THE CONVENIENT CATEGORY OF SEQUENTIAL SPACES

JOY GLENYS TILLOTSON

THE CONVENIENT CATEGORY

OF

SEQUENTIAL SPACES

by

JOY GLENYS TILLOTSON, B.A. ©

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science.

Department of Mathematics, Statistics and Computer Science
Memorial University of Newfoundland

DECEMBER 1977

St. John's Newfoundland

ABSTRACT

It is shown that sequential spaces are a convenient category in the terms of Steenrod's definition and that they have advantages over other such categories. The method of the Thesis is to define adjoint functors between the categories of all topological spaces and of sequential convergences. Sequential spaces are defined in terms of these functors and results proved for sequential convergences are used in proofs for sequential spaces. Initial and final topologies are used to generalize standard constructions and theorems in these categories. The fibred exponential law and the convergent sequence open topology are discussed in terms of sequential spaces.

ACKNOWLEDGEMENTS

I wish to acknowledge the invaluable assistance of my supervisor, Dr. P. Booth. Also to thank the typist, Elaine Boone.

TABLE OF CONTENTS

INTRODUCTION

In this thesis we show that sequential spaces form a convenient category and that, in addition, they have distinct advantages over other such categories.

The original and best known convenient category is that of compactly-generated spaces, also known as Hausdorff k-spaces [see 35 and 27 respectively]. Others include the categories of quasi-topological spaces [33], limit spaces [2, 17, 29], filter-merotopic spaces [26], convergence spaces [31], k-spaces [see 6, 12, and 36, example (ii)]. and the category Grill [1].

Quasi-topological spaces, limit spaces, filter-merotopic spaces, convergence spaces and Grill spaces have the disadvantage that they are not topological spaces and, as Vogt says [36, p. 545], "Many topologists dislike working with things that are not topological spaces".

One problem with the category of compactly generated spaces is that it is not closed under the formation of topological quotients, since the quotient in Top of a Hausdorff space need not be Hausdorff. Another difficulty is that it is not closed under the formation of various types of spaces of partial maps, including fibred mapping spaces [see 8, prop. 5.3]. Clark [12, see also 4 and 26] proposes a category to eliminate the "quotient" difficulty. This is the category of k-spaces, consisting of all topological spaces with the final topology with respect to all incoming maps from compact Hausdorff spaces. The difficulties with this category are largely aesthetic. The use of the class of all compact Hausdorff spaces to determine whether or not a single space is a k-space seems rather unreal even though the procedure is justifiable in terms of axiomatic set theory. Also the appropriate "Hausdorffness" condition i.e. that a k-space X is k-Hausdorff if the diagonal is closed in the k-space $X \times_k X$ might be thought to be relatively clumsy.

Sequential spaces avoid all of these problems and, in particular, have the advantage of requiring only one space in their definition, rather than a class of spaces. Another advantage is that "They are the spaces in which convergent sequences can do all the jobs for which convergent filters (or nets) are usually needed" [37, p.225].

The method of this thesis is analogous to Clark's paper [12, see also 6] and so is more categorical than previous approaches to sequential convergences and sequential spaces. Clark sets up adjoint functors between Top and the category of quasi-topological spaces. He uses these functors to define k-spaces. Similarly we define adjoint functors between Top and the category of sequential convergences and use them in definitions and proofs for sequential spaces. We thereby show the connection between these two categories in addition to obtaining results about sequential spaces. We also use the ideas of initial and final topologies to generalize standard constructions and theorems in these categories.

In chapter one we give preliminary definitions for, and some basic properties of, convenient categories and initial and final topologies. We also discuss the space $\mathbb{N}_\infty$ which is used in the definition of sequential convergences and sequential spaces.

The second chapter discusses the category of sequential convergences, shows its convenience and proves some results of use in chapter three.

The proof that sequential spaces are a convenient category occupies most of the third chapter. Also included in this section is a discussion of Hausdorffness in this category.

In chapter four we prove the fibred exponential law for sequential spaces. The last chapter discusses the convergent sequence-open topology for function spaces and a new topology on product spaces. They are found to be related by a corresponding exponential law. This then leads to a new proof of the exponential law for sequential spaces.

The exposition given is more or less self-contained insofar as it goes beyond results that can be found in textbooks.

CHAPTER I

Preliminaries

In this section we first discuss convenient categories, then we review the notions of initial and final topologies which are central to the approach used in chapters two and three. Finally we discuss the space $\mathbb{N}_\infty$ which is basic to our whole theory.

Convenient categories: Steenrod's paper [35] introduced the term "convenient category" and gave conditions which such a category should satisfy. The conditions are:

> "... first that it be large enough to contain all of the particular spaces arising in practice. Second, it must be closed under standard operations, these are the formation of subspaces, product spaces $X \times Y$, function spaces Y^X, decomposition spaces, unions of expanding sequences of spaces, and compositions of these operations. Third, the category should be small enough so that certain reasonable propositions about the standard operations are true. These state that the order of performing two operations can be reversed. We adopt the following as test propositions
>
> (1) $(Y \times Z)^X = Y^X \times Z^X$
>
> (2) $Z^{Y \times X} = (Z^Y)^X$
>
> (3) A product of decomposition spaces is a decomposition space of the product
>
> (4) A product of unions is a union of products
>
> (5) A decomposition space of a union is a union of decomposition spaces"[1]

The third condition is the one that eliminates the category of all topological spaces as a convenient category. In particular there is no topology on Z^Y making (2) true in general [33, Cor 5.2].

Remark: A Cartesian closed category C is a category equipped with:

(1) a terminal object

(2) a product $\times : C \times C \to C$

(3) a bifunctor $\mathrm{Mor} : C^0 \times C \to C$ such that there is a natural isomorphism

$$C(A \times B,\ C) \to C(A,\ \mathrm{Mor}(B, C))$$

for all A, B, C in C.

It is easily seen that convenient categories satisfy these conditions (the terminal object being a one point space).

Initial and final topologies: Given a set X, a family of spaces $\{X_a\}_{a \in A}$ and a family of functions

$$\{f_a : X \to X_a\}_{a \in A}$$

the initial (or strong) topology on X with respect to the $\{f_a\}_{a \in A}$ is the topology with subbasis:

$$\{f_a^{-1}(U) \mid U \text{ is open in } X_a \text{ for some } a \in A\}$$

This is the coarsest topology on X which makes all the f_a continuous. Some common spaces with initial topologies are subspaces, product spaces and fibred product spaces.

Initial topologies have the following universal property:

If X has the initial topology as above and $g : W \to X$ is a function then g is continuous if and only if $f_a \circ g : W \to X_a$ is continuous for all a in A.

It is easily seen that if the universal property is satisfied by a space X, then the topology on X is uniquely determined and coincides with that described above [see 9, p.153].

There is also a "transitive law" for initial topologies [9, p.154].

Lemma 1.1 Given a set X, a family

$$\{f_a : X \to X_a\}a \in A$$

of functions out of X and for each a in A a family

$$\{g_{ab} : X_a \to X_{ab}\}b \in B_a$$

of functions out of X_a. If X_a has the initial topology with respect to $\{g_{ab}\}b \in B_a$ then the initial topologies on X with respect to $\{f_a\}$ and $\{g_{ab} \circ f_a\}$ coincide.

Proof: Let $k : Y \to X$ be a function where Y is a topological space. Since X_a has the initial topology, the functions $g_{ab} \circ f_a \circ k$ are continuous if and only if $f_a \circ k$ is continuous. The result follows from the universal property.

Given a set X, a family $\{X_a\}a \in A$ of topological spaces and a family

$$\{f_a : X_a \to X\}a \in A$$

of functions then X has the final (or weak) topology with respect to the $\{f_a\}a \in A$ if either:

(a) if $U \subset X$, then U is open if and only if $f_a^{-1}(U)$ is open in X_a for each a in A

or (b) the same as (a) but with 'open' replaced by 'closed'.

This is the finest topology that makes the f_a continuous. Some standard examples of spaces with final topologies are identification spaces, topological sums, adjunction spaces and unions of expanding sequences of subspaces.

Final topologies also have a universal property:

Given a space X with the final topology with respect to the family of functions $\{f_a : X_a \to X\}a \in A$, then a function g from X to any topological space Y is continuous if and only if $g \circ f_a : X_a \to Y$ is continuous for each a in A.

As is the case for initial topologies, if a space satisfies the universal property then its topology is uniquely determined and coincides with the final topology.

We will need two lemmas about the combination of final topologies.

Notation: Let $M(X,Y)$ denote the set of all continuous functions $X \to Y$ and $M_c(X,Y)$ denote the same set with the compact-open topology.

Lemma 1.2 Let X be a set with the final topology with respect to a family of functions $\{f_a : X_a \to X\}_{a \in A}$ where the $\{X_a\}_{a \in A}$ are topological spaces. If X satisfies the condition that for each $x \in X$ there exists an $x' \in X_a$ such that $f_a(x') = x$ for some choice of $a \in A$ and if C is a locally compact Hausdorff space then $X \times C$ has the final topology with respect to the functions $\{f_a \times 1 : X_a \times C \to X \times C\}_{a \in A}$.

Proof: We show that $X \times C$ has the appropriate universal property. Let Y be any space and g a function $X \times C \to Y$. We have the following commutative diagram:

$$\begin{tikzcd}
X_a \times C \arrow[d, "f_a \times 1"'] \arrow[dr, "h_a"] & \\
X \times C \arrow[r, "g"] & Y
\end{tikzcd}$$

If g is continuous then h_a is continuous since it is the composite. If h_a is continuous we apply the exponential law, giving that the associated function $h_a' : X_a \to M_c(C,Y)$ is continuous. Clearly $g' : X \to \{\text{set of functions } C \to Y\}$ is well-defined. Also $g'(x) = h_a'(x')$

where $f_a(x') = x$ so $g'(x)$ is continuous. So we have the following commutative diagram:

$$\begin{tikzcd} X_a \arrow[d, "f_a"'] \arrow[dr, "h_a'"] & \\ X \arrow[r, "g'"'] & M_c(C,Y) \end{tikzcd}$$

Now the continuity of f_a and the universal property of final topologies imply the continuity of g' and hence the continuity of g. Therefore $X \times C$ has the required universal property and hence is a final topology in the appropriate sense.

Proposition 1.3 (Transitive Law [see 9, p.96])

Given a set X, a family

$$\{g_b : X_b \to Z\} b \in B$$

of functions into X and for each b in B a family

$$\{f_{ab} : X_{ab} \to X_b\} a \in A_b$$

of functions into X_b. If X_a has the final topology with respect to $\{f_{ab}\} a \in A_b$ then the final topologies on X with respect to $\{g_b\}$ and $\{g_b \circ f_{ab}\}$ coincide.

Proof: Suppose $h : X \to Y$ is a function. Then $h \circ g_b \circ f_{ab} : X_{ab} \to Y$ is continuous for all $a \in A$, $b \in B$ if and only if $h \circ g_b$ is continuous since X_b has the final topology with respect to the $\{f_{ab}\}$. The result follows from the universal property.

Corollary 1.4 Let $\{X_a\}a \in A$ be a family of disjoint topological spaces, each of which is equipped with an associated equivalence relation R_a. If X is the topological sum $\bigcup_{a \in A} X_a$ and R is the equivalence relation on X generated by $\{R_a\}a \in A$, then

$$X/R = \bigcup_{a \in A} X_a/R_a$$

Proof: The underlying sets are clearly the same for X/R and $\bigcup_{a \in A} X_a/R_a$. The proposition shows that both sets have the final topology with respect to the composite functions from the spaces X_a. The composite functions are the same so the result follows.

Remark: This result shows that Steenrod's fifth test proposition is true in the category of all topological spaces. This proposition will also be true in any topological style category that is closed under the formation of quotients and sums.

The space $\mathbb{N}_\infty$: The Alexandrov one-point compactification of the natural numbers, denoted by $\mathbb{N}_\infty$ is used to generate a convergence structure, analogous to Spanier's quasi-topology [33].

The topology on $\mathbb{N}_\infty$ can be simply described as follows:

Given $U \subseteq \mathbb{N}_\infty$, U is open if either:

(i) $\infty \notin U$

or (ii) $\infty \in U$ and $\mathbb{N}_\infty - U$ is finite.

$\mathbb{N}_\infty$ can be alternatively defined as the subspace $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\}$ of the unit interval with the elements renamed as $\{1, 2, 3, \ldots, \infty\}$. It is clear that in this last space U is open if it satisfies (i) or the following condition:

(ii)′ there exists an $n_0 \in \mathbb{N}$ such that
$U = \{n \in \mathbb{N}_\infty \mid n \geq n_0\} \cup \{\text{a finite subset of } \mathbb{N}\}$.

The equivalence of the topology defined by (i) and (ii) and the alternative definition using (i) and (ii)′ follows either from the uniqueness of the one-point compactification [13, p.246] or from the next result:

Lemma 1.5 $U \subseteq \mathbb{N}_\infty$ satisfies (ii) if and only if it satisfies (ii)′.

Proof: First assume U satisfies (ii).
Let $n_0 = \max(\mathbb{N}_\infty - U) + 1$. Then $\{n \in \mathbb{N}_\infty \mid n \geq n_0\} \subseteq U$. The remaining elements of U, if any, must be contained in $\{n \in \mathbb{N} \mid n < n_0\}$, which is a finite set, so U satisfies (ii)′.

Now assume U satisfies (ii)′. Then $\infty \in U$ and $(\mathbb{N}_\infty - U) \subseteq \{n \in \mathbb{N} \mid n \leq n_0\}$, which is a finite set. Therefore $(\mathbb{N}_\infty - U)$ is finite and (ii) is satisfied.

Note: $\mathbb{N}_\infty$ can also be viewed as the ordinal space $\omega + 1$ where ω is the first infinite ordinal [14, p.66, example 5]. It is easily seen that the topology of $\omega + \overline{1}$ coincides with that given by our second description of the $\mathbb{N}_\infty$ topology.

The following result describes a simple connection between convergent sequences and the space $\mathbb{N}_\infty$. This fact will be basic to our method of studying sequential spaces.

Lemma 1.6. For a topological space X, the sequence $\{x_n\}$ has limit $x \in X$ if and only if the corresponding function $f : \mathbb{N}_\infty \to X$ given by $f(n) = x_n$, $f(\infty) = x$, $n \in \mathbb{N}_\infty$ is continuous.

Proof: This follows immediately from the topology of $\mathbb{N}_\infty$.

Several other useful properties are noted here:

1. $\mathbb{N}_\infty$ is regular as it is essentially a subspace of the real line.
2. A subset of $\mathbb{N}_\infty$ is open unless it contains ∞ and its complement is infinite. A subset of $\mathbb{N}_\infty$ is closed unless it does not contain ∞ and its complement is infinite.
3. $\mathbb{N}_\infty \times \mathbb{N}_\infty$ is a first countable space since it can be regarded as a subspace of $\mathbb{R} \times \mathbb{R}$ which is metric and hence first countable. This will be important to our argument because it implies that $\mathbb{N}_\infty \times \mathbb{N}_\infty$ is a sequential space.

CHAPTER II

Sequential Convergences

In this section we discuss sequential convergences. The study of convergences was begun by Fréchet [20] and taken up again by Dudley [13]. We rewrite his definitions and proofs in a manner analogous to Spanier's quasi-topological spaces. This will be useful in our study of sequential spaces. We define a convergence structure for a set X then we can define the category of sequential convergences and demonstrate some of its basic properties.

A <u>convergence structure</u> $S(\mathbb{N}_\infty, X)$ on a set X is a set of functions $\mathbb{N}_\infty \to X$ satisfying the conditions:

(i) $S(\mathbb{N}_\infty, X)$ contains all constant functions $\mathbb{N}_\infty \to X$.

(ii) if $g : \mathbb{N}_\infty \to \mathbb{N}_\infty$ is an injective map and $f \in S(\mathbb{N}_\infty, X)$ then $f \circ g \in S(\mathbb{N}_\infty, X)$.

A <u>sequential convergence</u> (abbreviated hereafter to <u>convergence</u>) is a pair $(X, S(\mathbb{N}_\infty, X))$ where X is an arbitrary set and $S(\mathbb{N}_\infty, X)$ is a convergence structure on X.

The analagous definition in Dudley's paper is the following:

If S is any set, a sequential convergence C on S is a relation between sequences $\{s_n\}_{n=1}^\infty$ of members of S and members s of S denoted $s_n \to_c s$ such that

i) if $s_n = s$ for all n, $s_n \to_c s$

ii) if $s_n \to_c s$ and $\{r_m\}$ is any subsequence $\{s_{n_m}\}$ of $\{s_n\}$, then $r_m \to_c s$.

Our definition is slightly more restrictive than that of Dudley, as can be seen from Lemma 1.6 and the following example: the sequential convergence on the real line consisting of all constant sequences together with all subsequences of $s(n) = \frac{1}{n}$ satisfies Dudley's axioms but not ours.

Example: If X is a topological space and $S(\mathbb{N}_\infty, X)$ is defined as the set of continuous functions $\mathbb{N}_\infty \to X$, then $(X, S(\mathbb{N}_\infty, X))$ is a convergence, called the associated convergence.

If X and Y are convergences then a function $g : X \to Y$ is called sequentially continuous if $g \circ f : \mathbb{N}_\infty \to Y$ is in $S(\mathbb{N}_\infty, Y)$ for all $f \in S(\mathbb{N}_\infty, X)$. It follows easily that a composite of sequentially continuous functions is sequentially continuous.

Remark: If X and Y are topological spaces, it is usual to say that $f : X \to Y$ is sequentially continuous at a point $x \in X$ if and only if for every sequence $\{x_n\}$ in X converging to x, the sequence $\{f(x_n)\}$ in Y converges to $f(x)$. It follows by lemma 1.6 that $f : X \to Y$ is sequentially continuous in the usual sense if and only if f is sequentially continuous between the convergences associated to X and Y.

The category of convergences and sequentially continuous functions will be denoted Con. The isomorphisms in this category are sequential homeomorphisms, i.e. sequentially continuous bijections with sequentially continuous inverses.

Now let us define initial and final convergence structures, analogous to initial and final spaces, and prove their universal properties. Suppose we are given a set X, a family $\{X_a\}a \in A$ of convergences and for $a \in A$ a function

$$f_a : X \to X_a$$

$S(\mathbb{N}_\infty, X)$ is the <u>initial convergence structure</u> on X with respect to the $\{f_a\}$ if $f \in S(\mathbb{N}_\infty, X)$ if and only if $f_a \circ f \in S(\mathbb{N}_\infty, X_a)$ for all $a \in A$. It is easily checked that $S(\mathbb{N}_\infty, X)$ satisfies the conditions for a convergence structure. The pair $(X, S(\mathbb{N}_\infty, X))$ is then called an initial convergence. Notice that this convergence structure on X makes $f_a : X \to X_a$ sequentially continuous for all $a \in A$.

<u>Proposition 2.1</u> (Universal property of initial convergence). For an initial convergence X and any convergence Y, a function $k : Y \to X$ is sequentially continuous if and only if the composite $f_a \circ k : Y \to X_a$ is sequentially continuous for all $a \in A$.

<u>Proof</u>: If k is sequentially continuous then $f_a \circ k$ is sequentially continuous by composition.

If $f_a \circ k$ is sequentially continuous then $f_a \circ k \circ h \in S(\mathbb{N}_\infty, X_a)$ for all $h \in S(\mathbb{N}_\infty, Y)$. So $k \circ h \in S(\mathbb{N}_\infty, X)$ for all $h \in S(\mathbb{N}_\infty, Y)$ and hence k is sequentially continuous.

<u>Remark</u>: Initial convergences clearly satisfy a transitive law similar to that described for topological spaces in lemma 1.1.

<u>Example 1</u>: If A is a subset of a convergence X, then A can

be given the initial convergence structure with respect to the inclusion $i : A \to X$. A is then called a <u>subconvergence</u> of X. Notice that the inclusion is sequentially continuous and that A has a universal property analogous to that of subspaces, i.e. for any convergence Y and the following commutative diagram with functions f and g:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "g"'] & A \arrow[d, hook, "i"] \\ & X \end{tikzcd}$$

f is sequentially continuous if and only if g is sequentially continuous.

Note: This subconvergence definition is closely analagous to Dudley's definition [13, p.488]).

<u>Example 2</u>: Let X and Y be convergences. $X \times Y$ can be given the initial convergence structure with respect to the projections $pr_1 : X \times Y \to X$ and $pr_2 : X \times Y \to Y$. The product of more than two convergences is defined in a similar fashion. (This corresponds to Dudley's definition [12, p.487]). This convergence structure gives an associative product and also makes the projections, pr_i, sequentially continuous. The universal property, analogous to that of product spaces is:

for any convergence W and pair of functions $f_1 : W \to X$ and $f_2 : W \to Y$, (f_1, f_2) is sequentially continuous if and only if f_1 and f_2 are sequentially continuous.

Let X be a set, $\{X_a\}a \in A$ a family of convergences and $\{f_a : X_a \to X\}a \in A$ a family of functions with the property that, for each $x \in X$, there exists an $a \in A$ and an $x' \in X_a$ such that $f_a(x') = x$. $S(N_\infty, X)$ is the <u>final convergence structure</u> on X with respect to the $\{f_a\}$ when $f \in S(N_\infty, X)$ if and only if there exists an f_a and a $g(\in S(N_\infty, X_a))$ such that $f_a \circ g = f$. It is straightforward to verify that $S(N_\infty, X)$ satisfies the condition (i) for a convergence structure so we just check condition (ii). Let $h : N_\infty \to N_\infty$ be an injective map and $f \in S(N_\infty, X)$. Then there exists a g such that $f_a \circ g = f$ for some a, hence $f \circ h = f_a \circ g \circ h \in S(N_\infty, X)$ and the result follows. The pair $(X, S(N_\infty, X))$ is called a final convergence. Notice that this definition of the convergence structure makes the $\{f_a\}$ sequentially continuous.

<u>Proposition 2.2</u> (Universal property of final convergences)

Let X be a final convergence and Y be any convergence. A function $k : X \to Y$ is sequentially continuous if and only if the composite $k \circ f_a$ is sequentially continuous for all $a \in A$.

<u>Proof</u>: If k is sequentially continuous then $k \circ f_a$ is sequentially continuous by composition.

If $k \circ f_a$ is sequentially continuous for all f_a then $k \circ f_a \circ h \in S(\mathbb{N}_\infty, Y)$ for all $h \in S(\mathbb{N}_\infty, X_a)$. Now if $f \in S(\mathbb{N}_\infty, X)$ then $f = f_a \circ h$ for some choice of f_a and h. So $k \circ f = k \circ f_a \circ h \in S(\mathbb{N}_\infty, Y)$ for all $f \in S(\mathbb{N}_\infty, Y)$ and hence k is sequentially continuous.

Remark: Final convergences clearly satisfy a transitive law similar to that for final topologies (lemma 1.3).

Example 1: Let X be a convergence, Y a set and $p : X \to Y$ a surjective function. Then if we give Y the final convergence structure with respect to p, we call Y an identification convergence and p an identification function. (Note that when p is a quotient map this definition is equivalent to Dudley's quotient [13, p.488]). As in the general case, p is sequentially continuous and Y has the following universal property:

for a convergence W and functions $f : X \to W$ and $g : Y \to W$ such that $g \circ p = f$, f is sequentially continuous if and only if g is sequentially continuous.

Example 2: Let X and Y be disjoint convergences. We can give $X \cup Y$ the final convergence structure with respect to the inclusions of X and Y in $X \cup Y$. The sum of more than two convergences is similarly defined. The inclusions are sequentially continuous and the sum has the expected universal property:

for a convergence W and functions $f : X \to W$ and $g : Y \to W$, $f \cup g : X \cup Y \to W$ is sequentially continuous if and only if f and g are sequentially continuous.

One of the essential conditions for a convenient category is the existence of an exponential law. We now define function convergences and prove the appropriate exponential law.

If X and Y are convergences, let $F(X,Y)$ be the set of sequentially continuous functions $X \to Y$. To define a convergence structure on $F(X,Y)$ we first define the evaluation function:

$$e : F(X,Y) \times X \to Y$$

by $e(f,x) = f(x)$ for $f \in F(X,Y)$, $x \in X$. Then $f : \mathbf{N}_\infty \to F(X,Y)$ is in $S(\mathbf{N}_\infty, F(X,Y))$ if and only if $e \circ (f \circ k, g) \in S(\mathbf{N}_\infty, Y)$ for all $g \in S(\mathbf{N}_\infty, X)$ and for all injective maps $k : \mathbf{N}_\infty \to \mathbf{N}_\infty$.

Lemma 2.3 $S(\mathbf{N}_\infty, F(X,Y))$ is a well-defined convergence structure.

Proof: First we check condition (i). Suppose $f : \mathbf{N}_\infty \to F(X,Y)$ is a constant function, say $f(n) = t$ for all $n \in \mathbf{N}$. Then $e \circ (f \circ k, g) = t \circ g$ which is in $S(\mathbf{N}_\infty, Y)$ since t is sequentially continuous.

To check condition (ii), assume $f \in S(\mathbf{N}_\infty, F(X,Y))$. We have to show $e \circ (f \circ h \circ k, g) \in S(\mathbf{N}_\infty, Y)$ for an injective map $h : \mathbf{N}_\infty \to \mathbf{N}_\infty$. But $h \circ k$ is just an injective map $k' : \mathbf{N}_\infty \to \mathbf{N}_\infty$ and

$e \circ (f \circ k', g) \in S(\mathbb{N}_\infty, Y)$ for all injective maps $k' : \mathbb{N}_\infty \to \mathbb{N}_\infty$ by definition so $f \circ h \in S(\mathbb{N}_\infty, F(X,Y))$.

Lemma 2.4 The evaluation map, e, is sequentially continuous.

Proof: We have to show $e \circ h \in S(\mathbb{N}_\infty, Y)$ for all $h \in S(\mathbb{N}_\infty, F(X,Y) \times X)$. Since $h \in S(\mathbb{N}_\infty, F(X,Y) \times X)$ $pr_1 \circ h \in S(\mathbb{N}_\infty, F(X,Y))$ and $pr_2 \circ h \in S(\mathbb{N}_\infty, X)$. Since $pr_1 \circ h \in S(\mathbb{N}_\infty, F(X,Y))$ then $e \circ (pr_1 \circ h \circ k, g) \in S(\mathbb{N}_\infty, Y)$ for all $g \in S(\mathbb{N}_\infty, X)$ and all injective maps $k : \mathbb{N}_\infty \to \mathbb{N}_\infty$. In particular, then $e \circ (pr_1 \circ h, pr_2 \circ h) \in S(\mathbb{N}_\infty, Y)$. But $e \circ (pr_1 \circ h, pr_2 \circ h) = e \circ h$ so e is sequentially continuous.

Theorem 2.5 (Exponential Law) Let X, Y, Z be convergences. There is a sequential homeomorphism

$$\theta : F(Z \times X, Y) \to F(Z, F(X,Y))$$

which assigns to each sequentially continuous map $h : Z \times X \to Y$, the map $h' : Z \to F(X,Y)$ where $h(z,x) = h'(z)(x)$.

Proof: First we show that h' is sequentially continuous if and only if h is. Note that h can be written as $e \circ (h' \times 1)$. Now if h' is sequentially continuous then it follows by lemma 2.4 that $h = e \circ (h' \times 1)$ is sequentially continuous.

Now assume h is sequentially continuous. We have to check that $h'(z) \in F(X,Y)$. The composite

$$X \longrightarrow \{z\} \times X \xrightarrow{h|\{z\} \times X} Y$$

is sequentially continuous and is $h'(z)$ so $h'(z) \in F(X, Y)$. To show that h' is sequentially continuous, we must show that if $a \in S(\mathbb{N}_\infty, Z)$, then $h' \circ a \in S(\mathbb{N}_\infty, F(X,Y))$. By definition of $S(\mathbb{N}_\infty, F(X,Y))$, this will be so if for $g \in S(\mathbb{N}_\infty, X)$ and injective maps $k : \mathbb{N}_\infty \to \mathbb{N}_\infty$ $e \circ (h' \circ a \circ k, g) \in S(\mathbb{N}_\infty, Y)$. By definition of h', $e \circ (h' \circ a \circ k, g) = h \circ (a \circ k, g)$ so h' is sequentially continuous.

So we have shown that the relation $F(Z \times X, Y) \to F(Z, F(X,Y))$ given by $h \to h'$ is a bijection between the two sets. Call this function θ. We have to show that θ and θ^{-1} are sequentially continuous.

Let A be any convergence. It follows from the above argument that a function $f : A \to F(Z \times X, Y)$ is sequentially continuous if and only if the associated $f' : A \times (Z \times X) \to Y$ is sequentially continuous. By the associativity of the product and the above bijection, it follows that f' is sequentially continuous if and only if the corresponding $f'' : A \times Z \to F(X,Y)$ is sequentially continuous. Using the bijection once again, we find that the sequential continuity of f'' is equivalent to that of $f''' : A \to F(Z,F(X,Y))$.

Taking $A = F(Z \times X, Y)$ and f to be the identity map, it is routine to verify that the associated $f''' = \theta$ so θ is sequentially continuous.

Similarly let $A = F(Z \times X, Y)$ and f''' be the identity map. Then the sequential continuity of $f^{\sim}$ implies that of $f = \theta^{-1}$.

To relate convergences to ordinary and sequential topological spaces we need to define a functor $\underline{c}$ from $\underline{\text{Top}}$, the ordinary category of topological spaces, to $\underline{\text{Con}}$ and also a functor $\underline{t} : \underline{\text{Con}} \to \underline{\text{Top}}$.

The functor $\underline{c}$ assigns to a topological space Y the convergence with the underlying set Y and the associated convergence structure

$$S(\mathbb{N}_\infty, X) = \{f : \mathbb{N}_\infty \to Y \mid f \text{ is continuous}\}$$

as described previously. If $g : Y_1 \to Y_2$ is continuous, $\underline{c}g : \underline{c}Y_1 \to \underline{c}Y_2$ is sequentially continuous.

The functor $\underline{t} : \underline{\text{Con}} \to \underline{\text{Top}}$ assigns to a convergence X, the topological space with the same underlying set and the final topology with respect to all functions in $S(\mathbb{N}_\infty, X)$. If $g : X_1 \to X_2$ is sequentially continuous then $\underline{t}g : \underline{t}X_1 \to \underline{t}X_2$ is continuous. An equivalent description of this topology is given by Dudley which clarifies the fact that it is defined by convergent sequences. Rewritten in our notation, Dudley's description says that U is open in $\underline{t}X$ if whenever $x = f(\infty) \in U$ for some $f \in S(\mathbb{N}_\infty, X)$ then $f(n) \in U$ for n sufficiently large. The following proposition proves the equivalence of the two descriptions.

Proposition 2.6 The following conditions are equivalent on $U \subseteq X$:

(1) If $f(\infty) \in U$, there exists n_0 such that $f(n) \in U$ for all $n \geq n_0$

(2) $f^{-1}(U)$ is an open subset of $\mathbb{N}_\infty$ for all $f \in S(\mathbb{N}_\infty, X)$.

Proof: (1) $\Rightarrow$ (2) If $f(\infty) \notin U$, then $\infty \notin f^{-1}(U)$ so $f^{-1}(U)$ is open. If $f(\infty) \in U$, (1) implies that there exists an n_0 such that $n \in f^{-1}(U)$ for $n \geq n_0$ so by the definition of the topology of $\mathbb{N}_\infty$, $f^{-1}(U)$ is open.

(2) $\Rightarrow$ (1) If $f(\infty) \in U$, then $\infty \in f^{-1}(U)$ so by definition of the topology of $\mathbb{N}_\infty$, there exists an $n_0 \in \mathbb{N}$ such that

$$f^{-1}(U) = \{n \in \mathbb{N}_\infty \mid n \geq n_0\} \cup \{\text{a finite set}\}$$

so (1) follows immediately.

<u>Proposition 2.7</u> The functor $\underline{c} : \underline{Top} \to \underline{Con}$ is right adjoint to the functor $\underline{t} : \underline{Con} \to \underline{Top}$ i.e. the identity on underlying sets determines a natural bijection between $M(\underline{t}X, Y)$ and the set $F(X, \underline{c}Y)$, where X is a convergence, Y a topological space, and $M(\underline{t}X, Y)$ the set of all continuous functions $\underline{t}X \to Y$.

<u>Proof</u>: Let $f : X \to Y$ be a function on the underlying sets. The statements (1) $f : \underline{t}X \to Y$ is continuous and (2) $f : X \to \underline{c}Y$ is sequentially continuous are both equivalent to

(3) $g \in S(\mathbb{N}_\infty, X)$ implies that $f \circ g : \mathbb{N}_\infty \to Y$ is continuous.

(1) is equivalent to (3) because $\underline{t}X$ has the final topology with respect to all $g \in S(\mathbb{N}_\infty, X)$. (2) is equivalent to (3) because of the definition of sequential continuity and because a function $h \in S(\mathbb{N}_\infty, \underline{c}Y)$ if and only if $h : \mathbb{N}_\infty \to Y$ is continuous.

Therefore (1) is equivalent to (2) and the proof is complete.

Corollary 2.8 $\underline{c}Y = \underline{ctc}Y$ and $\underline{t}X = \underline{tct}X$ for Y a topological space and X a convergence.

Proof: Certainly $id : \underline{tc}Y \to \underline{tc}Y$ is continuous. Then by the adjointness property $id : \underline{c}Y \to \underline{ctc}Y$ is sequentially continuous. Also $id : \underline{c}Y \to \underline{c}Y$ is sequentially continuous so by adjointness $id : \underline{tc}Y \to Y$ is continuous and since $\underline{c}$ is a functor, $id : \underline{ctc}Y \to \underline{c}Y$ is sequentially continuous. Hence $\underline{c}Y = \underline{ctc}Y$.

Similarly $\underline{t}X = \underline{tct}X$.

Remark: A standard categorical style argument tells us that since $\underline{c}$ has a left adjoint it preserves initial topologies. For example, if X and Y are topological spaces $\underline{c}(X \times Y) = \underline{c}X \times \underline{c}Y$. Also since $\underline{t}$ has a right adjoint it preserves final topologies. For example if X is a convergence and R is an equivalence relation on its underlying set, then $\underline{t}(X/R) = (\underline{t}X)/R$.

CHAPTER III

Sequential Spaces

In this chapter we discuss the convenient category of sequential spaces. We show that it satisfies Steenrod's definition of convenience and demonstrate how it is related to CW-complexes and k-spaces. We close with a brief discussion of Hausdorffness in this category.

Sequential spaces were introduced by G. Birkhoff [3] and studied by S. P. Franklin [18, 19] and T. K. Boehme [4] (who calls them s-spaces). Our approach is different from the above mentioned papers in two ways. One is that we use the results we have proved on convergences to obtain the similar results for sequential spaces. This method was suggested by analogy with Clark's paper on k-spaces [12]. The other difference is that we use initial and final topologies to give a more comprehensive coverage of the constructions in the category.

Recalling the functors t and c from the previous section, we say that a topological space X is a <u>sequential space</u> (in future referred to as an s-space) if $X = tcX$.

The definition can also be stated using a functor $s : \underline{Top} \to \underline{Top}$. The functor s assigns to a topological space X, the space with the same underlying set and the final topology with respect to all continuous functions $N_\infty \to X$. It assigns to each continuous function $X \to Y$ the continuous function $sX \to sY$ with the same action on the underlying sets. It is clear that s and tc are the same functor, so we use whichever is more convenient in proofs. We denote the full subcategory of s-spaces in Top by Seq.

If a space X has the initial topology with respect to a family of functions $\{f_a : X \to X_a\}a \in A$ then X can be made into an initial s-space by retopologizing it as $\underline{s}X$. The functions $\{f_a\}$ are then continuous and $\underline{s}X$ has the expected universal property for s-spaces. To check this we need a preliminary lemma.

Lemma 3.1 (1) $\underline{s}X$ is an s-space.

(2) The identity $id : \underline{s}X \to X$ is continuous.

(3) If Y is an s-space and X is any topological space, a function $f : Y \to X$ is continuous if and only if $f : Y \to \underline{s}X$ is continuous.

Proof: (1) By corollary 2.8 we know that $\underline{t}(\underline{ctc}X) = \underline{t}(\underline{c}X)$ and the result follows.

(2) See proof of corollary 2.8.

(3) If $f : Y \to X$ is continuous then $f : \underline{s}Y \to \underline{s}X$ is continuous so $f : Y \to \underline{s}X$ is continuous since Y is an s-space. Given $f : Y \to \underline{s}X$ is continuous, then using part (2) $f : Y \to X$ is continuous since it is the composite $id \circ f$.

Proposition 3.2 (Universal property of initial s-spaces): If Y and $\{X_a\}a \in A$ are s-spaces and X has the standard initial topology with respect to $\{f_a : X \to X_a\}a \in A$ then $f : Y \to \underline{s}X$ is continuous if and only if $f_a \circ f$ is continuous for all $a \in A$.

Proof: The conditions that $f : Y \to sX$ and $f : Y \to X$ are continuous are equivalent by lemma 3.1 and the result follows from the universal property of initial topologies.

Remark: Initial $\underline{s}$-spaces satisfy a transitive law analogous to that for initial spaces (see lemma 1.1). The proofs are the same.

Example 1: If A is a subset of an $\underline{s}$-space X then the usual subspace A can be made into an $\underline{s}$-subspace by retopologizing it as $\underline{s}A$. $\underline{s}A$ is an $\underline{s}$-space (by lemma 3.1) and $\underline{s}A \to X$ is continuous.

Example 2: The product of two $\underline{s}$-spaces X_1, X_2 is made into an $\underline{s}$-space, denoted by $X_1 \times_s X_2$, by applying the function $\underline{s}$ to the usual product space. Note that this topology makes the projections continuous and that the $\underline{s}$-product has the usual universal property of products. Hence the $\underline{s}$-product is the product of $\underline{s}$-spaces in the categorical sense.

Example 3: If X, Y, B are $\underline{s}$-spaces, and there are maps $q : X \to B$ and $r : Y \to B$ then the set $X \sqcap Y = \{(x,y) \in X \times_s Y \mid q(x) = r(y)\}$ can be "s-topologized" as an $\underline{s}$-subspace of $X \times_s Y$. This space, denoted $X \sqcap_s Y$, will be called the $\underline{s}$-fibred-product space of X and Y over B. It is easily verified that $X \sqcap_s Y$, with the corresponding projections to X and Y, is the pullback of q and r in the category of $\underline{s}$-spaces.

The following propositions state properties of the product, some of which we need in later proofs.

Proposition 3.3 If X, Y and Z are $\underline{s}$-spaces and $\{*\}$ is a singleton space (and hence an $\underline{s}$-space) then the following functions are natural homeomorphisms.

(a) $X \times_s Y \to Y \times_s X$, $(x,y) \to (y,x)$, $x \in X$, $y \in Y$

(b) $X \to X \times_s \{*\}$, $x \to (x,*)$, and $X \to \{*\} \times_s X$, $x \to (*,x)$, $x \in X$.

(c) $(X \times_s Y) \times_s Z \to X \times_s (Y \times_s Z)$, $((x,y),z) \to (x,(y,z))$, $x \in X$, $y \in Y$, $z \in Z$.

Proof: These results follow easily from the fact that the $\underline{s}$-product is the product for the category of $\underline{s}$-spaces.

Proposition 3.4 If X is a convergence then $\underline{t}(X \times \underline{c}\,\mathbb{N}_\infty) = \underline{t}X \times_s \mathbb{N}_\infty = \underline{t}X \times \mathbb{N}_\infty$.

Proof: First we show that the identity maps $\underline{t}(X \times \underline{c}\,\mathbb{N}_\infty) \to \underline{t}X \times_s \mathbb{N}_\infty \to \underline{t}X \times \mathbb{N}_\infty$ are continuous. We know $X \times \underline{c}\,\mathbb{N}_\infty \to \underline{c}\underline{t}X \times \underline{c}\,\mathbb{N}_\infty$ is continuous so $\underline{t}(X \times \underline{c}\,\mathbb{N}_\infty) \to \underline{t}(\underline{c}\underline{t}X \times \underline{c}\,\mathbb{N}_\infty) = \underline{t}\underline{c}(\underline{t}X \times \mathbb{N}_\infty) = \underline{t}X \times_s \mathbb{N}_\infty$ is continuous. $\underline{t}X \times_s \mathbb{N}_\infty \to \underline{t}X \times \mathbb{N}_\infty$ is continuous by the universal property of products.

Then we show that the identity $\underline{t}X \times \mathbb{N}_\infty \to \underline{t}(X \times \underline{c}\,\mathbb{N}_\infty)$ is continuous. Suppose A is closed in $\underline{t}(X \times \underline{c}\,\mathbb{N}_\infty)$ and that (x,n) belongs to the complement of A. The function $f : \mathbb{N}_\infty \to X \times \underline{c}\,\mathbb{N}_\infty$ given by $f(n) = (x,n)$ belongs to $S(\mathbb{N}_\infty, X \times \underline{c}\,\mathbb{N}_\infty)$ since $pr_1 \circ f$ is constant, hence in $S(\mathbb{N}_\infty, X)$ and $pr_2 \circ f = 1\mathbb{N}_\infty$, hence is in $S(\mathbb{N}_\infty, \underline{c}\,\mathbb{N}_\infty)$. Therefore $f^{-1}(A)$ is closed in $\mathbb{N}_\infty$ by definition of $\underline{t}$. Since $(x,n) \notin A$, $n \notin f^{-1}(A)$ and therefore, since $\mathbb{N}_\infty$ is regular, n has an open neighbourhood U

such that $\overline{U} \cap f^{-1}(A) = \phi$. So $f(\overline{U} \cap f^{-1}(A)) = (x \times \overline{U}) \cap A = \phi$. Let B denote the projection of $(X \times \overline{U}) \cap A$ onto $\underline{t}X$. We shall see that B is closed in $\underline{t}X$.

Let $g \in S(\mathbb{N}_\infty, X)$. Then $(g,1) \in S(\mathbb{N}_\infty, X \times \underline{c}\mathbb{N}_\infty)$ and we have the commutative diagram:

$$\begin{tikzcd} \mathbb{N}_\infty \arrow[r, "(g,1)"] \arrow[dr, "g"'] & \underline{t}(X \times \underline{c}\,\mathbb{N}_\infty) \arrow[d, "pr_1"] \\ & \underline{t}X \end{tikzcd}$$

So $g^{-1}(B) = (g,1)^{-1}((X \times \overline{U}) \cap A)$ is closed in $\mathbb{N}_\infty$. So B is closed in $\underline{t}X$. Since $x \notin B$, we have $(x,n) \in (X-B) \times U \subset (\underline{t}X \times \mathbb{N}_\infty) - A$. Therefore $(\underline{t}X \times \mathbb{N}_\infty) - A$ is open in $\underline{t}X \times \mathbb{N}_\infty$ and so A is closed there.

Corollary 3.5 If X is an $\underline{s}$-space then $X \times_s \mathbb{N}_\infty = X \times \mathbb{N}_\infty$.

Proof: $X \times_s \mathbb{N}_\infty = \underline{tc}(X \times \mathbb{N}_\infty) = \underline{t}(\underline{c}X \times \underline{c}\,\mathbb{N}_\infty) = \underline{tc}X \times \mathbb{N}_\infty = X \times \mathbb{N}_\infty$.

Now we consider final $\underline{s}$-spaces, examples of which cover most of the remaining standard operations.

Proposition 3.6 If X has the final topology with respect to a family of functions $\{f_a : X_a \to X\}_{a \in A}$ where the $\{X_a\}_{a \in A}$ are $\underline{s}$-spaces then X is an $\underline{s}$-space.

Proof: We have to show that the identity $i : X \to \underline{s}X$ is continuous (the other direction was proved in lemma 3.1). The functions $f_a : X_a \to X$ are continuous, hence so are the functions $i \circ f_a : X_a \to \underline{s}X$ (lemma 3.1). The result then follows by the universal property of final topologies.

Clearly this X has the expected universal property for final topologies in the category of $\underline{s}$-spaces. Furthermore the transitive law for final topologies (lemma 1.3) also holds for final $\underline{s}$-topologies.

Example 1: If X is an $\underline{s}$-space and $p : X \to Y$ is an identification map then Y is an $\underline{s}$-space.

Example 2: If $\{X_a \mid a \in A\}$ is a set of disjoint $\underline{s}$-spaces then $X = \bigcup_{a \in A} X_a$ is an $\underline{s}$-space.

Example 3: If X and Y are $\underline{s}$-spaces and A is a closed $\underline{s}$-subspace of X we have

$$\begin{tikzcd} X \arrow[r, "\bar{f}"] & X \cup_f Y \\ A \arrow[u, hook, "i"] \arrow[r, "f"] & Y \arrow[u, "\bar{i}"'] \end{tikzcd}$$

then $X \cup_f Y$ is an $\underline{s}$-space ($X \cup_f Y$ has the final topology with respect to $\bar{f}$ and $\bar{i}$).

Example 4: If $X_1 \subseteq X_2 \subseteq X_3$ ... is an expanding sequence of s-spaces and $X = \bigcup_{i \in \mathbb{N}} X_i$ then X is an s-space.

Now let us investigate function spaces and the exponential law in Seq. We define the function space $M_s(X, Y)$ by $M_s(X, Y) = sM_c(X, Y)$. It is clearly an s-space (see lemma 3.1) so we just have to check the exponential law. We need two preliminary lemmas.

Lemma 3.7 If $f : \mathbb{N}_\infty \to X$ is continuous then the mapping $e_f : M_c(X, Y) \times \mathbb{N}_\infty \to Y$ given by $e_f(g, z) = g(f(z))$ is also continuous.

Proof: ef can be factored as follows:

$$\begin{array}{ccc} M_c(X, Y) \times \mathbb{N}_\infty & \xrightarrow{e_f} & Y \\ \downarrow{\scriptstyle f^* \times 1} & \nearrow & \\ M_c(\mathbb{N}_\infty, Y) \times \mathbb{N}_\infty & & \end{array}$$

where $f^* : M_c(X, Y) \to M_c(\mathbb{N}_\infty, Y)$ is the induced map defined by $f^*(g) = g \circ f$ for $g \in M_c(X, Y)$. Since f is continuous, $f^* \times 1$ is continuous. It is a standard result [see 20, p.74] that e is continuous since $\mathbb{N}_\infty$ is a locally compact regular space. Therefore e_f is continuous.

Lemma 3.8 If X, Y are s-spaces then $cM_c(X, Y) = F(cX, cY)$.

Proof: We need X to be an s-space so that $cM_c(X, Y)$ and

$F(\underline{c}X, \underline{c}Y)$ can be identified as sets. First we show that the identity $\underline{c}M_c(X, Y) \to F(\underline{c}X, \underline{c}Y)$ is sequentially continuous. We have to show that for all $f \in S(\mathbb{N}_\infty, \underline{c}M_c(X, Y))$, $f \in S(\mathbb{N}_\infty, F(\underline{c}X, \underline{c}Y))$. Let $g : \mathbb{N}_\infty \to X$ be continuous (so $g \in S(\mathbb{N}_\infty, \underline{c}X)$) and $h : \mathbb{N}_\infty \to \mathbb{N}_\infty$ be an injective map. Consider the commutative diagram:

$$\begin{tikzcd}
\mathbb{N}_\infty \arrow[r, "\Delta"] & \mathbb{N}_\infty \times \mathbb{N}_\infty \arrow[r, "h \times g"] \arrow[d, "(f \circ h) \times 1"] & \mathbb{N}_\infty \times X \arrow[d, "f \times 1"] & \\
& M_c(X,Y) \times \mathbb{N}_\infty \arrow[r, "1 \times g"'] & M_c(X,Y) \times X \arrow[r, "e"] & Y
\end{tikzcd}$$

Since $f \in S(\mathbb{N}_\infty, \underline{c}M_c(X,Y))$, $(f \circ h) \times 1$ is continuous. $e \circ (1 \times g)$ is continuous by the previous lemma. Therefore $e \circ (f \circ h \times g) \circ \Delta = e(f \circ h, g)$ is continuous and $f \in S(\mathbb{N}_\infty, F(\underline{c}X, \underline{c}Y))$.

Now we show $F(\underline{c}X, \underline{c}Y) \to \underline{c}M_c(X,Y)$ is sequentially continuous. Let $f \in S(\mathbb{N}_\infty, F(\underline{c}X, \underline{c}Y))$. Then the composite $e \circ (f \times 1) : \underline{c}\mathbb{N}_\infty \times \underline{c}X \to F(\underline{c}X, \underline{c}Y) \times \underline{c}X \to \underline{c}Y$ is sequentially continuous. Therefore $\mathbb{N}_\infty \times X \to Y$ is continuous, for $\underline{t}(\underline{c}\mathbb{N}_\infty \times \underline{c}X) = \mathbb{N}_\infty \times X$ (proposition 3.4) and Y is an $\underline{s}$-space. It follows from the exponential law in Top that $\mathbb{N}_\infty \to M_c(X,Y)$ is continuous so $f \in S(\mathbb{N}_\infty, \underline{c}M_c(X,Y))$ and the result follows.

We now see that $M_s(X,Y) = \underline{t}\underline{c}M_c(X,Y) = \underline{t}F(\underline{c}X, \underline{c}Y)$.

Theorem 3.9 (Exponential Law) If X, Y, Z are $\underline{s}$-spaces, then there is a homeomorphism

$$\theta_s : M_s(X \times_s Y, Z) \to M_s(X, M_s(Y, Z))$$

which assigns to each continuous function $f : X \times_s Y \to Z$ an $f' : X \to M_s(Y, Z)$ such that $f(x,y) = f'(x)(y)$.

Proof: We have

$$\begin{aligned} M_s(X,M_s(Y,Z)) &= \underline{t}\underline{c}M_c(X,M_s(Y,Z)) \\ &= \underline{t}F(\underline{c}X,\underline{c}M_s(Y,Z)) \\ &= \underline{t}F(\underline{c}X,\underline{c}\underline{t}\underline{c}M_c(Y,Z)) \\ &= \underline{t}F(\underline{c}X,F(\underline{c}Y,\underline{c}Z)) \end{aligned}$$

using lemma 3.8.

Also

$$\begin{aligned} M_s(X \times_s Y,Z) &= \underline{t}\underline{c}M_c(X \times_s Y,Z) \\ &= \underline{t}F(\underline{c}\underline{t}\underline{c}(X \times Y),\underline{c}Z) \\ &= \underline{t}F(\underline{c}X \times \underline{c}Y,\underline{c}Z) \end{aligned}$$

By the exponential law for convergences (Theorem 2.5) we have a sequential homeomorphism

$$\theta : F(\underline{c}X \times \underline{c}Y, \underline{c}Z) \to F(\underline{c}X, F(\underline{c}Y,\underline{c}Z))$$

where θ takes a sequentially continuous function $f : \underline{c}X \times \underline{c}Y \to \underline{c}Z$ to a function $f' : \underline{c}X \to F(\underline{c}Y,\underline{c}Z)$ where $f(x,y) = f'(x)(y)$ as required. So we set $\theta_s = \underline{t}\theta$ and the proof is complete.

The exponential law is the key property amongst Steenrod's test propositions; for once it is satisfied three of the four others follow as corollaries (for (1) see corollary 3.15 below, (3) see corollary 3.11 below and (4) see corollary 3.12 below).

Proposition 3.10 Let X be a final s-space with respect to maps $\{f_a\}a\in A$ from a family of s-spaces $\{X_a\}a\in A$ which satisfies the condition that for each $x \in X$ there exists an $x' \in X_a$ such that $f_a(x') = x$ for some choice of $a \in A$. Then for any s-space Y $X \times_s Y$ has the final topology with respect to the functions $\{f_a \times 1 : X_a \times_s Y \to X \times_s Y\}$.

Proof: The proof is formally the same as Lemma 1.1. We notice that the locally compact Hausdorff condition is no longer necessary.

Corollary 3.11 If $p : X \to A$ and $q : Y \to B$ are identification maps then $p \times q : X \times_s Y \to A \times_s B$ is again an identification map.

Proof: We can factor $p \times q$ as $(1_X \times q) \circ (p \times 1_Y)$, thus $p \times q$ is the composite of two identifications. The result follows from lemma 1.2.

Corollary 3.12 If X is the disjoint union of a family of s-spaces $\{X_a\}a \in A$ then $X \times_s Y$ is homeomorphic to $\bigcup_{a\in A} (X_a \times_s Y)$.

Proof: By the proposition $X \times_s Y$ has the final topology with respect to the functions $(i_a \times 1 : X_a \times_s Y \to X \times_s Y)$ where the i_a are the inclusions $X_a \to X$. $\bigcup_{a\in A} (X_a \times_s Y)$ and $X \times_s Y$ have the same underlying set and the same topology, so the result follows.

Proposition 3.13 Let Y be an initial $\underline{s}$-space with respect to maps $\{f_a : Y \to Y_a\}a \in A$. Then for any $\underline{s}$-space X, $M_s(X,Y)$ has the initial topology with respect to the induced maps $f_{a*}: M_s(X,Y) \to M_s(X,Y_a)\}a \in A$ where $f_{a*}(h) = f_a \circ h$, $h \in M_s(X,Y)$.

Proof: We check the universal property. Let W be any $\underline{s}$-space then $g : W \to M_s(X,Y)$ is continuous if and only if the associated $g' : W \times_s X \to Y$ is continuous. Since Y is an initial $\underline{s}$-space, the continuity of g' is equivalent to the continuity of $f_a \circ g' : W \times_s X \to Y_a$ for all $a \in A$. Applying the exponential law again, $f_a \circ g'$ is continuous if and only if the corresponding function $(f_a \circ g')' : W \to M_s(X,Y_a)$ is continuous. It is easily checked that $(f_a \circ g')' = f_a^* \circ g$ hence the universal property is satisfied and the result follows.

Corollary 3.14 If Y is an $\underline{s}$-subspace of Z, then $M_s(X,Y)$ is an $\underline{s}$-subspace of $M_s(X,Z)$.

Proof: The map induced by the inclusion is just the inclusion again. The result follows immediately from the proposition.

Corollary 3.15 If X, Y, Z are $\underline{s}$-spaces then there is a homeomorphism

$$\phi : M_s(X, Y \times_s Z) \to M_s(X,Y) \times_s M_s(Y,Z)$$

which takes $f \in M_s(X, Y \times_s Z)$ to $(pr_1 \circ f, pr_2 \circ f) \in M_s(X,Y) \times_s M_s(X,Z)$.

Proof: By the proposition $M_s(X, Y \times_s Z)$ has the initial topology with respect to $pr_1^* : M_s(X, Y \times_s Z) \to M_s(X,Y)$ and $pr_2^* : M_s(X, Y \times_s Z) \to M_s(X,Z)$. The product $M_s(X,Y) \times_s M_s(X,Z)$ has the initial topology with respect to the projections $pr_i (i = 1,2)$. It is easy to check that $pr_i \circ \phi = pr_i^*$ and $pr_i^* \circ \phi^{-1} = pr_i$ so ϕ and ϕ^{-1} are continuous and the result follows.

Proposition 3.16 If X is a final s-space with respect to maps $\{f_a : X_a \to X\}_{a \in A}$ then for any s-space Y, $M_s(X,Y)$ has the initial topology with respect to the induced maps $\{f_a^* : M_s(X,Y) \to M_s(X_a,Y)\}_{a \in A}$ where $f_a^*(h) = h \circ f_a$ for all $h \in M_s(X,Y)$.

Proof: To check the universal property let W be any s-space and $g : W \to M_s(X,Y)$ a function. We want to show g is continuous if and only if $h_a = f_a^* \circ g : W \to M_s(X_a,Y)$ is continuous for all $a \in A$, so we apply the exponential law to each function and get the following commutative diagram:

$$\begin{tikzcd} W \times_s X \arrow[r, "g'"] & Y \\ W \times_s X_a \arrow[u, "1 \times f_a"] \arrow[ur, "h_a'"'] & \end{tikzcd}$$

Now by proposition 3.10 $W \times_s X$ has the final topology with respect to the functions $\{1 \times f_a\}_{a \in A}$ so the continuity of g' is equivalent to that of h_a' and the result follows.

Corollary 3.17 If $X_1 \cup X_2$ is the disjoint topological sum of two s-spaces then for any s-space Y, $M_s(X_1 \cup X_2, Y)$ is homeomorphic to $M_s(X_1,Y) \times_s M_s(X_2,Y)$. The result also holds for a family $\{X_a\}a\in A$ of s-spaces.

Corollary 3.18 If $p : X \to B$ is an identification and Y is an s-space then $p^* : M_s(B,Y) \to M_s(X,Y)$ is essentially an inclusion (i.e. $M_s(B,Y)$ is homeomorphic to the corresponding subspace in $M_s(X,Y)$).

We have shown that our category Seq satisfies several basic requirements for convenience. Now let us consider whether there are enough useful spaces in Seq for it to be of practical value.

Proposition 3.19 If X is a first countable space (e.g. any metrizable space or any differentiable manifold) then X is an s-space.

Proof: We have only to show that $id : X \to sX$ is continuous. Since X is first countable id is continuous if and only if it is sequentially continuous [see 30, p.131]. To check sequential continuity recall that it is sufficient to check $cid : cX \to csX$ is sequentially continuous. But $cX = ctcX$ by corollary 2.8 so id is sequentially continuous and the result follows.

Proposition 3.20 If K is a CW-complex then K is an s-space.

Proof: K has the final topology with respect to the union of the expanding sequence of its skeletons, so we only have to show that the n-skeletons, K^n, are s-spaces for all n. We do this inductively. K^0 is a discrete space hence first countable and therefore an s-space. Assume K^{n-1} is an s-space. $K^n = \left(\bigcup_{a\in A} E^n_a\right) \cup_f K^{n-1}$. $\bigcup_{a\in A} E^n_a$ is a sum of s-spaces and so K^n is an adjunction of s-spaces and hence an s-space (see proposition 3.6 and examples following it).

Next we see the relationship of Seq to the category of k-spaces.

Proposition 3.21 (based on proposition 1.5 in Vogt [36]). If X is an s-space, then X is a k-space.

Proof: Let $U \subseteq X$ be a subset such that $f^{-1}(U)$ is open for all maps $f : C \to X$ where C is compact Hausdorff. Then in particular $f^{-1}(U)$ is open for all continuous functions $f : \mathbb{N}_\infty \to X$. Hence U is open in X and therefore X is a k-space.

On the other hand, not all k-spaces, or even all compact Hausdorff spaces, are s-spaces. S. P. Franklin [18] gives the example of the ordinal space $\Omega + 1 = \Omega \cup \{\Omega\}$ where Ω is the first uncountable ordinal [see 14, ex.5, p.66]. $\Omega + 1$ is compact Hausdorff since it has a complete order topology [see 34, p.69]. It follows that it is a k-space. The set $\{\Omega\}$ is open in $s(\Omega+1)$ since for any continuous function $f : \mathbb{N}_\infty \to \Omega + 1$, $f(n) \in \{\Omega\}$ for all $n \in \mathbb{N}_\infty$ (see proposition 2.6). However, $\{\Omega\}$ is not open in $\Omega + 1$ since its complement is open, so $\Omega + 1$ is not an s-space. Seq is therefore a proper subcategory of the category of k-spaces.

Hausdorffness in Seq: It is standard that, in the context of the usual category of topological spaces, a space X is Hausdorff if and only if the diagonal is closed in $X \times X$ [see for example, 9, p.55]. For a separation axiom in Seq we propose a weakened version of this condition i.e., the diagonal is closed in $X \times_s X$. This turns out to be equivalent to the property that convergent sequences in X have unique limits so s-spaces with this property will be called unique limit spaces. We will now show the equivalence of the two properties and demonstrate some basic facts about unique limit spaces.

Definition: For a topological space X, a subset F of X is sequentially closed if and only if no sequence in F converges to a point not in F.

Lemma 3.22 A subset F of X is sequentially closed if and only if F is closed in sX.

Proof: Assume F is closed in sX. Let $\{x_n\}$ be a sequence in F with limit $x \notin F$. Then there is a continuous function $h : \mathbb{N}_\infty \to X$ such that $h(n) = x_n$ for all $n \in \mathbb{N}$, $h(\infty) = x$. Also $\infty \notin h^{-1}(F)$ so $h^{-1}(F)$ is open in $\mathbb{N}_\infty$, hence F is not closed in sX. By this contradiction we see that $x \in F$ and F is sequentially closed.

Assume F is sequentially closed and suppose F is not closed in $\underline{s}X$. Then $h^{-1}(F)$ is not closed in $\mathbb{N}_\infty$ for some continuous function $h : \mathbb{N}_\infty \to X$. i.e., $h^{-1}(F)$ is an infinite subset of $\mathbb{N}_\infty$ not containing ∞. Therefore F contains a sequence converging to a point not in F. This is a contradiction so F must be closed in $\underline{s}X$.

Proposition 3.23 The following are equivalent for a topological space X:

i) convergent sequences in X have unique limits,

ii) Δ_X is a closed subset of $X \times_s X$.

Proof: i) $\Longrightarrow$ ii) Assume that if $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} s_n = s'$ then $s = s'$. If $\{x_n\} = \{(s_n, s_n)\}$ is a sequence in Δ_X then its limit (s,s) is in Δ_X so Δ_X is sequentially closed in $X \times X$. By lemma 3.22 Δ_X is closed in $\underline{s}(X \times X) = X \times_s X$.

ii) $\Longrightarrow$ i) Assume Δ_X is closed in $X \times_s X$. By lemma 3.22 Δ_X is sequentially closed in $X \times X$. If $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} s_n = s'$ then $\{(s_n, s_n)\}$ is a sequence in Δ_X so its limit (s, s') is in Δ_X. Hence $s = s'$ and X has unique sequential limits.

Proposition 3.24 Let X have the initial topology with respect to a family of functions $\{f_a : X \to X_a\}_{a \in A}$ where the $\{X_a\}$ are unique limit spaces and the following condition is satisfied: if $x, x' \in X$ and $f_a(x) = f_a(x')$ for all $a \in A$ then $x = x'$. Then X is a unique limit space.

is closed in $(X \times_s X) \cup (Y \times_s Y)$. But $(X \times_s X) \cup (Y \times_s Y)$ is closed in $(X \times_s X) \cup (Y \times_s Y) \cup (X \times_s Y) \cup (Y \times_s X)$ which is homeomorphic to $(X \cup Y) \times_s (X \cup Y)$ and the result follows.

Proposition 3.27 If Y is a unique limit space and X is an s-space, $M_s(X,Y)$ is a unique limit space.

Proof: Suppose $\{f_n\}$ is a sequence of continuous functions $X \to Y$ converging pointwise to f and f' i.e., $\lim_{n\to\infty} f_n(x) = f(x) = f'(x)$ for all $x \in X$. Then since Y is a unique limit space and $\{f_n(x)\}$ is a sequence in Y, $f(x) = f'(x)$ for all $x \in X$. Hence $f = f'$ and the result follows.

Proof: Suppose $\lim_{n\to\infty}(x_n) = x$ and $\lim_{n\to\infty}(x_n) = x'$, then $\lim_{n\to\infty} f_a(x_n) = f_a(x)$ and $\lim_{n\to\infty} f_a(x_n) = f_a(x')$ for all $a \in A$. Since the $\{X_a\}$ are unique limit spaces $f_a(x) = f_a(x')$ so $x = x'$ and X is a unique limit space.

We do not have a result of the same generality as proposition 3.24 for final topologies, or even for identification spaces. This is perhaps only to be expected as in the category of all spaces the conditions for the quotient of Hausdorff spaces to be Hausdorff are rather untidy [see 14, p.140]. We do, however, have the following useful property.

Proposition 3.25 If R is an equivalence relation on a unique limit space X and R is closed as a subset of $X \times_s X$, then X/R is a unique limit space.

Proof: If $p : X \to X/R$ denotes the quotient map then $p \times p : X \times_s X \to (X/R) \times_s (X/R)$ is a quotient map. (corollary 3.11). Then $\Delta X/R$ is closed in $X/R \times_s X/R$ because $(p \times p)^{-1}(\Delta X/R) = R$ is closed and $p \times p$ is an identification map.

There is also a result for topological sums.

Proposition 3.26 If X and Y are disjoint unique limit spaces, then their topological sum $X \widehat{\cup} Y$ is a unique limit space.

Proof: The diagonal $\Delta_{X \cup Y}$ is the same set as $\Delta_X \cup \Delta_Y$ which

CHAPTER IV

Fibred Exponential Law

We will now establish the existence of a fibred exponential law in the category of s-spaces over a fixed unique limit space, using a method analogous to that used for k-spaces in Booth and Brown [8]. The fibred exponential law will be derived from an exponential law for partial maps between sequential spaces. We will then define a sequential space topology on the fibred mapping space (YZ) and prove a fibred exponential law for s-spaces.

A closed domain partial map from X to Y is a map $A \to Y$ for some closed subspace A of X. Such a partial map is called a parc map. The set $P(X,Y)$ of all parc maps $X \to Y$ can be given a compact-open topology, i.e. the topology given by a subbasis of sets of the form:

$$W(K,U) = \{f \in P(X,Y) \mid \hat{f}(K) \subseteq U\}$$

for all compact subsets $K \subseteq X$ and open subsets $U \subseteq Y$ (where $\hat{f}(K)$ means $f(K \cap \text{domain } f)$).

To define the corresponding sequential space, denoted $P_s(X,Y)$, we use the space $Y^\sim$, which is the set $Y \cup \{w\}$ (where $w \notin Y$) with the topology in which C is closed in $Y^\sim$ if and only if $C = Y^\sim$ or C is closed in Y. Then $P_s(X,Y)$ is $P(X,Y)$ topologized in such a way that we have a homeomorphism $\lambda : M_s(X, sY^\sim) \to P_s(X,Y)$ where $\lambda(f) = f \mid f^{-1}(Y)$ for $f \in M_s(X, sY^\sim)$.

Theorem 4.1 (Exponential law for Parc Maps). Let X, Y, Z be s-spaces, then there is a homeomorphism

$$\theta : P_s(X \times_s Y, Z) \to M_s(X, P_s(Y,Z))$$

which assigns to each parc map $f : X \times_s Y \to Z$ a continuous function $f' : X \to P_s(Y,Z)$ such that $f(x,y) = f'(x)(y)$.

Proof: First we check that θ is a well defined bijection. For $f \in P_s(X \times_s Y, Z)$, the continuity of f is equivalent to the continuity of $\lambda^{-1}(f) : X \times_s Y \to sZ^{\sim}$ by definition of the topology on $P_s(X \times_s Y, Z)$. $\lambda^{-1}(f)$ is continuous if and only if its exponential correspondent, $g : X \to M_s(Y, sZ^{\sim})$ is continuous. Then g is continuous if and only if $\lambda \circ g$ is continuous and $\lambda \circ g$ is easily seen to be $\theta(f)$ so we can conclude that f is continuous if and only if $\theta(f)$ is continuous and that θ is well-defined.

To see that θ and θ^{-1} are continuous let f be a function from an arbitrary s-space A to $P_s(X \times_s Y,Z)$. Then by definition of the topology on $P_s(X \times_s Y, Z)$ f is continuous if and only if $\lambda^{-1} \circ f : A \to M_s(X \times_s Y, s(Z^{\sim}))$ is continuous. It follows by the exponential law in Seq (theorem 3.9) and the associativity of the product that the continuity of $\lambda^{-1} \circ f$ is equivalent to the continuity of $f' : (A \times_s X) \times_s Y \to sZ^{\sim}$, where f' is the exponential correspondent of $\lambda^{-1}f$. Using the exponential law once again f' is continuous if and only if $f'' : A \times_s X \to M_s(Y, sZ^{\sim})$ is continuous. Finally by the definition of $P_s(Y,Z)$ and Theorem 3.9 we find that the continuity of f'' is equivalent to that of $f''' : A \to M_s(X, P_s(Y,Z))$.

Now let $A = P_s(X \times_s Y, Z)$ and f be the identity. Since f is continuous, $f^{\sim}$ is continuous. But $f^{\sim} = \theta$ so θ is continuous. Similarly, let $A = M_s(X, P_s(Y,Z))$ and let $f^{\sim}$ be the identity. Then $f = \theta^{-1}$ is continuous.

We now consider a category we will call $\underline{Seq_B}$ where B is some fixed $\underline{s}$-space. The objects of this category are $\underline{s}$-spaces over B, i.e. continuous functions $p : X \to B$, $q : Y \to B$, where X, Y are $\underline{s}$-spaces. The morphisms $p \to q$ are continuous functions $h : X \to Y$ where $q \circ h = p$. The product in this category is the $\underline{s}$-fibred product projection, $p \sqcap_s q : X \sqcap_s Y \to B$ which takes a pair $(x,y) \in X \sqcap_s Y$ to $p(x) = q(y)$ (see example 3 following proposition 3.2).

The exponential law we will prove for $\underline{Seq_B}$ involves two mapping spaces. First we have $M_s(p,q) = \underline{s}M_c(p,q)$, where $M_c(p,q)$ is the set of all maps $p \to q$ with the compact-open topology. We also have a <u>fibred mapping space</u> $(YZ)_s$ defined as follows: given maps $q : Y \to B$ and $r : Z \to B$ then for each $b \in B$, let $Y_b = q^{-1}(b)$, $Z_b = r^{-1}(b)$. Then (YZ) is the set $\bigcup_{b \in B} M(Y_b, Z_b)$. Let $(qr) : (YZ) \to B$ be defined by $(qr)(f) = b$ for all $f \in M(Y_b, Z_b)$. Assume that B is T_1 so each fibre, Y_b, is closed. Then there is a function $i : (YZ) \to P(Y,Z)$ which sends a map $Y_b \to Z_b$ to the parc map $Y \to Z$ with the same domain and values. Now we define the <u>modified compact-open topology</u> on (YZ) as the initial topology with respect to the two functions i and (qr). Finally we define $(YZ)_s = \underline{s}(YZ)$.

To use our result on parc maps we need $X \sqcap_s Y$ to be closed subspace of $X \times_s Y$. We therefore assume that B is a unique limit space (hence $(p \times q)^{-1}\Delta_B = X \sqcap_s Y$ is closed in $X \times_s Y$ for all $p : X \to B$, $q : Y \to B$).

Now we can prove an exponential law for $\underline{\mathrm{Seq}}_B$ and hence can conclude that it is a cartesian closed category.

Theorem 4.2 (Fibred Exponential Law). Let X, Y, Z be $\underline{s}$-spaces, B a unique limit space and $p : X \to B$, $q : Y \to B$, $r : Z \to B$ continuous functions. Then there is a homeomorphism

$$\phi : M_s(p \sqcap_s q, r) \to M_s(p, (qr))$$

which assigns to each map $f : p \sqcap_s q \to r$ over B a map $f' : p \to (qr)$ over B such that $f(x,y) = f'(x)(y)$.

Proof: The main part of the proof is to show ϕ is well-defined. First assume $f : X \sqcap_s Y \to Z$ is continuous over B. We have to check that $f' : X \to (YZ)_s$ is a function over B. $f'(x)$ is the map $y \mapsto f(x,y)$. Now $y \in Y_{p(x)}$ since $y \in q^{-1}q(y) = q^{-1}(p(x))$ and $f(x,y) \in Z_{p(x)}$ since $r \circ f = p \sqcap_s q$ so $r^{-1}(p(x)) = r^{-1}(p \sqcap_s q(x,y)) = f(x,y)$. Therefore $f'(x)$ is a function $Y_{p(x)} \to Z_{p(x)}$ and hence $(qr) \circ f' = p$ so f' is a function over B.

Now we show that $f' : X \to (YZ)$ is continuous and hence conclude that the same function $X \to (YZ)_s$ is continuous by lemma 3.1. (YZ) has the initial topology with respect to i and (qr) so we have to show

$(qr) \circ f'$ and $i \circ f'$ are continuous. $(qr) \circ f' = p$ and hence is continuous. $i \circ f' = \theta(\overline{f})$ where $\overline{f}$ is the partial map $\hat{X} \times_s Y \to Z$ corresponding to f and θ is the exponential function for parc maps (Theorem 4.1). Hence $i \circ f'$ is continuous (note that we need the fact that $X \sqcap_s Y$ is closed in $X \times_s Y$).

Assume $f' : X \to (YZ)_s$ is a map over B. Then the corresponding function $f : X \sqcap_s Y \to Z$ takes $(x,y) \mapsto f'(x)(y)$, where $f'(x)(y) \in Z_{p(x)}$ so $r \circ f = p \sqcap_s q$ and f is a function over B. We notice that $i \circ f' : X \to P_s(Y,Z)$ is continuous and so $\theta^{-1}(i \circ f')$ is a continuous parc map $X \times_s Y \to Z$. But $f = \theta^{-1}(i \circ f')$ as functions, so f is continuous.

That ϕ and ϕ^{-1} are continuous follows by an argument similar to the one for parc maps. Let A be any s-space. Then a function $f : A \to M_s(X \sqcap_s Y, Z)$ is continuous if and only if $f : A \to P_s(X \times_s Y, Z)$ is continuous, since $X \sqcap_s Y$ is closed in $X \times_s Y$. By the exponential law for parc maps and the associativity of the product, the continuity of f is equivalent to that of its exponential correspondent $f' : (A \times_s X) \times_s Y \to Z$. Using the exponential law again, f' is continuous if and only if $f'' : A \times_s X \to P_s(Y,Z)$ is continuous. Using the definition of the topology on $(YZ)_s$ we see that the continuity of f'' is equivalent to the continuity of $i \circ f'' : A \times_s X \to (YZ)_s$. A final application of the exponential law in Seq yields the result that $i \circ f''$ is continuous if and only if $h = (i \circ f'')' : A \to M_s(X, (YZ)_s)$ is continuous. It is easily verified that $h(A) \subseteq M_s(p, (qr))$ and so we can regard h as a map $A \to M_s(p, (qr))$.

Now let $A = M_s(p \sqcap_s q, r)$. The continuity of the identity implies the continuity of h. It can easily be checked that $h = \phi$. Similarly, letting $A = M_s(p, (qr))$, we have proved the continuity of ϕ^{-1}.

Remarks:

1. If B is a single point, the fibred exponential law reduces to the usual exponential law, since $X \sqcap_s Y$ becomes $X \times_s Y$ and $(YZ)_s$ becomes $M_s(Y,Z)$.

2. If $X = B$ and $p = 1_B$ then Theorem 4.2 gives a bijective correspondence between maps $1_B \to (qr)$ and $1 \sqcap_s q \to r$ or in other words, between sections to (qr) and maps $q \to r$. This result has various applications in algebraic topology (see for example [7].

We can use the fibred exponential law to obtain a result on final topologies similar to proposition 3.13.

Proposition 4.3 Let B be a unique limit space and also a final $\underline{s}$-space with respect to a family of maps $\{p_a : X_a \to B\}_{a \in A}$ which satisfies the condition of proposition 3.10. If Y is any $\underline{s}$-space and $f : Y \to B$ is a map then Y has the final topology with respect to the induced projections $p_{af} : X_a \sqcap_s Y \to Y$.

Proof: We have to show that given an $\underline{s}$-space W, and a function

$g : Y \to W$, g is continuous if $h_a = g \circ p_{af}$ is continuous.

Using these functions and the easily verified fact that Y is homeomorphic to $B \sqcap_s Y$ (by a map $y \mapsto (f(y), y)$) we can form the following commutative diagram for each $a \in A$:

$$\begin{tikzcd}
X_a \sqcap_s Y \arrow[r, "h' = (f \sqcap_s p_a, h)"] \arrow[d] & B \times W \\
B \sqcap_s Y \arrow[ur, "q' = 1 \sqcap_s q"'] &
\end{tikzcd}$$

Noting that we have the projection map $B \times W \to B$, we see that this last diagram is a diagram of maps over B so we can use the fibred exponential law to obtain the following diagram:

$$\begin{tikzcd}
X_a \arrow[r, "\tilde{h}"] \arrow[d, "p_a"'] & (YB \times W)_s \\
B \arrow[ur, "\tilde{g}"'] &
\end{tikzcd}$$

Now B has the final topology with respect to the $\{p_a\}$ and the argument of proposition 3.13 assures us that $\tilde{g}$ is well-defined so the continuity of $\tilde{h}$ implies that of $\tilde{g}$ and hence of g and the result follows.

Corollary 4.4 If $p : X \to B$ is an identification map and $f : A \to B$ is a map then the induced projection $p_f : X \sqcap_s A \to A$ is an identification map.

Proof: The result follows immediately from the proposition.

Corollary 4.5 If $p : X \to B$ is an identification map and $A \subseteq B$ then $p \mid p^{-1}(A) : p^{-1}(A) \to A$ is an identification map.

Proof: This follows from the fact that $p^{-1}(A)$ is homeomorphic to $X \sqcap_s A$ since both satisfy the same universal property of pullbacks.

$$\begin{tikzcd}
p^{-1}(A) \arrow[drr] \arrow[ddr] & & \\
& X \sqcap_s A \arrow[r] \arrow[d] & X \arrow[d] \\
& A \arrow[r, hook] & B
\end{tikzcd}$$

This corollary arises, for example, in the theory of principal bundles.

CHAPTER V

The Convergent Sequence-Open Topology

Working in the category of all topological spaces, we can define a convergent sequence-open topology on the set of continuous functions $Y \to Z$ as follows:

This topology has a subbasis consisting of sets $W(f,U) = \{h : Y \to Z \mid h \circ f(N_\infty) \subseteq U\}$ for $f : \mathbb{N}_\infty \to Y$ continuous and U an open subset of Z.

We denote this function space by $M_{cs}(Y,Z)$.

This function space topology has been studied in [21] where it is pointed out that:

(i) it is proper, i.e., if $f : X \times Y \to Z$ is continuous then the associated $f^{\wedge} : X \to M_{cs}(Y,Z)$ is continuous (proposition 2), but
(ii) it is not admissible, i.e., if $g^{\wedge} : X \to M_{cs}(Y,Z)$ is continuous then it does not necessarily follow that the associated function $g : X \times Y \to Z$ is continuous (proposition 3).

Now similar results hold for the compact-open topology, and it is shown in [10] that if X, Y, Z are Hausdorff and the product topology on $X \times Y$ is replaced by another closely related topology, then proper and admissible conditions hold. In this chapter we establish analogous results for the convergent sequence-open topology. We also show that if X and Y are sequential spaces then our new product, $X \bar{\times} Y$, coincides with the s-product $X \times_s Y$. This leads to an alternative description of the s-function space and an alternative proof of the exponential law for s-spaces.

Our new product topology in Top is similar to R. Brown's "$\times_s$" in his paper on product topologies [11]. We define $X \overline{\times} Y$ to have the final topology with respect to all functions $1_X \times f : X \times N_\infty \to X \times Y$; where $f : N_\infty \to Y$ is continuous, and all inclusions $\{x\} \times Y \to X \times Y$ for all $x \in X$.

Lemma 5.1 The evaluation map, e, is continuous as a function $M_{cs}(X,Y) \overline{\times} X \to Y$.

Proof: By the universal property of the final topology on $M_{cs}(X,Y) \overline{\times} X$, we have to show that $\{f\} \times X \to Y$ is continuous for all $f \in M_{cs}(X,Y)$. This is immediate since $e(f,x) = f(x)$ and f is continuous.

We also have to show that $e_f : M_{cs}(X,Y) \times N_\infty \to Y$ is continuous for $f : N_\infty \to X$ continuous. Let U be an open subset of Y and let $(g,n) \in e_f^{-1}(U)$. Then, since N_∞ is regular, there exists an open neighbourhood V of $n \in N_\infty$ such that $\overline{V} \in f^{-1}g^{-1}(U)$. Now $(f/\overline{V})(N_\infty) = h(N_\infty)$ for the correct choice of h as follows:

Case 1: $\overline{V} = \{\infty\} \cup \{v_1, v_2, v_3, \ldots\}$; $h(n) = f(v_n)$, $h(\infty) = f(\infty)$.

Case 2: $\overline{V} = \{\infty\} \cup \{v_1, v_2, \ldots, v_n\}$; $h(i) = f(v_i)$ for $i \leq n$,
$h(i) = f(\infty)$ for $i > n$.

Case 3: $\overline{V} = \{v_1, \ldots, v_n\}$, $v_i \neq \infty$; $h(i) = f(v_i)$ for $i < n$
$h(i) = f(v_n)$ for $i \geq n$

h is continuous in each case so $W(h,U)$ is a subbasic open set for $M_{cs}(X,Y)$. Hence $(g,n) \in W(h,U) \times V$ which is open in $M_{cs}(X,Y) \times \mathbb{N}_\infty$ so e_f is continuous.

<u>Proposition 5.2</u> A function $f : X \bar{\times} Y \to Z$ is continuous if and only if the corresponding function $f' : X \to M_{cs}(Y,Z)$ is continuous (where $f(x,y) = f'(x)(y)$).

<u>Proof:</u> If $f' : X \to M_{cs}(Y,Z)$ is continuous then $X \bar{\times} Y \xrightarrow{f' \times 1} M_{cs}(Y,Z) \bar{\times} Y$ is continuous and by the lemma $X \bar{\times} Y \xrightarrow{f' \times 1} M_{cs}(X,Y) \bar{\times} Y \xrightarrow{e} Z$ is continuous. But this last composite is the required function f so f is continuous.

Let $W(g,U)$ be a subbasic open set for $M_{cs}(Y,Z)$. We prove $(f')^{-1}(W)$ is open in X. If $k = f(1 \times g) : X \times \mathbb{N}_\infty \to Z$ then k is continuous and therefore $k^{-1}(U)$ is open in $X \times \mathbb{N}_\infty$.

Let $x \in (f')^{-1}(W)$. Then $\{x\} \times \mathbb{N}_\infty \subseteq k^{-1}(U)$. Since $\mathbb{N}_\infty$ is compact, there exists an open set $V \subseteq X$ such that $x \in V$ and $V \times \mathbb{N}_\infty \subseteq k^{-1}(U)$ [a standard result on compact spaces, see [27], p.142]. This implies that $x \in V \subseteq (f')^{-1}(W)$ so $(f')^{-1}(W)$ is open.

So now we have part of the usual exponential law. To get the rest we need an associative product [see 28, Theorem 10]. Unfortunately it is not clear that the product defined above is associative. It can be seen that the identity $X \bar{\times} (Y \bar{\times} Z) \to (X \bar{\times} Y) \bar{\times} Z$ is continuous as follows:

$X \bar{\times} (Y \bar{\times} Z)$ has the final topology with respect to the maps $1 \times f : X \times \mathbb{N}_\infty \to X \bar{\times} (Y \bar{\times} Z)$, where $f : \mathbb{N}_\infty \to Y \bar{\times} Z$ is continuous, and the inclusions $i \times 1_{Y\times Z} : \{x\} \times (Y \bar{\times} Z) \to X \bar{\times} (Y \bar{\times} Z)$. Hence, using Lemma 1.3, it has the final topology with respect to the composite functions;

$$\begin{tikzcd}
X \times \mathbb{N}_\infty \arrow[dr, "1_X \times f"] & \\
& X \bar{\times} (Y \bar{\times} Z) \\
\{x\} \times Y \times \mathbb{N}_\infty \arrow[ur, "i \times 1_Y \times g"] & \\
\{x\} \times \{y\} \times Z \arrow[uur, "i \times i \times 1_Z"'] &
\end{tikzcd}$$

On the other hand $(X \bar{\times} Y) \bar{\times} Z$ has the final topology with respect to maps $1 \times f : (X \bar{\times} Y) \times \mathbb{N}_\infty \to (X \bar{\times} Y) \bar{\times} Z$, with $f : \mathbb{N}_\infty \to Z$ continuous, and the inclusions $\{x\} \times \{y\} \times Z$ into $(X \bar{\times} Y) \bar{\times} Z$. Hence it has the final topology illustrated below;

$$\begin{tikzcd}
X \times \mathbb{N}_\infty \times \mathbb{N}_\infty \arrow[dr, "1_X \times g \times h"] & \\
\{x\} \times Y \times \mathbb{N}_\infty \arrow[r, "i \times 1_Y \times h"] & (X \bar{\times} Y) \bar{\times} Z \\
\{x\} \times \{y\} \times Z \arrow[ur, "i \times i \times 1_Z"'] &
\end{tikzcd}$$

So $id : X \overline{\times} (Y \overline{\times} Z) \to (X \overline{\times} Y) \overline{\times} Z$ is continuous since $X \times \mathbb{N}_\infty \xrightarrow{1\times\Delta} X \times \mathbb{N}_\infty \times \mathbb{N}_\infty$ is continuous and therefore the commutativity of the diagram below gives the continuity of id.

$$\begin{tikzcd} X \times \mathbb{N}_\infty \arrow[r, "1\times\Delta"] \arrow[d, "1\times(f,g)"] & X \times \mathbb{N}_\infty \times \mathbb{N}_\infty \arrow[d, "1\times f\times g"] \\ X \overline{\times} (Y \overline{\times} Z) \arrow[r, "id"] & (X \overline{\times} Y) \overline{\times} Z \end{tikzcd}$$

The obvious approach to attempting to prove the continuity of $(X \overline{\times} Y) \overline{\times} Z \to X \overline{\times} (Y \overline{\times} Z)$ would be to find a continuous function $X \times \mathbb{N}_\infty \times \mathbb{N}_\infty \to X \times \mathbb{N}_\infty$ that makes the diagram commute. There is apparently no such function so we leave this question unanswered.

Therefore we cannot prove a complete exponential law only the following:

<u>Theorem 5.3</u> The bijection $\theta : M_{cs}(X, M_{cs}(Y,Z)) \to M_{cs}(X \overline{\times} Y, Z)$ is continuous.

<u>Proof:</u> By proposition 5.2 the continuity of the identity map $id : M_{cs}(X, M_{cs}(Y,Z)) \to M_{cs}(X, M_{cs}(Y,Z))$ implies the continuity of the corresponding function $id' : M_{cs}(X, M_{cs}(Y,Z)) \overline{\times} X \to M_{cs}(Y,Z)$. Using the same result, the continuity of id' implies the continuity of $id'' : (M_{cs}(X, M_{cs}(Y,Z)) \overline{\times} X) \times Y \to Z$. In the discussion of associativity we saw that the continuity of $M_{cs}(X, M_{cs}(Y,Z)) \overline{\times} (X \overline{\times} Y) \to Z$ follows from that of id''. A final application of proposition 5.2 yields the

result that $\mathrm{id}^{\sim\sim}: M_{cs}(X,M_c(Y,Z)) \to M_{cs}(X \overline{\times} Y,Z)$ is continuous. It is easily checked that $\theta = \mathrm{id}^{\sim\sim}$ and hence is continuous.

Using sequential spaces it turns out that the bijection is a homeomorphism. This is because the product above coincides with the product in Seq and hence is associative.

Proposition 5.4 If X, Y are s-spaces then $X \overline{\times} Y = X \times_s Y$.

Proof: If X is an s-space then $X \times \mathbf{N}_\infty$ is an s-space (corollary 3.5) and so has the final topology with respect to all maps from $\mathbf{N}_\infty$. If Y is an s-space then $\{x\} \times Y$ has the final topology with respect to all maps $1_x \times f : \{x\} \times \mathbf{N}_\infty \to \{x\} \times Y$. Then by composition of final topologies $X \overline{\times} Y$ has the final topology with respect to all maps $(f,g) : \mathbf{N}_\infty \to X \times Y$ and all maps $(1_x \times g) : \{x\} \times \mathbf{N}_\infty \to X \times Y$.

Now to see that $X \overline{\times} Y = X \times_s Y$ we first show that the identity $\mathrm{id} : X \overline{\times} Y \to X \times_s Y$ is continuous. $\mathrm{id} \circ (f,g) : \mathbf{N}_\infty \to X \times_s Y$ is continuous since $X \times_s Y$ is an s-space and $\mathrm{id} \circ (1_x \times g) : \{x\} \times \mathbf{N}_\infty \to X \times_s Y$ is continuous since these maps are just special cases of maps $\mathbf{N}_\infty \to X \times_s Y$. So by the universal property of final topologies $\mathrm{id} : X \overline{\times} Y \to X \times_s Y$ is continuous. Then we check that $\mathrm{id} : X \times_s Y \to X \overline{\times} Y$ is continuous. $\mathrm{id} \circ (f,g) : \mathbf{N}_\infty \to X \overline{\times} Y$ is continuous because of the final topology on $X \overline{\times} Y$. So $\mathrm{id} : X \times_s Y \to X \overline{\times} Y$ is continuous and the two products coincide.

It is not hard to check that the function space in Seq can be redefined as $\underline{s}M_{cs}(X,Y)$ and the proof of the exponential law goes through. Lemma 5.1 contains the proof of the first lemma we needed and the second lemma depended only on the first and the bijection $M_{cs}(X, M_{cs}(Y,Z)) \to M_{cs}(X \overline{\times} Y, Z)$. These two lemmas and the fact that $X \overline{\times} Y = X \times_s Y$ are sufficient to prove the exponential law. Using the exponential law on $\underline{s}M_{cs}(X,Y)$ we can show that $\underline{s}M_{cs}(X,Y) = M_s(X,Y)$.

Proposition 5.5 For sequential spaces Y, Z $\underline{s}M_{cs}(Y,Z) = M_s(Y,Z)$ (recall $M_s(Y,Z) = \underline{s}M_c(Y,Z)$).

Proof: We know that for an arbitrary $\underline{s}$-space X, that $X \times_s Y \to Z$ is continuous if and only if $X \to \underline{s}M_{cs}(Y,Z)$ is continuous and also if and only if $X \to M_s(Y,Z)$ is continuous. Let $X = \underline{s}M_{cs}(Y,Z)$. Then $\underline{s}M_{cs}(Y,Z) \times_s Y \to Z$ is continuous so $\underline{s}M_{cs}(Y,Z) \to M_s(Y,Z)$ is continuous. Similarly $M_s(Y,Z) \to \underline{s}M_{cs}(Y,Z)$ is continuous. Hence the two function spaces coincide. In the language of category theory, this is just a particular case of the result that right adjoints are unique.

BIBLIOGRAPHY

[1] Bentley, H.L. and Herrlich and Robertson, W.A.; Convenient categories for topologists, Comment. Math. Univ. Carolinae

[2] Binz, E. and Keller, H.H.; Funktionenraume inder kategorie der Limesraume, Ann. Acad. Sci. Fenn. Sec. AI 383(1966), 1-21

[3] Birkhoff, G.; On the Combination of Topologies; Fund. Math., 26(1936), 156-166

[4] Boehme, T.K.; Linear s-spaces; mimeographed notes, Symposium on Convergence Structures, University of Oklahoma, March 1965, unpublished

[5] Booth, P.I.; The Exponential Law of Maps II; Math. Z., 121(1971), 311-319

[6] __________; The Section Problem and the Lifting Problem; Math. Z., 121(1971), 273-287

[7] __________; A Unified Treatment of Some Basic Problems in Homotopy Theory; Bull. Amer. Math. Soc., 29, no. 2(1973), 331-336

[8] __________; and R. Brown; Spaces of Partial Maps, Fibred Mapping Spaces and the Compact-Open Topology; to appear in Gen. Top. Appl.

[9] Brown, R.; Elements of Modern Topology; McGraw-Hill Publishing Co. Ltd., London, 1968

[10] __________; Function Spaces and Product Topologies; Quart. J. Math. Oxford, 15, no. 59(1964), 238-250

[11] __________; Ten Topologies for X×Y; Quart. J. Math. Oxford, 14, no. 56(1963), 303-319

[12] Clark, A.; Quasitopology and Compactly Generated Spaces; mimeographed notes, Brown University, unpublished

[13] Dudley, R.M.; On Sequential Convergence; Trans. Amer. Math. Soc., 112(1964), 483-507

[14] Dugundji, J.; Topology; Allyn and Bacon, Inc., Boston, 1966

[15] Edgar, G.A.; A cartesian closed category for topology, General Top. and its Applics. 6(1976), 65-72

[16] Eilenberg, S. and G.M. Kelly; Closed Categories; Proceedings of the Conference on Categorical Algebra, ed. S. Eilenberg, D.K. Harrison, S. MacLane, H. Rohrl; Springer-Verlag, Berlin, 1966

[17] Fischer, H.R.; Limesraume, Math. Ann. 137(1959), 269-303

[18] Franklin, S.P.; Spaces in which Sequences Suffice; Fund. Math., 57(1965), 107-115

[19] __________; Spaces in which Sequences Suffice II; Fund. Math., 61(1967), 51-56

[20] Fréchet, M.; Sur la Notion de Voisinage dans les Ensembles Abstraits; Bull. Sci. Math., 42(1918), 138-156

[21] Guthrie, J.A.; A Characterization of $\aleph_0$-spaces; Gen. Top. Appl., 1(1971), 105-110

[22] Herrlich, H.; Cartesian closed topological categories, Math. Colloq. Univ. Cape Town, 9(1974), 1-16

[23] Herrlich, H.; Topological structures, Math. Centre Tract, 52(1974) 59-122

[24] Herrlich, H.; Some Topological Theorems which Fail to be True; Categorical Topology, ed. E. Binz and H. Herrlich, Springer-Verlag, Berlin, 1976

[25] Hu, S.T.; Homotopy Theory; Academic Press Inc., New York, 1959

[26] Katětov, J.; On Continuity Structures and Spaces of Mappings, Comment Math. Univ. Carolinae, 6(1965), 257-278

[27] Kelley, J.L.; General Topology; D. Van Nostrand Co., Inc., Princeton, N.J., 1955

[28] Kelly, G.M.; Tensor Products in Categories; J. Algebra, 2(1965), 15-37

[29] Kowalsky, J.J.; Limesräume und Komplettiening, Math. Nach., 12(1954), 301-340

[30] Lipschutz, S.; General Topology; McGraw-Hill Inc., New York, 1965

[31] Nel, L.D.; Initially Structured Categories and Cartesian Closedness, Can. J. Math., 27(1975), 1361-1377

[32] __________; Cartesian Closed Coreflective Nulls, Questiones Mathematicae, 2(1977), 269-283

[33] Spanier, E.; Quasi-topologies; Duke Math. J., 30(1963), 1-14

[34] Steen, L.A. and J.A. Seebach, Jr.; Counterexamples in Topology; Holt, Rinehart and Winston, Inc., New York, 1970

[35] Steenrod, N.E., A Convenient Category of Topological Spaces; Michigan Math. J., 14(1967), 133-152

[36] Vogt, R.M.; Convenient Categories of Topological Spaces for Homotopy Theory; Archiv der Math., 22(1971), 545-555

[37] Wyler, O.; Convenient Categories for Topology; Gen. Top. Appl., 3(1973), 225-242