HEREDITARY, CONTINUOUS, HOMOTOPY, ISOTOPY, PRODUCTIVE AND EXPANSIVE TOPOLOGICAL PROPERTIES

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by

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ABSTRACT

Several separation axioms, connectedness and compactness as well as a number of other general topological properties are studied to determine whether or not they are invariant with respect to heredity, closed heredity, open heredity, continuity, open continuity, closed continuity, divisibility, retractions, projections, homotopy and isotopy equivalences, as well as finite, countable, and arbitrary products, and also with respect to contractions and expansions of the topology of a space. Of the resulting four hundred eighty questions, all but seven can be answered. The resulting answers, found and discussed in the paper, are also tabulated in several tables.

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INTRODUCTION

This paper is the result of a study done on thirty topological properties in order to compile a table of invariances and non-invariances for all those properties with respect to homotopy and isotopy equivalences and certain constructions such as subspaces, continuous functions, product spaces and expansions of the topology of a space. The resulting four hundred eighty questions are studied in detail and the answers to all but seven of these questions are given in this paper and are compiled in Table I at the end of this introduction.

On the basis of results by S.T. Hu (28, 1961) and D. Gottlieb (23, 1964), we are able to conclude that most of those thirty properties are not homotopy properties but the majority are isotopy properties.

In order to fix our terminology, a definition of each of the thirty topological properties considered is given in Chapter 0.

In Chapter 1 the following types of properties for topological spaces are defined: hereditary, closed hereditary, open hereditary, retractive, projective, homotopy and isotopy topological properties. Two theorems from Hu, Gottlieb's theorem on closed hereditary and isotopy properties and the proof that path connectedness is a homotopy property are given and we show that, for any topological property, P, the following implications hold:

Hereditary $\Rightarrow$ retractive $\Rightarrow$ projective

Hereditary $\Rightarrow$ closed hereditary and open hereditary

Hereditary $\Rightarrow$ isotopy

Closed hereditary $\Rightarrow$ not isotopy $\Rightarrow$ not homotopy

Homotopy $\Rightarrow$ isotopy

$\Rightarrow$ not closed hereditary

$\Rightarrow$ not hereditary.

Chapter 2 gives the definitions of continuous, open continuous, closed continuous, divisible, contractive and open topological properties as well as Gottlieb's results on open, homotopy and isotopy topological properties and we show that the following implications, for any topological property, P, hold:

Continuous $\Rightarrow$ divisible $\Rightarrow$ retractive $\Rightarrow$ projective

Divisible $\Rightarrow$ open continuous and closed continuous

Continuous $\Rightarrow$ contractive

Open hereditary and open continuous $\Rightarrow$ open

Open and not hereditary $\Rightarrow$ not isotopy $\Rightarrow$ not homotopy

Homotopy and open continuous $\Rightarrow$ not open hereditary.

In Chapter 3 the invariance under finite, countable, and arbitrary products and under expansions of the topology of a space are considered, and the following implications hold:

$$\text{Arbitrarily productive} \Rightarrow \text{countably productive}$$
$$\Rightarrow \text{finitely productive.}$$

For completeness, Chapter 4 contains the proofs of some basic statements which were found in the literature without proof. Several counterexamples are also given, as well as four tables tabulating those counterexamples.

Table I

TABLE OF PROPERTIES

"+" indicates that the property is invariant.
"-" indicates that the property is not invariant.

	Property	Type of invariance															
		Hereditary	Closed hereditary	Open hereditary	Continuous	Divisible	Retractive	Projective	Open continuous	Closed continuous	Contractive	Homotopy	Isotopy	Arbitrarily productive	Countably productive	Finitely productive	Expansive
1	T_0	+	+	+	-	-	+	+	-	-	-	-	+	+	+	+	+
2	T_1	+	+	+	-	-	+	+	-	+	-	-	+	+	+	+	+
3	T_2	+	+	+	-	-	+	+	-	-	-	-	+	+	+	+	+
4	$T_{2\frac{1}{2}}$	+	+	+	-	-	+	+	-	-	-	-	+	+	+	+	+
5	T_3	+	+	+	-	-	+	+	-	-	-	-	+	+	+	+	-
6	$T_{3\frac{1}{2}}$	+	+	+	-	-	+	+	-	-	-	-	+	+	+	+	-
7	T_4	-	+	-	-	-	+	+	-	+	-	-	-	-	-	-	-
8	T_5	+	+	+	-	-	+	+	-	+	-	-	+	-	-	-	-
9	Regular	+	+	+	-	-	+	+	-	-	-	-	+	+	+	+	-
10	Completely regular	+	+	+	-	-	+	+	-	-	-	-	+	+	+	+	-
11	Normal	-	+	-	-	-	+	+	-	+	-	-	-	-	-	-	-
12	Completely normal	+	+	+	-	-	+	+	-	+	-	-	+	-	-	-	-
13	Connected	-	-	-	+	+	+	+	+	+	+	+	+	+	+	+	-
14	Path connected	-	-	-	+	+	+	+	+	+	+	+	+	+	+	+	-
15	Locally connected	-	-	+	-	+	+	+	+	+	-	-	-	-	-	+	-
16	Totally disconnected	+	+	+	-	-	+	+	-		-	-	+	+	+	+	+
17	Compact	-	+	-	+	+	+	+	+	+	+	-	-	+	+	+	?
18	Lindelöf	-	+	-	+	+	+	+	+	+	+	-	-	-	-	-	-
19	Locally compact	-	+	-	-	-	+	+	+	-	-	-	-	-	-	+	-
20	Countably compact	-	+	-	+	+	+	+	+	+	+	-	-	-	-	-	-
21	Paracompact	-	+	-	-	-	+	+	-	+	-	-	-	-	-	-	-
22	Separable	-	-	+	+	+	+	+	+	+	+	-	-	-	+	+	-
23	Second countable	+	+	+	-	-	+	+	+	-	-	-	+	-	+	+	-
24	First countable	+	+	+	-	-	+	+	+	-	-	-	+	-	+	+	-
25	Discrete	+	+	+	-	+	+	+	+	+	-	-	+	-	-	+	+
26	Indiscrete	+	+	+	+	+	+	+	+	+	+	-	+	+	+	+	-
27	Metrizable	+	+	+	-	-	+	+	-	-	-	-	+	-	+	+	-
28	Fixed point	-	-	-	-	-	+	+		-	-	-	-	-	-	-	-
29	Contractible	-	-	-	-	-	+	+		-		+	+	+	+	+	-
30	Locally contractible	-	-	+	-		+	+			-	-	-	-	-	+	-

CHAPTER 0

DEFINITIONS OF PROPERTIES

In this chapter we collect the definitions of basic properties for the convenience of the reader and to fix our terminology.

0.1: Definition:

A topological space (X,T) is a <u>T_0-space</u> if for any $a,b \in X$, $\exists$ an open set $U \in T$ s.t. either: $a \in U$, $b \notin U$ or: $a \notin U$, $b \in U$.

0.2: Definition:

A topological space (X,T) is a <u>T_1-space</u> if for any $a,b \in X$, $\exists$ open sets $U,V \in T$ s.t. $a \in U$, $b \in V$ and $a \notin V$, $b \notin U$.

0.3: Definition:

A topological space (X,T) is a <u>T_2-space</u> if for any $a,b \in X$, $\exists$ open sets $U,V \in T$ s.t. $a \in U$, $b \in V$ and $U \cap V = \emptyset$.

0.4: Definition:

A topological space (X,T) is a <u>$T_{2\frac{1}{2}}$-space</u> if for any $a,b \in X$, $\exists$ open sets $U,V \in T$ s.t. $a \in U$, $b \in V$ and $\bar{U} \cap \bar{V} = \emptyset$.

0.5: Definition:

A topological space (X,T) is a <u>T_3-space</u> if A is a closed set in X and $b \in X$ s.t. $b \notin A$; then $\exists$ open sets $U, V \in T$ s.t. $A \subset U$, $b \in V$ and $U \cap V = \emptyset$.

0.6: Definition:

A topological space (X,T) is a <u>$T_{3\frac{1}{2}}$-space</u> if A is a closed set in X and $b \in X$ s.t. $b \notin A$, then $\exists$ a continuous function $f: X \longrightarrow [0,1]$ s.t. $f(A)=0$ and $f(\{b\})=1$.

0.7: Definition:

A topological space (X,T) is a <u>T_4-space</u> if A,B are disjoint closed sets in X, then $\exists$ open sets $U, V \in T$ s.t. $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

0.8: Definition:

A topological space (X,T) is a <u>T_5-space</u> if A and B are separated sets in X (i.e. $\bar{A} \cap B = A \cap \bar{B} = \emptyset$), then $\exists$ $U, V \in T$ s.t. $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

0.9: Definition:

A topological space is a <u>regular space</u> if it is T_0 and T_3.

0.10: Definition:

A topological space is a <u>completely regular space</u> if it is T_0 and $T_{3\frac{1}{2}}$.

0.11: Definition:

A topological space is a <u>normal space</u> if it is T_1 and T_4.

0.12: Definition:

A topological space is a <u>completely normal space</u> if it is T_1 and T_5.

0.13: Definition:

A topological space X is a <u>connected space</u> if and only if it has no nontrivial separation, i.e. if U,V are non-empty open sets in X s.t. $U \cup V = X$, then $U \cap V = \emptyset$.

0.14: Definition:

A topological space X is a <u>path connected space</u> if it has only one path component, i.e. X is path-connected $\Longleftrightarrow$ for any two points $a, b \in X$, $\exists$ a path $p: I \longrightarrow X$ s.t. $p(0)=a$ and $p(1)=b$.

0.15: Definition:

A topological space X is a <u>locally connected space</u> if for every point $p \in X$ and for every neighbourhood U of p, $\exists$ a connected neighbourhood $V \subset U$ of p.

0.16: Definition:

A topological space X is a <u>totally disconnected space</u> if for each pair of points $p, q \in X$, $\exists$ a disconnection $U \cup V$ of X with $p \in U$ and $q \in V$. ($U \cup V$ is a disconnection of $X \Longleftrightarrow U \neq \emptyset$, $V \neq \emptyset$, $U \cap V = \emptyset$ and $U \cup V = X$.)

0.17: Definition:

A topological space X is a compact space if and only if every open cover of X has a finite subcover.

0.18: Definition:

A topological space X is a Lindelöf space if and only if every open cover of X has a countable subcover.

0.19: Definition:

A topological space X is a locally compact space if for every point $p \in X$, p has at least one compact neighbourhood in X.

0.20: Definition:

A topological space X is a countably compact space if and only if each countable open cover of X has a finite subcover.

0.21: Definition:

A topological space X is a paracompact space if and only if it is T_2 and every open cover of X has a locally finite open refinement, i.e. a refinement which is a locally finite open cover of X.

0.22: Definition:

A topological space X is a separable space if and only if it contains a countable subset which is dense in X.

0.23: Definition:

A topological space X is a second countable space if and only if its topology has a countable basis.

0.24: Definition:

A topological space X is a first countable space if and only if it has a countable local basis at each of its points.

0.25: Definition:

A topological space X is a discrete space if every subset of X is open.

0.26: Definition:

A topological space X is an indiscrete space if the only open sets are $\emptyset$ and X.

0.27: Definition:

A topological space X is a metrizable space if and only if $\exists$ a metric $d: X^2 \longrightarrow R$ which induces the topology of X.

0.28: Definition:

A topological space X has the <u>fixed point property</u> if every continuous function $f: X \rightarrow X$ is s.t. $f(x)=x$ for at least one $x \in X$.

0.29: Definition:

A topological space X is a <u>contractible space</u> if it has the same homotopy type as a singleton point.

0.30: Definition:

A topological space X is a <u>locally contractible space</u> if for every point $p \in X$ and for every neighbourhood U of $p \in X$ $\exists$ a neighbourhood $V \subset U$ of p which is contractible in U.

CHAPTER 1

HEREDITARY, ISOTOPY AND HOMOTOPY PROPERTIES

This chapter will present theorems by Hu (28) and Gottlieb (23) which allow us to conclude that sixteen of the thirty properties defined in Chapter 0 are hereditary properties, and hence are isotopy properties, but are not homotopy properties. Seven other properties, being closed hereditary properties, are not isotopy, and hence not homotopy properties. Counterexamples, to show that these seven properties are not hereditary properties, will be given in Chapter 4. Three further properties are homotopy properties, and hence are isotopy properties, but are not closed hereditary, and thus not hereditary properties. The remaining four properties, for which a conclusion with respect to hereditary, homotopy, and isotopy properties cannot be drawn from the theorems presented in this chapter, will be considered in Chapters 2 and 4.

1.1: Definition:

Property P of topological spaces is a <u>hereditary property</u> if and only if P is inherited by every subspace of a space which has P.

1.2: Definition:

Property P of topological spaces is a <u>closed hereditary</u>

property if and only if P is inherited by every closed subspace of a space which has P.

1.3: Definition:

Property P of topological spaces is an open hereditary property if and only if P is inherited by every open subspace of a space which has P.

1.4: Definition:

A map $f:X \longrightarrow Y$ from a given space X into a given space Y is said to be a homotopy equivalence if and only if there exists a map $g:Y \longrightarrow X$ from Y into X such that the composed maps $gof:X \longrightarrow X$ and $fog:Y \longrightarrow Y$ are homotopic to the identity maps on X and Y respectively. That is, there exist homotopies $H:X \times I \longrightarrow X$ and $K:Y \times I \longrightarrow Y$, where I is the closed unit interval, such that

$$H(x,0) = 1_X(x), \quad \forall x \in X$$
$$H(x,1) = gf(x), \quad \forall x \in X$$
$$K(y,0) = 1_Y(y), \quad \forall y \in Y$$
$$K(y,1) = fg(y), \quad \forall y \in Y$$

In this case, the map $g:Y \longrightarrow X$ is also a homotopy equivalence.

1.5: Definition:

Property P of topological spaces is a homotopy property if and only if P is inherited by every space that is homotopically equivalent to a space which has P.

1.6: Definition:

A map is a continuous function.

1.7: Definition:

If $f: X \longrightarrow Y$ is an injective map from a space X into a space Y which defines a homeomorphism from X onto $f(X)$, then f is an imbedding of X into Y.

1.8: Definition:

A homotopy $h_t: X \longrightarrow Y$, $(t \in I)$, is said to be an isotopy if, for each $t \in I$, h_t is an imbedding. Two imbeddings f, $g: X \longrightarrow Y$ are said to be isotopic if there exists an isotopy $h_t: X \longrightarrow Y$, $(t \in I)$, such that $h_0 = f$ and $h_1 = g$. An imbedding $f: X \longrightarrow Y$ is said to be an isotopy equivalence if there exists an imbedding $g: Y \longrightarrow X$ such that the composite imbeddings $g \circ f$ and $f \circ g$ are isotopic to the identity imbeddings on X and Y respectively. Two topological spaces X and Y are said to be isotopically equivalent (in symbol, $X \cong Y$) if there exists an isotopy equivalence $f: X \longrightarrow Y$. The relation $\cong$ among topological spaces is obviously an equivalence relation.(28, page 168).

1.9: Definition:

Property P of topological spaces is an isotopy property if and only if P is inherited by every space that is isotopically equivalent to a space which has P.

Hence the following implication holds trivially:

P is a homotopy property $\Rightarrow$ P is an isotopy property.

1.10: Definition:

If A is a subspace of the topological space X and $r: X \longrightarrow A$ is a continuous onto function such that $r \circ i = 1_A$, where i is the inclusion map $i: A \longrightarrow X$, then A is called a <u>retract</u> of X and the function $r: X \longrightarrow A$ is called a <u>retraction</u> of X onto A.

1.11: Definition:

Property P of topological spaces is a <u>retractive property</u> if and only if P is inherited by every retract of a space which has P.

1.12: Definition:

Property P of topological spaces is a <u>projective property</u> if and only if P is preserved by every projection of an arbitrary product space which has P.

Since every factor space, X_α, where α is from some arbitrary index set $\mathcal{A}$ (not necessarily finite or countably), obtained from a projection map, p, from a topological space X onto the factor space, is a retract of X, i.e. $p \circ i = 1_{X_\alpha}$, where i is the inclusion map $i: X_\alpha \longrightarrow X$, and every retract, open subspace, and closed subspace is a subspace of X, the following implications for any topological property hold trivially:

Hereditary $\Rightarrow$ retractive $\Rightarrow$ projective

Hereditary $\Rightarrow$ open hereditary

Hereditary $\Rightarrow$ closed hereditary

1.13: Theorem: (Hu's Theorem (28, page 170))

If P is a closed hereditary topological property such that every singleton space has P and that there exists a space X which does not have P, then P is not a homotopy property.

Proof:

Let X be a space which does not have P. Consider the cone, CX, over X which is the quotient space obtained by identifying the top, $X \times \{0\}$, of the cylinder, $X \times I$, to a single point, v, called the vertex of CX. X is homeomorphic with the bottom, $X \times \{1\}$, of CX since the inclusion map, $i: X \longrightarrow X \times \{1\}$ is continuous and bijective, the projection, $p: X \times \{1\} \longrightarrow X$ is continuous and $p = i^{-1}$. $X \times \{1\}$ is a closed subspace of CX since X is a closed subspace of X, $\{1\}$ is a closed subspace of I and the Cartesian product of closed sets is closed. Since P is a closed hereditary property which X does not have, it follows that CX cannot have P. Consider the inclusion map, $j: v \longrightarrow CX$, and the projection map, $q: CX \longrightarrow v$, which is a constant map.

Claim: j is a homotopy equivalence, that is, $j \circ q \simeq 1_{CX}$ and $q \circ j \simeq 1_v$. Now $q \circ j = 1_v \Longrightarrow q \circ j \simeq 1_v$ and $j \circ q$ = constant map, $c: CX \longrightarrow v \in CX$. Claim: there exists a homotopy, $h_t: CX \longrightarrow CX$, such that h_t connects $h_o = c$ with $h_1 = 1_{CX}$. Consider $h_t(x,s) = (x,st)$, $\forall s,t \in I$.

Now $h_o(x,s) = c(x,s) = (x,o) = v$

And $h_1(x,s) = 1_{CX}(x,s) = (x,s)$, $\forall s \in I$.

So CX and the singleton space, $\{v\}$, have the same homotopy type. Hence CX is contractible. Since the singleton space, $\{v\}$, has P and CX does not have P, P is not a homotopy property.

A contrapositive statement of Hu's theorem can be used in order to conclude that homotopy properties are not closed hereditary properties.

1.14: Theorem:(Corollary to Hu's Theorem)

If P is a homotopy topological property such that every singleton space has P and that there exists a space, X, which does not have P, then P is not a closed hereditary property and hence not a hereditary property.

1.15: Theorem:

Hereditary properties are isotopy properties.

Proof: (28, page 171)

Let P be any hereditary topological property of spaces. Assume that $f: X \longrightarrow Y$ is an isotopy equivalence and that the space, X, has the property P. It suffices to prove that Y also has P. By definition of an isotopy equivalence, there exists an imbedding, $g: Y \longrightarrow X$, such that the composed imbeddings, $g \circ f$ and $f \circ g$, are isotopic to the identity imbeddings on X and Y respectively. The image, $g(Y)$, is a subspace of X. Since P is hereditary, this implies that $g(Y)$ has P. As an imbedding, g is a homeomorphism of Y onto $g(Y)$. Since P is a topological property and $g(Y)$ has P, it follows that Y also has P.

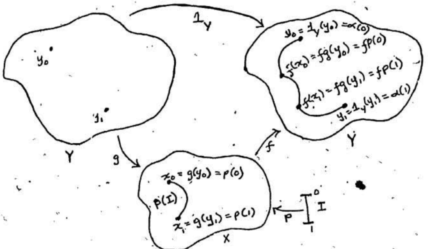
Since the singleton space has all the properties defined in Chapter 0 and for each property there exist spaces for which that property fails, then Hu's theorem can be applied to the properties in Chapter 0 whenever necessary.

1.16: Theorem:

"Being path connected" is a homotopy property.

Proof:

Let $f: X \longrightarrow Y$ be a homotopy equivalence with homotopy inverse $g: Y \longrightarrow X$ and let X be path connected. Let $y_0, y_1 \in Y$. We shall show that $\exists$ a path $\alpha: I \longrightarrow Y$ s.t. $\alpha(0) = y_0$ and $\alpha(1) = y_1$. Let $K: Y \times I \longrightarrow Y$ be a homotopy from f o g to 1_Y, i.e., $K(y,o) = f \circ g(y)$ and $K(y,1) = 1_Y(y), \forall y \in Y$. Let $p: I \longrightarrow X$ be a path s.t. for any $x_0, x_1 \in X$, $p(o) = x_0 = g(y_0)$ and $p(1) = x_1 = g(y_1)$.



Define $\alpha(t) = \begin{cases} K(y_0, 1-3t), & o \leq t \leq 1/3 \\ f \circ p(3t-1), & 1/3 \leq t \leq 2/3 \\ K(y_1, 3t-2), & 2/3 \leq t \leq 1 \end{cases}$

$\alpha(t)$ is well defined:

$\alpha(1/3) = K(y_0, 0) = f \circ g(y_0)$

$\alpha(1/3) = f \circ p(o) = f \circ g(y_0)$

$\alpha(2/3) = f \circ p(1) = f \circ g(y_1)$

$\alpha(2/3) = K(y_1, 0) = f \circ g(y_1)$

Since $K(y,t)$, $0 \leq t \leq 1$, and $f \circ p$ are continuous maps and α is continuous since $\alpha(o) = y_0$ and $\alpha(1) = y_1$, α is indeed a path in Y connecting y_0 with y_1, as required.

Hence Y is path connected and "being path connected" is a homotopy property and hence not a closed hereditary property and not a hereditary property.

1.17: Theorem: (23, page 562)

Let P be a closed hereditary topological property which holds on some space X and on X x I, where I is the closed unit interval. Let S be a subspace of X which does not have P. Then P is not an isotopy property.

Proof:

Let $Y = \{(x,t)/\ 0 \leq t \leq 1 \text{ or } x \in S\}$ be a subspace of X x I. Y does not possess P since $S \times \{1\} = \{(x,1)/\ x \in S\}$ is closed in Y and $S \times \{1\}$ is homeomorphic to S which does not have P. Y is isotopically equivalent to X x I since the maps

$i: Y \longrightarrow X \times I$ s.t. $i(x,t) = (x,t)$ and

$j: X \times I \longrightarrow Y$ s.t. $j(x,t) = (x,t/2)$

are imbeddings since i is the inclusion map of Y into X, $j(X \times I) = \{(x,t)/\ 0 \leq t \leq \frac{1}{2},\ x \in Y \subset X\}$ is the inclusion map of X x I into Y and inclusion maps are imbeddings. The maps $h_s: X \times I \longrightarrow X \times I$ s.t. $h_s(x,t) = (x, (1+s)t/2)$ and

$k_s: Y \longrightarrow Y$ s.t. $k_s(x,t) = (x, (1+s)t/2)$ are

isotopies since $h_0 = i \circ j$ and $h_1 = 1_{X \times I}$,

$k_0 = j \circ i$ and $k_1 = 1_Y$,

and i,j are imbeddings. Since X x I has P and Y does not have P, P is not an isotopy property.

Since the closed unit interval is T_4, normal, compact, Lindelöf, locally compact, countably compact and paracompact, the Tychonoff plank, T, has all of those properties and contains a subspace which does not have any of those properties, except for local compactness, and T x I has all of those properties, and since I x I is locally compact but the rationals are not, the above theorem can be used to conclude that none of those properties are isotopy properties.

Table II indicates which properties of Chapter 0, that can be derived from these theorems, are or are not hereditary, closed hereditary, open hereditary, retractive, projective, homotopy and isotopy properties. The "x" in each line indicates that the proof of invariance of that property is either given in this paper or referenced in the literature. The "+" indicates that the invariance is implied by the "x" in the same line. The "-" indicates that, as a result of Hu's and Gottlieb's theorems, non-invariance is implied by the "x" in the same line.

Table II

HEREDITARY, HOMOTOPY AND ISOTOPY PROPERTIES

Closed hereditary & not hereditary $\not\Rightarrow$ not isotopy
Hereditary $\Rightarrow$ closed hereditary $\Rightarrow$ not homotopy
Hereditary $\Rightarrow$ open hereditary
Hereditary $\Rightarrow$ retractive $\Rightarrow$ projective
Hereditary $\Rightarrow$ isotopy   Homotopy $\Rightarrow$ isotopy
Homotopy $\Rightarrow$ not closed hereditary $\not\Rightarrow$ not hereditary

	"x" indicates proof of invariance is either given in this paper or found in the literature. "+" indicates the invariance is implied "x" in the same line. "-" indicates the non-invariance is implied by "x" in the same line. Property	Hereditary	Closed hereditary	Open hereditary	Retractive	Projective	Homotopy	Isotopy	Reference to proof of "x". CN stands for completely normal.
1	T_0	x	+	+	+	+	-	+	4.1
2	T_1	x	+	+	+	+	-	+	44, p.79
3	T_2	x	+	+	+	+	-	+	27, p.55
4	$T_{2\frac{1}{2}}$	x	+	+	+	+	-	+	4.2
5	T_3	x	+	+	+	+	-	+	30, p.78
6	$T_{3\frac{1}{2}}$	x	+	+	+	+	-	+	30, p.78
7	T_4		x				-	-	30, p.78
8	T_5	x	+	+	+	+	-	+	24, p.75(CN=T_5)
9	Regular	x	+	+	+	+	-	+	30, p.78
10	Completely regular	x	+	+	+	+	-	+	30, p.78
11	Normal		x				-	-	30, p.78
12	Completely normal	x	+	+	+	+	-	+	4.3
13	Connected	-	-				x	+	44, p.229
14	Path connected	-	-				x	+	1.16
15	Locally connected			x					4.4
16	Totally disconnected	x	+	+	+	+	-	+	4.5
17	Compact		x				-	-	27, p.62
18	Lindelöf		x				-	-	30, p.78
19	Locally compact		x				-	-	27, p.66
20	Countably compact		x				-	-	30, p.158
21	Paracompact		x				-	-	56, p.148
22	Separable			x					41, p.258
23	Second countable	x	+	+	+	+	-	+	30, p.78
24	First countable	x	+	+	+	+	-	+	14, p.59
25	Discrete	x	+	+	+	+	-	+	4.6
26	Indiscrete	x	+	+	+	+	-	+	4.7
27	Metrizable	x	+	+	+	+	-	+	27, p.98
28	Fixed point								
29	Contractible	-	-				x	+	27, p.53
30	Locally contractible			x					4.8

CHAPTER 2

OPEN, ISOTOPY AND HOMOTOPY PROPERTIES

This chapter will present theorems by Gottlieb(23) which allow us to conclude that "being locally connected" and "being separable" are not isotopy properties and hence, not homotopy properties. Furthermore, "being locally contractible" is not a homotopy property. Also, connectedness, path connectedness and contractibility, which were found to be not closed hereditary properties, are not open hereditary properties either.

2.1: Definition:
Property P of topological spaces is a <u>continuous property</u> if and only if P is preserved by every onto map.

2.2: Definition:
Property P of topological spaces is an <u>open continuous property</u> if and only if P is preserved by every open, onto map.

2.3: Definition:
Property P of topological spaces is a <u>closed continuous property</u> if and only if P is preserved by every closed, onto map.

2.4: Definition:

A map $f: X \longrightarrow Y$ from a topological space X onto a topological space Y is a <u>quotient map</u> if and only if $O \subseteq Y$ is open in Y whenever $f^{-1}(O) \subset X$ is open in X.

2.5: Definition:

Property P of topological spaces is a <u>divisible property</u> if and only if P is preserved by every quotient map.

2.6: Definition:

Property P of topological spaces is a <u>contractive property</u> if and only if whenever T', T are topologies on a space X, $T' \subset T$ and (X,T) has P, then (X,T') also has P.

2.7: Definition:

Property P of topological spaces is an <u>open property</u> if and only if P is preserved by every open map and inherited by every open subspace of a space which has P.

2.8: Theorem:

Every retraction is a quotient map.

Proof:

Let $r: X \longrightarrow A$ be a retraction from a topological space X onto a retract A of X. Now r is continuous and onto. We need to show that a subset $O \subset A$ is open in A whenever $r^{-1}(O) \subset X$ is open in X. Assume that $r^{-1}(O) \subset X$ is open in X.

Define a relation $\sim$ on X s.t. $\forall$ $x,y \in X$, $x \sim y \Longleftrightarrow$ $r(x) = r(y)$. Clearly, $\sim$ is an equivalence relation. Let $p: X \longrightarrow X/\sim$ be the quotient map from X onto the topological space $X/\sim$, where $X/\sim$ is given the quotient topology by p defined by $p(x) = [x]$, where $[x] = \{y \in X / r(y) = r(x)\}$.

Then the map $q: X/\sim \longrightarrow A$, defined by $q([x]) = r(x)$ for $[x] \in X/\sim$, is well defined because $q \circ p(x) = r(x)$. Since r is onto, then q is onto. We shall show that q is 1-1. Assume that $r(a) = r(b)$ where $r(a), r(b) \in A$. Then by definition of q, $q([a]) = q([b])$.

$$\text{Therefore } [a] = \{y \in X/\ r(y) = r(a)\}$$
$$= \{y \in X/\ r(y) = r(b)\}$$
$$= [b]$$

Hence q is 1-1. Since q is 1-1 and onto, q is bijective and $q \circ q^{-1} = 1_A$. Now, since p is a quotient map, $r^{-1}(0) = p^{-1}(0') \subset X$ is open in X implies that $0' = q^{-1}(0) \subset X/\sim$ is open in $X/\sim$. Since q is bijective and continuous it is open, and $q(0') = q \circ q^{-1}(0) = 0 \subset A$ is open in A. Therefore r is a quotient map and every retraction is a quotient map.

2.9: Theorem:

If X is a non-empty set and T', T are topologies on X s.t. $T' \subset T$, then the function $1_X: (X,T) \longrightarrow (X,T')$ is a continuous function.

Proof:

Since $T' \subset T$, an open set $O \subset T'$ is open in T. Hence, $i_X^{-1}(O) \subset (X,T)$ is open in (X,T) whenever $O \subset (X,T')$ is open in (X,T'). Therefore i_X is a continuous function.

2.10: Theorem:

Every open, onto map is a quotient map.

Proof: (55 , page 103)

2.11: Theorem:

Every closed, onto map is a quotient map.

Proof: (55 , page 103)

Since projective maps are retractions and since retractions, open, onto maps and closed, onto maps are quotient maps, and quotient maps are continuous functions, and since, whenever a topological space is contractive, a continuous function can be defined from the space with topology T to the space with a coarser topology T', the following implications for any topological property, P, hold trivially:

Continuous $\Rightarrow$ divisible $\Rightarrow$ retractive $\Rightarrow$ projective

Continuous $\Rightarrow$ open continuous

Continuous $\Rightarrow$ closed continuous

Continuous $\Rightarrow$ contractive

Divisible $\Rightarrow$ open continuous and closed continuous

2.12: Theorem: (Gottlieb's Theorem (23, page 564))
Any open property P is not an isotopy property if there exists a space X and a subspace S such that X x I possesses P but S does not.

Proof:
Let S and X be chosen as in the hypothesis. Consider the subspace Y of X x I such that

$$Y = \{(x,t) /\ 0 \leq t \leq 3/4 \text{ or } x \in S\}.$$

Now S x (3/4, 1] is open relative to Y and would have P if Y had P. But S x (3/4 ,1] is also a product space and since S is the image of an open projection map from S x (3/4 ,1], we see that S would have P if S x (3/4 ,1] had P. Since S does not have P, S x (3/4 ,1] does not have P and hence, since P is open hereditary, Y does not possess P. The fact that Y is isotopically equivalent to X x I can be shown by taking the same maps i, j, h_s, and k_s as we take in theorem 1.17. Thus the theorem is proven.

2.13: Theorem: (23, page 565)
Let P be an open property such that X has P implies X x I has P. Then P is an isotopy property if and only if P is hereditary.

Proof:

If P is hereditary, it is an isotopy property by theorem 1.15. If P is not hereditary, there exists a space X enjoying P with a subspace S not possessing P. Since X x I has P, all the conditions of the preceding theorem are satisfied and so P is not an isotopy property.

2.14: Theorem:(23, page 564)

Open properties are not homotopy properties.

Proof:

Let P be an open property. The singleton space $\{v\}$ must have P, for there exists a space X with P and the constant map $c: X \longrightarrow \{v\}$ is open. Let S be a space without P. S x J, where J is the open unit interval, cannot have P since $S = \pi(S \times J)$, where π is the projection of S x J onto S. Since π is an open map, S without P implies that S x J does not possess P. Consider C(S), the cone over S. S x J is homeomorphic to an open subset of C(S), so C(S) does not possess P since P is open. Since C(S) is homotopically equivalent to $\{v\}$, P cannot be a homotopy property.

Gottlieb used the method of proof of the above theorem to arrive at a more general result.

2.15: Theorem:(23, page 564)

Let P be a property such that (i) the singleton space $\{v\}$

has P, (ii) P is an open hereditary property, and (iii) such that there exists a space S where S x J does not possess P, where J is the open unit interval. Then P is not a homotopy property.

Proof:

Let P be an open hereditary property such that the singleton space $\{v\}$ has P. Let S be a space where S x J does not possess P. S x J is homeomorphic to an open subset of C(S), the cone over S. Hence C(S) does not possess P. Since C(S) is homotopically equivalent to $\{v\}$, and $\{v\}$ has P, P is not a homotopy property.

Since X = Comb space (41, page 187) is not locally connected, it is not locally contractible. Then S x J is not locally contractible and the above theorem can be used to prove that "being locally contractible" is not a homotopy property.

A contrapositive statement to theorem 2.15 will be used in order to conclude that homotopy properties are not open hereditary properties and is stated as a corollary.

2.16: Theorem: (Corollary to theorem 2.15)

Let P be a homotopy property such that the singleton space $\{v\}$ has P and such that there exists a space S where S x J does not possess P. Then P is not an open hereditary property.

Remark:

The assumption that S x J does not possess P is, in fact, necessary, as the following example shows:

The singleton space $\{v\}$, the open unit interval J and the cone $C(\{v\}) = I$, the closed unit interval, are locally contractible. $\{v\} \times J$ is an open subspace of $C(\{v\})$ and $\{v\} \times J$ is locally contractible. "Being locally contractible" is an open hereditary property. Hence $C(\{v\})$ is locally contractible. $C(\{v\})$ and $\{v\}$ are homotopically equivalent . Therefore "being locally contractible" would be a homotopy property, which is not so.

Table III indicates which properties of Chapter 0 are continuous, divisible, retractive, projective, open continuous, closed continuous, and/or contractive properties. The "x" and/or "+" in a line indicate that the proof of invariance of that property is either given in this paper or referenced in the literature and a reference to the proof is given in the last column of the table. The "+" indicates that the invariance is implied by the "x" in the same line by one of the implications stated at the top of the table.

Table IV indicates that, by using theorem 2.12, two of the thirty properties defined in Chapter 0 are not isotopy properties and hence not homotopy properties. Also three of the properties are not open hereditary and hence not hereditary properties.The table also indicates that, by using theorem 2.16, three of the properties are not open properties. The table also indicates which of the properties are open continuous properties and which are open properties. The "x" and/or "*" in a line indicate that the proof of invariance of that property is either given in this paper or referenced in the literature and a reference to the proof is given in the last column of the table. The "+" and/or "@" in a line indicates that the invariance is implied by "x" in the same line by one of the implications stated at the top of the table. The "#" in a line indicates that the invariance is either implied by "x" and "*" in the same line or implied by "+" and "*" in the same line by one of the implications stated at the top of the table. (Note: "@" in a line does not imply anything whereas "+" in a line may help to imply "#" in the same line) The "-" in a line indicates that, as a result of theorems 2.12, 2.13, 2.14, 2.15 and 2.16, the non invariance is implied by "#" or "x".

Table III

CONTINUOUS PROPERTIES

Continuous ⇒ divisible ⇒ retractive ⇒ projective
Continuous ⇒ open continuous & closed continuous
Continuous ⇒ contractive
Divisible ⇒ open continuous & closed Continuous

"x" & "*" indicate that the proof of invariance is either given in this paper or found in the literature.

"+" indicates that the proof of invariance is implied by "x".

	Property	Continuous	Divisible	Retractive	Projective	Open continuous	Closed continuous	Contractive	Reference to proof of "x" and "*". T2 indicates that reference to proof is in Table II.
1	T_0			x	+				T2
2	T_1			x	+		*		x-T2 *-41,p194
3	T_2			x	+				T2
4	$T_{2 1/2}$			x	+				T2
5	T_3			x	+				T2
6	$T_{3 1/2}$			x	+				T2
7	T_4			x	+		*		x-4.9 *-41,p247
8	T_5			x	+		*		x-T2 *-24,p81
9	Regular			x	+				T2
10	Completely regular			x	+				T2
11	Normal			x	+		*		x-4.10 *-41,p247
12	Completely normal			x	+		*		x-T2 *-4.11
13	Connected	x	+	+	+	+	+	+	27,p78
14	Path connected	x	+	+	+	+	+	+	27,p85
15	Locally connected		x	+	+	+	+		56,p200
16	Totally disconnected			x	+				T2
17	Compact	x	+	+	+	+	+	+	27,p62
18	Lindelöf	x	+	+	+	+	+	+	24,p48
19	Locally compact			x	+	*			x-3,p19 *-56,p131
20	Countably compact	x	+	+	+	+	+	+	30,p158
21	Paracompact			x	+		*		x-4.12 *-15,p165
22	Separable	x	+	+	+	+	+	+	24,p48
23	Second countable			x	+	*			x-T2 *-15,p173
24	First countable			x	+	*			x-T2 *-41,p259
25	Discrete		x	+	+	+	+		31,p128
26	Indiscrete	x	+	+	+	+	+	+	4.13
27	Metrizable			x	+				T2
28	Fixed point			x	+				29,p20
29	Contractible			x	+				29,p25
30	Locally contractible			x	+				29,p26

Table IV

OPEN, ISOTOPY AND HOMOTOPY PROPERTIES

Open hereditary & open continuous ⇒ open
Open ⇒ not homotopy
Homotopy ⇒ isotopy
Homotopy ⇒ not open
Open hereditary ⇒ not homotopy
⇒ not open hereditary &/or not open continuous
Open and not hereditary ⇒ not isotopy ⇒ not homotopy

"x"&"*" indicate proof of invariance is either given in this paper or found in the literature. "+"&"@"indicate invariance is implied by "x". "#" indicates invariance is implied by "x"&"*" or "+"&"*". "-" indicates that non-invariance is implied by "#" or "x". Property		Hereditary	Open hereditary	Open continuous	Open	Homotopy	Isotopy	Reference to proof of "x" and "*". T2&T3 indicate that reference to proof is in Table II or Table III.
1	T_0	x	+			-	@	T2
2	T_1	x	+			-	@	T2
3	T_2	x	+			-	@	T2
4	$T_{2\frac{1}{2}}$	x	+			-	@	T2
5	T_3	x	+			-	@	T2
6	$T_{3\frac{1}{2}}$	x	+			-	@	T2
7	T_4							
8	T_5	x	+			-	@	T2
9	Regular	x	+			-	@	T2
10	Completely regular	x	+			-	@	T2
11	Normal							
12	Completely normal	x	+			-	@	T2
13	Connected	-	-	*	-	x	@	*-T3 x-T2
14	Path connected	-	-	*	-	x	@	*-T3 x-T2
15	Locally connected		x	*	#	-	-	x-T2 *-T3
16	Totally disconnected	x	+			-	@	T2
17	Compact			*				T3
18	Lindelöf			*				T3
19	Locally compact			*				T3
20	Countably compact			*				T3
21	Paracompact							
22	Separable		x	*	#	-	-	x-T2 *-T3
23	Second countable	x	+	*	#	-	@	x-T2 *-T3
24	First countable	x	+	*	#	-	@	x-T2 *-T3
25	Discrete	x	+	*	#	-	@	x-T2 *-T3
26	Indiscrete	x	+	*	#	-	@	x-T2 *-T3
27	Metrizable	x	+			-	@	T2
28	Fixed point							
29	Contractible	-	-			x	@	T2
30	Locally contractible		x			-		T2

CHAPTER 3

PRODUCTIVE AND EXPANSIVE PROPERTIES

In this chapter we will conclude that fourteen of the properties defined in Chapter 0 are arbitrarily productive properties, eighteen are countably productive properties and twenty-one are finitely productive properties. Also six of the properties are expansive properties; i. e. are preserved under refinements of the topological structure of a space.

3.1: Definition:
Property P of topological spaces is an arbitrarily productive property if and only if P is inherited by an uncountable product of spaces all of which have P.

3.2: Definition:
Property P of topological spaces is a countably productive property if and only if P is inherited by a countable product of spaces all of which have P.

3.3: Definition:
Property P of topological spaces is a finitely productive property if and only if P is inherited by a finite product of spaces all of which have P.

3.4: Definition:

Property P of topological spaces is a <u>expansive property</u> if and only if whenever T', T are topologies on a space X, $T' \subset T$ and (X, T') has P, then (X, T) also has P.

Since a finite product can be viewed as a special case of a countable product and a countable product can be viewed as a special case of an arbitrary product, the following implications for any topological property, P, hold trivially:

$$\text{Arbitrarily productive} \Rightarrow \text{countably productive} \Rightarrow \text{finitely productive}$$

Table V indicates which properties of Chapter 0 are arbitrarily productive, countably productive, finitely productive and/or expansive properties. The "x" and the "*" in a line indicate that the proof of invariance of that property is either given in this paper or referenced in the literature. The "+" in a line indicates that the invariance is implied by the "x" in the same line.

Table V

PRODUCTIVE AND EXPANSIVE PROPERTIES

Arbitrarily productive ⇒ countably productive
⇒ finitely productive

"x" & "*" indicate that the proof of invariance is either given in this paper or found in the literature.

"+" indicates that the proof of invariance is implied by "x".

	Property	Arbitrarily productive	Countably productive	Finitely productive	Expansive	Reference to proof of "x" and "*"
1	T_0	x	+	+	*	x-30,p86 *-49,p14
2	T_1	x	+	+	*	x-4.14 *-49,p14
3	T_2	x	+	+	*	x-27,p56 *-49,p14
4	$T_{2\frac{1}{2}}$	x	+	+	*	x-4.15 *-49,p14
5	T_3	x	+	+		30,p86
6	$T_{3\frac{1}{2}}$	x	+	+		30,p86
7	T_4					
8	T_5					
9	Regular	x	+	+		24,p80
10	Completely regular	x	+	+		30,p86
11	Normal					
12	Completely normal					
13	Connected	x	+	+		27,p79
14	Path connected	x	+	+		27,p85
15	Locally connected			x		41,p186
16	Totally disconnected	x	+	+	*	x-56,p210 *-4.19
17	Compact	x	+	+		27,p64
18	Lindelöf					
19	Locally compact			x		30,p86 or p165
20	Countably compact					
21	Paracompact					
22	Separable		x	+		41,p258
23	Second countable		x	+		30,p86
24	First countable		x	+		24,p46
25	Discrete			x	*	x-4.16 *-4.20
26	Indiscrete	x	+	+		4.17
27	Metrizable		x	+		27,p98
28	Fixed point					
29	Contractible	x	+	+		27,p50
30	Locally contractible			x		4.18

CHAPTER 4

APPENDIX ON SOME BASIC THEOREMS AND COUNTEREXAMPLES

For completeness, this chapter contains the proofs of a few basic facts for which we could find no explicit proof in the literature. A number of counterexamples are also given, in order to complete the four tables of counterexamples found at the end of the chapter.

4.1: Theorem:

Every subspace of a T_0-space is a T_0-space.

Proof:

Let (Y, T_Y) be a subspace of the T_0-space (X, T).

Let $a, b \in Y \subset X$ s.t. $a \neq b$. Since (X, T) is a T_0-space, $\exists$ an open set $U \in T$ s.t. either: $a \in U$, $b \notin U$ or: $a \notin U$, $b \in U$.

By the definition of subspace, $Y \cap U$ is a T_Y-open set.

Hence, either: $a \in Y,\ a \in U \Rightarrow a \in Y \cap U$

$b \in Y,\ b \notin U \Rightarrow b \notin Y \cap U$

or: $a \in Y,\ a \notin U \Rightarrow a \notin Y \cap U$

$b \in Y,\ b \in U \Rightarrow b \in Y \cap U$

Therefore (Y, T_Y) is a T_0-space

4.2: Theorem:

Every subspace of a $T_{2\frac{1}{2}}$-space is a $T_{2\frac{1}{2}}$-space.

Proof:

Let (Y, T_y) be a subspace of the $T_{2\frac{1}{2}}$-space (X, T).

Let $a, b \in Y \subset X$ s.t. $a \neq b$. Since (X, T) is a $T_{2\frac{1}{2}}$-space, $\exists$ open sets $U, V \in T$ s.t. $a \in U$, $b \in V$ and $\bar{U} \cap \bar{V} = \emptyset$.

By the definition of subspace, $Y \cap U$ and $Y \cap V$ are T_y-open sets. Hence, $a \in Y$, $a \in U \Longrightarrow a \in Y \cap U$

$b \in Y$, $b \in V \Longrightarrow b \in Y \cap V$

and $\overline{(Y \cap U)} \cap \overline{(Y \cap V)} \subset (\bar{Y} \cap \bar{U}) \cap (\bar{Y} \cap \bar{V}) = \bar{Y} \cap (\bar{U} \cap \bar{V})$

$= \bar{Y} \cap \emptyset$

$= \emptyset$

Thus, $\overline{(Y \cap U)} \cap \overline{(Y \cap V)} = \emptyset$.

Therefore (Y, T_y) is a $T_{2\frac{1}{2}}$-space.

4.3: Theorem:

Every subspace of a completely normal space is a completely normal space.

Proof:

Since a completely normal space is a T_5-space that is T_1, every subspace of a T_5-space is a T_5-space and every subspace of a T_1-space is a T_1-space, then every subspace of a completely normal space is a completely normal space.

4.4: Theorem:

Every open subspace of a locally connected space is a locally connected space.

Proof:

Let (X,T) be a locally connected space. Let (Y,T_y) be an open subspace of (X,T). Let $p \in Y$ be arbitrary. Since Y is a neighbourhood of p in X and (X,T) is a locally connected space, $\exists$ a neighbourhood V of p which is connected and $V \subset Y$. Therefore (Y,T_y) is a locally connected space.

4.5: Theorem:

Every subspace of a totally disconnected space is a totally disconnected space..

Proof:

Let (X,Y) be a totally disconnected space. Let (Y,T_y) be a subspace of (X,T). Let $a,b \in Y \subset X$ s.t. $a \neq b$.

Since (X,T) is a totally disconnected, $\exists$ a disconnection $U \cup V$ of X with $a \in U$ and $b \in V$.

By the definition of subspace, $Y \cap U$ and $Y \cap V$ are T_y-open sets.

Hence, $U \cup V = X \Rightarrow (Y \cap U) \cup (Y \cap V) = Y \cap (U \cup V)$.

$$= Y \cap X$$

$$= Y$$

$$U \cap V = \phi \Rightarrow (Y \cap U) \cap (Y \cap V) = Y \cap (U \cap V) = Y \cap \phi$$

$$= \phi$$

$$a \in Y,\ a \in U \Rightarrow a \in Y \cap U \Rightarrow Y \cap U \neq \phi$$

$$b \in Y,\ b \in V \Rightarrow b \in Y \cap V \Rightarrow Y \cap V \neq \phi\ .$$

Therefore $(Y \cap U) \cup (Y \cap V)$ is a disconnection of Y with $a \in Y \cap U$ and $b \in Y \cap V$. Therefore (Y,T_y) is a totally disconnected space.

4.6: Theorem:

Every subspace of a discrete space is a discrete space.

Proof:

Let (X,T) be a discrete space. Let (Y,T_y) be a subspace of (X,T). Since (X,T) is discrete, every subset of X is an open set and T = set of all subsets of X.

$= \{A: A \text{ is a subset of } X\}$.

By the definition of subspace, $Y \cap A$, $\forall A \in T$, is T_y-open.

Hence, $T_y = \{Y \cap A: Y \cap A \text{ is a subset of } Y\}$

= set of all subsets of Y.

Therefore (Y,T_y) is discrete.

4.7: Theorem:

Every subspace of an indiscrete space is an indiscrete space.

Proof:

Let (X,T) be an indiscrete space. Let (Y,T_y) be a subspace of (X,T). Since (X,T) is indiscrete, $T = \{\emptyset, X\}$.

By the definition of subspace, $Y \cap \emptyset$ and $Y \cap X$ are the only T_y-open sets. But $Y \cap \emptyset = \emptyset$ and $Y \cap X = Y$.

Hence $T_y = \{\emptyset, Y\}$. Therefore (Y,T_y) is indiscrete.

4.8: Theorem:

Every open subspace of a locally contractible space is a locally contractible space.

Proof:

Let (Y,T_Y) be an open subspace of the locally contractible space (X,T) and let $x \in Y$ be any point. Since Y is a neighbourhood of x in X and (X,T) is locally contractible, $\exists$ a neighbourhood U of x in X such that $U \subset Y$ and U is contractible in Y. Therefore (Y,T_Y) is locally contractible.

4.9: Theorem:

Every retract of a T_4-space is a T_4-space

(Note: In our definition of T_4-spaces, T_4-spaces are not assumed to be T_2.)

Proof:

Let X be a T_4-space. Let $r:X \longrightarrow A$ be a retraction from X onto the retract A of X. Let $M,N \subseteq A \subseteq X$ be disjoint, closed sets. Since X is a T_4-space, $\exists$ open sets $U,V \subset X$ s.t. $U \cap V = \emptyset$ and $M \subseteq U$, $N \subset V$. Since A is a retract of X, $A \cap U$ and $A \cap V$ are disjoint, open sets in A and
$M \subset A$, $M \subset U \Rightarrow M \subset A \cap U$, $\quad N \subset A$, $N \subset V \Rightarrow N \subset A \cap V$.
Hence A is a T_4-space.

4.10: Theorem:

"Being a normal space" is a retractive property.

Proof:

Since a normal space is T_4 and T_1 and since "being T_4" and "being T_1" are retractive properties, then "being a normal space" is a retractive property.

4.11: Theorem:

"Being a completely normal space" is a closed continuous property.

Proof:

Since a completely normal space is T_5 and T_1 and since "being T_5" and "being T_1" are closed continuous properties, then "being a completely normal space" is a closed continuous property.

4.12: Theorem:

Every retract of a paracompact space is a paracompact space.

Proof:

Since a paracompact space is a T_2-space and "being paracompact" is a closed hereditary property and since every retract of a T_2-space is closed (29, page 18), then every retract of a paracompact space is a paracompact space.

4.13: Theorem:

"Being an indiscrete space" is a continuous property.

Proof:

Clearly, a function from an indiscrete space X onto a topological space Y can only be continuous if Y is also indiscrete, therefore "being an indiscrete space" is a continuous property.

4.14 Theorem:

The arbitrary product of T_1-spaces is a T_1-space.

Proof:

Let $\{X_i : i \in I\}$ be a collection of T_1-spaces.

Let $X = \prod X_i$ be the product space.

Let $p = \langle a_i : i \in I \rangle$ and $q = \langle b_i : i \in I \rangle$ be distinct points in X. Then p and q must differ in at least one coordinate space, say X_{j_0}, i.e. $a_{j_0} \neq b_{j_0}$.

By hypothesis, X_{j_0} is a T_1-space, hence there exist open subsets U and V of X_{j_0} such that $a_{j_0} \in U$, $b_{j_0} \in V$ and $a_{j_0} \notin V$, $b_{j_0} \notin U$.

By the definition of the product space, the projection $\pi_{j_0} : X \longrightarrow X_{j_0}$ is continuous.

Accordingly, $\pi_{j_0}^{-1}[U]$ and $\pi_{j_0}^{-1}[V]$ are open sets of X and $p \in \pi_{j_0}^{-1}[U]$, $q \in \pi_{j_0}^{-1}[V]$ and $p \notin \pi_{j_0}^{-1}[V]$, $q \notin \pi_{j_0}^{-1}[U]$.

Hence X is also a T_1-space.

4.15: Theorem:

The arbitrary product of $T_{2\frac{1}{2}}$-spaces is a $T_{2\frac{1}{2}}$-space.

Proof:

Let $\{X_i : i \in I\}$ be a collection of $T_{2\frac{1}{2}}$-spaces.

Let $X = \prod_i X_i$ be the product space.

Let $p = \langle a_i : i \in I \rangle$ and $q = \langle b_i : i \in I \rangle$ be distinct points in X.

Then p and q must differ in at least one coordinate space, say X_{j_0}, i.e. $a_{j_0} \neq b_{j_0}$.

By hypothesis, X_{j_0} is a $T_{2\frac{1}{2}}$-space, hence there exist disjoint open sets U and V in X_{j_0} such that $a_{j_0} \in U$, $b_{j_0} \in V$ and $\bar{U} \cap \bar{V} = \emptyset$.

By the definition of the product space, the projection $\pi_{j_0} : X \longrightarrow X_{j_0}$ is continuous.

Accordingly, $\pi_{j_0}^{-1}[U]$ and $\pi_{j_0}^{-1}[V]$ are disjoint open sets in X such that $p \in \pi_{j_0}^{-1}[U]$, $q \in \pi_{j_0}^{-1}[V]$ and

$$\begin{aligned} \overline{\pi_{j_0}^{-1}[U]} \cap \overline{\pi_{j_0}^{-1}[V]} &\subset \pi_{j_0}^{-1}[\bar{U}] \cap \pi_{j_0}^{-1}[\bar{V}] \\ &= \pi_{j_0}^{-1}[\bar{U} \cap \bar{V}] \\ &= \pi_{j_0}^{-1}[\emptyset] \\ &= \emptyset \end{aligned}$$

Hence $\overline{\pi_{j_0}^{-1}[U]} \cap \overline{\pi_{j_0}^{-1}[V]} = \emptyset$.

Therefore X is also a $T_{2\frac{1}{2}}$-space.

4.16: Theorem:

The finite product of discrete spaces is a discrete space.

Proof:

Without loss of generality, let X and Y be discrete spaces.

Let $Z = X \times Y = \{(x,y) / x \in X, y \in Y\}$ be the product space.

By the definition of product space, the projections $p: Z \longrightarrow X$ and $q: Z \longrightarrow Y$ are continuous. Let $\{x_i\}$ and $\{y_j\}$ be any singleton open sets in X and Y respectively. Then $p^{-1}(\{x_i\}) = \{(x_i, y_0) \ldots (x_i, y_j) \ldots\}$ and $q^{-1}(\{y_j\}) = \{(x_0, y_j) \ldots (x_i, y_j) \ldots\}$ are open sets in Z and $p^{-1}(\{x_i\}) \cap q^{-1}(\{y_j\}) = \{(x_i, y_j)\}$ is any open singleton set in Z. Hence Z is a discrete space.

4.17: Theorem:

The arbitrary product of indiscrete spaces is an indiscrete space.

Proof:

Let $\{X_i : i \in I\}$ be a collection of indiscrete spaces.

Let $X = \prod_i X_i$ be the product space.

By the definition of product space, the projection $\pi_i : X \longrightarrow X_i$ is an open map.

Hence $\emptyset = \pi_i^{-1}(\emptyset)$ and $X = \pi_i^{-1}(X_i)$ are open sets of X.

$U \subset X$ is open in X implies that $\pi_i(U) \subset X_i$ is open in X_i and $\pi_i(U) = \emptyset$ or X_i.

Therefore U must be $\emptyset$ or X and X is an indiscrete space.

4.18: Theorem:

"Being locally contractible" is a finitely productive property.

Proof: (Based on proof of theorem 10.35 in 41, page 186)
Let $\{X_i\}$, $i = 1,2,\ldots,n$, be a finite family of locally contractible topological spaces. Let $\mathcal{B}_i$ be a contractible basis of X_i.
Then $\{B_1 \times B_2 \times \ldots \times B_n / B_i \in \mathcal{B}_i, i \quad N\}$ is a contractible basis of the product space.

4.19: Theorem:
Total disconnectedness is an expansive property.
Proof:
Let X be a non-empty set and let T', T be topologies on X such that $T' \subset T$ and (X,T') is totally disconnected. Since the identity map $1_X: (X,T') \longrightarrow (X,T)$ is a closed map, then any set, A, that is closed in T' is also closed in T. Also if $A \in T'$ is closed in T', then $X - A$ is open in T' and also in T since 1_X is also an open map. Hence, (X,T) is a totally disconnected space.

4.20: Theorem:
"Being discrete" is an expansive property.

Proof:
Let X be a set and T', T be topologies on X such that $T' \subset T$ and (X, T') is discrete. Then $T' = T$ and hence (X, T) is also discrete. Thus "being discrete" is an expansive property.

4.21: Counterexample:

The fixed point property is not closed hereditary.

Let $X = [0,1]$ and $Y = \{0\} \cup \{1\}$.
Y with the discrete subspace topology is a closed subspace of X. It is well known by the Brouwer fixed point theorem that X has the fixed point property. Y does not have the fixed point property, since the continuous function $f: Y \longrightarrow Y$ s.t. $f(0) = 1$ and $f(1) = 0$ is such that for any $y \in Y$, $f(y) \neq y$.

4.22: Counterexample:

The fixed point property is not open hereditary.

Let $X = [0,1]$ and $Y = (0,1)$.
Y is an open subspace of X. X has the fixed point property. Y does not have the fixed point property, since the continuous function $f: Y \longrightarrow Y$ s.t. $f(y) = y/2$ is such that for any $y \in Y$, $f(y) \neq y$.

4.23: Counterexample:

"Being locally contractible" is not closed hereditary.

Let $X = [0,1]$ and Y = the Cantor set(49, page 57).
X is locally contractible (26, page 191).
Y is a closed subspace of X and Y is not locally connected since it is totally separated. Hence Y is not locally contractible.

4.24: Counterexample:

Completely normal, T_5 and $T_{2\frac{1}{2}}$ are not open continuous topological properties.

Let X be the union of the lines y = 0 and y = 1 in R^2 with the usual topology. Let Y be the quotient space of X obtained by identifying each point (x,0), for $x \neq 0$, with the corresponding point (x,1). The resulting projection map $p:X \longrightarrow Y$ is continuous and open, but p(0,0) and p(0,1) are distinct points of Y which do not have disjoint neighbourhoods. Hence Y is not T_2 and thus not $T_{2\frac{1}{2}}$, T_5 and completely normal but X possesses all of those properties. (56, page 88; example 13.9(b))

4.25: Counterexample:

Total disconnectedness is not an open continuous topological property.

The rationals, Q, with the usual topology is a totally disconnected topological space (41, page 177). The set $\{0,1\}$ with the indiscrete topology is not a totally disconnected topological space. Let $a/b \in Q$ be in lowest terms where $b \neq 0$ and define a/b to be odd if a is an odd integer and even if a is an even integer. Let $f: Q \longrightarrow \{0,1\}$ be defined such that

$$f(a/b) = \begin{cases} 0, & \text{if } a/b \text{ is even} \\ 1, & \text{if } a/b \text{ is odd} \end{cases}$$

Then f is continuous since $\{0,1\}$ is indiscrete and f is open since the image of any open set in Q has to be the set $\{0,1\}$ or $\emptyset$.

4.26: Counterexample:

$T_{2\frac{1}{2}}$ is not a closed continuous topological property.

(56, page 87; example 13.9(a)) is an example of a closed continuous image of a $T_{2\frac{1}{2}}$-space which is not a $T_{2\frac{1}{2}}$-space.

4.27: Counterexample:

The fixed point property and "being contractible" are not closed continuous properties and hence not divisible properties.

Let X = [0,1] and Y = S^1.

It is well known that [0,1] is contractible and has the fixed point property and S^1 is not contractible and does not have the fixed point property. The map $f: X \longrightarrow Y$ s.t. $f(x) = (\cos 2\pi x, \sin 2\pi x)$ is a closed map (37, page 101).

4.28: Counterexample:

If property P is a T_0, T_1, T_2, $T_{2\frac{1}{2}}$, regular, completely regular, normal, completely normal, totally disconnected, discrete or metrizable topological property, then P is not a contractive property.

Let X be a finite set with at least two points and T^1, T be the indiscrete and discrete topologies on X respectively. $T' \subset T$. Then, by (49, pages 170-171), (X,T) has all of the above properties and (X,T') has none of those.

4.29: Counterexample:

If property P is a T_3, $T_{3\frac{1}{2}}$, T_4, T_5 or paracompact topological property, then P is not a contractive property.

Let X be an uncountable set . Let T be the discrete topology on X. Let T' be the countable complement topology on X. $T' \subset T$. Then, by (49, pages 170-171), (X,T) has all of the above properties and (X,T') has none of those.

4.30: Counterexample:

The fixed point property is not a contractive or expansive topological property.

[0,1] with the usual topology has the fixed point property but if it is given the discrete or the indiscrete topology it does not have the fixed point property for let $f:[0,1] \longrightarrow [0,1]$ be the function such that

$$f(x) = \begin{cases} x+\frac{1}{2}, & 0 \leq x \leq \frac{1}{2} \\ x-\frac{1}{2}, & \frac{1}{2} < x < 1 \\ 0, & x = 1 \end{cases}$$

Then f is continuous if the domain is discrete or the range is indiscrete and $f(x) \neq x$ for any $x \in [0,1]$.

4.31: Counterexample:

"Being locally contractible" is not a contractive topological property.

The set of rationals, Q, with the usual topology is not locally connected and hence, not locally contractible. Q with the discrete topology is locally contractible.

4.32: Counterexample:

The fixed point property is not an isotopy property and hence not a homotopy property.

[0,1] is isotopic to (0,1). (28, page 191; proposition 3.3)
[0,1] has the fixed point property and (0,1) does not.

4.33: Counterexample:

"Being locally contractible" is not an isotopy property.

(22, page 44)
Let X be the set of all points in the plane composing the square $\{(x,y) \ / \ 0 \leq x \leq 1, \ 0 \leq y \leq 1\}$.
Let $Y = X \cup \{(x,y) \ / \ 1 \leq y \leq 2$, x is rational and $0 \leq x \leq 1\}$.
X is locally contractible and Y is not locally contractible as any picture will show. X is isotopically equivalent to Y: define

$$f: X \longrightarrow Y: (x,y) \longrightarrow (x,y)$$
$$g: Y \longrightarrow X: (x,y) \longrightarrow (x,y/2)$$
$$h_t: X \longrightarrow X: (x,y) \longrightarrow (x,y/(2-t))$$

$k_t: Y \longrightarrow Y: (x,y) \longrightarrow (x, y/(2-t))$

$H: X \times I \longrightarrow X: (x,y,t) \longrightarrow h_t(x,y) = (x, y/(2-t))$

$K: Y \times I \longrightarrow Y: (x,y,t) \longrightarrow k_t(x,y) = (x, y/(2-t))$

Now all functions defined above are continuous since $P: R^{2'} \times I \longrightarrow R^2 \times I: (x,y,t) \longrightarrow (x, y/(2-t))$ is continuous since $\pi_1 \circ P$ and $\pi_2 \circ P$ are both continuous where π_1 and π_2 are the projections of $R^2 \longrightarrow R$. Then f, g, h_t, and k_t are imbeddings since they are clearly 1-1 and open on their images. By inspection $f \circ g = k_0$ and $1_Y = k_1$, $g \circ f = h_0$ and $1_X = h_1$, hence X and Y are homotopically equivalent.

4.34: Counterexample:

"Being locally contractible" is not a countably productive topological property.

The Cantor set is not locally contractible since it is not locally connected and it is the countable product of locally contractible spaces. ($\{0,2\}^N$, where $\{0,2\}$ is given the discrete topology, is homeomorphic to the Cantor set)

4.35: Counterexample:

"Being contractible" is not an expansive topological property.

The set, R, of reals, with the usual topology, is contractible but with the discrete topology is not connected hence, not contractible.

4.36: Counterexample:
"Being locally contractible" is not an expansive topological property.

The set, R, of reals, with the usual topology, is locally contractible but with the indiscrete rational(irrational) extension of R, which is finer than the usual topology on R, is not locally connected and hence is not locally contractible.

Table VI indicates the properties which are not hereditary and not closed hereditary and /or not open hereditary properties. The "=" in a line indicates that a counterexample is either given in this paper or referenced in the literature. The "-" in a line indicates the non-invariance is implied by the "=" in the same line, due to the contrapositive statements of the following:

Hereditary $\Rightarrow$ closed hereditary & open hereditary.

Table VII indicates the properties which are not continuous, not divisible and not open continuous and/or not closed continuous properties. The "=" in a line indicates that a counterexample is either given in this paper or referenced in the literature. The "-" in a line indicates the non-invariance is implied by the "=" in the same line due to the contrapositive statements of the following:

Continuous $\Rightarrow$ divisible

$\Rightarrow$ open continuous & closed continuous.

Table VIII indicates the properties which are not continuous and not contractive properties as well as those which are not isotopy and not homotopy properties. The "=" in a line indicates that a counterexample is either given in this paper or referenced in the literature. The "-" in a line indicates the non-invariance is implied by the "=" in the same line, due to the contrapositive statements of the following:

Continuous $\Rightarrow$ contractive.

Homotopy $\Rightarrow$ isotopy

Table IX indicates the properties which are not arbitrarily productive, not countably productive and/or not finitely productive properties as well as those which are not expansive properties. The "#" and/or "=" in a line indicate that a counterexample is either given in this paper or referenced in the literature. The "-" in a line indicates that the non-invariance is implied by the "=" in the same line due, the contrapositive statement of the following: Arbitrarily productive $\Rightarrow$ countably productive $\Rightarrow$ finitely productive.

Table VI

COUNTEREXAMPLES FOR NON-HEREDITARY PROPERTIES

Not closed hereditary $\Rightarrow$ not hereditary
Not open hereditary $\Rightarrow$ not hereditary

"=" indicates that a counterexample is either given in this paper or found in the literature.

"-" indicates that the non-invariance of the property is implied by "=".

N stands for normal.

CH stands for closed hereditary.

OH stands for open hereditary.

	Property	Hereditary	Closed hereditary	Open hereditary	Reference to "=".
1	T_0				
2	T_1				
3	T_2				
4	$T_{2\frac{1}{2}}$				
5	T_3				
6	$T_{3\frac{1}{2}}$				
7	T_4	-		=	41,p253 (T_4 is N)
8	T_5				
9	Regular				
10	Completely regular				
11	Normal	-		=	41,p253 (N is T_4)
12	Completely normal				
13	Connected				
14	Path connected				
15	Locally connected	-	=		41,p185
16	Totally disconnected				
17	Compact	-		=	41,p271-273
18	Lindelöf	-		=	41,p271-273
19	Locally compact	-		=	41,p280
20	Countably compact	-		=	41,p271-273
21	Paracompact	-		=	15,p162
22	Separable	-	=		30,p50
23	Second countable				
24	First countable				
25	Discrete				
26	Indiscrete				
27	Metrizable				
28	Fixed point	-	=	=	CH-4.21 OH-4.22
29	Contractible				
30	Locally contractible	-	=		4.23

Table VII

COUNTEREXAMPLES FOR NON-CONTINUOUS PROPERTIES

Not closed continuous ⇒ not divisible ⇒ not continuous
Not open continuous ⇒ not divisible ⇒ not continuous

"=" indicates that a counterexample is either given in this paper or found in the literature.

"-" indicates that the non-invariance of the property is implied by "=".

Reference to "=".

OC stands for open continuous.

CC stands for closed continuous.

	Property	Continuous	Divisible	Open continuous	Closed continuous	Reference to "="
1	T_0	-	-	=	=	OC & CC-41,p198
2	T_1	-	-	=		41,p198
3	T_2	-	-	=	=	OC-41,p198 CC-41,p254
4	$T_{2\frac{1}{2}}$	-	-	=	=	OC-4.24 CC-4.26
5	T_3	-	-	=	=	OC-41,p208 CC-41,p254
6	$T_{3\frac{1}{2}}$	-	-	=	=	OC-41,p243 CC-41,p254
7	T_4	-	-	=		41,p254
8	T_5	-	-	=		4.24
9	Regular	-	-	=	=	OC-41,p208 CC-41,p254
10	Completely regular	-	-	=	=	OC-41,p243 CC-41,p254
11	Normal	-	-	=		41,p254
12	Completely normal	-	-	=		4.24
13	Connected					
14	Path connected					
15	Locally connected					
16	Totally disconnected	-	-	=		4.25
17	Compact					
18	Lindelöf					
19	Locally compact	-	-		=	56,p133
20	Countably compact					
21	Paracompact	-	-	=		56,p149
22	Separable					
23	Second countable	-	-		=	34,p104
24	First countable	-	-		=	34,p104
25	Discrete					
26	Indiscrete					
27	Metrizable	-	-	=	=	OC-50,p697 CC-50,p691
28	Fixed point	-	-		=	CC-4.27
29	Contractible	-	-		=	CC-4.27
30	Locally contractible					

Table VIII

COUNTEREXAMPLES FOR NON-CONTRACTIVE AND NON-HOMOTOPY

PROPERTIES

Not contractive ⇏ not continuous
Not isotopy ⇏ not homotopy

"=" & "#" indicate that a counterexample is either given in this paper or found in the literature.

"-"&"*" indicate that the non-invariance of the property is implied by "="&"#" respectively.

	Property	Continuous	Contractive	Homotopy	Isotopy	Reference to "=" and "#".	
1	T_0	-	=			4.28	
2	T_1	-	=			4.28	
3	T_2	-	=			4.28	
4	$T_{2\frac{1}{2}}$	-	=			4.28	
5	T_3	-	=			4.29	
6	$T_{3\frac{1}{2}}$	-	=			4.29	
7	T_4	-	=			4.29	
8	T_5	-	=			4.29	
9	Regular	-	=			4.28	
10	Completely regular	-	=			4.28	
11	Normal	-	=			4.28	
12	Completely normal	-	=			4.28	
13	Connected						
14	Path connected						
15	Locally connected	-	=			41,p185	
16	Totally disconnected	-	=			4.28	
17	Compact						
18	Lindelöf						
19	Locally compact	-	=			41,p280	
20	Countably compact						
21	Paracompact	-	=			4.29	
22	Separable						
23	Second countable	-	=			41,p273	
24	First countable	-	=			41,p273	
25	Discrete	-	=			4.28	
26	Indiscrete						
27	Metrizable	-	=			4.28	
28	Fixed point	-	=	*	#	=-4.30	#-4.32
29	Contractible						
30	Locally contractible	-	=	*	#	=-4.31	#-4.33

Table IX

COUNTEREXAMPLES FOR NON-PRODUCTIVE AND NON-EXPANSIVE PROPERTIES

Not finitely productive ⇒ not countably productive
⇒ not arbitrarily productive

"#", and "=" indicate that a counterexample is either given in this paper or found in the literature.

"-" indicates that the non-invariance of the property is implied by "=".

	Property	Arbitrarily productive	Countably productive	Finitely productive	Expansive	Reference to "#" and "="
1	T_0					
2	T_1					
3	T_2					
4	$T_{2\frac{1}{2}}$					
5	T_3				#	49,p88-89
6	$T_{3\frac{1}{2}}$				#	49,p88-89
7	T_4	-	-	=	#	=-41,p254 #-49,p88-89
8	T_5	-	-	=	#	=-24,p80 #-49,p88-89
9	Regular				#	49,p88-89
10	Completely regular				#	49,p88-89
11	Normal	-	-	=	#	=-41,p254 #-49,p88-89
12	Completely normal	-	-	=	#	=-24,p80 #-49,p88-89
13	Connected				#	49,p41-43
14	Path connected				#	49,p41-43
15	Locally connected	-	=		#	=-41,p186 #-41,p185
16	Totally disconnected					
17	Compact				#	41,p271-273
18	Lindelöf	-	-	=	#	= & #-41,p271-273
19	Locally compact	-	=		#	=-49,p121 #-41,p280
20	Countably compact	-	-	=	#	=-30,p192 #-41,p271-3
21	Paracompact	-	-	=	#	=-24,p69 #-49,p88-89
22	Separable	=			#	=-49,p123 #-41,p271-3
23	Second countable	=			#	=-49,p123 #-41,p271-3
24	First countable	=			#	=-49,p123 #-41,p271-3
25	Discrete	-	=			49,p121
26	Indiscrete				#	49,p41-42
27	Metrizable	=			#	=-30,p100 #-49,p88-89
28	Fixed point	-	-	=	#	=-12,p977 #-4.30
29	Contractible				#	#-4.35
30	Locally contractible	-	=		#	=-4.34 #-4.36

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