SOME ANALYTIC PROPERTIES OF SEMIGROUPS OF POSITIVE MATRICES, TRANSITION MATRICES, P-FUNCTIONS AND SEMI-P-FUNCTIONS

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by

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ABSTRACT

In this dissertation we deal with four related mathematical entities: transition matrices and their generalization (semigroups of positive matrices), p-functions (which can arise from the diagonal elements of a transition matrix) and their generalization (semi-p-functions). In Chapter 1 we define these entities. In Chapter 2 we discuss continuity under the assumption of measurability alone. The remaining three chapters deal with standard entities only (i.e., those which enjoy a continuity condition at t=0) since most authors assume standardness anyway. Chapter 3 deals with continuity, Chapter 4 with differentiability at t=0, and Chapter 5 with differentiability at t>0. Counter examples are given for outstanding cases.

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Gordon Ezekiel

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INTRODUCTION

The purpose of this exposition is to explore some of the analytic properties of Markov transition functions and to see to what extent these properties hold when we make generalizations from

(i) the Markov situation to general regenerative phenomena;

(ii) stochastic semigroups to non-stochastic ones.

We ultimately consider a combination of the abstractions (i) (ii) embodied in semi-p-functions. We shall show that there is a considerable degree of common behaviour between the entities in (i) and (remarkably) between those in (ii).

Roughly speaking, the progress made in the study of Markov chains between 1930 and 1960 by Chung, Feller, Doeblin, Doob, Kolmogorov and others is set out in Chung's book [2]. Work on semigroups of non-negative matrices dates back to Jurkat [9] and has been further extended by Cornish [3]. The more recent theory of regenerative phenomena developed in the early 1960's and is due largely to Kingman [15]. Kingman has also been responsible for the theory of semi-p-functions [16] with additional results by Cornish [4].

None of the results presented here are original although many of the proofs are spelt out in greater detail and should be easier to follow that those of the original authors.

CHAPTER 1

Basic Concepts

In this chapter we define the four entities: transition matrices and their generalizations (semigroups of positive matrices); p-functions and their generalizations (semi-p-functions). In particular, we introduce the concept of a regenerative phenomenon by considering a diagonal element of a continuous time standard Markov chain. This establishes a link between transition functions and p-functions; namely, a diagonal member, $P_{ii}(t)$, of a transition matrix turns out to be a p-function.

In addition to the basic definition of a regenerative phenomenon, we give several equivalent characterizations due to Kingman [13], [15], along with the important Kingman inequalities for (semi)-p-functions. These results are needed in later chapters.

1.1 Some Definitions

In what follows T stands for the interval $[0,\infty)$, T^{o} stands for $(0,\infty)$ and I stands for a denumerable set of indices. An unspecified sum over the indices is over all I.

A semigroup of positive matrices [9] is a finite or denumerable array of functions $P(t) = (p_{ij}(t))$, $i, j \epsilon I$, defined on T^{o} and satisfying the following two conditions: for every $i, j \epsilon I$, $s, t \epsilon T^{o}$,

$$0 \leq p_{ij}(t) < \infty, \tag{1}$$

$$p_{ij}(s+t) = \sum_{k} p_{ik}(s)\, p_{kj}(t). \tag{2}$$

A semigroup of positive matrices is called standard if it satisfies the continuity condition

$$\lim_{t \to 0} p_{ij}(t) = \delta_{ij}, \text{ where } \delta_{ij} \text{ is the Kronecker symbol.} \tag{3}$$

A transition matrix is a semigroup of positive matrices satisfying

$$\sum_{k} p_{ik}(t) = 1, \quad t \epsilon T^{o}. \tag{4}$$

A transition matrix satisfying (3) is said to be standard.

Conditions (1) and (4) together may be expressed by saying that for each t, the matrix $(p_{ij}(t))$ is stochastic.

Condition (2) is often referred to as the Chapman-Kolmogorov equation or the semigroup property. If our semigroup of positive matrices (stochastic or otherwise) is standard, we shall extend the domain of definition from T^o to T by setting $p_{ij}(0) = \delta_{ij}$, $i, j \in I$.

Suppose we are given a probability triple (Ω, F, P) and a physical system with a countable or finite state space I. A stochastic process $\{X(t)\}$ ~~t≥0) with the~~ following properties is a Markov Chain:

$$P(X(0) = i) = p_i, \ i \in I \text{ where } p_i \geq 0 \text{ and } \sum_i p_i = 1. \qquad \text{(A)}$$

For any $0=t_0<t_1<\ldots<t_n$ and any states $i_o, i_1, \ldots, i_n$, (B) the joint distribution of $X(t_0), \ldots, X(t_n)$ is given by

$$P(X(t_0) = i_0, \ldots, X(t_n) = i_n) = p_{i0} p_{i0i_1}(t_1-t_0) \cdots \qquad (5)$$

$$p_{i_n, i_{n-1}}(t_n - t_{n-1}),$$

where the functions $(p_{ij}(t))$ are a transition matrix. The set $\{p_i, \ i \in I\}$ is called the initial distribution. The study of Markov Chains [2] is to a large extent the study of properties of their transition matrices.

1.2 Regenerative Phenomena and their p-functions

Let $(X(t); t \geq 0)$ be a continuous time standard Markov chain [2] on the (countable) state space I. Fix a state $a \in I$ and suppose that $X(0) = a$. Consider the process

$$Z(t) = \psi\{X(t)\} \tag{1}$$

where $\psi: I \to \{0,1\}$ satisfies $\psi(a)=1$, $\psi(i) = 0$, $i \neq a$. Then Z has the property that whenever

$0 = t_0 < t_1 < \ldots < t_n$,

$$P\{Z(t_1) = Z(t_2) = \ldots = Z(t_n) = 1\} \tag{2}$$

$$= P\{X(t_1) = X(t_2) = \ldots = X(t_n) = a \mid X(t_0) = a\}$$

$$= p_a\, p_{aa}(t_1-t_0)\, p_{aa}(t_2-t_1) \ldots p_{aa}(t_n-t_{n-1})/p_a$$

$$= \prod_{r=1}^{n} p_{aa}(t_r-t_{r-1}).$$

This property is crucial and leads to this definition: A stochastic process $(Z(t);\ t>0)$ taking values 0 and 1 is said to be a regenerative phenomenon [15] if there exists a function p on T^o (called the p-function of Z) such that whenever

$0=t_0<t_1<\ldots<t_n$

we have

$$P\{Z(t_1) = Z(t_2) = \ldots = Z(t_n) = 1\} = \prod_{r=1}^{n} p(t_r-t_{r-1}). \tag{3}$$

This discussion above shows that (1) is a regenerative phenomenon with p-function $p_{aa}(t)$. Since the underlying Markov chain is standard, we have

$$\lim_{t\to 0} p(t) = 1.$$

A p-function satisfying (4) is said to be standard, and the same adjective applies to the regenerative phenomenon. For the sake of simplicity we extend the domain of a standard p-function from T^o to T by setting $p(0)=1$.

The following proposition [13] gives an equivalent characterization of a regenerative phenomenon.

Proposition:

Z is a regenerative phenomenon iff given $0=t_o<t_1<\ldots<t_n$, then

$$P\{Z(t_1) = Z(t_2) = \ldots = Z(t_n) = 1\} =$$
$$P\{Z(t_1) = 1\}P\{Z(t_2-t_1) = \ldots = Z(t_n-t_1) = 1\}. \tag{5}$$

Proof:

Suppose Z is a regenerative phenomen. Putting n=1 in (3) we have for any $t_1>0$, $P\{Z(t_1)=1\} = p(t_1)$. So (3) may be written

$$P\{Z(t_1) = \ldots = Z(t_n) = 1\} = P\{Z(t_1) = 1\} P\{Z(t_2-t_1) = 1\}\ldots$$
$$P\{Z(t_n-t_{n-1}) = 1\}.$$

Then

$$P\{Z(t_1)=1\}P\{Z(t_2-t_1)=Z_3(t_3-t_2)=\ldots=Z(t_n-t_1)=1\}$$

$$= P\{Z(t_1)=1\}P\{Z(t_2-t_1)=1\}P\{Z[(t_3-t_1) - (t_2-t_1)]=1\}\ldots$$
$$P\{Z[(t_n-t_1) - (t_{n-1}-t_1)] = 1\}$$
$$= P\{Z(t_1) = 1\}P\{Z(t_2-t_1) = 1\}\ldots P\{Z(t_n-t_{n-1}) = 1\}$$

$$= \prod_{r=1}^{n} p(t_r - t_{r-1})$$

$$= P\{Z(t_1) = Z(t_2) = \ldots = Z(t_n) = 1\}.$$

Conversely, consider a process $\{Z(t);\ t>0\}$ taking values in $\{0,1\}$ such that, given

$0=t_0<t_1 \ldots <t_n,$

we have

$$P\{Z(t_1)=Z(t_2)=\ldots=Z(t_n)=1\}=P\{Z(t_1)=1\}P\{Z(t_2-t_1)=\ldots=Z(t_n-t_1)=1\}.$$

So

$$P\{Z(t_2-t_1)=\ldots=Z(t_n-t_1)=1\}=P\{Z(t_2-t_1)=1\}P\{Z[(t_3-t_1)-(t_2-t_1)]=$$
$$Z[(t_4-t_1)-(t_2-t_1)]=\ldots=Z[(t_n-t_1)-(t_2-t_1)] = 1\}$$
$$= P\{Z(t_2-t_1)=1\}P\{Z(t_3-t_2)=Z(t_4-t_2)=\ldots=Z(t_n-t_2)=1\}.$$

Inductively we have

$$P\{Z(t_1)=Z(t_2)=\ldots=Z(t_n)=1\}=P\{Z(t_1)=1\}P\{Z(t_2-t_1)=1\}\ldots$$
$$P\{Z(t_n-t_{n-1}) = 1\}. \tag{6}$$

Define $p:(0,\infty) \to [0,\infty)$ by

$$p(t) = p\{Z(t)=1\}. \tag{7}$$

Then (6) becomes

$$P\{Z(t_1)=Z(t_2)=\ldots=Z(t_n)=1\}= \prod_{r=1}^{n} p(t_r-t_{r-1}). \tag{8}$$

Hence Z is a regenerative phenomenon.

1.3 Kingman's inequalities and semi-p-functions.

Since a regenerative phenomenon only takes values

0 and 1, the defining relation (1.2.3) may be rewritten

$$E\left(\prod_{r=1}^{n}(Z(t_r))\right) = \prod_{r=1}^{n} p(t_r - t_{r-1}). \tag{1}$$

For any values of $t_1,\ldots,t_N$; $\alpha_1,\ldots,\alpha_N \in \{0,1\}$ consider the random variable $Y_r = (-1)^{\alpha_r+1}(Z(t_r) + \alpha_r - 1)$.

If $Z(t_r) = \alpha_r = 1$, then $Y_r = 1$;
if $Z(t_r) = \alpha_r = 0$, then $Y_r = 1$; or
if $Z(t_r) \neq \alpha_r$, then $Y_r = 0$.

So Y_r takes values 0 and 1. Therefore we have

$$E\left(\prod_{r=1}^{N} Y_r\right) = (1)\ P\left\{\prod_{r=1}^{N} Y_r = 1\right\} + (0)\ P\left\{\prod_{r=1}^{N} Y_r = 0\right\}$$

$$= P\{Y_r = 1;\ 1 \leq r \leq N\}.$$

In other words

$$P\{Z(t_r)=\alpha_r;\ 1\leq r\leq N\} = E\left(\prod_{r=1}^{N}(-1)^{\alpha_r+1}(Z(t_r) + \alpha_r - 1)\right). \tag{2}$$

The product on the right of (2) may be expanded to give a linear combination of values of Z, whose expectations are given by (1). Thus the finite-dimensional distributions of the process Z are expressible in terms of p by equations of the form

$$P\{Z(t_r)=\alpha_r,\ 1\leq r\leq N\} = \Phi(t_1,t_2,\ldots,t_N;\alpha_1,\ldots,\alpha_N;p) \tag{3}$$

where for given α_r, Φ is a polynomial in the numbers $p(t_r - t_s)$, $(0\leq s<r\leq N)$.

Fix a number T and let

$$f = \prod_{r=1}^{m} Z(t_r) \ , \ g = \prod_{s=1}^{n} Z(u_s)$$

where

$$0 = t_o < t_1 < \ldots < t_m < T < u_1 < \ldots < u_s .$$

Then, by (1) we have

$$E(fZ(T)g) = (\prod_{r=1}^{m} p(t_r - t_{r-1}))p(T-t_m)p(u_1-T) \prod_{s=2}^{n} p(u_s - u_{s-1})$$

$$= E(fZ(t))p(u_1-T) \prod_{s=2}^{n} p\{(u_s-T)-(u_{s-1}-T)\}$$

$$= E(fZ(T)) \ E(g')$$

where $g' = \prod_{s=1}^{n} (u_s - T)$.

We note that (4) holds if f and g are combinations of products of the above form. We have shown in particular that

$$P\{Z(t_r = \alpha_r \ (1 \leq r \leq m), Z(T)=1, \tilde{Z}(u_s) = \beta_s (1 \leq s \leq n)\}$$

$$= P\{Z(t_r) = \alpha_r (1 \leq r \leq m), Z(T)=1\} \quad P\{Z(u_s - T) = \beta_s (1 \leq s \leq n)\}, \tag{5}$$

$\alpha_r, \epsilon \beta_s \ \{0,1\}$.

It follows from the Kolmogorov extension theorem that if A is any member of the smallest σ-field with respect to which Z(t) is measurable for all t<T, and if B belongs to the smallest σ-field with respect to which Z(t) is measurable for all t>T, then

$$P\{A\cap\{Z(T)=1\}\cap B\} = P\{A\cap\{Z(T)=1\}\}P\{B'\} \tag{6}$$

where B' is obtained from B by the shift $u\to u-T$. Taking A to be the space on which Z is defined, (6) takes the form

$$P(\{Z(T)=1\}\cap B\} = P\{Z(T)=1\}P\{B'\}. \tag{7}$$

We have established the following theorem.

Theorem 1:

Z is a regenerative phenomenon iff for any $T > 0$ with $P(Z(T)=0)$, the processes $\{Z(T);\ t>0\}$ and $\{Z(T+t)|Z(T)=1;\ t>0\}$ have the same distributions. The probabilities (3) are of course non-negative, so that any p-function must satisfy the inequalities

$$\Phi(t_1, t_2, \ldots, t_N; \alpha_1, \ldots, \alpha_N; p) \geq 0. \tag{8}$$

By varying choices of the α_r we get 2^N different inequalities for each N. So (8) gives rise to an infinite family of inequlaities. But in view of (5) if $\alpha_k=1$ for some $k\neq N$, then

$$\Phi(t_1,t_2,\ldots,t_N;\alpha_1,\alpha_2,\ldots,\alpha_N;p) =$$

$$\Phi(t_1,\ldots,t_k;\alpha_1,\ldots,\alpha_k;p) \times \Phi(t_{k+1}-t_k,\ldots,t_n-t_k;\alpha_{k+1},\ldots,\alpha_N;p).$$

In these cases we see that the inequality

$$\Phi(t_1,\ldots,t_N;\alpha_1,\ldots,\alpha_N;\ p) \geq 0$$

is implied by the inequalities

$$\Phi(t_1,\ldots,t_k;\alpha_1,\ldots,\alpha_k;p)\geq 0 \text{ and}$$

$$\Phi(t_{k+1}-t_k,\ldots,t_N-t_k;\alpha_{K+1},\ldots,\alpha_N;p) \geq 0$$

In view of this (8) only gives new information if $\alpha_1=\alpha_2=\ldots=\alpha_{N-1} = 0$.

Define $F_N = F(t_1,t_2,\ldots,t_N;p)=\phi(t_1,\ldots,t_N;0,0,\ldots,1;\ p)$.

For any given $t_1,\ldots,t_N$ we have

$$\sum P\{Z(t_r) = \alpha_r;\ 1\le r\le N\} = 1, \qquad (9)$$

where this sum is over all possible combinations of the α_r.

Expanding (9) explicitly we have

$$\begin{aligned} &P(Z(t_1) = 1) \\ &+ P(Z(t_1) = 0,\ Z(t_2) = 1 \\ &+ \ldots \\ &+ P(Z(t_1) = \ldots = Z(t_{N-1})=0,\ Z(t_N)=1) \\ &+ P(Z(t_1) = \ldots + Z(t_N) = 0) = 1. \end{aligned}$$

In other words,

$$\sum_{n=1}^{N} F_n+\phi(t_1,t_2,\ldots,t_N;\ 0,0,\ldots,0;\ p) = 1. \qquad (10)$$

Putting $G(t_1,t_2,\ldots,t_N;\ p)=\phi(t_1,t_2,\ldots,t_N;\ 0,\ 0,\ldots,0;\ p)$ we have from (10)

$$G(t_1,\ldots,t_N;\ p) = 1 - \sum_{n=1}^{N} F_n .$$

Therefore the inequalities (8) are equivalent to the inequalities

$$F(t_1,\ldots,t_N;\ P) \ge 0, \quad \sum_{n=1}^{N} F(t_1,\ldots,t_N;\ p) \le 1.$$

Using (2) we find explicit expressions for the polynomials F:

$$F(t_1,\ldots,t_N;\ p) = p(t_N) - \sum_{1\leq i<N} p(t_i)p(t_N-t_i)$$

$$+ \sum_{1\leq i<j<N}\sum p(t_i)p(t_j-t_i)p(t_N-t_j)$$

$$+ \ldots + (-1)^{N-1}p(t_1)p(t_2-t_1)\ldots p(t_N-t_{N-1}).$$

These results give another characterization of regenerative phenomenon [15]:

For a given function p on $(0,\infty)$ there exists a regenerative phenomenon Z with p-function p iff p satisfies the inequalities

$$F(t_1,\ldots,t_N;\ p) \geq 0, \tag{13}$$

$$G(t_1,\ldots,t_N;\ p) \geq 0 \tag{14}$$

whenever $0<t_1<\ldots<t_N$, F is given by (12) and $G(t_1,\ldots,t_N;\ p)= 1-\sum_{n=1}^{N} F(t_1,\ldots,t_N;\ p)$.

A semi-p-function [16] is a more general p that enjoys property (13) but not necessarily property (14).

When N=1, (13) and (14) merely assert that

$$0\leq p(t)\leq 1. \tag{15}$$

The inequalities with N=2 are less trivial. From (2) we obtain

$$F_2 = \phi(t_1, t_2;\ 0,1;\ p)$$

$$= E\ (-1)(Z(t_1)-1)Z(t_2)$$

$$= -E(Z(t_1)Z(t_2)) + E(Z(t_2)$$

$$= p(t_2) - p(t_1)\ p(t_2-t_1) \geq 0.$$

Putting $s=t_1$, $t=t_2-t_1$ we have

$$F_1 = p(s) \geq 0\ ,$$

$$F_2 = p(s+t) - p(s)p(t) \geq 0.$$

Since $F_1+F_2 \leq 1$, the inequalities take the form

$$p(s)p(t) \leq p(s+t) \leq p(s)p(t) + 1 - p(s). \qquad (16)$$

CHAPTER 2

Measurability and Continuity

The point of this chapter is to highlight the startling fact that if any of the four entities that we are considering enjoys the property of Lebesgue measurability then, roughly speaking, it has to have continuity.

Theorem 2.1 is due to Doob [6] but the proof given here is an improved version due to Chung [2]. Both Theorem 2.2 and 2.3 are due to Cornish [3], [4]. Theorem 2.2 reveals that if we dispose of the property of stochasticity then Theorem 2.1 still holds, although we have to use the idea of approximate continuity to prove it. Finally, Theorem 2.3 shows how the "diagonal" case comes out.

Let (p_{ij}) be a semigroup of positive matrices. If each p_{ij} is a (Lebesgue) measurable function in $(0,\infty)$ then the matrix is said to be measurable.

Theorem 2.1

If (p_{ij}) is a measurable transition matrix, then each element p_{ij} is a continuous function in $(0,\infty)$.

Proof:

For $0<s<t$ and fixed $h>0$ we have

$$\sum_j |p_{ij}(t+h) - p_{ij}(t)| = \sum_j \left|\sum_k p_{ik}(s+h)p_{kj}(t-s) - \sum_k p_{ik}(s)p_{kj}(t-s)\right|$$

$$= \sum_j \left|\sum_k (p_{ik}(s+h) - p_{ik}(s))p_{kj}(t-s)\right|$$

$$\leq \sum_j\sum_k |p_{ik}(s+h) - p_{ik}(s)|p_{kj}(t-s)$$

$$= \sum_k |p_{ik}(s+h - p_{ik}(s)|\sum_j p_{kj}(t-s)$$

$$= \sum_k |p_{ik}(s+h) - p_{ik}(s)| \qquad (1)$$

Since the matrix is measurable, upon integrating over $[0,\delta]$, $0<\delta\leq t$ we obtain

$$\frac{1}{\delta}\int_0^\delta \sum_j |p_{ij}(t+h) - p_{ij}(t)|ds \leq \frac{1}{\delta}\int_0^\delta \sum_k |p_{ik}(s+h) - p_{ik}(s)|ds$$

whence, by Monotone Convergence,

$$\sum_j |p_{ij}(t+h) - p_{ij}(t)| \leq \sum_k \frac{1}{\delta}\int_0^\delta |p_{ik}(s+h) - p_{ik}(s)|ds \quad . (2)$$

Now, for $0\le h\le\delta$

$$\int_0^\delta |p_{ik}(s+h) - p_{ik}(s)|ds \le \int_0^\delta (p_{ik}(s+h) + p_{ik}(s))ds$$

$$= \int_0^\delta p_{ik}(s+h)ds + \int_0^\delta p_{ik}(s)\,ds$$

$$= \int_h^{\delta+h} p_{ik}(s)ds + \int_0^\delta p_{ik}(s)\,ds$$

$$= \int_h^\delta p_{ik}(s)\,ds + \int_\delta^{\delta+h} p_{ik}(s)ds + \int_0^\delta p_{ik}(s)ds$$

$$\le \int_0^\delta p_{ik}(s)ds + \int_\delta^{2\delta} p_{ik}(s)ds + \int_0^\delta p_{ik}(s)ds$$

$$\le 2\int_0^{2\delta} p_{ik}(s)ds \quad .$$

Hence the second series in (2) is dominated by

$$\sum_k 2/\delta \int_0^{2\delta} p_{ik}(s)\,ds$$

which converges. Consequently the series

$$\sum_k \frac{1}{\delta}\int_0^\delta |p_{ik}(s+h) - p_{ik}(s)|ds$$

converges uniformly in $h\in[0,\delta]$. But by a well-known theorem (see e.g. TITCHMARSH [1; p. 377]), we have for each k,

$$\lim_{h\to 0}\int_0^\delta |p_{ik}(s+h) - p_{ik}(s)|ds = 0 \quad . \qquad (3)$$

From (2) we have, for $t\ge\delta$,

$$0 \leq \limsup_{h \to 0} \sum_j |p_{ij}(t+h) - p_{ij}(t)|$$

$$\leq \limsup_{h \to 0} \sum_k \frac{1}{\delta} \int_0^\delta |p_{ik}(s+h) - p_{ik}(s)| \, ds$$

$$= \sum_k \frac{1}{\delta} \lim_{h\to 0} \int_0^\delta |p_{ik}(s+h) - p_{ik}(s)| \, ds$$

$$= 0,$$

the interchange of summation and limit being justified by uniform convergence. We conclude that

$$\sum_k |p_{ik}(t+h) - p_{ik}(t)|$$

tends to zero uniformly in $t \in [\delta, \infty]$ as h tends to zero. An immediate consequence is that each p_{ik} is uniformly continuous in $t \in [\delta, \infty]$. Thus the theorem is proved.

Let P be a semigroup of positive matrices. Denote by F the set of states i for which either $p_{ij}(t) \equiv 0$ for all j or $p_{ji} \equiv 0$ for all j. The following generalization of theorem 2.1 is due to Cornish [3].

Theorem 2.2:

Let P be a semigroup of positive matrices. If P is measurable then any $p_{ij}(t)$ is continuous for $t>0$ unless both $i, j \in F$.

Proof:

The result is trivially true if $p_{ij} \equiv 0$. Assume $p_{ij}(t)$ is not identically zero and that $j \notin F$. Then $p_{j\ell}(t_o) > 0$ for some ℓ and some $t_o > 0$. A measurable function is approximately continuous almost everywhere (see e.g. HOBSON [II; p. 257]). So, given $t>0$ we can find a positive number $t_1 < t_o$ such that all the p_{ij}'s are approximately continuous at t_1 and t_1+t. From the semigroup property

$$p_{j\ell}(t_o) = \sum_m p_{jm}(t_1)\; p_{m\ell}(t_o-t_1)$$

we deduce that there is a state m such that $p_{jm}(t_1) > 0$. Since p_{jm} is approximately continuous at t_1, given $\varepsilon > 0$ there exists a set G_1 having metric density 1 at t_1 such that for all $g\epsilon G_1$,

$$|p_{jm}(t_1) - p_{jm}(g)| \leq \varepsilon/M, \tag{4}$$

where M is chosen so that $p_{jm}(t_1) - \varepsilon/M > 0$. From (4) we have, for all $g\epsilon G_1$,

$$p_{jm}(g) \geq p_{jm}(t_1) - \varepsilon/M > 0. \tag{5}$$

Similarly, since p_{im} is approximately continuous at t_1+t, for $\varepsilon>0$, there is a set G_2 such that, for all $g\epsilon G_2$,

$$0 \leq p_{im}(g) \leq \varepsilon + p_{im}(t_1+t) . \tag{6}$$

Let $S_1 = G_1 - t_1$, $S_2 = G_2 - (t_1+t)$. Both S_1 and S_2 have metric density 1 at 0. Using the measure theoretic identity

$$m(S_1 \cap S_2) + m(S_1 \cup S_2) = m(S_1) + m(S_2)$$

we can easily deduce that $S_1 \cap S_2$ also has metric density 1 at 0.

Put $S = S_1 \cap S_2$, $c = \varepsilon + p_{im}(t_1+t)$, $d = p_{jm}(t_1) - \varepsilon/\mu > 0$ then we have found a measurable set $S \subset (-t_1,\infty)$ having metric density 1 at 0, and constants c and d, such that for $s \in S$ we have

$$0 \leq p_{im}(t_1+t+s) \leq c, \quad p_{jm}(S+t_1) \geq d > 0 \quad . \tag{7}$$

For $s \in S$, $s<t$ we have

$$p_{km}(t_1+t) \geq p_{kj}(t-s)\, p_{jm}(t_1+s) \geq d\, p_{kj}(t-s) \quad . \tag{8}$$

Since S has metric density 1 at 0 we can choose δ such that $0 < \delta < \min(\tfrac{1}{2}t_1, \tfrac{1}{2}t)$ and such that $m(S \cap [0,2\delta]) \geq 7/4\,\delta$ (where m is Lebesgue measure), then $m(S \cap [\tfrac{1}{2}\delta, 3/2\,\delta]) \geq 3/4\delta$.

For $h \in [-\tfrac{1}{4}\delta, \tfrac{1}{4}\delta]$ let

$$S_h = \{s \in S \cap [\tfrac{1}{2}\delta, 3/2\delta] \mid s+h \in S\} \quad .$$

Then

$$mS_h \geq 3/4\delta - \tfrac{1}{4}\delta = \tfrac{1}{2}\delta \quad . \tag{9}$$

Now, for $h \in [-\tfrac{1}{2}\delta, \tfrac{1}{2}\delta]$, $s \in (\tfrac{1}{2}\delta, 3/2\delta)$,

$$\begin{aligned} |p_{ij}(t+h) - p_{ij}(t)| &= \left|\sum_k p_{ik}(s+h)\, p_{kj}(t-s) - \sum_k p_{ik}(s)\, p_{kj}(t-s)\right| \\ &= \left|\sum_k (p_{ik}(s+h) - p_{ik}(s))\, p_{kj}(t-s)\right| \\ &\leq \sum_k |p_{ik}(s+h) - p_{ik}(s)|\, p_{kj}(t-s) \quad . \end{aligned}$$

The above inequality holds for all $s \in S_h$. Integrating with respect to s over S_h we obtain, using (8) and (9),

$$|p_{ij}(t+h) - p_{ij}(t)| \leq \frac{1}{mS_h} \sum_k \int_{S_h} |p_{ik}(s+h) - p_{ik}(s)| p_{kj}(t-s) ds$$

$$\leq \sum_k \frac{2p_{km}(t_1+t)}{\delta d} \int_{S_h} |p_{ik}(s+h) - p_{ik}(s)| ds$$

$$= \sum_k \frac{2p_{km}(t_1+t)}{\delta d} \int_{\frac{1}{2}\delta}^{3/2\delta} |p_{ik}(s+h) I_S(s+h) - p_{ik}(s) I_S(s)| ds, \qquad (10)$$

where I_S is the indicator function of S. But

$$\int_{\frac{1}{2}\delta}^{3/2\delta} |p_{ik}(s+h) I_S(s+h) - p_{ik}(s) I_S(s)| ds$$

$$\leq \int_{\frac{1}{2}\delta}^{3/2\delta} p_{ik}(s+h)\ I_S(s+h)\ ds + \int_{\frac{1}{2}\delta}^{3/2\delta} p_{ik}(s) I_S(s) ds$$

$$\leq \int_{\frac{1}{2}\delta+h}^{3/2h+\delta} p_{ik}(s) I_S(s)\ ds + \int_{\frac{1}{2}\delta}^{3/2\delta} p_{ik}(s)\ I_S(s)\ ds$$

$$\leq 2 \int_0^{2\delta} p_{ik}(s)\ I_S(s)\ ds$$

$$= 2 \int_{S \cap [0,2\delta]} p_{ik}(s)\ ds.$$

Hence the series on the right in (10) is dominated by

$$\sum_k \frac{4p_{km}(t_1+t)}{\delta d} \int_{S \cap [0,2\delta]} p_{ik}(u)\ du$$

$$= \frac{4}{\delta d} \int_{S \cap [0,2\delta]} p_{im}(t_1+t+u)\ du$$

$$\leq \frac{4}{\delta d} \cdot 2\delta c < \infty,$$

and consequently converges uniformly in $h \in [-\frac{1}{2}\delta, \frac{1}{2}\delta]$. But for each k,

$$\lim_{h \to 0} \int_{\frac{1}{2}\delta}^{3/2\delta} |p_{ik}(s+h) I_S(s+h) - p_{ik}(s) I_S(s)| ds = 0,$$

so that, using uniform convergence in (10),

$$|p_{ij}(t+h) - p_{ij}(t)| \to 0 \text{ as } h \to 0.$$

The result, if $i \notin F$, follows at once by considering the transpose of P.

Cornish [3] gives an example of a function p_{ij}, $i, j \in F$, which is not continuous for all $t>0$.

For p-functions and semi-p-functions we have the following analogue of theorems 1 and 2, due to Cornish [4].

THEOREM 2.3:

A measurable (semi)-p-function is either continuous or zero almost everywhere.

PROOF:

Suppose p is a measurable semi-p-function and let $S = \{t>0; p(t)>0\}$ have positive measure. Given $x>0$ we shall show that p is continuous at x. Writing $t_1=s$, $t_2=s+t$, $t_3=s+t+u$, the first three Kingman F-inequalities become

$$p(s) \geq 0, \qquad (11)$$

$$p(s+t) \geq p(s)\, p(t), \qquad (12)$$

$$p(s+t+u) + p(s)\, p(t)\, p(u) \geq p(s)\, p(t+u) + p(s+t)\, p(u). \qquad (13)$$

Now if $s, t \in S$ then (12) implies that $s + t \in S$. Hence S is a semi-module in the sense of Hille and Phillips (see e.g. HILLE and PHILLIPS [1; p. 237]). Furthermore (12) implies that $-\log p(s)$ is finite and subadditive on S. By theorems of Hille and Phillips [7; theorems 7.3.2, 7.4.2] there exists $a \in (x,\infty)$ such that $(a,\infty) \subseteq S$ and p is bounded on all closed subintervals of $(2a,\infty)$. Given $\varepsilon > 0$, define for $n=1, 2, 3, \ldots,$

$$S_n = \{y \in (3a,4a) : \frac{1}{p(y-x)} \leq e^{na},\ \frac{p(y)}{p(y-x)} < \varepsilon e^{nx},\ \frac{p(2y-x)}{p^2(y-x)} < \varepsilon e^{nx}\}. \tag{14}$$

For $y \in (3a, 4a)$ put $p(y-x) = \alpha$; $p(2y-x) = \beta$; $p(y) = \gamma$. We can find integers n_1, n_2 and n_3 such that

$$\frac{1}{\alpha} < e^{n_1 a}\ ,\ \frac{\gamma}{\alpha} < \varepsilon\, e^{n_2 x}\ ,\ \text{and}\ \frac{\beta}{\alpha^2} < \varepsilon\, e^{n_3 x}.$$

Letting $n = \max(n_1, n_2, n_3)$ we have $y \in S_n$. This and (14) imply that

$$\bigcup_{n=1}^{\infty} S_n = (3a, 4a)\ ,$$

so there exists a positive integer N such that $m(S_N) > 0$. Now define

$$p_1(t) = e^{-Nt} p(t)\ ,\ t > 0.$$

An easy calculation shows that $p_1(t)$ is a semi-p-function. Also, if $y \in S_N$, we have from (14) that

$$\frac{1}{p_1(y-x)} < e^{5Na} , \frac{p_1(y)}{p_1(y-x)} < \varepsilon , \frac{p_1(2y-x)}{p_1^2(y-x)} < \varepsilon . \tag{15}$$

Let $h \in (-x,x)$ and $y \in S_N$. From (12) we have

$$p_1(y+h) \geq p_1(x+h)\, p_1(y-x)$$

whence

$$\begin{aligned} p_1(x+h) - p_1(x) &\leq \frac{p_1(y+h)}{p_1(y-x)} - p_1(x) \\ &\leq \frac{p_1(y+h)}{p_1(y-x)} \\ &= \frac{1}{p_1(y-x)} [p_1(y+h) - p_1(y)] + \frac{p_1(y)}{p_1(y-x)} . \end{aligned} \tag{16}$$

Applying (13) with $s=u=y-x$, $t=x+h$ we obtain $p_1(2y-x-h) + p_1^2(y-x)\, p_1(x+h) \geq p_1(y-x)\, p_1(y+h) + p_1(y+h)\, p_1(y-x)$

whence

$$p_1(x+h) - p_1(x) \geq \frac{2p_1(y+h)}{p_1(y-x)} - \frac{p_1(2y-x-h)}{p_1^2(y-x)} - p_1(x) . \tag{17}$$

But $p_1(y) = p_1(y-x+x) \geq p_1(y-x)\, p_1(x)$ which implies

$$\frac{p_1(y)}{p_1(y-x)} \geq p_1(x)$$

So (17) becomes

$$\begin{aligned} p_1(x+h) - p_1(x) &\geq \frac{2}{p_1(y-x)} [p_1(y+h) - p_1(y)] \\ &- \frac{1}{p_1^2(y-x)} [p_1(2y-x+h) - p_1(2y-x)] - \frac{p_1(2y-x)}{p_1^2(y-x)} . \end{aligned} \tag{18}$$

Using the algebraic result that $a \leq x \leq b \rightarrow |x| \leq |a| + |b|$ we obtain from (15), (16) and (18) that for all $h \in (-x,x)$ and all $y \in S_N$,

$$|p_1(x+h) - p_1(x)| \leq 3e^{5Na}|p_1(y+h) - p_1(y)| + e^{10Na} |p_1(2y-x+h) - p_1(2y-x)| + 2\varepsilon. \quad (19)$$

Integrating (19) over S_N with respect to y we obtain

$$\begin{aligned}|p_1(x+h) - p_1(x)| &\leq \frac{3e^{5Na}}{m(S_N)}\int_{S_N} |p_1(y+h) - p_1(y)|\,dy \\ &+ \frac{e^{10Na}}{m(S_N)}\int_{S_N} |p_1(2y-x+h) - p_1(2y-x)| + 2\varepsilon \\ &\leq \frac{3e^{5Na}}{m(S_N)}\int_{3a}^{4a} |p_1(y+h) - p_1(y)|\,dy \\ &+ \frac{e^{10Na}}{2m(S_N)}\int_{6a-x}^{8a-x} |p_1(y+h) - p_1(y)|\,dy + 2\varepsilon \\ &\leq \frac{3e^{10Na}}{m(S_N)}\int_{3a}^{8a} |p_1(y+h) - p_1(y)|\,dy \\ &+ \frac{3e^{10Na}}{m(S_N)}\int_{3a}^{8a} |p_1(y+h) - p_1(y)|\,dy + 2\varepsilon \\ &= \frac{6e^{10Na}}{m(S_N)}\int_{3a}^{8a} |p_1(y+h) - p_1(y)|\,dy + 2\varepsilon. \quad (20)\end{aligned}$$

But $x<a$ implies that $(3a-x,\ 8a+x) \subset (2a,\infty)$. Thus p_1 is bounded on $(3a-x,\ 8a+x)$ and consequently integrable. Then

$$\lim_{h\to 0} \int_{3a}^{8a} |p_1(y+h) - p_1(y)|\ dy = 0.$$

From (20) we obtain

$$0 \le \limsup_{h\to 0} |p_1(x+h) - p_1(x)|$$

$$\le \limsup_{h\to 0} \frac{6e^{10Na}}{m(S_N)} \int_{3a}^{8a} |p_1(y+h) - p_1(y)|\ dy + 2\varepsilon$$

$$= 2\varepsilon\ .$$

Since $\varepsilon>0$ is arbitrary, it follows that p_1, and hence p, is continuous for $x>0$.

NOTE: Kingman [14] first proved the result for p-functions only. His proof used the powerful representation theorem for p-functions (see e.g. KINGMAN [15 ; Chapter 3]). These ideas take us outside the scope of this thesis and his original proof will not be presented here.

CHAPTER 3

Standardness and Continuity

This chapter surveys the situation for "standard" entities. Most authors are concerned only with such entities, and Chung [2], in the case of transition functions, gives a formal justification, based on probabilistic grounds, for imposing such a continuity condition at $t=0$. Although it may appear that this chapter is repetitive, it turns out that standardness is a stronger property than measurability. In the stochastic case standardness implies uniform continuity on $[0,\infty)$. In Theorem 3.1 we give the obvious simplification of Theorem 2.1 and set out the corresponding proof for p-functions in Theorem 3.3.1. In the nonstochastic case standardness implies continuity without exception. Theorem 3.2.1 is due to Jurkat [9] and Theorem 3.4.1 is due to Kingman [16].

3.1 Standard Transition Functions

Theorem 3.1

If $P(t)$ is a standard transition matrix then the functions $p_{ij}(t)$ are uniformly continuous for $t\epsilon[0,\infty)$, $j\epsilon S$.

Proof:

For $s>0$, $t>0$,

$$p_{ij}(t+s) - p_{ij}(t) = \sum_k p_{ik}(s)\, p_{kj}(t) - \sum_k p_{ik}(0)\, p_{kj}(t)$$

$$= \sum_k (p_{ik}(s) - p_{ik}(0))\, p_{kj}(t)\ .$$

Then,

$$|p_{ij}(t+s) - p_{ij}(t)| = |\sum_k (p_{ik}(s) - p_{ik}(0))\, p_{kj}(t)|$$

$$\leq \sum_k |p_{ik}(s) - p_{ik}(0)|\, p_{kj}(t)$$

$$\leq \sum_k |p_{ik}(s) - p_{ik}(0)|$$

$$= |p_{ii}(s) - p_{ii}(0)| + \sum_{k\neq i} |p_{ik}(s) - p_{ik}(0)|$$

$$= 1 - p_{ii}(s) + \sum_{k\neq i} p_{ik}(s)$$

$$\leq 2(1-p_{ii}(s))\ , \qquad (1)$$

since P is a transition matrix. But $p_{ii}(t)$ is standard, so

(1) implies that

$$\lim_{s\downarrow 0} p_{ij}(t+s) = p_{ij}(t) , \qquad (2)$$

i.e. $p_{ij}(t)$ is continuous from the right for all $t\geq 0$.

In (1) let $s=t_o-t$, $0<t<t_o$, so that $s+t = t_o$. Then, from (1) it follows that

$$|p_{ij}(t_o) - p_{ij}(t)| \leq 2(1-p_{ii}(t_o-t) ,$$

whence

$$\lim_{t\uparrow t_o} p_{ij}(t) = p_{ij}(t_o) . \qquad (3)$$

Both (2) and (3) together imply that $p_{ij}(t)$ is continuous for all $t\geq 0$. Furthermore, in view of (1), we have that $p_{ij}(t)$ is uniformly continuous on $[0,\infty)$ and for $j\in S$.

3.2 Semigroups of Positive Matrices

Let $P(t)$ be a semigroup of positive matrices such that

$$\lim_{t\to 0} P(t) = I \text{ ; i.e. } \lim_{t\to 0} p_{ij}(t) = \delta_{ij} , \qquad (1)$$

where δ_{ij} is the Kronecker symbol.

Lemma: $p_{ii}(t) > 0$, for all $t \geq 0$, $\forall i\in S$.

Proof:

From the semigroup property we have

$$p_{ii}(s+t) \geq p_{ii}(s)\, p_{ii}(t) . \qquad (2)$$

Repeated use of (2) implies

$$p_{ii}(t) \geq p_{ii}(\tfrac{t}{n})^n, \quad \forall n \geq 1.$$

Since $p_{ii}(t) \to 1$ as $t \to 0$, $\exists T$ such that $p_{ii}(t) > \frac{1}{2}$ for all $t < T$. For $t > 0$, choose n large enough so that $t/n < T$. Then

$$p_{ii}(t) \geq p_{ii}(\tfrac{t}{n})^n > (\tfrac{1}{2})^n > 0 .$$

Furthermore,

$$\liminf_{s \uparrow t} p_{ii}(s) \geq \liminf_{s \uparrow t} p_{ii}(\tfrac{s}{n})^n \geq 0.$$

Theorem 3.2.1

All $p_{ij}(t)$ are continuous for $t \geq 0$.

Proof:

For $x \geq 0$ define

$$\alpha = \liminf_{s \downarrow 0} p_{ij}(x+s) \; , \; \beta = \limsup_{t \downarrow 0} p_{ij}(x+t) \; . \tag{3}$$

There exist positive sequences $\{s_n\}$, $\{t_n\}$ converging to zero with $0 < t_n < s_n$ such that

$$\alpha = \lim_{n\to\infty} p_{ij}(x+s_n) \; , \; \beta = \lim_{n\to\infty} p_{ij}(x+t_n). \tag{4}$$

Now,

$$\begin{aligned} p_{ij}(x+s_n) &= p_{ij}(s_n-t_n+t_n+x) \\ &\geq p_{ii}(S_n-t_n)\, p_{ij}(t_n+x) \; , \end{aligned} \tag{5}$$

whence, using (1) and (4)

$$\alpha = \lim_{n\to\infty} (x+s_n) \geq \lim_{n\to\infty} p_{ii}(s_n - t_n)\ p_{ij}(x+t_n) \tag{6}$$

$$= \beta \quad .$$

We infer that the limits $p_{ij}(x+)$ exist (possibly infinite) $\forall x \geq 0$. A similar argument shows that for $x > 0$, $p_{ij}(x-)$ exists.

For $x \geq 0$, we have, by the semigroup property

$$p_{ij}(x+s) \geq p_{ii}(s)\ p_{ij}(x)$$

whence

$$p_{ij}(x+) \geq p_{ij}(x) \quad . \tag{7}$$

Also, for $x > 0$, $0 < t < x$, we have

$$p_{ij}(x) = p_{ij}(x+t-t)$$

$$\geq p_{ii}(t)\ p_{ij}(x-t) \quad .$$

Letting $t \to 0$ we obtain

$$p_{ij}(x-) \leq p_{ij}(x) < \infty \quad . \tag{8}$$

For $x \geq 0$, $0 < t < s$, s fixed, we have

$$p_{ij}(x+s) = p_{ij}(x+s-t+t)$$

$$\geq p_{ii}(s-t)\ p_{ij}(x+t) \quad .$$

Letting $t \downarrow 0$, we have

$$p_{ij}(x+s) \geq p_{ii}(s-)\ p_{ij}(x+) \quad . \tag{9}$$

Since $p_{ii}(t) \to 1$ as $t \to 0$ $\exists\, \delta > 0$ such that $p_{ii}(t) > \frac{1}{2}$ for all $0 < t < \delta$. Then

$$p_{ii}(t-) = \lim_{s \uparrow t} p_{ii}(s) > \tfrac{1}{2} .$$

Together with (9) this implies

$$p_{ij}(x+) < \infty. \tag{10}$$

Altogether we have from (7), (8), and (10)

$$0 \leq p_{ij}(x-) \leq p_{ij}(x) \leq p_{ij}(x+) < \infty \tag{11}$$

for $x \geq 0$ as far as these quantities are defined. Such a function has at most a countable set of discontinuities (see e.g. SAKS [1; P. 261]). Let D be the set where some p_{ij} has a discontinuity. Then D is countable.

Given any $t > 0$ there exists an s such that $0 < s < t$ and $s \notin D$. If $0 < \epsilon < s$ then

$$p_{ij}(t-\epsilon) = p_{ij}(t-s+s-\epsilon)$$

$$= \sum_k p_{ik}(t-s)\, p_{kj}(s-\epsilon) \quad .$$

Hence by Fatou's lemma,

$$\liminf_{\epsilon \downarrow 0} p_{ij}(t-\epsilon) \geq \sum_k \liminf_{\epsilon \downarrow 0} p_{ik}(t-s)\, p_{kj}(s-\epsilon)$$

$$= \sum_k p_{ik}(t-s)\, p_{kj}(s) \quad \text{(since } s \notin D)$$

$$= p_{ij}(t) \ ;$$

i.e. $p_{ij}(t-) \geq p_{ij}(t)$.

In view of (8) we have for all $t > 0$, that the p_{ij}'s are left continuous at t.

Assume D is non-empty, then there exists a $t_o \in D$ such that

$$p_{ij}(t_o+) > p_{ij}(t), \tag{12}$$

for some p_{ij}. Then for $s > 0$,

$$\begin{aligned}
p_{ij}(t_o+s+) &= \lim_{\delta \downarrow 0} p_{ij}(t_o + s + \delta) \\
&= \lim_{\delta \downarrow 0} \sum_k p_{ik}(s)\ p_{kj}(t_o+\delta) \\
&= \lim_{\delta \downarrow 0} p_{ii}(s)\ p_{ij}(t_o+\delta) + \lim_{\delta \downarrow 0} \sum_{k \neq i} p_{ik}(t_o+\delta)\, p_{ik}(s) \\
&\geq p_{ii}(s)\ p_{ij}(t_o+) + \liminf_{\delta \downarrow 0} \sum_{k \neq i} p_{ik}(s)\ p_{kj}(t_o+\delta) \\
&\geq p_{ii}(s)\ p_{ij}(t_o+) + \sum_{k \neq i} p_{ik}(s)\ \liminf_{\delta \downarrow 0} p_{kj}(t_o+\delta) \\
&= p_{ii}(s)\ p_{ij}(t_o+) + \sum_{k \neq i} p_{ik}(s)\ p_{kj}(t_o+) .
\end{aligned} \tag{13}$$

But both $p_{ii}(s)$ and $p_{ij}(t_o+)$ are positive, so from (12) and (13) we obtain

$$\begin{aligned}
p_{ij}(t_o+s+) &> p_{ii}(s)\ p_{ij}(t) + \sum_{k \neq i} p_{ik}(s)\ p_{kj}(t_o) \\
&= p_{ij}(s+t_o).
\end{aligned} \tag{14}$$

This implies that D is uncountable, a contradiction. Hence D is empty and continuity of P_{ij} is established.

We conclude that each p_{ij} is continuous on $[0,s)$. The example $p_{ij}(t) = \delta_{ij}e^{t}$ rules out uniform continuity on $[0,\infty)$.

3.3. Continuity of p-functions

We have shown in Chapter 2 that if the transition probabilities $p_{ij}(t)$ of a Markov chain are measurable functions of t then they are necessarily continuous. It is not true, however, that measurable p-functions are continuous. For example, let $\{u_n\}$ be a renewal sequence. It is well known [15] that there exists a Markov chain and a state a such that $u_n=p_{aa}(n)$. Then the function

$$p(n) = u_n \;,\; p(t) = 0, \text{ (t non-integral)} \tag{1}$$

is a p-function which is measurable but not continuous. This example really arises from a discrete time Markov chain. Therefore, to discuss continuity of p-functions we shall impose a standardness condition; namely, we shall require that

$$\lim_{t\to 0} p(t) = 1 \tag{2}$$

Theorem:

If p is a standard p-function then p is uniformly continuous on $[0,\infty)$.

Proof:

For s, t > 0, we have by Kingman's second inequality

$$p(s)\ p(t) \leq p(s+t) \leq p(s)\ p(t) + 1 - p(s). \tag{3}$$

Subtracting p(t) we get

$$p(s)\ p(t) - p(t) \leq p(s+t) - p(t) \leq p(s)\ p(t) + 1 - p(s) - p(t),$$

whence

$$-p(t)\ \{1-p(s)\} \leq p(s+t) - p(t) \leq (1-p(s))\ (1-p(t)). \tag{4}$$

Since $0 \leq p(t) \leq 1$, (4) implies

$$|p(s+t) - p(t)| \leq 1-p(s). \tag{5}$$

For t_1, $t_2 \in [0,\infty)$ with $t_1 < t_2$ put $t = t_1$, $s = t_2-t_1$. Then (5) becomes

$$|p(t_2) - p(t_1)| \leq 1 - p(t_2-t_1)\ .$$

Letting $t_1 \uparrow t_2$ and noting that p is standard we have that p is uniformly continuous.

3.4 Continuity of semi-p-functions

Let p(t) be a standard semi-p-function; i.e.

$$\lim_{t\downarrow 0} p(t) = 1\ . \tag{1}$$

Lemma 1: for $t \geq 0$, $p(t) > 0$.

Proof:

From Kingman's second inequality we have

$$p(s+t) \geq p(s)\, p(t) .$$

This implies that, for any $n \geq 1$,

$$p(t) = p(n \cdot \tfrac{t}{n}) \geq p(\tfrac{t}{n})^n .$$

Since $p(t) \to 1$ as $t \downarrow 0$ we must have $p(t) > 0$ for all $t < t_o$, for some $t_o > 0$. If we take $n > \frac{t}{t_o}$, we have $p(t) > 0$.

Lemma 2:

$$p(t+u+v) - p(t)\, p(u+v) - p(t+u)\, p(v) + p(t)p(u)p(v) \geq 0 \qquad (2)$$

for all $t, u, v \geq 0$.

Proof:

From the Kingman F-inequalities established in Chapter 1 we have, for N=3,

$$p(t_3) - p(t_1)p(t_3-t_1) - p(t_2)p(t_3-t_2) + p(t_1)p(t_2)p(t_3) \geq 0 .$$

Putting $t_1=t$, $t_2=t+u$, $t_3=t+u+v$ we have the lemma.

Theorem:

A standard semi-p-function is continuous on $[0,\infty)$.

Proof:

For $t > 0$ define

$$\alpha = \liminf_{s \downarrow 0} p(t+s) , \quad \beta = \limsup_{u \downarrow 0} p(t+u) . \qquad (3)$$

By properties of lim inf and lim sup there exist positive sequences $\{S_n\}$ and $\{u_n\}$ converging to zero with $0 < u_n < S_n$ such that

$$\alpha = \lim_{n\to\infty} p(t+s_n) \quad , \quad \beta = \lim_{n\to\infty} p(t+u_n) . \tag{4}$$

By Kingman's second F-inequality we have

$$\begin{aligned} p(t+s_n) &= p(t+s_n-u_n+u_n) \\ &\geq p(t+u_n)\, p(s_n-u_n) \quad . \end{aligned}$$

This implies, using (1) and (4), that

$$\begin{aligned} \alpha = \lim_{n\to\infty} p(t+s_n) &\geq \lim_{n\to\infty} p(t+u_n)\, p(s_n-u_n) \\ &= \beta . \end{aligned} \tag{5}$$

Hence $p(t+) = \lim_{s\downarrow 0} p(t+s)$ exists, and similarly so does $p(t-) = \lim_{s\downarrow 0} p(t-s)$, $0 < s < t$.

From $p(t+s) \geq p(t)\, p(s)$, letting $s \to 0$ we have immediately that

$$p(t+) \geq p(t) . \tag{6}$$

Furthermore, for $t > 0$, $0 < s < t$ we have

$$\begin{aligned} p(t) &= p(t - s + s) \\ &\geq p(t - s)\, p(s) \; . \end{aligned}$$

Letting $s \to 0$ we have

$$p(t-) \leq p(t) \leq \infty \quad . \tag{7}$$

Altogether we have

$$p(t-) \leq p(t) \leq p(t+) \quad , \; t > 0 \; . \tag{8}$$

Notice that $p(0+) = p(0) = 1$ by (1).

Since p has left- and righthand limits everywhere, its discontinuity set

$$D = \{t; \; p(t-) < p(t)\}$$

is at most countable (see SAKS [1; p. 261]). Consider now Kingman's 3rd F-inequality

$$p(t+u+v) - p(t+u)p(v) + p(t)p(u)p(v) \geq p(t)p(u+v) \; .$$

Letting $v \to 0$ we have

$$p(t+u+) - p(t+u) + p(t)p(u) \geq p(t)p(u+) \quad ,$$

which can be rewritten as

$$p(t+u+) - p(t+u) \geq p(t) \; \{p(u+) - p(u)\} \quad . \tag{9}$$

Since D is countable we may choose t such that $t + u \notin D$. Then (9) becomes

$$0 \geq p(t) \; \{p(u+) - p(u)\} \quad .$$

By lemma (1) $p(t) > 0$, so

$$p(u+) = p(u) < \infty \; . \tag{10}$$

Now write $s = u + v$; fix s and let $v \to 0$. Then (2) and (10) give.

$$p(t+s) - p(t)p(s) - p(t+s-) \geq p(t)p(s-) \quad ,$$

which can be rewritten as

$$p(t+s) - p(t+s-) \geq p(t)\{p(s) - p(s-)\} \quad .$$

By arguments similar to those above we must have

$$p(s) = p(s-) \quad . \tag{11}$$

Combining (10 and (11), we have the theorem.

CHAPTER 4

Differentiability at t=0

This chapter examines the question of (right-hand) differentiability at the special point t=0. In all (standard) cases it is shown that these derivatives do exist even though they may possibly be infinite. The diagonal case (including p-functions and semi-p-functions) hinges upon a result [4.1 Theorem 1] of Hille and Phillips [7] in subadditive function theory. This case is dealt with in Theorem 4.2.1 and Theorem 4.3.1. The (denumerable) off-diagonal situation defied the efforts of probabilists for a couple of decades. For a stochastic matrix, Doeblin [5] proved the result in the finite-dimensional case. Doob [4] extended this result to the (denumerably) infinite-dimensional case, under the hypothesis that $P_{ii}'(0+)$ was finite. Finally Kolmogorov [17] produced the remarkable but somewhat technical proof given in Theorem 4.2.2.

4.1 A property of subadditive functions

Definition:

A function f defined on a set T is called subadditive if $f(t_1+t_2) \leq f(t_1) + f(t_2)$, $\forall t_1, t_2, t_1+t_2 \in T$.

Theorem 1:

If f(t) is a finite-valued subadditive function defined on $(0,\infty)$ satisfying

$$\lim_{t\downarrow 0} f(t) = 0 \tag{1}$$

and if $B = \sup_{t>0} \frac{f(t)}{t}$,

then

$$\lim_{t\downarrow 0} \frac{f(t)}{t} = B. \tag{2}$$

Proof:

Clearly $B > -\infty$

If $B < \infty$, there exists an s and $\varepsilon > 0$ such that $\frac{f(s)}{s} > B-\varepsilon$.

For each t write $s=nt+\delta$ where n is a positive integer and $0 \leq \delta < t$. Then

$$B - \varepsilon < \frac{f(s)}{s} \leq \frac{f(nt+\delta)}{s}$$

$$= \frac{nf(t) + f(\delta)}{s}$$

$$= \frac{nt}{s}\frac{f(t)}{t} + \frac{f(\delta)}{s}$$

As $t \to 0$, $\frac{nt}{s} \to 1$ and $f(\delta) \to 0$ by (1).

Hence

$$B-\varepsilon \leq \liminf_{t \downarrow 0} \left(\frac{nt}{t}\frac{f(t)}{s} + \frac{f(\delta)}{s}\right)$$

$$= \liminf_{t \downarrow 0} \frac{f(t)}{t}$$

$$\leq \limsup_{t \downarrow 0} \frac{f(t)}{t} \leq B$$

Since ε was arbitrary, we have

$$\lim_{t \downarrow 0} \frac{f(t)}{t} = B.$$

If $B = \infty$, for any large M there exists an s s.t. $M < \frac{f(s)}{s}$. A similar argument to the above will show

$$M \leq \liminf_{t \downarrow 0} \frac{f(t)}{t} \leq \limsup_{t \downarrow 0} \frac{f(t)}{t} = \infty \quad .$$

Letting M tend to infinity we find

$$\lim_{t \downarrow 0} \frac{f(t)}{t} = \infty.$$

Hence the theorem is established.

4.2 Transition matrices

Theorem 1:

If $P(t)$ is a standard transition matrix, then for each i, the limit

$$p_{ii}(0+) = \lim_{t \downarrow 0} \frac{p_{ii}(t)-1}{t} \qquad (1)$$

exists, but may be infinite.

Proof:

We know from theorem 3.2.1 that $p_{ii}(t) > 0$ for all t. Let

$$\psi(t) = -\log p_{ii}(t) \quad , t > 0. \qquad (2)$$

From the Chapman - Kolmogorov equation we have

$$p_{ii}(t+s) \geq p_{ii}(s)\, p_{ii}(t) \quad .$$

From this we conclude the subadditive property of ψ:

$$\psi(s+t) \leq \psi(s) + \psi(t) \quad . \qquad (3)$$

Put $q_i = \sup_{t>0} \frac{\psi(t)}{t}$. Then applying theorem 4.1,

$$q_i = \lim_{t \downarrow 0} \frac{\psi(t)}{t} . \qquad (4).$$

We notice $0 \leq q_i \leq \infty$.

Now Taylor's theorem gives

$$\log(1-x) = -x - \frac{x^2}{2} \left(\frac{1}{1-\xi}\right)^2 \text{, where } \xi \text{ is between 0 and x.}$$

So

$$\frac{-\log(1-(1-p_{ii}(t)))}{t} = \frac{1-p_{ii}(t)}{t} + \frac{(1-p_{ii}(t))^2}{2t} \left(\frac{1}{1-\xi}\right)^2 ,$$

$1-\xi$ between 1 and $p_{ii}(t)$.

We have

$$\frac{-\log p_{ii}(t)}{t} = \frac{1-p_{ii}(t)}{t}\left[1 + \frac{1-p_{ii}(t)}{2}\cdot\left(\frac{1}{1-\xi}\right)^2\right].$$

Letting $t \to 0$ and noting $\xi \to 0$ we have

$$\lim_{t\to 0} - \frac{\log p_{ii}(t)}{t} = \lim_{t\to 0}\frac{1-p_{ii}(t)}{t}$$

Hence, the limits (1) exist in $[-\infty,0]$.

Theorem 2:

If $P(t)$ is a standard transition matrix, then for each $i\neq j$, $p_{ij}'(0+)$ exists and $0 \leq p_{ij}'(0+) < \infty$.

Proof:

The Chapman-Kilmogorov equation can be written in matrix form as

$$P(s+t) = P(s)\,P(t) \qquad (6)$$

Repeated use of (6) gives

$$P(ns) = P(s)^n \qquad (7)$$

therefore

$$p_{ij}(ns) = \sum_{k_1}\cdots\sum_{k_{n-1}} p_{ik_1}(s)\,p_{k_1k_2}(s)\,\cdots\,p_{k_{n-1}j}(s)\,. \qquad (8)$$

Group these terms according to the first occurrence of j.

Put

$$P_1 = p_{ij}(s),$$

$$P_2 = \sum_{k\neq j} p_{ik}(s)\, p_{kj}(s) \quad ,$$

$$\vdots$$

$$P_n = \sum_{k_1\neq j,\ldots,k_{n-1}\neq j} p_{ik_1}(s)\ldots p_{k_{n-1}j}(s) \quad .$$

Then (8) becomes

$$p_{ij}(ns) = \sum_{m=1}^{n} P_m p_{jj}[(n-m)s] \quad . \tag{A}$$

From (8) we have

$$p_{ii}(ms) = \sum_{k_1} \cdots \sum_{k_{m-1}} p_{ik_1}(s) \ldots p_{ik_{m-1}}(s) \quad .$$

Let Q_m be the set of terms of above sum in which j does not occur.

Then

$$p_{ii}(ms) = Q_m + \sum_{\nu=1}^{m-1} P_\nu p_{ji}[(m-\nu)s] \quad . \tag{B}$$

Furthermore, from the definition of P_m and Q_m we have

$$P_m \geq Q_{m-1}\, p_{ij}(s). \tag{C}$$

Given $\varepsilon>0, \exists\, T$ such that $p_{ii}(t) \geq 1-\varepsilon$, $p_{jj}(t) \geq 1-\varepsilon$, $\forall t\in[0,T]$.

Hence, since $\sum_k p_{ik}(t) = 1$, for $k\neq i$ we have

$$p_{ik}(t) = 1 - \sum_{\ell \neq k} p_{i\ell}(t)$$

$$\leq 1 - p_{ii}(t)$$

$$\leq \varepsilon \qquad \forall t \leq T,\ \forall k \neq i. \tag{9}$$

Similarly, we have

$$p_{jk}(t) \leq \varepsilon,\ \forall t \leq T,\ \forall k \neq j.$$

Now, consider any $s > 0$ with $ns \leq T$.

From (A), $p_{ij}(ns) \geq (1-\varepsilon) \sum_{m=1}^{n} P_m = (1-\varepsilon) P$, say, where

$$P = \sum_{m=1}^{n} P_m,$$

which implies $P \leq \frac{p_{ij}(ns)}{1-\varepsilon} \leq \frac{\varepsilon}{1-\varepsilon}$.

Using (B), we have (for $m \leq n$),

$$p_{ii}(ms) \leq Q_m + \varepsilon \sum_{\nu=1}^{m-1} P_\nu$$

$$\leq Q_m + \varepsilon P$$

$$\leq Q_m + \frac{\varepsilon^2}{1-\varepsilon}.$$

This implies $Q_m \geq p_{ii}(ms) - \frac{\varepsilon^2}{1-\varepsilon}$

$$\geq (1-\varepsilon) - \frac{\varepsilon^2}{1-\varepsilon}$$

$$= \frac{1-2\varepsilon}{1-\varepsilon} \quad (m=1,2,\ldots,n-1).$$

From (C) we derive

$$P_m \geq \frac{(1-2\varepsilon)}{1-\varepsilon} \, p_{ij}(s).$$

Using (A) once more, we arrive at

$$p_{ij}(ns) \geq \frac{1-2\varepsilon}{1-\varepsilon} \, p_{ij}(s) \sum_{m=1}^{n} p_{jj}(n-m)s$$

$$\geq \frac{1-2\varepsilon}{1-\varepsilon} \, p_{ij}(s) \sum_{m=1}^{n} (1-\varepsilon)$$

$$= n(1-2\varepsilon) \, p_{ij}(s).$$

Consider any t (fixed for now) in $[0,T]$ and put $t = ns + \sigma$, where $0 \leq \sigma < s$. Then

$$p_{ij}(t) = p_{ij}(ns + \sigma)$$

$$\geq p_{ii}(\sigma) \, p_{ij}(ns)$$

$$\geq (1-\varepsilon) \, n \, (1-2\varepsilon) \, p_{ij}(s)$$

$$= n(1-3\varepsilon+2\varepsilon^2) \, p_{ij}(s) \quad (\varepsilon < \tfrac{1}{2}).$$

This implies that

$$\frac{p_{ij}(s)}{s} \leq \frac{p_{ij}(t)}{ns(1-3\varepsilon+2\varepsilon^2)}$$

$$= \frac{t}{ns} \cdot \frac{1}{(1-3\varepsilon+2\varepsilon^2)} \cdot \frac{p_{ij}(t)}{t}$$

Since $t = ns + \sigma$ we note that

$$\frac{t}{ns} \to 1 \quad \text{as} \quad s \to 0.$$

Now let $s \to 0$.

$$\limsup_{s \to 0} \frac{p_{ij}(s)}{s} \leq \frac{1}{(1-3\varepsilon+2\varepsilon^2)} \frac{p_{ij}(t)}{t} < \infty, \forall \epsilon \leq T.$$

Letting $t \to 0$ we have

$$\limsup_{s \to 0} \frac{p_{ij}(s)}{s} \leq \frac{1}{(1-3\varepsilon+\varepsilon^2)} \liminf_{t \to 0} \frac{p_{ij}(t)}{t}, \forall \varepsilon < \tfrac{1}{2}.$$

If we now let $\varepsilon \to 0$ we get

$$\limsup_{s \to 0} \frac{p_{ij}(s)}{s} \leq \limsup_{t \to s} \frac{p_{ij}(t)}{t} .$$

So $\lim_{t \to 0} \frac{p_{ij}(t)}{t}$ exists and is finite.

This establishes the theorem.

4.3 Semigroups of positive matrices.

Theorem 1:

If $P(t)$ is a semi-group of positive matrices satisfying

$$\lim_{t \to 0} p_{ij}(t) = \delta_{ij}, \forall i,j \in s, \tag{1}$$

then

$$\dot{p}_{ii}(0+) = \lim_{t \to 0} \frac{p_{ii}(t)-1}{t} \tag{2}$$

exists, but may be infinite.

Proof:

We know that $p_{ii}(t) > 0, \forall t$, and the semi-group property tells us that

$$p_{ii}(s+t) \geq p_{ii}(s)\, p_{ii}(t) \quad .$$

Using the transformation $\psi(t) = -p_{ii}(t)$, (3) becomes the sub-additive property of ψ. The proof of the existence of the limits (2) follows as in theorem 4.2.1.

Theorem 2:

Let P(t) be as in theorem 1. Then for $i \neq j$, the derivatives $p_{ij}{}'(0+)$ exists in $[0,\infty)$.

Proof:

An examination of theorem 4 reveals that the Markovian property of P(t) was used in equation (9) only to establish $p_{ik}(t) \leq \varepsilon$.

In this more general case, fix $i \neq j$. Given $0 < \varepsilon < \frac{1}{2}$, we can determine $\delta > 0$ (because P(t) is standard), such that $p_{ij}(t) \geq 1-\varepsilon$, $p_{kk}(t) \geq 1-\varepsilon$, $p_{ij}(t) \leq \varepsilon$ and $p_{ji}(t) \leq \varepsilon$. The proof of theorem 2 is now exactly similar to the proof of the corresponding theorem for transition matrices.

4.4 (Semi)-p-functions.

Theorem 1:

Let $p(t)$ be a standard (semi)-p-function. Then $p'(0+)$ exists in $[-\infty, 0]$.

Proof:

From lemma 1, section 3.4 we know that $p(t) > 0$, for $t \geq 0$. Kingman's 2nd-order F-inequality implies

$$p(s+t) \geq p(s)\, p(t) \quad . \tag{1}$$

Using the transformation

$$\psi(t) = -p(t) \tag{2}$$

we get the subadditive property

$$\psi(s+t) \leq \psi(s) + \psi(t).$$

Hence theorem 4.1.1 and the techniques of theorem 4.2.1 may be used to establish the theorem.

CHAPTER 5

Differentiability at $t > 0$

This chapter is concerned with the complex question of the existence of derivatives for $t > 0$. In the Markov case, several proofs of the existence of a continuous $p_{ij}'(t)$ were given (Austin [1], Chung [2], Yuškevic [23]) under the hypothesis of a finite $p_{ii}'(0+)$. Ornstein [19] first proved the result for transition matrices without any restrictions. To Jurkat [10] is due the remarkable observation that the assumption of stochasticity is not needed. We set out in detail this tour de force of Jurkat in Theorem 5.1.1. We include also a number of interesting corollaries of Jurkat's results. A beautiful example of Yuskevic [23], [18] is given which dispels the possibility that second derivatives necessarily exist even in the Markov case.

Paragraph 2 spells out Kingman's work on the corresponding situation for p-functions. It shows that $p'(t)$ exists except possibly on a null set. An example, Kingman [15], of a p-function which is not everywhere differentiable is included.

That semi-p-functions are also differentiable almost everywhere turns on an astonishing result of Kingman [16] which we detail in Theorem 5.3.1. It is shown that over any finite interval [0,T] a standard semi-p-function can be expressed as the product of an exponential and a standard p-function.

5.1 Semigroups of positive matrices

Theorem:

Let P(t) be a semigroup of positive matrices satisfying

$$\lim_{t\to 0} P(t) = I = P(0); \text{ i.e. } \lim_{t\to 0} p_{ij}(t)=\delta_{ij}, \ \forall i,j \in S. \quad (1)$$

Then for each i, j, $p_{ij}(t)$ has a continuous derivative at each $t > 0$.

Proof:

Step 1.

By (1) we may choose small $\varepsilon>0$ (to be fixed later) and $T_i>0$ such that

$$1-\varepsilon < p_{ii}(t) < 1+\varepsilon, \ 0 \leq t \leq T_i. \quad (2)$$

Fix i and $\alpha > 0$ and define

$$f_n = \sum_{h,\dots,h_{n-1}\neq i} p_{ih_1}(\alpha)\, p_{h_1h_2}(\alpha)\cdots p_{h_{n-1}i}(\alpha); \ n\geq 1 \quad (3)$$

Note that f_1 is just $p_{ii}(\alpha)$.

Write P_n for $p_{ii}(n\alpha)$. Then it follows from the semi-group property and (1) above that

$$P_o=1; \ P_n = \sum_{m=1}^{n} P_{n-m} f_m, \quad (n\geq 1). \quad (4)$$

Also define $g_o = 1$, $g_n = 1-\sum_{1}^{n} f_n \ (n\geq 0)$.

Note that the g_n's are non-decreasing but may assume negative values. Then we have $g_n - g_{n-1} = -f_n$. Substituting into (4) we get

$$P_n = -\sum_{m=1}^{n} (g_m - g_{m-1}) P_{n-m}$$

or

$$\sum_{m=0}^{n} P_{n-m} g_m = \sum_{m=0}^{n-1} P_{(n-1)-m} g_m \quad (\text{since } g_o=1). \qquad (6)$$

This means that the sum (6) does not depend on the value of n. Since $g_o P_o = 1$ we have therefore

$$P_n + \sum_{m=1}^{n} P_{n-m} g_m = 1. \qquad (7)$$

Now consider suffices m, n, ..., ranging up to at most N, with $N\alpha \leq T_i$. From (7) we have

$$(P_{n+1} + \sum_{m=1}^{n+1} P_{n+1-m} g_m) - (P_n + \sum_{m=1}^{n} P_{n-m} g_m) = 0$$

or

$$(P_{n+1} - P_n) + g_{n+1} + \sum_{m=1}^{n} (P_{n+1-m} - P_{n-m}) g_m = 0.$$

Define $V = \sum_{n=0}^{N-1} |P_{n+1} - P_n|$. Then from (8)

$$V \leq \sum_{n=0}^{N-1} |g_{n+1}| + \sum_{n=0}^{N-1} \sum_{m=1}^{n} |P_{n+1-m} - P_{n-m}| \; |g_m|$$

$$= \sum_{r=1}^{N} |g_r| + \sum_{m=1}^{N-1} |g_m| \; (\sum_{n=m}^{N-1} |P_{n+1-m} - P_{n-m}|$$

$$\leq \sum_{r=1}^{N} |g_r| \; (1 + V). \qquad (9)$$

Now we estimate $\sum_{r=1}^{N} |g_r|$.

If $g_n \geq 0$, so is $g_m \geq 0$ for $m \leq n$; then from (7) and (2) we get

$$\sum_{m=1}^{n} P_{n-m} \, g_m = 1-P_n \leq \varepsilon \quad ,$$

which implies

$$\varepsilon \geq \sum_{m=1}^{n} P_{n-m} \, g_m$$

$$\geq (1-\varepsilon) \sum_{m=1}^{n} g_m .$$

So

$$\sum_{m=1}^{n} g_m \leq \frac{\varepsilon}{1-\varepsilon} \text{ if } g_n \geq 0 . \qquad (10)$$

If $g_n < 0$, take $n_o < n$ so that $g_m \geq 0$ if $m \leq n_o$, and $g_m < 0$ for $m > n_o$. By (10)

$$\sum_{m=1}^{n_o} g_m \leq \varepsilon/1-\varepsilon. \tag{11}$$

Furthermore, since from (7)

$$P_n + \sum_{m=1}^{n_o} P_{n-m}g_m + \sum_{m=n_o+1}^{n} P_{n-m}\, g_m = 1$$

then

$$\sum_{m=n_o+1}^{n} P_{n-m}(-g_m) = P_n - 1 + \sum_{1}^{n_o} P_{n-m}\, g_m$$

$$\leq \varepsilon + (1+\varepsilon)\frac{\varepsilon}{1-\varepsilon} = \frac{2\varepsilon}{1-\varepsilon} ;$$

so $\frac{2\varepsilon}{1-\varepsilon} \geq \sum_{m=n_o+1}^{n} P_{n-m}(-g_m) \geq (1-\varepsilon) \sum_{m=n_o+1}^{n} (-g_m)$.

This implies

$$\sum_{m=n_o+1}^{n} (-g_m) \leq \frac{2\varepsilon}{(1-\varepsilon)^2} . \tag{12}$$

Combining (10) and (12) we get

$$\sum_{m=1}^{n} |g_m| = \sum_{m=1}^{n_o} g_m + \sum_{m=n_o+1}^{n} (-g_m) \tag{13}$$

$$\leq \frac{\varepsilon}{1-\varepsilon} + \frac{2\varepsilon}{(1-\varepsilon)^2} = \frac{3\varepsilon-\varepsilon^2}{(1-\varepsilon)^2} = \theta, \text{ say.}$$

Now fix $\varepsilon = 1/5$, then $\theta = 7/8$. We note, from (9), that if

$\sum_{r=1}^{N} |g_r| \leq \theta < 1$, then $V \leq \theta/1-\theta$. In our case we have $\theta = 7/8$ which implies

$$V = \sum_{n=0}^{N-1} |p_{ii}(n+1)\alpha - p_{ii}(n\alpha)| \leq 7 \text{ provided } N\alpha \leq T_i, \text{ where}$$

$4/5 < p_{ii}(t) < 6/5$ on $[0,T_i]$. Since $p_{ii}(t)$ is continuous, we may conclude that

$$\underset{[0,T_i]}{\text{var}}\ p_{ii} \leq 7 \text{ if } 4/5 \leq p_{ii}(t) \leq 6/5 \text{ on } [0,T_i]. \tag{14}$$

Step 2:

Let $j \neq i$ and try to estimate var p_{ij}. Define

$$a_n = \sum_{h,\ldots h_{n-1} \neq i} p_{ih_1}(\alpha)\, p_{h_1 h_2}(\alpha) \ldots p_{h_{n-1}j}(\alpha) \quad (n \geq 1) \tag{15}$$

taking a_1 to be $p_{ij}(\alpha)$. Then, as before,

$$p_{ij}(n\alpha) = \sum_{m=1}^{n} p_{ii}\,(\overline{n-m}\,\alpha)\, a_m. \tag{16}$$

Assuming we stay in $[0,T_i]$, (16) implies

$$\sum_{m=1}^{n} a_m \leq p_{ij}(n\alpha)/1-\varepsilon. \tag{17}$$

Also

$$p_{ij}(\overline{n+1}\,\alpha) - p_{ij}(n\alpha) = \sum_{m=1}^{n+1} p_{ii}\,(\overline{n+1-m}\,\alpha)\, a_m - \sum_{m=1}^{n} p_{ii}(n-m\,\alpha)\, a_m$$

$$= a_{n+1} + \sum_{m=1}^{n} \{p_{ii}(\overline{n+1-m}\,\alpha) - p_{ii}(\overline{n-m}\,\alpha)\}\, a_m.$$

Defining V as in Step 1 we arrive at

$$\sum_{n=0}^{N-1} |p_{ij}(\overline{n+1}\ \alpha) - p_{ij}(n\alpha)| \leq \sum_{r=1}^{N} a_r(1 + V). \tag{18}$$

Setting $\varepsilon = 1/5$, using (17) and the bound $V \leq 7$ from (14), we have

$$\operatorname*{var}_{[0,t]} p_{ij} \leq 10\ p_{ij}(t) \text{ for } 0 \leq t \leq T_i, \quad \forall j. \tag{19}$$

[Note; (19) also holds for j=1 since $10\ p_{ii} \geq 4/5\ (10) > 7$].

Now the transposed matrix P* also is a semigroup of positive matrices, so we can deduce

$$\operatorname*{var}_{[0,t]} p_{ji} \leq 10\ p_{ji}(t) \text{ for } 0 \leq t \leq T_i, \quad \forall j. \tag{19'}$$

We may extend (19) and (19') to any interval [r, r+t] of length $t \leq T_i$ as follows:

$$p_{ij}(r+u) = \sum_k p_{ik}(u)\ p_{kj}(r).$$

$$p_{ij}(r+\overline{n+1}\ \alpha) - p_{ij}(r+n\alpha) = \sum_k \{p_{ik}(\overline{n+1}\ \alpha) - p_{ik}(n\alpha)\}p_{kj}(r)$$

$$\sum_{n=0}^{N-1} |p_{ij}(r+\overline{n+1}\ \alpha) - p_{ij}(r+n\alpha)| \leq \sum_{n=0}^{N-1}\sum_k |p_{ik}(\overline{n+1}\ \alpha) - p_{ik}(n\alpha)|p_{kj}(r)$$

$$= \sum_k \sum_{n=0}^{N-1} |p_{ik}(\overline{n+1}\ \alpha - p_{ik}(n\alpha)|\ p_{kj}(r)$$

$$\leq \sum_k \operatorname*{var}_{[0,t]} p_{ik} \cdot p_{kj}(r)$$

$$\leq 10 \sum p_{ik}(t) p_{kj}(r)$$

$$= 10\, p_{ij}(r+t),$$

taking $N\alpha = t$ as before.

Hence, we have

$$\underset{[r,r+t]}{\text{var}}\; p_{ij} \leq 10\, p_{ij}(r+t)\;,\; t \leq T_i \;;\quad \forall j. \qquad (20)$$

$$\underset{[r,r+t]}{\text{var}}\; p_{ji} \leq 10\, p_{ji}(r+t)\;;\; t \leq T \;;\quad \forall j \qquad (20')$$

Step 3:

Fix i, j, and let $T = T_{ij} = \min(T_i, T_j)$.

Consider the interval $[0,S]$ where $S = qT$, q an interger ≥ 1.

This interval can be written as

$$[0,S] = \bigcup_{r=0}^{q-1} [rT, \overline{r+1}\,T]\,.$$

Applying (20) and (20') to each subinterval $[rT, \overline{r+1}\,T]$ we get

$$\underset{[0,S]}{\text{var}}\; p_{ik} \leq 10 \sum_{S=1}^{q} p_{ik}(sT)\quad, \text{ for all } k \qquad (21)$$

and

$$\underset{[0,S]}{\text{var}}\; p_{kj} \leq 10 \sum_{s'=1}^{q} p_{kj}(s'T)\quad, \text{ for all } k. \qquad (22)$$

So

$$\sum_k \operatorname*{var}_{[0,s]} p_{ik} \cdot \operatorname*{var}_{[0,s]} p_{kj} \le \sum_k (10 \sum_{s=1}^{q} p_{ik}(sT)(10 \sum_{s'=1}^{q} p_{kj}(s'T)$$

$$= 100 \sum_{s=1}^{q} \sum_{s'=1}^{q} \sum_k p_{ik}(sT)\, p_{kj}(s'T)$$

$$= 100 \sum_{s=1}^{q} \sum_{s'=1}^{q} p_{ij}(\overline{s+s'}\ T) = C<\infty, \quad (23)$$

where C depends on i, j, and S.

Now, for $t \in [0,S]$, we have

$$\operatorname*{var}_{[0,t]} p_{ik} \ge |p_{ik}(t) - p_{ik}(0)| . \quad (24)$$

But $p_{ik}(0) = \delta_{ik}$ from (1). If $i \ne k$ (24) becomes simply

$$p_{ik}(t) \le \operatorname*{var}_{[0,t]} p_{ik} . \quad (25)$$

If $i=k$, we have

$$\operatorname*{var}_{[0,t]} p_{ik} \ge |p_{ik}(t) - 1|$$

which implies

$$p_{ik}(t) \le 1 + \operatorname*{var}_{[0,t]} p_{ik} .$$

Both (25) and (26) imply, since t was an arbitrary element of [0,S], that

$$\sup_{t\in[0,S]} p_{ik}(t) \le \delta_{ik} + \operatorname*{var}_{[0,S]} p_{ik} .$$

So it follows from (19), (23), and (27) that

$$\sum_k \sup_{t\epsilon[0,S]} p_{ik}(t) \cdot \operatorname*{var}_{[0,S]} p_{kj}$$

$$\leq \sum_k (\delta_{ik} + \operatorname*{var}_{[0,S]} p_{ik}) \operatorname*{var}_{[0,S]} p_{kj}$$

$$= \operatorname*{var}_{[0,S]} p_{ij} + \sum_k \operatorname*{var}_{[0,S]} p_{ik} \cdot \operatorname*{var}_{[0,S]} p_{kj}$$

$$\leq 10\, p_{ij}(S) + C < \infty, \qquad (28)$$

Let $C_k = \sup_{[0,S]} p_{ik}$ and $Q_k(x) = C_k\, p_{kj}(x)$. We have

$\operatorname*{var}_{[0,S]} Q_k = C_k \operatorname{var} p_{kj}$ and hence, by (28)

$$\sum_k \operatorname*{var}_{[0,S]} Q_k = \sum_k C_k \operatorname*{var}_{[0,S]} p_{kj} < \infty. \qquad (29)$$

We now need the following lemma:

Lemma 1.

If $\sum_m \operatorname*{var}_{[a,b]} Q_m < \infty$ holds then

$$\sum_m \left| \frac{Q_m(x+h) - Q_m(x)}{h} - Q_m'(x) \right| \to 0 \quad \text{as } h \to 0$$

for almost all $x \in [a,b]$.

Proof:

Since $\sum_m \operatorname*{var}_{[a,b]} Q_m < \infty$, then $Q_m(x)$ is of bounded variation for each m. Then it is well known (see, for example, TITCHMARSH [1; p. 355]) that Q_m can be written as

$$Q_m(x) - Q_m(a) = P_m(x) - N_m(x),$$

where $P_m(x)$ and $N_m(x)$ are respectively the positive and negative variations of Q_m in $[a,x]$.

Now
$$\sum_m |P_m(x) - N_m(x)| \le \sum_m (P_m(x) + N_m(x))$$
$$= \sum_m \operatorname*{var}_{[a,x]} Q_m$$
$$\le \sum_m \operatorname*{var}_{[a,b]} Q_m < \infty.$$

This implies that both $\sum_m P_m(x)$ and $\sum_m N_m(x)$ converge for all $x \in [a,b]$. Furthermore both $P_m(x)$ and $N_m(x)$ are bounded non-decreasing functions of x. Putting $P(x) = \sum_m P_m(x)$ and $N(x) = \sum_m N_m(x)$ we have, by a theorem of Fubini (see e.g. SAKS [1; p 117]) that relations

$$P'(x) = \sum_m P'_m(x) \text{ and } N'(x) = \sum_m N'_m(x)$$

hold for almost all $x \in [a,b]$. But

$$P'(x) - N'(x) = \sum_m (P'_m(x) - N'_m(x))$$
$$= \sum_m Q'_m(x) < \infty.$$

That is, for almost all $x \in [a,b]$, $\sum_m Q'_m(x)$ converges.

In other words

$$\sum_m \left| \frac{Q_m(x+h) - Q_m(x)}{h} - Q'_m(x) \right| \to 0 \quad \text{as} \quad h \to 0.$$

Thus the lemma is proven.

From (29) we see that

$$Q_k(x) = \sup_{s\in[0,S]} p_{ik}(s)\, p_{kj}(x)$$

satisfies the hypothesis of the lemma.

We may conclude that there is a null set $N(i,j,q)$ such that, for $x \notin N(i,j,q)$, $x \in [0,S]$, all $p_{kj}'(x)$ (k varying, j fixed) exist, and

$$\sum_k \sup_{s\in[0,S]} p_{ik}(s) \left| \frac{p_{kj}(x+h) - p_{kj}(x)}{h} - p_{kj}'(x) \right| \to 0 \text{ as } h \to 0. \quad (30)$$

In particular,

$$\sum_k p_{ik}(s) \left| \frac{p_{kj}(x+h) - p_{kj}(x)}{h} - p_{kj}'(x) \right| \to 0 \text{ as } h \to 0$$

for each $s \in [0,S]$. (31)

Also, we have

$$\left| \sum_k p_{ik}(s) \left\{ \frac{p_{kj}(x+h) - p_{kj}(x)}{h} - p_{kj}'(x) \right\} \right|$$

$$\leq \sum_k p_{ik}(s) \left| \frac{p_{kj}(x+h) - p_{kj}(x)}{h} - p_{kj}'(x) \right| \; ; \quad (32)$$

that is $\sum_k p_{ik}(s)\, p_{kj}'(x)$ converges absolutely in view of (31) and uniformly in s, by (30), for each $s \in [0,S]$. Finally, for $s \in [0,S]$, we have

$$\left|\frac{p_{ij}(s+x+h) - p_{ij}(s+x)}{h} - \sum_k p_{ik}(s)\, p_{kj}'(x)\right|$$

$$= \left|\frac{\sum_k p_{ik}(s)\, p_{kj}(x+h) - \sum_k p_{ik}(s)\, p_{kj}(x)}{h} - \sum_k p_{ik}(s) p_{kj}'(x)\right|$$

$$= \left|\sum_k p_{ik}(s) \left\{\frac{p_{kj}(x+h) - p_k(x)}{h} - p_{kj}'(x)\right\}\right|$$

$$\leq \sum_k p_{ik}(s) \left|\frac{p_{kj}(x+h) - p_{kj}(x)}{h} - p_{kj}'(x)\right|$$

which tends to 0 as $h \downarrow 0$;

that is

$D^+(s+x)$ exists and equals $\sum_k p_{ik}(s)\, p_{kj}'(x)$,

where D^+ is the upper Dini derivate. But by (32) the series $\sum_k p_{ik}(s)\, p_{kj}'(x)$ converges uniformly in $s \in [0,S]$; hence it is continuous. By a theorem of Dini (see e.g. SAKS [1, p 204]) $p_{ij}'(s+x)$ exists. We have, therefore, for $s \in [0,S]$, $x \notin N(i,j,q)$

$$p_{ij}'(s+x) = \sum_k p_{ij}(s)\, p'_{jk}(x) \quad . \tag{33}$$

We may now enlarge the exceptional set to

$$N = \bigcup_{i,j,q} N(i,j,q).$$

We have therefore proved the following: there is a null set N such that, for $t > 0$, and for all p_{ij}, $p_{ij}'(t)$ exists for $t \notin N$. For any $s \geq 0$ and $t \notin N$, $p_{ij}'(s+t)$ exists and equals $\sum_k p_{ik}(s)\, p_{kj}'(t)$. This latter series, for fixed $t \notin N$, converges absolutely and uniformly for s on any finite interval $[0,S]$. To complete the proof we need only show N is empty. To do this, we check that the differential equation

$$p'_{ij}(s+t) = \sum_k p_{ik}(s)\, p'_{kj}(t),$$

known to exist for $s \geq 0$, $t \notin N$, holds for $s \geq 0$ and <u>all</u> $t>0$. For $s \geq 0$, $t>0$, choose $t'<t$, $t' \notin N$. Then

$$\begin{aligned} p_{ij}(s+t) &= p_{ij}'(\overline{s+t-t'} + t') \\ &= \sum_k p_{ik}(s+t-t')\, p_{kj}'(t') \\ &= \sum_k \Big(\sum_\ell p_{i\ell}(s)\, p_{\ell k}(t-t')\Big)\, p_{kj}'(t') \\ &= \sum_\ell p_{i\ell}(s) \Big(\sum_k p_{\ell k}(t-t')\, p_{kj}'(t')\Big) \\ &= \sum_\ell p_{i\ell}(s)\, p_{\ell j}'(t-t'+t') \\ &= \sum_\ell p_{i\ell}(s)\, p_{\ell j}'(t) \end{aligned}$$

where the interchange of summation is justified by the absolute convergence of the series. Since $t > 0$ was arbitrary, N is empty. By (32) we have that $p_{ij}'(t)$ is continuous. End of proof.

Corollary 1:

The generalized Kolmogorov equations hold.

i.e.

$$p'_{ij}(s+t) = \sum_k p_{ik}(s)\, p'_{kj}(t),\ s\geq 0,\ t>0. \tag{34}$$

and

$$p'_{ij}(s+t) = \sum_k p'_{ik}(s)\, p_{kj}(t),\ s>0,\ t\geq 0. \tag{35}$$

Proof:

We have already established (34). Applying the theorem to the transposed matrix gives (35).

Corollary 2:

If $P(t) = p_{ij}(t)$ is a transition matrix, than for $t>0$, $p_{ij}'(t)$ exists and is continuous. Note: Although we have shown only that the p_{ij}'s have bounded variation, the continuity of the derivatives imply that the p_{ij}'s are absolutely continuous (See i.e. TITCHMARSH, p 368).

Corollary 3:

$p_{ij}(t)$ is absolutely continuous on every finite interval $[a,b]$, $a\geq 0$, $b>0$.

Proof:

If $a>0$ then $p_{ij}'(t)$ exists and is finite for all $t\epsilon[a,b]$. Since $p_{ij}'(t)$ is continuous throughout $[a,b]$ it is bounded there. Hence (see e.g. TITCHMARSH [1; p. 368]) $p_{ij}(t)$ is absolutely continuous.

If $a=0$, $p_{ii}'(0+)$ may be infinite. We note that $p_{ij}(t)$ has bounded variation on $[0,s]$ for any $s>0$. Then $p_{ij}'(t)$ is equivalent in $[a,b]$ to a summable function (see e.g. KESTELMAN [1; p. 188]). But $p_{ij}(t)$ is continuous throughout $[0,S]$ and $p_{ij}'(t)$ exists and is finite everywhere except possibly at $t=0$. From this we may conclude (see e.g. KESTELMAN[1; p. 183]) that

$$\int_0^s p_{ij}'(t)\,dt = p_{ij}(s) - p_{ij}(0), \tag{36}$$

which of course means that $p_{ij}(t)$ is absolutely continuous on $[0,s]$.

By a known theorem (see e.g. HOBSON [1, p. 605]) it follows that

$$\underset{0\leq t\leq s}{\text{var}}\; p_{ij}(t) = \int_0^s |p_{ij}'(t)|\,dt < \infty. \tag{37}$$

Now, for $t'>0$

$$\int_0^t \{\sum_m \int_0^{s'} |p_{im}'(s)|\,ds \cdot |p_{mj}'(t)|\}\,dt$$

$$= \int_0^{t'} \sum_m \underset{0\leq s\leq s'}{\text{var}}\; p_{im}(s) \cdot |p_{mj}'(t)|\,dt$$

$$= \sum_m \operatorname*{var}_{0\le s\le s'} p_{im}(s) \int_0^{t'} |p_{mj}'(t)\, dt$$

$$= \sum_m \operatorname*{var}_{0\le s\le s'} p_{im}(s) \operatorname*{var}_{0\le t\le t'} p_{mj}(t) < \infty$$

where the interchange of the integral and summation signs is allowed by the monotone convergence theorem and the finiteness of the last series follows from (23). We have, therefore

$$\int_0^{t'} \{\sum_m \int_0^{s'} |p_{im}'(s) \quad |ds \cdot |p_{mj}'(t)\}\, dt < \infty. \qquad (38)$$

But the integrand is positive so it must be finite for almost all $t>0$; i.e. for all i, j, $s'>0$, and for almost all $t>0$ we have

$$\sum_m \int_0^{s'} |p_{im}'(s)|ds\, |p_{mj}'(t)| < \infty, \qquad (39)$$

and hence

$$p_{ij}'(s'+t) - p_{ij}'(t) = \sum_m p_{im}(s')\, p_{mj}'(t) - p_{ij}'(t)$$

$$= \sum_m (p_{im}(s') - p_{im}(0))\, p_{mj}'(t)$$

$$= \sum_m \int_0^{s'} p_{im}(s)\, ds \cdot p_{mj}'(t)$$

$$= \int_0^{s'} \sum_m p_{im}'(s)\, p_{mj}'(t)\, ds \qquad (40)$$

where the last interchange is allowed by Lebesque's Dominated Convergence Theorem (the dominating function is $g(s) = \sum_m |p_{im}'(s)||p_{mj}'(t)|$ which is integrable by (39)).

Hence, from (40) we have

$$p_{ij}''(s'+t) = \sum_m p_{im}'(s')\, p_{mj}'(t) \quad \text{(almost all pairs } s', t>0)$$

and the series is absolutely convergent for those pairs. We have proved that there is a null set N such that $p_{ij}''(r)$ exists for $r>0$, $r \notin N$.

The following example (due to Yuskevitch [14] shows that the exceptional set N need not be empty.

Example:

There exists a Markov chain with transition matrix P such that $p_{ij}(t)$ does not have a finite second derivative for some $t>0$.

Index the states of a countable state space as follows:

0,1; and (n,k), $(k=1,\ldots,n)$; $(n=1,2,\ldots)$. Define a matrix $Q = (q_{ij})$ as follows:

$$q_{00} = -1$$

$$q_{0(n,1)} = a_n, \quad \sum_{n=1}^{\infty} a_n = 1,$$

$$q_{(n,k)', (n,k+1)} = n-1,$$

$$q_{(n,n)', 1} = n-1,$$

$$q_{(n,k)', (n,k)} = -(n-1),$$

all other q_{ij}'s are zero. All states all stable (i.e. $q_{ii} > -\infty$ for all i) and Q gives rise to a minimal process [1] $P(t) = (p_{ij}(t)) \leq 1$ satisfying

$$P'(t) = QP(t) = P(t)Q, \quad t>0.$$

Consider the Laplace transform [18]

$$R(\lambda) = \int_0^\infty e^{-\lambda t} P(t)\, dt, \qquad \lambda>0.$$

It is easy to show that

$$R(\lambda) \geq 0; \; \lambda R(\lambda) \cdot I \leq 1,$$

and

$$R(\lambda) - R(\mu) + (\lambda-\mu) R(\lambda) R(\mu) = 0.$$

Consider $P'(t) = Qp(t)$. Taking Laplace transforms of both sides we get

$$\lambda R(\lambda) - I = QR(\lambda).$$

From this equation we can get expressions for $r_{ij}(\lambda)$. For example, let i=1, j=1. Then

$$\lambda r_{11}(\lambda) - 1 = \sum_k q_{1k}\, r_{k1}(\lambda).$$

But $q_{1k}=0$ for all k. So

$$\lambda r_{11}(\lambda) - 1 = 0$$

from which we imply $r_{11}(\lambda) = 1/\lambda$ and this implies $p_{11}(t) = 1$, $t>0$; i.e. 1 is an abosrbing state. Let $i = (n,1)$, $j=1$.

$$\lambda r_{(n,1),1}(\lambda) = \sum_k q_{(n,1),k}\, r_{k,1}(\lambda)$$

$$= q_{(n,1),(n,1)} r_{(n,1),1}(\lambda) + q_{(n,1),(n,2)} r_{(n,2),1}(\lambda)$$

$$= -(n-1) r_{(n,1),1}(\lambda) + (n-1r)_{(n-2),1}(\lambda)$$

or

$$(\lambda+n-1)\, r_{(n,1),1}(\lambda) = (n-1) r_{(n-2),1}(\lambda).$$

Similarly we can solve for $r_{(n,k),1}(\lambda)$ $(1<k<n)$ to get

$$\left(\frac{\lambda+n-1}{n-1}\right)^{n-1} r_{(n,1),1}(\lambda) = r_{(n,n),1}.$$

But

$$\lambda r_{(n,n),1}(\lambda) = \sum_k q_{(n,n),k}\, r_{k,1}(\lambda)$$

$$= q_{(n,n),(n,n)} r_{(n,n),1}(\lambda) + q_{(n,n),1} r_{11}(\lambda)$$

$$= -(n-1) r_{(n,n)}(\lambda) + (n-1)/\lambda.$$

Finally we obtain

$$r_{(n,1),1}(\lambda) = \frac{(n-1)^n}{\lambda[\lambda+n-1]^n}.$$

Taking inverse Laplace transforms we get

$$p_{(n,1),1}(t) = \frac{(n-1)^n}{(n-1)!} \int_0^t e^{-(n-1)s} s^{n-1} ds$$

which is $\Gamma(n, 1/n-1)$. We now show that $p_{01}(t)$ has an infinite second derivative for $t=1$.

$$p'_{01}(t) = \sum_k q_{0k} p_{k1}(t) = \sum_n Q_n\, p_{(n,1),1}(t) - p_{01}(t).$$

So

$$\frac{p_{01}'(t+s) - p_{01}'(t)}{s} = \frac{-(p_{01}(t+s) - p(s))}{s} +$$

$$\sum_n a_n \frac{1}{s} \frac{(n-1)^n}{(n-1)!} \int_t^{t+s} e^{-(n-1)\mu} \mu^{n-1}\, d\mu.$$

Taking $\liminf_{s \downarrow 0}$ and using Fatou's lemma we get

$$\liminf_{s \downarrow 0} \frac{p_{01}'(t+s)-p_{01}'(t)}{s} \geq -p_{01}'(t) + \sum_n a_n \frac{(n-1)^n}{(n-1)!}$$

$$\liminf_{s \to 0} \frac{1}{s} \int_t^{t+s} e^{-(n-1)\mu} \mu^{n-1} \, d\mu$$

$$= -p_{01}'(t) + \sum_n a_n \frac{(n-1)^n}{(n-1)!} e^{-(n-1)t} t^{n-1}.$$

We conclude by showing that for t=1 the series deverges to ∞.

Rewriting we get

$$\sum_n a_n \frac{(n-1)^{n-1} e^{-(n-1)}}{(n-1)!} \sqrt{2\pi(n-1)} \sqrt{\frac{n-1}{2\pi}} (te^{1-t})^{n-1} = \sum a_n f_n(t),$$

Now te^{1-t} increases from 0 to 1 as t increases from 0 to 1 and decreases to 0 as to increases from 1 to ∞. Using Stirling's formula we see that $f_n(1) \to \infty$ as $n \to \infty$, $f_n(t) \to 0$ as $n \to \infty$, $t \neq 1$. There exists a sequence $\{n_k\} \uparrow \infty$ such that $f_{n_k}(1) \geq 2^k$ $(k=1,2,\ldots)$. Define

$$a_n = \begin{cases} 2^{-k} & \text{if } n=n_k \\ 0 & \text{if } n=n_k \end{cases}$$

We have $\sum_n a_n = 1$ but $p_{01}''(1) \geq \sum_{n=1}^{\infty} 1 = \infty$.

5.2 Differentiability of p-functions

Theorem 5.2.1

Let p(t) be a standard p-function then p is of bounded variation in every finite interval, and is thus differentiable almost everywhere in t>0.

Proof:

We carry out the proof in two stages.

1. Let $\{u_n\}$ be any renewal sequence and consider the generating function $U(z) = \sum_{n=0}^{\infty} u_n z^n$. Then it is well known [12], that for $|z| < 1$

$$U(z) = 1 + U(z)\ F(z) \tag{1}$$

where

$$F(z) = \sum_{n=1}^{\infty} f_n z^n,\ f_n \geq 0,\ F(1) \leq 1. \tag{2}$$

Fix λ in $0 < \lambda < 1$ and write, for any power series

$$A(z) = a_0 + a_1 z + a_2 z^2 + \dots,$$

$$||A|| = \sum_{n=0}^{\infty} |a_n| \lambda^n \text{ ; which may be } +\infty.$$

Then clearly, for any two power series A and B, we have

$$||A + B|| \leq ||A|| + ||B||. \tag{3}$$

and

$$||A \cdot B|| \leq ||A||\,||B||, \tag{4}$$

where AB is the usual Cauchy product of series. Hence (writing $u_{-1}=0$)

$$\nabla = \sum_{n=0}^{\infty} |u_n - u_{n-1}| \lambda^n$$

$$= ||(1-z)U(z)||$$

$$= \left|\left| \frac{(1-z)}{1-F(z)} \right|\right| \text{ (from (1))}$$

$$= ||\{1-\frac{F(z)-z}{1-z}\}^{-1}||.$$

Now, if $A = (1-B)^{-1}$, then

$A = 1+AB$ and so, from (3) and (1)

$$||A|| \leq 1+||A||\,||B||$$

or

$$||A||(1-||B||) \leq 1.$$

In the event that $||B||<1$, then

$$||A|| \leq \frac{1}{1-||B||}.$$

Hence, so long as

$$M = ||\frac{F(z)-z}{1-z}|| < 1,$$

we must have

$$V \leq 1/(1-M).$$

Now write $f_\infty = 1-F(1) = 1 - \sum_{n=1}^{\infty} f_n$

$$M = ||\frac{F(z)-z}{1-z}||$$

$$= \left|\left|\frac{\sum_{n=1}^{\infty} f_n z^n - z(f_\infty + \sum_{n=1}^{\infty} f_n)}{1-z}\right|\right|$$

$$= \left|\left|\sum_{n=1}^{\infty} f_n \frac{(z^n-z)}{1-z} - f_\infty \frac{z}{1-z}\right|\right|$$

$$\leq \sum_{n=1}^{\infty} f_n \left|\left|\frac{(z^n-z)}{1-z}\right|\right| + f_\infty \left|\left|\frac{z}{1-z}\right|\right|$$

$$= \sum_{n=1}^{\infty} f_n ||z+z^2+\ldots+z^{n-1}|| + f_\infty ||z+z^2+\ldots||$$

$$= (\sum_{n=1}^{\infty} f_n)(\lambda+\lambda^2+\ldots+\lambda^{n-1}) + f_\infty(\lambda+\lambda^2+\ldots)$$

$$= (\sum_{n=1}^{\infty} f_n)\ (\frac{\lambda^n-\lambda}{\lambda-1}) + f_\infty(\frac{\lambda}{1-\lambda})$$

$$= \frac{\sum_{n=1}^{\infty} f_n\lambda^n + \lambda \sum_{n=1}^{\infty} f_n + \lambda - \lambda(\sum_{n=1}^{\infty} f_n)}{1-\lambda}$$

$$= \frac{\lambda-F(\lambda)}{1-\lambda}.$$

So $M \leq \frac{\lambda-F(\lambda)}{1-\lambda} < 1$ implies

$V \leq 1/(1-M)$; i.e. $V \leq \frac{1-\lambda}{1-2\lambda+F(\lambda)}$ so long as $1-2\lambda+F(\lambda) > 0$.

Finally we obtain, for any renewal sequence $\{u_n\}$

$$V = \sum_{n=0}^{\infty} |u_n-u_{n-1}|\ \lambda^n \leq \frac{1-\lambda}{1-2\lambda+F(\lambda)}$$

$$= \frac{1-\lambda}{1-2\lambda+\frac{u(\lambda)-1}{u(\lambda)}} \quad \text{(from (1))}$$

$$= \frac{(1-\lambda)u(\lambda)}{2(1-\lambda)u(\lambda)-1}$$

for all λ in $0 < \lambda < 1$ such that $1 - 2\lambda + F(\lambda) > 0$.

In other words, for any renewal sequence $\{u_n\}$

$$\sum_{n=0}^{\infty} |u_n - u_{n-1}| \lambda^n \le \frac{(1+\lambda)\upsilon(\lambda)}{\sqrt{2(1-\lambda)\upsilon(\lambda)-1}} \tag{5}$$

for all λ in $0 < \lambda < 1$ such that $2(1-\lambda)\upsilon(\lambda) > 1$.

Step 2:

By a wellknown theorem (see e.g. KINGMAN [12; p. 35]) $\{p(nh)\}$ is a renewal sequence. Putting $u_n = p(nh)$, $\lambda = e^{-\theta h}$ in (5) we have

$$\sum_{n=0}^{\infty} |p(nh) - p(\overline{n-1}\,h)| \; e^{-n\theta h} \le \frac{\psi(\theta,h)}{2\psi(\theta,h)-1} \tag{6}$$

so long as $\psi(\theta,h) = (1-e^{-\theta h}) \sum_{n=0}^{\infty} p(nh) e^{-\theta nh} > \frac{1}{2}$.

Since $p(t)$ is continuous and bounded, for fixed positive θ we have

$$\lim_{h\to 0} h \sum_{n=0}^{\infty} p(nh)\, e^{-\theta nh} = \int_0^{\infty} p(t)\, e^{-\theta t}\, dt \tag{7}$$

by the definition of the integral. Also, by L'Hôpital Rule we have

$$\lim_{h\to 0} \frac{(1-e^{-\theta h})}{h} = \theta. \tag{8}$$

So, from (7) and (8), for fixed positive θ, we have

$$\lim_{h\to 0} \psi(\theta,h) = \lim_{h\to 0} \frac{(1-e^{-\theta h})}{h} \cdot h \sum_{n=0}^{\infty} p(nh) e^{-\theta h} \tag{9}$$

$$= \theta \int_0^{\infty} p(t)\, e^{-\theta t} dt$$

$$= \psi(\theta) \quad , \text{ say.}$$

Since $0 < p(t) \leq 1$ for all t we have

$$\psi(t) = \theta \int_0^\infty p(t)\, e^{-\theta t}\, dt \qquad (10)$$

$$\leq \theta \int_0^\infty e^{-\theta t}\, dt$$

$$= 1.$$

Furthermore, since $\lim_{t \downarrow 0} p(t) = 1$, for given $\varepsilon > 0$, there exists $\delta > 0$ such that $p(t) \geq 1 - \varepsilon$ for $0 < t < \delta$. This implies that, for $\theta > 0$

$$\psi(t) = \theta \int_0^\infty p(t)\, e^{-\theta t}\, dt$$

$$\geq \theta \int_0^\delta p(t)\, e^{-\theta t}\, dt$$

$$\geq \theta \int_0^\delta (1-\varepsilon)\, e^{-\theta t}\, dt$$

$$= (1-\varepsilon)(1-e^{-\theta\delta}).$$

Hence

$$\liminf_{\theta \to \infty} \psi(t) \geq 1 - \varepsilon. \qquad (11)$$

Since ε was arbitrary, we get

$$\liminf_{\theta \to \infty} \psi(t) \geq 1. \qquad (12)$$

But (10) implies that

$$\limsup_{\theta \to \infty} \psi(t) \leq 1$$ (13)

and from (12) and (13) we finally deduce that

$$\lim_{\theta \to \infty} \theta \int_0^\infty p(t)\, e^{-\theta t} = 1.$$ (14)

Now fix θ such that $\psi(\theta) \geq 3/4$. Since $\lim_{h \downarrow 0} \psi(\theta, h) = \psi(\theta)$, we can find some positive b such that $\psi(\theta, h) \geq 2/3$ for all $h < b$ then (6) shows that, for a certain value of $\theta > 0$, and all $0 < h < b$,

$$\sum_{n=0}^{\infty} |p(nh) - p(\overline{n-1}\, h)|\, e^{-\theta nh} \leq 2.$$ (15)

Now let a be any positive number. Then, from (15) it follows that, for all $N > a/b$

$$\sum_{n=0}^{\infty} \left|p\left(\frac{na}{N}\right) - p\left(\frac{n-1}{N}\, a\right)\right| e^{-\frac{\theta na}{N}} \leq 2.$$ (16)

In particular, we have

$$\sum_{n=1}^{N} \left|p\left(\frac{n}{N}\, a\right) - p\left(\frac{n-1}{N}\, a\right)\right| e^{-\theta \frac{n}{N} a} \leq 2.$$ (17)

But $e^{-\theta a} \leq e^{-\theta \frac{n}{N} a}$ for $n \leq N$. So, from (17) we obtain

$$\sum_{n=1}^{N} \left|p\left(\frac{n}{N}\, a\right) - p\left(\frac{n-1}{N}\, a\right)\right| \leq 2e^{\theta a}$$ (18)

Since this upperbound is independent of N, and since p is (uniformly) continuous in (0,a), it follows that p is of bounded variation in (0,a). Since a was arbitrary, the proof is complete.

We complete this section by giving an example of a p-function which is not differentiable everywhere.

Consider independent random variables $X_1, X_2, \ldots$ distributes as follows:

$$P(X_n \leq x) = A(x) \text{ if } n \text{ is odd,}$$
$$= B(x) \text{ if } n \text{ is even.}$$

Put $S_0=0$, $S_n = X_1 + \ldots + X_n$, $(n \geq 1)$.

Define a process $(Z(t);\ t > 0)$ by

$$Z(t) = 1 \quad (S_{2n} < t \leq S_{2n+1})$$

$$Z(t) = 0 \quad (S_{2n+1} < t \leq S_{2n+2}),$$

(for $n=0, 1, 2, \ldots$).

Such a process, which takes values 1 and 0 alternately on intervals of random length is called an alternating renewal process and is in general, not a regenerative phenomenon.

If we write $p(t) = P(Z(t) = 1)$ then

$$p(t) = \sum_{n=0}^{\infty} P(S_{2n} < t \leq S_{2n+1})$$

$$= \sum_{n=0}^{\infty} \{P(S_{2n} < t) - P(S_{2n+1} < t)\}$$

So that

$$p(t) = \sum_{n=0}^{\infty} (-1)^n P(S_n < t). \tag{19}$$

Taking Laplace transforms; for $\theta > 0$,

$$\int_0^\infty p(t)\, e^{-\theta t}\, dt = \int_0^\infty E\{Z(t)\}\, e^{-\theta t}\, dt$$

$$= E \int_0^\infty Z(t)\, e^{-\theta t}\, dt$$

$$= E \sum_{n=0}^{\infty} \int_{S_{2n}}^{S_{2n+1}} e^{-\theta t}\, dt$$

$$= E \sum_{n=0}^{\infty} \theta^{-1} (e^{-S_{2n}} - e^{-S_{2n+1}})$$

$$= \theta^{-1} \sum_{n=0}^{\infty} \{E(e^{-S_{2n}}) - E(e^{-S_{2n+1}})\}$$

where the interchanges of summation are allowed (since all quantities are positive) by monotone convergence.

So we have

$$\int_0^\infty p(t) e^{-\theta t} dt = \theta^{-1} \sum_{n=0}^{\infty} \{E(e^{-S_{2n}}) - E(e^{-S_{2n+1}})\}. \tag{20}$$

If we put

$$\alpha(\theta) = \int_0^\infty e^{-\theta x} dA(x) \tag{21}$$

and

$$\beta(\theta) = \int_0^\infty e^{-\theta x} dB(x) \tag{22}$$

then

$$E(e^{-S_{2n}}) = E(e^{-(X_1 + X_2 + \cdots + X_{2n})})$$

$$= \prod_{i=1}^{2n} E(e^{-X_i}) \quad \text{(by independence)}$$

$$= \alpha(\theta)^n \beta(\theta)^n.$$

Similarly, $E(e^{-S_{2n+1}}) = \alpha(\theta)^{n+1} \beta(\theta)^n$, so (20) becomes

$$\int_0^\infty p(t)\, e^{-\theta t} dt = \theta^{-1} \sum_{n=0}^{\infty} \{\alpha(\theta)^n \beta(\theta)^n - \alpha(\theta)^{n+1} \beta(\theta)^n\}$$

$$= \theta^{-1} \cdot \frac{(1-\alpha(\theta))}{1-\alpha(\theta)\beta(\theta)} ,$$

since the series in (22) is geometric.

Now let A be the negative exponential distribution.

$$A(x) = 1-e^{-ax}, \quad (x \geq 0,\ a>0), \tag{23}$$

and suppose that, for some $T>0$, $Z(T) = 1$ and let N be the integer $S_{2n} < T < S_{2N+1}$.

$$P(S_{2N+1} - T \geq t \mid Z(T) = 1) = P(X_{2n+1} \geq T + t - S_{2n} \mid X_{2n=1} \geq T - S_{2n})$$

$$= \frac{P(X_{2n+1} \geq T + t - S_{2n})}{P(X_{2n+1} \geq T - S_{2n})}$$

$$= e^{-a(T+t-S_{2n})} / e^{-a(T-S_{2n})}$$

$$= e^{-at}.$$

In other words, $S_{2N+1} - T$ has the same negative exponential distribution, and so the process $Z(T+t,\ t\geq 0)$ has the same structure as $Z(t)$. By theorem 1.3.1, Z is a regenerative phenomenon with p-function p.

Since $p(t) = P(Z(t) = 1)$

$$= \sum_{n=0}^{\infty} P(S_{2n} < t \leq S_{2n+1})$$

$$\geq P(S_0 < t \leq S_1)$$

$$= P(X_1 \geq t)$$

$$= e^{-at} ,$$

which implies $\lim_{t\downarrow 0} p(t) \geq 1$; that is, p(t) is a standard p-function. So p is continuous and hence uniquely determined by its Laplace transform. From (20) we have

$$\alpha(\theta) = a(a+\theta)^{-1},$$

so, with A(x) the negative exponential given in (23), (22) takes the form

$$\int_0^{\infty} p(t)\ e^{-\theta t}dt = \theta^{-1}\ \frac{(1-\alpha(\theta))}{1-\alpha(\theta)\beta(\theta)}$$

$$= \theta^{-1}\ \frac{(1-a(a+\theta)^{-1})}{1-a(a+\theta)^{-1}\beta(\theta)}$$

Simplifying, we obtain

$$\int_0^{\infty} p(t)\ e^{-\theta t}dt = (a+\theta-a\ \beta(\theta))^{-1} . \qquad (24)$$

Expanding the right hand side of (24) in a series we get

$$\sum_{n=0}^{\infty} \frac{a^n \beta(\theta)^n}{(a+\theta)^{n+1}}$$

which is the Laplace transform of

$$p(t) = \sum_{n=0}^{\infty} \int_0^t \Pi_n \{a(t-\mu)\} \, d\, B_n(u)$$

where B_n is the n-fold Stieltjes convolution of B with itself and

$$\Pi_n(x) = \frac{e^{-x} x^n}{n!} .$$

From (25) we see that by varying B we may construct many p-functions. For our purpose let us consider the case where B is degenerate, at the point b, say. That is, take

$$B(x) = 0 , \; x < b \qquad (26)$$
$$= 1 , \; x \geq b$$

Then an easy calculation shows that (25) becomes

$$p(t) = \sum_{n=0}^{[t/b]} \Pi_n \{a(t-nb)\}. \qquad (27)$$

At the point t=b, the function has right and left derivatives

$$D_+ \, p(b) = a-a \, e^{-ab} , \; D_- \, p(b) = -a \, e^{-ab}. \qquad (28)$$

We noted in Chapter 1 that many p-functions arise from diagonal elements of a standard Markov chain. But the elements of a standard transition matrix are everywhere continuously differentiable in $(0,\infty)$. Accordingly, (28) debars the function (27) from being of the form $p_{aa}(t)$ for any chain.

5.3 Semi-p-functions

Theorem 5.3.1

If p is a standard semi-p-function, and $0 < t < \infty$, then there exists a number λ and a p-function $\overline{p}$ such that

$$p(t) = e^{\lambda t}\,\overline{p}(t) \quad (0 \leq t \leq T). \tag{1}$$

Proof:

Consider the set $E = \{t \geq 0;\ Tp(t) = (t-T)p(2T)\}$. Since p is continuous, E is closed. E is non-empty since $2T \in E$. Furthermore, since $p(t) > 0$, then $t \in E$ implies $(t-T)p\text{-}2T > 0$; i.e. $t > T$. Hence $E \subset (T,\infty)$. Let S denote the greatest lowerbound of E. We have, since E is closed, $S \in E$. Also, since $T \notin E$ but $2T \in E$, then $T < S \leq 2T$.

Consider the continuous function

$$f(t) = T\,p(t) - (t-T)\,p(2T).$$

Since $S = g \cdot l \cdot b\ E$ then f is non-zero on $[0,S]$. But f is non-zero on $[0,T]$. It follows from the Intermediate Value Theorem of Calculus that

$$T\,p(t) - (t-T)\,p(2T) > 0 \quad (0 \leq t < S).$$

Since $T\,p(S) - (S-T)\,p(2T) > 0$ we have

$$\frac{p(t)}{p(S)} > \frac{(t-T)}{(S-T)} \tag{2}$$

Consider the function

$$\phi(t) = \frac{\log p(S) - \log p(t)}{S-t}, \quad t \in [0,S].$$

ϕ is continuous on $[0,S]$ so it is bounded above on $[0,T]$. Also, if $T < t < S$, then

$$\limsup_{t \uparrow s} \phi(t) = \limsup_{t \uparrow s} \frac{\log p(S) - \log p(t)}{S-t}$$

$$= \limsup_{t \uparrow s} \frac{\log (p(S)/p(t))}{s-t} \quad (3)$$

It follows from (2) that if $T < t < S$, then

$$\frac{p(S)}{p(t)} < \frac{S-T}{t-T} \quad (4)$$

Hence, from (3) and (4) we obtain

$$\limsup_{t \uparrow s} \phi(t) < \limsup_{t \uparrow s} \frac{\log \left(\frac{S-T}{t-T}\right)}{S-t}$$

$$= \frac{1}{S-T} < \infty.$$

Hence ϕ is bounded above on $[0,S)$. We may therefore choose λ so large that

$$\phi(t) \leq \lambda \quad (0 \leq t < S)$$

whence

$$p(s)\, e^{-\lambda S} \leq p(t)\, e^{-\lambda t} \quad (0 \leq t \leq S). \quad (5)$$

For any positive integer N define

$f_n = e^{-\lambda hn} F(h, 2h, \ldots, nh;\, p)$ where $h = S/N$ and F is as in section I.3. It is well known [16] that f_n is the f-sequence associated with the generalised renewal sequence $e^{\lambda nh} p(nh)$; hence $f_n \geq 0$. Define a generalised renewal sequence u such that

$$f_n = e^{-\lambda nh} F(h, 2h, \ldots, nh; p) \quad (1 \le n \le N)$$

$$= 0 \quad (n > N).$$

Then

$$u_n = e^{-\lambda nh} p(nh) \quad (0 \le n \le N), \tag{6}$$

and it follows from (5) that

$$u_N \le u_n \quad (0 \le n \le N).$$

Thus

$$u_N = \sum_{r=1}^{N} f_r u_{N-r}$$

$$\ge \sum_{r=1}^{N} f_r u_N$$

which shows that

$$\sum_{r=1}^{N} f_r \le 1.$$

But $f_n = 0$, $n > N$. Hence $\sum_{r=1}^{\infty} f_r \le 1$ and u is a renewal sequence.

It is well known (see e.g. KINGMAN [15; p. 34])

that the function

$$p_N(t) = \sum_{n=0}^{\infty} u_n \frac{e^{-t/h}(t/h)^n}{n!} \tag{7}$$

is a p-function. Fix $t \in [0,S]$. Then, putting $\Pi_n(t) = e^{-t}t^n/n!$, $n \ge 0$, we have

$$|p(t)\ e^{-\lambda t} - P_N(t)|$$

$$= |\sum_{n=0}^{\infty} p(t)\ e^{-\lambda t}\ \Pi_n\ (\tfrac{t}{h}) - \sum_{n=0}^{\infty} u_n\ \Pi_n(\tfrac{t}{h})|$$

$$\leq \sum_{n=0}^{\infty} |p(t)\ e^{-\lambda t} - u_n|\Pi_n(\tfrac{t}{h})$$

$$= \sum_{n=0}^{N} |p(t)\ e^{-\lambda t} - e^{-\lambda nh}p(nh)|\Pi_n(\tfrac{t}{h})$$

$$+ \sum_{n=N+1}^{\infty} |p(t)\ e^{-\lambda t} - u_n|\Pi_n(\tfrac{t}{h})$$

$$\leq \sum_{n=0}^{N} |p(t)\ e^{-\lambda t} - p(nh)\ e^{-\lambda nh}|\Pi_n(\tfrac{t}{h})$$

$$+ (p(t)\ e^{-\lambda t} + 1) \sum_{n=N+1}^{\infty} \Pi_n(\tfrac{t}{h})\ ,$$

since $0 \leq u_n \leq 1$. But $\sum_{n=0}^{\infty} \Pi_n(\frac{t}{h}) = 1$, so given $\varepsilon > 0$, $\exists$ an integer n_0 such that

$$\sum_{n=m}^{\infty} \Pi_n(\tfrac{t}{h}) < \frac{\varepsilon}{3(p(t)\ e^{-\lambda t} + 1)} \quad \text{if } m > n_0.$$

Also, by the continuity of $p(t)\ e^{-\lambda t}$ we may choose $\delta>0$ such that $|p(t)\ e^{-\lambda t} - p(s)\ e^{-\lambda s}| < \frac{\varepsilon}{3}$ if $|s-t| < \delta$. Then, if $N > n_0$

$$|p(t)e^{-\lambda t}-p_N(t)| \leq \sum_{n=0}^{N} |p(t)e^{-\lambda t}-p(nh)e^{-\lambda nh}|\ \Pi_n(\tfrac{t}{h}) + \frac{\varepsilon}{3}$$

$$= \sum_{|t-nh|<\delta} |p(t)e^{-\lambda t}-p(nh)e^{-\lambda nh}|\Pi_n(\tfrac{t}{h})$$

$$+ \sum_{|t-nh|\geq\delta} |p(t)e^{-\lambda t}-p(nh)e^{-\lambda nh}| \Pi_n(\tfrac{t}{h}) + \frac{\varepsilon}{3}$$

$$\leq \sum_{|t-nh|<\delta} \frac{\varepsilon}{3}\Pi_n(\tfrac{t}{h}) + \Delta \sum_{|t-nh|\geq\delta} \Pi_n(\tfrac{t}{h}) + \frac{\varepsilon}{3},$$

where $\Delta = 2 \max_{t\in[0,S]} p(t)e^{-\lambda t}$.

Hence, using Tchebychev's inequality we obtain

$$|p(t)e^{-\lambda t}-p_N(t)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \Delta \sum_{|\frac{t}{h}-n|\geq\frac{\delta}{h}} \Pi_n(\tfrac{t}{h})$$

$$< \frac{2\varepsilon}{3} + \frac{\Delta th}{\delta^2}.$$

But $h = S/N$. Hence if $N > \max(n_0, \frac{3\Delta S^2}{\varepsilon\delta^2}) = N_0$

we have

$$|p(t)e^{-\lambda t}-p_N(t)| < \varepsilon, \ t\in[0,S], \quad N>N_0.$$

Now $p(t)e^{-\lambda t}\to 1$ as $t\downarrow 0$, so $\exists S'$ such that

$|p(t)e^{-\lambda t}-1| < \varepsilon$, for all $t\in[0,S']$, from which it follows that $|p_N(t) - 1| < 2\varepsilon$, $t\in[0,S']$, $N>N_0$. Also, $p_N(t) = e^{-\frac{tN}{S}} \sum_{n=0}^{\infty} \frac{u_n}{n!} (\frac{tN}{S})^n$. By Monotone Convergence theorem it follows that

$$\lim_{t\downarrow 0} p_N(t) = 1.$$

So $\exists S_1,\ldots,S_{N_0}$ such that $|p_N(t) - 1| < 2\varepsilon$, for all $t\in[0,S_N]$, $N=1,\ldots,N_0$. If we let $\delta = \min\{S_1,\ldots,S_N,S'\}$ then we have

$|p_N(t) - 1| < 2\varepsilon, \ \forall t \in [0,\delta], \forall N.$

This means that the family $\{p_N\}$ is equicontinuous at $t=0$. It has been shown [11] that the set P of standard p-functions forms a metrisable space in which the relatively compact sets are precisely the sets of functions which are equicontinuous at $t=0$. Thus $\{p_N\}$ is a relatively compact sequence in P. So $\{p_N\}$ has a limit point, $\overline{p}(t)$, in P. So $\exists$ a subsequence $\{p_{N_i}\}$ of $\{p_N\}$ such that $p_{N_i}(t) \to \overline{p}(t)$, for all t.

We conclude that $\hat{p}(t)\, e^{-\lambda t} = \overline{p}(t)$ on $[0,S]$.

Corollary:

If p is a semi-p-function then p is differentiable almost everywhere.

Proof:

This follows immediately from the theorem and the corresponding results for p-functions.

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