SOME FIXED POINT THEOREMS IN METRIC SPACES

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BY

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A THESIS
SUBMITTED TO THE COMMITTEE ON
GRADUATE STUDIES IN PARTIAL
FULFILLMENT OF THE REQUIREMENT
FOR THE DEGREE OF MASTER
OF SCIENCE

MEMORIAL UNIVERSITY OF NEWFOUNDLAND
ST. JOHN'S, NEWFOUNDLAND

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ABSTRACT

The main object of this thesis is to study contractive type mappings on a complete metric space. These mappings are generalizations of the wellknown Banach contraction and have the property that each such mapping has a unique fixed point.

In the beginning we give some definitions of mappings of this type and theorems which give the conditions necessary to guarantee, for different definitions, the existence of a unique fixed point. We also consider common fixed points of pairs of mappings which satisfy a contractive type condition between the pair.

In Chapter II, special consideration is given to mappings T which satisfy a condition involving $\phi : R^+ \to R^+$ such that $d(Tx,Ty) < \phi(d(x,y))$. This idea has been extended to a pair of maps so that $\phi : (R^+)^5 \to R^+$ and ϕ acts upon the five terms: $d(x,y)$, $d(x,Tx)$, $d(y,Ty)$, $d(x,Ty)$, and $d(y,Tx)$. Also mappings which satisfy ~~a commuting condition are con~~sidered.

In the end sequences of mappings are considered. These mappings or their limiting mappings satisfy contractive type conditions and theorems are given which contain the conditions necessary for the sequence of fixed points to converge to the fixed point of the limiting mapping.

ACKNOWLEDGEMENTS

I would like to express my sincere thanks to my supervisor, Dr. S.P. Singh, for his invaluable help throughout the preparation of this thesis and especially for his interest and encouragement; and to Mrs. Hilda Tiller for typing the manuscript.

CHAPTER I

Contractive Mappings

The aim of this chapter is to give preliminary definitions of some terms and to discuss various definitions of contractive type mappings which are generalizations of the well known Banach contraction.

1.1 Preliminary Definitions

Definition 1.1.1. Let X be a set and let R^+ denote the set of non-negative real numbers. A distance function $d : X \times X \to R^+$ is defined to be a metric if the following conditions are satisfied for all $x,y,z \in X$:

(i) $d(x,y) = 0$ if and only if $x = y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,z) \leq d(x,y) + d(y,z)$.

The set X with metric d is called a metric space and is denoted (X,d).

Definition 1.1.2 A sequence $\{x_n\}$ in a metric space X is said to converge to a point x belonging to X, if given any $\varepsilon > 0$, there exists a positive integer N such that for all $n \geq N$, we have $d(x_n,x) < \varepsilon$ or $\lim_{n\to\infty} d(x_n,x) = 0$.

It can easily be proved that a convergent sequence has a unique limit.

Definition 1.1.3 A sequence $\{x_n\}$ of points of a metric space X is called a Cauchy sequence if given any $\varepsilon > 0$, there exists a positive integer N, such that for all $n,m \geq N$ we have $d(x_n,x_m) < \varepsilon$ or $\lim_{n,m\to\infty} d(x_n,x_m) = 0$.

Remark: A convergent sequence is always a Cauchy sequence.

Definition 1.1.4 A metric space X is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 1.1.5 A metric space X is said to be compact if every open covering of X has a finite subcover.

Definition 1.1.6 Let T be a mapping of a set X into itself. A point $x_o \in X$ is called a fixed point of T if $Tx_o = x_o$.

Definition 1.1.7 A mapping T of a metric space X into a metric space Y is said to be continuous at a point $x_o \in X$ if and only if $\lim_{n\to\infty} Tx_n = Tx_o$ for all sequences in X with $\lim_{n\to\infty} x_n = x_o$. If it is true for all $x_o \in X$, then T is continuous on X.

Definition 1.1.8 If T is a real-valued function on an interval $J \subset R$ we say that T is nondecreasing on J if $T(x) \leq T(y)$ whenever $x < y$, $x,y \in J$. We say that T is nonincreasing on J if $T(x) \geq T(y)$ whenever $x < y$, $x,y \in J$.
We say that T is monotone if T is either nondecreasing or nonincreasing.

Definition 1.1.9 An extended real-valued function f is called upper semicontinuous at the point y if $f(y) \neq +\infty$ and $f(y) \geq \overline{\lim}_{x\to y} f(x)$. (where $\overline{\lim}_{x\to y} f(x) = \inf_{\delta>0} \sup_{0<|x-y|<\delta} f(x)$).

Definition 1.1.10 Let X be a compact metric space with metric d, and let F be a non-empty set of continuous real or complex functions defined on X. F is said to be equicontinuous if given $\varepsilon > 0$ there exists $\delta > 0$ such that for every $f \in F$,

$$d(x,y) < \delta \Rightarrow |fx - fy| < \varepsilon .$$

<u>Definition 1.1.11</u> Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set E. We say that $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the function f on E if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for all } x \in E.$$

<u>Definition 1.1.12</u> Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on a set E. We say that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the function f on E if given $\varepsilon > 0$ there exists a positive integer N such that

$$|f_n(x) - f_n(y)| < \varepsilon \quad \text{for all } n \geq N; \ x \in E.$$

<u>Remark (1)</u>: If $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on E, then $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f on E.

(2): The pointwise convergence of an equicontinuous sequence of functions on a compact set implies the uniform convergence of the sequence.

1.2 <u>Definitions of contractive type mappings</u>

A theorem proven by S. Banach [1], known as the Banach Contraction Principle is widely used to prove the existence and uniqueness of solutions of differential and integral equations.

<u>Definition 1.2.1</u> A mapping $T : X \to X$ is called a contraction map if $d(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$ and $0 \leq k < 1$.

<u>Remark</u>: If T is a contraction mapping on a metric space X, then T is continuous on X.

Theorem 1.2.1. Let T be a contraction mapping of a complete metric space X into itself. Then T has a unique fixed point.

Proof: Choose any $x_o \in X$ and define the sequence $\{x_n\}$ in X inductively by

$$x_1 = Tx_o$$
$$x_2 = Tx_1 = T^2 x_o$$
$$\vdots$$
$$x_n = Tx_{n-1} = T^n x_o .$$

We have to show that $\{x_n\}$ is a Cauchy sequence; that is,
$\lim\limits_{n,m\to\infty} d(x_m,x_n) = 0$.

Since T is a contraction mapping, we have

$$d(x_1,x_2) = d(Tx_o,Tx_1) \le kd(x_o,x_1)$$
$$d(x_2,x_3) = d(Tx_1,Tx_2) \le kd(x_1,x_2) \le k^2 d(x_o,x_1).$$
$$\vdots$$
$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) \le k^n d(x_o,x_1).$$

Consider $d(x_n,x_m)$, $m > n$.

Then
$$\begin{aligned} d(x_n,x_m) &\le d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \dots + d(x_{m-1},x_m) \\ &\le k^n d(x_o,x_1) + k^{n+1} d(x_o,x_1) + \dots + k^{m-1} d(x_o,x_1) \\ &= k^n d(x_o,x_1)[1 + k + \dots + k^{m-n-1}] \\ &< \frac{k^n}{1-k} d(x_o,x_1). \end{aligned}$$

Since $k < 1$, $d(x_n,x_m) \to 0$ as $n \to \infty$.

Thus $\{x_n\}$ is a Cauchy sequence, and since X is complete $\{x_n\}$ converges to a point $z \in X$. Since a contraction mapping is continuous, T is continuous and we have,

$$Tz = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z .$$

Therefore z is a fixed point of T.

To see that z is a unique fixed point of T, let y and z be two fixed points of T, $y \neq z$.

Then $d(y,z) = d(Ty,Tz) \leq kd(y,z)$. Since $y \neq z$, then $d(y,z) > 0$ and thus dividing both sides by $d(y,z)$ we get $k \geq 1$, a contradiction.

Thus T has a unique fixed point.

Some writers have attempted to generalize the Banach contraction principle by replacing the Lipschitz constant k by some real-valued function whose values are less than 1. The following definition of a contractive type mapping is due to Rakotch [25].

Definition 1.2.2 There exists a monotone decreasing function $\alpha : (0,\infty) \to [0,1)$ such that for each $x,y \in X$, $x \neq y$

$$d(Tx,Ty) \leq \alpha(d(x,y))d(x,y).$$

Rakotch [25] gives the following theorem.

Theorem 1.2.2 Let X be a complete metric space. If $T : X \to X$ satisfies Definition 1.2.2, then T has a unique fixed point.

Contractive mappings, defined as follows, are also more general than contraction mappings.

Definition 1.2.3 A mapping $T : X \to X$ is called contractive if $d(Tx,Ty) < d(x,y)$ for all $x,y \in X$, $x \neq y$.

Like contractions, contractive mappings are continuous. Completeness of the space, however, is not enough to ensure the existence of a fixed point. Edelstein [11] has shown that compactness of X will guarantee a unique fixed point for a contractive mapping on X. This follows as a corollary to the following theorem of Edelstein [11].

Theorem 1.2.3 Let T be a contractive self-mapping on a metric space X and $x \in X$ such that the sequence of iterates $\{T^n x\}$ has a subsequence $\{T^{n_i} x\}$ which converges to a point $z \in X$. Then z is the unique fixed point of T.

Remark. If X is compact then each sequence has a convergent subsequence. Thus if X is compact and $T : X \to X$ is contractive, T has a unique fixed point.

Nonexpansive mappings are more general than contractive mappings.

Definition 1.2.4 A mapping $T : X \to X$ is said to be nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in X$.

Some of the useful properties of contractive mappings do not carry over to the nonexpansive case. The existence of a fixed point does not ensure its uniqueness; for example, the identity map on a metric space has every point fixed. The sequence of iterates need not converge to a fixed point, even in a compact space.

Cheney and Goldstein [8] have given the following result for a nonexpansive mapping in a metric space.

Theorem 1.2.4 Let X be a metric space and $T : X \to X$ a mapping such that

(i) T is nonexpansive

(ii) If $Tx \neq x$, then $d(T^2x,Tx) < d(x,Tx)$

(iii) for each $x \in X$, the sequence $\{x_n\} = \{T^n x\}$ has a cluster point.

Then for each x, the sequence $\{x_n\}$ converges to a fixed point of T.

The previous definitions involve only the term $d(x,y)$ on the right. Kannan [20] studied the following mapping which involves the terms $d(x,Tx)$ and $d(y,Ty)$.

Definition 1.2.5 There exists a number a, $0 < a < \frac{1}{2}$, such that for each $x,y \in X$

$$d(Tx,Ty) \leq a[d(x,Tx) + d(y,Ty)].$$

A mapping satisfying Definition 1.2.5 need not be continuous. For example $T : [0,1] \to [0,1]$ defined by $Tx = \begin{cases} \frac{x}{4}, & x \in [0, \frac{1}{2}) \\ \frac{x}{5}, & x \in [\frac{1}{2}, 1], \end{cases}$

satisfies Definition 1.2.5, but is discontinuous at $x = \frac{1}{2}$.

Kannan [20] showed that if T is a selfmapping of a complete metric space X satisfying Definition 1.2.5, then T has a unique fixed point.

Further definitions using $d(x,Tx)$ and $d(y,Ty)$ have been considered by Bianchini [2].

Definition 1.2.6 There exists a number h, $0 \leq h \leq 1$ such that for each $x,y \in X$

$$d(Tx,Ty) \leq h \max \{d(x,Tx), d(y,Ty)\}.$$

Definition 1.2.7 For each $x,y \in X$, $x \neq y$

$$d(Tx,Ty) < \max \{d(x,Tx), d(y,Ty)\}.$$

Reich [28] considered the following definition which involves all three of $d(x,y)$, $d(x,Tx)$ and $d(y,Ty)$, Definition 1.2.8 There exist nonnegative numbers a_1, a_2, a_3 satisfying $\sum_{i=1}^{3} a_i < 1$ such that for each $x, y \in X$

$$d(Tx,Ty) \leq a_1 d(x,Tx) + a_2 d(y,Ty) + a_3 d(x,y).$$

This definition can be further generalized by replacing the constants a_i by nonnegative functions α_i.

Definition 1.2.9 There exist monotonically decreasing functions $\alpha_1, \alpha_2, \alpha_3$ from $(0,\infty)$ to $[0,1)$ satisfying $\sum_{i=1}^{3} \alpha_i(t) < 1$ such that for each $x, y \in X$, $x \neq y$

$$d(Tx,Ty) \leq \alpha_1 d(x,Tx) + \alpha_2 d(y,Ty) + \alpha_3 d(x,y)$$

where $\alpha_i = \alpha_i(d(x,y))$.

Another definition involving these three terms has been considered by Sehgal [32].

Definition 1.2.10 For each $x, y \in X$, $x \neq y$

$$d(Tx,Ty) < \max\{d(x,Tx), d(y,Ty), d(x,y)\}.$$

The terms $d(x,Ty)$ and $d(y,Tx)$ are introduced in the following definitions which have been studied by Chatterjea [7].

Definition 1.2.11 There exist a number a, $0 < a < \frac{1}{2}$ such that for each $x, y \in X$,

$$d(Tx,Ty) \leq a[d(x,Ty) + d(y,Tx)].$$

Definition 1.2.12 For each $x, y \in X$, $x \neq y$

$$d(Tx,Ty) < \max\{d(x,Ty), d(y,Tx)\}.$$

He has also given definitions similar to Definitions 1.2.8 and 1.2.9 considered by Reich [28] and Definition 1.2.10 considered by Sehgal [32].

replacing $d(x,Tx)$ and $d(y,Ty)$ by $d(x,Ty)$ and $d(y,Tx)$.

Hardy and Rogers [12] have given a definition using all five of $d(x,y)$, $d(x,Tx)$, $d(y,Ty)$, $d(x,Ty)$ and $d(y,Tx)$.

Definition 1.2.13 There exist nonnegative constants a_i satisfying $\sum_{i=1}^{5} a_i < 1$ such that for each $x,y \in X$

$$d(Tx,Ty) \leq a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty) + a_4 d(x,Ty) + a_5 d(y,Tx).$$

Ćirić [10] has generalized this definition by replacing the constants a_i by functions α_i.

Definition 1.2.14 There exist monotonically decreasing functions $\alpha_i : (0,\infty) \to [0,1)$ satisfying $\sum_{i=1}^{5} \alpha_i(t) < 1$ such that for each $x,y \in X$ $x \neq y$,

$$d(Tx,Ty) \leq \alpha_1 d(x,Tx) + \alpha_2 d(y,Ty) + \alpha_3 d(x,Ty) + \alpha_4 d(y,Tx) + \alpha_5 d(x,y)$$

where $\alpha_i = \alpha_i(d(x,y))$.

He has also given a definition involving five terms corresponding to Definition 1.2.10 given for three terms.

1.3 Generalizations of definitions of contractive type mappings.

Further generalizations of the definitions of Section 1.2 can be obtained by considering functions T with the property that some iterate of T satisfies one of the preceding definitions. For example, the following is a generalization of Definition 1.2.1 (contraction).

Definition 1.3.1 There exists a positive integer n and a number k, $0 \leq k < 1$ such that for each $x,y \in X$

$$d(T^n x, T^n y) \leq k d(x,y).$$

Chu and Diaz [9] have given the following generalization of the Banach contraction principle.

Theorem 1.3.1 If T is a mapping of a complete metric space X into itself satisfying Definition 1.3.1, then T has a unique fixed point.

Proof: Since T^n is a contraction, by the Banach contraction principle T^n has a unique fixed point.
Let $T^n x_o = x_o$. Then $TT^n x_o = Tx_o$.
i.e. $T^n Tx_o = Tx_o$. Thus Tx_o is a fixed point of T^n,
but T^n has a unique fixed point. Therefore $Tx_o = x_o$.
Thus x_o is a unique fixed point of T. It is unique since any fixed point of T is also a fixed point of T^n.

Remark: It is easily seen that there is no need to assume that T^n is a contraction and is defined on a complete metric space. All that is needed in obtaining the conclusion of the theorem is that T^n has a unique fixed point.

The following example illustrates the theorem.

Let $T : R \to R$ be defined by $Tx = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

Then $T^2 x = 1$ for all x. Thus T^2 is a contraction map and hence T^2 has a unique fixed point $(T^2(1) = 1)$ and thus T has a unique fixed point.

The following is a generalization of Kannan's [20] Definition 1.2.5, which was considered by Singh [34].

Definition 1.3.2 There exists a positive integer n and a number a, $0 < a < \frac{1}{2}$, such that for each $x, y \in X$

$$d(T^n x, T^n y) \leq a[d(x,T^n x) + d(y,T^n y)].$$

It may happen that the particular iterate of T depends on the point in the space, which will give further generalizations of the definitions in Section 1.2. The following generalization of Definition 1.2.1 (contraction) has been considered by Sehgal [31].

Definition 1.3.3 There exist a positive integer $n = n(x)$ for each $x \in X$ and a number $k < 1$ such that

$$d(T^n x, T^n y) \leq kd(x,y).$$

Sehgal [31] has given the following theorem.

Theorem 1.3.2 Let X be a complete metric space and $T : X \to X$ a continuous mapping satisfying Definition 1.3.3. Then T has a unique fixed point.

Remark: The condition of continuity, however, is not a necessary one.

The iterate of T may depend on both x and y, giving more general definitions. A further generalization has been given by Holmes [14].

Definition 1.3.4 There exists a positive integer $n = n(x,y)$ such that $d(T^n x, T^n y) \leq kd(x,y)$.

Theorem 1.3.3 (Holmes [14]) If $T : X \to X$ is a continuous function on a complete metric space X satisfying Definition 1.3.4, then T has a unique fixed point.

1.4 Pairs of mappings

Eldon Dyer in 1954, Allen Shields in 1955, and Lester Dubins in 1956, considered the question of when two maps have a common fixed point. They considered $f : [0,1] \to [0,1]$ and $g : [0,1] \to [0,1]$ where f and g were continuous. The question: "If $fg(x) = gf(x)$ for all $x \in [0,1]$, do f and g have a common fixed point?" was a source of inspiration for a decade. However, the conjecture that they did have a fixed point was disproved independently by Boyce [4] and Huneke [15] in 1967.

A number of definitions exist which define a contractive type condition between a pair of maps $T,S : X \to X$. Applying this idea, generalizations of the definitions for one map in Section 1.2 are obtained. The definitions for one map are special cases of the definitions for two maps, obtained by taking $S = T$ in the new definitions. Results have been given for common fixed points of two maps using these definitions.

For example, Kannan [20] considered two maps S and T which satisfy the following condition.

Definition 1.4.1 There exists a number a, $0 < a < \frac{1}{2}$, such that for each $x,y \in X$,

$$d(Sx,Ty) \leq a[d(x,Sx) + d(y,Ty)].$$

The following theorem is due to Kannan [20].

Theorem 1.4.1 If S and T are two mappings of a complete metric space X into itself satisfying Definition 1.4.1, then S and T have a unique common fixed point.

Proof: We define a sequence of elements $\{x_n\}$ on X as follows:

Let x be any element of X. Let $x_1 = Sx$; $x_2 = Tx_1$, $x_3 = Sx_2$, $x_4 = Tx_3$ and so on.

Then $d(x_1,x_2) = d(Sx,Tx_1) \le a[d(x,Sx) + d(x_1,Tx_1)]$

$$= a[d(x,Sx) + d(x_1,x_2)]$$

Therefore $d(x_1,x_2) \le \frac{a}{1-a} d(x,Sx)$.

$$d(x_2,x_3) = d(Tx_1,Sx_2) \le a[d(x_1,Tx_1) + d(x_2,Sx_2)]$$

$$= a[d(x_1,x_2) + d(x_2,x_3)]$$

Therefore $d(x_2,x_3) \le \frac{a}{1-a} d(x_1,x_2) \le (\frac{a}{1-a})^2 d(x,Sx)$.

Similarly, $d(x_3,x_4) \le (\frac{a}{1-a})^3 d(x,Sx)$

In general, $d(x_n,x_{n+1}) \le (\frac{a}{1-a})^n d(x,Sx)$.

Therefore $d(x_n,x_{n+p}) \le d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \dots + d(x_{n+p-1},x_{n+p})$

$$\le (r^n + r^{n+1} + \dots + r^{n+p-1}) d(x,Sx)$$

where $r = \frac{a}{1-a}$

$$< \frac{r^n}{1-r} d(x,Sx).$$

Since $0 < a < \frac{1}{2}$ we have $0 < r < 1$.

Therefore $d(x_n,x_{n+p}) \to 0$ as $n \to \infty$.

Therefore the sequence is Cauchy. Since X is complete, there exists an element $z \in X$ which is the limit of this sequence, i.e. $z = \lim_{n\to\infty} x_n$.

We now show $Sz = z = Tz$.

Now $d(z,Sz) \le d(z,x_n) + d(x_n,Sz)$

$$= d(z,x_n) + d(Tx_{n-1},Sz)$$

where we choose n to be an even positive integer.

Therefore $d(z,Sz) \leq d(z,x_n) + a[d(x_{n-1},Tx_{n-1}) + d(z,Sz)]$.

Therefore $(1 - a)d(z,Sz) \leq d(z,x_n) + ad(x_{n-1},x_n)$. (1)

If $\varepsilon > 0$ is arbitrary then for all sufficiently large values of n we have $d(z,x_n) < \varepsilon \frac{1 - a}{1 + a}$ and $d(x_{n-1},x_n) < \varepsilon \frac{1 - a}{1 + a}$.

From (1), for all sufficiently large even values of n,

$$d(z,Sz) \leq \frac{d(z, x_n) + ad(x_{n-1},x_n)}{1 - a}$$

$$< \frac{\varepsilon}{1 + a} + \frac{a\varepsilon}{1 + a} = \varepsilon.$$

Therefore $z = Sz$ since $\varepsilon > 0$ is arbitrary.

In a similar way it can be shown that $z = Tz$.

Therefore $Sz = Tz = z$.

It is easy to see that z is the unique common fixed point of S and T. For if y is a point of X such that $y = Sy = Ty$ then

$$d(z,y) = d(Sz,Ty) \leq a[d(z,Sz) + d(y,Ty)] = 0.$$

Therefore $z = y$.

<u>Remark</u>: Under the conditions of the theorem, S and T have only one fixed point, namely z. For if y is any point of X with $Sy = y$ then

$d(z,y) = d(Tz,Sy)$

$\leq a[d(z,Tz) + d(y,Sy)] = 0.$

Therefore $y = z$. Similarly for T.

Kannan [20] also showed that a unique common fixed point for S and T can be obtained under different conditions.

<u>Theorem 1.4.2</u> Let S and T be mappings of a complete metric space X into itself such that

(i) $d(Sx,Ty) \leq ad(x,y)$ for all $x,y \in X$, $x \neq y$, $0 \leq a < 1$.

(ii) T is a contraction mapping.

(iii) there exists some $x \in X$ such that if $x_1 = Sx, x_2 = Tx_1$, $x_3 = Sx_2$ and so on, if $r \neq s$ then $x_r \neq x_s$.

Then S and T have a common fixed point.

This theorem has been improved by Singh [34] by omitting condition (ii).

In a similar vein, Srivastava and Gupta [40] have considered pairs of mappings S and T which satisfy the following definition.

Definition 1.4.2 There exist positive numbers a_1 and a_2, $a_1 + a_2 < 1$ such that for each $x,y \in X$,

$$d(S^p x, T^q y) \leq a_1 d(x, S^p x) + a_2 d(y, T^q y)$$

where p and q are positive integers.

Theorem 1.4.3 (Srivastava and Gupta [40]) If S and T are two mappings of a complete metric space into itself satisfying Definition 1.4.2, then S and T have a unique and common fixed point.

C.S. Wong [41] has considered a definition for two maps which generalizes Hardy and Rogers [12] Definition 1.2.13.

Definition 1.4.3 There exist nonnegative constants a_i satisfying $\sum_{i=1}^{5} a_i < 1$ and $a_1 = a_2$ or $a_3 = a_4$ such that for each $x,y \in X$

$$d(Sx,Ty) \leq a_1 d(x,Sx) + a_2 d(y,Ty) + a_3 d(x,Ty) + a_4 d(y,Sx) + a_5 d(x,y).$$

Wong [41] has given the following theorem.

Theorem 1.4.4 Let S and T be mappings of a complete metric space (X,d) into itself satisfying Definition 1.4.3, then S and T have a common unique fixed point.

Wong [41] has generalized this further by following the same line as Ciric's Definition 1.2.14.

Definition 1.4.4 There exist decreasing functions $\alpha_i : (0,\infty) \to [0,1)$ satisfying

(a) $\sum_{i=1}^{5} \alpha_i < 1$

(b) $\alpha_1 = \alpha_2$ or $\alpha_3 = \alpha_4$

(c) $\lim_{t \downarrow 0} (\alpha_2 + \alpha_3) < 1$ and $\lim_{t \downarrow 0} (\alpha_1 + \alpha_4) < 1$

such that for each $x, y \in X$, $x \neq y$,

$$d(Sx,Ty) \leq \alpha_1 d(x,Sx) + \alpha_2 d(y,Ty) + \alpha_3 d(x,Ty) + \alpha_4 d(y,Sx) + \alpha_5 d(x,y)$$

where $\alpha_i = \alpha_i(d(x,y))$.

Theorem 1.4.5 (Wong [41]) Let S and T be functions on a nonempty complete metric space (X,d) satisfying Definition 1.4.4. Then at least one of S and T has a fixed point. If both S and T have fixed points, then each of S,T has a unique fixed point and these two fixed points coincide.

1.5 Mappings of a different nature and fixed points

M.S. Khan (private communication) has considered mappings of the following type.

Definition 1.5.1 There exists a number k, $0 \leq k < 1$, such that for each $x, y \in X$,

$$d(Tx,Ty) \leq k \left[\frac{d(x,Tx)d(x,Ty) + d(y,Tx)d(y,Ty)}{d(x,Ty) + d(y,Tx)}\right], \text{ denominator} \neq 0.$$

We give some results for fixed points of mappings satisfying Definition 1.5.1.

Theorem 1.5.1 Let X be a complete metric space and let $T : X \to X$ satisfy Definition 1.5.1. Then T has a unique fixed point.

Proof: Let x_o be an arbitrary point of X. Construct a sequence $\{x_n\}$ setting $x_{n+1} = Tx_n$. We show that $\{x_n\}$ is a Cauchy sequence.
Since T satisfies Definition 1.5.1, we have

$$\begin{aligned} d(x_1,x_2) &= d(Tx_o,Tx_1) \\ &\leq k\left[\frac{d(x_o,Tx_o)d(x_o,Tx_1) + d(x_1,Tx_o)d(x_1,Tx_1)}{d(x_o,Tx_1) + d(x_1,Tx_o)}\right] \\ &= k\left[\frac{d(x_o,x_1)d(x_o,x_2) + d(x_1,x_1)d(x_1,x_2)}{d(x_o,x_2) + d(x_1,x_1)}\right] \\ &= k\left[\frac{d(x_o,x_1)d(x_o,x_2)}{d(x_o,x_2)}\right] \\ &= kd(x_o,x_1). \end{aligned}$$

Similarly, $d(x_2,x_3) \leq kd(x_1,x_2) \leq k^2 d(x_o,x_1)$, $d(x_3,x_4) \leq k^3 d(x_o,x_1)$ and so on.

In general, $d(x_n,x_{n+1}) \leq k^n d(x_o,x_1)$. (1)

Consider $d(x_n,x_m)$ where $m > n$. Using the triangle inequality and (1), we have

$$d(x_n,x_m) < \frac{k^n}{1-k}\, d(x_o,x_1).$$

Since $k < 1$, $d(x_n,x_m) \to 0$ as $n \to \infty$.

Thus $\{x_n\}$ is a Cauchy sequence, and since X is complete $\{x_n\}$ converges to a point $y \in X$.

We show that y is a fixed point of T. Using the triangle inequality and Definition 1.5.1, we have,

$$\begin{aligned} d(y,Ty) &\leq d(y,T^{n+1}y) + d(T^{n+1}x,Ty) \\ &\leq d(y,T^{n+1}y) + k\left[\frac{d(T^n x,T^{n+1}x)d(T^n x,Ty) + d(y,T^{n+1}x)d(y,Ty)}{d(T^n x,Ty) + d(y,T^{n+1}x)}\right]. \end{aligned}$$

As $n \to \infty$, we have

$$\begin{aligned} d(y,Ty) &\leq d(y,y) + k\left[\frac{d(y,y)d(y,Ty) + d(y,y)d(y,Ty)}{d(y,Ty) + d(y,y)}\right] \\ &= 0, \end{aligned}$$

i.e. $d(y,Ty) \to 0$ as $n \to \infty$ and thus $Ty = y$.

To see that y is the unique fixed point, let z be another fixed point of T.

$$\begin{aligned} \text{Then } d(z,y) = d(Tz,Ty) &\leq k\left[\frac{d(z,Tz)d(z,Ty) + d(y,Tz)d(y,Ty)}{d(z,Ty) + d(y,Tz)}\right], \\ &= 0. \end{aligned}$$

Therefore $z = y$.

A corresponding definition to Definition 1.5.1 can be given for a pair of maps.

<u>Definition 1.5.2</u> There exists an integer k, $0 \leq k < 1$ such that for each $x,y \in X$.

$$d(Sx,Ty) \leq k\left[\frac{d(x,Sx)d(x,Ty) + d(y,Sx)d(y,Ty)}{d(x,Ty) + d(y,Sx)}\right].$$

We give the following theorem.

Theorem 1.5.2 If X is a complete metric space and $S,T : X \to X$ are maps satisfying Definition 1.5.2, then S and T have a unique common fixed point.

Proof: Let x_o be an arbitrary point of X. Let $x_1 = Sx_o$, $x_2 = Tx_1$, $x_3 = Sx_2$ and so on.

Then $d(x_1,x_2) = d(Sx_o,Tx_1)$

$$\leq k\left[\frac{d(x_o,Sx_o)d(x_o,Tx_1) + d(x_1,Sx_o)d(x_1,Tx_1)}{d(x_o,Tx_1) + d(x_1,Sx_o)}\right]$$

$$= k\left[\frac{d(x_o,x_1)d(x_o,x_2) + d(x_1,x_1)d(x_1,x_2)}{d(x_o,x_2) + d(x_1,x_1)}\right]$$

$$= k\left[\frac{d(x_o,x_1)d(x_o,x_2)}{d(x_o,x_2)}\right]$$

$$= kd(x_o,x_1).$$

Similarly $d(x_2,x_3) \leq kd(x_1,x_2)$

i.e. $d(x_2,x_3) \leq k^2 d(x_o,x_1)$.

In general, $d(x_n,x_{n+1}) \leq k^n d(x_o,x_1)$.

Therefore $d(x_n,x_{n+p}) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \ldots + d(x_{n+p-1},x_{n+p})$

$$\leq (k^n + k^{n+1} + \ldots + k^{n+p-1})d(x_o,x_1)$$

$$< \frac{k^n}{1-k}\, d(x_o,x_1).$$

Since $0 \leq k < 1$, $d(x_n,x_{n+p}) \to 0$ as $n \to \infty$.

Thus $\{x_n\}$ is a Cauchy sequence, and since X is complete $\{x_n\}$ converges to a point $y \in X$.

We show that y is a fixed point of S.

Let n be even. Using the triangle inequality we have,

$$d(y,Sy) \le d(y,x_n) + d(x_n,Sy)$$
$$= d(y,x_n) + d(Tx_{n-1},Sy).$$

Using Definition 1.5.2, we have,

$$d(y,Sy) \le d(y,x_n) + k\left[\frac{d(y,Sy)d(y,Tx_{n-1}) + d(x_{n-1},Sy)d(x_{n-1},Tx_{n-1})}{d(y,Tx_{n-1}) + d(x_{n-1},Sy)}\right].$$

Therefore $d(y,Sy) - \dfrac{kd(y,Sy)d(y,x_n)}{d(y,x_n) + d(x_{n-1},Sy)} \le d(y,x_n) + \dfrac{kd(x_{n-1},Sy)d(x_{n-1},x_n)}{d(y,x_n) + d(x_{n-1},Sy)}$.

Multiplying by $d(y,x_n) + d(x_{n-1},Sy)$ we have,

$$d(y,Sy)[d(y,x_n) + d(x_{n-1},Sy) - kd(y,x_n)] \le [d(y,x_n)]^2 + d(y,x_n)d(x_{n-1},Sy) + kd(x_{n-1},Sy)d(x_{n-1},x_n).$$

Thus,

$$d(y,Sy) \le \frac{[d(y,x_n)]^2 + d(y,x_n)d(x_{n-1},Sy) + kd(x_{n-1},Sy)d(x_{n-1},x_n)}{d(y,x_n) + d(x_{n-1},Sy) - kd(y,x_n)}. \qquad (1)$$

Since $d(x_{n-1},x_n) \to 0$ as $n \to \infty$ and $d(y,x_n) \to 0$ as $n \to \infty$, using (1) we have $d(y,Sy) \to 0$ as $n \to \infty$,

i.e. $y = Sy$.

Similarly, y is a fixed point of T.

To see that y is a unique common fixed point of S and T, let z be another common fixed point.

Then
$$d(z,y) = d(Sz,Ty)$$
$$\le k\left[\frac{d(z,Sz)d(z,Ty) + d(y,Sz)d(y,Ty)}{d(z,Ty) + d(y,Sz)}\right]$$
$$= k\left[\frac{d(z,z)d(z,y) + d(y,z)d(y,y)}{d(z,y) + d(y,z)}\right]$$
$$= 0.$$

Therefore $z = y$.

We now consider mappings of the following type.

Definition 1.5.3 $T : X \to X$ is a mapping satisfying

$$d(Tx,Ty) < \frac{d(x,Tx)d(x,Ty) + d(y,Tx)d(y,Ty)}{d(x,Ty) + d(y,Tx)}$$

for all $x,y \in X$, $x \neq y$.

We give the following theorem:

Theorem 1.5.3 Let X be a metric space and $T : X \to X$ be a continuous mapping satisfying Definition 1.5.3.
Let $x_o \in X$ be such that the sequence of iterates $\{T^n x_o\}$ has a convergent subsequence $\{T^{n_i} x_o\}$ converging to a point $z \in X$. Then z is the unique fixed point of T.

Proof: Since $\{T^{n_i} x_o\}$ converges to $z \in X$ and T is continuous, then the sequence $\{T^{n_i+1} x_o\}$ converges to Tz and consequently the sequence $\{T^{n_i+2} x_o\}$ converges to T^2z. Consider the sequence $\{d(T^n x_o, T^{n+1} x_o)\}$ of non-negative real numbers. Since T satisfies Definition 1.5.3, we have, $d(x_1,x_2) = d(Tx_o,Tx_1)$

$$< \frac{d(x_o,Tx_o)d(x_o,Tx_1) + d(x_1,Tx_o)d(x_1,Tx_1)}{d(x_o,Tx_1) + d(x_1,Tx_o)}$$

$$= \frac{d(x_o,x_1)d(x_o,x_2) + d(x_1,x_1)d(x_1,x_2)}{d(x_o,x_2) + d(x_1,x_1)}$$

$$= d(x_o,x_1).$$

Similarly, $d(x_2,x_3) < d(x_1,x_2)$.
Therefore $d(x_2,x_3) < d(x_o,x_1)$.
In general, $d(x_n,x_{n+1}) < d(x_{n-1},x_n) \leq \ldots < d(x_o,x_1)$

i.e. $d(T^n x_o, T^{n+1} x_o) < \ldots < d(x_o,Tx_o)$

Thus $\{d(T^n x_o, T^{n+1} x_o)\}$ is a sequence of real numbers, monotone decreasing, bounded below by 0 and hence it has a limit, say α.

Now, $d(z,Tz) = \lim_{n\to\infty} d(T^{n_i}x_o, T^{n_i+1}x_o)$

$= \alpha$

$= \lim_{n\to\infty} d(T^{n_j+1}x_o, T^{n_i+2}x_o)$

$= d(Tz, T^2z)$.

If $z \neq Tz$, then $d(Tz,T^2z) < \dfrac{d(z,Tz)d(z,T^2z) + d(Tz,Tz)d(Tz,T^2z)}{d(z,T^2z) + d(Tz,Tz)}$

$= d(z,Tz)$.

Thus $d(z,Tz) < d(z,Tz)$, a contradiction.

Therefore $z = Tz$.

To see that z is unique, let y be another fixed point of T, $y \neq z$. Then

$$d(y,z) = d(Ty,Tz) < \frac{d(y,Ty)d(y,Tz) + d(z,Ty)d(z,Tz)}{d(y,Tz) + d(z,Ty)}$$

$$= \frac{d(y,y)d(y,z) + d(z,y)d(z,z)}{d(y,z) + d(z,y)}$$

$$= 0,$$

i.e. $d(y,z) < 0$, a contradiction. Therefore $y = z$.

Corollary 1.5.1 If X is a compact metric space and T as in theorem, then T has a unique fixed point.

CHAPTER II

Extension of Contraction Principle and Fixed Point Theorems

In the first section of this chapter we look at generalizations of the Banach contraction principle involving the introduction of a non-decreasing function $\phi : R^+ \to R^+$. This type of condition has been studied by Rakotch [25], Browder [6], and Boyd and Wong [5]. In section two, this idea has been extended to pairs of mappings and at the end, a result of Husain and Sehgal [16] has been improved. In section three, we look at fixed points for mappings with a commuting condition. At the end of this section, a result of Jungck [19] has been improved.

2.1 Non-linear contractions and fixed points

Rakotch [25] generalized the Banach contraction principle by replacing the Lipschitz constant k by some real-valued function whose values are less than one. At first only one constant was involved; but as new terms $d(x,Tx)$, $d(y,Ty)$, $d(x,Ty)$ and $d(y,Tx)$ were introduced, new definitions involving constants multiplied by these terms could be generalized using the idea of replacing the constant by some real-valued function with appropriate properties.

In a similar vein, Browder [6] and Boyd and Wong [5] consider contraction type mappings given by the following definition.

Definition 2.1.1. There exists a right-continuous, non-decreasing function ϕ satisfying $\phi(t) < t$ for $t > 0$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \phi(d(x,y)).$$

The following theorem has been given by Browder [6].

Theorem 2.1.1. Let X be a complete metric space, M a bounded subset of X, T a mapping of M into M. Suppose that there exists a monotone decreasing function $\phi(t)$ for $t \geq 0$, with ϕ continuous on the right such that $\phi(t) < t$ for all $t > 0$, while for all $x, y \in M$, $d(Tx,Ty) \leq \phi(d(x,y))$. Then for each x_0 in M, $T^n x_0$ converges to an element z of X, independent of x_0 and $d(T^n x_0, z) \leq \phi^n(d_0)$ where d_0 is the diameter of M, ϕ^n is the n^{th} iterate of ϕ and $d_n = \phi^n(d_0) \to 0$ as $n \to +\infty$.

Corollary 2.1.1. Under the hypotheses of Theorem 2.1.1, T can be extended in one and only one way to a continuous mapping of the closure of M in X into itself and z is the unique fixed point of this extended mapping.

Boyd and Wong [5] proved the following theorem.

Theorem 2.1.2. Let X be a complete metric space and let T be a mapping from X into itself such that $d(Tx,Ty) \leq \phi(d(x,y))$ where $\phi : \bar{P} \to [0,\infty)$ is upper-semicontinuous from the right on $\bar{P}$ ($\bar{P}$ is the closure of $P = \{d(x,y) \mid x,y \in X\}$ and satisfies $\phi(t) < t$ for all $t \in P - \{0\}$.) Then T has a unique fixed point x_0 and $T^n x \to x_0$ as $n \to \infty$ for each $x \in X$.

Proof. Given $x \in X$, define $c_n = d(T^n x, T^{n-1} x)$.
Then c_n decreases to 0. For, since $d(Tx,Ty) \leq \phi(d(x,y))$, c_n is decreasing and hence has a limit c. But if $c > 0$, we have $c_{n+1} \leq \phi(c_n)$ so that

$$c \leq \limsup_{t \to c+} \phi(t) \leq \phi(c)$$

which is a contradiction.

Now, we show that for each $x \in X$, $\{T^n x\}$ is a Cauchy sequence. This will complete the proof, since the limit of this sequence is a fixed point of T which is clearly unique. Suppose that $\{T^n x\}$ is not a Cauchy sequence. Then there is an $\varepsilon > 0$ and sequences of integers $\{m(k)\}$, $\{n(k)\}$, with $m(k) > n(k) \geq k$ and such that

$$d_k = d(T^m x, T^n x) \geq \varepsilon \quad \text{for } k = 1,2,3, \ldots \tag{1}$$

We may also assume that $d(T^{m-1}x, T^n x) < \varepsilon$ (2)

by choosing $m(k)$ to be the smallest number exceeding $n(k)$ for which (1) holds.

Using $c_n = d(T^n x, T^{n-1} x)$, we have

$$\begin{aligned} d_k &\leq d(T^m x, T^{m-1} x) + d(T^{m-1} x, T^n x) \leq c_m + \varepsilon \\ &\leq c_k + \varepsilon . \end{aligned} \tag{3}$$

Hence $d_k \to \varepsilon^+$ as $k \to \infty$.

But now,

$$\begin{aligned} d_k &= d(T^m x, T^n x) \\ &\leq d(T^m x, T^{m+1} x) + d(T^{m+1} x, T^{n+1} x) + d(T^{n+1} x, T^n x) \\ &\leq 2c_k + \phi(d(T^m x, T^n x)) \\ &= 2c_k + \phi(d_k) \end{aligned} \tag{4}$$

Thus as $k \to \infty$ in (4) we obtain $\varepsilon \leq \phi(\varepsilon)$ which is a contradiction for $\varepsilon > 0$.

Remark: Boyd and Wong [5] have also shown that if X is metrically convex (that is, $x,y \neq z$ for each $x,y \in X$, $d(x,y) = d(x,z) + d(z,y)$) then the upper-semicontinuous condition can be dropped from the above theorem.

Remark: If $\phi(t) = \alpha(t)t$ where α is decreasing and $\alpha(t) < 1$ for $t > 0$, then we have Theorem 1.2.2 due to Rakotch [25].

The following example, due to Meir and Keeler [21], shows that the condition "$d(Tx,Ty) \leq \phi(d(x,y))$ where ϕ is defined on the closure of the range of d and $\phi(t) < t$ for all $t > 0$" can be satisfied in a complete metric space without T having a fixed point.

Example 2.1.1 Let $S_n = \sum_{k=1}^{n} (1 + \frac{1}{k})$ and let $X = \{S_n\}$. Then X is complete. Let $T(S_n) = S_{n+1}$ for all n.
Then $d(Tx,Ty) \leq \phi(d(x,y))$ with $\phi(1 + \frac{1}{n}) = 1 + \frac{1}{n+1}$.
However, T has no fixed point.

2.2 Pairs of Mappings

Extensions of Definition 2.1.1 have been considered for two maps S and T and for three terms: $d(x,y)$, $d(x,Tx)$, $d(y,Sy)$ and for five terms: $d(x,y)$, $d(x,Tx)$, $d(y,Sy)$, $d(x,Sy)$ and $d(y,Tx)$.

Sehgal [33] considers pairs of maps of the following type.

Definition 2.2.1 There exists some real-valued function ϕ defined on a subset of $R^+ \times R^+ \times R^+$ such that

$$d(Tx,Sy) \leq \phi(d(x,Tx), d(y,Sy), d(x,y)) \text{ for all } x,y \in X.$$

Let (X,d) be a complete metric space, Q the closure of the set $\{d(x,y) \mid x,y \in X\}$ and $P = Q \times Q \times Q$.

Definition 2.2.2 A function $\phi : P \to R^+$ is right continuous iff $(a_{n_1}, a_{n_2}, a_{n_3})$, $(a_1,a_2,a_3) \in P$ and $a_{n_k} \downarrow a_k$ $k = 1,2,3$, ($\downarrow$ = decreasing) then

$$\phi(a_{n_1}, a_{n_2}, a_{n_3}) \to \phi(a_1, a_2, a_3).$$

Definition 2.2.3 The function ϕ will be called symmetric iff $\phi(a,b,c) = \phi(b,a,c)$ for all $(a,b,c) \in P$.

Definition 2.2.4 The mappings T_i $(i = 1,2)$ satisfy a (I_1, I_2, ϕ, k) functional inequality if and only if for each $i (i = 1,2)$ there is a mapping $I_i : T_i \times X \to I^+$ (positive integers) such that if $n(x) = I_1(T_1, x)$ and $m(x) = I_2(T_2, x)$ then

$$d(T_1^{n(x)}x, T_2^{m(y)}y) \le k\phi(d(x, T^{n(x)}x), d(y, T_2^{m(y)}y), d(x,y))$$

for all $x, y \in X$, where k is some real constant and $\phi : P \to R^+$ is a symmetric right-continuous function.

Sehgal [33] proves the following theorems:

Theorem 2.2.1 Let the mappings $T_i : X \to X$ $(i = 1,2)$ satisfy a (I_1, I_2, ϕ, k) functional inequality for some $k < 1$. If (i) $\phi(a,b,a) \le \max\{a,b\}$, $(a,b,a) \in P$, then there exists a $z \in X$ such that $T_1^{n(z)}z = T_2^{m(z)}z = z$. If (ii) $\phi(0,0,a) \le a$ for each $a \in Q$, then z is unique.

Corollary 2.2.1 Let the mappings $T_i : X \to X$ $(i = 1,2)$ satisfy either of the following conditions:

(1) $d(T_1^{n(x)}x, T_2^{m(y)}y) \le k \max\{d(x, T_1^{n(x)}x), d(y, T^{m(y)}y), d(x,y)\}$ for some $k < 1$

(2) $d(T_1^{n(x)}x, T_2^{m(y)}y) \le a_1 d(x, T_1^{n(x)}x) + a_2 d(y, T_2^{m(y)}y) + a_3 d(x,y)$, for some nonnegative reals $a_i (i=1,2,3)$ satisfying $a_1 + a_2 + a_3 < 1$.

Then there exists a unique $z \in X$ such that $T_1^{n(z)}z = T_2^{m(z)}z = z$.

Theorem 2.2.2 Let for some positive integers m and n, the mappings $T_i : X \to X$ $(i = 1,2)$ satisfy for all $x,y \in X$

$$d(T_1^n x, T_2^m y) \leq k\phi(d(x,T_1^n x),\ d(y,T_2^m y),\ d(x,y))$$

where $k < 1$ and the function $\phi : P \to R^+$ is symmetric and right continuous. If ϕ satisfies condition (i) and (ii) of Theorem 2.2.1, then T_i $(i = 1,2)$ have a unique common fixed point $z \in X$.

Corollary 2.2.2 Let for some positive integers m and n, the mappings $T_i : X \to X$ satisfy the condition:

$$d(T_1^n x, T_2^m y) \leq k \max\{d(x,T_1^n x),\ d(y,T_2^m y),\ d(x,y)\}$$

for $k < 1$ and for all $x,y \in X$. Then T_i $(i = 1,2)$ have a unique common fixed point in X.

Corollary 2.2.3 If for some positive integers m and n, the mappings $T_i : X \to X$ $(i = 1,2)$ satisfy the inequality

$$d(T_1^n x, T_2^m y) \leq a_1 d(x,T_1^n x) + a_2 d(y,T_2^m y) + a_3 d(x,y)$$

for some non-negative reals a_1, a_2, a_3 with $a_1 + a_2 + a_3 < 1$, then T_i $(i = 1,2)$ have a unique common fixed point in X.

Remark 1: If $n = m = 1$ and $T_1 = T_2 = T$ in Corollary 2.2.3, we obtain Definition 1.2.8 considered by Reich [28].

Remark 2: If $a_3 = 0$ in Corollary 2.2.3, we get a result of Srivastava and Gupta [40] (Theorem 1.4.3).

Continuing in this line, Husain and Sehgal [16] have given a result with ϕ acting upon five terms.

Let (X,d) be a complete metric space and let ψ denote a family of mappings such that each $\phi \in \psi, \phi : (R^+)^5 \to R^+$ and ϕ is continuous and non-decreasing in each coordinate variable.

Theorem 2.2.3 (Husain and Sehgal). Let S,T be self mappings of X. Suppose there exists a $\phi \in \psi$ such that for all $x,y \in X$

(1) $d(Sx,Ty) \leq \phi(d(x,Sx), d(y,Ty), d(x,Ty), d(y,Sx), d(x,y))$

where ϕ satisfies the condition: for any $t > 0$,

(2) $\phi(t,t,a_1t,a_2t,t) < t$ with $a_1 + a_2 = 2$.

Then there exists a $z \in X$ such that

(a) $Sz = Tz = z$ and

(b) z is the unique fixed point of each S and T.

Proof: Define a sequence $\{x_n\}$ in X as follows.
Let $x_o \in X$, $x_1 = Sx_o$, $x_2 = Tx_1$, and inductively, for each $n \in I^+$, $x_{2n-1} = Sx_{2n-2}$, $x_{2n} = Tx_{2n-1}$.
Let $d_n = d(x_n, x_{n+1})$. Since $d(x_{2n-1}, x_{2n+1}) \leq d_{2n-1} + d_{2n}$ it follows by (1) of the Theorem that, for each $n \in I^+$,

$$d_{2n} = d(Sx_{2n}, Tx_{2n-1}) \leq \phi(d_{2n}, d_{2n-1}, 0, d_{2n-1} + d_{2n}, d_{2n-1}) \quad (3)$$

Now, if for some $n \in I^+$, $d_{2n} > d_{2n-1}$, then (3) will imply that $d_{2n} \leq \phi(d_{2n}, d_{2n}, 0, 2d_{2n}, d_{2n}) < d_{2n}$, a contradiction.
Hence $d_{2n} \leq d_{2n-1}$. Similarly it follows that $d_{2n+1} \leq d_{2n}$ for each $n \in I^+$. Consequently $\{d_n\}$ is a nonincreasing sequence in R^+ and hence there is a $r \in R^+$ such that $d_n \to r$. Clearly $r = 0$, for otherwise by (3),

$$r \leq \phi(r,r,0,2r,r) < r,$$

a contradiction. Thus

$$d_n = d(x_n, x_{n+1}) \to 0. \tag{4}$$

We show that $\{x_n\}$ is a Cauchy sequence in X. In view of (4) it suffices to show that the sequence $\{x_{2n}\}$ is Cauchy. Suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then there is an $\varepsilon > 0$ such that for each even integer $2k$, $k \in I^+$, there exist integers $2n(k)$ and $2m(k)$ with $2k \leq 2n(k) < 2m(k)$ such that

$$d(x_{2n(k)}, x_{2m(k)}) > \varepsilon . \tag{5}$$

Let for each $2k$, $k \in I^+$, $2m(k)$ be the least integer exceeding $2n(k)$ satisfying (5); that is

$$d(x_{2n(k)}, x_{2m(k)-2}) \leq \varepsilon \text{ and } d(x_{2n(k)}, x_{2m(k)}) > \varepsilon. \tag{6}$$

Then for each integer $2k$, $k \in I^+$,

$$\varepsilon < d(x_{2n(k)}, x_{2m(k)}) \leq d(x_{2n(k)}, x_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}.$$

Therefore by (4) and (6) we obtain

$$d(x_{2n(k)}, x_{2m(k)}) \to \varepsilon \quad \text{as} \quad k \to \infty .$$

It now follows immediately from the triangular inequality that

$$|d(x_{2n(k)}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})| \leq d_{2m(k)-1}$$

and

$$|d(x_{2n(k)+1}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)}| \leq d_{2m(k)-1} + d_{2n(k)}$$

and hence, by (6) as $k \to \infty$,

$$d(x_{2n(k)}, x_{2m(k)-1}) \to \varepsilon, \; d(x_{2n(k)+1}, x_{2m(k)-1}) \to \varepsilon \tag{8}.$$

Let $r(2k) = d(x_{2n(k)}, x_{2m(k)})$,

$s(2k) = d(x_{2n(k)}, x_{2m(k)-1})$,

and

$t(2k) = d(x_{2n(k)+1}, x_{2m(k)-1})$.

Then, since $r(2k) \le d_{2n(k)} + d(Sx_{2n(k)}, Tx_{2m(k)-1})$,

it follows by (1) that

$$r(2k) \le d_{2n(k)} + \phi(d_{2n(k)}, d_{2m(k)-1}, r(2k), t(2k), s(2k)),$$

and hence, it follows by (2), (7) and (8) that

$$\varepsilon \le \phi(0,0,\varepsilon,\varepsilon,\varepsilon) < \varepsilon,$$

contradicting the existence of an $\varepsilon > 0$.

Consequently, $\{x_n\}$ is a Cauchy sequence and hence, by completeness, there is a $z \in X$ such that $x_n \to z$.

We show that $Sz = Tz = z$.

Now, since $x_{2n} = Tx_{2n-1}$,

$$d(Sx, x_{2n}) \le \phi(d(z,Sz), d_{2n-1}, d(z,x_{2n}), d(x_{2n-1}, Sz), d(x_{2n-1},z)).$$

Therefore as $n \to \infty$ in the above inequality, we obtain

$$d(Sz,z) \le \phi(d(z,Sz), 0,0,d(z,Sz), 0) \quad ,$$

and hence, by the nondecreasing property of ϕ, it follows that $Sz = z$. A similar argument applied to $d(x_{2n+1},Tz)$ yields $Tz = z$. This proves (a). To prove (b) suppose there is a $y \neq z$ for which $Ty = y$. Let $r = d(z,y) > 0$. Then

$$r = d(Sz,Ty) \le \phi(0,0,r,r,r) < r,$$

contradicting $r > 0$. Thus $y = z$. A similar argument shows that z is the unique fixed point of S also. This proves (b).

The condition that ϕ should be continuous can be replaced by a semicontinuity condition. Moreover, a different proof can be given, the last part of which requires no continuity or semicontinuity condition whatsoever.

Let ψ denote a family of mappings such that each $\phi \in \psi$; $\phi : (R^+)^5 \to R^+$ and ϕ is upper semicontinuous and nondecreasing in each co-ordinate variable. Also let $\gamma(t) = \phi(t,t,a_1t,a_2t,t)$ where γ is a function $\gamma : R^+ \to R^+$ where $a_1 + a_2 = 3$.

<u>Lemma 2.2.1</u> For every $t > 0$,

$$\gamma(t) < t \quad \text{iff} \quad \lim_{n\to\infty} \gamma^n(t) = 0.$$

<u>Proof</u>: (Necessity) Since ϕ is upper semicontinuous, then γ is upper-semicontinuous. Assume $\lim_{n\to\infty} \gamma^n(t) = A$ where $A \neq 0$. Then

$A = \lim_{n\to\infty} \gamma^{n+1}(t) \leq \gamma \lim_{n\to\infty} \gamma^n(t) = \gamma(A) < A,$ a contradiction.

Therefore $A = 0$.

(Sufficiency). Since ϕ is nondecreasing, then γ is nondecreasing.

Given $\lim_{n\to\infty} \gamma^n(t) = 0$, assume

$\gamma(t) > t$ for some $t > 0$.

Then $\gamma^n(t) > t$ for some $t > 0$ for $n = 1,2,3, \ldots$

Thus $\lim_{n\to\infty} \gamma^n(t) \not\to 0$, a contradiction.

Also if $\gamma(t) = t$ for some $t > 0$, then $\lim_{n\to\infty} \gamma^n(t) \not\to 0$. Hence for all $t > 0$, $\gamma(t) < t$.

<u>Theorem 2.2.4</u> Let (X,d) be a complete metric space and let S and T be selfmappings of X. Suppose there exists a $\phi \in \psi$ such that for all $x,y \in X$, $d(S(x),T(y)) \leq \phi(d(x,y), d(x,S(x)), d(x,T(y)), d(y,S(x)), d(y,T(y))$

where ϕ satisfies the condition: for any $t > 0$, (1)

$\phi(t,t,a_1t,a_2t,t) < t$ where $a_1 + a_2 = 3$. (2)

Then there exists a $z \in X$ such that z is a unique common fixed point of S and T.

Proof: Since $\gamma(t) = \phi(t,t,t,2t,t) < t$, then by Lemma 2.2.1, $\lim_{n\to\infty} \gamma^n(t) = 0$. Now let $x_o \in X$ be any point. Then define a sequence of iterates $\{x_n\}$ in the following way:

$$x_1 = S(x_o),\ x_2 = T(x_1),\ x_3 = S(x_2) \ldots\ x_{2n} = T(x_{2n-1}),\ x_{2n+1} = S(x_{2n}) \ldots$$

Claim: $d(x_1,x_2) \le d(x_o,x_1)$

Assume $d(x_o,x_1) < d(x_1,x_2)$. Then using the triangular inequality $d(x_o,x_2) < 2d(x_1,x_2)$.

Let $r = d(x_1,x_2)$. Then

$$r \le \phi(d(x_o,x_1), d(x_1,x_2), d(x_1,x_1), d(x_o,x_2), d(x_o,x_1)) \le \phi(r,r,r,2r,r)$$

By (2), $\phi(r,r,r,2r,r) < r$ and thus $r < r$, a contradiction. Therefore, $d(x_1,x_2) \le d(x_o,x_1)$ and $d(x_1,x_2) \le \phi(d(x_o,x_1), d(x_o,x_1), 0, 2d(x_o,x_1), d(x_o,x_1)) \le \gamma(d(x_o,x_1))$.

Similarly, $d(x_2,x_3) \le \gamma(d(x_1,x_2)) \le \gamma^2(d(x_o,x_1))$ and in general, $d(x_n,x_{n+1}) \le \gamma^n(d(x_o,x_1))$. Since $\lim_{n\to\infty} \gamma^n(t) = 0$ for $t > 0$, therefore

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{3}$$

In order to show $\{x_n\}$ is Cauchy, it is sufficient to show $\{x_{2n}\}$ is Cauchy. Using the same procedure as in Theorem 2.2.3, we assume that $\{x_{2n}\}$ is not Cauchy and derive a contradiction. It is not necessary to take ϕ continuous, instead the upper semicontinuity condition on ϕ gives the required contradiction.

Therefore $\{x_n\}$ is a Cauchy sequence and hence by completeness, there is a $z \in X$ such that $x_n \to z$. We show that z is a common fixed point of S and T.

Since $\{x_n\}$ converges to z, therefore $\{x_{2n}\}$ and $\{x_{2n+1}\}$ both converge to z. Let $d(Sz,z) = \varepsilon > 0$. Thus we have for $n \geq N$,

$$d(x_{2n},z) \leq \frac{\varepsilon - \gamma(\varepsilon)}{4} \text{ and } d(x_{2n},x_{2n+1}) \leq \frac{\varepsilon - \gamma(\varepsilon)}{4}$$

Therefore using the triangle inequality,

$$\begin{aligned} d(z,x_{2n-1}) &\leq d(z,x_{2n}) + d(x_{2n},x_{2n-1}) \\ &\leq \frac{\varepsilon - \gamma(\varepsilon)}{4} + \frac{\varepsilon - \gamma(\varepsilon)}{4} \\ &= \frac{\varepsilon - \gamma(\varepsilon)}{2}, \end{aligned}$$

i.e. $d(z,x_{2n-1}) < \varepsilon$ since $\gamma(\varepsilon) < \varepsilon$. (4)

Using the triangle inequality and (4) we have,

$$d(x_{2n-1},S(z)) \leq d(x_{2n-1},z) + d(z,S(z)) < \varepsilon + \varepsilon$$

i.e. $d(x_{2n-1},S(z)) < 2\varepsilon$ (5)

Now, $\varepsilon = d(S(z),z) \leq d(S(z),T(x_{2n-1})) + d_{2n} + d(x_{2n+1},z)$.

By (1), $\varepsilon \leq \phi(d(z,x_{2n-1}), d(z,S(z)), d(z,x_{2n}), d(x_{2n-1},S(z)), d_{2n-1}) + d_{2n} + d(x_{2n+1},z)$.

By (4) and (5), $\varepsilon \leq \phi(\varepsilon,\varepsilon,\varepsilon,2\varepsilon,\varepsilon) + \frac{\varepsilon-\gamma(\varepsilon)}{4} + \frac{\varepsilon-\gamma(\varepsilon)}{4}$.

i.e. $\varepsilon \leq \gamma(\varepsilon) + \frac{\varepsilon-\gamma(\varepsilon)}{2} = \frac{\gamma(\varepsilon) + \varepsilon}{2}$.

Since $\gamma(\varepsilon) < \varepsilon$, then $\frac{\gamma(\varepsilon) + \varepsilon}{2} < \varepsilon$

i.e. $\varepsilon < \varepsilon$, a contradiction.

Therefore, $z = S(z)$. Similarly $z = T(z)$.

It remains to show that z is a unique common fixed point.

Let $z \neq y$ be two common fixed points of S and T.

Then $d(z,y) = d(S(z),T(y)) \leq \phi(d(z,y),\ d(z,S(z)),\ d(z,T(y)),\ d(y,S(z)),\ d(y,T(y)))$,

$$= \phi(d(z,y), 0, d(z,y), d(z,y), 0)$$
$$< \gamma(d(z,y))$$
$$< d(z,y).$$

Therefore $z = y$ and S and T have a unique common fixed point.

Remark. Taking ϕ continuous we get a slightly revised version of Theorem 2.2.3 as a corollary. Instead of $a_1 + a_2 = 2$ in Theorem 2.2.3, we require $a_1 + a_2 = 3$.

2.3 Commuting Maps and Fixed Points

Gerald Jungck [18] has examined the interdependence between the commuting mapping and the fixed point concepts.

Proposition 2.3.1 Let g be a mapping of a set X into itself. Then g has a fixed point if and only if there is a constant map $h : X \to X$ which commutes with g.

i.e. $h(g(x)) = g(h(x))$ for all $x \in X$.

Proof. (Sufficiency) By hypothesis there exists $a \in X$ and $h : X \to X$ such that $h(x) = a$ and $h(g(x)) = g(h(x))$ for all $x \in X$. We can therefore write $g(a) = g(h(a)) = h(g(a)) = a$ so that a is a fixed point of g.

(Necessity) There exists some $a \in X$ such that $g(a) = a$. Define $h : X \to X$ by $h(x) = a$ for all $x \in X$.

Then $h(g(x)) = a$ and $g(h(x)) = g(a) = a$. Therefore $h(g(x)) = g(h(x))$ for all $x \in X$ and h commutes with g.

Jungck [18] gives the following theorem and corollary which has the Banach contraction principle as a consequence.

<u>Theorem 2.3.1</u> Let g be a continuous mapping of a complete metric space (X,d) into itself.

Then g has a fixed point in X if and only if there exists $\alpha \in (0,1)$ and a mapping $f : X \to X$ which commutes with g and satisfies $f(X) \subset g(X)$ and $d(fx,fy) \leq \alpha d(gx,gy)$ for all $x,y \in X$. Indeed g and f have a unique common fixed point.

<u>Proof.</u> (Necessity) If $g(a) = a$ for some $a \in X$, then there exists a constant map $f(x) = a$ which commutes with g. Moreover, $f(x) = a = g(a)$ for all $x \in X$ so that $f(X) \subset g(X)$. Finally, for any $\alpha \in (0,1)$ we have for all x,y in X:

$$d(fx,fy) = d(a,a) = 0 \leq \alpha d(gx,gy).$$

(Sufficiency) Suppose there is a mapping f of X into itself which commutes with g and for which $f(X) \subset g(X)$ and $d(fx,fy) \leq \alpha d(gx,gy)$ for all $x,y \in X$.

To show that g and f have a unique common fixed point let $x_0 \in X$ and let x_1 be such that $g(x_1) = f(x_0)$.

In general, choose x_n so that $g(x_n) = f(x_{n-1})$.

We can do this since $f(X) \subset g(X)$.

Since $d(gx_{n+1}, gx_n) = d(fx_n, fx_{n-1}) \leq \alpha d(gx_n, gx_{n-1})$, then $\{gx_n\}$ is a Cauchy sequence. Completeness of X implies that there exists $t \in X$ such that $gx_n \to t$. Then $fx_n \to t$.

Now since g is continuous, $d(fx, fy) \leq \alpha d(gx, gy)$ implies that f is continuous. Hence $f(gx_n) \to f(t)$ and $g(fx_n) \to g(t)$. But g and f commute so that $f(gx_n) = g(fx_n)$ for all n. Thus $g(t) = f(t)$ and consequently $g(gt) = g(ft) = f(gt) = f(ft)$.

Therefore,

$$d(ft, f(ft)) \leq \alpha d(gt, g(ft)) = \alpha d(ft, f(ft)).$$

Hence, $d(ft, f(ft))(1 - \alpha) \leq 0$. Since $\alpha \in (0,1)$, we get $f(t) = f(ft)$. We now have $f(t) = f(ft) = g(ft)$, i.e. $f(t)$ is a common fixed point of g and f.

To see that g and f have only one common fixed point, suppose that $x = gx = fx$ and $y = gy = fy$. Then

$$d(x,y) = d(fx, fy) \leq \alpha d(gx, gy) = \alpha d(x,y).$$

i.e. $d(x,y)(1 - \alpha) \leq 0$. Since $\alpha < 1$, $x = y$.

<u>Corollary 2.3.1</u> Let g and f be commuting mappings of a complete metric space (X,d) into itself. Suppose that g is continuous and $f(X) \subset g(X)$. If there exists $\alpha \in (0,1)$ and a positive integer k such that $d(f^k x, f^k y) \leq \alpha d(gx, gy)$ for all $x, y \in X$, then g and f have a unique common fixed point.

<u>Proof:</u> Clearly f^k commutes with g and $f^k(X) \subset f(X) \subset g(X)$. Using Theorem 2.3.1, there is a unique $a \in X$ such that $a = ga = f^k a$. Since g and f commute, we can write $fa = g(fa) = f^k(fa)$, i.e. $f(a)$ is a common fixed point of g and f^k. The uniqueness of a implies $a = fa = ga$.

Remark: We obtain the Banach contraction principle as a consequence of Corollary 2.3.1 if we set $k = 1$ and let g be the identity map.

Jungck [19] also gives a theorem and corollary which has Edelstein's [ ] theorem for contractive maps as a consequence.

Theorem 2.3.2 Let g be a continuous mapping of a compact metric space (X,d) into itself.
Then g has a fixed point if and only if there is a mapping $f : X \to g(X)$ which commutes with g and satisfies:

$$d(fx,fy) < d(gx,gy) \ , \quad gx \neq gy.$$

Corollary 2.3.2 If g is a continuous mapping of a compact metric space (X,d) into itself such that $d(g^2x,g^2y) \leq d(gx,gy)$, $gx \neq gy$, then g has a unique fixed point.

If g is contractive, i.e. $d(gx,gy) < d(x,y)$ then g satisfies $d(g^2x,g^2y) < d(gx,gy)$. Therefore we have Edelstein's theorem as a consequence of the corollary.

To see that Corollary 2.3.2 is indeed a generalization of Edelstein's theorem, Jungck [19] has given the following example:

Example. Let $X = [0,1]$ and let d be the absolute value metric.
Define $g : X \to X$ by

$$g(x) = \begin{cases} 3x + \frac{1}{4} & , \quad 0 \leq x \leq \frac{1}{4} \ ; \\ 1 & , \quad \frac{1}{4} < x \leq 1 \ . \end{cases}$$

Then $g(x) \geq \frac{1}{4}$ for all x, so that $g^2x = 1$ for all $x \in X$.
Thus $|gx - gy| > 0 = |g^2x - g^2y|$ if $gx \neq gy$, so that the condition of Corollary 2.3.2 is satisfied. On the other hand $d(gx,gy) \nless d(x,y)$ when

$x,y \in [0,\frac{1}{4}]$ and $d(gx,gy) \nless d(x,y)$ if $x,y \in [\frac{1}{4},1]$.

Theorem 2.3.2 can be proven under considerably weaker conditions. Jungck required continuity of the mapping and compactness of the space which are not required in the following theorem. Also the metric required by Jungck can be replaced by a function $F : X \times X \to R^+$ such that $F(x,y) = 0$ iff $x = y$. We give the following.

Theorem 2.3.3 Let g be a mapping of a space X into itself such that $F(gx,g^2x)$ has a minimum value at some $a \in X$, where F is a function $F : X \times X \to R^+$ such that $F(x,y) = 0$ if and only if $x = y$. Then g has a fixed point if and only if there is a mapping $f : X \to g(X)$ which commutes with g and satisfies

$$F(fx,fy) < F(gx,gy),\ g(x) \neq g(y). \tag{1}$$

Proof: (Necessity) If $g(b) = b$ for some $b \in X$, then there exists a constant map $f(x) = b$ which commutes with g. Moreover, $f(x) = b = g(b)$ for all $x \in X$ so that $f(X) \subset g(X)$.
Finally, $g(x) \neq g(y)$ implies $F(g(x),g(y)) > 0$.
$F(f(x),f(y)) = F(b,b) = 0$ implies $F(g(x),g(y)) > F(f(x),f(y))$ so that condition (1) of the theorem holds.

(Sufficency) By hypothesis, $F(g(x),g^2(x))$ has a minimum value at some $a \in X$, i.e.

$$F(g(a),g^2(a)) \leq F(g(x),g^2(x)),\quad x \in X. \tag{i}$$

Claim: $g(a) = g^2(a)$, i.e. $g(a)$ is a fixed point of g. If not,

$$F(f(a),g(f(a))) = F(f(a),f(g(a))) < F(g(a),g^2(a)) \tag{ii}$$

But $f(X) \subset g(X)$ by hypothesis; hence there exists $c \in X$ such that $g(c) = f(a)$. Then (ii) implies $F(g(c), g^2(c)) < F(g(a), g^2(a))$ which contradicts (i).

Corollary 2.3.3 Let g be a mapping of a space X into itself such that $F(g(x), g^2(x))$ has a minimum value at some $a \in X$. (F defined as in theorem). If $F(g^2(x), g^2(y)) < F(g(x), g(y))$, $g(x) \neq g(y)$ (2) then g has a unique fixed point.

Proof: Substitute g^2 for f in the statement of the theorem. Since g^2 and g commute and since $g^2(X) \subset g(X)$, then g has a fixed point c by the theorem. To see that c is unique, suppose that $g(b) = b$. If $c \neq b$, then $g(c) \neq g(b)$, so that
$F(c,b) = F(g^2(c), g^2(b)) < F(g(c), g(b)) = F(c,b)$, i.e. $F(c,b) < F(c,b)$.

Remark. If (X,d) is a metric space, then F can be replaced by d in the statement of the theorem.

Remark. Let g be continuous and X compact. Then if F is lower semi-continuous, $F(g(x), g^2(x))$ has a minimum value at some $a \in X$ and the theorem holds.

Remark. Let g be continuous and (X,d) a compact metric space. Then $d(g(x), g^2(x))$ has a minimum value at some $a \in X$. Thus the result of Jungck [19] follows as a corollary.

Remark. If (X,d) is compact and g satisfies $d(g(x), g(y)) < d(x,y)$ $x \neq y$ then g is continuous and satisfies $d(g^2(x), g^2(y)) < d(g(x), g(y))$. Thus we have Edelstein's theorem [11] as a consequence of the corollary with $F = d$.

To see that continuity of g is not required and compactness of X is not required, consider the following example:

Example. Let $X = \mathbb{R}$ and let F be the absolute value metric. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0, & x \in \text{rationals}, \\ 1, & x \in \text{irrationals}. \end{cases}$$

Then $g^2(x) = 0$, $x \in \mathbb{R}$, so that $F(g(x), g^2(x)) = |g(x)| = \begin{cases} 0, & x \in \text{rationals} \\ 1, & x \in \text{irrationals}, \end{cases}$

i.e. $F(g(x), g^2(x))$ has a minimum value.

This mapping satisfies the condition (2) since

$$F(g^2(x), g^2(y)) = 0 < F(g(x), g(y)), \; g(x) \neq g(y).$$

Therefore g has a unique fixed point.

This corollary can also be used to show that a mapping of the type $g(x) = ax + b$ where $0 < a < 1$, defined on $\mathbb{R}$, has a unique fixed point. Since $d(g^2(x), g^2(y)) = a^2 |x - y|$ and $d(g(x), g(y)) = |x - y|$, then $0 < a < 1$ implies $a^2|x - y| < a |x - y|$. Therefore condition (2) is satisfied (where F is the absolute value metric). To see that $d(g(x), g^2(x))$ has a minimum value:

$d(g(x), g^2(x)) = |a^2x + (ab + b) - (ax + b)| = a \, |(a - 1)x + b|$. Since $d(g(x), g^2(x)) \geq 0$ and for $x = \frac{b}{1 - a}$,

$d(g(x), g^2(x)) = 0$, then $d(g(x), g^2(x))$ has a minimum value, i.e. the corollary is satisfied and g has a unique fixed point.

CHAPTER III

Sequences of Mappings and Fixed Points

3.1 Convergence of sequences and subsequences of fixed points

Several mathematicians have investigated the conditions under which the convergence of a sequence of contraction mappings to a mapping T of a metric space into itself implies the convergence of their fixed points to a fixed point of T.

A partial solution to this problem has been given by Bonsall [3] as follows:

Theorem 3.1.1 Let (X,d) be a complete metric space. Let T_n $(n = 1,2, \ldots)$ and T be contraction mappings of X into itself with the same Lipschitz constant $k < 1$, and with fixed points u_n and u respectively. Suppose that $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$. Then $\lim_{n\to\infty} u_n = u$.

There is no need to assume that T is a contraction mapping since this follows from the conditions in the theorem itself. This was shown by Singh and Russell [39] in the following result:

Lemma 3.1.1 Let (X,d) be a complete metric space and let T_n $(n = 1,2, \ldots)$ be contraction mappings of X into itself with the same Lipschitz constant $k < 1$. Suppose $\lim_{n\to\infty} T_n x = Tx$ for each $x \in X$, where T is a mapping from X into itself. Then T is a contraction mapping.

Proof: Since $k < 1$ is the same Lipschitz constant for all n, $d(Tx,Ty) = \lim_{n\to\infty} d(T_n x, T_n y) < kd(x,y)$.

Thus T is a contraction mapping with contraction constant k, and as such has a unique fixed point.

The following modified version of Theorem 3.1.1 is due to Singh [35].

Theorem 3.1.2 Let X be a complete metric space and let $\{T_n\}$, $n = 1,2, \ldots$ be a sequence of contraction mappings with the same Lipschitz constant $k < 1$, and with fixed points u_n $(n = 1,2, \ldots)$. Suppose that $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$, where T is a mapping from X into itself. Then T has a unique fixed point u and $\lim_{n\to\infty} u_n = u$.

Proof: From Lemma 3.1.1 it follows that T has a unique fixed point u. Since the sequence of contraction mappings converges to T, there exists, for a given $\varepsilon > 0$, an N such that $n \geq N$ implies

$$d(T_n u, Tu) \leq (1 - k)\varepsilon$$

where k is the contraction constant. Now for $n \geq N$,

$$\begin{aligned} d(u,u_n) &= d(Tu, T_n u_n) \\ &\leq d(Tu, T_n u) + d(T_n u, T_n u_n) \\ &\leq (1 - k)\varepsilon + kd(u,u_n). \end{aligned}$$

Thus $(1 - k)d(u,u_n) \leq (1 - k)\varepsilon$.
Since $0 \leq k < 1$, we have $d(u,u_n) \leq \varepsilon$, $n \geq N$, and so $\lim_{n\to\infty} u_n = u$.

As pointed out by Nadler Jr. [23], the restriction that all contraction mappings have the "same Lipschitz constant $k < 1$" is very strong; for one can easily construct a sequence of contraction mappings from the reals into the reals which converges uniformly to the zero mapping but whose Lipschitz constants tend to one.

A modification of Theorem 3.1.1 has been given by Singh [37] where the restriction that all contractions have the same Lipschitz constant has been relaxed in the following way:

Theorem 3.1.3 Let (X,d) be a complete metric space and let $T_n : X \to X$ be a contraction mapping with Lipschitz constant k_n and with fixed point u_n for each $n = 1,2, \ldots$. Furthermore, if $k_{n+1} \leq k_n$ for $n = 1,2, \ldots$ and $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$, where T is a mapping of X into itself. Then T has a unique fixed point and sequence $\{u_n\}_{n=1}^{\infty}$ of fixed points converges to the fixed point of T.

Proof: Since T_n is a contraction with Lipschitz constant k_n,

$$d(T_n x, T_n y) \leq k_n d(x,y) \quad \text{for all } x,y \in X.$$

Thus,

$$\lim_{n\to\infty} d(T_n x, T_n y) \leq \lim_{n\to\infty} k_n d(x,y).$$

Since $k_{n+1} \leq k_n < 1$ for each n, it follows that $\lim_{n\to\infty} k_n < 1$. Hence $\lim_{n\to\infty} T_n x = Tx$ is a contraction mapping. Moreover, k_1 serves the purpose of a Lipschitz constant for all $T_n (n = 1,2, \ldots)$. Thus the proof follows from Theorem 3.1.1 on replacing k by k_1.

The theorem may be illustrated by taking the following example.

Example 3.1.1 Let $T_n : [0,1] \to [0,1]$ be defined by $T_n x = 1 - \frac{1}{n+1} x$ for all $x \in [0,1]$, $n = 1,2, \ldots$.

Obviously T_n is a contraction mapping of $[0,1]$ into itself, with Lipschitz constant $k_n = \frac{1}{n+1}$ for each $n = 1,2, \ldots$. As we observe $k_{n+1} \leq k_n < 1$ for each n, $k_1 = \frac{1}{2}$ will serve the purpose of Lipschitz constant for all the mappings. The unique fixed point for T_n is $u_n = \frac{n}{n+1}$ for each $n = 1,2, \ldots$. The limiting function T is given by

$$Tx = \lim_{n\to\infty} T_n x = 1 \quad \text{for every } x \in [0,1].$$

Now, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$, where 1 is the unique fixed point for T.

If the Lipschitz constants are such that $k_{n+1} \geq k_n$ for each n, the theorem is, in general, false. To justify this, the following example has been given by Russell [30]:

Example 3.1.2 Let $T_n : E^1 \to E^1$ be defined by
$T_n x = p + \frac{n}{n+1} x$ $(n = 1,2, \ldots)$, $p > 0$
for all $x \in E^1$, where $E^1 = (-\infty, +\infty)$.

We see that T_n is a contraction mapping, with Lipschitz constant $k_n = \frac{n}{n+1}$ and with fixed point $u_n = (n+1)p$ for each $n = 1,2, \ldots$.

Now $Tx = \lim_{n\to\infty} T_n x = p + x$ for every $x \in E^1$.

Thus under the mapping T, every point of E^1 has been translated by a distance p and therefore T has no fixed point. Moreover,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (n+1)p = \infty \notin E^1.$$

Remark: Theorem 3.1.3 has been further modified by Singh [37] by replacing the condition $k_{n+1} \leq k_n < 1$ by $k_n \to k < 1$.

Another modification of Theorem 3.1.1 was given by Nadler [23], who considered separately the uniform convergence and pointwise convergence of a sequence of contraction mappings.

<u>Theorem 3.1.4</u> Let (X,d) be a metric space; let $T_i : X \to X$ be a function with at least one fixed point u_i for each $i = 1,2, \ldots$, and let $T : X \to X$ be a contraction mapping with fixed point u. If the sequence $\{T_i\}_{i=1}^{\infty}$ converges uniformly to T, then the sequence $\{u_i\}_{i=1}^{\infty}$ of fixed points converges to u.

<u>Proof</u>: Since $\{T_i\}_{i=1}^{\infty}$ converges uniformly to T, therefore for $\varepsilon > 0$, there is a positive integer N such that $i \geq N$ implies $d(T_i x, Tx) < \varepsilon(1 - \alpha)$ for all $x \in X$, where $\alpha < 1$ is a Lipschitz constant for T.

We have,

$$\begin{aligned} d(u_i,u) &= d(T_i u_i, Tu) \\ &\leq d(T_i u_i, Tu_i) + d(Tu_i, Tu) \\ &\leq d(T_i u_i, Tu_i) + \alpha d(u_i,u), \end{aligned}$$

i.e. $(1 - \alpha)d(u_i,u) \leq d(T_i u_i, Tu_i)$.

Therefore, for $i \geq N$, $(1 - \alpha)d(u_i,u) < \varepsilon(1 - \alpha)$,

i.e. $d(u_i,u) < \varepsilon$ since $0 \leq \alpha < 1$.

This proves that $\{u_i\}_{i=1}^{\infty}$ converges to u.

<u>Theorem 3.1.5</u> Let (X,d) be a locally compact metric space; let $A_i : X \to X$ be a contraction mapping with fixed point a_i for each $i = 1,2, \ldots$ and let $A_o : X \to X$ be a contraction mapping with fixed point a_o. If the sequence $\{A_i\}_{i=1}^{\infty}$ converges to A, then the sequence $\{a_i\}_{i=1}^{\infty}$ converges to a_o.

Next, we consider the convergence of a sequence of contractive mappings to a mapping T on compact and locally compact spaces.

The following theorem was given by Nadler [23].

Theorem 3.1.6 Let (X,d) be a compact metric space and $T_n: X \to X$ be a sequence of contractive mappings of X into itself. Suppose the sequence $\{T_n\}$ converges uniformly to T, a contraction mapping of X into itself. Then the sequence $\{T_n\}_{n=1}^{\infty}$ has unique fixed points $\{u_n\}_{n=1}^{\infty}$ and the sequence $\{u_n\}_{n=1}^{\infty}$ converges to u, a unique fixed point of T.

Singh [36] has given the following theorem for a locally compact metric space.

Theorem 3.1.7 Let (X,d) be a locally compact space, and let $T_n : X \to X$ be a contractive mapping with fixed point u_n for each $n = 1,2,\ldots$ and let $T : X \to X$ be a contraction mapping with fixed point u. If the sequence converges pointwise to T, then the sequence $\{u_n\}_{n=1}^{\infty}$ of fixed points converges to u.

Results of the same type have been given for maps other than contractions or contractive maps. For example, Ray [26] has given the following theorems for maps which satisfy a condition similar to that considered by Kannan [20]. (Definition 1.2.5).

Theorem 3.1.8 Let (X,d) be a metric space and let $T_n : X \to X$ be a map with fixed point u_n, for $n = 0,1,2, \ldots,$ and let $\{T_n\}$ converge uniformly to T_o. If $d(T_o x,T_o y) \leq a[d(x,T_o x) + d(y,T_o y)]$, where a is any positive number and $x,y \in X$, then $\{u_n\}$ converges to u_o.

Theorem 3.1.9 Let (X,d) be a metric space, and $T_n : X \to X$ be a map with a fixed point u_n such that $d(T_n x,T_n y) \leq a[d(x,T_n x) + d(y,T_n y)]$ where a is any positive number and $n = 1,2,3, \ldots$.

If $\{T_n\}$ converges pointwise to a map T_o that maps X into itself with $T_o(u_o) = u_o$, then $\{u_n\}$ converges to u_o.

Theorem 3.1.10 Let (X,d) be a metric space and $T_n : X \to X$ be a map with fixed point u_n such that $d(T_n x, T_n y) \leq a[d(x,T_n x) + d(y,T_n y)]$, where $0 < a < 1$ and $n = 1,2,3, \ldots$. If $\{T_n\}$ converges pointwise to T_o that maps X into itself and if u_o is the limit point of $\{u_n\}$, then u_o is a fixed point of T_o.

Iseki [17] has considered mappings $T : X \to X$ which satisfy the following condition:

$$d(Tx,Ty) \leq a[d(x,Tx) + d(y,Ty)] + b[d(x,Ty) + d(y,Tx)] + cd(x,y) \qquad (*)$$

where a,b,c are nonnegative and $2a + 2b + c < 1$.

The following theorems are due to Iseki [17].

Theorem 3.1.11 Let $\{T_n\}$ be a sequence of mappings of a complete metric space X into itself. Let u_n be a fixed point of T_n $(n = 1,2, \ldots)$ and suppose that T_n converges uniformly to T_o. If T_o satisfies the condition $(*)$, then $\{u_n\}$ converges to the fixed point u_o of T_o.

Theorem 3.1.12 Let T_n $(n = 1,2, \ldots)$ be a sequence of mappings with fixed point u_n of a metric space X into itself. Suppose T_n satisfies condition $(*)$. If $\{T_n\}$ converges to a mapping T_o and u_o is an accumulation point of $\{u_n\}$ then u_o is the fixed point of T_o.

Remark: Husain and Sehgal [16] have generalized Theorem 3.1.11 and Theorem 3.1.12 by replacing condition $(*)$ with the following condition

$$d(Tx,Ty) \leq \phi(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), d(x,y))$$

where $\phi : (R^+)^5 \to R^+$ and ϕ is continuous and nondecreasing in each co-ordinate variable. Also, ϕ satisfies $\phi(t,t,a_1t,a_2t,t) < t$ where $a_1 + a_2 = 2$.

Let u_n be a fixed point of the mapping T_n. If T is the limit mapping of the sequence $\{T_n\}$, can one conclude that there exists a fixed point for T from subsequential convergence of $\{u_n\}$? This question was considered by Ng [24] and he gives a partial answer to this question in the following theorem.

Theorem 3.1.13 Suppose

(i) $\{T_n\}_{n=1}^{\infty}$ is an equicontinuous sequence of mappings from X into X each of which has a fixed point u_n.

(ii) $\{T_n\}$ converges pointwise to a mapping $T : X \to X$

(iii) $\{u_n\}$ has a convergent subsequence $\{u_{n_i}\}$ whose limit is u.

Then u is the fixed point of T.

Let H be the family of all functions $\alpha : (0,\infty) \to [0,1]$ such that α is monotonically decreasing. Hillam [13] proved the following theorem:

Theorem 3.1.14 For $n = 1,2, \ldots$ let $T_n : X \to X$ be a sequence of functions each of which has at least one fixed point u_n. Let $T : X \to X$ be a function with a unique fixed point u such that for all $x \in X$

(1) $d(Tx,u) \leq \alpha(d(x,u))d(x,u)$, $\alpha \in H$. Then if $T_n \to T$ uniformly on X, $u_n \to u$.

Diaz and Metcalf [22] have considered functions where $d(Tx,F(T)) < d(x,F(T))$, where $F(T)$ is the fixed point set of the function T. Hillam [13] has shown that if (1) in Theorem 3.1.14 is replaced by $d(Tx,F(T)) \leq \alpha(d(x,F(T))d(x,F(T))$ then the sequence of fixed points might not converge but the subsequential limit points are fixed points.

Ray [27] has considered a sequence of functions $\{T_n\}$ that map a complete metric space (X,d) into itself and satisfy the following definition:

<u>Definition 3.1.1</u> Let $F(T_n)$ denote the set of fixed points of T_n. There exist monotonically decreasing functions $\alpha,\beta : (0,\infty) \to [0,1)$ such that for all $x \notin F(T_n)$ and for all n, T_n satisfies
$d(T_n x,F(T_n)) \leq \alpha(d(x,F(T_n)))d(x,F(T_n)) + \beta(d(x,F(T_n)))d(x,T_n x)$
where $\alpha(d(x,F(T_n))) + 2\beta(d(x,F(T_n))) < 1$.

Before stating the theorem we give the following definition.

<u>Definition 3.1.2</u> Let (X,d) be a metric space and for $n = 1,2,3,\ldots$, let $K_n \subset X$ be a sequence of nonempty sets. We define $L(\{K_n\})$ to be the set of all possible subsequential limit points of all possible sequences $\{k_j\}$ where $k_j \in K_j$.

The following theorem is due to Ray [27].

<u>Theorem 3.1.15</u> For $n = 1,2,3,\ldots$, let $T_n : X \to X$ be a sequence of functions such that $F(T_n)$ is nonempty. Suppose T_n satisfies Definition 3.1.1. Let $T_o : X \to X$ be a continuous function and suppose $T_n \to T_o$ uniformly, then $L(\{F(T_n)\})$ is nonempty. Furthermore,

$L(\{F(T_n)\}) = F(T_o)$ and $F(T_o) = \lim \{F(T_n)\}$.

For the special case that for every integer n, $F(T_n) = \{a_n\}$ and α, β are defined to be $\alpha(t) = k_1$ and $\beta(t) = k_2$ such that $k_1 + 2k_2 < 1$, T_o need not be continuous. This is given in the following theorem due to Ray [27].

Theorem 3.1.16 For $n = 1,2, \ldots,$ let $T_n : X \to X$ be a sequence of functions such that $F(T_n) = \{a_n\}$.
Suppose there exist strictly positive k_1 and k_2 with $k_1 + 2k_2 < 1$ such that for all $x \in X - \{a_n\}$ and for all n,

$$d(T_n x, a_n) \le k_1 d(x, a_n) + k_2 d(x, T_n x).$$

Then if $T_o : X \to X$ is a function such that $T_n \to T_o$ uniformly, then $F(T_o)$ is nonempty.

The following theorem due to Ray [27] gives conditions that ensure that $F(T_o)$ is compact.

Theorem 3.1.17 For $n = 1,2,3, \ldots,$ let $T_n : X \to X$ be a sequence of functions such that $F(T_n)$ is nonempty and compact. Suppose T_n satisfies Definition 3.1.1. Let $T_o : X \to X$ be a continuous function and suppose that $T_n \to T_o$ uniformly. Then $F(T_o)$ is nonempty and compact.

Ray [27] has also given a theorem in which the limit mapping is a contractive map and a subsequence of fixed points converges.

Theorem 3.1.18 Let $T_n : X \to X$ be a sequence of mappings with fixed point u_n for each $n = 1,2, \ldots,$ and let $T_o : X \to X$ be a contractive mapping with fixed point u_o. If the sequence $\{T_n\}$ converges uniformly to T_o and if a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ converges to a point $z \in X$, then $z = u_o$.

Proof: Let $\varepsilon > 0$. There is a positive integer N such that $i \geq N$ implies $d(u_{n_i}, z) < \frac{\varepsilon}{2}$. Therefore,

$$d(u_{n_i}, T_o z) \leq d(u_{n_i}, T_o u_{n_i}) + d(T_o u_{n_i}, T_o z)$$

$$= d(T_{n_i} u_{n_i}, T_o u_{n_i}) + d(T_o u_{n_i}, T_o z).$$

Since T_o is contractive, we have

$$d(u_{n_i}, T_o z) < d(T_{n_i} u_{n_i}, T_o u_{n_i}) + d(u_{n_i}, z)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon.$$

Thus $\{u_{n_i}\}$ converges to $T_o z$ and $z = T_o z$. The uniqueness of the fixed point u_o of T_o implies $z = u_o$.

3.2 Convergence of sequence of mappings

We now consider a sequence of maps $\{T_n\}$ such that T_n satisfies Definition 1.5.1. First we give the following lemma.

Lemma 3.2.1: Let (X,d) be a complete metric space, T_n $(n = 1,2, \ldots)$ be mappings of X into itself satisfying Definition 1.5.1 with the same k.

Suppose $\lim_{n\to\infty} T_n x = Tx$ for each $x \in X$, where T is a mapping from X into itself. Then T has a unique fixed point.

Proof: Since $k < 1$ is the same for all n,

$$d(Tx,Ty) = \lim_{n\to\infty} d(T_n x, T_n y)$$

$$\leq \lim_{n\to\infty} k \cdot \frac{d(x,T_n x)d(x,T_n y) + d(T_n x,y)d(y,T_n y)}{d(x,T_n y) + d(y,T_n x)}$$

$$= k \quad \frac{d(x,Tx)d(x,Ty) + d(Tx,y)d(y,Ty)}{d(x,Ty) + d(y,Tx)}$$

Thus T satisfies Definition 1.5.1 and by Theorem 1.5.1 has a unique fixed point.

We give the following theorems.

Theorem 3.2.1 Let (X,d) be a complete metric space and let $\{T_n\}$ be a sequence of mappings satisfying Definition 1.5.1 with the same k. Let u_n be the fixed point of T_n. Suppose $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$, where T is a mapping from X into itself. Then T has a unique fixed point u and $\lim_{n\to\infty} u_n = u$.

Proof: From Lemma 3.2.1, it follows that T has a unique fixed point u. Since the sequence of mappings converges to T, there exists for a given $\varepsilon > 0$, an N such that $n \geq N$ implies

$$d(T_n u, Tu) \leq \frac{\varepsilon}{1 + k}.$$

Now for $n \geq N$, we have

$$d(u,u_n) = d(Tu, T_n u_n)$$

$$\leq d(Tu,T_n u) + d(T_n u, T_n u_n)$$

$$\leq d(Tu,T_n u) + k \left[\frac{d(u,T_n u)d(u,T_n u_n) + (u_n,T_n u)d(u_n,T_n u_n)}{d(u,T_n u_n) + d(u_n,T_n u)}\right]$$

$$= d(Tu,T_n u) + k\frac{d(u,T_n u)d(u,u_n)}{d(u,u_n) + d(u_n,T_n u)}$$

$$\leq d(Tu,T_n u) + k\frac{d(u,T_n u)d(u,u_n)}{d(u,u_n)}$$

$$= d(Tu,T_n u) + kd(Tu,T_n u)$$

$$= (1 + k)d(Tu,T_n u)$$

$$\leq (1 + k)\frac{\varepsilon}{(1 + k)}$$

$$= \varepsilon .$$

i.e. $d(u,u_n) \leq \varepsilon$ for $n \geq N$. Thus $\lim_{n\to\infty} u_n = u$.

Theorem 3.2.2 Let (X,d) be a metric space. Let $T_n : X \to X$ be a mapping with at least one fixed point u_n for each $n = 1,2, \ldots$ and let $T : X \to X$ be a mapping satisfying Definition 1.5.1 with fixed point u. If the sequence $\{T_n\}_{n=1}^{\infty}$ converges uniformly to T, then the sequence $\{u_n\}_{n=1}^{\infty}$ of fixed points converges to u.

Proof: $\{T_n\}_{n=1}^{\infty}$ converges uniformly to T, therefore for $\varepsilon > 0$, there exists a positive integer N such that $n \geq N$ implies $d(T_n x,Tx) < \frac{\varepsilon}{1 + k}$ for all $x \in X$ where k is the constant for T in Definition 1.5.1.

We have,

$$d(u_n,u) = d(T_n u_n,Tu)$$
$$\leq d(T_n u_n,Tu_n) + d(Tu_n,Tu).$$

Using Definition 1.5.1, we have

$$d(u_n,u) \le d(T_n u_n, Tu_n) + k\left[\frac{d(u_n,Tu_n)d(u_n,Tu) + d(u,Tu_n)d(u,Tu)}{d(u_n,Tu) + d(u,Tu_n)}\right]$$

Thus, $d(u_n,u) \le d(T_n u_n, Tu_n) + \dfrac{kd(u_n,Tu_n)d(u_n,Tu)}{d(u_n,Tu) + d(u,Tu_n)}$

$$< d(T_n u_n, Tu_n) + \frac{kd(u_n Tu_n)d(u_n,Tu)}{d(u_n,Tu)}$$

$$= d(T_n u_n, Tu_n) + kd(u_n,Tu_n)$$

$$= d(T_n u_n, Tu_n) + kd(T_n u_n, Tu_n)$$

$$= (1+k)d(T_n u_n, Tu_n)$$

$$< (1+k)\frac{\varepsilon}{(1+k)}$$

$$= \varepsilon .$$

Thus, $\{u_n\}_{n=1}^{\infty}$ converges to u.

Theorem 3.2.3 Let (X,d) be a complete metric space and let $T_n : X \to X$ be a map satisfying Definition 1.5.1 with constant k_n and fixed point u_n for each $n = 1,2, \ldots$. Furthermore, if $k_{n+1} \le k_n$ for $n = 1,2, \ldots$ and $\lim_{n\to\infty} T_n x = Tx$ for every $x \in X$, where T is a mapping of X into itself. Then T has a unique fixed point and sequence $\{u_n\}_{n=1}^{\infty}$ of fixed points converges to the fixed point of T.

Proof: Since T_n satisfies Definition 1.5.1

$$d(T_n x, T_n y) \le k_n\left[\frac{d(x,T_n x)d(x,T_n y) + d(y,T_n x)d(y,T_n y)}{d(x,T_n y) + d(y,T_n x)}\right]$$

Therefore,

$$\lim_{n\to\infty} d(T_n x, T_n y) \le \lim_{n\to\infty} k_n\left[\frac{d(x,T_n x)d(x,T_n y) + d(y,T_n x)d(y,T_n y)}{d(x,T_n y) + d(y,T_n x)}\right]$$

i.e. $$d(Tx,Ty) \le \lim_{n\to\infty} k_n \left[\frac{d(x,Tx)d(x,Ty) + d(y,Tx)d(y,Ty)}{d(x,Ty) + d(y,Tx)}\right].$$

Since $k_{n+1} < k_n < 1$ for each n, it follows that $\lim_{n\to\infty} k_n < 1$ and hence T_ω satisfies Definition 1.5.1.

Thus the proof follows from Theorem 3.2.1 on replacing k by k_1.

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