$ and $\mu_X:\mathcal{A}\epsilon P^2(X)\to\bigcup_{A\epsilon\mathcal{A}}A\,\epsilon P(X)$. Now let us check that $P=(P,\iota,\mu)$ is a monad: first of all we have that for every $A \in P(X)$ $\mu_X \iota_{P(X)}(A) = \mu_X(\{A\}) = \bigcup_{A \in \{A\}} A = A$ and $\mu_X P(\iota_X)(A) = \mu_X(\iota_X^{\to}(A)) = \mu_X(\{\{x\}; x \in A\}) = \bigcup_{x \in A} \{x\} = A$; moreover we have that for every $\mathcal{B} \in P^3(X)$

$\mu_X \mu_{P(X)}(\mathcal{B}) = \mu_X(\bigcup_{\mathcal{A} \in \mathcal{B}} \mathcal{A}) = \bigcup_{A \in \bigcup_{\mathcal{A} \in \mathcal{B}} \mathcal{A}} A$ and

$\mu_X P(\mu_X)(\mathcal{B}) = \mu_X(\{\bigcup_{A \in \mathcal{A}} A; \mathcal{A} \in \mathcal{B}\}) = \bigcup_{\mathcal{A} \in \mathcal{B}} \bigcup_{A \in \mathcal{A}} A$. Now $\bigcup_{A \in \bigcup_{\mathcal{A} \in \mathcal{B}} \mathcal{A}} A = \bigcup_{\mathcal{A} \in \mathcal{B}} \bigcup_{A \in \mathcal{A}} A$

because $x \in \bigcup_{A \in \bigcup_{\mathcal{A} \in \mathcal{B}} \mathcal{A}} A$ <=> there exists $A \in \bigcup_{\mathcal{A} \in \mathcal{B}} \mathcal{A}$ such that $x \in A$ <=> there exist $\mathcal{A} \in \mathcal{B}$ and $A \in \mathcal{A}$ such that $x \in A$ <=> there exists $\mathcal{A} \in \mathcal{B}$ such that $x \in \bigcup_{A \in \mathcal{A}} A$ <=> $x \in \bigcup_{\mathcal{A} \in \mathcal{B}} \bigcup_{A \in \mathcal{A}} A$ and hence $\mu_X \mu_{P(X)}(\mathcal{B}) = \mu_X P(\mu_X)(\mathcal{B})$.

A P-algebra is a pair (X, ϕ) with X a set and $\phi: P(X) \to X$ a function such that $\phi(\{x\}) = x$ for every $x \in X$ and such that $\phi(\{\phi(A) \mid A \in \mathcal{A}\}) = \phi(\bigcup_{A \in \mathcal{A}} A)$ for every collection $\mathcal{A} \in P^2(X)$ of subsets of X. We mention a result due to E. Manes [32] which states that each P-algebra (X, ϕ) is a complete semi-lattice, when $x \leq y$ is defined by $\phi(x, y) = y$ and $\sup A = \phi(A)$, for each $A \subseteq X$; conversely, every complete semi-lattice is a P-algebra in this way.

(iii) Fix a semigroup G, that is, a set with an associative operation (multiplicatively denoted) and a neutral element e. Consider the functor $T: \mathrm{Set} \to \mathrm{Set}$ defined on objects by $T(X) = G \times X$ and on morphisms $f: X \to Y$ by $T(f): (g, x) \in G \times X \to (g, f(x)) \in G \times Y$. Natural transformations $\iota: 1_{\mathrm{Set}} \to T$ and $\mu: T^2 \to T$ are defined by $\iota_X: x \in X \to (e, x) \in G \times X$ and

$\mu_X:(g_1,(g_2,x))\varepsilon G\times(G\times X) \to (g_1g_2,x)\varepsilon G\times X$. It is easy to check that (T,ι,μ) is a monad on Set, that is the following diagrams commute

$$\begin{tikzcd} T(X) \arrow[r, "\iota_{T(X)}"] \arrow[dr, equal] & T^2(X) \arrow[d, "\mu_X"] & T(X) \arrow[l, "T(\iota_X)"'] \arrow[dl, equal] \\ & T(X) & \end{tikzcd} \qquad \begin{tikzcd} T^3(X) \arrow[r, "\mu_{T(X)}"] \arrow[d, "T(\mu_X)"'] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"] & T(X) \end{tikzcd}$$

In fact $\mu_X\iota_{T(X)}$ is the composition $(g,x)\varepsilon G\times X \to (e,(g,x))\varepsilon G\times(G\times X) \to (eg,x)=(g,x)\varepsilon G\times X$ and $\mu_XT(\iota_X)$ is the composition $(g,x)\varepsilon G\times X \to (g,(e,x))\varepsilon G\times(G\times X) \to (ge,x)=(g,x)\varepsilon G\times X$. With regard to the second diagram we have that $\mu_X\mu_{T(X)}$ is the composition

$(g_1,(g_2,(g_3,x)))\varepsilon T^3(X)\to(g_1g_2,(g_3,x))\varepsilon T^2(X)\to((g_1g_2)g_3,x)\varepsilon T(X)$

and that $\mu_XT(\mu_X)$ is the composition

$(g_1,(g_2,(g_3,x)))\varepsilon T^3(X)\to(g_1,(g_2g_3,x))\varepsilon T^2(X)\to(g_1(g_2g_3),x)\varepsilon T(X)$

and hence $\mu_X\mu_{T(X)}=\mu_XT(\mu_X)$, since the multiplication in G is associative.

Now let us find out what the T-algebras are. They are couples (X,ϕ) with $\phi:G\times X \to X$ a function such that the following diagrams commute

$$
\begin{tikzcd}
X \arrow[r, "\iota_X"] \arrow[dr, equal] & G\times X \arrow[d, "\phi"] \\
 & X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
G\times(G\times X) \arrow[r, "T(\phi)"] \arrow[d, "\mu_X"'] & G\times X \arrow[d, "\phi"] \\
G\times X \arrow[r, "\phi"] & X
\end{tikzcd}
$$

Commutativity of the first diagram means that $\phi(e,x)=x$ and commutativity of the second diagram means that $\phi(g_1,\phi(g_2,x))=\phi(g_1g_2,x)$, that is, using the notation $g.x=\phi(g,x)$, $e.x=x$ and $g_1.(g_2.x)=(g_1g_2).x$. Hence T-algebras are just G-sets.

The free T-algebras are those G-sets of the form $G\times X$, for some X, where the action ϕ is given by the multiplication in G, that is $\phi:(g_1,(g_2,x))\in G\times(G\times X)\to(g_1g_2,x)\in G\times X$. We observe that in this case free T-algebras are "free" in the same sense of the meaning of free groups, free modules, free algebras, etc. In fact X is canonically embedded in $G\times X$ and for any G-set Y there is a one-to-one correspondence between (set-theoretic) functions $f:X\to Y$ and G-equivariant functions $\bar{f}:G\times X\to Y$ given by $\bar{f}(g,x)=g.f(x)$. In fact $\bar{f}$ so defined is G-equivariant since $\bar{f}(g_1.(g_2,x))=\bar{f}(g_1g_2,x)=(g_1g_2).f(x)=g_1.(g_2.f(x))=g_1.\bar{f}(g_2,x)$ and moreover if $\bar{f}$ and $\bar{\bar{f}}$ are G-equivariant and

$\bar{f}|\{e\}\times X=\bar{\bar{f}}|\{e\}\times X$ then $\bar{f}(g,x)=\bar{f}(g.(e,x))=g.\bar{f}(e,x)=g.\bar{\bar{f}}(e,x)=\bar{\bar{f}}(g.(e,x))=\bar{\bar{f}}(g,x)$, that is $\bar{f}=\bar{\bar{f}}$.

(iv) Let $(X,\leq)$ be a preordered set (i.e. $\leq$ is reflexive, transitive but not necessarily antisymmetric) and let $\mathcal{X}$ be the category associated to $(X,\leq)$; that is, $|\mathcal{X}|=X$ and $\mathcal{X}(x,y)=\{1_{xy}=(x,y)\}$, if $x\leq y$, and empty otherwise. It is known that functors $T:\mathcal{X}\to\mathcal{X}$ are in one-to-one correspondence with monotone functions $t:X\to X$, that is $t(x)\leq t(y)$ whenever $x\leq y$. It is easy to see that for a given functor $T:\mathcal{X}\to\mathcal{X}$ there exist natural transformations $\iota:1_{\mathcal{X}}\to T$ and $\mu:T^2\to T$ such that (T,ι,μ) is a monad if and only if $x\leq t(x)$ and $t^2(x)\leq t(x)$ for every $x\in X$. In such a case, T-algebras can be identified with the elements $x\in X$ such that $t(x)\leq x$.

When X is partially ordered (i.e. $\leq$ is antisymmetric), then from $x\leq t(x)$ and the monotonicity of t it follows that $t(x)\leq t^2(x)$, which, combined with $t^2(x)\leq t(x)$, gives $t^2(x)=t(x)$. Hence, if X is partially ordered, the monads on $\mathcal{X}$ are in one-to-one correspondence with the closure operators on X (i.e. $x\leq t(x)$ and $t^2(x)=t(x)$). Moreover if $(x,1_{t(x),x})$ is a T-algebra, then from $x\leq t(x)$ and $t(x)\leq x$ it follows that $x=t(x)$. Hence, T-algebras can be identified with the elements of X which are closed. Observe that in this case T-algebras and free T-algebras coincide.

Particular examples of the general situation described above are: (a) take X=R with the natural ordering and $t:R \to R$ with $t(x)=[x]$, where $[x]$ is the least integer greater or equal than x; in this case T-algebras can be identified with the integers; (b) take any topological space (S,τ), $(X,\leq)=(2^S,\subseteq)$ and $t:X \to X$ the closure operator $A \mapsto \bar{A}$; in this case T-algebras can be identified with the closed subsets of S.

After having illustrated the concept of monad, we need a digression on Moore paths before explaining the relation between monads and fibrations. A Moore (or measured) path in B is a continuous function $\alpha:[0,r] \to B$ where $r\geq 0$; $\alpha(0)$ is called the origin of α, $\alpha(r)$ the end of α and r the length of α and denoted by $\ell(\alpha)$. For every $b\in B$ and $r\in[0,\infty[$, r_b will denote the path of length r constant at b; so in particular 0_b will denote the path of length 0 determined by b. We denote by MB the set of all Moore paths in B, so that $MB = \bigcup_{r\geq 0} B^{[0,r]}$. The subsets of MB consisting of all paths having $b_0\in B$ as origin and of all paths having b_0 as end will be denoted by $M(B,b_0)$ and $M'(B,b_0)$ respectively. The elements of $\Lambda(B,b_0)=M(B,b_0)\cap M'(B,b_0)$ are called the Moore loops based at b_0.

We topologize MB in the following way. For each Moore path α in B we define its extension $\bar{\alpha}:[0,\infty[\to B$ by $\bar{\alpha}(t)=\alpha(t)$, if $0\leq t\leq \ell(\alpha)$, and $\bar{\alpha}(t)=\alpha(\ell(\alpha))$, if $t\geq \ell(\alpha)$. A function $e:MB \to B^{[0,\infty[}\times[0,\infty[$ can then be defined by

$e(\alpha)=(\bar{\alpha},\ell(\alpha))$; e is injective, since $e(\alpha)=e(\beta)$ means $(\bar{\alpha},\ell(\alpha))=(\bar{\beta},\ell(\beta))$ and so α and β have the same length and coincide on their common domain of definition; furthermore, $\mathrm{Im}\, e = \{(\gamma,r)\in B^{[0,\infty[}\times[0,\infty[: \gamma(t)=\gamma(r)$ for every $t>r\}$. Once we topologize $B^{[0,\infty[}$ with the compact-open topology and $B^{[0,\infty[}\times [0,\infty[$ with the product topology, we topologize MB with the initial topology with respect to e, that is, we view MB as a subspace of $B^{[0,\infty[}\times[0,\infty[$.

<u>Proposition 1</u> (i) The initial topology on MB with respect to e induces on each $B^{[0,r]}\subseteq MB$ the compact-open topology; (ii) if B is Hausdorff, then each $B^{[0,r]}$ is a closed subspace of MB.

<u>Proof</u> (i) We first observe that the topology that MB induces on $B^{[0,r]}$ coincides with the topology induced by the function $e: \alpha\in B^{[0,r]} \to (\bar{\alpha},r)\in B^{[0,\infty[}\times\{r\}$. This is a consequence of the general observation that if $f:(X,A)\to (Y,B)$ is a function of pairs, X a set and Y a space, and $f_0:A\to B$ denotes the restriction of f to A, then the restriction to A of the topology induced by f on X coincides with the topology induced by f_0. Indeed $A\cap f^{-1}(U)=f_0^{-1}(B\cap U)$ for any open set U of Y. Since $B^{[0,\infty[}\times\{r\}$ is homeomorphic to $B^{[0,\infty[}$ we can consider just $e:\alpha\in B^{[0,r]}\to\bar{\alpha}\in B^{[0,\infty[}$ and prove that the topology induced by e on $B^{[0,r]}$ coincides with the compact-open topology on $B^{[0,r]}$. Let k denote the compact-open topology and $\tau(e)$ the topology induced by e.

<u>$k\subseteq\tau(e)$</u> We must prove that for any $U\subseteq B^{[0,r]}$ open in the

compact-open topology there is some $U'\subseteq B^{[0,\infty[}$ open such that $U=e^{-1}(U')$, or equivalently $e(U)=\mathrm{Ime}\cap U'$. First let U be a subbasic set, that is $U=\langle K,A\rangle$ with $K\subseteq[0,r]$ compact and $A\subseteq B$ open. Then $e(\langle K,A\rangle)=\{\gamma\in B^{[0,\infty[}:\gamma(t)=\gamma(r)$ if $t>r$, and $\gamma(K)\subseteq A\}=\mathrm{Ime}\cap\langle K,A\rangle^+$, where $\langle K,A\rangle^+$ denotes the corresponding subset in $B^{[0,\infty[}$. Now if U is a general open set in the compact-open topology of $B^{[0,r]}$ it will be a union of finite intersection of subbasic sets, that is $U=\bigcup_{j\in J}\langle K_1^j,A_1^j\rangle\cap\dots\cap\langle K_{n_j}^j,A_{n_j}^j\rangle$; hence

$$e\Big(\bigcup_{j\in J}\langle K_1^j,A_1^j\rangle\cap\dots\cap\langle K_{n_j}^j,A_{n_j}^j\rangle\Big)=$$

$$\bigcup_{i\in J}e(\langle K_1^j,A_1^j\rangle\cap\dots\cap\langle K_{n_j}^j,A_{n_j}^j\rangle)= \qquad \text{(e is injective)}$$

$$\bigcup_{i\in J}e(\langle K_1^j,A_1^j\rangle)\cap\dots\cap e(\langle K_{n_j}^j,A_{n_j}^j\rangle)=$$

$$\bigcup_{i\in J}(\mathrm{Ime}\cap\langle K_1^j,A_1^j\rangle^+)\cap\dots\cap(\mathrm{Ime}\cap\langle K_{n_j}^j,A_{n_j}^j\rangle^+)=$$

$$\bigcup_{i\in J}\mathrm{Ime}\cap\langle K_1^j,A_1^j\rangle^+\cap\dots\cap\langle K_{n_j}^j,A_{n_j}^j\rangle^+=$$

$$\mathrm{Ime}\cap\bigcup_{i\in J}\langle K_1^j,A_1^j\rangle^+\cap\dots\cap\langle K_{n_j}^j,A_{n_j}^j\rangle^+,$$ as required.

<u>$k\supseteq\tau(e)$</u> Let $\langle K,A\rangle^+$ be a subbasic set in $B^{[0,\infty[}$ with $K\subseteq[0,\infty[$ compact and $A\subseteq B$ open; define

$$K'=\begin{cases}K\cap[0,r]\cup\{r\} & \text{if } r\notin K \text{ and } K\cap]r,\infty[\neq\emptyset\\ K\cap[0,r] & \text{otherwise}\end{cases}$$

We claim that $e^{-1}(\langle K,A\rangle^{+})=\langle K',A\rangle$. To this end, we first observe that $e^{-1}(\langle K,A\rangle^{+})=\{\alpha\in B^{[0,r]}:\ \bar{\alpha}(K)\subseteq A\}$ and that $\bar{\alpha}(K)=\bar{\alpha}(K\cap[0,r])\cup\bar{\alpha}(K\cap]r,\infty[)=\alpha(K\cap[0,r])\cup\bar{\alpha}(K\cap]r,\infty[)$. Now if $K\cap]r,\infty[\neq\emptyset$ and $r\notin K$, then $\bar{\alpha}(K\cap]r,\infty[)=\{\alpha(r)\}$; otherwise, we have that $\bar{\alpha}(K\cap]r,\infty[)\subseteq\alpha(K\cap[0,r])$ since, if $K\cap]r,\infty[=\emptyset$, then $\bar{\alpha}(K\cap]r,\infty[)=\emptyset$ and, if $K\cap[r,\infty[\neq\emptyset$ and $r\in K$, then $\bar{\alpha}(K\cap]r,\infty[)=\{\alpha(r)\}\subseteq\alpha(K\cap[0,r])$. Hence

$$\bar{\alpha}(K) = \begin{cases}\alpha(K\cap[0,r]\cup\{r\}) & \text{if } r\notin K \text{ and } K\cap]r,\infty[\neq\emptyset\\ \alpha(K\cap[0,r]) & \text{otherwise}\end{cases}$$

and so $e^{-1}(\langle K,A\rangle^{+})=\{\alpha\in B^{[0,r]}:\ \bar{\alpha}(K)\subseteq A\}=\langle K',A\rangle$. If U' is a general open set in $B^{[0,\infty[}$ then

$U'=\bigcup_{j\in J}\langle K_1^j,A_1^j\rangle^{+}\cap\ldots\cap\langle K_{n_j}^j,A_{n_j}^j\rangle^{+}$ and therefore

$e^{-1}(U')=e^{-1}(\bigcup_{j\in J}\langle K_1^j,A_1^j\rangle^{+}\cap\ldots\cap\langle K_{n_j}^j,A_{n_j}^j\rangle^{+})=$

$\bigcup_{j\in J}e^{-1}(\langle K_1^j,A_1^j\rangle^{+}\cap\ldots\cap\langle K_{n_j}^j,A_{n_j}^j\rangle^{+})=$

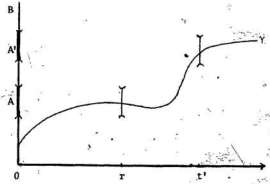
$\bigcup_{j\in J}e^{-1}(\langle K_1^j,A_1^j\rangle^{+})\cap\ldots\cap e^{-1}(\langle K_{n_j}^j,A_{n_j}^j\rangle^{+})=$

$\bigcup_{j\in J}\langle K_1'^j,A_1^j\rangle\cap\ldots\cap\langle K_{n_j}'^j,A_{n_j}^j\rangle$

which is an open set in the compact-open topology of $B^{[0,r]}$.

(ii) For any $r\in[0,\infty[$ let $B_r^{[0,\infty[}=\{\gamma\in B^{[0,\infty[}:\ \gamma(t)=\gamma(r)\ t>r\}$. Now $B^{[0,r]}=e^{-1}(B_r^{[0,\infty[}\times\{r\})$ and so $B^{[0,r]}$ is closed in MB if $B_r^{[0,\infty[}$ is closed in $B^{[0,\infty[}$. Let $\gamma\in B^{[0,\infty[}-B_r^{[0,\infty[}$; then there is some $t'>r$ with $\gamma(t')\neq\gamma(r)$. Let

A be an open neighbourhood of $\gamma(r)$ and let A' be an open neighbourhood of $\gamma(t')$ with A and A' disjoint (B is Hausdorff).



Then $\langle\{r\},A\rangle^+ \cap \langle\{t'\},A'\rangle^+$ is a neighbourhood of γ disjoint from $\overset{*}{B}{}_r^{[0,\infty[}$; indeed, if $\gamma' \in \langle\{r\},A\rangle^+ \cap \langle\{t'\},A'\rangle^+$ then $\gamma'(r) \in A$ and $\gamma'(t) \in A'$ and hence $\gamma'(r) \neq \gamma'(t')$ because A and A' are disjoint. Thus $B_r^{[0,\infty[}$ is closed in $B^{[0,\infty[}$.

As for the unitary path space B^I, the Moore path space gives rise to a (covariant) functor M: Top $\to$ Top which associates to any space B its Moore path space MB and to any map $p: B \to C$ the map $Mp: MB \to MC$ defined by $Mp(\alpha) = p\alpha$. The continuity of Mp is a consequence of the commutativity of the following diagram (filled arrows denote maps)

$$\begin{tikzcd} MB \arrow[r, "e"] \arrow[d, "Mp"'] & B^{[0,\infty[}\times[0,\infty[\arrow[d, "p_*\times 1_{[0,\infty[}"] \\ MC \arrow[r, "e"] & C^{[0,\infty[}\times[0,\infty[\end{tikzcd}$$

Indeed, since MC has the initial topology with respect to e, Mp is continuous if and only if e Mp is continuous. But $e\,Mp=(p_*\times 1_{[0,\infty[})e$, which is continuous, and so Mp is continuous.

Although MB is not, strictly speaking, a function space (i.e. $MB=Y^X$ for some X and Y) we have the following result.

Propositon 2 Let X be any space, $d:X\to[0,\infty[$ any map and $X_d=\{(x,t)\in X\times[0,\infty[:\ 0\leq t\leq d(x)\}$. Then for any map $f:X_d\to B$ the function $f:x\in X\to f(x)\in MB$, where $f(x):t\in[0,d(x)]\to f(x,t)\in B$, is continuous.

Proof Since MB has the initial topology with respect to the function $e:MB\to B^{[0,\infty[}\times[0,\infty[$, the continuity of $f:X\to MB$ is equivalent to the continuity of ef. To prove that ef is continuous, let $X_d'=\{(x,t)\in X\times[0,\infty[:\ t\geq d(x)\}$ and observe that X_d and X_d' are closed subspaces of $X\times[0,\infty[$ such that $X\times[0,\infty[=X_d\cup X_d'$. Then define $F:X\times[0,\infty[\to B$ by $F(x,t)=f(x,t)$, if $0\leq t\leq d(x)$, and $F(x,t)=f(x,d(x))$, if $t\geq d(x)$; F is continuous because its restriction to X_d is f and its restriction to X_d'

is equal to the composition $(x,t)\in X_d' \xrightarrow{(pr_1, d\circ pr_1)} (x,d(x))\in X_d \xrightarrow{f}$ $f(x,d(x))\in B$. Let $F:X \to B^{[0,\infty[}$ denote the adjoint of F. Then $ef=(F,d)$ and so ef is continuous.

As in the context of unitary paths, we have that the functions $\ell:\alpha\in MB \to \ell(\alpha)\in[0,\infty[$, $\pi_0:\alpha\in MB \to \alpha(0)\in B$ and $\pi_1:\alpha\in MB \to \alpha(\ell(\alpha))\in B$ are continuous. In fact, ℓ can be identified with the composition $MB \xrightarrow{e} B^{[0,\infty[}\times[0,\infty[\xrightarrow{pr_2} [0,\infty[$, π_0 with the composition $MB \xrightarrow{e} B^{[0,\infty[}\times[0,\infty[\xrightarrow{pr_1} B^{[0,\infty[} \xrightarrow{\omega_0} B$ and π_1 with the composition $MB \xrightarrow{e} B^{[0,\infty[}\times[0,\infty[\xrightarrow{\omega} B$.

The following result is the analogue for Moore paths of the example (iii) in section I.2. Its proof can be found in [46; p.108].

<u>Proposition 3</u> For any space B the map $\pi:\alpha\in MB \to (\alpha(0),\alpha(\ell(\alpha)))\in B\times B$ is a fibration.

<u>Corollary 4</u> For any space B the maps $\pi_0:\alpha\in MB \to \alpha(0)\in B$ and $\pi_1:\alpha\in MB \to \alpha(\ell(\alpha))\in B$ are fibrations.

<u>Corollary 5</u> For any space B and $b_0\in B$ the maps $p:\alpha\in M(B,b_0) \to \alpha(\ell(\alpha))\in B$ and $p':\alpha\in M'(B,b_0) \to \alpha(0)\in B$ are fibrations with fibre over b_0 equal to the Moore loop space $\Lambda(B,b_0)$.

The next proposition is relevant to relating notions defined using Moore paths to analogous notions defined using unitary paths.

Proposition 6 There is a canonical fibre homotopy equivalence over $B\times B$ between the fibration $\pi:\alpha\in MB \to (\alpha(0),\alpha(\ell(\alpha)))\in B\times B$ and the fibration $\pi:\alpha\in B^I \to (\alpha(0),\alpha(1))\in B\times B$.

Proof We claim that the inclusion map $i:B^I \to MB$ is a fibre homotopy equivalence over $B\times B$. To this end, define the fibre map $d:MB \to B^I$ over $B\times B$ by $d(\alpha)(t)=\alpha(\ell(\alpha)t)$; d is continuous because it is equal to the composition

$MB \xrightarrow{e} B^{[0,\infty[\times[0,\infty[} \xrightarrow{1\times m} B^{[0,\infty[\times[0,\infty[^{[0,1]}} \xrightarrow{T} B^{[0,1]}$, where $m:[0,\infty[\to [0,\infty[^{[0,1]}$ is the adjoint of the multiplication $m:(r,t)\in[0,\infty[\times[0,1] \to rt\in[0,\infty[$ and T is given by composition of maps. d is a left inverse for i and a right fibre homotopy inverse for i. Indeed, $H:MB\times I \to MB$, defined by letting $H(\alpha,t)$ be the Moore path $s\in[0,(1-\ell(\alpha))t+\ell(\alpha)]\to \alpha(\ell(\alpha)s/[(1-\ell(\alpha))t+\ell(\alpha)])\in B$, is a vertical homotopy from the identity of MB to the composite id. The continuity of H is proved considering the map

$K:(\alpha,t,s)\in\{(\alpha,t,s)\in MB\times I\times[0,\infty[: 0\leq s\leq(1-\ell(\alpha))t+\ell(\alpha)\} \to \alpha(\ell(\alpha)s/[(1-\ell(\alpha))t+\ell(\alpha)])\in B$ and then applying proposition 2, observing that $H=K$.

Remark 7 Since B^I is included in MB and $\mathrm{Im}H_1\subseteq B^I$, the statement of proposition 6 is equivalent to the statement that B^I is a strong deformation retract of MB via a deformation which is vertical over $B\times B$, that is, fixing the end points of the paths during the deformation.

Corollary 8 For any space B and $b_0 \in B$, the fibrations $p:M(B,b_0) \to B$ and $p:P(B,b_0) \to B$ have the same fibre homotopy type over B. Furthermore, $M(B,b_0)$ is a contractible space and the loop spaces $\Lambda(B,b_0)$ and $\Omega(B,b_0)$ have the same homotopy type. An analogous statement holds for the fibrations $p':M'(B,b_0) \to B$ and $p':P'(B,b_0) \to B$.

Proof The first statement follows from proposition 6 and I.1.4, since $p:M(B,b_0) \to B$ and $p:P(B,b_0) \to B$ can be identified with the pullbacks of $\pi:MB \to B\times B$ and $\pi:B^I \to B\times B$, respectively, along the map $f:b\in B \to (b_0,b)\in B\times B$. The remaining assertions are obvious.

We now define an addition operation $\mu:MB\underline{\times}MB \to MB$ where $MB\underline{\times}MB$ denotes the subspace of $MB\times MB$ consisting of all couples (α,β) such that the end of α is equal to the origin of β. μ is defined by setting $\mu(\alpha,\beta):[0,\ell(\alpha)+\ell(\beta)] \to B$ with $\mu(\alpha,\beta)(t)=\alpha(t)$, if $0\leq t\leq\ell(\alpha)$, and $\mu(\alpha,\beta)(t)=\beta(t-\ell(\alpha))$, if $\ell(\alpha)\leq t\leq\ell(\alpha)+\ell(\beta)$. We write $\mu(\alpha,\beta)=\alpha+\beta$. Observe that if α, β and γ are paths with $\alpha(\ell(\alpha))=\beta(0)$ and $\beta(\ell(\beta))=\gamma(0)$ then the above addition is strictly associative, that is, $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$, and that for any path α we have that $0_{\alpha(0)}+\alpha = \alpha+0_{\alpha(\ell(\alpha))}=\alpha$. We now prove that μ is continuous on $MB\underline{\times}MB$. To this end, we define an addition

$\mu' : (B^{[0,\infty]} \times [0,\infty[) \underline{\times} (B^{[0,\infty[} \times [0,\infty[) \to B^{[0,\infty[} \times [0,\infty[$ where $(B^{[0,\infty[} \times [0,\infty[) \underline{\times} (B^{[0,\infty[} \underline{\times} [0,\infty[)$ denotes the subspace consisting of all couples $(\alpha, r; \beta, q)$ such that $\alpha(r) = \beta(0)$; μ' is defined by $\mu(\alpha, r; \beta, q) = (\gamma, r+q)$ where $\gamma(t) = \alpha(t)$, if $0 \leq t \leq r$, and $\gamma(t) = \beta(t-r)$, if $t \geq r$. It is easy to see that the restriction of μ' to $MB \underline{\times} MB$ is just μ, or to be more precise, that $\mu'(e(\alpha), e(\beta)) = e(\mu(\alpha, \beta))$; in fact both are equal to $(\gamma, \ell(\alpha) + \ell(\beta))$ where $\gamma(t) = \alpha(t)$, if $0 \leq t \leq \ell(\alpha)$, $\gamma(t) = \beta(t - \ell(\alpha))$, if $\ell(\alpha) \leq t \leq \ell(\alpha) + \ell(\beta)$, and $\gamma(t) = \beta(\ell(\beta))$, if $t \geq \ell(\alpha) + \ell(\beta)$. Thus it is sufficient to prove the continuity of μ'. Now the second component $pr_2\mu'$ can be identified with the composition $(\alpha, r; \beta, q) \in (B^{[0,\infty[} \times [0,\infty[) \underline{\times} (B^{[0,\infty[} \times [0,\infty[) \to (r, q) \in [0,\infty[\times [0,\infty[\to r+q \in [0,\infty[$, which is continuous. For the first component $pr_1\mu'$ we observe that, since $[0,\infty[$ is Hausdorff and locally compact, $pr_1\mu'$ is continuous if and only if its adjoint $(\alpha, r; \beta, q; t) \in (B^{[0,\infty[} \times [0,\infty[) \underline{\times} (B^{[0,\infty[} \times [0,\infty[) \times [0,\infty[\to$
$\to \begin{cases} \alpha(t) & \text{if } 0 \leq t \leq r \\ \beta(t-r) & \text{if } t \geq r \end{cases} \in B$ is continuous. Let $S_1 = \{(\alpha, r; \beta, q; t) \mid 0 \leq t \leq r\}$ and $S_2 = \{(\alpha, r; \beta, q; t) \mid t \geq r\}$. Then $\{S_1, S_2\}$ is a cover of $(B^{[0,\infty[} \times [0,\infty[) \underline{\times} (B^{[0,\infty[} \times [0,\infty[) \times [0,\infty[$ by closed sets. The above map restricted to S_1 is the composition $(\alpha, r; \beta, q; t) \in S_1 \to$
$(\alpha, r; \beta, q; t) \in (B^{[0,\infty[} \times [0,\infty]) \underline{\times} (B^{[0,\infty[} \times [0,\infty[) \times [0,\infty[\to$
$\to (\alpha, t) \in B^{[0,\infty[} \times [0,\infty[\xrightarrow{\omega} \alpha(t) \in B$ and restricted to S_2 is the

composition $(\alpha,r;\beta,q;t)\in S_2 \to (\alpha,r;\beta,q;t)\in(B^{[0,\infty[}\times[0,\infty[)\underline{\times}$
$(B^{[0,\infty[}\times[0,\infty[)\times[0,\infty[\to (\beta,|t-r|)\in B^{[0,\infty[}\times[0,\infty[\xrightarrow{\omega} \beta(|t-r|)\in B$.
Since both compositions are continuous the adjoint of $pr_1\mu'$ is continuous, as required.

In section I.2 we saw how Hurewicz fibrations can be characterized intrinsically by the existence of a lifting function with respect to unitary paths. Now we discuss lifting functions in the context of Moore paths.

For any map $p:E \to B$ let $\Gamma_p=\{(e,\alpha)\in E\times MB:\ \alpha(0)=p(e)\}$ and define $\bar{p}:\Gamma_p \to B$ by $\bar{p}(e,\alpha)=\alpha(\ell(\alpha))$. Let $\rho:ME \to \Gamma_p$ be the map defined by $\rho(\alpha)=(\alpha(0),p\alpha)$. Then a <u>Moore (global) lifting function</u> for p is a map $\tau:\Gamma_p \to ME$ such that $\tau(e,\alpha)(0)=e$ and $p\circ\tau(e,\alpha)=\alpha$; in other words τ is a section of ρ. Given a Moore lifting function τ for p, there is associated to each Moore path α in B a <u>translation map along</u> α, $\tau_\alpha:F_{\alpha(0)} \to F_{\alpha(\ell(\alpha))}$, defined by $\tau_\alpha(e)=\tau(e,\alpha)(\ell(\alpha))$. τ is said to have <u>transitive translation maps</u> if for any $\alpha,\beta\in MB$ with $\alpha(\ell(\alpha))=\beta(0)$ the relation $\tau_{\alpha+\beta}=\tau_\beta\tau_\alpha$ holds. We say that τ is <u>transitive</u> [33;p.288] if for any $\alpha,\beta\in MB$ with $\alpha(\ell(\alpha))=\beta(0)$ we have that $\tau(e,\alpha+\beta)=\tau(e,\alpha)+\tau(\tau(e,\alpha)(\ell(\alpha)),\beta)$. Of course a transitive lifting function has transitive translation maps since
$\tau_{\alpha+\beta}(e)=\tau(e,\alpha+\beta)(\ell(\alpha+\beta))=[\tau(e,\alpha)+\tau(\tau(e,\alpha)(\ell(\alpha)),\beta)](\ell(\alpha+\beta))=$
$=\tau(\tau(e,\alpha)(\ell(\alpha)),\beta)(\ell(\beta))=\tau_\beta(\tau(e,\alpha)(\ell(\alpha)))=\tau_\beta\tau_\alpha(e)$.

A <u>Moore end-point lifting function</u> for p is a map

$\xi: \Gamma_p \to E$ such that $p\xi(e,\alpha)=\alpha(\ell(\alpha))$ and $\xi(e,0_{p(e)})=e$. Given a Moore end-point lifting function ξ for p, there is associated to each Moore path α a <u>translation map along</u> α, $\xi_\alpha: F_{\alpha(0)} \to F_{\alpha(\ell(\alpha))}$, defined by $\xi_\alpha(e)=\xi(e,\alpha)$. ξ is said <u>transitive</u> [33, p.11] if for any $\alpha, \beta \varepsilon MB$ such that $\alpha(\ell(\alpha))=\beta(0)$ the relation $\xi_{\alpha+\beta}=\xi_\beta \xi_\alpha$ holds.

We now study the relationship between Moore global lifting functions and Moore end-point lifting functions. We denote by G the set of all Moore global lifting functions for p, by $G' \subseteq G$ the subset of those with transitive translation maps, by $G'' \subseteq G$ the subset of those which are transitive, by T the set of all Moore end-point lifting functions for p and by $T' \subseteq T$ the subset of those which are transitive. There is a function $f: G \to T$ which associates to each global lifting function τ the end-point lifting function $f(\tau):(e,\alpha)\varepsilon\Gamma_p \to \tau(e,\alpha)(\ell(\alpha))\varepsilon E$. A right inverse $g: T \to G$ for f can be constructed in the following way. Let $MB_\ell = \{(\alpha,t)\varepsilon MB\times[0,\infty[: 0\leq t\leq \ell(\alpha)\}$ and for each $(\alpha,t)\varepsilon MB_\ell$ let α_t denote the Moore path of length t defined by $\alpha_t(s)=\alpha(s)$. Applying proposition 2 to the map $(\alpha,t,s)\varepsilon\{(\alpha,t,s)\varepsilon MB\times[0,\infty[^2: 0\leq s\leq t\leq \ell(\alpha)\} \to \alpha(s)\varepsilon B$ we have that the function $L:(\alpha,t)\varepsilon MB_\ell \to \alpha_t \varepsilon MB$ is continuous. Now define $g: T \to G$ by the rule $g(\xi)(e,\alpha)(t)=\xi(e,\alpha_t)$. For every $(e,\alpha)\varepsilon\Gamma_p$ $g(\xi)(e,\alpha)$ is equal to the composition $t\varepsilon[0,\ell(\alpha)] \to (\alpha,t)\varepsilon MB_\ell \xrightarrow{L} \alpha_t \varepsilon MB \to (e,\alpha_t)\varepsilon\Gamma_p \xrightarrow{\xi} E$ and satisfies

$g(\xi)(e,\alpha)(0)=\xi(e,\alpha_0)=\xi(e,0_{p(e)})=e$ and $p[g(\xi)(e,\alpha)(t)]=$ $p\xi(e,\alpha_t)=\alpha_t(\ell(\alpha_t))=\alpha(t)$; so $g(\xi)(e,\alpha)$ is a Moore path of length $\ell(\alpha)$. Now applying proposition 2 to the continuous composition $(e,\alpha,t)\varepsilon\{(e,\alpha,t)\varepsilon\Gamma_p\times[0,\infty[:0<t<\ell(\alpha)\}\xrightarrow{(pr_1,L)}$ $(e,\alpha_t)\varepsilon\Gamma_p\xrightarrow{\xi}\xi(e,\alpha_t)\varepsilon E$ we have that for every $\xi\varepsilon T$, $g(\xi)$ is continuous and hence a Moore global lifting function for p. It is easy to see that $fg=1_T$ because $[fg(\xi)](e,\alpha)=$ $g(\xi)(e,\alpha)(\ell(\alpha))=\xi(e,\alpha_{\ell(\alpha)})=\xi(e,\alpha)$, and hence f is onto. For every $\tau\varepsilon G$, $f(\tau)$ has the same translation maps as τ; indeed $f(\tau)_\alpha(e)=f(\tau)(e,\alpha)=\tau(e,\alpha)(\ell\alpha))=\tau_\alpha(e)$. It follows that $\tau\varepsilon G$ has transitive translation maps if and only if $f(\tau)\varepsilon T$ is transitive. f is injective on G"; indeed, if τ is a transitive global lifting function then $\tau(e,\alpha)(t)=$ $[\tau(e,\alpha_t)+\tau(\tau(e,\alpha_t)(t),\alpha_t')](t)=\tau(e,\alpha_t')(t)=f(\tau)(e,\alpha_t)$ where $\alpha_t':s\,\varepsilon[0,\ell(\alpha)-t]\to\alpha(t+s)\varepsilon B$, which shows that τ is completely determined by $f(\tau)$. Furthermore, if $\xi\varepsilon T$ is transitive then $g(\xi)$ is a transitive global lifting function; indeed

$$g(\xi)(e,\alpha+\beta)(t)=\xi(e,(\alpha+\beta)_t)=\begin{cases}\xi(e,\alpha_t) & \text{if } 0<t<\ell(\alpha)\\ \xi(e,\alpha+\beta_{t-\ell(\alpha)}) & \text{if } \ell(\alpha)<t\leq\ell(\alpha+\beta)\end{cases}$$

and

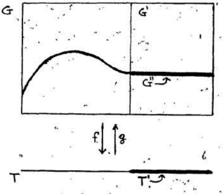
$$[g(\xi)(e,\alpha)+g(\xi)(g(\xi)(e,\alpha)(\ell(\alpha)),\beta)](t)=$$

$$=\begin{cases}g(\xi)(e,\alpha)(t) & \text{if } 0<t<\ell(\alpha)\\ g(\xi)(\xi)(e,\alpha),\beta(t-\ell(\alpha)) & \text{if } \ell(\alpha)<t<\ell(\alpha+\beta)\end{cases}=$$

$$=\begin{cases}\xi(e,\alpha_t) & \text{if } 0<t<\ell(\alpha)\\ \xi(\xi(e,\alpha),\beta_{t-\ell(\alpha)}) & \text{if } \ell(\alpha)<t<\ell(\alpha+\beta)\end{cases}=$$

$$=\begin{cases}\xi(e,\alpha_t) & \text{if } 0<t<\ell(\alpha)\\ \xi(e,\alpha+\beta_{t-\ell(\alpha)}) & \text{if } \ell(\alpha)<t<\ell(\alpha+\beta)\end{cases}$$

which shows that $g(\xi)(e,\alpha+\beta)(t)=$
$[g(\xi)(e,\alpha)+g(\xi)(g(\xi)(e,\alpha)(\ell(\alpha)),\beta](t)$, for every $t\in[0,\ell(\alpha+\beta)]$,
and so $g(\xi)(e,\alpha+\beta)=[g(\xi)(e,\alpha)+g(\xi)(g(\xi)(e,\alpha)(\ell(\alpha)),\beta]$.



We can summarize the above observations as follows.

Proposition 9 The following properties hold:

(i) for any Moore end-point lifting functions ξ there exists at least one Moore global lifting function having the same translation maps as ξ;

(ii) a Moore global lifting function has transitive translation maps if and only if its associated end-point lifting function is transitive;

(iii) for any Moore global lifting function with transitive translation maps there is exactly one transitive global lifting function having the same translation maps.

The following result is the analogue of proposition I.2.14 in the context of Moore paths.

Proposition 10 A map $p:E \to B$ is a fibration if and only if it admits a Moore global lifting function or, equivalently, a Moore end-point lifting function.

Proof Suppose $p:E \to B$ is a fibration. Let $[0,\infty]$ denote the space obtained from $[0,\infty[$ adding a point ∞ and topologizing as follows: $U\subseteq[0,\infty]$ is open if and only if either $U\subseteq[0,\infty[$ is open or $U=A\cup\{\infty\}$, where $A\subseteq[0,\infty[$ is open and $A\supseteq]t,\infty[$ for some $t>0$. The space $[0,\infty]$ is homeomorphic to the unit interval $[0,1]$. For example the map $h:t\in[0,1[\to \tan\pi t/2\in[0,\infty[$ can be extended to the closed interval $[0,1]$ by defining $h(1)=\infty$; h is continuous at 1 because $\lim_{t\to 1}\tan\pi t/2=\infty$, yielding a homeomorphism. We further observe that for any space X there is a natural map $w:MX\times[0,\infty]\to X$ defined by $w(\alpha,t)=\alpha(t)$, if $0\leq t\leq \ell(\alpha)$, and $w(\alpha,t)=\alpha(\ell(\alpha))$, if $\ell(\alpha)\leq t\leq\infty$. To prove the continuity of w consider the two closed sets $D_1=\{(\alpha,t)\in MX\times[0,\infty]:\ 0\leq t\leq\ell(\alpha)\}$ and $D_2=\{(\alpha,t)\in MX\times[0,\infty]:\ \ell(\alpha)\leq t\leq\infty\}$ which cover $MX\times[0,\infty]$ and observe that w restricted to D_1 is the composition

$(\alpha,t)\in D_1 \to (\bar{\alpha},t)\in X^{[0,\infty[}\times[0,\infty[\xrightarrow{\omega} \bar{\alpha}(t)\in X$ and that w restricted to D_2 is the composition

$(\alpha,t)\in D_2 \to (\alpha,\ell(\alpha))\in D_1 \xrightarrow{w|D_1} \alpha(\ell(\alpha))\in X$. Now consider the following commutative diagram (ignore the dotted arrow)

$$\begin{tikzcd} \Gamma_p \arrow[r, "pr_1"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ \Gamma_p\times[0,\infty] \arrow[r, "H"] \arrow[ur, dashed, "\widetilde{H}"] & B \end{tikzcd}$$

where H is the composition $\Gamma_p\times[0,\infty] \xrightarrow{pr_2\times 1} MB\times[0,\infty] \xrightarrow{\psi} B$. Since $[0,\infty]$ is homeomorphic to $[0,1]$ and p is a fibration, we can find a map $\widetilde{H}\colon \Gamma_p\times[0,\infty] \to E$ extending pr_1 and lifting H. The restriction of $\widetilde{H}$ to $\{(e,\alpha,t)\in\Gamma_p\times[0,\infty]\colon 0\le t\le l(\alpha)\}$ gives rise by proposition 2 to a map $\Gamma_p \to ME$ which is a Moore global lifting function for p, as required.

Now, suppose τ is a Moore global lifting function for p. Let $f\colon X \to E$ be any map and let $H\colon X\times I \to B$ be a homotopy of pf. Consider the following commutative diagram, where $\hat{H}\colon X \to B^I \subseteq MB$ is the adjoint of H

$$\begin{tikzcd} X \arrow[dr, dashed, "{(f,\hat{H})}"] \arrow[drr, "f"] \arrow[ddr, "\hat{H}"'] & & \\ & \Gamma_p \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ & MB \arrow[r, "\pi_0"] & B \end{tikzcd}$$

Since Γ_p is the pullback of p along π_0, we have the map $(f,\hat{H}):X \to \Gamma_p$ which when composed with the lifting function $\tau:\Gamma_p \to ME$ takes values in $E^I \subseteq ME$. Taking the adjoint of $\tau\circ(f,\hat{H})$ we obtain a homotopy of f which lifts H, as required.

As an application of proposition 10 we show that for any map $p:E \to B$ the map $\bar{p}:\Gamma_p \to B$ is a fibration. To this end, observe that $\Gamma_{\bar{p}}=\{(e,\alpha,\beta)\varepsilon E\times MB\times MB: \alpha(0)=p(e)$ and $\alpha(\ell(\alpha))=\beta(0)\}$ and define $\xi:\Gamma_{\bar{p}} \to \Gamma_p$ by $\xi(e,\alpha,\beta)=(e,\alpha+\beta)$. ξ is well defined because $(\alpha+\beta)(0)=p(e)$; ξ is continuous being equal to the map $\Gamma_{\bar{p}} \xrightarrow{(pr_1,\,\mu\circ pr_{23})} \Gamma_p$, where $pr_{23}:(e,\alpha,\beta)\varepsilon\Gamma_{\bar{p}} \to (\alpha,\beta)\varepsilon MB\underline{\times}MB$ and $\mu:MB\underline{\times}MB \to MB$ is the addition of Moore paths previously defined; furthermore ξ satisfies the relations $\bar{p}\xi(e,\alpha,\beta)=\bar{p}(e,\alpha+\beta)=(\alpha+\beta)(\ell(\alpha+\beta))=\beta(\ell(\beta))$, $\xi(e,\alpha,0_{\bar{p}(e,\alpha)})=$ $=\xi(e,\alpha,0_{\alpha(\ell(\alpha))})=(e,\alpha+0_{\alpha(\ell(\alpha))})=(e,\alpha)$ and $\xi(e,\alpha,\beta+\gamma)=$ $=(e,\alpha+(\beta+\gamma))=(e,(\alpha+\beta)+\gamma)=\xi(e,\alpha+\beta,\gamma)=\xi(\xi(e,\alpha,\beta),\gamma)$. This shows that ξ is a transitive Moore end-point lifting function for $\bar{p}$ and hence $\bar{p}$ is a fibration.

Furthermore we have that the map $\iota:e\varepsilon E\to(e,0_{p(e)})\varepsilon\Gamma_p$ is a homotopy equivalence. Indeed the map $\nu:(e,\alpha)\varepsilon\Gamma_p \to e\varepsilon E$ is a left inverse of ι and a right homotopy inverse of ι. To prove this latter assertion, consider first the map

$f:(\alpha,t,s)\varepsilon\{(\alpha,t,s)\varepsilon MB\times I\times[0,\infty[:\ 0\leq s\leq \ell(\alpha)t\}\to\alpha(s)\varepsilon B$ and apply proposition 2 to get a homotopy $f:MB\times I\to MB$ from $\iota\nu$ to the identity of MB; then define $H:(e,\alpha,t)\varepsilon\Gamma_p\times I\to(e,f(\alpha,t))\varepsilon\Gamma_p$ which gives a homotopy from $\iota\nu$ to the identity of Γ_p. So the above maps $\iota:E\to\Gamma_p$ and $\bar{p}:\Gamma_p\to B$ give an alternative way to factorize a map as a homotopy equivalence followed by a fibration.

We are now ready to discuss the connection of monads to fibrations. We will show using Moore paths that the standard procedure of factorizing any map $p:E\to B$ as the homotopy equivalence $\iota:E\to\Gamma_p$ followed by the fibration $\bar{p}:\Gamma_p\to B$ gives rise to a monad on Top_B and that the algebras for this monad are essentially fibrations with a specified transitive Moore end-point lifting function. We first discuss the situation in the context of unitary paths. We will see the problems that arise there and how the algebraic behavior of Moore paths allows us to overcome these problems.

Let $\Lambda:Top_B\to Top_B$ be the functor which associates to each object $p:E\to B$ of Top_B the map $\Lambda(p)=\bar{p}:\Lambda_p\to B$, that is $\Lambda(p)(e,\alpha)=\alpha(1)$, and to each morphism $f:p\to p'$ the morphism $\Lambda(f):\Lambda(p)\to\Lambda(p')$ defined by $\Lambda(f)(e,\alpha)=(f(e),\alpha)$. There are natural transformations $\iota:1_{Top_B}\to\Lambda$ and $\mu:\Lambda^2\to\Lambda$ defined by $\iota_p(e)=(e,\overline{p(e)})$ and $\mu_p(e,\alpha,\beta)=(e,\alpha.\beta)$; μ_p is well-defined because $\beta(0)=\Lambda(p)(e,\alpha)=\alpha(1)$.

$$\begin{tikzcd}
E \arrow[r, "\iota_p"] \arrow[d, "p"'] & \Lambda_p \arrow[dl, "\Lambda(p)"] & \Lambda_{\Lambda(p)} \arrow[l, "\mu_p"'] \arrow[dll, "\Lambda^2(p)"] \\
B & &
\end{tikzcd}$$

Let us see if $\Lambda=(\Lambda,\iota,\mu)$ is a monad. For the diagram

$$\begin{tikzcd}
\Lambda(p) \arrow[r, "\Lambda(\iota_p)"] \arrow[dr, equal] & \Lambda^2(p) \arrow[d, "\mu_p"] & \Lambda(p) \arrow[l, "\iota_{\Lambda(p)}"'] \arrow[dl, equal] \\
& \Lambda(p) &
\end{tikzcd}$$

we have that $\mu_p\Lambda(\iota_p)(e,\alpha)=\mu_p(e,\widetilde{p(e)},\alpha)=(e,\widetilde{p(e)}.\alpha)$ and $\mu_p\iota_{\Lambda(p)}(e,\alpha)=\mu_p(e,\alpha,\widetilde{\alpha(1)})=(e,\alpha.\widetilde{\alpha(1)})$; so the diagram is not commutative except up to homotopy. Similarly for the diagram

$$\begin{tikzcd}
\Lambda^3(p) \arrow[r, "\mu_{\Lambda(p)}"] \arrow[d, "\Lambda(\mu_p)"'] & \Lambda^2(p) \arrow[d, "\mu_p"] \\
\Lambda^2(p) \arrow[r, "\mu_p"] & \Lambda(p)
\end{tikzcd}$$

we have $\mu_p \mu_{\Lambda(p)}(e, \alpha, \beta, \gamma) = \mu_p(e, \alpha, \beta . \gamma) = (e, \alpha . (\beta . \gamma))$ and that $\mu_p \Lambda(\mu_p)(e, \alpha, \beta, \gamma) = \mu_p(e, \alpha . \beta, \gamma) = (e, (\alpha . \beta) . \gamma)$ and since $\alpha . (\beta . \gamma)$ is generally different from $(\alpha . \beta) . \gamma$, we again get commutativity only up to homotopy.

The problem with unitary paths is that the operation of addition does not have a strict unit, but rather a homotopy unit, and is only homotopy associative.

Let us now perform the same construction using Moore paths. Let $\Gamma : Top_B \to Top_B$ be the functor defined on objects $p : E \to B$ by $\Gamma(p) = \bar{p} : \Gamma_p \to B$ and on morphisms $f : p \to p'$ by $\Gamma(f) : (e, \alpha) \in \Gamma_p \to (f(e), \alpha) \in \Gamma_{p'}$. There are natural transformations $\iota : 1_{Top_B} \to \Gamma$ and $\mu : \Gamma^2 \to \Gamma$ defined by $\iota_p(e) = (e, 0_{p(e)})$ and $\mu_p(e, \alpha, \beta) = (e, \alpha + \beta)$. Now for any $p \in |Top_B|$ the diagrams

$$\begin{tikzcd} \Gamma(p) \arrow[r, "\Gamma(\iota_p)"] \arrow[dr, equal] & \Gamma^2(p) \arrow[d] & \Gamma(p) \arrow[l, "\iota_{\Gamma(p)}"'] \arrow[dl, equal] \\ & \Gamma(p) & \end{tikzcd} \qquad \begin{tikzcd} \Gamma^3(p) \arrow[r, "\mu_{\Gamma(p)}"] \arrow[d, "\Gamma(\mu_p)"'] & \Gamma^2(p) \arrow[d, "\mu_p"] \\ \Gamma^2(p) \arrow[r, "\mu_p"] & \Gamma(p) \end{tikzcd}$$

commute because the Moore paths of length zero are strict units for the addition and moreover addition of Moore paths is strictly associative. Hence $\Gamma=(\Gamma,\iota,\mu)$ is a monad on Top_B.

The following result is due to P. Malraison [34].

<u>Theorem 11</u> Let $\Gamma=(\Gamma,\iota,\mu)$ be the monad on Top_B as defined above. Then the Γ-algebras are precisely the pairs $(p:E\to B,\xi)$ where p is a fibration and $\xi:\Gamma_p\to E$ is a Moore end-point transitive lifting function for p.

<u>Proof</u> According to the definition, the pair $(p:E\to B,\xi)$ is a Γ-algebra if ξ is a morphism in Top_B from $\Gamma(p)$ to p making the following diagrams in Top_B commute

$$\begin{tikzcd} p \arrow[r, "\iota_p"] \arrow[dr, equal] & \Gamma(p) \arrow[d, "\xi"] \\ & p \end{tikzcd} \qquad \begin{tikzcd} \Gamma^2(p) \arrow[r, "\Gamma(\xi)"] \arrow[d, "\mu_p"'] & \Gamma(p) \arrow[d, "\xi"] \\ \Gamma(p) \arrow[r, "\xi"] & p \end{tikzcd}$$

This means that ξ must be a fibre map from $\bar{p}$ to p over B such that the following diagrams in Top commute

$$\begin{tikzcd} E \arrow[r, "\iota_p"] \arrow[dr, equal] & \Gamma_p \arrow[d, "\xi"] \\ & E \end{tikzcd} \qquad \begin{tikzcd} \Gamma_p^2 \arrow[r, "\Gamma(\xi)"] \arrow[d, "\mu_p"'] & \Gamma_p \arrow[d, "\xi"] \\ \Gamma_p \arrow[r, "\xi"] & E \end{tikzcd}$$

In other words ξ must satisfy the relations $p\xi(e,\alpha)=\alpha(\ell(\alpha))$ for every $(e,\alpha)\varepsilon\Gamma_p$; $\xi(e,0_{p(e)})=e$ for every $e\varepsilon E$ and $\xi(\xi(e,\alpha),\beta)=\xi(e,\alpha+\beta)$ for every $(e,\alpha)\varepsilon\Gamma_p$ and $\beta\varepsilon MB$ with $\beta(0)=\alpha(\ell(\alpha))$. These are just the requirements for a Moore end-point transitive lifting function for p. By proposition 10 p is a fibration.

BIBLIOGRAPHY

1. Allaud, G., De-looping homotopy equivalences, Arch. Math. 23(1972), 167-169.

2. Arnold, J., Local to global theorems in the theory of Hurewicz fibrations, Trans. Amer. Math. Soc. 164(1972), 179-188.

3. Bauer, F. W. and Dugundji, J., Categorical homotopy and fibrations, Trans. Amer. Math. Soc. 140(1969), 239-256.

4. Booth, P., The exponential law of maps I, Proc. London Math. Soc. (3)20(1970), 179-192.

5. Booth, P., The exponential law of maps II, Math. Z. 121(1971), 311-319.

6. Booth, P., The section problem and the lifting problem, Math. Z. 121(1971), 273-287.

7. Booth, P., A unified treatment of some basic problems in homotopy theory, Bull. Amer. Math. Soc. 79(1973), 331-336.

8. Booth, P. and Brown, R., On the application of fibred mapping spaces to exponential laws for bundles, ex-spaces and other categories of maps, Gen. Top. and Appl. 8(1978), 165-179.

9. Booth, P. and Brown, R., Spaces of partial maps, fibred mapping spaces and the compact open topology, Gen. Top. and Appl. 8(1978), 181-195.

10. Booth, P., Heath, P. and Piccinini, R., Fibre preserving maps and functional spaces, Springer Lecture notes 673(1978).

11. Brown, E. H., Twisted tensor products I, Ann. of Math. (2) 69(1959), 223-246.

12. tom Dieck, T., Partitions of unity in homotopy theory, Comp. Math. 23(1971), 159-167.

13. Dold, A., Partitions of unity in the theory of fibrations, Ann. of Math. 78(1963), 223-255.

14. Dold, A., Local extension properties in topology, Proc. Adv. Study Inst. Alg. Top., Aarhus 1970.

15. Dold, A., and Lashof, R., Principal quasi-fibrations and fibre homotopy equivalence of bundles, Illinois J. Math. 3(1959), 285-305.

16. Dold, A. and Thom, R., Une généralisation de la notion d'espace fibré. Application aux produits symmetrique infinis, C. R. Acad. Sci. Paris 242(1956), 1680-1682.

17. Dold, A. and Thom, R., Quasifaserungen and unendliche symmetrische produkte, Ann. of Math. 67(1958), 239-281.

18. Dugundji, J., Topology, Allyn and Bacon 1966.

19. Dyer, E. and Kahn, D. S., Some spectral sequences associated with fibrations, Trans. Amer. Math. Soc. 145(1969), 397-437.

20. Eilenberg, S. and Moore, J. C., Adjoint functors and triples, Illinois J. Math. 9(1965), 381-398.

21. Godement, R., Topologie algébrique et théories des faisceaux, Hermann 1958.

22. Halmos, P. R., Naive Set Theory, Van Nostrand Reinhold Company, New York, 1960.

23. Heath, P., Groupoid operations and fibre homotopy equivalence I, Math. Z. 130(1973), 207-233.

24. Hilton, P., Catégories non-abéliennes, Les Presses de l'Univ. de Montréal 1964.

25. Huber, P. J., Homotopy theory in general categories, Math. Ann. 144(1961), 361-385.

26. Hurewicz, W., On the concept of fiber spaces, Proc. Nat. Acad. Sci. U.S.A 41(1955), 956-961.

27. James, I., and Thomas, E., Note on the classification of cross-sections, Topology 4(1966), 351-359.

28. Kamps, K. H., Faserungen and Cofaserungen in Kategorien mit Homotopiesystem, Dissertation, Saarbrucken (1968).

29. Kamps, K. H., Über einige formale Eigenschaften von Faserungen and h-Faserungen, Manuscripta math. 3(1970), 237-255.

30. Kleisli, H., Every standard construction is induced by a pair of adjoint functors, Proc. Amer. Math. Soc. 16(1965), 544-546.

31. MacLane S., Categories for the working mathematician, Springer 1971.

32. Manes, E., A triple miscellaney: some aspects of the theory of algebras over a triple, Ph.D. thesis, Wesleyan Univ. 1967.

33. May, P. J., Classifying spaces and fibrations, Amer. Math. Soc. Memoirs 155 (1975).

34. Malraison, P., Fibrations and triple algebras, J. Pure and Applied Algebra 3 (1973), 287-293.

35. C. Morgan, Characterizations of F-fibrations, Proc. Amer. Math. Soc. 88 (1983), 169-172.

36. Ringel, C. N., Faserungen und Homotopie in Kategorien, Math. Ann. 190 (1971), 215-230.

37. Schlesinger, J. W., On transitive translation functions, Proc. Amer. Math. Soc. 9 (1958), 507-510.

38. Seip, U., Categorical fibrations, Canad. Math. Bull. 10 (1967), 5-17.

39. Siebenmann, L., Le fibré tangent, Faculté des Sciences d'Orsay, 1966.

40. Spanier, E., Algebraic topology, McGraw Hill 1966.

41. Strom, A., Note on cofibrations, Math. Scand. 19 (1966), 11-14.

42. Strom, A., Note on cofibrations II, Math. Scand. 22 (1968), 130-142.

43. Tulley, P., On regularity in Hurewicz fiber spaces, Trans. Amer. Math Soc. 116 (1965), 126-134.

44. Varadarajan, K., On fibrations and category, Math. Zeitschr. 88 (1965), 267-273.

45. Weinzweig, A. I., Fibre spaces and fibre homotopy equivalences, Colloq. on Algebraic Topology, Aarhus 1962.

46. Whitehead, G. W., Elements of homotopy theory, Springer 1978.

5.7