ON THE GROUPS OF AUTOMORPHISMS OF PRINCIPAL AND FIBRE BUNDLES

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by

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ABSTRACT

If $p: X \to B$ is a principal G-bundle, an automorphism of p is an equivariant map of X to itself over B. The set $\mathfrak{G}(p)$ of all such automorphisms inherits, in a natural way, a topological group structure. Similarly we can define, for a fibre bundle $p^F: X \times_G F \to B$, the group $\mathfrak{F}(p^F)$ of automorphisms of p^F and, under suitable conditions, this is also a topological group. The purpose of this thesis is to obtain information on the homotopy properties of $\mathfrak{G}(p)$ and $\mathfrak{F}(p^F)$. This is accomplished by using known relations between two bundles in order to determine corresponding relations between their groups of automorphisms.

Having shown that $\mathfrak{F}(p^F)$ is, algebraically, a quotient of $\mathfrak{G}(p)$ classified by the subgroup of G which acts trivially on F, we prove that such classification is often also topological. Moreover if $h: G \to K$ is a topological group morphism and p^h is the bundle induced from p by h, there is a homomorphism $\Pi^h_p: \mathfrak{G}(p) \to \mathfrak{G}(p^h)$, with image $\mathfrak{F}(p^h)$, which is a fibration if h is [illegible], or an n-equivalence if h has similar properties. This generates information on the fibre bundle problem and also on

the effect of an enlargement of the structure group of p on $\mathcal{G}(p)$. Several computations are given, especially when the structure group is a classical group.

The already known relation between $\mathcal{G}(p)$ and the space $\Omega Map(B,B_G;k)$, where k is a classifying map for p, is then interpreted as a natural transformation connecting Π_p^h and the map induced by h, in the obvious way, between the corresponding loop spaces. We also outline a theory analogous to that of the main body of the thesis; in it a change of the base space replaces the change of the structure group or fibre.

Finally, we give a non-standard construction of fibre bundles and associated principal bundles which leads to a simple proof of the equivalence between the categories of principal G-bundles over a space B and of fibre bundles over B with fibre an effective G-space F.

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LIST OF ABBREVIATIONS

BG	Classifying space for G (th. 4.1).
C	Set of complex numbers.
C(H,G)	Normal core of H in G (def. 2.4).
e	Unit of a given topological group.
$\mathcal{F}_B^F(G)$	Category of fibre bundles with fibre F, base space B and structure group G (prop. 1.30).
$\mathfrak{F}(p^F)$, $\mathfrak{F}(p^h)$	Group of fibre bundle automorphisms (ch. 2 sec. 4)
GL(n,C)	General linear group over the complex.
GL(n,R)	General linear group over the reals.
$\mathcal{G}(p)$, $\mathcal{G}(p^h)$	Group of principal bundle automorphisms (ch. 2 sec. 1).
$\mathcal{G}(p;H)$	Subgroup of $\mathcal{G}(p)$ generated by H (ch. 2 sec. 2).
HLP	Homotopy lifting property.
I	Closed unit interval [0,1].
Map(X,Y)	Space of maps from X to Y (ch. 0).
$Map_*(X,Y)$	Space of based maps from X to Y.
$M_G(X,Y)$	Space of G-maps from X to Y (def.1.6).

$M_\lambda(X,X)$	Space of action maps of X (1.3).
N	Set of natural numbers.
O	Infinite orthogonal group.
O(n)	Orthogonal group.
P	Category of principal bundles (1.19).
P^G	Category of principal G-bundles (1.19)
P_B	Category of principal bundles over B (1.19).
P_B^G	Category of principal G-bundles over B (1.19).
PB	Space of paths on B.
p^F	Fibre bundle associated to p with fibre F (1.26).
p^h	Principal bundle induced from p by h (def. 1.31).
$p_G\colon E_G \to B_G$	Milnor's universal G-bundle (th. 4.1)
Q_F	Quotient group of G acting effectively on F (1.4).
R	Set of real numbers.
sec(p)	Space of sections of a map p (ch.0)
S^i	Sphere of dimension i, i≥0.
SO	Infinite special orthogonal group.
SO(n)	Special orthogonal group.
SU	Infinite special unitary group.
SU(n)	Special unitary group.

U	Infinite unitary group.
$U(n)$	Unitary group.
$\{(U,\phi_U)\}$	Locally trivial structure for a map p (def. 1.15).
$\{(U,\chi_U)\}$	Locally trivial structure for a functional bundle (1.49).
$\{(U_i^G,\phi_i^G)\}$	Locally trivial structure for Milnor's universal G-bundle (4.2 and 4.3).
X/G	Quotient of X by G.
$(X\ Y)$, $(X\ Y)_G$	
$(X\ X)_\lambda$, $(X\ X)_{G;H}$	Functional spaces (1.45, 1.51, 1.56 and 2.8).
$X \sqcap Y$	Pullback space (ch. 0).
Z	Set of integer numbers.
Γ_p^F, Γ_p^h	Maps of fibre bundle automorphisms (2.19).
$\lambda\colon G\times F \to F$	Left action of G on F (ch.1 sec.1).
$\lambda^*\colon G \to M_\lambda(F,F)$	Map associated to the left action of G on F (1.3).
$\pi_i(X)$	i-th homotopy group of X.
Π_p^h	Map of principal bundle automorphisms (2.28).
$\rho\colon X\times G \to X$	Right action of G on X (ch.1 sec.1).
Ψ^F, Ψ_p^F	Maps induced by the fibre bundle construction (1.54 and 2.15).

Ψ^h , Ψ^h_p	Maps induced by a topological group morphism (1.31, 1.54 and 2.15).
ΩB	Space of loops on B.

I N T R O D U C T I O N

If $p: X \to B$ is a principal G-bundle, then an automorphism of p is an equivariant map of X onto itself over B. The set of all such automorphisms can be given the structure of a topological group and will be denoted by $\mathcal{G}(p)$. Similarly if $p^F: X \times_G F \to B$ is the fibre bundle associated to p and with fibre a G-space F, an automorphism of p^F is a map of the form $f \times_G 1 : X \times_G F \to X \times_G F$, where f is an automorphism of p. The space of all automorphisms of p^F is denoted by $\mathfrak{J}(p^F)$. It has a group structure and, under suitable conditions, it is a topological group.

In recent years several authors have obtained results concerning the homotopy properties of $\mathcal{G}(p)$ and $\mathfrak{J}(p^F)$, the interest in such matters being increased by the occurrence of the group $\mathcal{G}(p)$ within the context of the "gauge theories" of theoretical physics. Even without this, however, the topological properties involved are sufficiently interesting to justify a detailed study and this thesis examines only these mathematical aspects.

What seems to be the first work on the subject was carried out by I.M. James ([J] page 47) who, identifying $\mathcal{G}(p)$ with a certain subspace of the mapping space Map(X,G), showed, for example, that if G is abelian then $\mathcal{G}(p)$ is homeomorphic to Map(B,G).

D.H. Gottlieb used a different approach (see, e.g., [Go1], [Go2]), showing that when p is numerable, $\mathcal{G}(p)$ has the same weak homotopy type as the space of loops on the mapping space $Map(B,B_G;k)$, where $k: B \to B_G$ is a classifying map for p. He also looked at the space $\mathcal{F}(p^F)$ ([Go2] prop. 6.1), but only in the case in which the fibre F is effective.

New proofs of modified forms of Gottlieb's result, obtained within the context of a general theory of fibrations, have been obtained by C.Morgan ([Mo] chapter 3) and P.Booth, P.Heath, C.Morgan and R.Piccinini ([BHMP1] and [BHMP2]) together with some computations of $\pi_i(\mathcal{G}(p))$ in particular cases.

Other work has also been done for the cases where p carries a differentiable structure (see, e.g., [CM]).

One feature of the methods used by these authors is that they provide information about $\mathcal{G}(p)$ on the basis of data concerning only the bundle p. A distinguishing element of the approach used in this thesis is that we start off with known relations between a pair of bundles, p and q; for instance q might be obtained from p by an enlargement of the structure group, or, alternatively, q is a fibre bundle and p is the principal bundle to which it is associated. We then construct homomorphisms of topological groups between the corresponding groups of automorphisms and may, in suitable circumstances, use them to transform data about $\mathcal{G}(p)$ into data about $\mathcal{G}(q)$, or $\mathcal{F}(q)$, and vice versa. For example,we are able to use the computations of $\pi_i(\mathcal{G}(p))$ given in [Mo] in the case where p is a principal U-bundle over a sphere to compute many homotopy groups of $\mathcal{G}(q)$ in the case where the base space of q is a sphere and the structure group one of U(n), SU(n), SU or GL(n,C).

A basic technical tool used is a homeomorphism that exists between $\mathcal{G}(p)$ and the space of sections of a certain functional bundle $(p\ p)_G$ associated to the principal G-bundle p. This homeomorphism, and its analogue for $\mathcal{F}(p^F)$, reduce the study of the topology of $\mathcal{G}(p)$ and of $\mathcal{F}(p^F)$ to that of a more familiar type of space and allows us to view the homomorphisms between automorphisms groups, that we have referred to, as being induced by certain maps between the total spaces of appropriate functional bundles. Now these inducing maps are frequently locally trivial and this facilitates various proofs. Another important point is the fact that the relevant homomorphisms are natural with respect to the weak homotopy equivalences of Gottlieb, thus they correspond, in a strong sense, to induced maps between loop spaces on the corresponding mapping spaces.

The thesis is structured as follows. After a brief description of the convenient category of topological spaces used, chapter 1) contains an exposition of the definitions and properties of principal and fibre bundles that are needed later. Also we introduce the functional bundle construction used and derive a few of its properties.

In the first part of chapter 2) $\mathfrak{G}(p)$ and $\mathfrak{F}(p^F)$ are defined and some of their features analyzed. Certain normal subgroups of $\mathfrak{G}(p)$, determined by subgroups of G, are described: they will be seen to play a basic role later. Also the homeomorphism between $\mathfrak{G}(p)$ and the space of sections to the functional bundle $(p\ p)_G$ is described, as well as its restrictions to these relevant subgroups. This is used immediately to provide a generalization of the already mentioned result of James. The analogous homeomorphism for $\mathfrak{F}(p^F)$ is also discussed.

In the later part of the chapter we describe how a G-space F and a topological group morphism $h: G \to K$ determine, respectively, continuous homomorphisms $\Gamma_p^F: \mathcal{G}(p) \to \mathcal{F}(p^F)$ and $\Pi_p^h: \mathcal{G}(p) \to \mathcal{G}(p^h)$, where p^h is the principal K-bundle determined by p and h. In the special case where h is the inclusion of a subgroup in a group, p^h is the principal bundle obtained from p by the corresponding enlargement of the structure group; this leads to several applications later. Using the homomorphism Γ_p^F we prove that $\mathcal{F}(p^F)$ is, algebraically, a quotient group of $\mathcal{G}(p)$, the kernel being determined by the subgroup of G which acts trivially on F. This classification, under suitable conditions, is also valid topologically. We also show that the space $\mathcal{F}(p^F)$, in favourable circumstances and for a convenient choice of h, can be identified with the image of the homomorphism Π_p^h mentioned before. In the last section we show that if p is a principal $(G_1 \times \ldots \times G_n)$-bundle, then $\mathcal{G}(p)$ is isomorphic, as a topological group, to the product $\mathcal{G}(p_1) \times \ldots \times \mathcal{G}(p_n)$, where p_i is a corresponding G_i-bundle $(i=1,\ldots,n)$.

The homomorphism Π_p^h can become a (Hurewicz or Serre) fibration, provided h has similar properties. This idea is formalized and exploited in chapter 3), where different situations, leading to different kinds of fibrations, are analyzed. One consequence is that, under suitable conditions, we are able to identify $\mathfrak{F}(p^F)$ with a subgroup of the group of complete path components of $\mathcal{G}(p^h)$, where h is chosen as before. Thus the study of the homotopy groups of the spaces $\mathfrak{F}(p^F)$ reduces, except in dimension zero, to the study of the same problem for groups of automorphisms of principal bundles. Another important homotopy property of h is shown to be reflected by the induced homomorphism Π_p^h; namely if h induces isomorphisms of homotopy groups in a range of dimensions, then so does Π_p^h, even though the two ranges, depending on the dimension of the space B, may be different. The results of this chapter can be considered as central to the thesis, in that they provide the means of obtaining most of our applications and main theoretical results.

In chapter 4 we use functional bundles to outline the proof of a strenghtened version of Gottlieb's result. The method in question is taken from [BHMP2] and is used also to prove a key result of this chapter: the homomorphism Π_p^h corresponds, in a natural fashion, to the map $\Omega(h_*)_B$, between the corresponding loop spaces of mapping spaces, induced by $h_* : B_G \to B_K$. Putting this in categorical terms, there is a natural isomorphism between the functor which associates to h the homomorphism Π_p^h and the functor which associates to h the H-map $\Omega(h_*)_B$ An important consequence of these results is that, in suitable conditions, $\mathfrak{F}(p^F)$ can be identified, up to weak homotopy equivalence, or even homotopy equivalence, with the H-space consisting of a group of path components of a loop space on a certain mapping space.

The results of chapters 3 and 4 are applied in chapter 5, where we compute a variety of homotopy groups of the groups $\mathcal{G}(p)$. Some of the cases for which this is done have been mentioned previously and results for orthogonal groups, analogous to those described before for unitary groups, are given. We also obtain conditions under which $\mathfrak{F}(p^F)$ can be identified with the corresponding group $\mathcal{G}(p^h)$ rather than with a proper subgroup of its group of path components.

The work is completed by two appendices. The first one outlines a theory analogous to that of the main part of the thesis, in which the technique of inducing a principal bundle p^h from p by means of a homomorphism h is replaced by the familiar method of inducing a principal bundle p_f from p by means of a map f between the base spaces. The homomorphisms between bundle automorphism groups obtained with this approach have properties that are often analogous to those described earlier.

In the second appendix we give another application of functional bundles. Namely, we show that associated principal and fibre bundles can be obtained one from the other as associated functional bundles. This construction has the advantage of symmetry and of providing, in the case of an effective fibre F, a simple proof of the equivalence between the category of principal G-bundles over a space B and the category of fibre bundles with fibre F, and structure group G, over B.

C H A P T E R   0

B A S I C   D E F I N I T I O N S

A N D   N O T A T I O N

The spaces analyzed in this thesis are spaces of maps and it is well known that several interesting properties of such spaces are valid only under certain conditions. If one works in the category *Top* of all topological spaces, this means stating such conditions each time they are needed, with the consequent loss of simplicity in the exposition.

In order to avoid this complication we shall make two basic assumptions which will be adhered to throughout the thesis, without being recalled. These assumptions do not rule out any of the interesting cases that one meets in practice (e.g., CW-complexes, manifolds etc.) and allow us to use those needed properties without any problem. Clearly, there are alternative assumptions which can be made to serve the same purpose and we are certain that the reader will easily be able to determine how to maintain the validity of our results under those different conditions.

The first and main assumption is as follows.

ASSUMPTION 1 We shall work within the convenient category $\underline{K}$ of k-spaces.

This category (see [V]) is a full subcategory of *Top*, the objects of which are topological spaces having the final topology with respect to all incoming maps from compact Hausdorff spaces. There is a functor

$$K : \underline{Top} \to \underline{K}$$

right adjoint to the inclusion functor, which assigns to each topological space X its "*k-ification*", that is, the same underlying set retopologized with the final topology with respect to all maps from compact Hausdorff spaces to X.

With this in mind, we now begin to give some definitions and results which will often be used in this thesis without being explicitely quoted. All the proofs will be omitted, but can easily be obtained or found in the literature.

A *space* is an object of the category *K*.

A *subspace* U of a space X is a subset of X endowed with the "k-ification" of the relative topology. Notice that if U is closed or open in X, then the subspace topology and the relative topology coincide.

Given two spaces X and Y, the *product* X×Y is the "k-ification" of the usual cartesian product in *Top*.

A *map* is a morphism in $\underline{K}$. The notation

$$f\colon X \to Y\colon x \leadsto y$$

means that f is a map (or, sometimes, just a function) from X to Y, which associates to the typical element $x \in X$ the element $y \in Y$. For a space X, the identity map on X will be denoted by 1_X , or simply by 1 , whenever no confusion arises.

The *initial topology* on a space X with respect to a family of outgoing maps can be defined as in *Top* (see [Br], page 153).

The notion of *final topology* , and, in particular, of *quotient space* , coincide with the analogous constructions in *Top* ([Br] section 4.2), since the functor K preserves colimits. We shall often use the fact that, if X is a space and $\{U_j\}_{j \in J}$ is an open cover of X, then X has the final (or, in this case, *weak*) topology with respect to the inclusions $\{j\colon U_j \to X\}_{j \in J}$.

Given two spaces X and Y, the set map(X,Y) consists of all maps from X to Y. When endowed with the "k-ification" of the compact open topology ([Br] page 155), it becomes a *mapping space* , denoted by Map(X,Y). Notice that for each compact subspace C of X and each open subspace U of Y, the set

$$((C,U)) = \{f \in \mathrm{Map}(X,Y) \mid f(C) \subset U\}$$

which is subbasic for the compact open topology, is still open in Map(X,Y).

The following theorem will often be referred to as the *exponential law* and constitutes one of the main advantages of working in the category $\underline{K}$.

THEOREM Given spaces X, Y and Z there is a homeomorphism

$$\Phi\colon \mathrm{Map}(X\times Y,Z) \to \mathrm{Map}(X,\mathrm{Map}(Y,Z))$$

defined by the relation $\Phi(f)(x)(y) = f(x,y)$.

Denoting by $f\cdot g$ the composite of the maps $g\colon X \to Y$ and $f\colon Y \to Z$, we then have:

COROLLARY For any spaces X, Y and Z, the following functions are continuous:

$$ev\colon \mathrm{Map}(X,Y)\times X \to Y\colon (f,x) \rightsquigarrow f(x)$$

$$c\colon \mathrm{Map}(Y,Z)\times \mathrm{Map}(X,Y) \to \mathrm{Map}(X,Z)\colon (f,g) \rightsquigarrow f\cdot g$$

If $p\colon X \to B$ is a map and U (resp., b) is a subset (element) of B, then $X|U$ $(X|b)$ will denote the subspace $p^{-1}(U)$ $(p^{-1}(b))$ of X.

If $p\colon X \to B$ is a map, a *section* of p is a map $s\colon B \to X$ such that $p \cdot s = 1_B$. The subspace of $\mathrm{Map}(B,X)$ consisting of all sections of p will be denoted by $\sec(p)$. When p is a bundle or a fibration we shall also say that X is the *total space* and B the *base space* of p.

The results which we are now going to quote require the condition that certain spaces be Hausdorff. Since they will be used quite frequently in the thesis, our second assumption will be as follows.

ASSUMPTION 2 All basic spaces that we shall consider, that is, all spaces not obtained from others through some construction of sort, are assumed to be Hausdorff. Moreover, all quotient spaces that we shall consider are assumed to be Hausdorff.

This hypothesis, as we shall see, forces all the spaces that we shall construct to be Hausdorff, a condition perhaps excessive, but which will greatly simplify the exposition. We also like to notice that a Hausdorff k-space is also "compactly generated", that is, a subspace of such a space X is closed if and only if its intersection with each compact subspace of X is closed.

If $p: X \to B$ and $q: Y \to B$ are two maps with the same range space, a map $f: X \to Y$ is <u>*over B*</u> if $q \cdot f = p$. In this case the subspace of $\mathrm{Map}(X,Y)$ consisting of maps over B will be denoted by $M(p,q)$. Also, for a given map $f \in M(p,q)$ and a given subspace U of B, the notation $f|U$ will denote the restriction of f to $X|U$.

The following result is one of those which depend mainly on the fact that B is assumed to be Hausdorff.

<u>LEMMA</u> Given maps $p: X \to B$ and $q: Y \to B$, then $M(p,q)$ is closed in $\mathrm{Map}(X,Y)$.

Still in the same situation, the <u>*pullback*</u> of p and q is the (closed) subspace $X \sqcap Y$ of $X \times Y$ defined by:

$$X \sqcap Y = \{(x,y) \mid p(x) = q(y)\}$$

and there exist maps:

$$p_q : X \sqcap Y \to Y : (x,y) \rightsquigarrow y$$

$$q_p : X \sqcap Y \to X : (x,y) \rightsquigarrow x$$

$$p \sqcap q : X \sqcap Y \to B : (x,y) \rightsquigarrow p(x)$$

and the "<u>*universal property*</u>" described by the following theorem.

<u>THEOREM</u> Given a space Z and two maps $f\colon Z \to X$ and $g\colon Z \to Y$ such that $p\cdot f = q\cdot g$, the map

$$(f,g)\colon Z \to X \sqcap Y\colon z \rightsquigarrow (f(z), g(z))$$

is the unique element in the intersection of $M(f, q_p)$ with $M(g, p_q)$.

In general a commutative diagram of maps of the form

$$\begin{array}{ccc} W & \xrightarrow{p'} & Y \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{p} & B \end{array}$$

is a *pullback diagram* if the corresponding "universal property" holds; that is, if for any space Z and any two maps $f\colon Z \to X$ and $g\colon Z \to Y$ such that $p\cdot f = q\cdot g$, there is a unique map $h\colon Z \to W$ such that $q'\cdot h = f$ and $p'\cdot h = g$.

A *topological group* is a space G which has also a multiplicative group structure such that the multiplication and inverse functions:

$$\mu\colon G\times G \to G\colon (g,g') \rightsquigarrow gg'$$
$$i\colon G \to G\colon g \rightsquigarrow g^{-1}$$

are both continuous. If G is a topological group and H is a closed subspace of G which is also an algebraic subgroup of G, we shall say that H is a *topological subgroup* of G, or that H is a *subgroup of the topo-*

logical group G. When another expression will be used, this will refer to a situation different from the one just described and clearly identifiable from the context. The notion of a *normal* subgroup will always refer to the algebraic meaning of the word and will never carry any topological significance.

A continuous homomorphism $h: G \to K$ between topological groups will often be referred to as a topological group *morphism*. An *isomorphism* of topological groups is a morphism which is also a homeomorphism.

We conclude this introductory chapter with a few remarks on the notation.

First of all, we shall quite often consider collections of sets (e.g. open covers of a space) whose indexing set has no relevance at all in the discussion. In those cases the indexing set will be omitted completely, in order to simplify the notation, and only the typical element of the collection, enclosed in curly brackets, will identify the collection. So, for instance, $\{U\}$ will stand for $\{U_j\}_{j\in J}$ whenever no confusion arises. The use of curly brackets to describe sets, like

$$G = \{g \mid g \in G\}$$

will be clearly distinguishable from the context.

The symbol $\underset{=}{\sim}$ will denote a topological homeomorphism and the symbol $\sim$ an algebraic isomorphism. If X and Y are spaces, the notation $X \underset{-}{\sim} Y$ indicates that X and Y are homotopy equivalent, while if f and g are maps, $f \underset{-}{\sim} g$ means that f and g are homotopic. The symbol // will signal the end of a proof, or, when appearing at the end of the statement of a result, that the proof is omitted because of its simplicity.

When quoting a piece of literature, an abbreviation in square brackets will identify the paper or book quoted, according to the list presented, in alphabetical order, at the end of the thesis and will be followed by the exact location, within that work, of the needed result (notice that such convention has already been used in this chapter). So, for instance, [Hus] th. 3.2 page 42 will refer to theorem 3.2, on page 42, of the book by D. Husemoller on "Fibre bundles".

For any other symbols or terms used, we refer the reader to the general literature.

CHAPTER 1

PRINCIPAL, FIBRE AND FUNCTIONAL BUNDLES

1) G-SPACES AND G-MAPS

Let X be a space and G a topological group with unit e. We say that X is a *right G-space* if there is a map $\rho: X\times G \to X$, called the *action* of G on X, such that:

1) for every $x\in X$, $\rho(x,e) = x$,

2) for every $x\in X$ and every $g,g'\in G$, $\rho(x,gg')=\rho(\rho(x,g),g')$.

Similarly we say that X is a *left* G-space if there is a map $\lambda: G\times X \to X$, also called the action of G on X, such that:

1) for every $x\in X$, $\lambda(e,x) = x$,

2) for every $x\in X$ and every $g,g'\in G$, $\lambda(gg',x)=\lambda(g,\lambda(g',x))$.

Usually we shall denote the element $\rho(x,g)$ (resp. $\lambda(g,x)$) as xg (gx). Also, for a given subset $U\subset X$ and a given $g\in G$, Ug (gU) will denote the set $\{xg|x\in U\}$ ($\{gx|x\in U\}$). There is a bijective correspondence between left and right actions; namely, given a right action ρ and a left action λ of G on X, we say that ρ and λ are *associated* if for any $x\in X$ and $g\in G$ $\lambda(g,x) = \rho(x,g^{-1})$. Properties enjoyed by a given action have a corresponding analogue for its associated action. So whenever that analogy is evident we shall only discuss the case of interest to us, but we shall be more specific when significant differences will arise.

1.1 EXAMPLES

a) Let G be a topological group and let H be a subgroup of G. Then multiplication defines both a right and left action of H on G. However the two actions are not associated in general.

b) Let E be the closed interval $[-1,1]$ and S^0 be the discrete group $\{1,-1\}$. Then the map

$$\lambda: S^0 \times E \to E: (a,t) \rightsquigarrow at$$

makes E into a left S^0-space.

c) Let $h: G \to K$ be a topological group morphism. Then there is a left action $\lambda(h)$ of G on K defined by the relation

$$\lambda(h)(g,k) = h(g)k \tag{1.2}$$

Similarly we get a right action $\rho(h)$ by letting $\rho(h)(k,g) = k(h(g))$. In general $\lambda(h)$, $\rho(h)$ and their associated actions determine four distinct actions of G on K. However we shall only use $\lambda(h)$.

Notice that for a given left G-space X and a given $g \in G$ the restriction of the action λ to $\{g\} \times X$ determines a homeomorphism

$$g^* : X \to X : x \rightsquigarrow gx$$

which will be said to be an *action map*. In fact using an exponential law we can associate to λ a map

1.3 $$\lambda^* : G \to \mathrm{Map}(X,X) : g \rightsquigarrow g^*$$

Its image, i.e. the subspace of Map(X,X) consisting of action maps, will be denoted by $M_\lambda(X,X)$.

1.4 PROPOSITION Let $\lambda : G \times X \to X$ be a left action. Then $M_\lambda(X,X)$ has a group structure, given by map composition, with respect to which $\lambda^* : G \to M_\lambda(X,X)$ is a continuous epimorphism.

PROOF It suffices to notice that, for each $g \in G$, $(g^{-1})^*$ is the inverse of g^* .//

The kernel of λ^* is a subgroup of the topological group G; it will be referred to as the *kernel of the action λ* and will be denoted by Kr_λ or Kr_X according to whether it will be more important to specify the action or the G-space. The quotient G/Kr_X will often be denoted by Q_X.

In examples 1.1 a) and 1.1 b) the kernel of the action is trivial, while in example 1.1 c) $Kr_{\lambda(h)}$ coincides with the kernel of h.

The analogous construction for the case in which X is a right G-space gives rise to a map $\rho^* : G \to M_\rho(X,X)$ which is not a homomorphism, but has the property that for any $g,g' \in G$ $\rho^*(gg') = \rho^*(g')\rho^*(g)$. Hence it is still true that ρ^* is injective if and only if its kernel, Kr_ρ, is trivial.

1.5 DEFINITION Let ν be a (left or right) action of G on X. Then X is said to be *effective* if ν^* is injective, *free* if $\nu^*(e)$ is the only map in $M_\nu(X,X)$ that has a fixed point (or points), *admissible* if ν^* is an identification map.

In example 1.1 a) G is a free H-space and we shall show later that it is also admissible; in example 1.1 b) E is effective but not free, since $(-1)^*(0) = 0 = (1)^*(0)$; in example 1.1 c) K being effective and K being free are both equivalent to h being injective. It is easy to construct examples of G-spaces which are free but not admissible, however if X is admissible then $M_\nu(X,X)$, being a quotient group of G, is itself a topological group. If X is a free G-space then it is also effective, while if it is not effective then the action ν of G induces an action

$$\nu': X\times Q_X \to X: (x, gKr_\nu) \rightsquigarrow xg$$

(and similarly for left G-spaces) with respect to which X becomes an effective Q_X-space. Finally we notice that if G is compact then any G-space is admissible.

1.6 DEFINITION A map $f: X \to Y$ between right G-spaces is a *G-map* if, for every $x \in X$ and $g \in G$, $f(xg)=f(x)g$.

The notion of G-map can be easily extended to the case in which X and Y are left spaces ($f(gx)=gf(x)$) or one is a left and the other a right G-space ($f(xg)=g^{-1}f(x)$ or $f(gx)=f(x)g^{-1}$), but these are used less frequently. If X and Y are G-spaces the subspace of Map(X,Y) consisting of G-maps will be denoted by $M_G(X,Y)$. In the particular case when $X = Y = G$ the notation $M_G(G,G)$ will refer to the right action of G on itself determined by multiplication.

1.7 LEMMA Let X and Y be two G-spaces. Then $M_G(X,Y)$ is closed in Map(X,Y).

PROOF If X and Y are right G-spaces, let $f \in \mathrm{Map}(X,Y)$

and suppose that, for given $x \in X$ and $g \in G$, $f(xg) \neq f(x)g$. Choose open sets U and V in Y such that $f(x)g \in U$, $f(xg) \in V$ and $U \cap V = \phi$. Then $((\{x\}, Ug^{-1})) \cap ((\{xg\}, V))$ is open in Map(X,Y), contains f, but no G-maps. Thus the complement of $M_G(X,Y)$ is open and this proves the claim. The other cases can be treated in the same way.//

If X is a right G-space, we can also consider, for each $x \in X$, the restriction of the action ρ to $\{x\} \times G$. this gives rise to a map

$$x_* : G \to X : g \leadsto xg$$

which, considering G as a right G-space, is a G-map. In fact we can associate to the action ρ a map

1.8 $$\rho_* : X \to \mathrm{Map}(G,X) : x \leadsto x_*$$

and it is easy to see that the image of ρ_* is $M_G(G,X)$.

1.9 <u>LEMMA</u> For every G-space X, the map $\rho_* : X \to M_G(G,X)$ is a homeomorphism.

<u>PROOF</u> Evaluation at e provides an inverse for ρ_*.//

In the case of a left action generated by a topological group morphism $h: G \to K$ (example 1.1 c)), every element of $M_{\lambda(h)}(K,K)$ is also an element of $M_K(K,K)$.

1.10 <u>**PROPOSITION**</u> If $h: G \to K$ is a topological group morphism and $\mu: K\times K \to K$ is the multiplication, then $M_{\lambda(h)}(K,K)$ coincides with the subspace $\mu_*(h(G))$ of $M_K(K,K)$ and is closed in $Map(K,K)$ if $h(G)$ is closed in K.

<u>**PROOF**</u> For any $k \in K$ $\mu_*(k)$ is left multiplication by k, so it is in $M_{\lambda(h)}(K,K)$ if and only if k is in the image of h. Lemmas 1.7 and 1.9 prove that if $h(G)$ is closed in K then $M_{\lambda(h)}(K,K)$ is closed in $Map(K,K)$.//

1.11 <u>**COROLLARY**</u> For any normal subgroup H of a topological group G the quotient group G/H is an admissible G-space with respect to the left action $\lambda(\pi)$ generated by the canonical projection $\pi: G \to G/H$.

<u>**PROOF**</u> We only need to notice that in this case the map $\lambda(\pi)^*: G \to M_{\lambda(\pi)}(G/H,G/H)$ is, up to homeomorphism, the canonical projection itself.//

1.12 <u>**COROLLARY**</u> Let H be a subgroup of the topological group G. Then G is an admissible left H-space with respect to the action $\lambda(\iota)$ generated by the inclusion $\iota: H \rightarrowtail G$.

<u>**PROOF**</u> Again the map $\lambda(\iota)^*: H \to M_{\lambda(\iota)}(G,G)$ is, up to

homeomorphism, the identity on H.//

§ 2) PRINCIPAL G-BUNDLES

If X is a right G-space there is an equivalence relation $\mathcal{O}$ on X obtained setting $x\mathcal{O}y$ if, and only if, there exists an element $g \in G$ such that $y = xg$. The equivalence class of an element $x \in X$ will be called the *orbit* of x and denoted by xG. Consequently the quotient space of X determined by $\mathcal{O}$ will be called the *orbit space* of X and denoted by X/G, $\pi: X \to X/G$ being the canonical projection. When, in particular, X is free there is a function τ from the space $X \sqcap X$, obtained as a pullback of π with itself, to G, defined by

$$\tau: X \sqcap X \to G: (x, xg) \rightsquigarrow g \tag{1.13}$$

1.14 **DEFINITION** The function τ described by 1.13 is called the *translation function* for X and if it is continuous X is said to be a *principal* G-space.

1.15 **DEFINITION** A map $p: X \to B$ is said to be *locally trivial with fibre* F if there exists an open cover $\{U\}$ of B and, for each $U\in\{U\}$, a homeomorphism $\phi_U: U\times F \to p^{-1}(U)$ such that the composite $p\cdot\phi_U$ is the projection on U. In this case the cover $\{U\}$ will be said to be a *locally trivial cover* for p, each ϕ_U will be a *local homeomorphism* and the collection $\{(U,\phi_U)\}$ a *locally trivial structure* for p.

Notice that for any topological group G and any space B the product $B\times G$ is a free right G-space, the action being:

$$1\times\mu: B\times G\times G \to B\times G: (b,g,g') \rightsquigarrow (b,gg')$$

With this in mind we can now define the basic objects of our study.

1.16 **DEFINITION** A map $p: X \to B$ is said to be a (right) *principal G-bundle* if X is a right G-space, p is locally trivial with fibre G and the local homeomorphisms are all G-maps.

The analogous notion of *left* principal G-bundle is rarely considered in the literature. We shall never use it and hence the word "right" will be omitted from now on.

1.17 EXAMPLES

a) Let B be a space and G a topological group. Then the projection pr: $B \times G \to B$ is a principal G-bundle with only one local homemorphism, given by the identity map. We shall call this a *product* principal G-bundle.

b) If H is a subgroup of the topological group G and G has a local section at H ([Sw] def. 4.12), the projection π: $G \to G/H$ is a principal H-bundle (use, for instance, [Sw] th. 4.13).

The following result justifies the terminology used.

1.18 PROPOSITION If p: $X \to B$ is a principal G-bundle, then X is a principal G-space and B is homeomorphic to X/G.

PROOF Let $\{(U,\phi_U)\}$ be the locally trivial structure for p and assume that there exist $x \in X$ and $g \in G$ such that $xg = x$. Then there must exist $U \in \{U\}$ and $a \in G$ so that $p(x) \in U$ and $\phi_U^{-1}(x) = (p(x),a)$. But now we have

$$(p(x),a) = \phi_U^{-1}(x) = \phi_U^{-1}(xg) = \phi_U^{-1}(x)g = (p(x),ag)$$

thus implying that $g = e$. This means that X is free and hence has a translation function τ. Using the fact that for any $U \in \{U\}$

$$(X \sqcap X)|U \cong (X|U) \sqcap (X|U) \cong U \times G \times G$$

we obtain, for each $U \in \{U\}$, a commutative diagram

$$\begin{array}{ccc} (X \sqcap X)|U & \xrightarrow{\tau|U} & G \\ \cong \updownarrow & & \updownarrow 1 \\ U \times G \times G & \xrightarrow{\sigma} & G \end{array}$$

where $\sigma(b,g,g') = g^{-1}g'$. The obvious continuity of σ proves the local, and hence global, continuity of the translation function. To prove the second part we observe that by hypothesis p is surjective and open and so it is an identification. Using again the local homeomorphisms one can easily see that two points of X have the same image under p if and only if they are in the same orbit. Combining these two facts it is immediate to obtain the homeomorphism between B and X/G. //

The following definition will allow us to consider some categories whose objects are principal bundles.

1.19 <u>DEFINITION</u> Given a principal G-bundle $p: X \to B$ and a principal G'-bundle $p': X' \to B'$, a *principal bundle morphism* from p to p' is a triple (f, f_o, h) where $f: X \to X'$ and $f_o: B \to B'$ are maps, $h: G \to G'$

is a topological group morphism and the following diagrams are commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ p\downarrow & & \downarrow p' \\ B & \xrightarrow{f_\circ} & B' \end{array} \qquad \begin{array}{ccc} X\times G & \xrightarrow{\rho} & X \\ f\times h\downarrow & & \downarrow f \\ X'\times G' & \xrightarrow{\rho'} & X' \end{array}$$

The category of principal bundles and principal bundle morphisms, denoted by P , can now be defined using the obvious composites and identities. We shall be interested mainly in some of its subcategories. Namely, for a given space B we shall denote by P_B the subcategory of P consisting of principal bundles over B and morphisms of the form $(f, 1_B, h)$. Also, given a topological group G, P^G will denote the subcategory of P consisting of principal G-bundles and morphisms of the form $(f, f_\circ, 1_G)$. Finally P_B^G will denote the intersection of P_B and P^G; in this category a morphism is a G-map $f: X \to X'$ over B and will often be denoted by $f: p \to p'$.

1.20 <u>THEOREM</u> Every morphism of P_B^G is an isomorphism.

<u>PROOF</u> See [Hus] th. 3.2 page 42.//

1.21 <u>DEFINITION</u> An object of P_B^G is said to be *trivial* if it is isomorphic to the product principal G-bundle over B.

1.22 **PROPOSITION** Given a principal G-bundle $p: X \to B$ the following statements are equivalent:

a) p is trivial,

b) sec(p) is not empty,

c) $M_G(X,G)$ is not empty.

PROOF The map

$$s: B \to B\times G: b \to (b,e)$$

provides a section to the product principal G-bundle over B and this suffices to prove that a) implies b). Now if $s \in sec(p)$ and $B \sqcap X$ denotes the pullback of the identity on B with p, the composite

$$X \xrightarrow{(p,1)} B\sqcap X \xrightarrow{s\times 1} X\sqcap X \xrightarrow{\tau} G$$

is an element of $M_G(X,G)$ and therefore b) implies c). Finally if $f \in M_G(X,G)$ the map $(p,f): X \to B\times G$ is a morphism in P_B^G. By theorem 1.20 this proves that c) implies a).//

We shall now describe two standard constructions which enable us to produce new principal bundles using the pullback construction.

1.23 <u>PROPOSITION</u> Let $p: X \to B$ be a principal G-bundle and $f: A \to B$ be a map. Then the map $p_f: A \sqcap X \to A$ is a principal G-bundle.

<u>PROOF</u> See [Hus] prop. 4.1 page 43.//

1.24 <u>DEFINITION</u> Let $p: X \to B$ be a principal G-bundle and $p': X' \to B$ a principal G'-bundle. Then the map $p \sqcap p': X \sqcap X' \to B$ is called the <u>*Whitney sum*</u> of p and p'.

1.25 <u>PROPOSITION</u> The Whitney sum $p \sqcap p': X \sqcap X' \to B$ of a principal G-bundle $p: X \to B$ and a principal G'-bundle $p': X' \to B$ is a principal $(G \times G')$-bundle.

<u>PROOF</u> The map

$\rho \sqcap \rho': (X \sqcap X') \times (G \times G') \to X \sqcap X': (x, x', g, g') \rightsquigarrow (xg, x'g')$

is a well defined action of $G \times G'$ on $X \sqcap X'$. Let now $\{(U, \phi_U)\}$ and $\{(U, \phi'_U)\}$ be the locally trivial structures for p and p' respectively. (We can assume, without loss of generality, that the locally trivial covers coincide) Then, for each $U \in \{U\}$, the map

$\phi_U \sqcap \phi'_U: U \times G \times G' \to (X \sqcap X')|U: (b, g, g') \rightsquigarrow (\phi_U(b, g), \phi'_U(b, g'))$

defines the local homeomorphism needed to complete the proof.//

§ 3) FIBRE BUNDLES

Let $p: X \to B$ be a principal G-bundle and let F be a left G-space. Then there is a right action of G on $X \times F$ defined by the relation

$$(x,y)g = (xg, g^{-1}y)$$

and we can consider the resulting orbit space, denoted by $X \times_G F$, and the projection

1.26 $$p^F: X \times_G F \to B: (x,y)G \rightsquigarrow p(x)$$

An element $(x,y)G$ of $X \times_G F$ will be denoted by $[x,y]$ whenever no confusion arises.

1.27 DEFINITION A *fibre bundle* associated to the principal G-bundle $p: X \to B$ and with fibre F is a pair (q,h) in which $q: Y \to B$ is a map and $h: Y \to X \times_G F$ is a homeomorphism over B.

We will usually consider fibre bundles of the form $(p^F, 1)$, where 1 is the identity on $X \times_G F$, and they will be denoted simply by p^F. Even in other cases the homeomorphism h will, for simplicity of exposition, be ignored.

Since for any space B $(B \times G) \times_G F \cong B \times F$, it follows that a locally trivial structure $\{(U, \phi_U)\}$ for p

induces a locally trivial structure $\{(U,\phi_U^F)\}$ on p^F in which the locally trivial cover is unchanged and, for a given $U\in\{U\}$ the local homeomorphism ϕ_U^F is given by

1.28 $\quad \phi_U^F\colon U\times F \to (X\times_G F)|U\colon (b,y) \rightsquigarrow [\phi_U(b,e),y]$

Hence for each $b\in B$ the fibre $(p^F)^{-1}(b)$ is homeomorphic to F.

Let now $f\colon p \to p'$ be a morphism in P_B^G, where $p\colon X \to B$ and $p'\colon X' \to B$. Then the map $f\times 1\colon X\times F \to X'\times F$ induces a map

$$f^F\colon X\times_G F \to X'\times_G F\colon [x,y] \rightsquigarrow [f(x),y]$$

1.29 <u>**DEFINITION**</u> Let (q,h) and (q',h') be fibre bundles with fibre F associated to the principal G-bundles p and p' respectively. If $f\colon p \to p'$ is a morphism in P_B^G the map $(h')^{-1}\cdot f^F\cdot h$ is said to be a <u>*fibre bundle morphism*</u>.

1.30 <u>**PROPOSITION**</u> Given a space B, a topological group G and a left G-space F there is a category $F_B^F(G)$ whose objects are all fibre bundles associated to objects of P_B^G and with fibre F and whose morphisms are fibre bundle morphisms. Moreover there is a functor $\Gamma^F: \mathcal{P}_B^G \to \mathcal{F}_B^F(G)$ defined by setting $\Gamma^F(p) = p^F$ and $\Gamma^F(f) = f^F$.//

It is also possible to construct, in a similar way, a category $\mathcal{F}^F(G)$ associated to $\mathcal{P}^G$, but it will not be used in this thesis.

§ 4) PRINCIPAL BUNDLES INDUCED BY A TOPOLOGICAL GROUP MORPHISM

Every principal G-bundle $p: X \to B$ is a fibre bundle associated to itself with fibre G, since, viewing G as a left G-space with respect to its multiplication, $X \times_G G$ is canonically homeomorphic to X ([Hus] cor. 1.2 page 70). The converse is not true in general, that is, not every fibre bundle is a principal bundle, since not every left G-space is a topological group. However the following definition gives rise to an important exception which will be used extensively.

1.31 DEFINITION Let $p: X \to B$ be a principal G-bundle and $h: G \to K$ a topological group morphism. The fibre bundle $p^K: X \times_G K \to B$, obtained with respect to the action $\lambda(h)$, is said to be *induced from* p *by* h. In this case we shall use the notation $p^h\colon X\times_h K \to B$.

This different notation will be extended in the obvious way to local homeomorphisms, fibre bundle morphisms and so on; it is justified by the fact that the action of G on K depends mostly on h and by the following result.

1.32 THEOREM Let $p\colon X \to B$ be a principal G-bundle and $h\colon G \to K$ a topological group morphism. Then the map $p^h\colon X\times_h K \to B$ is a principal K-bundle, the action being:

$$\rho^h\colon (X\times_h K)\times K \to X\times_h K\colon ([x,k],k') \rightsquigarrow [x,kk']$$

PROOF The function ρ^h is continuous since $X\times_h K$ is a quotient of $X\times K$ and K is a topological group. Moreover the local homeomorphisms for p^h, as given in 1.28, are of the form

$$\phi_U^h\colon U\times K \to (X\times_h K)|U\colon (b,k) \rightsquigarrow [\phi_U(b,e),k]$$

and therefore are K-maps. So by definition 1.16 p^h is a principal K-bundle.//

We also observe that the translation function for p^h is given by:

1.33 $\tau\colon (X\times_h K)\sqcap(X\times_h K) \to K\colon ([x,k],[xg,k']) \rightsquigarrow k^{-1}\, h(g)\, k'$

1.34 THEOREM Given a space B and a topological group morphism $h\colon G \to K$ there is a functor $\Pi^h\colon P_B^G \to P_B^K$ defined by $\Pi^h(p) = p^h$ and $\Pi^h(f) = f^h$ for objects and morphisms respectively.

PROOF We only need to prove that for any morphism f of P_B^G the map f^h is a K-map, since identities and composites are clearly preserved. To that end notice that for any $[x,k]\in X\times_h K$ and $k'\in K$, if $f(x) = xg$ we have

$$f^h([x,k]k') = f^h[x,kk'] = [xg,kk'] = $$
$$= [xg,k]k' = f^h([x,k])k'$$

thus proving the claim.//

1.34 EXAMPLE Let H be a normal subgroup of the topological group G and consider the canonical projection $\pi\colon G \to G/H$. In this case $X\times_\pi(G/H) \cong X/H$, where the action of H on X is just the restriction of the action of G ([Hus] th. 1.1 page 70). The homeomorphism is obtained associating to an element $[x,gH]\in X\times_\pi(G/H)$ the element $xgH\in X/H$. For simplicity of notation we shall often consider X/H as the total space of p^π. Notice that if H is not normal in G the homogeneous space G/H is still a left G-space, so we can define $p^{G/H}$, but this will be only a fibre bundle.

1.35 <u>DEFINITION</u> Let $p: X \to B$ be a principal G-bundle and $\iota: G \to K$ an inclusion. Then the principal K-bundle p^{ι} is said to be an <u>*extension*</u> of p and p is said to be a <u>*restriction*</u> of p^{ι}.

1.36 <u>PROPOSITION</u> Let $p: X \to B$ be a principal G-bundle and $h: G \to K$ a topological group morphism. If p is trivial or h is trivial then p^h is trivial.

<u>PROOF</u> Using theorem 1.20 we see that the map

$$t: (B\times G)\times_h K \to B\times K: [(b,g),k] \rightsquigarrow (b,h(g)k)$$

proves the claim in the case where p is trivial, while the map

$$t': X\times_h K \to B\times K: [x,k] \rightsquigarrow (p(x),k)$$

proves it in the case where h is trivial.//

Given a principal G-bundle $p: X \to B$ and a topological group morphism $h: G \to K$ there is a map

1.37 $$j(h): X \to X\times_h K: x \rightsquigarrow [x,e]$$

1.38 <u>LEMMA</u> For each $U\in\{U\}$ there is a commutative diagram

$$\begin{array}{ccc} X|U & \xrightarrow{j(h)|U} & (X\times_h K)|U \\ \phi_U \downarrow & & \downarrow \phi_U^h \\ U\times G & \xrightarrow{1\times h} & U\times K \end{array}$$

and hence if $h(G)$ is closed in K then $j(h)(X)$ is closed in $X\times_h K$ and if h is an inclusion then so is $j(h)$.//

1.39 LEMMA Let $p: X \to B$ be a principal G-bundle and $q: Y \to B$ a principal K-bundle. Then q is isomorphic, in $\mathcal{P}_B^K$, to p^h, for some topological group morphism $h: G \to K$, if and only if there exists, in $\mathcal{P}_B$, a morphism from p to q of the form (f,h).

PROOF If such a morphism exists, then the function

$$f': X\times_h K \to Y: [x,k] \rightsquigarrow f(x)k$$

is well defined and continuous and, being a K-map over B, defines an isomorphism, in $\mathcal{P}_B^K$, between p^h and q. Conversely, let $f': X\times_h K \to Y$ be an isomorphism in $\mathcal{P}_B^K$ between p^h and q. Then the pair (f,h), where $f = f'\cdot j(h)$, is a morphism in $\mathcal{P}_B$, as proved by the commutativity of the diagram

$$\begin{array}{ccccc} X\times G & \xrightarrow{j(h)\times 1} & (X\times_h K)\times G & \xrightarrow{f'\times h} & Y\times K \\ \phi_X \downarrow & & & & \downarrow \phi_Y \\ X & \xrightarrow{j(h)} & X\times_h K & \xrightarrow{f'} & Y \end{array}$$

which can be easily verified.//

1.40 PROPOSITION Let $p: X \to B$ be a principal K-bundle and $h: G \to K$ a topological group morphism. If there exists a principal G-bundle $q: Y \to B$ such that q^h is isomorphic to p then the fibre bundle $p^{K/h(G)}$ has a section.

PROOF We can assume, without loss of generality, that $q: Y \to B$ is such that $q^h = p$. Then the composite

$$Y \xrightarrow{j(h)} X \xrightarrow{\pi} X/h(G)$$

is constant on each fibre of q and hence defines a map

$$s: B \to X/h(G): b \rightsquigarrow [j(h)(q^{-1}(b))]$$

which is the required section of $p^{K/h(G)}$.//

In general proposition 1.40 does not have a converse. The following counterexample uses some results which will be mentioned only in a later chapter, but which can be easily found in the literature. Since the real line R is contractible, every principal R-bundle over S^2 is trivial. The Hopf bundle $\eta: S^3 \to S^2$ (see, e.g., [Sp] 2.7.6) is a non trivial principal S^1-bundle and therefore, by proposition 1.36, it cannot be induced from any principal R-bundle by the projection $\pi: R \to S^1$. Nevertheless the fibre bundle $p^{K/h(G)}$ in this case is η^{S^1/S^1}, which is the identity on S^2 and hence has a section.

When h is an inclusion, however, such converse can be obtained, as proved in [Hus] th.2.3 page 71.

The following result will provide a converse to proposition 1.25.

1.41 PROPOSITION Let $p: X \to B$ be a principal $(G^1 \times G^2)$-bundle. Then denoting by $\pi^i: G^1 \times G^2 \to G^i$ $(i=1,2)$ the canonical projections, p is isomorphic to the Whitney sum $p^{\pi^1} \sqcap p^{\pi^2}$.

PROOF The universal property of pullbacks ensures that the projections $\theta^i: X \to X/G^i$ induce a map

$$f: X \to (X/G^2) \sqcap (X/G^1) : x \rightsquigarrow (xG^2, xG^1)$$

Now for any $x \in X$, $g^1 \in G^1$, $g^2 \in G^2$ we have

$$\begin{aligned} f(x(g^1,g^2)) &= (x(g^1,g^2)G^2, x(g^1,g^2)G^1) = \\ &= (x(g^1,e^2)G^2, x(e^1,g^2)G^1) = \\ &= (xG^2, xG^1)(g^1,g^2) \end{aligned}$$

so f is a $(G^1 \times G^2)$-map over B and hence an isomorphism between p and $p^{\pi^1} \sqcap p^{\pi^2}$.//

1.42 COROLLARY A principal G-bundle $p: X \to B$ is isomorphic to the Whitney sum of a principal H-bundle and a principal H'-bundle if and only if G is isomorphic to $H \times H'$.//

We conclude this section by mentioning a result whose proof can be easily obtained.

1.43 PROPOSITION Let $p: X \to B$ be a principal G-bundle and let $h: G \to G'$, $h': G' \to G''$ be topological group morphisms. Then $(p^h)^{h'}$ is isomorphic to $p^{h'\cdot h}$ in $P_B^{G''}$.//

1.44 COROLLARY Let $p: X \to B$ be a principal G-bundle, $h: G \to K$ a topological group morphism with kernel H and $h': G/H \to K$ the induced monomorphism. Then $X\times_h K$ and $X/H\times_{h'} K$ are canonically homeomorphic over B.//

5) FUNCTIONAL BUNDLES

We now recall a construction of Booth and Brown which will be heavily used in this thesis.

Let X and Y be topological spaces. We denote by $P(X,Y)$ the space of maps having a closed subspace of X as domain and Y as range. Its topology is the "k-ification" of the topology having as a subbasis all sets of the form $((C,U))$, where C is a compact subspace of X, U is an open subspace of Y and

$$((C,U)) = \{f \in P(X,Y) \mid \forall x \in (C \cap \mathrm{dom}(f)),\ f(x) \in U\}$$

We draw the reader's attention to the fact that the "empty map" $\phi\colon \phi \to Y$ also belongs to $P(X,Y)$.

If we have two maps $p\colon X \to B$ and $q\colon Y \to B$ we can define a set

1.45 $$(X\ Y) = \bigcup_{b \in B} \operatorname{map}(X|b, Y|b)$$

and two functions

1.46 $$\begin{aligned} (p\ q)&\colon (X\ Y) \to B\colon (\alpha\colon X|b \to Y|b) \rightsquigarrow b \\ j&\colon (X\ Y) \to P(X,Y)\colon (\alpha\colon X|b \to Y|b) \rightsquigarrow i_b \cdot \alpha \end{aligned}$$

where $i_b\colon Y|b \to Y$ is the inclusion. The initial topology on $(X\ Y)$ with respect to $(p\ q)$ and j is called the *modified compact-open topology* ([BB1] def. 2.1).

1.47 DEFINITION The space $(X\ Y)$ is said to be the *functional space associated to* p *and* q and the map $(p\ q)$ the *functional bundle* associated to p and q.

The result concerning functional bundles which we shall mostly use is as follows.

1.48 THEOREM Let $p\colon X \to B$ and $q\colon Y \to B$ be maps. Then there is a homeomorphism $\Phi\colon M(p,q) \to \sec(p\ q)$

between the space of maps from X to Y over B and the space of sections to $(p\ q)$, defined by

$$\Phi(f)(b) = f|p^{-1}(b)$$

PROOF see [BB1] cor. 3.7 and th. 7.3.//

In the cases in which we are interested, the maps p and q are locally trivial and a significant simplification occurs. In fact, suppose that p and q have fibres F and G respectively and locally trivial structures $\{(U,\phi_U)\}$ and $\{(U,\psi_U)\}$ respectively (again it is easily arranged, by restriction, that the two structures have a common locally trivial cover). Then, for each $U\in\{U\}$, there is a map

1.49 $\chi_U\colon U\times\mathrm{Map}(F,G) \to (X\ Y)|U\colon (b,f) \leadsto \psi_{U,b}\cdot f\cdot\phi_{U,b}^{-1}$

where $\phi_{U,b}$ and $\psi_{U,b}$ are the restrictions of ϕ_U and ψ_U to $\{b\}\times F$ and $\{b\}\times G$ respectively.

1.50 PROPOSITION All the maps χ_U are homeomorphisms and hence $(p\ q)$ is a locally trivial map with fibre $\mathrm{Map}(F,G)$ and locally trivial structure $\{(U,\chi_U)\}$.

PROOF see [BB2] th. 2.1 and remark 8.2.//

We notice that this proposition is very important also because we want to work with Hausdorff spaces. In fact $(X\ Y)$ in general is not Hausdorff (see [BB1] section 5), but if p and q are locally trivial then proposition 1.50 together with the Hausdorffness of G proves that $(X\ Y)$ is Hausdorff.

In the remainder of this thesis we shall be interested mainly in two particular cases of this construction. First of all let $p\colon X \to B$ and $p'\colon X' \to B$ be principal G-bundles. Then we can consider the subspace $(X\ X')_G$ of $(X\ X')$ consisting of G-maps. Since the local homeomorphisms for $(p\ p')$ are given by the maps

$$\chi_U\colon U{\times}\mathrm{Map}(G,G) \to (X\ X')|U\colon (b,f) \rightsquigarrow \phi'_{U,b}\cdot f\cdot \phi^{-1}_{U,b}$$

it follows that $\chi_U(b,f)$ is a G-map if and only if $f \in M_G(G,G)$. Using lemma 1.9 we can therefore state the following proposition.

1.51 <u>PROPOSITION</u> The restriction $(p\ p')_G$ of $(p\ p')$ to $(X\ X')_G$ is a locally trivial map with fibre G, each local homeomorphism χ_U being defined by the relation $\chi_U(b,g)(\phi_U(b,g')) = \phi'_U(b,gg')$.//

1.52 <u>COROLLARY</u> The subspace $(X\ X')_G$ is closed in $(X\ X')$.//

A similar construction can now be done for fibre bundles. Let $p\colon X \to B$ and $p'\colon X' \to B$ be principal G-bundles and let F be a left G-space. Then we can associate to the fibre bundles p^F and p'^F the functional bundle $(p^F\ p'^F)$ and observe that its local homeomorphisms are of the form

$$\chi_U^F\colon U\times \mathrm{Map}(F,F) \to (X\times_G F\ \ X'\times_G F)|U$$

and are defined by the relation

1.53 $$\chi_U^F(b,f)[\phi_U(b,e),y] = [\phi'_U(b,e),f(y)].$$

Define now a function

1.54 $$\Psi^F\colon (X\ X')_G \to (X\times_G F\ \ X'\times_G F)$$

by setting $\Psi^F(\alpha)[x,y] = [\alpha(x),y]$

1.55 <u>LEMMA</u> The function Ψ^F is well defined and continuous.

<u>PROOF</u> For each $U\in\{U\}$ the diagram

$$\begin{array}{ccc}
(X\ X')_G|U & \xrightarrow{\Psi^F|U} & (X\times_G F\ \ X'\times_G F)|U \\
\chi_U \uparrow & & \uparrow \chi_U^F \\
U\times G & \xrightarrow{1\times\lambda^*} & U\times \mathrm{Map}(F,F)
\end{array}$$

is commutative, since for any $(b,g)\in U\times G$ and $y\in F$
$\Psi^F(\chi_U(b,g))[\phi_U(b,e),y] = [\phi'_U(b,g),y] = [\ \phi'_U(b,e),gy] =$

$= \chi_U^F(b,g^*)[\phi_U(b,e),y]$. This is sufficient to prove the claim.//

The diagram used in the last proof also illustrates the fact that an element $\chi_U^F(b,f)$ is in the image of Ψ^F if and only if f is an action map. We shall denote such image by $(X\times_G F \;\; X'\times_G F)_\lambda$, since locally it consists of action maps.

1.56 <u>**PROPOSITION**</u> The restriction $(p^F \; p'^F)_\lambda$ of $(p^F \; p'^F)$ to $(X\times_G F \;\; X'\times_G F)_\lambda$ is a locally trivial map with fibre $M_\lambda(F,F)$.//

The maps $(p \; p')_G$ and $(p^F \; p'^F)_\lambda$ will play a fundamental role in the following chapters.

CHAPTER 2

HOMOMORPHISMS BETWEEN GROUPS OF BUNDLE AUTOMORPHISMS

§ 1) AUTOMORPHISMS OF PRINCIPAL G-BUNDLES

Given an object $p: X \to B$ of P_B^G we can consider, with the usual categorical notation, the set $P_B^G(p,p)$ of all endomorphisms of p in P_B^G. Since, by theorem 1.20, P_B^G is a groupoid, $P_B^G(p,p)$ is a group under map composition; for simplicity we shall denote it by $\mathcal{G}(p)$ and its elements will be called the *automorphisms* of p, so that $\mathcal{G}(p)$ will be referred to as the *group of automorphisms* of p. We can also give a topology to $\mathcal{G}(p)$ by regarding it as a subspace of $\mathrm{Map}(X,X)$.

2.1 LEMMA $\mathcal{G}(p)$ is a closed subspace of $\mathrm{Map}(X,X)$.

PROOF We know already that $M(p,p)$ and $M_G(X,X)$ are closed in $\mathrm{Map}(X,X)$, so we only need to notice that $\mathcal{G}(p) = M(p,p) \cap M_G(X,X)$.//

2.2 PROPOSITION $\mathcal{G}(p)$ is a topological group.

PROOF Since we are in a convenient category, composition of maps is a continuous operation. To prove that the function

$$i: \mathcal{G}(p) \to \mathcal{G}(p): f \rightsquigarrow f^{-1}$$

is continuous we use the fact that X is a principal

G-space and that for each $f \in \mathcal{G}(p)$ and $x \in X$ we have

$$x = f(x)\,\tau(f(x),x)$$

so that $f^{-1}(x) = x\,\tau(f(x),x)$. This means that the map

$$i' \colon X \times \mathcal{G}(p) \to X \colon (x,f) \rightsquigarrow x\,\tau(f(x),x) = f^{-1}(x)$$

is continuous, as a composite of continuous functions. A simple application of the exponential law can now complete the proof.//

It is easy to verify that if p and p' are isomorphic objects of P_B^G then $\mathcal{G}(p)$ and $\mathcal{G}(p')$ are isomorphic topological groups. Also it is easily seen that if p is trivial then $\mathcal{G}(p)$ is homeomorphic to the space $\mathrm{Map}(B,G)$. The same result holds when G is abelian ([J] page 47); we shall obtain both results as particular cases of a more general theorem. The situation is more complicated in general and, as we shall see in chapter 4, there are cases in which $\mathcal{G}(p)$ and $\mathrm{Map}(B,G)$ even have different homotopy groups in certain dimensions.

The following result shows that when the space B is not path connected the problem of studying $\mathcal{G}(p)$ reduces to the study of the groups $\mathcal{G}(p_j)$ where the p_j's are the restrictions of p over the various path components of B. Thus from now on we can assume, without loss of generality, that B is path connected.

2.3 **THEOREM** Let $p: X \to B$ be a principal G-bundle and assume that $B = \bigcup_{j\in J} B_j$, where each B_j is one path component of B. Let $p_j: X_j \to B_j$ be the restriction of p over B_j. Then $\mathcal{G}(p)$ and $\prod_{j\in J} \mathcal{G}(p_j)$ are isomorphic topological groups.

PROOF By proposition 1.23 each p_j is a principal G-bundle and it is clear that the function

$$\Pi: \mathcal{G}(p) \to \prod_{j\in J} \mathcal{G}(p_j): f \rightsquigarrow \{f|X_j\} = \{f_j\}$$

is a well defined group isomorphism. Its continuity follows, using an exponential law, from the continuity of the maps

$$e_i: \mathcal{G}(p)\times X_i \to X_i: (f,x) \rightsquigarrow f(x)$$

To check the continuity of Π^{-1} notice that each X_i, being the inverse image of an open set of B, is open in X. Hence the continuity of the composite

$$\prod_{j\in J} \mathcal{G}(p_j)\times X_i \to \mathcal{G}(p_i)\times X_i \to X_i: (\{f_j\},x) \rightsquigarrow f_i(x)$$

for each $i\in J$, proves the continuity of the function

$$e: \prod_{j\in J} \mathcal{G}(p_j)\times X \to X: (\{f_j\},x) \rightsquigarrow f_i(x)$$

where $x\in X_i$. This, again together with an exponential law, suffices to justify the claim.//

§ 2) SOME SUBGROUPS OF $\mathcal{G}(p)$

Let H be a subgroup of G and define a subset $\mathcal{G}(p;H)$ of $\mathcal{G}(p)$ by setting

$$\mathcal{G}(p;H) = \{f \in \mathcal{G}(p) \mid (\forall x \in X) \quad \tau(x, f(x)) \in H\}$$

In other words, while every element of $\mathcal{G}(p)$ is required to keep every point of X within its G-orbit, an element of $\mathcal{G}(p;H)$ is required to keep any point within its H-orbit. So, for instance, $\mathcal{G}(p;\{e\}) = 1_X$, $\mathcal{G}(p;G) = \mathcal{G}(p)$ and if $H \subset K \subset G$ then $\mathcal{G}(p;H) \subset \mathcal{G}(p;K) \subset \mathcal{G}(p)$. Therefore a filtration

$$\{e\} \subset H_1 \subset H_2 \subset \ldots \subset H_n \subset G$$

gives rise to a filtration

$$1_X \subset \mathcal{G}(p;H_1) \subset \mathcal{G}(p;H_2) \subset \ldots \subset \mathcal{G}(p;H_n) \subset \mathcal{G}(p)$$

2.4 DEFINITION Let H be a subgroup of the group G. The *normal core* of H in G, denoted by C(H,G), is the subgroup of G defined by

$$C(H,G) = \bigcap_{g \in G} g^{-1} H g$$

It is easy to see that C(H,G) is a normal subgroup of G, indeed it is the largest normal subgroup of G contained in H (compare, for example, [St] page 42).

2.5 <u>PROPOSITION</u> Let $p\colon X \to B$ be a principal G-bundle. Then for any subgroup H of G, $\mathcal{G}(p;H)$ and $\mathcal{G}(p;C(H,G))$ coincide.

<u>PROOF</u> Since $C(H,G) \subset H$, $\mathcal{G}(p;C(H,G)) \subset \mathcal{G}(p;H)$. On the other hand, if $f \in \mathcal{G}(p;H)$, $x \in X$ and $\tau(x,f(x)) = h$, for each $g \in G$ there exists $h' \in H$ such that $f(xg) = xgh'$. But $f(xg) = f(x)g = xhg$ so that $gh' = hg$, and hence $h \in gHg^{-1}$. Given the arbitrary choice of g and x, this proves that $f \in \mathcal{G}(p;C(H,G))$, as we wanted.//

2.6 <u>THEOREM</u> For any subgroup H of G, $\mathcal{G}(p;H)$ is a normal subgroup of $\mathcal{G}(p)$ and if H is closed in G, then $\mathcal{G}(p;H)$ is closed in $\mathcal{G}(p)$.

<u>PROOF</u> First of all, if H is closed in G, the subspace of Map(X,X) consisting of maps which move each point of X within its H-orbit is closed in Map(X,X), hence $\mathcal{G}(p;H)$, which is its intersection with $\mathcal{G}(p)$, is closed in $\mathcal{G}(p)$. $\mathcal{G}(p;H)$ is certainly a subgroup of $\mathcal{G}(p)$, so we only have to check its normality. To that end let $f \in \mathcal{G}(p)$, $\phi \in \mathcal{G}(p;H)$, $x \in X$ be arbitrary and suppose that $f(x) = xg$ and $\phi(xg) = xgh$ for some $g \in G$ and $h \in H$. Then

$$f^{-1}\cdot\phi\cdot f(x) = f^{-1}\cdot\phi(xg) = f^{-1}(xgh) = xg^{-1}gh = xh$$

which means that $\tau(x,f^{-1}\cdot\phi\cdot f(x)) \in H$. By the arbitrary choice of f, ϕ and x it follows that $\mathcal{G}(p;H)$ is normal in $\mathcal{G}(p)$.//

2.7 LEMMA If H is abelian then so is $\mathcal{G}(p;H)$.

PROOF Let $f, f' \in \mathcal{G}(p;H)$ and $x \in X$. Then $f(x) = xh$ and $f'(x) = xh'$ for some $h, h' \in H$ and we have

$$f \cdot f'(x) = f(xh') = xhh'$$
$$f' \cdot f(x) = f'(xh) = xh'h$$

This, by the hypothesis on H, proves the claim.//

§ 3) RELATIONS BETWEEN $\mathcal{G}(p)$ AND $(p\ p)_G$

We shall now reduce the study of $\mathcal{G}(p)$ to that of the space of sections to the functional bundle $(p\ p)_G$ and the study of $\mathcal{G}(p;H)$ to that of the space of sections to a suitable restriction $(p\ p)_{G;H}$ of $(p\ p)_G$. This is useful because it identifies the group in question with spaces of a relatively familiar type and this opens up new avenues for their study.

Let $p: X \to B$ be a principal G-bundle. Then for each subgroup H of G we can define a subspace $(X\ X)_{G;H}$ of $(X\ X)_G$ by considering only maps which keep each point in their domain within its H-orbit. Recall that if $\{(U,\phi_U)\}$ is the locally trivial structure for p, then the associated local homemorphisms for $(p\ p)_G$, as defined in proposition 1.51, are denoted by $\{\chi_U\}$.

2.8 PROPOSITION The restriction $(p\ p)_{G;H}$ of $(p\ p)_G$ to $(X\ X)_{G;H}$ has the locally trivial structure $\{(U,\chi_{U;H})\}$, where $\chi_{U;H}$ is the restriction of χ_U to $U\times C(H,G)$. Moreover if H is closed in G, then $(X\ X)_{G;H}$ is closed in $(X\ X)_G$.

PROOF If H is closed in G, C(H,G), as intersection of closed subspaces, is closed in G, so it will be sufficient to prove the first part of the proposition. To that end notice that for any $(b,g)\in U\times G$ and $g'\in G$

$$\chi_U(b,g)(\phi_U(b,g')) = \phi_U(b,gg')$$

So if $\chi_U(b,g)\in(X\ X)_{G;H}$, for any $g'\in G$ there is an $h\in H$ such that $gg' = g'h$ and this implies that $g\in C(H,G)$ and hence that $(X\ X)_{G;H}|U \subset \chi_U(U\times C(H,G))$. Vice versa if $g\in C(H,G)$, for any $g'\in G$ $g = g'h(g')$ for some $h\in H$. Therefore $gg' = g'h$ and $\chi_U(b,g)\in(X\ X)_{G;H}$. This completes the proof.//

2.9 THEOREM For any subgroup H of G there is a homeomorphism $\Phi\colon \mathcal{G}(p;H) \to \sec(p\ p)_{G;H}$ defined by the relation $\Phi(f)(b) = f|p^{-1}(b)$.

PROOF The homeomorphism described here is obtained by suitably restricting the homeomorphism

$$\Phi : M(p,p) \to \sec(p\ p)$$

of theorem 1.48.//

2.10 COROLLARY There is a homeomorphism

$$\Phi : \mathcal{G}(p) \to \sec(p\ p)_G$$

defined by setting $\Phi(f)(b) = f|p^{-1}(b)$.

PROOF Take $H = G$ in theorem 2.9.//

Clearly corollary 2.10 can be proved independently from theorem 2.9, by a direct application of theorem 1.48.

As a first application we prove the already mentioned generalization of the homeomorphism which occurs in some particular cases between $\mathcal{G}(p)$ and $Map(B,G)$.

2.11 THEOREM Let $p: X \to B$ be a principal G-bundle, with respect to the action $\rho: X\times G \to X$, and let H be a normal subgroup of G. If there is a left action $\lambda: H\times X \to X$ such that

a) X is a principal H-space with respect to λ;

b) For any $x\in X$, $g\in G$ and $h\in H$

$$\lambda(h,\rho(x,g)) = \rho(\lambda(h,x),g)$$

or, with our usual notation, $h(xg) = (hx)g$;

c) Two points of X are in the same H-orbit with respect to ρ if and only if they are in the same H-orbit with respect to λ;

then $\mathcal{G}(p;H)$ is homeomorphic to Map(B,H).

PROOF Define a left action η of H on $(X\ X)_{G;H}$ by setting

$$\eta\colon H\times(X\ X)_{G;H} \to (X\ X)_{G;H}\colon (h,\alpha) \rightsquigarrow h\alpha$$

where $h\alpha$ has the same domain as α and is defined by $(h\alpha)(x) = h(\alpha(x))$. To prove that η is well defined first notice that using condition b) we have, for any $g\in G$,

$$(h\alpha)(xg) = h(\alpha(xg)) = h(\alpha(x)g) = (h\alpha(x))g = (h\alpha)(x)g$$

so that $h\alpha$ is a G-map. Moreover if $\alpha(x) = xh'$, where $h'\in H$, denoting by τ the translation function for ρ we have

$$h\alpha(x) = hxh' = x\tau(x,hx)h'$$

Now condition c) implies that $\tau(x,hx)$ exists and is in H, so that $h\alpha\in(X\ X)_{G;H}$.

To prove the continuity of η notice that for each $U\in\{U\}$, $b\in U$, $h,h'\in H$ and $g\in G$

$$h\chi_{U;H}(b,h')(\phi_U(b,g)) = (h\chi_{U;H}(b,h'))(\phi_U(b,e))g =$$

$$= (h\phi_U(b,h'))g = \phi_U(b,h')h''g = \phi_U(b,h'h''g) =$$

$$= \chi_U(b,h'h'')(\phi_U(b,g))$$

where $h'' = \tau(\phi_U(b,h'),h\phi_U(b,h'))$ also exists and is in H by condition c). This means that locally η becomes

$\eta|U\colon H\times U\times H \to U\times H\colon (h,b,h') \rightsquigarrow (b,h'\tau(\phi_U(b,h'),h\phi_U(b,h')))$

which is continuous, being the composite of continuous functions, and therefore η itself is continuous.
It is now easy to see that, because of condition a), $(X\ X)_{G;H}$ is a free left H-space. Moreover for any $U\in\{U\}$, $b\in U$, $h,h'\in H$ the following relation is valid:

$$\tau'(\phi_U(b,h),\phi_U(b,h'))\chi_U(b,h) = \chi_U(b,h')$$

where τ' is the translation function for λ. Thus the continuity of τ',again assured by condition a), implies that the translation function for η is locally, and hence globally, continuous. Therefore $(X\ X)_{G;H}$ is a principal left H-space and $(p\ p)_{G;H}$ is, up to homeomorphism, the projection onto its orbit space. Since $(p\ p)_{G;H}$ has a section, determined by the identity automorphism of p, it follows, by [Hus] cor.8.3 page 48, that there is a homeomorphism over B between $(X\ X)_{G;H}$ and $H\times B$ and hence that $\sec(p\ p)_{G;H}$ and Map(B,H) are homeomorphic. Theorem 2.9 can now be used to complete the proof.//

We remark that if H is not normal one can apply the theorem to C(H,G) and obtain conditions ensuring that $\mathcal{G}(p;H)$ and Map(B,C(H,G)) are homeomorphic.

2.12 <u>COROLLARY</u> Let $p: X \to B$ be a principal G-bundle and let H be a normal topological subgroup of G. If one of the following conditions is satisfied then $\mathcal{G}(p;H)$ and $Map(B,H)$ are homeomorphic:

a) H is contained in the centre of G;

b) p is trivial;

c) X is a topological group having G and H as normal subgroups and the action is given by group multiplication.

<u>PROOF</u> In these cases we can define left actions of H on X by, respectively,

a) $\eta(h,x) = xh$;

b) $\eta(h,(b,g)) = (b,hg)$;

c) $\eta(h,x) = hx$.

It is easy to see that each of these actions satisfies the conditions of theorem 2.11.//

2.13 <u>COROLLARY</u> Let $p: X \to B$ be a principal G-bundle. If one of the following conditions is satisfied then $\mathcal{G}(p)$ is homeomorphic to $Map(B,G)$:

a) G is abelian;

b) p is trivial;

c) X is a topological group having G as a normal subgroup.//

Once again we remark that cases a) and b) of corollary 2.13 had been analyzed in [J], while case c) was proved by a different method in [Mo], page 95.

2.14 <u>COROLLARY</u> Let $p: X \to B$ be a principal G-bundle and H a normal subgroup of G. If p and H satisfy the conditions of theorem 2.11, then for any map $f: A \to B$ the space Map(A,H) is homeomorphic to $\mathcal{G}(p_f;H)$, where $p_f: A \sqcap X \to A$ is the induced principal G-bundle.

<u>PROOF</u> The principal left action of H on X induces a principal left action of H on $A \sqcap X$ which also satisfies the conditions of the theorem.//

REMARK: In view of the result of corollary 2.14, one may be led to analyze, for a given topological group G, the possibility of the existence of a left action of the type described in theorem 2.11 on the total space of Milnor's universal G-bundle (see Chapter 4 or [Mi2]). It is easy to verify, however, that such an action exists only under the same general assumptions of corollary 2.12 and cannot be constructed in general.

§ 4) AUTOMORPHISMS OF FIBRE BUNDLES

Let $p: X \to B$ be a principal G-bundle and F a left G-space. Then the fibre bundle $p^F: X\times_G F \to B$ is an object of the category $\mathcal{F}_B^F(G)$, so we can consider the set $\mathcal{F}_B^F(G)(p^F,p^F)$ of all endomorphisms of p^F in $\mathcal{F}_B^F(G)$. Because of theorem 1.20 and definition 1.29, $\mathcal{F}_B^F(G)$ is a groupoid and hence $\mathcal{F}_B^F(G)(p^F,p^F)$ is a group with respect to map composition. In analogy with the definitions of section 1), we shall denote this group by $\mathfrak{F}(p^F)$, call it the *group of automorhisms* of p^F and call each of its elements an *automorphism* of p^F. We can give a topology to $\mathfrak{F}(p^F)$ by viewing it as a subspace of the space $Map(X\times_G F, X\times_G F)$ and we shall see later that in many favourable cases this topology makes $\mathfrak{F}(p^F)$ into a topological group.

It is clear that if (q,h) is any other fibre bundle associated to p and with fibre F, the group of automorphisms of (q,h) is isomorphic as a group and homeomorphic as a space to $\mathfrak{F}(p^F)$ and hence we shall restrict our attention to fibre bundles of the form p^F.

Still in analogy with what has been done for principal G-bundles, $\mathfrak{F}(p^F)$ is easily shown to be homeomorphic to a space of sections of a certain functional bundle. Recall that, if λ denotes the action of G on F, there is a map $\Psi^F\colon (X\ \ X)_G \to (X\times_G F\ \ X\times_G F)_\lambda$ over B (see 1.54) and this induces a map

2.15 $$\Psi^F_p\colon \sec(p\ \ p)_G \to \sec(p^F\ \ p^F)_\lambda : s \leadsto \Psi^F\cdot s$$

2.16 <u>THEOREM</u> There is a homeomorphism

$$\Phi^F\colon \quad \mathfrak{F}(p^F) \to \sec_p(p^F\ \ p^F)_\lambda$$

where $\sec_p(p^F\ \ p^F)_\lambda$ is the image of Ψ^F_p and Φ^F is defined by the relation $\Phi^F(f^F)(b) = f^F|\{b\}$.

<u>PROOF</u> Since $(X\times_G F\ \ X\times_G F)_\lambda$ is a subspace of the space $(X\times_G F\ \ X\times_G F)$, $\sec_p(p^F\ \ p^F)_\lambda$ is a subspace of $\sec(p^F\ \ p^F)$, so we only have to check that the image of $\mathfrak{F}(p^F)$ under the homeomorphism $\Phi\colon M(p^F,p^F) \to \sec(p^F\ \ p^F)$ of theorem 1.48 is $\sec_p(p^F\ \ p^F)_\lambda$. But this is an easy consequence of the definitions of a fibre bundle morphism and of the map Ψ^F.//

We notice that the homeomorphism defined in theorem 2.16 will be a fundamental tool in the rest of the thesis. Later on we shall give examples in which the map Ψ^F_p is not surjective and conditions which ensure that it is surjective. Now we are going to use theorem 2.16 to give a condition for $\mathfrak{F}(p^F)$ to be a topological group.

2.17 <u>PROPOSITION</u> Let $p\colon X \to B$ be a principal G-bundle and F a left G-space with respect to the action λ. If $M_\lambda(F,F)$ is a topological group then so is $\mathfrak{J}(p^F)$.

<u>PROOF</u> Since we are in a convenient category, map composition is a continuous operation, so the only problem is to show that the map

$$i\colon \mathfrak{J}(p^F) \to \mathfrak{J}(p^F)\colon f^F \rightsquigarrow (f^F)^{-1}$$

is continuous. To that end notice that the function

$$i'\colon (X\times_G F \quad X\times_G F)_\lambda \to (X\times_G F \quad X\times_G F)_\lambda\colon \alpha \rightsquigarrow \alpha^{-1}$$

is continuous, since locally it is just the product of the identity on an open set of B with the map

$$i_\lambda\colon M_\lambda(F,F) \to M_\lambda(F,F)\colon g^* \rightsquigarrow (g^*)^{-1}$$

which, by hypothesis, is continuous. This implies that the function

$$i''\colon \sec_p(p^F \quad p^F)_\lambda \to \sec_p(p^F \quad p^F)_\lambda\colon s \rightsquigarrow i'\cdot s$$

is continuous and, by theorem 2.16, this suffices to complete the proof.//

2.18 <u>COROLLARY</u> Let $p\colon X \to B$ be a principal G-bundle and F a left G-space with respect to the action λ. If one of the following conditions is satisfied, then $\mathfrak{J}(p^F)$ is a topological group:

a) F is admissible (see 1.5);

b) F is a topological group and $\lambda = \lambda(h)$ is induced by a topological group morphism $h: G \to F$.

<u>PROOF</u> In case a) $M_\lambda(F,F)$ is a quotient group of G, while in case b) $M_\lambda(F,F)$ is, up to homeomorphism, a subgroup and a subspace of F. So in both cases $M_\lambda(F,F)$ is a topological group.//

§ 5) HOMOMORPHISMS INDUCED BY THE FIBRE BUNDLE CONSTRUCTION

Let G be a topological group and F a left G-space. Then the functor $\Gamma: \mathcal{P}_B^G \to \mathcal{F}_B^F(G)$ defined in proposition 1.30 determines, for each principal G-bundle $p: X \to B$, by restriction, a homomorphism

2.19 $$\Gamma_p^F: \mathcal{G}(p) \to \mathfrak{F}(p^F): f \rightsquigarrow f^F .$$

2.20 <u>LEMMA</u> The diagram

$$\begin{array}{ccc} \mathcal{G}(p) & \xrightarrow{\ \Gamma_p^F\ } & \mathfrak{F}(p^F) \\ \Phi \downarrow & & \downarrow \Phi^F \\ \sec(p\ p)_G & \xrightarrow{\ \Psi_p^F\ } & \sec_p(p^F\ p^F)_\lambda \end{array}$$

is commutative and therefore the homomorphism Γ_p^F is continuous.

PROOF The result is an immediate consequence of the definition of the maps which form the diagram.//

2.21 THEOREM Let $p: X \to B$ be a principal G-bundle and F a left G-space with respect to the action λ. Then $\Gamma_p^F: \mathcal{G}(p) \to \mathcal{F}(p^F)$ is a continuous epimorphism with kernel $\mathcal{G}(p;Kr_\lambda)$.

PROOF Γ_p^F is a group epimorphism by definition and is continuous by lemma 2.20. To determine its kernel, let $f \in Ker(\Gamma_p^F)$, $x \in X$ and $f(x) = xg$. Then, for any $y \in F$, we have

$$\Gamma_p^F(f)[x,y] = [f(x),y] = [x,y]$$

and this means that (xg,y) and (x,y) are in the same orbit and hence that $y = gy$. Since x and y are arbitrary, it follows that $Ker(\Gamma_p^F) \subset \mathcal{G}(p;Kr_\lambda)$. Conversely, if $f \in \mathcal{G}(p;Kr_\lambda)$, for each $[x,y] \in X \times_G F$ there is an $h \in Kr_\lambda$ so that

$$\Gamma_p^F(f)[x,y] = [f(x),y] = [xh,y] = [x,hy] = [x,y]$$

and hence $\mathcal{G}(p;Kr_\lambda) \subset Ker(\Gamma_p^F)$.//

The kernel of any action of a topological group G is always a normal topological subgroup of G and, vice-versa, for any normal topological subgroup H of G, the quotient group G/H is a G-space (corollary 1.11), the kernel of the corresponding action being H. This gives us a canonical way of choosing a fibre bundle associated to a given principal G-bundle and to a particular kernel. Therefore, in view of theorem 2.21, the problem of studying the groups $\mathfrak{J}(p^F)$ can be reduced to the study of examples belonging to a particular family, as described in the following corollary.

2.22 <u>COROLLARY</u> Let $p\colon X \to B$ be a principal G-bundle, F a left G-space, Kr_F the kernel of the action of G on F and $\pi\colon G \to Q_F$ the canonical projection. Then $\mathfrak{J}(p^F)$ is isomorphic, as a group, to $\mathfrak{J}(p^\pi)$.//

We shall now look for conditions which will make the classification provided by corollary 2.22 also valid topologically. To that end let F and F' be left G-spaces, the actions being λ and λ' respectively, and suppose that $Kr_\lambda = Kr_{\lambda'}$. There is then a unique bijection $t\colon M_\lambda(F,F) \to M_{\lambda'}(F',F')$ such that the diagram

2.23
$$\begin{array}{ccc} G & \xrightarrow{\lambda^*} & M_\lambda(F,F) \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle t} \\ G & \xrightarrow{\lambda'^*} & M_{\lambda'}(F',F') \end{array}$$

is commutative.

2.24 <u>LEMMA</u> If the function $t\colon M_\lambda(F,F) \to M_{\lambda'}(F',F')$ is continuous, for each principal G-bundle $p\colon X \to B$ there is a continuous bijection $t_*\colon (X\times_G F \;\; X\times_G F)_\lambda \to$ $\to (X\times_G F' \;\; X\times_G F')_{\lambda'}$ making the diagram

$$\begin{array}{ccc} (X \;\; X)_G & \xrightarrow{\Psi^F} & (X\times_G F \;\; X\times_G F)_\lambda \\ {\scriptstyle 1}\downarrow & & \downarrow {\scriptstyle t_*} \\ (X \;\; X)_G & \xrightarrow{\Psi^{F'}} & (X\times_G F' \;\; X\times_G F')_{\lambda'} \end{array}$$

commutative. In particular if t is a homeomorphism, then so is t_*.//

<u>PROOF</u> In order for the given diagram to be commutative t_* has to be defined by the relation

$$t_*(\Psi^F(\alpha)) = \Psi^{F'}(\alpha)$$

and by the hypothesis on the kernels this can be done. Then locally the diagram becomes

$$\begin{array}{ccc} U\times G & \xrightarrow{1\times\lambda^*} & U\times M_\lambda(F,F) \\ {\scriptstyle 1}\downarrow & & \downarrow {\scriptstyle 1\times t} \\ U\times G & \xrightarrow{1\times\lambda'^*} & U\times M_{\lambda'}(F',F') \end{array}$$

thus showing the continuity of t_*. If t is a homeomorphism, by reversing the given argument we can prove that t_* is also a homeomorphism.//

2.25 <u>COROLLARY</u> Given any two admissible left G-spaces such that the kernels of the respective actions coincide, for any principal G-bundle $p\colon X \to B$ the map

$t_*: (X\times_G F \quad X\times_G F)_\lambda \to (X\times_G F' \quad X\times_G F')_{\lambda'}$, is a homeomorphism.

PROOF In this case λ^* and λ'^* are identifications and therefore t is a homeomorphism.//

Recall, as an example, that if G is compact, any G-space is admissible.

The last two results are useful because they enable us to derive analogous statements for the groups of automorphisms; in particular our next result is a kind of uniqueness theorem for $\mathfrak{F}(p^F)$, showing that it depends only on p and the kernel of the action.

2.26 THEOREM Let $p: X \to B$ be a principal G-bundle and assume that F and F' are left G-spaces such that $Kr_F = Kr_{F'}$. Then there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{G}(p) & \xrightarrow{\Gamma_p^F} & \mathfrak{F}(p^F) \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle t_{**}} \\ \mathfrak{G}(p) & \xrightarrow{\Gamma_p^{F'}} & \mathfrak{F}(p^{F'}) \end{array}$$

where t_{**} is a bijection which is continuous (resp., a homeomorphism) if $t: M_\lambda(F,F) \to M_{\lambda'}(F',F')$ is continuous (resp., a homeomorphism).

PROOF Under the given hypothesis, the map t_* of lemma 2.24 is a continuous bijection over B, thus inducing a map $t_*': \sec(p^F\ p^F)_\lambda \to \sec(p^{F'}\ p^{F'})_\lambda$, such that $t_*' \cdot \Psi_p^F = \Psi_p^{F'}$. Now, restricting the map t_*' to the space $\sec_p(p^F\ p^F)_\lambda$ and using theorem 2.16, we obtain the desired map t_{**}. If t is a homeomorphism, one can use the same argument, applied to t^{-1}, to prove that t_{**} is a homeomorphism.//

2.27 <u>COROLLARY</u> Let $p: X \to B$ be a principal G-bundle and F a left G-space. Then there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{G}(p) & \xrightarrow{\Gamma_p^\pi} & \mathfrak{J}(p^\pi) \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle t_{**}} \\ \mathfrak{G}(p) & \xrightarrow{\Gamma_p^F} & \mathfrak{J}(p^F) \end{array}$$

where $\pi: G \to Q_F$ is the canonical projection and t_{**} is a continuous bijection or, if F is admissible, a homeomorphism.

<u>PROOF</u> This follows from corollary 1.11 and theorem 2.26.//

REMARK: For a given fibre bundle $p^F: X \times_G F \to B$, the group $\mathfrak{J}(p^F)$ depends on p in an essential way. For example, by proposition 1.43 (or [Hus] th.3.1 page 72), we can see that if the principal G-bundle $p: X \to B$ associated to p^F has a restriction $q: Y \to B$ to Kr_F, then $X \times_G F$ and $Y \times_{Kr_F} F$ are homeomorphic over B. However, if $Kr_F \neq G$,

$\mathfrak{J}(q^F) \sim \mathcal{G}(q)/\mathcal{G}(q;Kr_F) = 0$, but $\mathfrak{J}(p^F) = \mathcal{G}(p)/\mathcal{G}(p;Kr_F)$ is not trivial in general.

§ 6) HOMOMORPHISMS INDUCED BY A TOPOLOGICAL GROUP MORPHISM

Given a principal G-bundle $p: X \to B$ and a topological group morphism $h: G \to K$, we can obtain, by restricting the functor $\Pi^h: P_B^G \to P_B^K$ described in theorem 1.34, a group homomorphism

2.28 $$\Pi_p^h: \mathcal{G}(p) \to \mathcal{G}(p^h): f \rightsquigarrow f^h$$

Since for each $f \in \mathcal{G}(p)$, $\Pi_p^h(f) = \Gamma_p^h(f)$, we can expect to obtain for Π_p^h many of the results which are valid for Γ_p^h. The following proposition clarifies this relation.

2.29 PROPOSITION Let $p: X \to B$ be a principal G-bundle and $h: G \to K$ a topological group morphism. Then $(X\times_h K \;\; X\times_h K)_{\lambda(h)}$ is a subset of $(X\times_h K \;\; X\times_h K)_K$, (closed if $h(G)$ is closed in K) contains $(X\times_h K \;\; X\times_h K)_{K;h(G)}$ and coincides with it if $h(G)$ is normal in K.

PROOF We only need to notice that locally $(X\times_h K \;\; X\times_h K)_{\lambda(h)}$ corresponds to $U\times M_{\lambda(h)}(K,K) \cong U\times h(G)$, by proposition 1.10, while $(X \times_h K \;\; X \times_h K)_K$ corresponds to $U \times K$ and $(X \times_h K \;\; X \times_h K)_{K;h(G)}$ corresponds to $U \times C(h(G);K)$.//

We can now use the same tecniques as in section 5) to obtain:

2.30 <u>THEOREM</u> The function $\Pi_p^h \colon \mathfrak{G}(p) \to \mathfrak{G}(p^h)$ is a continuous homomorphism with kernel $\mathfrak{G}(p;\mathrm{Ker}(h))$ and image $\mathfrak{J}(p^h)$. Moreover if $h(G)$ is normal in K, then $\mathfrak{J}(p^h) \subset \mathfrak{G}(p^h;h(G))$.

<u>PROOF</u> The continuity of Π_p^h is ensured by the continuity of $\Psi^h \colon (X \; X)_G \to (X \times_h K \;\; X \times_h K)$. The fact that its image is $\mathfrak{J}(p^h)$ is obvious. The last statement is obtained using proposition 2.29 and the fact that an element of $\mathfrak{J}(p^h)$ corresponds to a section of $(p^h \; p^h)_{\lambda(h)}$ and an element of $\mathfrak{G}(p^h;h(G))$ corresponds to a section of $(p^h \; p^h)_{K;h(G)}$.//

So when $h(G)$ is normal in K, $\mathfrak{G}(p^h;h(G))$ corresponds to the space of all sections of $(p^h \; p^h)_{\lambda(h)}$ while $\mathfrak{J}(p^h)$ consists of those sections which can be lifted to a section of $(p \; p)_G$. It is therefore natural to look for conditions under which these two spaces coincide. The following proposition, which is proved without assuming that h(G) is normal in K, will give us a first answer.

2.31 PROPOSITION If $\iota: G \to K$ is an inclusion, then $\mathfrak{G}(p^\iota;G) \subset \mathfrak{F}(p^\iota)$.

PROOF Working locally it is easy to see that, under the given hypothesis, the map $\Psi^\iota: (X\ X)_G \to (X\times_\iota K\ \ X\times_\iota K)_{\lambda(\iota)}$ is a homeomorphism. Since $(X\times_\iota K\ \ X\times_\iota K)_{K;G}$ is contained in $(X\times_\iota K\ \ X\times_\iota K)_{\lambda(\iota)}$, every element of $\mathfrak{G}(p^\iota;G)$ determines an element of $\sec(p^\iota\ p^\iota)_{K;G}$ and hence of $\sec(p^\iota\ p^\iota)_{\lambda(\iota)}$. The fact that Ψ^ι is a homeomorphism suffices to complete the proof.//

2.32 COROLLARY Let $p: X \to B$ be a principal G-bundle and $\iota: G \to K$ an inclusion of a normal subgroup. Then $\mathfrak{F}(p^\iota) = \mathfrak{G}(p^\iota;G)$.//

An immediate application of this result allows us to obtain, in a particular case, a stronger version of corollary 2.27 (cf. [Go2] prop. 6.1).

2.33 THEOREM Let $p: X \to B$ be a principal G-bundle and F an admissible and effective left G-space. Then $\mathfrak{F}(p^F)$ is homeomorphic to $\mathfrak{G}(p)$.

PROOF By corollary 2.27, $\mathfrak{F}(p^F) = \mathfrak{F}(p^\pi)$, but in this case π is the identity on G, so by corollary 2.32, $\mathfrak{J}(p^F) \cong \mathfrak{J}(p) = \mathcal{G}(p;G) = \mathcal{G}(p)$.//

The next result is central to our theory; it describes the way in which the enlargement of the structure group of a principal bundle has the effect of enlarging the corresponding group of automorphisms.

2.34 THEOREM Let $p: X \to B$ be a principal G-bundle and $\iota: G \to K$ an inclusion of a topological subgroup. Then $\Pi_p^\iota: \mathcal{G}(p) \to \mathcal{G}(p^\iota)$ is also an inclusion of a topological subgroup; moreover if G is normal in K, $\Pi_p^\iota(\mathcal{G}(p)) = \mathcal{G}(p^\iota;G)$.

PROOF Under the given hypothesis the map

$$\Psi^\iota: (X\ X)_G \to (X\times_\iota K\ \ X\times_\iota K)_K$$

is also an inclusion of a closed subspace and hence so is the induced map Ψ_p^ι: $\sec(p\ p)_G \to \sec(p^\iota\ p^\iota)_K$. The second part follows easily from the fact that K is an effective and admissible G-space (corollary 1.12), corollary 2.32 and theorem 2.33.//

Because of corollaries 2.22 and 2.27, the case in which the morphism h is the projection $\pi: G \to G/H$ of G onto one of its quotient groups will be basic to our study of the groups of automorphisms of fibre bundles. In this situation $X \times_{\pi} G/H$ is canonically homeomorphic to X/H and $(X/H \;\; X/H)_{\lambda(\pi)} = (X/H \;\; X/H)_{G/H}$. Hence from the previous results we get:

2.35 <u>THEOREM</u> Let H be a normal topological subgroup of G, $\pi\colon G \to G/H$ be the canonical projection and $p\colon X \to B$ a principal G-bundle. Then $\Pi_p^{\pi}\colon \mathfrak{G}(p) \to \mathfrak{G}(p^{\pi})$ is a continuous homomorphism with kernel $\mathfrak{G}(p;H)$ and image $\mathfrak{J}(p^{\pi})$.//

2.36 <u>COROLLARY</u> Let H be a normal topological subgroup of G, $\pi\colon G \to G/H$ be the projection and $p\colon X \to B$ be a principal G-bundle admitting a restriction to a principal H-bundle $q\colon Y \to B$. Then there is a continuous homomorphism $\Pi_*\colon \mathfrak{G}(p) \to \mathrm{Map}(B,G/H)$ with kernel isomorphic to $\mathfrak{G}(q)$.

<u>PROOF</u> In this case p^{π} is trivial and $\mathfrak{G}(p;H) = \mathfrak{G}(q)$.//

§ 7) THE GROUP OF AUTOMORPHISMS OF A WHITNEY SUM

Let $p: X \to B$ be a principal G-bundle and $p': X' \to B$ a principal G'-bundle. We shall now look for the relation existing among $\mathcal{G}(p)$, $\mathcal{G}(p')$ and $\mathcal{G}(p \sqcap p')$, where $p \sqcap p'$ denotes the Whitney sum of p and p' (definition 1.24).

First of all we notice that for any $f \in \mathcal{G}(p)$ and $f' \in \mathcal{G}(p')$, the map

$$f \sqcap f': X \sqcap X' \to X \sqcap X': (x,x') \rightsquigarrow (f(x), f'(x'))$$

is an element of $\mathcal{G}(p \sqcap p')$. With this in mind we can now prove the following result (cf. [BHMP1] th.5).

2.37 THEOREM The function

$$h: \mathcal{G}(p) \times \mathcal{G}(p') \to \mathcal{G}(p \sqcap p'): (f,f') \rightsquigarrow f \sqcap f'$$

is a topological group isomorphism.

PROOF It follows immediately from the definition that h is a well defined monomorphism of groups. To prove that it is surjective, let $g \in \mathcal{G}(p \sqcap p')$ and $p'_p: X \sqcap X' \to X$ be the projection. Consider now the function

$$f: X \to X: x \rightsquigarrow p'_p \cdot g(x,x')$$

where $x' \in X'$ is any point such that $(x,x') \in X \sqcap X'$. To check that f is well defined notice that, if (x,x') and (x,x'') are both in $X \sqcap X'$, then $x'' = x'g'$ for some

$g' \in G'$, so, if $g(x,x') = (y,y')$, we have

$$p'_p \cdot g(x,x'g') = p'_p \cdot g((x,x')(e,g')) = p'_p((y,y')(e,g')) =$$
$$= p'_p(y,y'g') = y = p'_p \cdot g(x,x') \quad .$$

The continuity of f follows from the fact that p'_p is an identification, being locally trivial (proposition 1.23). Moreover f is a G-map over B by definition and hence $f \in \mathcal{G}(p)$. In a similar way we can define a map $f' \in \mathcal{G}(p')$ such that $h(f,f') = g$, thus proving the surjectivity of h. The continuity of h follows easily from that of

$$ev \times ev\colon\ \mathcal{G}(p) \times X \times \mathcal{G}(p') \times X' \to X \sqcap X'\colon$$
$$\colon (f,x,f',x') \leadsto (f(x),f'(x'))$$

and an exponential law. To prove the continuity of h^{-1}, define a function over B:

$$h_1\colon (X \sqcap X' \quad X \sqcap X')_{G \times G'} \to (X \quad X)_G$$

by setting $h_1(\alpha)(x) = p'_p \cdot \alpha(x,x')$, where, as before, $(x,x') \in X \sqcap X'$. This function is well defined for the same reasons which make h^{-1} well defined. Moreover, for each U in the common locally trivial cover of p and p', the diagram

$$\begin{array}{ccc}
(X \sqcap X' \quad X \sqcap X')_{G \times G'}|U & \xrightarrow{\ h_1|U\ } & (X \quad X)_G|U \\
X_U \sqcap X'_U \downarrow & & \downarrow X_U \\
U \times G \times G' & \xrightarrow{\ pr\ } & U \times G
\end{array}$$

where pr denotes the projection, is commutative, thus proving the continuity of h_1. In the same way we can construct a map $h_2\colon (X\cap X' \;\; X\cap X')_{G\times G'} \to (X' \;\; X')_{G'}$ and hence a map

$$h_{12}\colon (X\cap X' \;\; X\cap X')_{G\times G'} \to (X \;\; X)_G \times (X' \;\; X')_{G'}\colon$$
$$\colon \alpha \rightsquigarrow (h_1(\alpha), h_2(\alpha))$$

over B. In turn h_{12} determines a map

$$h_* \colon \mathrm{Map}(B, (X\cap X' \;\; X\cap X')_{G\times G'}) \to$$
$$\to \quad \mathrm{Map}(B, (X \;\; X)_G) \times \mathrm{Map}(B, (X' \;\; X')_{G'})$$

whose restriction to the respective spaces of sections makes the following diagram commutative

$$\begin{array}{ccc} \sec(p\cap p' \;\; p\cap p')_{G\times G'} & \xrightarrow{h_*} & \sec(p \;\; p)_G \times \sec(p' \;\; p')_{G'} \\ \Phi \downarrow & & \downarrow \Phi \\ \mathcal{G}(p\cap p') & \xrightarrow{h^{-1}} & \mathcal{G}(p) \times \mathcal{G}(p') \end{array}$$

and therefore proves the continuity of h^{-1}.//

2.38 <u>COROLLARY</u> Let $p\colon X \to B$ be a principal $G\times G'$-bundle. Then $\mathcal{G}(p)$ is isomorphic, as a topological group, to $\mathcal{G}(p^{\pi}) \times \mathcal{G}(p^{\pi'})$, where $\pi\colon G\times G' \to G$ and $\pi'\colon G\times G' \to G'$ are the projections.

<u>PROOF</u> It follows from proposition 1.41.//

2.39 <u>COROLLARY</u> If $p\colon X \to B$ is a principal $G\times G'$-bundle, then $\mathcal{G}(p)$ is isomorphic, as a topological group, to $\mathcal{G}(p;G) \times \mathcal{G}(p;G')$.

PROOF It is easy to verify that, under the isomorphism of corollary 2.38, $\mathcal{G}(p^{\pi})$ corresponds to $\mathcal{G}(p;G')$ and $\mathcal{G}(p^{\pi'})$ to $\mathcal{G}(p;G)$.//

2.40 COROLLARY Let $p: X \to B$ be a principal G-bundle and assume that G is isomorphic, as a topological group, to $G_1 \times G_2 \times \ldots \times G_n$. Then

$$\begin{aligned}\mathcal{G}(p) &\cong \mathcal{G}(p^{\pi_1}) \times \mathcal{G}(p^{\pi_2}) \times \ldots \times \mathcal{G}(p^{\pi_n}) \\ &\cong \mathcal{G}(p;G_1) \times \mathcal{G}(p;G_2) \times \ldots \times \mathcal{G}(p;G_n)\end{aligned}$$

where π_i is the projection from G onto the product of all its factors except the i-th.//

We close this section noticing that its results may give rise to a simplification of the general problem of computing $\mathcal{G}(p)$ in those cases in which the structure group of p can be expressed as a product of groups having "nice" topological properties (e.g. compact groups or classical groups).

C H A P T E R 3

S O M E R E S U L T S I N T H E

H O M O T O P Y T H E O R Y O F

G R O U P S O F B U N D L E

A U T O M O R P H I S M S

In this chapter we shall analyze the homotopy lifting properties (see, e.g., [Sp] page 66) of the maps Γ_p^F and Π_p^h defined in chapter 2, in order to obtain information on the homotopy groups of the groups of automorphisms involved. The abbreviation HLP will stand for "homotopy lifting property" and, with the usual terminology, a map is a *Serre fibration* if it has the HLP for all CW-complexes or a *Hurewicz fibration* if it has the HLP for all spaces.

§ 1) BASIC TOOLS

The results of this chapter will rely basically on a well known result of A. Dold ([D] th. 4.8) which we now recall.

3.1 THEOREM If $q: Y \to Z$ is a map having the HLP over every set of a) a numerable cover, resp. b) an open cover of Z, then q has the HLP for a) all spaces, resp. b) all paracompact spaces.//

In particular we obtain:

3.2 COROLLARY If $q: Y \to Z$ is locally trivial then it has the HLP for all paracompact spaces and so, in particular, it is a Serre fibration. Moreover, if the locally trivial cover of B is numerable, then q is a Hurewicz fibration.//

We shall say that a space X is *pericompact* if $X \times Y$ is paracompact whenever Y is paracompact. It is well known that a compact space is pericompact and it is clear that a pericompact space is paracompact.

3.3 PROPOSITION If $q: Y \to Z$ is a map having the HLP for all paracompact spaces and B is pericompact, then the induced map $q_*: \mathrm{Map}(B,Y) \to \mathrm{Map}(B,Z): f \rightsquigarrow q \cdot f$ has the HLP for all paracompact spaces. If $q: Y \to Z$ is a Hurewicz fibration and B is any space, then $q_*: \mathrm{Map}(B,Y) \to \mathrm{Map}(B,Z)$ is a Hurewicz fibration.

PROOF Using an exponential law, we can associate to each diagram of the form

$$\begin{array}{ccc} A \times \{0\} & \xrightarrow{\quad f \quad} & \mathrm{Map}(B,Y) \\ \downarrow & & \downarrow q_* \\ A \times I & \xrightarrow{\quad F \quad} & \mathrm{Map}(B,Z) \end{array}$$

where I is the unit interval $[0,1]$, a diagram

$$\begin{array}{ccc} A\times B\times\{0\} & \xrightarrow{f'} & Y \\ \downarrow & & \downarrow q \\ A\times B\times I & \xrightarrow{F'} & Z \end{array}$$

Now the existence of a lifting for F' is guaranteed by theorem 3.1,in the first case,and by the hypothesis in the second case. Using again an exponential law, we can therefore obtain the required lifting of F.//

3.4 <u>COROLLARY</u> Let $p: X \to B$ and $q: Y \to B$ be maps and $f: X \to Y$ be a map over B. Then

a) if f has the HLP for all paracompact spaces and B is pericompact, then $f_*: \sec(p) \to \sec(q)$ has the HLP for all paracompact spaces,

b) if f is a Hurewicz fibration, then $f_*:\sec(p) \to$ $\to \sec(q)$ is a Hurewicz fibration.

<u>PROOF</u> The diagram

$$\begin{array}{ccc} \sec(p) & \longrightarrow & \mathrm{Map}(B,X) \\ f_* \downarrow & & \downarrow f_* \\ \sec(q) & \longrightarrow & \mathrm{Map}(B,Y) \end{array}$$

is a pullback and this suffices to prove the claim.//

§ 2) THE HLP FOR THE MAP Γ_p^F

3.5 DEFINITION If $p\colon X \to B$ is a principal G-bundle or a fibre bundle, we say that p is *numerable* if its locally trivial cover is numerable.

We can now apply the results of the previous section to our problem in the following way.

3.6 PROPOSITION Let $p\colon X \to B$ be a principal G-bundle, F a left G-space and $\Psi^F\colon (X\ X)_G \to (X\times_G F\ \ X\times_G F)_\lambda$ the map defined in 1.54. Then we have:

a) if $\lambda^*\colon G \to M_\lambda(F,F)$ is locally trivial, then Ψ^F is locally trivial;

b) if $\lambda^*\colon G \to M_\lambda(F,F)$ is a Hurewicz fibration, then Ψ^F has the HLP for all paracompact spaces;

c) if $\lambda^*\colon G \to M_\lambda(F,F)$ is a Hurewicz fibration and p is numerable, then Ψ^F is a Hurewicz fibration.

PROOF We know that for each U in the locally trivial cover for p, there is a commutative diagram

$$\begin{array}{ccc}
(X\ X)_G|U & \xrightarrow{\ \Psi^F|U\ } & (X\times_G F\ \ X\times_G F)_\lambda|U \\
\Big\uparrow{\scriptstyle \chi_U} & & \Big\uparrow{\scriptstyle \chi_U^F} \\
U\times G & \xrightarrow{\ 1\times\lambda^*\ } & U\times M_\lambda(F,F)
\end{array}$$

where χ_U and χ_U^F are local homeomorphisms. So, if $\{V\}$ is the locally trivial cover for λ^*, Ψ^F has a locally trivial structure whose cover is given by

$$\{\chi_U^F(U\times V) \mid U\in\{U\},\ V\in\{V\}\}$$

thus proving case a). Case b) follows directly from theorem 3.1 and case c) from the same theorem and the observation that, if $\{U\}$ is a numerable cover of B, then $\{U\times M_\lambda(F,F)\}$ is a numerable cover of $(X\times_G F \;\; X\times_G F)_\lambda$ ([D] page 226).//

With the hope of making the exposition less cumbersome, we now state two conditions, denoted by C1) and C2), which will be often quoted later. So, for a given principal G-bundle $p: X \to B$ and a left G-space F, we say that the pair (p,F) satisfies:

C1) if $\lambda^* : G \to M_\lambda(F,F)$ is locally trivial and B is pericompact;

C2) if $\lambda^* : G \to M_\lambda(F,F)$ is a Hurewicz fibration and p is numerable.

Combining proposition 3.6 and corollary 3.4 we can obtain the following result.

3.7 <u>LEMMA</u> Let $p: X \to B$ be a principal G-bundle and F a left G-space. If (p,F) satisfies C1) then the map Ψ_p^F: $\sec(p\ p)_G \to \sec(p^F\ p^F)_\lambda$ has the HLP for all paracompact spaces; if (p,F) satisfies C2), then Ψ_p^F is a Hurewicz fibration.//

The next theorem follows from the fact that if a map $q: Y \to A$ has the HLP for a certain class of spaces, then so does the map $q: Y \to q(Y)$; it will prove very useful in the study of the homotopy of $\mathfrak{F}(p^F)$.

3.8 THEOREM Let $p: X \to B$ be a principal G-bundle and F a left G-space. If (p,F) satisfies C1), the map Γ_p^F: $\mathfrak{G}(p) \to \mathfrak{F}(p^F)$ has the HLP for all paracompact spaces, so, in particular, it is a Serre fibration. If (p,F) satisfies C2), then Γ_p^F is a Hurewicz fibration.//

3.9 COROLLARY In the situation of theorem 3.8, if (p,F) satisfies C1) or C2), there is a long exact sequence

$$\ldots \to \pi_i(\mathfrak{G}(p;Kr_F)) \to \pi_i(\mathfrak{G}(p)) \to \pi_i(\mathfrak{F}(p^F)) \to$$
$$\to \pi_{i-1}(\mathfrak{G}(p;Kr_F)) \to \ldots \to \pi_0(\mathfrak{G}(p)) \to \pi_0(\mathfrak{F}(p^F)) \to 0$$

where Kr_F is the kernel of λ^*.//

These results raise the problem of finding conditions under which a G-space F can give rise to pairs (p,F) satisfying C1) or C2); i.e. of determining the respective conditions under which λ^* is either locally trivial or a fibration. This problem can be completely solved for C1) by solving it just for the quotient groups of G. In fact if the map λ^* is locally trivial, then it is an identification. This means that F must be admissible and $M_\lambda(F,F)$ homeomorphic to Q_F. Therefore we have the following result.

3.10 LEMMA A left G-space F can give rise to a pair (p,F) satisfying C1) if and only if one of the following conditions is satisfied:

a) F is admissible and G has a local section at Kr_F;

b) $\lambda^*: G \to M_\lambda(F,F)$ is a principal Kr_F-bundle.//

Notice that G satisfies condition a) of the lemma when, for instance, $G \cong Kr_F \times Q_F$ or G is a Lie group or Kr_F is a compact Lie group ([St] page 33). Also, if F is effective, conditions a) and b) are both equivalent to the requirement for F to be admissible.

In the case of C2), F is not necessarily forced to be admissible, but the existence of a commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\pi} & Q_F \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle t} \\ G & \xrightarrow{\lambda^*} & M_\lambda(F,F) \end{array}$$

where π is the canonical projection and t is the continuous bijection obtained from 2.23 and corollary 1.11, implies that λ^* is a fibration only if π is such. Hence:

3.11 LEMMA Let F be a left G-space. If $\pi: G \to Q_F$ is not a Hurewicz fibration, F cannot give rise to a pair (p,F) satisfying C2); moreover, if F is admissible, then it can give rise to such a pair if and only if Q_F can.//

Lemmas 3.10 and 3.11 provide another example of a context in which properties of the groups of automorphisms of fibre bundles are determined by properties of the structure group G and of its normal subgroups.

§ 3) THE HLP FOR THE MAP Π_p^h

The results of section 2) are clearly valid also in the special case in which the fibre bundle is, in fact, a principal bundle induced by a topological group morphism. In this case, however, we can use the same techniques to obtain information on the HLP for the map Π_p^h, that is, to relate $\mathcal{G}(p)$ and $\mathcal{G}(p^h)$. Moreover this will give us some extra information about the groups $\mathfrak{J}(p^F)$.

We have seen that if $h\colon G \to K$ is a topological group morphism, then the map $\lambda(h) : G \to M_{\lambda(h)}(K,K)$ is, up to homeomorphism, the restriction of h to $h\colon G \to h(G)$ (proposition 1.10). Moreover if $p\colon X \to B$ is a principal G-bundle, for each U in the locally trivial cover for p, the diagram

$$\begin{array}{ccccc}
(X\ X)_G|U & \xrightarrow{\psi^h} & (X\times_h K\ \ X\times_h K)_\lambda|U & \xrightarrow{j} & (X\times_h K\ \ X\times_h K)_K|U \\
\uparrow \chi_U & & \uparrow \chi_U^h & & \uparrow \chi_U^h \\
U\times G & \xrightarrow{1\times h} & U\times h(G) & \xrightarrow{1\times j} & U\times K
\end{array}$$

where the maps denoted by j are inclusions, is commutative (propositions 1.56 and 2.29). Now, in analogy with conditions C1) and C2), we shall say that a pair (p,h), where $p\colon X \to B$ is a principal G-bundle and $h\colon G \to K$ is a topological group morphism, satisfies:

C1') if $h\colon G \to K$ is locally trivial and B is pericompact;

C2') if $h\colon G \to K$ is a Hurewicz fibration and p is numerable.

Now, using the same technique as in section 2), we can prove the following result.

3.12 <u>THEOREM</u> Let $p: X \to B$ be a principal G-bundle and $h: G \to K$ a topological group morphism. If the pair (p,h) satisfies C1'), then $\Pi_p^h: \mathcal{G}(p) \to \mathcal{G}(p^h)$ has the HLP for all paracompact spaces; if (p,h) satisfies C2'), then Π_p^h is a Hurewicz fibration.//

3.13 <u>COROLLARY</u> Given p and h as in theorem 3.12, if (p,h) satisfies C1') or C2'), there is a long exact sequence of homotopy groups

$$\ldots \to \pi_i(\mathcal{G}(p;\mathrm{Ker}(h))) \to \pi_i(\mathcal{G}(p)) \to \pi_i(\mathcal{G}(p^h)) \to$$
$$\to \pi_{i-1}(\mathcal{G}(p;\mathrm{Ker}(h))) \to \ldots \to \pi_0(\mathcal{G}(p)) \to \pi_0(\mathcal{G}(p^h))$$

generated by the fibration Π_p^h.//

The following standard result will provide a link between theorem 3.8 and theorem 3.12.

3.14 <u>LEMMA</u> Let $q: Y \to A$ be a map having the HLP for the singleton space. Then the image of q consists of a set of complete path components of A. If, in particular, q is a topological group morphism, then it induces a homomorphism between the groups of path components of Y and A.//

3.15 PROPOSITION Let $p: X \to B$ be a principal G-bundle and $h: G \to K$ a topological group morphism. If (p,h) satisfies C1') or C2'), then $\mathfrak{J}(p^h)$ consists of a group of complete path components of $\mathcal{G}(p^h)$.

PROOF Under the given hypothesis the map $\Pi_p^h: \mathcal{G}(p) \to \mathcal{G}(p^h)$ has the HLP for the singleton space, is a topological group morphism and its image is $\mathfrak{J}(p^h)$. By lemma 3.14 the result follows.//

Notice that if (p,h) satisfies C1') or C2'), then $h(G)$ consists of a group of path components of K. However there are cases in which $h(G)$ does not have this property, but still the restriction $h: G \to h(G)$ satisfies C1') or C2'). We are now going to see how our results can be generalized to these cases.

To that end, let $G_h = h^{-1}(C(h(G),K)$ and let us say that the pair (p,h) satisfies:

C1") if $h: G_h \to C(h(G),K)$ is locally trivial and B is pericompact;

C2") if $h: G_h \to C(h(G),K)$ is a Hurewicz fibration and p is numerable.

Now, applying the usual technique to the restriction

$$\Psi^h: (X\ X)_{G;G_h} \to (X\times_h K\ \ X\times_h K)_{K;h(G)}$$

we can obtain the following result.

3.16 <u>PROPOSITION</u> If (p,h) satisfies C1"), then the restriction Π_p^h: $\mathcal{G}(p;G_h) \to \mathcal{G}(p^h;h(G))$ has the HLP for all paracompact spaces; if (p,h) satisfies C2"), then such restriction is a Hurewicz fibration with fibre $\mathcal{G}(p;Ker(h))$.//

A result corresponding to corollary 3.13 can also be obtained. Notice that if $h(G)$ is normal in K, then conditions C1") and C2") simplify, since, in this case, $G_h = G$.

§ 4) RELATING FIBRE AND PRINCIPAL BUNDLE AUTOMORPHISMS GROUPS

In corollary 2.27 we have seen how to reduce the problem of studying the spaces $\mathfrak{J}(p^F)$ to the problem of studying the topological groups $\mathfrak{J}(p^\pi)$, where $\pi: G \to Q_F$ is the canonical projection. Now, using that relation and some of the results of this chapter, we are going to relate, in some particular cases, the space $\mathfrak{J}(p^F)$ with the topological groups $\mathfrak{G}(p^\pi)$, thus reducing the study of the group of automorphisms of a fibre bundle to that of the group of automorphisms of a principal bundle.

3.17 <u>THEOREM</u> Let $p: X \to B$ be a principal G-bundle, F a left G-space and $\pi: G \to Q_F$ the canonical projection. If the pair (p,F) satisfies C1), then the map

$$t_{**}: \mathfrak{J}(p^\pi) \to \mathfrak{J}(p^F)$$

of corollary 2.27 defines a homeomorphism,between $\mathfrak{J}(p^F)$ and a space consisting of a group of complete path components of $\mathfrak{G}(p^\pi)$, which is also a topological group isomorphism. If (p,F) satisfies C2), then t_{**} is a weak homotopy equivalence between these two spaces.

<u>PROOF</u> The first part of the theorem can be verified by

combining lemma 3.10, corollary 2.27 and proposition 3.15 applied to the pair (p,π). To prove the second part, notice that, by lemma 3.11, the pair (p,Q_F) also satisfies C2), if (p,F) does, and hence the maps $\Gamma_p^F\colon \mathfrak{G}(p) \to \mathfrak{F}(p^F)$ and $\Gamma_p^\pi\colon \mathfrak{G}(p) \to \mathfrak{F}(p^\pi)$ are both Hurewicz fibrations. So the commutative diagram

$$
\begin{array}{ccccc}
\mathfrak{G}(p;\mathrm{Kr}_F) & \longrightarrow & \mathfrak{G}(p) & \xrightarrow{\Gamma_p^\pi} & \mathfrak{F}(p^\pi) \\
\downarrow 1 & & \downarrow 1 & & \downarrow t_{**} \\
\mathfrak{G}(p;\mathrm{Kr}_F) & \longrightarrow & \mathfrak{G}(p) & \xrightarrow{\Gamma_p^F} & \mathfrak{F}(p^F)
\end{array}
$$

generates an exact ladder of homotopy groups in all dimensions, except, possibly, for the lack of a group structure on $\pi_0(\mathfrak{F}(p^F))$. Applying the "five lemma" to this ladder we can prove that t_{**} induces isomorphisms of homotopy groups in all positive dimensions. Finally, combining the facts that t_{**} is a bijection and that Γ_p^F is a fibration, one can easily check that also $\pi_0(t_{**})$ is a bijection.//

3.18 <u>COROLLARY</u> Let $p\colon X \to B$ be a principal G-bundle, F a left G-space and $\pi\colon G \to Q_F$ the canonical projection. If the pair (p,F) satisfies C1) or C2), then $\pi_i(\mathfrak{F}(p^F)) \sim \pi_i(\mathfrak{F}(p^\pi))$ for all $i \geq 0$. Moreover

$$\pi_i(\mathfrak{F}(p^F)) = \begin{cases} \pi_i(\mathfrak{G}(p^\pi)) & i>0 \\ \dfrac{\pi_0(\mathfrak{G}(p))}{\mathrm{Ker}(\pi_0(\Gamma_p^\pi))} & i=0 \end{cases}$$

if the same condition holds.//

As an illustration, let $p\colon B\times G \to B$ be a trivial principal G-bundle and F a left G-space such that the pair (p,F) satisfies C2) (e.g., let G be a compact Lie group and B a CW-complex). Then $p^\pi\colon B\times Q_F \to B$ is also trivial and the morphism $\Pi_p^\pi\colon \mathfrak{G}(p) \to \mathfrak{G}(p^\pi)$ reduces to the map:

$$\pi_*\colon \mathrm{Map}(B,G) \to \mathrm{Map}(B,Q_F)\colon f \rightsquigarrow \pi\cdot f$$

which is a fibration by proposition 3.3. So we have, for any $i>0$,

$$\pi_i(\mathfrak{F}(p^F)) \sim \pi_i(\mathrm{Map}(B,Q_F))$$

On the other hand, $\pi_0(\mathrm{Map}(B,G))$ is just the set $[B,G]$ of free homotopy classes of maps from B to G and $\pi_0(\Gamma_p^\pi)$ is defined by assigning to each class $[f]\in[B,G]$ the class $[\pi\cdot f]\in[B,Q_F]$. Hence we can describe $\mathrm{Ker}(\pi_0(\Gamma_p^\pi))$ as corresponding to the set of all homotopy classes of maps $[f]\in[B,G]$ such that $\pi\cdot f$ is null-homotopic. Also, in this case, $\pi_0(\mathfrak{F}(p^F))$ corresponds to the image of the function $\pi_0(\pi_*)\colon [B,G] \to [B,Q_F]$, so that Π_p^π is surjective if and only if $\pi_0(\pi_*)$ is.

This enables us to give examples for which such condition is satisfied (e.g., $G = Q_F \times Kr_F$) as well as examples in which it is not. In fact, if G is the real line R and Kr_F is the subgroup of integers, Q_F is the circle S^1 and clearly the function $\pi_*: [S^1,R] \to [S^1,S^1]$ is not surjective.

In chapter 5 we shall examine other conditions sufficient to ensure the surjectivity of Π^h_p.

§ 5) RECOGNIZING n-EQUIVALENCES BETWEEN AUTOMORPHISM GROUPS OF PRINCIPAL BUNDLES

In what follows we shall use the following definition of an n-equivalence (cf. [Sw] def. 3.17).

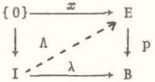
3.19 DEFINITION A map $f: X \to Y$ is said to be an *n-equivalence* , $0 \leq n \leq \infty$, if, for all $x \in X$, the induced function $\pi_i(f): \pi_i(X,x) \to \pi_i(Y,f(x))$ is a bijection for $0 \leq i < n$ and, if $n < \infty$, a surjection for $i = n$.

So, in particular, the concepts of ∞-equivalence and of weak homotopy equivalence coincide.

We now recall two lemmas which will be used to show that, under suitable conditions, if $h: G \to K$ is a topological group morphism which is an n-equivalence, then the induced homomorphism $\Pi_p^h: \mathcal{G}(p) \to \mathcal{G}(p^h)$ is also an equivalence in a certain range of dimensions.

In the first of them, due to P. Heath, $p: E \to B$ will be a map having the HLP for the spaces S^n and $S^n \times I$, $n \geq 0$, b will be the base point of B and $j: F = p^{-1}(b) \to E$ the inclusion. Also, if x is an element of F, then $\underline{x}$ will denote the path component of x in F and $\underline{\underline{x}}$ the path component of x in E.

3.20 <u>LEMMA</u> Let $\underline{x}, \underline{y} \in \pi_0(F)$. Then $\pi_0(j)(\underline{x}) = \pi_0(j)(\underline{y})$ if and only if there is a loop $\lambda: I \to S^1 \to B$, based at b, such that, if $\Lambda: I \to E$ is a path completing the commutative diagram



then $\underline{\Lambda(1)} = \underline{y}$.

<u>PROOF</u> See [H] cor. 3.11.//

We now use lemma 3.20 to prove the following needed result.

3.21 <u>THEOREM</u> Let $p: E \to B$ and $p': E' \to B$ be two (Serre or Hurewicz) fibrations over a path connected space B. If $f: E \to E'$ is a map over B such that, for each $b \in B$, the restriction $f|b: p^{-1}(b) \to (p')^{-1}(b)$ is an n-equivalence, then f is an n-equivalence.

<u>PROOF</u> Let x be any element of E, $b = p(x)$, $x' = f(x)$, $F = p^{-1}(b)$ and $F' = (p')^{-1}(b)$. Then, using "diagram chasing" techniques on the homotopy ladder generated by p, p' and f, it is easy to prove that $\pi_i(f)$ is injective for $1 \le i \le (n-1)$ and surjective for $1 \le i \le n$. The hypothesis on B guarantees the surjectivity of $\pi_0(f)$, so we only have to check that, if $n>0$, $\pi_0(f)$ is injective. To that end, let $y, z \in E$ be such that, with the same notation of lemma 3.20,

$$\underline{f(y)} = \underline{f(z)}$$

Since B is path connected, there must exist two elements $v, w \in F$ such that

$$\pi_0(j)(\underline{v}) = \underline{y} \quad \text{and} \quad \pi_0(j)(\underline{w}) = \underline{z}$$

Now we have

$$\pi_0(j')(\underline{f(v)}) = \underline{f(y)} = \underline{f(z)} = \pi_0(j')(\underline{f(w)})$$

so that we can use lemma 3.20 and find a loop $\lambda: I \to B$ satisfying the property described there for $\underline{f(v)}$ and $\underline{f(w)}$. Since lemma 3.20 does not depend on the choice of the lifting of λ, we can choose this lifting to be $f \cdot \Lambda$, where $\Lambda\colon I \to E$ is a path completing the commutative diagram

$$\begin{array}{ccc} \{0\} & \xrightarrow{\;v\;} & E \\ \downarrow & \Lambda \nearrow & \downarrow p \\ I & \xrightarrow{\;\lambda\;} & B \end{array}$$

With this choice the lemma states that

$$\pi_0(f|b)(\underline{\Lambda(1)}) = \underline{f\cdot\Lambda(1)} = \underline{f(w)} = \pi_0(f|b)(\underline{w})$$

and the bijectivity of $\pi_0(f|b)$ implies that $\underline{\Lambda(1)} = \underline{w}$. This means, since $\underline{\Lambda(0)} = \underline{v}$, that we can apply the same lemma to the fibration p and the path components $\underline{v}$ and $\underline{w}$ and obtain

$$\pi_0(j)(\underline{v}) = \pi_0(j)(\underline{w})$$

thus proving that $\pi_0(f)$ is injective, as we wanted. The arbitrary choice of $x \in E$ completes the proof.//

The second lemma is a result given in [Mo] (lemma 3.1.7).

3.22 <u>LEMMA</u> Let $p\colon E \to B$ and $p'\colon E' \to B$ be Hurewicz fibrations over a CW-complex B and let $f\colon E \to E'$ be a map over B which is an n-equivalence, n finite or infinite. If both $\sec(p)$ and $\sec(p')$ are non empty and $\dim(B) = m \leq n$, then f induces a map

$$f_*\colon \sec(p) \to \sec(p')\colon s \rightsquigarrow f\cdot s$$

which is an $(n-m)$-equivalence. In the case $n = \infty$, the map f_* is an ∞-equivalence and there is no condition on the dimension of B.//

We can now state and prove the main result of this section.

3.23 <u>THEOREM</u> Let $p: X \to B$ be a principal G-bundle and $h: G \to K$ a topological group morphism. If h is an n-equivalence and B is a path connected CW-complex of dimension $m \leq n$, then $\Pi_p^h: \mathcal{G}(p) \to \mathcal{G}(p^h)$ is an (n-m)-equivalence. In the case $n = \infty$, Π_p^h is an ∞-equivalence and there is no restriction on the dimension of B.

<u>PROOF</u> Since B is a CW-complex, both $(p\ p)_G$ and $(p^h\ p^h)_K$ are, by theorem 3.1, Hurewicz fibrations and, by corollary 2.10, both have a section, corresponding to the identity automorphisms of p and p^h respectively. The hypotheses on h and B allow us to use theorem 3.21 and claim that the map $\Psi^h: (X\ X)_G \to (X\times_h K\ \ X\times_h K)_K$ is an n-equivalence. Now lemma 3.22 implies that the induced map $\Psi_p^h: \sec(p\ p)_G \to \sec(p^h\ p^h)_K$ is an (n-m)-equivalence. This, together with lemma 2.20, suffices to prove the claim.//

CHAPTER 4

APPROXIMATING GROUPS

OF BUNDLE AUTOMORPHISMS

BY LOOP SPACES

All the results of chapter 3 have been obtained under the hypothesis that the bundles considered were numerable. In this case, however, one can use a different approach to study the homotopy properties of the groups of bundle automorphisms. This approach was originally developed, for principal bundles, by D.H.Gottlieb in [Go1] and [Go2], where he constructed, for a given numerable principal G-bundle $p: X \to B$, a Serre fibration Φ having as base space the path component $Map(B,B_G;k)$ of $Map(B,B_G)$ containing a classifying map $k: B \to B_G$ for p and, as fibre, a space homeomorphic to $\mathcal{G}(p)$. By showing that the total space of Φ is "essentially contractible", Gottlieb deduced the existence of a weak homotopy equivalence between $\Omega Map(B,B_G;k)$ and $\mathcal{G}(p)$.

When B is a CW-complex, one can use functional bundles to refine this result and obtain a Hurewicz fibration with a contractible total space, $Map(B,B_G;k)$ as base space and $sec(p\ p)_G$ as fibre, thus deriving a homotopy equivalence between $\Omega Map(B,B_G;k)$ and $\mathcal{G}(p)$.

It is this second method, proposed in [Mo] and [BHMP2], that we shall recall and use, together with our previous results, to obtain further properties of the homotopy equivalence in question and to generalize it to the case in which we consider spaces of fibre bundle automorphisms.

The first two sections will be devoted to some needed background material.

§ 1) THE MILNOR CONSTRUCTION

In [Mi2] J.Milnor showed that, given a topological group G, there exists a numerable principal G-bundle $p_G\colon E_G \to B_G$ which classifies numerable principal G-bundles, in the sense of the following theorem.

4.1 THEOREM Given a numerable principal G-bundle $p\colon X \to B$, there exists a map $k\colon B \to B_G$, called a *classifying map* for p, and a G-map $k'\colon X \to E_G$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{k'} & E_G \\ p \downarrow & & \downarrow p_G \\ B & \xrightarrow{k} & B_G \end{array}$$

is a pullback diagram. Moreover, two numerable principal G-bundles over B are isomorphic if and only if their respective classifying maps are homotopic.//

We remind the reader that E_G is the contractible space obtained as a join of a countable number of copies of G. Therefore an element of E_G is an equivalence class of sequences of the form $(t_i g_i)_{i\in N}$, where N is the set of natural numbers, $t_i \in I$, $g_i \in G$, $t_i \neq 0$ for at most a finite number of indices, the sum of all the t_i equals 1 and the equivalence relation is obtained identifying two sequences $(t_i g_i)$ and $(t'_i g'_i)$ if $t_i = t'_i$ for every $i \in N$ and, for each $i \in N$ such that $t_i \neq 0$, $g_i = g'_i$. (For the details of the topology see [Hus] page 53, or [Mi2]) Denoting the equivalence class of a sequence $(t_i g_i)_{i\in N}$ by $\{t_i g_i\}$, the action of G on E_G is

$$\rho_G\colon E_G \times G \to E_G\colon (\{t_i g_i\}, a) \rightsquigarrow \{t_i g_i a\}$$

The space B_G is then the orbit space of E_G, its numerable cover $\{U_i^G\}$ is determined by setting, for each $j \in N$,

4.2 $$U_j^G = \{\{t_i g_i\}G \mid t_j \neq 0\}$$

and the local homeomorphisms are given by

4.3 $$\phi_j^G\colon U_j^G \times G \to E_G | U_j^G \colon (\{t_i g_i\}G, a) \rightsquigarrow \{t_i g_i g_j^{-1} a\}$$

The numerable principal G-bundle $p_G\colon E_G \to B_G$ will be referred to as _*Milnor's universal G-bundle*_

Given a topological group morphism $h\colon G \to K$, there are maps

4.4
$$h^* : E_G \to E_K : \{t_i g_i\} \rightsquigarrow \{t_i h(g_i)\}$$
$$h_* : B_G \to B_K : \{t_i g_i\}G \rightsquigarrow \{t_i h(g_i)\}G$$

such that, for any $i \in N$ and $b \in U_i^G$, the diagram

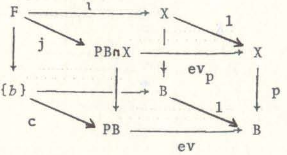
4.5
$$\begin{array}{ccc} G & \xrightarrow{\;h\;} & K \\ \phi^G_{i,b}\downarrow & & \downarrow \phi^K_{i,h_*(b)} \\ E_G & \xrightarrow{\;h^*\;} & E_K \\ p_G \downarrow & & \downarrow p_K \\ B_G & \xrightarrow{\;h_*\;} & B_K \end{array}$$

is commutative. If $h: G \to G'$ and $h': G' \to G''$ are two topological group morphisms, then $h'^* \cdot h^* = (h' \cdot h)^*$ and $h'_* \cdot h_* = (h' \cdot h)_*$. Moreover for any $x \in E_G$ and $g \in G$, $h^*(xg) = h^*(x)h(g)$.

§ 2) THE HOMOTOPY SEQUENCE OF A FIBRATION

Let $p: X \to B$ be a Hurewicz fibration, $\iota: F \to X$ be the inclusion of the fibre over the base point $b \in B$ and $x \in F$ be the base point of both F and X. Moreover, let PB be the space of paths in B, starting at b, and
$$ev: PB \to B : \ell \rightsquigarrow \ell(1)$$
be the usual evaluation fibration ([Sp] cor. 2.8.8).

Consider now the commutative diagram



where both the front and back squares are pullback diagrams and, denoting by c_b the constant path at b,

$$j: F \to PB\sqcap X: y \rightsquigarrow (c_b, y)$$

Since PB is contractible, the map c, defined by $c(b) = c_b$, is a homotopy equivalence, as well as the identities on X and B. By [BrH] th. 1.2, it follows that also j is a homotopy equivalence and we can consider the maps

$$\gamma: \Omega B \to PB\sqcap X: \ell \rightsquigarrow (\ell, x)$$

4.6
$$\delta_p = j^{-1}\cdot\gamma: \Omega B \to PB\sqcap X \to F$$

where j^{-1} is the homotopy inverse of j.

Using this construction it is possible to obtain a sequence of maps

$$\ldots \to \Omega E \xrightarrow{\Omega p} \Omega B \xrightarrow{\delta_p} F \xrightarrow{\iota} X \xrightarrow{p} B$$

which we shall call the <u>*homotopy sequence of the fibration*</u> p (cf. [CJ]) and which can be used to derive the exact sequence of homotopy groups of a fibration. However, we are now only interested in the map δ_p and, in particular, in the following two results related to it.

4.7 LEMMA If $p: X \to B$ is a Hurewicz fibration and X is contractible, then δ_p is a homotopy equivalence.

PROOF The map ev_p is a fibration, since ev is, and γ is the inclusion of its fibre. So, by easy standard arguments, the result follows.//

4.8 PROPOSITION If, in the commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{f|b} & F' \\ \iota \downarrow & & \downarrow \iota' \\ X & \xrightarrow{f} & X' \\ p \downarrow & & \downarrow p' \\ B & \xrightarrow{g} & B \end{array}$$

p and p' are Hurewicz fibrations, ι and ι' are the inclusions of the fibres and the horizontal maps are base point preserving, then the diagram

$$\begin{array}{ccc} \Omega B & \xrightarrow{\Omega g} & \Omega B' \\ \delta_p \downarrow & & \downarrow \delta_{p'} \\ F & \xrightarrow{f|b} & F' \end{array}$$

is homotopy commutative.

PROOF Define a map

$$r\colon PB\sqcap X \to PB'\sqcap X'\colon (\ell,x) \rightsquigarrow (g\cdot\ell, f(x))$$

and observe that, if b and b' denote the base points of B and B' respectively, we have, for any $x\in F$,

$$r\cdot j(x) = r(c_b,x) = (g\cdot c_b, f(x)) = (c_{b'}, f(x)) = j'\cdot f(x)$$

Therefore $(j')^{-1}\cdot r \simeq (f|b)\cdot j^{-1}$, where, as in 4.6, j^{-1} and $(j')^{-1}$ denote the homotopy inverses of j and j'. Using this fact we obtain

$$\delta_{p'}\cdot\Omega g = (j')^{-1}\cdot\gamma'\cdot\Omega g = (j')^{-1}\cdot r\cdot\gamma \simeq (f|b)\cdot j^{-1}\cdot\gamma =$$
$$= (f|b)\cdot\delta_p$$

thus proving the claim.//

Notice that the map j^{-1}, and hence δ_p, are defined only up to homotopy. However this will not affect the discussion of the following sections.

§ 3) RELATING GROUPS OF PRINCIPAL BUNDLE AUTOMORPHISMS AND LOOP SPACES

Let $p\colon X \to B$ be a numerable principal G-bundle and

$$\begin{array}{ccc} X & \xrightarrow{k'} & E_G \\ p\downarrow & & \downarrow p_G \\ B & \xrightarrow{k} & B_G \end{array}$$

be a pullback diagram, so that k is a classifying map for p. We can now construct the functional space $(X\times B_G \quad B\times E_G)_G$ associated to the principal G-bundles

$$p\times 1\colon X\times B_G \to B\times B_G\colon (x,b) \rightsquigarrow (p(x),b)$$

$$1\times p_G\colon B\times E_G \to B\times B_G\colon (b,x) \rightsquigarrow (b,p_G(x))$$

and we shall denote the projection $(p\times 1 \quad 1\times p_G)_G$ simply by $p*p_G$. Since both p and p_G are locally trivial over numerable covers, then so is $p*p_G$ and hence, by theorem 3.1, it is a Hurewicz fibration with fibre G. By proposition 3.3, the induced map

$$(p*p_G)_B\colon \mathrm{Map}(B,(X\times B_G \quad B\times E_G)_G) \to \mathrm{Map}(B,B\times B_G)$$

is a Hurewicz fibration, as well as the map $p*_1p_G$, obtained composing $p*p_G$ with the projection on B. Since the diagram

$$\begin{array}{ccc} \sec(p*_1p_G) & \longrightarrow & \mathrm{Map}(B,(X\times B_G \quad B\times E_G)_G) \\ \Phi\downarrow & & \downarrow (p*p_G)_B \\ \mathrm{Map}(B,B_G) \cong \{1_B\}\times\mathrm{Map}(B,B_G) & \longrightarrow & \mathrm{Map}(B,B\times B_G) \end{array}$$

in which the horizontal maps are inclusions and Φ is the obvious restriction of $(p*p_G)_B$, is a pullback diagram, it follows that Φ is also a Hurewicz fibration. Notice that an element $s\in\sec(p*_1p_G)$ is in $\Phi^{-1}(k)$ if and only if it lands in $(p*p_G)^{-1}(\mathrm{Im}(1_B,k))$, where

$$(1_B,k)\colon B \to B\times B_G\colon b \rightsquigarrow (b,k(b))$$

Using the fact that $Im(1_B,k)$ is homeomorphic to B and working locally as done in the previous chapters (it may be necessary to refine the covers), it is easy to see that the function

$$j\colon (X\ X)_G \to (p * p_G)^{-1}(Im(1_B,k))$$

defined, for a given $\alpha \in (p\ p)_G^{-1}(b)$, by the relation

$$j(\alpha)(x,k(b)) = (b,k'(\alpha(x)))$$

is a homeomorphism over B and hence that the induced map

4.9 $$j_B\colon sec(p\ p)_G \to \Phi^{-1}(k)\colon s \rightsquigarrow j \cdot s$$

is a homeomorphism.

4.10 <u>THEOREM</u> Let $p\colon X \to B$ be a principal G-bundle over a CW-complex B. Then the space $sec(p*_1p_G)$ is contractible.

<u>PROOF</u> The map $p*_1p_G$ is a Hurewicz fibration and, using [BB2] th. 1.1 and the fact that, for every principal G-bundle $q\colon Y \to B$, the map

$$f\colon (B\times G\ \ Y)_G \to Y\colon \alpha \rightsquigarrow \alpha((pr\ \ p)_G(\alpha),e)$$

where $pr\colon B\times G \to B$ is the projection, is a homeomorphism over B, it is easy to see that the fibre of $p*_1p_G$ is homeomorphic to E_G and hence homotopy equivalent to a CW-complex. Since B is a CW-complex, by [Sc] th.2 we have that also the total space of $p*_1p_G$ has the homotopy type of a CW-complex and so that $p*_1p_G$ is a homotopy equivalence. This implies that the induced fibration

$$(p*_1p_G)_B\colon \mathrm{Map}(B,(X\times B_G \quad B\times E_G)_G) \to \mathrm{Map}(B,B)$$

is a homotopy equivalence and that its fibre, $\sec(p*_1p_G)$, is contractible.//

Theorem 4.10 says, in particular, that $\sec(p*_1p_G)$ is path connected and, since $\Phi^{-1}(k) \neq \emptyset$, it follows that the image of Φ is $\mathrm{Map}(B,B_G;k)$ and that

$$\Phi\colon \sec(p*_1p_G) \to \mathrm{Map}(B,B_G;k)$$

is a Hurewicz fibration with fibre $\sec(p\ p)_G \cong \mathcal{G}(p)$.

4.11 THEOREM Let $p\colon X \to B$ be a principal G-bundle over a CW-complex B. Then there is a homotopy equivalence $\delta_p\colon \Omega\mathrm{Map}(B,B_G;k) \to \mathcal{G}(p)$ which is also an H-homomorphism.

PROOF The map δ_p is obtained applying lemma 4.7 to the fibration $\Phi\colon \sec(p*_1p_G) \to \mathrm{Map}(B,B_G;k)$ and using the homeomorphism between $\sec(p\ p)_G$ and $\mathcal{G}(p)$, as described in corollary 2.10. For the last statement see [BHMP2], th. 4.2.//

We have seen in chapter 2 that, in favourable circumstances, $\mathcal{G}(p)$ is homeomorphic to Map(B,G) and we also claimed there that this is not always so. We are now ready to give a counterexample.

In [Mil] J.Milnor proved that for any simplicial complex B, it is possible to construct a topological group G and a numerable principal G-bundle $p: E \to B$ which is universal, having a contractible total space. So let $p: E^4 \to S^4$ be the principal G-bundle constructed in that way and let G be its structure group. If $p_k: S^4 \sqcap E^4 \to S^4$ is the numerable principal G-bundle induced from p by a map $k: S^4 \to S^4$ of degree n, then, by theorem 4.11,

$$\pi_2(\mathcal{G}(p_k)) \sim \pi_2(\Omega \mathrm{Map}(S^4,S^4;k)) \sim \pi_3(\mathrm{Map}(S^4,S^4;k))$$

The last group is isomorphic by [K], lemma 3.10, to

$$Z_{24|n|} \oplus Z_{12}$$

and so it depends on the degree of k. This means that, taking maps of different degrees, we obtain principal bundles with the same base space and structure group, but with groups of automorphisms having different homotopy groups in dimension 2. Hence $\mathcal{G}(p_k)$ can be homeomorphic to Map(B,G) only for one of these bundles, clearly the trivial one, induced by the constant map. This shows that the problem of computing $\mathcal{G}(p)$ in the general case is non trivial.

Notice that one can use this same construction and some more of the results contained in [K] to give other counterexamples.

§ 4) LOOP SPACES AND HOMOMORPHISMS BETWEEN GROUPS OF AUTOMORPHISMS

4.12 LEMMA Let $p: X \to B$ be a numerable principal G-bundle classified by the map $k: B \to B_G$ and let $h: G \to K$ be a topological group morphism. Then the map $h_* \cdot k: B \to B_K$ is a classifying map for p^h.

PROOF Under the given hypothesis, there are pullback diagrams

$$\begin{array}{ccc} X & \xrightarrow{k'} & E_G \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_G} \\ B & \xrightarrow{k} & B_G \end{array} \qquad\qquad \begin{array}{ccc} B \sqcap E_K & \xrightarrow{q'} & E_K \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p_K} \\ B & \xrightarrow{h_* \cdot k} & B_K \end{array}$$

where $q = (p_K)_{h_* \cdot k}$ and $q' = (h_* \cdot k)_{p_K}$. Now the composite

$$f: X \times K \xrightarrow{k' \times 1} E_G \times K \xrightarrow{h^\# \times 1} E_K \times K \xrightarrow{\rho_K} E_K$$

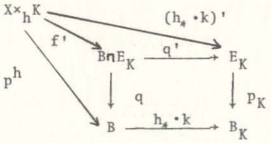
has the property that for each $g \in G$, $x \in X$ and $k \in K$,

$$f(xg, h(g)^{-1}k) = h^\#(k'(x))h(g)h(g)^{-1}k = h^\#(k'(x))k = f(x,k)$$

and hence defines a map

$$(h_* \cdot k)': X \times_h K \to E_K : [x,k] \rightsquigarrow h^\#(k'(x))k$$

which is a K-map and is such that $p_K \cdot (h_* \cdot k)' = h_* \cdot k \cdot p^h$. Therefore there is a unique map f' completing the diagram



and this is a K-map and hence an isomorphism in $\mathcal{P}_B^K$ between p^h and q. This means that the four outer maps in the diagram form a pullback diagram and hence that $h_* \cdot k$ is a classifying map for p^h.//

4.13 <u>COROLLARY</u> Under the conditions of lemma 4.12, there is a homotopy equivalence $\delta_{(p^h)}: \Omega \mathrm{Map}(B, B_K; h_* \cdot k) \to \mathcal{G}(p^h)$.

<u>PROOF</u> Apply theorem 4.11 to the principal K-bundle p^h.//

Since the map h_* induces, by composition, a map

$$\Omega(h_*)_B: \Omega \mathrm{Map}(B, B_G; k) \to \Omega \mathrm{Map}(B, B_K; h_* \cdot k)$$

it is natural to ask what properties the diagram

$$\begin{array}{ccc} \Omega \mathrm{Map}(B, B_G; k) & \xrightarrow{\delta_p} & \mathcal{G}(p) \\ {\scriptstyle \Omega(h_*)_B} \downarrow & & \downarrow {\scriptstyle \pi_p^h} \\ \Omega \mathrm{Map}(B, B_K; h_* \cdot k) & \xrightarrow{\delta_{(p^h)}} & \mathcal{G}(p^h) \end{array}$$

has and, in particular, whether it is homotopy commutative, in which case Π_p^h and $\Omega(h_*)_B$ can be identified up to homotopy. By proposition 4.8 and lemma 2.20, it will be sufficient to show that there exists a base point preserving map Ψ_* completing the diagram

4.14
$$\begin{array}{ccc}
\mathrm{sec}(p\ \ p)_G & \xrightarrow{\ \Psi_p^h\ } & \mathrm{sec}(p^h\ \ p^h)_K \\
{\scriptstyle j_B}\downarrow & & \downarrow{\scriptstyle j_B'} \\
\mathrm{sec}(p*_1 p_G) & \overset{\Psi_*}{- - - \rightarrow} & \mathrm{sec}(p^h *_1 p_K) \\
{\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\
\mathrm{Map}(B,B_G;k) & \xrightarrow{\ (h_*)_B\ } & \mathrm{Map}(B,B_K;h_*\cdot k)
\end{array}$$

Notice that the base points in $\mathrm{sec}(p\ \ p)_G$ and $\mathrm{sec}(p^h\ \ p^h)_K$ are the sections corresponding to the respective identities, so the base points in $\mathrm{sec}(p*_1 p_G)$ and $\mathrm{sec}(p^h *_1 p_K)$ will be defined, respectively, by the relations:

$$s(b)(x,k(b)) = (b,k'(x))$$
$$s^h(b)([x,e],h_*\cdot k(b)) = (b,(h_*\cdot k)'([x,e]))$$

for any $b \in B$ and $x \in X$. In this way all the already existing maps in diagram 4.14 are base point preserving.

Let us therefore define a function

$$\Psi\colon (X\times B_G\ \ B\times E_G)_G \to ((X\times_h K)\times B_K\ \ B\times E_K)_K$$

by setting, for a given $(b,c)\in B\times B_G$ and a given G-map $\alpha\colon p^{-1}(b)\times\{c\}\to\{b\}\times p_G^{-1}(c)$,

$$\Psi(\alpha)\colon (p^h)^{-1}(b)\times\{h_*(c)\}\to\{b\}\times p_K^{-1}(h_*(c))\colon$$

$$\colon ([x,k],h_*(c))\rightsquigarrow (b,h^*(\alpha(x))k)$$

(we identify the given α with the corresponding map $\alpha\colon p^{-1}(b)\to p_G^{-1}(c)$). In other words, $\Psi(\alpha)$ is obtained by composing $\alpha\times 1_K$ with a suitable restriction of $h^*\times 1_K$ and with the action of K on E_K and then quotienting out the action of G. By the definition of h^*, this makes $\Psi(\alpha)$ properly defined and continuous. Moreover, since $\Psi(\alpha)$ preserves the action of K, Ψ itself is well defined. Notice also that

$$(p^h * p_K)\cdot\Psi = (p * p_G)\cdot(1\times h_*)$$

so, in particular, as a map from $p *_1 p_G$ to $p^h *_1 p_K$, Ψ is over B. It is also easy to see that, for any U_i^G in the standard numerable cover for p_G, $h_*(U_i^G)$ is contained in the corresponding U_i^K, so that to prove the continuity of Ψ it will be sufficient to prove the continuity of each of the restrictions

$$\Psi|U\times U_i^G\colon (X\times B_G \quad B\times E_G)_G|U\times U_i^G \to ((X\times_h K)\times B_k \quad B\times E_K)_K|U\times U_i^K$$

where U is in the numerable cover for p and $i\in N$.

4.15 <u>LEMMA</u> For any $U\in\{U\}$ and $i\in N$, the diagram

$$\begin{array}{ccc}
(X\times_{B_G} B\times E_G)_G|U\times U_i^G & \xrightarrow{\Psi|U\times U_i^G} & ((X\times_h K)\times_{B_K} B\times E_K)_K|U\times U_i^K \\
\chi_{U,i}\Big\uparrow & & \Big\uparrow \chi_{U,i}^h \\
U\times U_i^G\times G & \xrightarrow{1\times h_*\times h} & U\times U_i^K\times K
\end{array}$$

where $\chi_{U,i}$ and $\chi_{U,i}^h$ are the local homeomorphisms for $p*p_G$ and p^h*p_K respectively, is commutative.

<u>PROOF</u> If $\{(U,\phi_U)\}$ is the locally trivial structure for p, it follows by 1.49 that, for any $(b,c,g)\in U\times U_i^G\times G$ and $a\in G$,

$$\chi_{U,i}(b,c,g)(\phi_U(b,a),c) = (b,\phi_i^G(c,ga))$$

and hence, using the commutativity of diagram 4.5,

$$((\Psi|U\times U_i^G)\cdot\chi_{U,i}(b,c,g))([\phi_U(b,e),k],h_*(c)) =$$
$$= (b,h^*(\chi_{U,i}(b,c,g)(\phi_U(b,e))k) =$$
$$= (b,h^*(\phi_i^G(c,g))k) = (b,\phi_i^K(h_*(c),h(g))k)$$

On the other hand

$$(\chi_{U,i}^h\cdot(1\times h_*\times h)(b,c,g))([\phi_U(b,e),k],h_*(c)) =$$
$$= (\chi_{U,i}^h(b,h_*(c),h(g)))([\phi_U(b,e),k],h_*(c)) =$$
$$= (b,\phi_i^K(h_*(c),h(g)k)) = (b,\phi_i^K(h_*(c),h(g))k)$$

and this proves the result.//

4.16 **PROPOSITION** The function Ψ is continuous and therefore induces a continuous function

$$\Psi_* : \sec(p*_1 p_G) \to \sec(p^h *_1 p_K)$$

which is base point preserving and completes the diagram 4.14.

PROOF The continuity of Ψ follows from lemma 4.15. Moreover if $s \in \sec(p*_1 p_G)$ is the base point, then for any $b \in B$,

$$\Psi_*(s)(b) : (p^h)^{-1}(b) \times \{h_* \cdot k(b)\} \to \{b\} \times p_K^{-1}(h_* \cdot k(b)) :$$

$$: ([x,k], h_* \cdot k(b)) \leadsto (b, h^*(k'(x))k)$$

and, since $h^* \cdot k'(x) = (h_* \cdot k)'([x,e])$, it follows that $\Psi_*(s) = s^h$, that is, that Ψ_* is base point preserving. A similar analysis proves the commutativity of the lower square of diagram 4.14. The commutativity of the upper square follows from that of the diagram

$$\begin{array}{ccc} (X \quad X)_G & \xrightarrow{\Psi^h} & (X \times_h K \quad X \times_h K)_K \\ j \downarrow & & \downarrow j' \\ (X \times B_G \quad B \times E_G)_G & \xrightarrow{\Psi} & ((X \times_h K) \times B_K \quad B \times E_K)_K \end{array}$$

which can be easily checked.//

We can now summarize these results in a theorem.

4.17 <u>THEOREM</u> Given a principal G-bundle $p\colon X \to B$ over a CW-complex classified by a map $k\colon B \to B_G$ and a topological group morphism $h\colon G \to K$, there exists a diagram, commutative up to homotopy

$$\begin{array}{ccc} \Omega\mathrm{Map}(B,B_G;k) & \xrightarrow{\ \delta_p\ } & \mathcal{G}(p) \\ {\scriptstyle \Omega(h_*)_B}\downarrow & & \downarrow{\scriptstyle \Pi_p^h} \\ \Omega\mathrm{Map}(B,B_K;h_*\cdot k) & \xrightarrow{\ \delta_{(p^h)}\ } & \mathcal{G}(p^h) \end{array}$$

where δ_p and $\delta_{(p^h)}$ are H-homotopy equivalences.

<u>PROOF</u> Proposition 4.16 and proposition 4.8 ensure the existence of a commutative diagram

$$\begin{array}{ccc} \Omega\mathrm{Map}(B,B_G;k) & \xrightarrow{\ \delta_p\ } & \mathrm{sec}(p\ p)_G \\ {\scriptstyle \Omega(h_*)_B}\downarrow & & \downarrow{\scriptstyle \Pi_p^h} \\ \Omega\mathrm{Map}(B,B_K;h_*\cdot k) & \xrightarrow{\ \delta_{(p^h)}\ } & \mathrm{sec}(p^h\ p^h)_K \end{array}$$

which, together with lemma 2.20, proves the claim.//

We can now combine the results of this section with theorem 3.17 to obtain the following generalization, to the group of fibre bundle automorphisms, of theorem 4.11.

4.18 <u>THEOREM</u> Let $p\colon X \to B$ be a principal G-bundle over a CW-complex B, F a left G-space and $\pi\colon G \to Q_F$

the canonical projection. If $k: B \to B_G$ is a classifying map for p and the map $\lambda^*: G \to M_\lambda(F,F)$ is a Hurewicz fibration, then $\mathfrak{J}(p^F)$ is weakly homotopy equivalent to the subspace of $\Omega Map(B,B_{Q_F};\pi_* \cdot k)$ consisting of the group of path components in the image of the homomorphism

$$\pi_0(\Omega(\pi_*)_B): \pi_0(\Omega Map(B,B_G;k)) \to \pi_0(\Omega Map(B,B_{Q_F};\pi_* \cdot k))$$

Moreover, if F is admissible such a correspondence is a homotopy equivalence.

PROOF Under the given hypotheses, we can combine theorem 3.17 and theorem 4.17, applied to the projection π, to obtain the result.//

§ 5) SOME CATEGORICAL CONSIDERATIONS

Given a CW-complex B, let us define a category PW_B as follows. The objects are pairs of the form (p,k), where p is a principal G-bundle over B, for some topological group G, and $k: B \to B_G$ is a classifying map for p. Given two objects (p,k) and (p',k') in PW_B, such that p is an object of P_B^G and p' of P_B^K, the morphisms from (p,k) to (p',k') are those pairs (f,h) which are morphisms in P_B from p to p' and are such that $k' = h_* \cdot k$.

Composites and identities can now be obtained as in P_B (definition 1.19) and these ensure that PW_B is a well defined category.

Let now TGr be the category of topological groups and define a functor

$$\mathcal{G}: PW_B \to TGr$$

by setting, for an object (p,k),

$$\mathcal{G}(p,k) = \mathcal{G}(p)$$

and, for a morphism (f,h) from (p,k) to (p',k'),

$$\mathcal{G}(f,h) = f^* \cdot \Pi_p^h : \mathcal{G}(p) \to \mathcal{G}(p^h) \to \mathcal{G}(p')$$

where f^* is the natural isomorphism induced by the map

$$f^h: X \times_h K \to X' : [x,k] \rightsquigarrow f(x)k$$

To be sure that $\mathcal{G}$ is well defined we only have to show that it respects composition of morphisms, since it is obvious that it preserves identities. But if (f,h) and (f',h') are two composable morphisms, then we have that $\mathcal{G}((f',h')\cdot(f,h)) = \mathcal{G}(f'\cdot f, h'\cdot h)$ is given by

$$\mathcal{G}(p) \xrightarrow{\Pi_p^{h'\cdot h}} \mathcal{G}(p^{h'\cdot h}) \xrightarrow{(f'\cdot f)^*} \mathcal{G}(p'')$$

while $\mathcal{G}(f',h')\cdot\mathcal{G}(f,h)$ is given by

$$\mathcal{G}(p) \xrightarrow{\Pi_p^h} \mathcal{G}(p^h) \xrightarrow{f^*} \mathcal{G}(p') \xrightarrow{\Pi_{p'}^{h'}} \mathcal{G}(p'^{h'}) \xrightarrow{(f')^*} \mathcal{G}(p'')$$

and, by the definitions of the various maps involved, it is easy to see that they are equal and hence that $\mathcal{G}$ is a functor.

Let now $\mathcal{H}$ be the category of H-groups and H-homomorphisms, as defined in [Sp] page 35, and define a functor

$$\Omega_B : \mathcal{PW}_B \to \mathcal{H}$$

by setting, for an object (p,k) of $\mathcal{PW}_B$,

$$\Omega_B(p,k) = \Omega \mathrm{Map}(B,B_G;k)$$

and, for a morphism (f,h) from (p,k) to $(p',h_* \cdot k)$,

$$\Omega_B(f,h) = \Omega(h_*)_B : \Omega\mathrm{Map}(B,B_G;k) \to \Omega\mathrm{Map}(B,B_K;h_* \cdot k)$$

It is then easy to see, using standard results about loop spaces, that Ω_B is a well defined functor.

Notice that if we had chosen, as morphisms in $\mathcal{PW}_B$ from (p,k) to (p',k'), pairs (f,h) satisfying only the condition $k' \simeq h \cdot k$, the functor Ω_B could not have been properly defined, since there is no canonical choice of a path from k' to $h_* \cdot k$ in $\mathrm{Map}(B,B_K;k')$, in general.

Finally, let $\mathcal{Hh}$ be the homotopy category of $\mathcal{H}$, that is, the category having the same objects as $\mathcal{H}$ and, as morphisms, homotopy classes of morphisms of $\mathcal{H}$. We then have two obvious "projection" functors:

$$J : H \to Hh$$
$$J' : TGr \to Hh$$

and we are now ready for the main result of this section.

4.19 THEOREM There exists a natural isomorphism

$$\delta : J\cdot\Omega_B \stackrel{\cdot}{\to} J'\cdot\mathcal{G}$$

defined by the relation

$$\delta(p,k) = [\delta_p] : \Omega\mathrm{Map}(B,B_G;k) \to \mathcal{G}(p)$$

PROOF By theorem 4.11, δ is well defined and $\delta(p,k)$ is an isomorphism for every object (p,k) of PW_B, so we only have to check the naturality of δ. To that end, notice that, by theorem 4.17, given a morphism (f,h) from (p,k) to (p',k'), the diagram

$$\begin{array}{ccc}
\Omega\mathrm{Map}(B,B_G;k) & \xrightarrow{\delta_p} & \mathcal{G}(p) \\
{\scriptstyle \Omega(h_*)_B}\downarrow & & \downarrow{\scriptstyle \Pi_p^h} \\
\Omega\mathrm{Map}(B,B_K;h_*\cdot k) & \xrightarrow{\delta_{(p^h)}} & \mathcal{G}(p^h)
\end{array}$$

is homotopy commutative. Now the same techniques used in the proof of that theorem can be applied to the homeomorphism $f^h: X\times_h K \to X' : [x,k] \rightsquigarrow f(x)k$ to obtain a homeomorphism from $\mathrm{sec}(p^h *_1 p_K)$ to $\mathrm{sec}(p' *_1 p_K)$ which is over the identity on $\mathrm{Map}(B,B_K;h_*\cdot k)$ and therefore induces a homotopy commutative diagram

$$\begin{array}{ccc} \Omega\mathrm{Map}(B,B_K;h_*\cdot k) & \xrightarrow{\delta_{(p^h)}} & \mathcal{G}(p^h) \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle f^*} \\ \Omega\mathrm{Map}(B,B_K;h_*\cdot k) & \xrightarrow{\delta_{p'}} & \mathcal{G}(p') \end{array}$$

which, together with the previous one shows the naturality of δ.//

CHAPTER 5

APPLICATIONS AND COMPUTATIONS

In this chapter we shall give a few computations based on results obtained earlier in the thesis. There is, of course, no claim of completeness for such computations, neither with respect to cases analyzed,nor to techniques used.

Throughout the chapter we shall assume that the base space of the bundles considered is a path connected CW-complex, and thus that the bundles themselves are numerable.

In the first two sections we consider the effect on $\mathcal{G}(p)$ of enlarging the structure group of p, say H, into a larger group G. In the first section we consider cases where H is a normal subgroup of G, in the second one cases where the inclusion $\iota : H \to G$ is an n-equivalence, for some value of n.

§ 1) EXTENSIONS: NORMAL SUBGROUP CASE

Let $p: X \to B$ be a principal H-bundle, $\iota : H \to G$ be the inclusion of H as a normal subgroup of a larger topological group G and assume that the canonical projection $\pi: G \to Q=G/H$ is a Hurewicz fibration. In the light of the results of chapters 1,2 and 3, it is clear that this is a particularly favourable situation. In fact, letting $q = p^{\iota}$, by proposition 3.15, there is a topological group morphism

$$\Pi_q^{\pi}: \mathcal{G}(q) \to \mathcal{G}(q^{\pi})$$

which is also a Hurewicz fibration with image $\mathcal{J}(q^{\pi})$ and, by theorem 2.34, with fibre $\mathcal{G}(p)$. Since $\pi\cdot\iota$ is trivial, by propositions 1.36 and 1.43, q^{π} is trivial and so, by corollary 2.13, $\mathcal{G}(q^{\pi}) \cong \mathrm{Map}(B,Q)$.
By corollary 3.13, we obtain a long exact sequence of homotopy groups

$$\ldots \to \pi_i(\mathcal{G}(p)) \to \pi_i(\mathcal{G}(q)) \to \pi_i(\mathrm{Map}(B,Q)) \to \pi_{i-1}(\mathcal{G}(p)) \to$$
$$\to \pi_0(\mathcal{G}(q)) \to \pi_0(\mathrm{Map}(B,Q))$$

from which we may extract information in several cases. One simple case is when, for two given integers n and m, $\pi_i(\mathrm{Map}(B,Q)) = 0$ for $n \le i \le m$. In fact this implies immediately that

$$\pi_i(\mathcal{G}(p)) \sim \pi_i(\mathcal{G}(q)) \qquad \text{for } n \le i < m,$$

while $\pi_m(\Pi_p^{\iota}): \pi_m(\mathcal{G}(p)) \to \pi_m(\mathcal{G}(q))$ is surjective and $\pi_{n-1}(\Pi_p^{\iota}): \pi_{n-1}(\mathcal{G}(p)) \to \pi_{n-1}(\mathcal{G}(q))$ is injective (this last statement being meaningful only if $n>0$).

This situation occurs, for instance, when $Q = S^1$, or a torus, and we let $n = 2$ and $m = \infty$. This can be easily seen using the exact homotopy sequence of the evaluation fibration

$$ev\colon \mathrm{Map}(B,S^1) \to S : f \leadsto f(b)$$

where $b \in B$ is the base point. In fact, in this case, both the base space, S^1, and the fibre, the space $\mathrm{Map}_*(B,S^1)$ of based maps, have vanishing homotopy groups in all dimensions bigger than 1 and so $\pi_i(\mathrm{Map}(B,S^1))=0$ for $i \geq 2$. Moreover

$$\pi_1(\mathrm{Map}_*(B,S^1)) \sim [B,\Omega S^1]_* \sim 0$$
$$\pi_0(\mathrm{Map}_*(B,S^1)) \sim [B,S^1]_* \sim H^1(B)$$

so that, if we assume that B is simply connected, we obtain the following extra data:

$$\pi_1(\mathrm{Map}(B,S^1)) \sim Z$$
$$\pi_0(\mathrm{Map}(B,S^1)) \sim 0$$

which allow us to establish, for $Q = (S^1)^n$, the following exact sequence:

$$0 \to \pi_1(\mathcal{G}(p)) \to \pi_1(\mathcal{G}(q)) \to Z^n \to \pi_0(\mathcal{G}(p)) \to$$
$$\to \pi_0(\mathcal{G}(q)) \to 0 .$$

As an application we obtain:

5.1 <u>THEOREM</u> Let $p\colon X \to B$ be a principal $SU(n)$-bundle, where $1 \leq n \leq \infty$, and let $\iota\colon SU(n) \to U(n)$ be the inclusion.

Then, for all $i \geq 2$,

$$\pi_i(\mathcal{G}(p)) \sim \pi_i(\mathcal{G}(p^1))$$

and, moreover, if B is simply connected, the sequence

$$0 \to \pi_1(\mathcal{G}(p)) \to \pi_1(\mathcal{G}(p^1)) \to Z \to \pi_0(\mathcal{G}(p)) \to \pi_0(\mathcal{G}(p^1)) \to 0$$

is exact.

PROOF We only need to notice that $U(n)$ is a Lie group and so the projection $\pi: U(n) \to U(n)/SU(n) \cong S^1$ is a fibration.//

Theorem 5.1 tells us that the problem of computing the higher homotopy groups of $\mathcal{G}(p)$ for p a principal $SU(n)$-bundle over a CW-complex can be solved by solving the same problem for principal $U(n)$-bundles.

When $n = \infty$, the classifying spaces BU and BSU are H-groups (cf. [Sw] page 213) and this implies, as shown in [Mo], th. 3.3.4, that, for a principal SU-bundle p, $\mathcal{G}(p)$ is homotopy equivalent to Map(B,SU) and $\mathcal{G}(p^1)$ is homotopy equivalent to Map(B,U). In particular, if $B = S^n$, we can use [K], th. 2.2, to obtain

$$\pi_i(\mathcal{G}(p)) \sim \pi_i(SU) \oplus \pi_{i+n}(SU)$$

$$\pi_i(\mathcal{G}(p^1)) \sim \pi_i(U) \oplus \pi_{i+n}(U)$$

(for the details, see [Mo] page 92). Since for $i \geq 2$ $\pi_i(U) \sim \pi_i(SU)$, we get, for this particular case, an alternative verification of the general result of the theorem.

An even more convenient situation occurs when Q is a discrete group, since, in this case, $\mathrm{Map}(B,Q) \cong Q$ and hence $\pi_i(\mathrm{Map}(B,Q)) = 0$ for $i>0$ and $\pi_0(\mathrm{Map}(B,Q)) \sim Q$. The isomorphism

$$\pi_i(\mathcal{G}(p)) \sim \pi_i(\mathcal{G}(p^\iota))$$

then holds for $i\geq 1$.

We give two examples of this situation.

5.2 THEOREM Let $p: X \to B$ be a principal $SO(n)$-bundle, $1\leq n\leq\infty$, and $\iota: SO(n) \to O(n)$ be the inclusion. Then, for $i\geq 1$, $\pi_i(\mathcal{G}(p)) \sim \pi_i(\mathcal{G}(p^\iota))$ and the function $\pi_0(\Pi_p^\iota): \pi_0(\mathcal{G}(p)) \to \pi_0(\mathcal{G}(p^\iota))$ is a monomorphism.

PROOF Again, we only need to notice that the projection $\pi: O(n) \to O(n)/SO(n) \cong Z_2$ is a fibration.//

As for theorem 5.1, this theorem also reduces the study of the higher homotopy groups, and of the fundamental group, of $\mathcal{G}(p)$ for principal $SO(n)$-bundles to the same problem for principal $O(n)$-bundles. Moreover, for $n = \infty$ we can again use the H-group structure of BO and BSO and obtain, for an SO-bundle over a sphere S^m, the following isomorphisms:

$$\pi_i(\mathcal{G}(p)) \sim \pi_i(SO) \oplus \pi_{i+m}(SO)$$

$$\pi_i(\mathcal{G}(p^\iota)) \sim \pi_i(O) \oplus \pi_{i+m}(O)$$

which confirm the result of theorem 5.2.

For the second example, let G be a compact, connected Lie group, T a maximal torus of G and N_T its normalizer. It is well known that the quotient N_T/T is a finite discrete group ([Hus] page 172) called the *Weyl group* of G. So we have:

5.3 THEOREM Let $p: X \to B$ be a principal T-bundle and let $\iota: T \to N_T$ be the inclusion. Then $\pi_i(\mathcal{G}(p^\iota))$ is trivial for $i>1$; moreover if B is simply connected, $\pi_1(\mathcal{G}(p^\iota)) \sim Z^n$, where $n = \dim(T)$, and $\pi_0(\mathcal{G}(p^\iota))$ is finite.

PROOF Since T is abelian, by corollary 2.13, $\mathcal{G}(p) \simeq \mathrm{Map}(B,T)$ and so $\pi_i(\mathcal{G}(p)) \sim (\pi_i(\mathrm{Map}(B,S^1)))^n = 0$ for $i>1$. Now, using the isomorphism between $\pi_i(\mathcal{G}(p))$ and $\pi_i(\mathcal{G}(p^\iota))$ we can prove the first part of the theorem. Moreover we have seen earlier that, if B is simply connected, $\pi_1(\mathrm{Map}(B,S^1)) \sim Z$ and $\pi_0(\mathrm{Map}(B,S^1))$ is trivial, so, in this case, we obtain an exact sequence:

$$0 \to Z^n \to \pi_1(\mathcal{G}(p^\iota)) \to 0 \to 0 \to \pi_0(\mathcal{G}(p^\iota)) \to N_T/T$$

which proves the second part.//

In the proof of theorem 5.3 the computations of $\pi_i(\mathcal{G}(p^1))$ were simplified by the fact that the group T is abelian and hence $\mathcal{G}(p)$ was detectable. This suggests the following generalization:

5.4 <u>THEOREM</u> Let $p: X \to B$ be a principal G-bundle and let H be a normal abelian subgroup of G such that the projection $\pi: G \to Q=G/H$ is a Hurewicz fibration. If p has a restriction to a principal H-bundle q and $\pi_i(\mathrm{Map}(B,H)) = 0$ for $n \le i \le m$, then $\pi_i(\mathcal{G}(p)) \sim \pi_i(\mathrm{Map}(B,G))$ for $n \le i < m$ and the function $\pi_m(\Pi_p^\pi): \pi_m(\mathcal{G}(p)) \to \pi_m(\mathrm{Map}(B,Q))$ is injective.

<u>PROOF</u> The long exact sequence generated by the map $\Pi_p^\pi: \mathcal{G}(p) \to \mathrm{Map}(B,Q)$ and the homeomorphism between $\mathcal{G}(q)$ and $\mathrm{Map}(B,H)$ show that $\pi_i(\mathcal{G}(p)) \sim \pi_i(\mathrm{Map}(B,Q))$ for $n \le i < m$ and that $\pi_m(\Pi_p^\pi)$ is injective. However, in our situation there is a fibration

$$\pi_* : \mathrm{Map}(B,G) \to \mathrm{Map}(B,Q): f \rightsquigarrow \pi \cdot f$$

whose fibre is clearly Map(B,H). Now the hypothesis on Map(B,H) allows us to complete the proof.//

5.5 <u>COROLLARY</u> Let $p: X \to B$ be a principal G-bundle and let H be a normal abelian subgroup of G which is either a) discrete, or b) a torus (or a union of tori). If the projection $\pi: G \to G/H$ is a Hurewicz fibration and p admits a restriction to a principal H-bundle, then $\pi_i(\mathcal{G}(p)) \sim \pi_i(\mathrm{Map}(B,G))$ for a) $i \geq 1$, or b) $i \geq 2$.//

Notice that if G is connected, any discrete normal subgroup is abelian ([P] th. 15) and that, if G is compact and finite dimensional, then its center is a torus (or a union of tori) and hence corollary 5.5 can be applied.

§ 2) EXTENSIONS: n-EQUIVALENCE CASE

In this section we shall analyze some cases in which theorem 3.23 may be applied.

5.6 THEOREM Let $p: X \to B$ be a principal $U(n)$-bundle, where $0 \leq n \leq \infty$, and assume that $\dim(B) = m \leq 2n$. If $\iota: U(n) \to U(n+q)$ denotes the inclusion, for $q \geq 0$, then $\Pi_p^{\iota}: \mathcal{G}(p) \to \mathcal{G}(p^{\iota})$ is a $(2n-m)$-equivalence.

PROOF In this case the inclusion ι is a $(2n)$-equivalence and the result follows from theorem 3.23.//

Theorem 5.6 remains valid if we consider SU(n) rather than U(n).

5.7 <u>THEOREM</u> Let $p: X \to B$ be a principal $O(n)$-bundle, where $0 \le n \le \infty$, and assume that $\dim(B) = m \le (n-1)$. If $\iota: O(n) \to O(n+q)$ denotes the inclusion, for $q \ge 0$, then $\Pi_p^{\iota}: \mathcal{G}(p) \to \mathcal{G}(p^{\iota})$ is an $(n-m-1)$-equivalence.

<u>PROOF</u> We only need to notice that in this case the inclusion ι is an $(n-1)$-equivalence.//

Also for theorem 5.7,the same result holds if we consider SO(n) rather than O(n).

Theorems 5.6 and 5.7, together with the results of section 1, allow us to compile the following list of values for the homotopy groups of $\mathcal{G}(p)$ in the case in which $p: X \to S^m$ is a principal bundle over a sphere with a classical group G as structure group.

1) $G = U$, $i \ge 0$, $m \ge 1$:

$$\pi_i(\mathcal{G}(p)) \sim \pi_i(U) \oplus \pi_{i+m}(U)$$

2) $G = SU$, $i \ge 0$, $m \ge 1$:

$$\pi_i(\mathcal{G}(p)) \sim \pi_i(SU) \oplus \pi_{i+m}(SU)$$

3) $G = U(n)$, $1 \leq m \leq 2n$, $0 \leq i \leq 2n-m-1$:

$$\pi_i(\mathcal{G}(p)) \sim \pi_i(U) \oplus \pi_{i+m}(U) \sim$$
$$\sim \pi_i(U(n)) \oplus \pi_{i+m}(U(n))$$

4) $G = SU(n)$, $1 \leq m \leq 2n$, $0 \leq i \leq 2n-m-1$:

$$\pi_i(\mathcal{G}(p)) \sim \pi_i(SU) \oplus \pi_{i+m}(SU) \sim$$
$$\pi_i(SU(n)) \oplus \pi_{i+m}(SU(n))$$

5) $G = O$, $i \geq 0$, $m \geq 1$:

$$\pi_i(\mathcal{G}(p)) \sim \pi_i(O) \oplus \pi_{i+m}(O)$$

6) $G = SO$, $i \geq 0$, $m \geq 1$:

$$\pi_i(\mathcal{G}(p)) \sim \pi_i(SO) \oplus \pi_{i+m}(SO)$$

7) $G = O(n)$, $1 \leq m \leq n-1$, $0 \leq i \leq n-m-2$

$$\pi_i(\mathcal{G}(p)) \sim \pi_i(O) \oplus \pi_{i+m}(O) \sim$$
$$\sim \pi_i(O(n)) \oplus \pi_{i+m}(O(n))$$

8) $G = SO(n)$, $1 \leq m \leq n-1$, $0 \leq i \leq n-m-2$:

$$\pi_i(\mathcal{G}(p)) \sim \pi_i(SO) \oplus \pi_{i+m}(SO) \sim$$
$$\sim \pi_i(SO(n)) \oplus \pi_{i+m}(SO(n))$$

It is only to be expected, now, that the strong relation existing between $U(n)$ and $GL(n,C)$ or between $O(n)$ and $GL(n,R)$ will give rise to a similarly strong relation between corresponding groups of bundle automorphisms, as the following theorem shows.

5.8 <u>THEOREM</u> Let $p: X \to B$ be a principal $O(n)$-bundle (resp., $U(n)$-bundle) and let $\iota: O(n) \to GL(n,R)$ (resp., $\iota: U(n) \to GL(n,C)$) be the inclusion. Then the map $\Pi_p^\iota: \mathcal{G}(p) \to \mathcal{G}(p^\iota)$ is a weak homotopy equivalence; moreover if B is compact, then Π_p^ι is a homotopy equivalence.

<u>PROOF</u> The first part of the theorem can be proved as in theorem 5.7. The second part follows from the fact that, under the given hypotheses, $\mathcal{G}(p)$ and $\mathcal{G}(p^\iota)$ have the homotopy type of a CW-complex (see [BHMP2] th.4.7).//

The following result, given in [Hoc], page 180 th.3.1, allows us to give a more general version of theorem 5.8.

5.9 <u>PROPOSITION</u> Let G be a connected Lie group. Then G contains a maximal compact subgroup H which is a strong deformation retract of G.//

5.10 <u>THEOREM</u> Let $p: X \to B$ be a principal G-bundle, where G is a connected Lie group and H be a maximal compact subgroup of G which is a strong deformation retract of G. If p admits a restriction q to H, then $\mathcal{G}(q)$ and $\mathcal{G}(p)$ are weakly homotopy equivalent; moreover if B is compact, then they are homotopy equivalent.//

§ 3) IDENTIFYING GROUPS OF FIBRE BUNDLE AUTOMORPHISMS WITH GROUPS OF PRINCIPAL BUNDLE AUTOMORPHISMS

In view of theorem 3.17 and corollary 3.18, it is clear that the results of sections 1 and 2 can give rise, under suitable conditions, to analogous statements concerning the groups of automorphisms of fibre bundles. Since these derived results can be obtained very easily, we shall not list them. Instead, we shall analyze the only point which is not clarified by theorem 3.17, that is, the relation between the path components of the spaces $\mathfrak{F}(p^F)$ and $\mathcal{G}(p^\pi)$ considered there.

To that end, let $p: X \to B$ be a principal G-bundle, F a left G-space with respect to the action λ and assume that the map $\lambda^*: G \to M_\lambda(F,F)$ is a Hurewicz fibration. Since we are assuming that B is a CW-complex, we can use theorem 3.17 and obtain a weak homotopy equivalence between $\mathfrak{F}(p^F)$ and $\mathfrak{F}(p^\pi)$, where $\pi: G \to Q_F$ is the canonical projection. Moreover we know that, in these conditions, the map $\Pi^{\pi}_{p}\colon \mathcal{G}(p) \to \mathcal{G}(p^{\pi})$ is also a Hurewicz fibration, so that $\mathfrak{F}(p^{\pi})$ is a group of complete path components of $\mathcal{G}(p^{\pi})$ and the map Π^{π}_{p} is surjective if and only if $\pi_0(\Pi^{\pi}_{p})$ is.

With this in mind we can prove the following results.

5.11 PROPOSITION Let $p\colon X \to B$ be a principal G-bundle over a CW-complex B of dimension n and let H be an (n-1)-connected normal subgroup of G. If the projection $\pi\colon G \to G/H$ is a Hurewicz fibration, then $\mathfrak{F}(p^{\pi}) = \mathcal{G}(p^{\pi})$.

PROOF Since π is a topological group epimorphism, the condition on H and lemma 3.14 imply that π is an n-equivalence. By theorem 3.23, $\Pi^{\pi}_{p}\colon \mathcal{G}(p) \to \mathcal{G}(p^{\pi})$ is a 0-equivalence, that is, it induces a surjection between path components. This is exactly the condition we need and so the claim is proved.//

5.12 COROLLARY Let $p\colon X \to B$ be a principal G-bundle over a CW-complex B of dimension n and F a left G-space such that the pair (p,F) satisfies C2). If Kr_F is (n-1)-connected, then there is a bijective map $t_{**}\colon \mathcal{G}(p^{\pi}) \to \mathfrak{F}(p^{F})$, where $\pi\colon G \to Q_F$ is the projection, which is a weak homotopy equivalence; moreover, if F is admissible, then t_{**} is a homeomorphism.//

A similar result can be obtained using the following generalization of [Sp], th. 7.8.12, as given in [Bo], page 334.

5.13 <u>LEMMA</u> Let B be a based, n-connected, CW-complex and $p_i\colon X_i \to B$ be fibrations with fibres F_i $(i=1,2)$. Let $f\colon X_1 \to X_2$ be a base point preserving map over B such that $f|b\colon F_1 \to F_2$ induces isomorphisms of homotopy groups in all dimensions bigger than, or equal to, n. Then f induces a bijection $f_*\colon [\sec(p_1)] \to [\sec(p_2)]$, where $[\sec(p_i)]$ denotes the set of based homotopy classes of base point preserving sections of p_i.//

5.14 <u>THEOREM</u> Let $p\colon X \to B$ be a principal G-bundle over an n-connected CW-complex B and let H be a normal subgroup of G such that $\pi_i(H) = 0$ for $i \geq (n-1)$ and $\pi\colon G \to G/H$ is a Hurewicz fibration. Then $\mathfrak{J}(p^\pi) = \mathcal{G}(p^\pi)$.

<u>PROOF</u> Fix a base point $b \in B$ and, given $s \in \sec(p^\pi\ p^\pi)_{G/H}$, let $s(b)$ be the base point of $(X/H\ \ X/H)_{G/H}$. Since, in this case, $\Psi^\pi\colon (X\ X)_G \to (X/H\ \ X/H)_{G/H}$ is surjective, we can select a point in $(\Psi^\pi)^{-1}(s(b))$ as base point for

$(X\ X)_G$. We can now apply lemma 5.13 to the fibrations $(p\ p)_G$ and $(p^\pi\ p^\pi)_{G/H}$ and find a section s' of $(p\ p)_G$ such that $\Psi^\pi \cdot s' \simeq s$. This means, as we need, that $\pi_0(\Psi^\pi_p)\colon \pi_0(\sec(p\ p)_G) \to \pi_0(\sec(p^\pi\ p^\pi)_{G/H})$ is surjective and hence $\mathfrak{J}(p^\pi) = \mathfrak{G}(p^\pi)$.//

5.15 <u>COROLLARY</u> Let $p\colon X \to B$ be a principal G-bundle over an n-connected CW-complex B and F a left G-space such that the pair (p,F) satisfies C2). If $\pi_i(Kr_F)=0$ for $i \geq (n-1)$ and $\pi\colon G \to Q_F$ denotes the projection, then there is a continuous bijection $t_{**}\colon \mathfrak{G}(p^\pi) \to \mathfrak{J}(p^F)$ which is a weak homotopy equivalence; moreover if F is admissible, then t_{**} is a homeomorphism.//

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APPENDIX A

AN OUTLINE OF AN

ANALOGOUS THEORY

Throughout the thesis we have been concerned with the problem of looking at how the groups of principal bundle automorphisms are affected by a change in the structure group, assuming that this change is induced by a topological group morphism; i.e., if $p: X \to B$ is a principal G-bundle and $h: G \to K$ is a topological group morphism, then what relations exist between $\mathcal{G}(p)$ and $\mathcal{G}(p^h)$?

There is now an analogous question which can be analyzed, that is, how are the groups of principal bundle automorphisms affected by a change of the base space induced by a continuous function? More specifically, if $p: X \to B$ is a principal G-bundle, $h: A \to B$ a map and $p_h: A \sqcap X \to A$ the induced principal G-bundle, what relations exist between $\mathcal{G}(p)$ and $\mathcal{G}(p_h)$?

Answering this question in full falls outside the scope of this thesis, however we now outline the way in which the techniques that we have used in the main body of the thesis can be applied to this question in order to obtain similar results. The details of the proofs will be omitted, as they will find a better location in a separate and more complete treatment elsewhere.

We like to point out that a partial approach to this question has been described in [Mo] , in the case where h is the inclusion of a base point, and in [Go2] and [BHMP2], where the authors give a generalization, to the relative case, of the relation, described in chapter 4, between $\mathcal{G}(p)$ and loop spaces on mapping spaces. We shall see later that the relative groups obtained there play a role analogous to that played by the groups $\mathcal{G}(p;H)$ in the main portion of the thesis.

First of all, the following analogue of theorem 1.34 can be easily verified.

A.1 <u>THEOREM</u> Let G be a topological group and $h: A \to B$ a map. Then, there is an induced functor $\Pi_h: P_B^G \to P_A^G$ defined, for principal G-bundles $p: X \to B$ and $p': X' \to B$ and morphisms f from p to p', by the relations

$$\Pi_h(p) = p_h \quad ; \quad \Pi_h(f) = 1\times f$$

where

$$1\times f: A\sqcap X \to A\sqcap X': (a,x) \rightsquigarrow (a,f(x)) \quad .//$$

A technique similar to that of theorem 2.37 gives rise to the following result.

A.2 THEOREM Let $p: X \to B$ be a principal G-bundle and $h: A \to B$ be a map. Then there is an induced continuous homomorphism

$$\Pi_h^p: \mathcal{G}(p) \to \mathcal{G}(p_h)$$

defined by the relation

$$\Pi_h^p(f)(a,x) = (a,f(x)) \quad .//$$

We notice that if h is an inclusion, then Π_h^p has the effect of restricting an automorphism of p to the inverse image of A.

As in the case of Π_p^h, the map Π_h^p is not surjective in general and its kernel can be described in terms of subspaces of B, in a fashion similar to that of theorem 2.30.

A.3 DEFINITION Let $p: X \to B$ be a principal G-bundle and let U be a subspace of B. Then $\mathcal{G}(p/U)$ will denote the subgroup of $\mathcal{G}(p)$ consisting of those automorphisms which fix all points $x \in X$ such that $p(x) \in U$.

If $U = \{b\}$, where b is the base point of B, $\mathcal{G}(p/U)$ is just the group $\mathcal{G}^1(p)$ of [Mo]. In view of the ongoing analogy, the following results will not be surprising.

A.4 <u>THEOREM</u> Let $p: X \to B$ be a principal G-bundle and U a subspace of B. Then $\mathcal{G}(p/U)$ is a normal topological subgroup of $\mathcal{G}(p)$ and coincides with $\mathcal{G}(p/\overline{U})$, where $\overline{U}$ is the closure of U in B.//

A.5 <u>THEOREM</u> Let $p: X \to B$ be a principal G-bundle and $h: A \to B$ a map. Then the kernel of the homomorphism $\Pi_h^p: \mathcal{G}(p) \to \mathcal{G}(p_h)$ is $\mathcal{G}(p/\mathrm{im}(h))$.//

The spaces $\mathcal{G}(p)$, $\mathcal{G}(p_h)$ and $\mathcal{G}(p/U)$ can all be viewed as spaces of sections to functional bundles, as described in the following theorem.

A.6 <u>THEOREM</u> Let $p:X \to B$ be a principal G-bundle, $h: A \to B$ be a map and U be a subspace of B. Then $\mathcal{G}(p)$ is homeomorphic to $\mathrm{sec}(p\ p)_G$, while $\mathcal{G}(p/U)$ is homeomorphic to the subspace of $\mathrm{sec}(p\ p)_G$ consisting of those sections s such that, for every $b \in U$, $s(b)$ is the identity map on $(p\ p)_G^{-1}(b)$. Moreover, since, by [BB2] th. 1.1 and 8.2, $(A \sqcap X\ \ A \sqcap X)_G$ and $A \sqcap (X\ X)_G$ are homeomorphic over B, $\mathcal{G}(p_h)$ is homeomorphic to $\mathrm{sec}(((p\ p)_G)_h)$.//

The map $\Psi_h^p: \mathrm{sec}(p\ p)_G \to \mathrm{sec}(p_h\ p_h)_G$ corresponding to the homomorphism $\Pi_h^p\colon \mathcal{G}(p) \to \mathcal{G}(p_h)$ can now be defined using the universal property of pullbacks and the reader will notice that the fibre of this map between spaces of sections is the subspace of $\sec(p\ p)_G$ corresponding, according to theorem A.6, to $\mathcal{G}(p/\mathrm{im}(h))$. However, in contrast with the map Ψ_p^h, the map Ψ_h^p is not directly induced , in general, by a map between the total spaces of the functional bundles in question.

The next step is to determine conditions under which the map Π_h^p is a fibration. This, as one would expect, requires some condition on h.

A.7 <u>THEOREM</u> Let $p\colon X \to B$ be a principal G-bundle and $h\colon A \to B$ be a closed cofibration. Then the map $\Pi_h^p\colon \mathcal{G}(p) \to \mathcal{G}(p_h)$ is a fibration and hence maps $\mathcal{G}(p)$ onto a group of complete path components of $\mathcal{G}(p_h)$.//

The naturality, in this context, of the weak homotopy equivalence between $\Omega\mathrm{Map}(B,B_G;k)$ and $\mathcal{G}(p)$ is described in the following result.

A.8 <u>THEOREM</u> Let $p\colon X \to B$ be a numerable principal G-bundle classified by the map $k\colon B \to B_G$ and let $h\colon A \to B$ be a map. Then there exists a homotopy commutative diagram

$$\begin{array}{ccc} \Omega\mathrm{Map}(B,B_G;k) & \xrightarrow{\quad\delta\quad} & \mathcal{G}(p) \\ \Omega h^* \downarrow & & \downarrow \Pi^p_h \\ \Omega\mathrm{Map}(A,B_G;k\cdot h) & \xrightarrow{\quad\delta_h\quad} & \mathcal{G}(p_h) \end{array}$$

where δ and δ_h are the H-weak homotopy equivalences described in chapter 4.//

This result can conceivably be formulated in categorical terms as stating the existence of a natural isomorphism between two functors.

Finally, the following observation indicates a way of connecting the main theory of the thesis with the one we have just described.

A.9 <u>THEOREM</u> Let $p: X \to B$ be a principal G-bundle, $h: G \to K$ a topological group morphism and $f:A \to B$ a map. Then $(p^h)_f$ and $(p_f)^h$ are isomorphic objects of P^K_A.//

The connection between the two theories may prove fruitful both for its theoretical aspects and for its applications.

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A P P E N D I X   B

P R I N C I P A L   A N D

F I B R E   B U N D L E S   A S

F U N C T I O N A L   B U N D L E S

In chapter 1, proposition 1.30, we have seen how to construct a functor $\Gamma^F: P_B^G \to F_B^F(G)$ when we are given a space B, a topological group G and a left G-space F. Still in chapter 1, section 5, we have seen how to use the functional construction to obtain information on the group of automorphisms of a principal or fibre bundle.

In this appendix we shall use that functional construction to obtain a functor $\Theta^F: P_B^G \to F_B^F(G)$, naturally equivalent to Γ^F, and, whenever F is admissible and effective, another functor $\Xi^F: F_B^F(G) \to P_B^G$ such that $\Theta^F \cdot \Xi^F$ and $\Xi^F \cdot \Theta^F$ are naturally equivalent to the respective identity functors.

We like to notice that the construction on which the functor Ξ^F is based was first suggested, even though in a less rigorous form, in [St], page 39, while in [BB2], example 3.3, functional bundles were explicitly indicated as an effective formal way to obtain associated principal bundles. However the symmetric construction, used here to define the functor Θ^F, was never considered in the original papers of Booth and Brown, and we notice that its basic idea reminds us of the Ehresmann-Feldbau definition of a bundle (see, e.g., [St], def. 5.2).

§ 1) FIBRE BUNDLES

First of all recall that, given a right G-space X and a left G-space F, we can consider the notion of a "G-map" $f\colon X \to F$, by requiring that, for every $x \in X$ and $g \in G$, $f(xg) = g^{-1}f(x)$ (see the comment following definition 1.6).

If $p\colon X \to B$ is a principal G-bundle, F a left G-space and $t\colon B\times F \to B$ the projection, we can form the functional bundle $(p\ t)\colon (X\ B\times F) \to B$ and, if $\{(U,\phi_U)\}$ is the locally trivial structure for p, then $\{(U,\chi_U)\}$ is a locally trivial structure for $(p\ t)$, where

$$\chi_U\colon U\times \mathrm{Map}(G,F) \to (X\ \ B\times F)|U$$

is defined by the relation

$$\chi_U(b,f)(\phi_U(b,g)) = (b,f(g)) \quad .$$

Let us now consider the restriction $(p\ t)_G$ of $(p\ t)$ to the subspace $(X\ \ B\times F)_G$ of $(X\ \ B\times F)$ consisting of G-maps. Then it is easy to see, using the homeomorphism between $M_G(G,F)$ and F described in lemma 1.9, that $(p\ t)_G$ has a locally trivial structure given by $\{(U,\chi_U)\}$, where, with a little abuse of notation,

$$\chi_U\colon U\times F \to (X\ \ B\times F)_G|U$$

is defined by the relation

$$\chi_U(b,y)(\phi_U(b,g)) = (b,g^{-1}y)$$

B.1 <u>THEOREM</u> Given a principal G-bundle $p\colon X \to B$ and a left G-space F, there is a homeomorphism

$$T_p^F\colon (X \quad B\times F)_G \to X\times_G F$$

over B, defined, for a given $\alpha\colon p^{-1}(b) \to \{b\}\times F \simeq F$, by the relation $T_p^F(\alpha) = [x,\alpha(x)]$, where x is any point in $p^{-1}(b)$. Therefore the pair $((p\ \ t)_G, T_p^F)$ is a fibre bundle with fibre F associated to p.

<u>PROOF</u> If x and y are two points in $p^{-1}(b)$, then $y = xg$ for some $g\in G$ and hence

$$[y,\alpha(y)] = [xg,\alpha(xg)] = [xg,g^{-1}\alpha(x)] = [x,\alpha(x)]$$

proving that T_p^F is well defined. Moreover, for each $U\in\{U\}$, there is a commutative diagram

$$\begin{array}{ccc}
(X \quad B\times F)_G|U & \xrightarrow{T_p^F|U} & X\times_G F|U \\
\uparrow{\scriptstyle \chi_U} & & \uparrow{\scriptstyle \phi_U^F} \\
U\times F & \xrightarrow{\quad 1 \quad} & U\times F
\end{array}$$

which proves that T_p^F is a homeomorphism. The last statement follows from definition 1.27.//

Let now f be a morphism in P_B^G from the principal bundle $p\colon X \to B$ to the principal bundle $p'\colon X' \to B$ and define a function

$$f_*\colon (X \quad B\times F)_G \to (X' \quad B\times F)_G\colon \alpha \rightsquigarrow \alpha\cdot(f^{-1}|b)$$

where $b = (p\ t)_G(\alpha)$. Since P_B^G is a groupoid, this can be done; moreover the commutativity of the diagram

B.2
$$\begin{array}{ccc} (X \quad B\times F)_G & \xrightarrow{\quad f^* \quad} & (X' \quad B\times F)_G \\ {\scriptstyle T_p^F}\big\downarrow & & \big\downarrow{\scriptstyle T_{p'}^F} \\ X\times_G F & \xrightarrow{\quad f^F \quad} & X'\times_G F \end{array}$$

ensures that f^* is a homeomorphism and so, by definition 1.29, that it is a fibre bundle morphism.

B.3 <u>**THEOREM**</u> Given a space B, a topological group G and a left G-space F, there is a functor

$$\Theta^F : P_B^G \to F_B^F(G)$$

defined by $\Theta^F(p) = (p\ t)_G$ and $\Theta^F(f) = f^*$, which is naturally isomorphic to the functor Γ^F defined in proposition 1.30.

<u>**PROOF**</u> The functorial properties of Θ^F can be easily verified. Since, for every fibre bundle (q,h) associated to a principal bundle p, the map h is a morphism in $F_B^F(G)$ from (q,h) to $(p^F,1)$, diagram B.2 proves that there is a natural isomorphism $T^F : \Theta^F \dot{\to} \Gamma^F$ determined, for each object p of P_B^G, by setting $T^F(p) = T_p^F$.//

REMARK: It is possible to retrieve most, if not all, of the general theory of fibre bundles by using the functor Θ^F to define a fibre bundle and then applying the results of [BB1] and [BB2]. For instance, [BB1] th. 1.1, plus the natural homeomorphism between $A\sqcap(B\times F)$ and $A\sqcap F$ for any map $k: A \to B$, prove [Hus] cor. 6.4 page 46. Also [BB1] cor 3.7, plus the obvious homeomorphism between $(X \; B\times G)_G$ and X, imply the fact that a principal G-bundle is trivial if and only if it has a section, etc.

§ 2) ASSOCIATED PRINCIPAL BUNDLES

We now give a construction,similar to the one of section 1, in order to obtain a principal bundle associated to a given fibre bundle.

So, let $q: Y \to B$ and $h: Y \to X\times_G F$ be such that (q,h) is a fibre bundle with fibre F associated to the principal G-bundle $p: X \to B$ and consider the functional bundle $(t \; q): (B\times F \;\; Y) \to B$, where $t: B\times F \to B$ is the projection. Again, if $\{(U,\phi_U^F)\}$ is the locally trivial structure for q, then $(t \; q)$ has a locally trivial structure $\{(U,\chi_U^F)\}$, where

$$\chi_U^F\colon U\times \mathrm{Map}(F,F) \to (B\times F \quad Y)|U$$

is defined by the relation

$$\chi_U^F(b,f)(b,y) = \phi_U^F(b,f(y)) \quad .$$

Consider now the restriction $(t \quad q)_\lambda$ of $(t \quad q)$ to the subspace $(B\times F \quad Y)_\lambda$ of $(B\times F \quad Y)$ consisting of maps of the form

$$\phi_{U,b}^F \cdot g^*$$

where g^* is an action map, $b\in U\in\{U\}$ and $\phi_{U,b}^F$ is the restriction of ϕ_U^F to $\{b\}\times F$. Then we have:

B.4 <u>LEMMA</u> For each $U\in\{U\}$, $\chi_U^F(U\times M_\lambda(F,F)) = (t \quad q)_\lambda^{-1}(U)$.

<u>PROOF</u> If $(b,g^*)\in U\times M_\lambda(F,F)$, then, by definition, $\chi_U^F(b,g^*) = \phi_{U,b}^F\cdot g^*$, hence $\chi_U^F(U\times M_\lambda(F,F)) \subset (t \quad q)_\lambda^{-1}(U)$. To prove the other inclusion, let $b\in U$, $V\in\{U\}$ be an open set containing b and $\phi_{V,b}^F\cdot g^*$ be an element of $(t \quad q)_\lambda^{-1}(b)$. We can now write

$$\phi_{V,b}^F\cdot g^* = \phi_{U,b}^F\cdot(\phi_{U,b}^F)^{-1}\cdot\phi_{V,b}^F\cdot g^*$$

and it is easy to verify that the composite

$$(\phi_{U,b}^F)^{-1}\cdot\phi_{V,b}^F$$

is an action map, say a^*. Therefore we have

$$\phi_{V,b}^F\cdot g^* = \phi_{U,b}^F\cdot(ag)^* \in \chi_U^F(U\times M_\lambda(F,F))$$

thus proving the claim.//

So, with another useful abuse of notation, we can say that the local homeomorphisms for $(B\times F\ \ Y)_\lambda$ are given by the maps:

$$\chi_U^F\colon U\times M_\lambda(F,F) \to (B\times F\ \ Y)_\lambda|U$$

where $\chi_U^F(b,g^*) = \phi_{U,b}^F\cdot g^*$. If F is admissible, then $M_\lambda(F,F)$ is homeomorphic to $Q_F=G/Kr_F$ (definition 1.5) and hence we can write the local homeomorphisms as

$$\chi_U^F\colon U\times Q_F \to (B\times F\ \ Y)_\lambda|U$$

where $\chi_U^F(b,gKr_F) = \phi_{U,b}^F\cdot g^*$.

B.5 THEOREM If F is admissible, then $(t\ q)_\lambda$ is a principal Q_F-bundle.

PROOF Define an action of Q_F on $(B\times F\ \ Y)_\lambda$ by $\nu\colon (B\times F\ \ Y)_\lambda\times Q_F \to (B\times F\ \ Y)_\lambda\colon (\phi_{U,b}^F\cdot g^*, aKr_F) \rightsquigarrow \phi_{U,b}^F\cdot(ga)^*$
This action is clearly well defined; it is continuous because, for each $U\in\{U\}$, the composite

$$U\times M_\lambda(F,F)\times G \xrightarrow{1\times1\times\lambda^*} U\times M_\lambda(F,F)\times M_\lambda(F,F) \xrightarrow{1\times\varsigma} U\times M_\lambda(F,F)$$

is continuous. Moreover the χ_U^F are clearly Q_F-maps, so, by definition, $(t\ q)_\lambda$ is a principal Q_F-bundle.//

Recall now that (q,h) is associated to the principal G-bundle $p\colon X \to B$ and that, by corollary 1.44,

$X\times_G F$ is naturally homeomorphic to $(X/Kr_F)\times_G F$. Then the following proposition will show that q can also be considered as a fibre bundle associated to $(t\ q)_\lambda$.

B.6 <u>PROPOSITION</u> If F is admissible and $\pi: G \to Q_F$ is the canonical projection, then the principal Q_F-bundles $(t\ q)_\lambda: (B\times F\ \ Y)_\lambda \to B$ and $p^\pi: X/Kr_F \to B$ are isomorphic.

<u>PROOF</u> Define, for each $U\in\{U\}$, a map

$$S_U^F: (X/Kr_F)|U \to (B\times F\ \ Y)_\lambda|U$$

as the composite

$$S_U^F: (X/Kr_F)|U \xrightarrow{(\phi_U^\pi)^{-1}} U\times Q_F \xrightarrow{\chi_U^F} (B\times F\ \ Y)_\lambda|U$$

Each of these maps is a Q_F-homeomorphism over U, so if we can glue them together, we shall obtain the desired isomorphism $S^F: (X/Kr_F) \to (B\times F\ \ Y)_\lambda$. To see that this is indeed possible, observe that, for any U and V in $\{U\}$ such that $U\cap V \neq \emptyset$, we have a diagram:

$$\begin{array}{ccc}
(U\cap V)\times Q_F & \xrightarrow{\phi_U^\pi} & (X/Kr_F)|(U\cap V) \\
{\scriptstyle \chi_U^F}\downarrow & & \downarrow{\scriptstyle (\phi_V^\pi)^{-1}} \\
(B\times F\ \ Y)_\lambda|(U\cap V) & \xrightarrow{(\chi_V^F)^{-1}} & (U\cap V)\times Q_F
\end{array}$$

in which, for any $(b, gKr_F) \in (U\cap V)\times Q_F$, we have:

$$(\chi_V^F)^{-1}\cdot\chi_U^F(b,g\mathrm{Kr}_F) = (\chi_V^F)^{-1}(\phi_{U,b}^F\cdot g^*) =$$

$$= (\chi_V^F)^{-1}(\phi_{V,b}^F\cdot(\phi_{V,b}^F)^{-1}\cdot\phi_{U,b}^F\cdot g^*) =$$

$$= (\chi_V^F)^{-1}(\phi_{V,b}^F\cdot(\phi_{V,b}^{-1}\cdot\phi_{U,b}(e))^*\cdot g^*) =$$

$$= (b,(\phi_{V,b}^{-1}\cdot\phi_{U,b}(e))\ g\mathrm{Kr}_F)$$

while

$$(\phi_V^\pi)^{-1}\cdot\phi_U^\pi(b,g\mathrm{Kr}_F) = (b,(\phi_{V,b}^{-1}\cdot\phi_{U,b}(g))\mathrm{Kr}_F)$$

Since the maps ϕ_U are G-maps, it follows that the diagram is commutative and hence that the isomorphism S^F exists.//

If F is effective, the kernel Kr_F is trivial and we are able to identify Q_F with G, thus obtaining the following result.

B.7 <u>PROPOSITION</u> Let $p: X \to B$ and $p': X' \to B$ be two principal G-bundles, F an admissible and effective left G-space and $f^F = \Gamma^F(f)$ a morphism in $\mathcal{F}_B^F(G)$ from p^F to p'^F. Then the function

$$f_*^F: (B\times F\ \ Y)_\lambda \to (B\times F\ \ Y')_\lambda: \alpha \rightsquigarrow f_b^F\cdot\alpha$$

where $b = (t\ q)_\lambda(\alpha)$ and $f_b^F = f^F|b$, is a morphism of the category $\mathcal{P}_B^G$.

<u>PROOF</u> To show that f_*^F is well defined, let $\alpha = \phi_{U,b}^F \cdot a^*$ and notice that, since we are assuming $Y = X \times_G F$ and $Y' = X' \times_G F$, for every $y \in F$,

$$f_b^F \cdot \phi_{U,b}^F(y) = [f \cdot \phi_U(b,e), y]$$

So, if $f \cdot \phi_U(b,e) = \phi_U'(b,g)$, then

$$f_b^F \cdot \phi_{U,b}^F(y) = [\phi_U'(b,e), gy] = \phi_{U,b}'^F \; g^*(y)$$

where g does not depend on y. Hence

$$f_b^F \cdot \alpha = \phi_{U,b}'^F \cdot (ga)^*$$

is an element of $(B \times F \;\; Y')_\lambda$. Since f_*^F works by composition on the left, it preserves the action of G, so, to prove that it is a morphism of P_B^G, we only have to prove its continuity. But this follows at once from the commutativity of the diagram

$$\begin{array}{ccc} (B\times F \;\; Y)_\lambda & \xrightarrow{\;\; f^F \;\;} & (B\times F \;\; Y')_\lambda \\ S^F \uparrow & & \uparrow S'^F \\ X & \xrightarrow{\;\; f \;\;} & X' \end{array}$$

which can be verified considering that, by a previous remark , for an element $\phi_U(b,e) \in X$, we have

$$f_*^F \cdot S^F(\phi_U(b,e)) = f_*^F(\phi_{U,b}^F) = \phi_{U,b}'^F \cdot g^* =$$

$$= S'^F(\phi_U'(b,g)) = S'^F(f \cdot \phi_U(b,e))$$

and, moreover, that all the maps in the diagram are G-maps.//

B.8 <u>THEOREM</u> If F is an admissible and effective left G-space, there exists a functor $\Xi^F: \mathcal{F}_B^F(G) \to \mathcal{P}_B^G$ defined, for a given object (q,h) and a given morphism $(h')^{-1}\cdot f^F \cdot h$ from (q,h) to (q',h'), by the relations:

$$\Xi^F(q,h) = (t\ q)_\lambda$$

$$\Xi^F((h')^{-1}\cdot f^F\cdot h) = (h')_*^{-1}\cdot f_*^F\cdot h_*$$

where h_* and $(h')_*^{-1}$ are the maps induced in the obvious way between the corresponding functional spaces.

<u>PROOF</u> We have seen that Ξ^F determines a well defined function between objects and.it follows from proposition B.7 that it is well defined also for morphisms. Its functoriality can be easily verified.//

B.9 <u>THEOREM</u> If F is an admissible and effective left G-space, then the composites $\Xi^F\cdot\Theta^F$ and $\Theta^F\cdot\Xi^F$ are naturally isomorphic to the respective identity functors and so the categories $\mathcal{P}_B^G$ and $\mathcal{F}_B^F(G)$ are equivalent.

<u>PROOF</u> Define a natural isomorphism $\Delta: 1 \stackrel{\bullet}{\to} \Xi^F\cdot\Theta^F$ by setting, for a given object $p: X \to B$ of $\mathcal{P}_B^G$,

$$\Delta(p): X \xrightarrow{S^F} (B\times F \quad X\times_G F)_\lambda \xrightarrow{(T_p^F)_*^{-1}} (B\times F \quad (X \quad B\times F)_G)_\lambda$$

where $(T_p^F)_*^{-1}$ is induced in the usual way by $(T_p^F)^{-1}$.

Since $(T_p^F)_*^{-1}$ works by composition on the left and the action of G on $(B{\times}F\ \ Y)_\lambda$ by composition on the right, it follows that $(T_p^F)_*^{-1}$, and hence $\Delta(p)$, are G-maps and, in particular, $\Delta(p)$ is an isomorphism. The naturality of Δ follows from the commutativity of the diagram

$$\begin{array}{ccccc} X & \xrightarrow{S^F} & (B{\times}F\ \ X{\times}_G F)_\lambda & \xrightarrow{(T_p^F)_*^{-1}} & (B{\times}F\ \ (X\ \ B{\times}F)_G)_\lambda \\ {\scriptstyle f}\downarrow & & {\scriptstyle f_*^F}\downarrow & & \downarrow{\scriptstyle (f^*)_*} \\ X' & \xrightarrow{S'^F} & (B{\times}F\ X'{\times}_G F)_\lambda & \xrightarrow{(T_{p'}^F)_*^{-1}} & (B{\times}F\ (X'\ \ B{\times}F)_G)_\lambda \end{array}$$

which can be derived from our earlier results.
Similar considerations on the natural isomorphism $\nabla\colon \Theta^F\cdot\Xi^F \dot{\to} 1$, defined by setting, for a given object (q,h)

$$\nabla(q,h)\colon ((B{\times}F\ Y)_\lambda\ \ B{\times}F)_G \xrightarrow{T_r^F} (B\ \ F{\times}Y)_\lambda{\times}_G F \longrightarrow$$

$$\xrightarrow{(S^F)^{-1}{\times}_G 1} X{\times}_G F \xrightarrow{h^{-1}} Y$$

where $r = (t\ \ q)_\lambda$, complete the proof of the theorem.//

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