EMBEDDING THEOREMS FOR CLOSED CATEGORIES

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ABSTRACT

Let us consider the following symmetric monoidal closed categories:

(i) $\mathcal{S}_M$, the category of sets under the action of a commutative monoid M; in short, a category of M-sets;

(ii) $\mathcal{S}_G$, the category of G-sets, where G is an abelian group;

(iii) $\mathcal{M}_K$, the category of moduloids over a commutative semiring K (a moduloid is basically a monoid acted on by a semiring);

(iv) Mod_K, the category of modules over a commutative ring K;

(v) $\mathcal{V}_F$, the category of vector spaces over a field F.

Let $\mathcal{C}$ be an arbitrary closed category. We are concerned with the following question:

What conditions have to be imposed on $\mathcal{C}$ to ensure that it can be embedded (in some canonical way) into one or more of the above categories?

The basic category theory needed in this thesis is provided in chapters I and II. In chapter I we have provided the details of how, in a category with biproducts, the set hom(A,B) can be given the structure of a commutative monoid (under addition). Chapter II gives a summary of the standard definitions and results leading up to the concept of a symmetric monoidal closed category.

Since the properties of categories (i) and (iii) are not so well known, these categories are discussed in some detail in chapters III and IV. It is shown that each of the categories is in fact a symmetric monoidal closed category.

In chapter V we answer our original question by establishing five embedding theorems.

Each of these theorems gives sufficient conditions for a closed category to be embeddable in one of the above categories. Fairly elementary examples are given to illustrate each of the theorems.

In the appendix a detailed example is given to show that these embeddings are not, in general, full embeddings.

CONTENTS

CHAPTER ONE

CATEGORIES AND BIPRODUCTS

It is well known that in an Abelian category the set hom(A,B) of morphisms from A to B can be enriched with an Abelian group structure. In this chapter we will provide some basic category theory and show, by a fairly standard argument, that in a category with biproducts, the set hom(A,B) can be given the structure of a commutative monoid.

1. Categories

Definition 1.1

A category $\mathcal{C}$ consists of

(i) a class of objects A,B,C ... ;

(ii) for each pair (A,B) of objects a set hom(A,B) the elements of which are called morphisms from A to B of $\mathcal{C}$, with domain A and codomain B. (We write $x:A \longrightarrow B$ or $A \xrightarrow{x} B$ for each $x \in$ hom(A,B)

(iii) for each triple (A,B,C) of objects a function

$$\text{hom}(A,B) \times \text{hom}(B,C) \longrightarrow \text{hom}(A,C)$$

called composition of morphisms;

these data being subject to the two axioms

(1) If $x \in$ hom(A,B), $y \in$ hom(B,C), $z \in$ hom(C,D) then $z \circ (y \circ x) = (z \circ y) \circ x$

(2) For each object A there exists an element $1_A \in$ hom(A,A) called an identity morphism such that if $x \in$ hom(A,B) then $x \circ 1_A = x$; $1_B \circ x = x$

Remark: The morphism 1_A whose existence is required by (2) is uniquely defined; because if $1'_A$ is a second morphism with the same property then $1'_A \circ 1_A = 1'_A = 1_A$

During the course of this thesis we will frequently refer to the following categories:

$\mathcal{S}$, the category of all sets;

$\mathcal{S}_*$, the category of pointed sets;

$\mathcal{S}_M$, the category of sets under the action of a commutative monoid M;

$\mathcal{S}_G$, the category of sets under the action of an Abelian group G;

Mod_K, the category of modules over a commutative ring K;

M_K, the category of moduloids over a commutative semiring K (the terms moduloid and semiring will be defined in chapter 4);

$\mathcal{V}_F$, the category of vector spaces over a field F.

Because the above categories have objects with underlying sets, that is there is a faithful functor $\mathcal{C} \longrightarrow \mathcal{S}$, they are more specifically referred to as concrete categories. However, it can be easily shown that every concrete category is a category.

A category $\mathcal{C}'$ is a <u>subcategory</u> of $\mathcal{C}$ under the following conditions

(1) $\mathrm{Ob}\ \mathcal{C}' \subset \mathrm{Ob}\ \mathcal{C}$

(2) $\hom_{\mathcal{C}'}(A,B) \subset \hom_{\mathcal{C}}(A,B)$ for all $(A,B) \in \mathcal{C}' \times \mathcal{C}'$

(3) the composition of any two morphisms in $\mathcal{C}'$ is the same as their composition in $\mathcal{C}$

(4) 1_A is the same in $\mathcal{C}'$ as in $\mathcal{C}$ for all $A \in \mathcal{C}'$

If furthermore $\hom_{\mathcal{C}'}(A,B) = \hom_{\mathcal{C}}(A,B)$ for all $(A,B) \in \mathcal{C}' \times \mathcal{C}'$ we say that C' is a full subcategory of C. For example, the category of Abelian groups is a full subcategory of the category of all groups.

Definition 1.2

For every category C we define the dual category C^* as follows:

(i) $\text{Ob } C^* = \{A^* \mid A \in \text{Ob } C\}$

(ii) $\text{Mor } C^* = \{x^* \mid x \in \text{Mor } C\}$ where $x^* \circ y^* = (y \circ x)^*$

That is the objects of C^* are the same as the objects of C and a morphism $A \longrightarrow B$ in C^* is a morphism $B \longrightarrow A$ in C.

Definition 1.3

For each pair of categories C, C', there exists a product category $C \times C'$. An object of this product is an ordered pair (A,A') of objects of C and C' respectively; a morphism $(A,A') \longrightarrow (B,B')$ with the indicated domain and codomain is an ordered pair (f,f') of morphisms $f:A \longrightarrow B$, $f':A' \longrightarrow B'$. The composite of morphisms is defined term-wise; Thus (f,f') as above and a second such ordered pair $(g,g'): (B,B') \longrightarrow (D,D')$ have the composite $(g,g') \circ (f,f') = (g \circ f, g' \circ f'): (A,A') \longrightarrow (D,D')$.

Definition 1.4

A morphism $x:A \longrightarrow B$ is invertible (is an isomorphism) in C iff there is a morphism $x':B \longrightarrow A$ in C with both $x' \circ x = 1_A$ and $x \circ x' = 1_B$.

A familiar argument shows that if such a morphism exists, it is unique; hence it is usually written $x' = x^{-1}$. Two objects A and B are equivalent (i.e. isomorphic) in C if there is an invertible morphism $x:A \longrightarrow B$.

2. Functors

Definition 1.5

A covariant functor (or simply functor) from a category A to a category B is a function $F:A \longrightarrow B$ which assigns to every object A of A an object F(A) of B and to every morphism $x:A_1 \longrightarrow A_2$ in A a morphism $F(x) : F(A_1) \longrightarrow F(A_2)$ in B such that

$$(1) \quad F(1_A) = 1_{F(A)}$$

$$(2) \quad F(x \circ y) = F(x) \circ F(y)$$

Remark: If condition (2) is replaced by

$$(2') \quad F(x \circ y) = F(y) \circ F(x)$$

we speak of a contravariant functor $F:A \longrightarrow B$ which assigns to every morphism $x:A_2 \longrightarrow A_1$ in A a morphism $F(x) : F(A_1) \longrightarrow F(A_2)$ in B

Examples (The Standard hom Functors)

Let S be the category of sets, A an arbitrary category and A an object in A.

(1) The functor $\hom(A,-) : A \longrightarrow S$ defined as follows: For $B \in A$ $\hom(A,-)(B) = \hom(A,B)$. For $x:B_1 \longrightarrow B_2 \in A$, $\hom(A,-)(x)$ is the function $\hom(A,B_1) \xrightarrow{\hom(A,x)} \hom(A,B_2)$ defined by $\hom(A,x)(y) = x \circ y$ where $y:A \longrightarrow B_1$; is covariant.

(2) The functor $\hom(-,A) : A \longrightarrow S$ defined as follows: For $B \in A$ $\hom(-,A)(B) = \hom(B,A)$. For $x:B_1 \longrightarrow B_2 \in A$ $\hom(-,A)(x)$ is the function $\hom(B_2,A) \xrightarrow{\hom(x,A)} \hom(B_1,A)$ defined by $\hom(x,A)(y) = y \circ x$ where $y: B_2 \longrightarrow A$; is contravariant.

Definition 1.6

Given a functor $F:A \longrightarrow B$ between two categories A and B, we write F_{AB}: hom(A,B) $\longrightarrow$ hom(FA,FB) for the associated functions on the sets of morphisms. The functor F is called faithful if each F_{AB} is injective.

That is if F is a faithful functor, then the morphism $f:A \longrightarrow B$ is completely determined by $Ff:FA \longrightarrow FB$ since $f,g:A \longrightarrow B$ with $Ff = Fg : FA \longrightarrow FB$ implies that $f = g$.

Lemma 1.1

If $F:A \longrightarrow B$ is faithful, then a diagram of morphisms in A commutes iff F of it commutes in B..

Proof (Easy)

Definition 1.7

A functor $\theta: A \times B \longrightarrow C$ on a product category $A \times B$ to another category C is called a bifunctor on A and B to C.

For example, the functor $\hom: A^* \times A \longrightarrow S$, called the usual hom functor to sets, is a bifunctor.

Definition 1.8

Let $F:C \longrightarrow S$ be a functor (covariant) from a category C to the category of sets S. A universal element for F is a pair (u,R) consisting of an object R of C and an element $u \in F(R)$ with the following property. To every object A of C and every element $s \in F(A)$ there is exactly one morphism $f:R \longrightarrow A$ with $F(f)(u) = s$.

<u>Remark</u>: A universal element (v,R) for a <u>contravariant</u> functor $K:C \longrightarrow S$ consists of an object R of C and an element $v \in K(R)$, such that for each element $a \in K(A)$ there is exactly one morphism $f:A \longrightarrow R$ with $K(f)(v) = a$. Since K is contravariant, $K(f)$ is a function $K(f) : K(R) \longrightarrow K(A)$.

3. Natural Transformations

Definition 1.9

If $F,H:A \longrightarrow B$ are functors, a <u>natural transformation</u> $\theta:F \longrightarrow H$ from F to H is a function which assigns to each object A of A a morphism $\theta_A: F(A) \longrightarrow H(A)$ of B in such a way that every morphism $f:A \longrightarrow A'$ of A yields a commutative diagram

$$\begin{tikzcd} F(A) \arrow[r, "\theta_A"] \arrow[d, "F(f)"] & H(A) \arrow[d, "H(f)"] \\ F(A') \arrow[r, "\theta_{A'}"'] & H(A') \end{tikzcd}$$

A natural transformation $\theta:F \longrightarrow H$ is also called a "morphism of functors".

Definition 1.10

If each θ_A is an isomorphism in category B, we call $\theta:F \longrightarrow H$ a <u>natural isomorphism</u> or a <u>natural equivalence</u>.

<u>Remark:</u> A generalization of the notion of a natural transformation has been given by Eilenberg and Kelly [6]. Rather than presenting a detailed account of this generalization, we will give the particular details for each situation in which this generalized notion is used.

4. Zero Objects

Definition 1.11

An object 0 is called a zero object in a category $\mathcal{C}$ if for each object A in $\mathcal{C}$, $\hom(A,0)$ and $\hom(0,A)$ contain exactly one element.

Proposition 1.1

Any two zero objects are isomorphic.

Proof: Assume that 0 and $0'$ are distinct zero objects in $\mathcal{C}$. Since 0 is a zero object we have $\hom(0',0)$ and $\hom(0,0')$ each containing exactly one element. Thus, $0' \xrightarrow{x} 0 \xrightarrow{y} 0'$ implies that $y \circ x = 1_{0'}$. Similarly, $x \circ y = 1_0$ and consequently 0 and $0'$ are isomorphic.

Definition 1.12

If $\mathcal{C}$ has a zero object then the map $A \xrightarrow{a} B$ is called zero if it factors through the zero object.

Proposition 1.2

$\hom(A,B)$ contains exactly one zero map.

Proof: Consider $A \xrightarrow{u} 0 \xrightarrow{v} B$ where $a = v \circ u$. Since 0 is a zero object for $\mathcal{C}$, u and v are unique. Therefore $a = v \circ u$ is unique. Write $a = 0$.

Proposition 1.3

$x \circ 0 = 0 \circ x = 0$

<u>Proof</u>: Consider $C \xrightarrow{x} A \xrightarrow{u} 0 \xrightarrow{v} B$

$0 \circ x = (v \circ u) \circ x = v \circ (u \circ x) = 0$ since $C \longrightarrow B$ factors through 0. Similarly $x \circ 0 = 0$.

5. Products and Sums

Definition 1.13

Given a pair of objects A, B of a category C, we say that the object P is a <u>product</u> of A and B if there exist morphisms $P \xrightarrow{P_1} A$ and $P \xrightarrow{P_2} B$ such that for every pair of morphisms $X \longrightarrow A$ and $X \longrightarrow B$ there is a unique $X \longrightarrow P$ such that

$$\begin{tikzcd} & A \\ X \arrow[ur] \arrow[r, dashed] \arrow[dr] & P \arrow[u, "P_1"'] \arrow[d, "P_2"] \\ & B \end{tikzcd}$$

commutes.

It can be easily shown that P, the product of A and B, is unique up to isomorphism. P is written as $A \times B$.

If

$$\begin{tikzcd} & A_1 \\ A \arrow[ur, "f_1"] \arrow[r, dashed, "f"] \arrow[dr, "f_2"'] & A_1 \times A_2 \arrow[u, "P_1"'] \arrow[d, "P_2"] \\ & A_2 \end{tikzcd}$$

we write $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} p_1 f \\ p_2 f \end{bmatrix}$

We define the <u>diagonal map</u> $\Delta : A \longrightarrow A \times A$ by $\Delta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, i.e.

$$\begin{tikzcd} & A \\ A \arrow[ur, "1"] \arrow[r, dashed, "\Delta"] \arrow[dr, "1"'] & A \times A \arrow[u, "P_1"'] \arrow[d, "P_2"] \\ & A \end{tikzcd}$$

Definition 1.14

Given a pair of objects A and B, we say that an object S is a sum (or coproduct) of A and B if there exist morphisms $i_1: A \longrightarrow S$ and $i_2: B \longrightarrow S$ such that for every pair of morphisms $A \longrightarrow X$ and $B \longrightarrow X$ there is a unique morphism $S \longrightarrow X$ such that

$$\begin{tikzcd} A \arrow[d, "i_1"'] \arrow[dr] & \\ S \arrow[r, dashed] & X \\ B \arrow[u, "i_2"] \arrow[ur] & \end{tikzcd}$$

commutes.

It can be easily shown that S, the sum of A and B is unique up to isomorphism. S is written $A + B$

If

$$\begin{tikzcd} A \arrow[d, "i_1"'] \arrow[dr, "f_1"] & \\ A_1 + A_2 \arrow[r, dashed] & A \\ A \arrow[u, "i_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

we write $f = [f_1, f_2]$
$= [fi_1, fi_2]$

We define the <u>folding map</u> or <u>codiagonal map</u> $\nabla: A + A \longrightarrow A$ by $\nabla = [1,1]$, i.e.

$$\begin{tikzcd} A \arrow[d, "i_1"'] \arrow[dr, "1"] & \\ A + A \arrow[r, dashed, "\nabla"] & A \\ A \arrow[u, "i_2"] \arrow[ur, "1"'] & \end{tikzcd}$$

<u>Remark:</u> Maps $A_1 + A_2 \longrightarrow B_1 \times B_2$ can be written as matrices $\begin{bmatrix} f_{11}, & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$

where $f_{\alpha\beta} = P_\alpha f i_\beta \qquad \alpha,\beta = 1,2$

This can be seen by considering the following diagrams

$$\begin{tikzcd}
 & B_1 \\
A_1 + A_2 \arrow[ur, "P_1 f"] \arrow[r, dashed, "f"'] \arrow[dr, "P_2 f"'] & B_1 \times B_2 \arrow[u, "P_1"'] \arrow[d, "P_2"] \\
 & B_2
\end{tikzcd} \qquad \therefore \quad f = \begin{bmatrix} P_1 f \\ P_2 f \end{bmatrix}$$

$$\begin{tikzcd}
A_1 \arrow[d, "i_1"'] \arrow[dr, "P_1 f i_1"] & \\
A_1 + A_2 \arrow[r, dashed, "P_1 f"'] & B_1 \\
A_2 \arrow[u, "i_2"] \arrow[ur, "P_1 f i_2"'] &
\end{tikzcd} \qquad \therefore \quad P_1 f = [P_1 f i_1, P_1 f i_2]$$

In a similar way $P_2 f = [P_2 f i_1, P_2 f i_2]$

This result is invariant in the sense that

$$f = [fi_1, fi_2] = \begin{bmatrix} \begin{bmatrix} P_1 f i_1 \\ P_2 f i_2 \end{bmatrix}, & \begin{bmatrix} P_1 f i_2 \\ P_2 f i_2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} [P_1 f i_1, & P_1 f i_2] \\ [P_2 f i_2, & P_2 f i_2] \end{bmatrix}$$

To show this invariance we use $P_\alpha \begin{bmatrix} P_1 f \\ P_2 f \end{bmatrix} = P_\alpha f$ where $\alpha = 1$ or 2

and $[fi_1, fi_2] i_\beta = fi_\beta$ where $\beta = 1$ or 2.

Take $h = \begin{bmatrix} [P_1 fi \ , & P_1 f i_2] \\ [P_2 f i_1, & P_2 f i_2] \end{bmatrix} = \begin{bmatrix} P_1 f \\ P_2 f \end{bmatrix}$

$$P_\alpha h i = \left(P_\alpha \begin{bmatrix} P_1 f \\ P_2 f \end{bmatrix} \right) i_\beta = (P_\alpha f) i_\beta$$

Take $k = \left[\begin{bmatrix} P_1 fi_1 \\ P_2 fi_1 \end{bmatrix} \quad \begin{bmatrix} P_1 f_2 \\ P_2 fi_2 \end{bmatrix}\right] = \left[fi_1, fi_2\right]$

$$P_\alpha k i_\beta = P_\alpha ([fi_1, fi_2] i_\beta) = P_\alpha (fi_\beta)$$

Therefore $P_\alpha h i_\beta = P_\alpha k i_\beta$ for $\alpha, \beta \in \{1,2\}$

Thus $h = k$.

6. Biproducts

Definition 1.15

If C is a category with a zero object then a biproduct of A_1 and A_2 is a diagram

$$\begin{tikzcd} A_1 \arrow[dr, "i_1"] & & & & A_1 \\ & & A \arrow[urr, "P_1"'] \arrow[drr, "P_2"] & & \\ A_2 \arrow[ur, "i_2"'] & & & & A_2 \end{tikzcd}$$

such that

(i) (i_1, i_2) is a sum

(ii) (P_1, P_2) is a product

(iii) $P_1 i_1 = 1_{A_1}$; $P_2 i_2 = 1_{A_2}$; $P_1 i_2 = 0$; $P_2 i_1 = 0$

We write $A = A_1 \oplus A_2$

Let C be a category with a zero object such that any two objects have a biproduct.

Definition 1.16

If $x, y \in \hom(A,B)$ we define

$$x +_L y : A \longrightarrow B \quad \text{by} \quad A \xrightarrow{\Delta} A + A \xrightarrow{[x,y]} B$$

and $$x +_R y : A \longrightarrow B \quad \text{by} \quad A \xrightarrow{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]} B + B \xrightarrow{\nabla} B$$

Proposition 1.4

$$0 +_L x = x +_L 0 = x$$

$$0 +_R y = y +_R 0 = y.$$

Proof: $0 +_L x$ is by definition the composite

$$A \xrightarrow{\Delta} A + A \xrightarrow{[0,x]} B$$

That is

$$\begin{tikzcd}
 & A \arrow[dr, "0"] \arrow[d, shift left, "i_1"] & \\
A \arrow[ur, "1"] \arrow[r, "\Delta"] \arrow[dr, "1"'] & A + A \arrow[r, "{[0,x]}"] \arrow[u, shift left, "P_1"] \arrow[d, shift left, "P_2"] & B \\
 & A \arrow[ur, "x"'] \arrow[u, shift left, "i_2"] &
\end{tikzcd}$$

By the definition of sum $[0,x]i_1 = 0$ and $[0,x]i_2 = x$. But $xP_2i_1 = x.0 = 0$ and $xP_2i_2 = x.1 = x$. Therefore, $xP_2 = [0,x]$. So $[0,x]\Delta = xP_2\Delta = x1 = x$.

The other results may be obtained similarly and by duality.

Proposition 1.5

(i) Given morphisms $A \underset{y}{\overset{x}{\rightrightarrows}} B \quad B \xrightarrow{u} C$ then $ux +_L uy = u(x +_L y)$

(ii) If $z : C \longrightarrow A$ is a morphism, then $xz +_R yz = (x +_R y)z$

Proof: (i) By definition $u(x +_L y)$ is the composite

$$A \xrightarrow{\Delta} A + A \xrightarrow{[x,y]} B \xrightarrow{u} C$$

That is

$$\begin{tikzcd}
 & A \arrow[dr, "x"] \arrow[d, shift left, "i_1"] & & \\
A \arrow[ur, "1"] \arrow[r, "\Delta"] \arrow[dr, "1"'] & A + A \arrow[r, "{[x,y]}"] \arrow[u, shift left, "P_1"] \arrow[d, shift left, "P_2"] & B \arrow[r, "u"] & C \\
 & A \arrow[ur, "y"'] \arrow[u, shift left, "i_2"] & &
\end{tikzcd}$$

By definition $ux +_L uy$ is

$$\begin{tikzcd}
 & A \arrow[dr, "ux"] \arrow[d, shift left, "i_1"] & \\
A \arrow[ur, "1"] \arrow[r, "\Delta"] \arrow[dr, "1"'] & A + A \arrow[u, shift left, "P_1"] \arrow[d, shift right, "P_2"'] \arrow[r, "{[ux,uy]}"] & C \\
 & A \arrow[ur, "uy"'] \arrow[u, shift right, "i_2"'] &
\end{tikzcd}$$

Thus it is sufficient to prove that $u[x,y] = [ux,uy]$. If $u[x,y]$ replaces $[ux,uy]$ in the second diagram, the triangles on the right side of this diagram will still commute. Thus by the uniqueness property

$$u[x,y] = [ux,uy]$$

The other result is obtained in a similar way.

Proposition 1.6

(i) $+_L$ and $+_R$ are the same (written $+$).

(ii) $\langle \hom(A,B), + \rangle$ is a commutative monoid.

Proof: (i) Consider the four morphisms

$$A \xrightarrow[x,y,z,w]{} B$$

$$A \xrightarrow{\Delta} A \oplus A \xrightarrow[\theta = \begin{bmatrix} w,x \\ y,z \end{bmatrix}]{} B \oplus B \xrightarrow{\nabla} B$$

Remember that $\begin{bmatrix} w,x \\ y,z \end{bmatrix} = \begin{bmatrix} [w,x] \\ [y,z] \end{bmatrix} = \left[\begin{bmatrix} w \\ y \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \right]$

Therefore, $\theta\Delta = \left[\begin{bmatrix} w \\ y \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \right] \Delta = \begin{bmatrix} w \\ y \end{bmatrix} +_L \begin{bmatrix} x \\ z \end{bmatrix}$

and
$$\begin{aligned} \nabla(\theta\Delta) &= \nabla\left(\begin{bmatrix} w \\ y \end{bmatrix} +_L \begin{bmatrix} x \\ z \end{bmatrix} \right) \\ &= \nabla \begin{bmatrix} w \\ y \end{bmatrix} +_L \nabla \begin{bmatrix} x \\ z \end{bmatrix} \\ &= (W +_R y) +_L (x +_R z) \end{aligned}$$

In a similar way

$$\begin{aligned} (\nabla\theta)\Delta &= [w,x]\Delta +_R [y,z]\Delta \\ &= (w +_L x) +_R (y \cdot +_L z) \end{aligned}$$

Therefore $(w +_R y) +_L (x +_R z) = (w +_L x) +_R (y +_L z)$

(i) Put $x = y = 0$

Then $w +_L z = w +_R z$ and we write $+$ for $+_R$ and $+_L$

(ii) Put $y = 0$. Then $w + (x + z) = (w + x) + z$

Put $w = z = 0$. Then $y + x = x + y$

Thus $\langle \hom(A,B), + \rangle$ is a commutative monoid.

Proposition 1.7

Given a biproduct of A_1 and A_2 in $\mathcal{C}$, then

$$i_1P_1 + i_2P_2 = 1_{A_1 \oplus A_2}$$

Proof: Consider the following diagram

$$\begin{tikzcd}
 & A_1 \oplus A_2 & \\
A_1 \arrow[ur, "i_1"] \arrow[dr, "i_1"'] & & A_2 \arrow[ul, "i_2"'] \arrow[dl, "i_2"] \\
 & A_1 \oplus A_2 \arrow[uu, "x"'] &
\end{tikzcd}$$

Since i_1, i_2 is a sum, there exists a unique morphism $x: A_1 \oplus A_2 \longrightarrow A_1 \oplus A_2$ such that

$$xi_1 = i_1 \quad \text{and} \quad xi_2 = i_2$$

Clearly $x = 1_{A_1 \oplus A_2}$ is such a morphism.

But
$$\begin{aligned}(i_1P_1 + i_2P_2)i_2 &= i_1P_1i_1 + i_2P_2i_1 \\ &= i_1 + 0 \\ &= i_1\end{aligned}$$

and
$$\begin{aligned}(i_1P_1 + i_2P_2)i_2 &= i_1P_1i_2 + i_2P_2i_2 \\ &= 0 + i_2 \\ &= i_2\end{aligned}$$

Therefore by the uniqueness property

$$i_1P_1 + i_2P_2 = 1_{A_1 \oplus A_2}$$

CHAPTER TWO

CLOSED CATEGORIES

1. Closed Categories with Faithful Basic Functor

Definition 2.1

A closed category $\mathcal{C} = (\mathcal{C}_o, F, H, K, \pi, \theta, L)$ consists of the following seven data :

(i) a category $\mathcal{C}_o$;

(ii) a functor $F : \mathcal{C}_o \longrightarrow \mathcal{S}$;

(iii) a functor $H : \mathcal{C}_o^* \times \mathcal{C}_o \longrightarrow \mathcal{C}$;

(iv) an object K of $\mathcal{C}_o$;

(v) a natural isomorphism $\pi = \pi_A : A \longrightarrow H(K,A)$ in $\mathcal{C}_o$;

(vi) a natural transformation $\theta = \theta_A : K \longrightarrow H(A,A)$ in $\mathcal{C}_o$;

(vii) a natural transformation

$$L = L^A_{BC} : H(B,C) \longrightarrow H(H(A,B),\ H(A,C)) \ .$$

These data are to satisfy the following six axioms :

CC0. The following diagram of functors commutes:

$$\begin{tikzcd} \mathcal{C}_o^* \times \mathcal{C}_o \arrow[r, "H"] \arrow[dr, "\mathrm{hom}"'] & \mathcal{C} \arrow[d, "F"] \\ & \mathcal{S} \end{tikzcd}$$

CC1. The following diagram commutes:

$$\begin{tikzcd} H(B,B) \arrow[rr, "L^A"] & & H(H(A,B),\ H(A,B)) \\ & K \arrow[ul, "\theta"] \arrow[ur, "\theta"'] & \end{tikzcd}$$

CC2. The following diagram commutes:

$$\begin{tikzcd}
H(A,C) \arrow[r, "L^A"] \arrow[dr, "\pi"'] & H(H(A,A),\ H(A,C)) \arrow[d, "H(\theta,1)"] \\
& H(K,\ H(A,C))
\end{tikzcd}$$

CC3. The following diagram commutes:

$$\begin{tikzcd}
H(C,D) \arrow[rr, "L^B"] \arrow[d, "L^A"] & & H(H(B,C),\ H(B,D)) \arrow[d, "H(1,L^A)"] \\
H(H(A,C),\ H(A,D)) \arrow[d, "L^{H(A,B)}"] & & H(H(B,C),\ H(H(A,B),\ H(A,D))) \\
H(H(H(A,B),\ H(A,C)),\ H(H(A,B),\ H(A,D))) \arrow[urr, "H(L^A,1)"'] & &
\end{tikzcd}$$

CC4. The following diagram commutes:

$$\begin{tikzcd}
H(B,C) \arrow[r, "L^K"] \arrow[dr, "H(1,\pi)"'] & H(H(K,B),\ H(K,C)) \arrow[d, "H(\pi,1)"] \\
& H(B,\ H(K,C))
\end{tikzcd}$$

CC5. The map

$$F\pi_{H(A,A)} : FH(A,A) \longrightarrow FH(K,H(A,A)) \quad ,$$

which by CC0 may also be written

$$F\pi_{H(A,A)} : \hom(A,A) \longrightarrow \hom(K,H(A,A)) \quad ,$$

sends $1_A \in \hom(A,A)$ to $\theta_A \in \hom(K,H(A,A))$

Remark (i) For a closed category $\mathcal{C}$, $\mathcal{C}_o$ is called the underlying category, $F:\mathcal{C}_o \to \mathcal{S}$ the basic functor and H the internal Hom-functor. To simplify notation $\mathcal{C}$ will be used for both a closed category and its underlying category

(ii) The word natural is used in condition (vi) of a closed category in a generalized sense. Eilenberg and Kelly [6] have given a detailed account of this generalized notion of a natural transformation, which requires that the following diagram commutes:

$$\begin{tikzcd}
\hom(A,B) \arrow[r, "H(-,B)"] \arrow[d, "H(A,-)"] & \hom(H(B,B),\ H(A,B)) \arrow[d, "\hom(\theta_B,1)"] \\
\hom(H(A,A),\ H(A,B)) \arrow[r, "\hom(\theta_A,1)"'] & \hom(K,H(A,B))
\end{tikzcd}$$

Evaluating this diagram at $f \in \hom(A,B)$, the requirement is that the following diagram commutes:

$$\begin{tikzcd}
K \arrow[r, "\theta_A"] \arrow[d, "\theta_B"'] & H(A,A) \arrow[d, "H(A,f)"] \\
H(B,B) \arrow[r, "H(f,B)"'] & H(A,B)
\end{tikzcd}$$

Note that by CC0 $FH(A,B) = \hom(A,B)$

and $FH(f,g) = \hom(f,g)$.

Proposition 2.1

In the presence of CC0 and CC5, the axiom CC1 is equivalent to any of the following:

(a) $FL^{A}_{BC}\ (f) = H(1,f) \in \hom(H(A,B),\ H(A,C))$ for $f \in \hom(B,C)$

(b) $(FL^{A}_{BB})(1_B) = 1_{H(A,B)}$

(c) $FL^{A}_{BC} = H(A,-) : \hom(B,C) \longrightarrow \hom(H(A,B),\ H(A,C))$

Proof: (See [5] page 430).

Proposition 2.2

If the basic functor F is faithful, the axioms CC2, CC3 and CC4 are consequences of CC0, CC1 and CC5.

Proof: (See [5] page 432).

Also with the simplification that the basic functor F is faithful, it is not necessary to assume that $L^{A}_{BC}: H(B,C) \longrightarrow H(H(A,B),\ H(A,C))$ is natural in B and C. This will be shown to be a consequence of the faithfulness of F.

Lemma 2.3

Given $A'' \xrightarrow{f'} A' \xrightarrow{f} A$ and $B \xrightarrow{g} B' \xrightarrow{g'} B''$ in any closed category $\mathcal{C}$ with faithful basic functor F. Then $H(f \circ f', g' \circ g) : H(A,B) \longrightarrow H(A'',B'')$ is equal to $H(f',g') \circ H(f,g)$.

Proof:
$$\begin{aligned} FH(f \circ f', g' \circ g) &= \hom(f \circ f', g' \circ g) \\ &= \hom(f',g') \circ \hom(f,g) \\ &= FH(f',g') \circ FH(f,g) \\ &= F(H(f',g') \circ H(f,g)) \end{aligned}$$

Since F is faithful,

$$H(f \circ f', g' \circ g) = H(f',g') \circ H(f,g)$$

Proposition 2.4

L_A is a natural transformation in B and C.

Proof: Consider the following diagram

$$\begin{tikzcd} FH(B,C) \arrow[r, "FL^A_{BC}"] \arrow[d, "FH(p,q)"'] & FH(H(A,B),\ H(A,C)) \arrow[d, "FH(H(1,p),\ H(1,q))"] \\ FH(M,N) \arrow[r, "FL^A_{MN}"] & FH(H(A,M),\ H(A,N)) \end{tikzcd}$$

where $p:M \longrightarrow B$ and $q:C \longrightarrow N$.

If $f \in FH(B,C)$, then by Proposition 2.1 (part (a)) $FL^A_{BC}(f) = H(1,f)$. Therefore the above diagram commutes if $H(1, q \circ f \circ p) = H(1,q) \circ H(1,f) \circ H(1,p)$.

This equality follows immediately from Lemma 2.3. Since F is faithful, the naturality of L^A is assured by Lemma 1.1.

Proposition 2.5

For any $f:H(K,K) \longrightarrow X$ in $\mathcal{C}$, the composite $K \xrightarrow[\pi]{} H(K,K) \xrightarrow[f]{} X$ is the image of $1 \in hom(K,K)$ under the composite map

$$hom(K,K) \xrightarrow[Ff]{} FX \xrightarrow[F\pi]{} hom(K,X) \ .$$

Proof: Evaluate at $1 \in hom(K,K)$ the diagram

$$\begin{tikzcd} hom(K,K) \arrow[r, "Ff"] \arrow[d, "F\pi_{H(K,K)}"'] & FX \arrow[d, "F\pi_X"] \\ hom(K,H(K,K)) \arrow[r, "hom(1,f)"'] & hom(K,X) \end{tikzcd}$$

which commutes by the naturality of π. Note that

$F\pi_{H(K,K)} = FH(1,\pi) = \hom(1,\pi)$. Therefore

$\hom(1,f)(\hom(1,\pi)(1)) = f \circ \pi$

Corollary 1

The following diagram commutes for each $f \in \hom(K,K)$

$$\begin{tikzcd} K \arrow[r, "\pi"] \arrow[d, "\pi"'] & H(K,K) \arrow[d, "H(1,f)"] \\ H(K,K) \arrow[r, "H(1,f)"'] & H(K,K) \end{tikzcd}$$

Proof: In proposition 3.6 replace f by $H(1,f)$ and X by $H(K,K)$.

Then $H(1,f) \circ \pi = F\pi(FH(1,f)(1))$, where $1 \in \hom(K,K)$

$= F\pi(f \circ 1 \circ 1)$

$= F\pi(f)$

and $H(f,1) \circ \pi = F\pi(FH(f,1)(1))$

$= F\pi(1 \circ 1 \circ f)$

$= F\pi(f)$

That is, $H(1,f) \circ \pi = H(f,1) \circ \pi$.

Corollary 2

For $f \in \hom(K,K)$

$$H(1,f) = H(f,1) : H(K,K) \longrightarrow H(K,K)$$

Proof: The result follows because π is an isomorphism.

Corollary 3.

The monoid $\hom(K,K)$ of endomorphisms of K is commutative.

Proof: Applying F to corollary 2 gives

$$\hom(1,f) = \hom(f,1) : \hom(K,K) \longrightarrow \hom(K,K).$$

Evaluating at $g \in \hom(K,K)$ now gives $f \circ g = g \circ f$.

2. Monoidal Categories

Definition 2.2

A monoidal category $\mathcal{C} = (\mathcal{C}_o, \otimes, K, r, \ell, a)$ consists of the following six data:

(i) a category $\mathcal{C}_o$;

(ii) a functor $\otimes : \mathcal{C}_o \times \mathcal{C}_o \longrightarrow \mathcal{C}_o$ (written between its arguments and called the tensor product of $\mathcal{C}$);

(iii) an object K of $\mathcal{C}_o$;

(iv) a natural isomorphism $r = r_A : A \otimes K \longrightarrow A$;

(v) a natural isomorphism $\ell = \ell_A : K \otimes A \longrightarrow A$;

(vi) a natural isomorphism $a = a_{ABC} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$.

These data are to satisfy the following five axioms:

MC1. The following diagram commutes:

$$\begin{tikzcd}
(K \otimes A) \otimes B \arrow[rr, "a"] \arrow[dr, "\ell \otimes 1"'] & & K \otimes (A \otimes B) \arrow[dl, "\ell"] \\
& A \otimes B &
\end{tikzcd}$$

MC2. The following diagram commutes:

$$\begin{tikzcd}
(A \otimes K) \otimes B \arrow[rr, "a"] \arrow[dr, "r \otimes 1"'] & & A \otimes (K \otimes B) \arrow[dl, "1 \otimes \ell"] \\
& A \otimes B &
\end{tikzcd}$$

MC3. The following diagram commutes:

$$\begin{tikzcd}
((A \otimes B) \otimes C) \otimes D \arrow[r, "a"] \arrow[d, "a \otimes 1"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "a"] & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "a"] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "1 \otimes a"']
\end{tikzcd}$$

MC4. The following diagram commutes:

$$\begin{tikzcd}
(A \otimes B) \otimes K \arrow[rr, "a"] \arrow[dr, "r"'] & & A \otimes (B \otimes K) \arrow[dl, "1 \otimes r"] \\
& A \otimes B &
\end{tikzcd}$$

MC5. $\ell_K = r_K : K \otimes K \longrightarrow K$.

Remark: (1) The above axioms are not independent. It has been shown (Kelly [13]) that MC1, MC4 and MC5 are consequences of MC2 and MC3.

(2) Natural isomorphisms such as a, r, ℓ are said to be coherent if, roughly speaking, all diagrams made by their use alone (with their inverses, 1 and $\otimes$) such as the diagrams of MC1 - MC5, commute. It has been shown (MacLane [15]), that MC1 - MC5 imply that the isomorphisms a, r, ℓ are coherent.

(3) In the terminology of Bénabou [1], a monoidal category is a category avec multiplication .

3. Monoidal Closed Categories

Definition 2.3

A monoidal closed category (or equally closed monoidal category) $\mathcal{C} = ({}^m\mathcal{C},\rho,{}^c\mathcal{C})$ consists of the following three data:

(i) a monoidal category ${}^m\mathcal{C} = (\mathcal{C}_o,\otimes,K,r,\ell,a)$;

(ii) a closed category ${}^c\mathcal{C} = (\mathcal{C}_o,F,H,K,\pi,\theta,L)$ with the same $\mathcal{C}_o$ and K as in ${}^m\mathcal{C}$;

(iii) a natural isomorphism $\rho = \rho_{ABC} : H(A \otimes B,C) \longrightarrow H(A,H(B,C))$.

These data are to satisfy the following four axioms:

MCC1. The following diagram commutes:

$$\begin{tikzcd}
H(K \otimes A,B) \arrow[rr, "\rho"] \arrow[dr, "H(\ell,1)"'] & & H(K, H(A,B)) \\
& H(A,B) \arrow[ur, "\pi_{H(A,B)}"'] &
\end{tikzcd}$$

MCC2. The following diagram commutes:

$$\begin{tikzcd}
H((A \otimes B) \otimes C,D) \arrow[r, "\rho"] & H(A \otimes B, H(C,D)) \arrow[r, "\rho"] & H(A, H(B,H(C,D))) \\
H(A \otimes (B \otimes C),D) \arrow[u, "H(a,1)"] \arrow[rr, "\rho"'] & & H(A, H(B \otimes C,D)) \arrow[u, "H(1,\rho)"']
\end{tikzcd}$$

MCC3. The following diagram commutes:

$$\begin{tikzcd}
H(C,D) \arrow[rr, "L^{A \otimes B}"] \arrow[d, "L^B"] & & H(H(A \otimes B,C), H(A \otimes B,D)) \arrow[dd, "H(1,\rho)"] \\
H(H(B,C), H(B,D)) \arrow[d, "L^A"] & & \\
H(H(A,H(B,C)), H(A,H(B,D))) \arrow[rr, "H(\rho,1)"] & & H(H(A \otimes B,C), H(A,H(B,D)))
\end{tikzcd}$$

MCC4. The following diagram commutes:

$$\begin{tikzcd}
H(A \otimes K, B) \arrow[rr, "\rho"] & & H(A, H(K,B)) \\
& H(A,B) \arrow[ul, "H(r,1)"] \arrow[ur, "H(1,\pi)"'] &
\end{tikzcd}$$

Remark (i) We shall denote the monoidal category ${}^{m}\mathcal{C}$ and the closed category ${}^{c}\mathcal{C}$ by the same symbol $\mathcal{C}$ as the monoidal closed category, except when it is necessary to distinguish between the three structures.

(ii) Both the data and the axioms for a monoidal closed category contain redundancies. The interconnections have been shown by Eilenberg and Kelly [5] pp. 477-489. These interconnections lead to an economical way of giving a monoidal closed category. Consider the so called "basic situation" ([5] pp. 477) in which we are given a category $\mathcal{C}_o$, functors $\otimes : \mathcal{C}_o \times \mathcal{C} \longrightarrow \mathcal{C}_o$ and $H : \mathcal{C}_o^* \times \mathcal{C}_o \longrightarrow \mathcal{C}_o$, a natural isomorphism

$P = P_{ABC}$; $\hom(A \otimes B, C) \longrightarrow \hom(A, H(B,C))$, and a functor $F : \mathcal{C}_o \longrightarrow S$ satisfying CC0.

Since P is a natural isomorphism, the Yoneda representation theorem [5] shows that the commutativity of the diagram

$$(2.1)\qquad \begin{tikzcd} \hom(A\otimes K,B) \arrow[rr, "P"] & & \hom(A,H(K,B)) \\ & \hom(A,B) \arrow[ul, "\hom(r,1)"] \arrow[ur, "\hom(1,\pi)"'] & \end{tikzcd}$$

sets up a bijection between natural isomorphisms $r:A\otimes K\longrightarrow A$ and natural isomorphisms $\pi:A\longrightarrow H(K,A)$. Putting $B = A$ and evaluating at 1 gives $Pr = \pi$.

In the same way commutativity of the diagram

$$(2.2)\qquad \begin{tikzcd} \hom((A\otimes B)\otimes C,D) \arrow[r, "P"] & \hom(A\otimes B,H(C,D)) \arrow[r, "P"] & \hom(A,H(B,H(C,D))) \\ \hom(A\otimes(B\otimes C),D) \arrow[u, "\hom(a,1)"] \arrow[rr, "P"'] & & \hom(A,H(B\otimes C,D)) \arrow[u, "\hom(1,\rho)"'] \end{tikzcd}$$

sets up a bijection between natural isomorphisms

$$a:(A\otimes B)\otimes C\longrightarrow A\otimes(B\otimes C)$$

and natural isomorphisms $\rho:H(B\otimes C,D)\longrightarrow H(B,H(C,D))$.

<u>Proposition 2.6</u>

Given (i) a category $\mathcal{C}_o$,

(ii) a functor $\otimes:\mathcal{C}_o\times\mathcal{C}_o\longrightarrow\mathcal{C}_o$,

(iii) a functor $H:\mathcal{C}_o^*\times\mathcal{C}_o\longrightarrow\mathcal{C}_o$,

(iv) a functor $F:\mathcal{C}_o\longrightarrow\mathcal{S}$ such that $FH = \hom$,

(v) an object K of $\mathcal{C}_o$,

(vi) a natural isomorphism

$$\pi = \pi_A:A\longrightarrow H(K,A) .$$

and (vii) a natural isomorphism

$$\rho = \rho_{ABC} : H(A \otimes B, C) \longrightarrow H(A, H(B,C)).$$

Then these data can be completed to give a monoidal closed category if and only if the r and a defined by diagrams 2.1 and 2.2, where $P = F\rho$, satisfy MCC4 and MCC2. Moreover, if F is faithful, the satisfaction of MCC4 and MCC2 is automatic.

Proof: ([5] p. 495)

The examples considered in this paper will be such that the basic functor F is faithful. In this case it is sufficient to establish the existence of the seven pieces of data of Proposition 2.6 to show that a category is closed monoidal. This is so, since the faithfulness of F automatically gives us the "basic situation", which in turn gives from diagrams 2.1 and 2.2 the natural isomorphisms r and a; these automatically satisfying MCC4 and MCC2.

4. Symmetric Monoidal Closed Categories

Definition 2.4

A symmetry for a monoidal category $\mathcal{C}$ consists of a natural isomorphism $C = C_{AB} : A \otimes B \longrightarrow B \otimes A$ in $\mathcal{C}$ satisfying the following two axioms:

MC6. $C_{BA}C_{AB} = 1 : A \otimes B \longrightarrow B \otimes A.$

MC7. The following diagram commutes:

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "a"] \arrow[d, "C \otimes 1"'] & A \otimes (B \otimes C) \arrow[r, "C"] & (B \otimes C) \otimes A \arrow[d, "a"] \\
(B \otimes A) \otimes C \arrow[r, "a"'] & B \otimes (A \otimes C) \arrow[r, "1 \otimes C"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

A monoidal closed category $\mathcal{C}$ together with a symmetry C for ${}^{m}\mathcal{C}$ is called a symmetric monoidal closed category.

<u>Remark:</u> A monoidal category, even a closed one, may admit several distinct symmetries. For example, Eilenberg and Kelly [5] have shown that the closed monoidal category GK of graded K-modules admits one symmetry for every $k \in K$ with $k^2 = 1$. However, it has also been shown (Eilenberg and Kelly [5]) that if the basic functor F is faithful, then the monoidal closed category $\mathcal{C}$ admits at most one symmetry.

<u>To show that a category $\mathcal{C}$ with faithful basic functor is a symmetric monoidal closed category, we shall have to establish the seven data of Proposition 2.6 plus the existence of a natural isomorphism $C = C_{AB} : A \otimes B \longrightarrow B \otimes A$ in $\mathcal{C}$, satisfying MC6 and MC7.</u>

CHAPTER THREE

M-SETS

In this chapter we will give a fairly detailed account of the category of sets under the action of a commutative monoid M, in short, a category of M-sets. The objective is to show that any category of M-sets is a symmetric monoidal closed category.

1. Definitions and Examples

Definition 3.1

A monoid M acts on a set X when there is a given function $M \times X \longrightarrow X$ written $(m,x) \longmapsto mx$ and called the "action" of $m \in M$ on $x \in X$, such that for all $x \in X$ and $m \in M$

$$1x = x \quad \text{and} \quad (m_1 m_2)x = m_1(m_2 x)$$

Any pair $(X, M \times X \longrightarrow X)$ consisting of a set X together with an "action" of M on X is called an M-set. If $M = G$, a group, then the pair $(X, G \times X \longrightarrow X)$ is the well known G-set.

Examples

(i) Every set X is an M-set where $M = \{1\}$ is the trivial monoid; the action being defined by $1x = x$ for all $x \in X$.

(ii) Every pointed set X_* is an M-set where $M = \{0,1\}$; the action being defined by $1x = x$ and $0x = *$ for all $x \in X$.

(iii) A monoid M is an M-set with the obvious action.

(iv) If T is a transformation group consisting of permutations t of X, the assignment $(t,x) \longmapsto t(x)$ defines an action of T on X.

(v) More generally, any representation $h:G \longrightarrow T$ of a group G gives by $(g,x) \longmapsto h(g)(x)$ an action of G on X.

Consider for a particular monoid M any two M-sets X and Y.

Definition 3.2

A morphism of M-sets consists of a function $f:X \longrightarrow Y$ such that $f(mx) = mf(x)$ for all $x \in X$, $m \in M$.

A morphism $f:X \longrightarrow Y$ from the M-set X to the M-set Y is said to be a monomorphism iff it is injective; f is said to be an epimorphism iff it is surjective. A bijective morphism of M-sets is called an isomorphism. Clearly composition of morphisms of M-sets is a morphism of M-sets. Also, the identity function is obviously a morphism of M-sets.

We now restrict our discussion to M-sets where M is a commutative monoid and more generally to categories of M-sets, denoted $\mathcal{S}_M$ for each fixed monoid M. An alternate formulation of Definition 3.2 is that a morphism of M-sets X and Y is a function $f:X \longrightarrow Y$ such that the following diagram commutes:

$$\begin{array}{ccc} M \times X & \longrightarrow & X \\ \big\downarrow {\scriptstyle 1 \times f} & & \big\downarrow {\scriptstyle f} \\ M \times Y & \longrightarrow & Y \end{array}$$

Note that the category $\mathcal{S}$ of sets is the category of $\{1\}$-sets, whereas the category $\mathcal{S}_*$ of pointed sets is a full subcategory of the category of $\{0,1\}$-sets. The latter statement is easily verified since the objects of $\mathcal{S}_*$ are pairs $(X, M \times X \longrightarrow X)$ where, as indicated in example (ii) above, the action $M \times X \longrightarrow X$ is fixed for each pointed set X_*.

2. Quotient M-Sets.

Let X be an arbitrary M-set and let E be an equivalence relation on X such that $x\,E\,x'$ implies $mx\,E\,mx'$ for all $m \in M$; that is, the equivalence relation is compatible with the action on X. Two obvious examples of such a compatible equivalence relation are the trivial relation $X \times X$ (all elements of X are related) and the equality relation I_X (two elements are related if and only if they are the same).

Consider the quotient set $\frac{X}{E}$. We define a function on this set as follows

$$\begin{aligned} M \times \frac{X}{E} &\longrightarrow \frac{X}{E} \\ (m,[x]) &\longmapsto [mx] \end{aligned}$$

This function is well-defined since for all $m \in M$, $[x] = [x']$ implies that $[mx] = [mx']$ by the compatibility of E with the action on X.

Clearly this function is an action on $\frac{X}{E}$, and $(\frac{X}{E}, M \times \frac{X}{E} \rightarrow \frac{X}{E})$ is an M-set called a quotient M-set of X.

3. M-Sets of Morphisms

If X and Y are two M-sets, define for all $f \in \hom(X,Y)$ $m \in M$

$$\begin{aligned} M \times \hom(X,Y) &\longrightarrow \hom(X,Y) \\ (m,\ f) &\longmapsto mf \end{aligned}$$

such that $(mf)(x) = f(mx)$ for all $x \in X$.

Since M is commutative, it can be readily shown that mf is a morphism of M-sets; therefore, the above function is well defined. Also for all $x \in X$, $m_1, m_2 \in M$ and $f \in \hom(X,Y)$

$$\begin{aligned}((m_1 m_2)f)(x) &= f((m_1 m_2)x)\\ &= f((m_2 m_1)x)\\ &= f(m_2(m_1 x))\\ &= m_2 f(m_1 x)\\ &= m_1(m_2 f)(x)\end{aligned}$$

and $(1f)(x) = f(1x) = f(x)$

Therefore the set $\hom(X,Y)$ together with the above function $M \times \hom(X,Y) \longrightarrow \hom(X,Y)$ is an M-set and is written $\hom_M(X,Y)$.

Now let $f:X' \longrightarrow X$ and $g:Y \longrightarrow Y'$ denote arbitrarily given morphisms of M-sets and consider the M-sets $\hom_M(X,Y)$ and $\hom_M(X',Y')$. Define a function

$$\phi : \hom_M(X,Y) \longrightarrow \hom_M(X',Y')$$

by taking $\phi(h) = g \circ h \circ f$ for every $h \in \hom_M(X,Y)$. It is a routine exercise to show that ϕ is a morphism of M-sets. Denote ϕ by $\hom_M(f,g)$.

<u>Proposition 3.1</u>

For any set X acted on by a commutative monoid M $\pi = \pi_X : X \longrightarrow \hom_M(M,X)$ defined by $\pi(x) = f_x$ such that $f_x(m) = mx$ for all $x \in X$, $m \in M$ is a natural isomorphism.

<u>Proof:</u> π is well-defined since for $n,m \in M$

$$f_x(nm) = (nm)x = n(mx) = nf_x(m)$$

Also π is a morphism of M-sets since for all $n,m \in M$ $x \in X$

$$\begin{aligned}\pi(nx)(m) = f_{nx}(m) &= m(nx)\\ &= (mn)x\\ &= (nm)x\end{aligned}$$

$$= n(mx)$$
$$= n\pi(x)(m)$$

To show that π is a monomorphism of M-sets, consider $x_1, x_2 \in X$ such that $x_1 \neq x_2$.

For $1 \in M$, $f_{x_1}(1) = x_1 \neq x_2 = f_{x_2}(1)$; i.e. $f_{x_1} \neq f_{x_2}$

To show that π is an epimorphism of M-sets, consider $f \in \hom_M(M,X)$. For all $m \in M$

$$f(m) = f(m1) = mf(1) = mx' \quad \text{where} \quad x' = f(1)$$

Therefore $\pi(x') = f$.

A routine check shows that the above isomorphism is natural in X.

4. M-Bimorphisms

Let X and Y denote arbitrarily given M-sets where M is a commutative monoid and consider the Cartesian product of the sets X and Y. A function $g:X \times Y \longrightarrow Z$ from $X \times Y$ to an M-set Z is called an M-bimorphism iff

$g(mx,y) = g(x,my) = mg(x,y)$ for all $x \in X$, $y \in Y$, $m \in M$.

Let Bimorph (X,Y:Z) denote the set of all M-bimorphisms $h:X \times Y \longrightarrow Z$. If $t:Z \longrightarrow W$ is a morphism of M-sets, the composite $t \circ h:X \times Y \longrightarrow W$ is an M-bimorphism. For fixed M-sets, X and Y, the formulae

$F(Z) = \text{Bimorph}(X,Y;Z)$, $\quad F(t)(h) = t \circ h$, $\quad h \in F(Z)$

define a functor F from the category S_M of sets acted on by a commutative monoid M to the category S of sets.

A universal element h_o for this functor F is called a "universal M-bimorphism" on $X \times Y$. That is $h:X \times Y \longrightarrow Z$ is a universal M-bimorphism iff for every M-bimorphism $h:X \times Y \longrightarrow W$ there exists a unique morphism of M-sets $t:Z \longrightarrow W$ such that the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "h_o"] \arrow[dr, "h"'] & Z \arrow[d, dashed, "t"] \\ & W \end{tikzcd}$$

4. Tensor Product of M-sets

For any two M-sets X and Y acted on by a commutative monoid M we shall construct a universal M-bimorphism on $X \times Y$. That is we construct a new M-set, $X \times Y$ and an M-bimorphism $X \times Y \longrightarrow X \otimes Y$ which is universal among M-bimorphisms from $X \times Y$ to an arbitrary M-set.

To define the tensor product of M-sets X and Y we must take the "biggest possible" quotient M-set $\frac{X \times Y}{E}$ so that $X \times Y \longrightarrow \frac{X \times Y}{E}$ is an M-bimorphism. To do this let R be the following relation on $X \times Y$:

$$(mx,y) \;\; R \;\; (x,my) \quad \text{for all } x \in X,\ y \in Y,\ m \in M.$$

We will now construct the "finest" equivalence relation E on $X \times Y$ which contains R. That is we construct an equivalence relation $E \supset R$ such that any other equivalence relation $E' \supset R$ must have $E \subset E'$.

Putting $m = 1$ clearly shows that R is reflexive. Let $T = R \cup R^{-1}$. Then T is both reflexive and symmetric, and furthermore has the following property:

Lemma 3.2

If $(x,y)\ T\ (x',y')$, then (i) $(mx,my)\ T\ (mx',my')$,

(ii) $(mx,y)\ T\ (x,my)\ T\ (x',my')$,

and (iii) $(mx,y)\ T\ (mx',y')\ T\ (x',my')$

for all $m \in M$.

Proof: $(x,y)\ T\ (x',y')$ implies that $(x,y)\ R\ (x',y')$ or $(x',y')\ R\ (x,y)$

If $(x,y)\ R\ (x',y')$ then for some $n \in M$

$$x = nx' \quad \text{and} \quad y' = ny$$

For all $m \in M$ $\quad mx = m(nx') = (mn)x' = (nm)x' = n(mx')$

$$\text{and} \quad my' = m(ny) = (mn)y = (nm)y = n(my) \quad .$$

Using these equations and the definition of R, we have

$$(mx,my)\ R\ (mx',my')$$

$$(mx,y)\ R\ (x,my)\ R\ (x',my')$$

$$(mx,y)\ R\ (mx',y')\ R\ (x',my').$$

Since $R \subset T$ these above statements give (i) (ii) and (iii).

If $(x',y')\ R\ (x,y)$, then a similar argument completes the proof.

We are now able to define the required relation E.

Definition 3.3

For (x,y), (x',y') in $X \times Y$

$(x,y)\ E\ (x',y')$ iff there exists a finite sequence

$$(x,y)\ T\ (x_1,y_1),\ (x_1,y_1)\ T\ (x_2,y_2),\ \dots\ ,\ (x_n,y_n)\ T\ (x',y')$$

for $x_i \in X$, $y_i \in Y$, $i = 1,2,3, \dots , n$.

Proposition 3.3

The relation E defined above is the finest equivalence relation on $X \times Y$ containing R.

Proof: The reflexivity and symmetry of T ensures that E is reflexive and symmetric. If $(x,y)\ E\ (x',y')$ and $(x',y')\ E\ (x'',y'')$ then juxtaposition of the two implied finite sequences gives another finite sequence which implies that $(x,y)\ E\ (x'',y'')$. Thus E is also transitive and is therefore an equivalence relation on $X \times Y$.

Clearly $R \subset E$. To show that E is the finest such equivalence relation, suppose that F is any other equivalence relation with $R \subset F$. Since T is essentially R with symmetry,

$$R \subset T \subset F.$$

If $(x,y)\ E\ (x',y')$, then there exists a finite sequence

$(x,y)\ T\ (x_1,y_1),\ (x_1,y_1)\ T\ (x_2,y_2)\ , \ldots ,\ (x_n,y_n)\ T\ (x',y')$

which gives the following sequence

$(x,y)\ F\ (x_1,y_1),\ (x_1,y_1)\ F\ (x_2,y_2)\ , \ldots ,\ (x_n,y_n)\ F\ (x',y')$

By the transitivity of F we have $(x,y)\ F\ (x',y')$.

Therefore $E \subset F$.

We shall now show that the quotient set $\frac{X \times Y}{E}$ can be given the structure of an M-set, which we shall call the tensor product of the M-sets X and Y.

Define $M \times \frac{X \times Y}{E} \longrightarrow \frac{X \times Y}{E}$ by

$$(m,[(x,y)]) \longmapsto [(mx,y)] = [(x,my)]$$

where $[(mx,y)] = [(x,my)]$ since $R \subset E$.

<u>Proposition 3.4</u>

$$M \times \frac{X \times Y}{E} \longrightarrow \frac{X \times Y}{E} \text{ is well defined.}$$

<u>Proof:</u> If $(x,y)\ E\ (x',y')$, then there exists a finite sequence

$(x,y)\ T\ (x_1,y_1),\ (x_1,y_1)\ T\ (x_2,y_2),\ \ldots ,\ (x_n,y_n)\ T\ (x',y')$

By Lemma 3.2

$(mx,y)\ T\ (mx_1,y_1),\ (mx_1,y_1)\ T\ (mx_2,y_2)\ ,\ \ldots\ ,\ (mx_n,y_n)\ T\ (mx',y')$

for all $m \in M$.

Therefore $[(x,y)] = [(x\ ,y')]$ implies that

$[(mx,y)] = [(mx',y')]$ for all $m \in M$. That is, the above function is well defined.

It follows easily that the set $\frac{X \times Y}{E}$ with the above action is an M-set.

Definition 3.4

The tensor product $X \otimes Y$ of the M-sets X and Y is the M-set $\frac{X \times Y}{E}$

Note that writing $[(x,y)]$ as $x \otimes y$ we have

$m(x \otimes y) = (mx) \otimes y = x \otimes (my)$ for all $m \in M$.

The function $\otimes : X \times Y \longrightarrow \frac{X \times Y}{E}$ is evidently an M-bimorphism. We shall now show that this function is universal among M-bimorphisms from $X \times Y$ to any M-set.

Proposition 3.5 (Universal Bimorphism Property)

To each M-bimorphism $h : X \times Y \longrightarrow Z$ there is exactly one morphism of M-sets $t : X \otimes Y \longrightarrow Z$ such that $t(x \otimes y) = h(x,y)$.

Proof: In the following diagram we are given the solid arrows

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[dr, "h"'] & \frac{X \times Y}{E} = X \otimes Y \arrow[d, dashed, "t"] \\ & Z \end{tikzcd}$$

To show that t is well defined, consider $(x,y)\ E\ (x',y')$

Then there exists a finite sequence

$(x,y)\ T\ (x_1,y_1),\ (x_1,y_1)\ T\ (x_2,y_2),\ \ldots\ ,\ (x_n,y_n)\ T\ (x',y')$

for $x_i \in X$ $y_i \in Y$ and $n \in \mathbb{N}$

But $(x,y)\ T\ (x_1,y_1)$ implies $(x,y)\ R\ (x_1,y_1)$ or $(x_1,y_1)\ R\ (x,y)$

Thus $x = ax_1$ and $y_1 = ay$ for some $a \in M$.

or $x_1 = bx$ and $y = by_1$ for some $b \in M$.

That is
$$\begin{aligned} h(x,y) &= h(ax_1,y) & \text{or} \quad h(x,y) &= h(x,by_1) \\ &= ah(x_1,y) & &= bh(x,y_1) \\ &= h(x_1,ay) & &= h(bx,y_1) \\ &= h(x_1,y_1) & &= h(x_1,y_1) \end{aligned}$$

Since a similar result obviously holds for each of the remaining terms of the above sequence, we have

$$h(x,y) = h(x_1,y_1) = h(x_2,y_2) = \ldots = h(x_n,y_n) = h(x',y')$$

Therefore $(x,y)\ E\ (x',y')$ implies that $t(x \otimes y) = t(x' \otimes y')$

For all $x \in X$, $y \in Y$, $m \in M$

$$\begin{aligned} t(m(x \otimes y)) &= t(mx \otimes y) = h(mx,y) \\ &= mh(x,y) = mt(x \otimes y) \end{aligned}$$

Hence t is a morphism of M-sets which is obviously unique for each M-bimorphism h.

For M-sets X,Y and Z consider the set Bimorph $(X,Y;Z)$ of all M-bimorphisms $X \times Y \longrightarrow Z$.

Define $M \times \text{Bimorph}\ (X,Y;Z) \longrightarrow \text{Bimorph}\ (X,Y;Z)$

by $(m,\ f) \longmapsto mf$

such that $(mf)(x,y) = m.f(x,y)$ for all $x \in X$, $y \in Y$, $m \in M$, and $f \in$ Bimorph $(X,Y;Z)$. To show that the above function is well defined we have to show that mf is an M-bimorphism from $X \times Y$ to Z. For all

$n \in M$.

$$\begin{aligned}(mf)(nx,y) = mf(nx,y) &= m(nf(x,y)) \\ &= (mn)f(x,y) \\ &= (nm)f(x,y) \\ &= n(mf(x,y)) \\ &= n(mf)(xy)\end{aligned}$$

Similarly $(mf)(x,ny) = n(mf)(x,y)$ and the above function is well defined.

It follows readily that Bimorph $(X,Y;Z)$ with the above action is an M-set.

We now show that

$$\text{Bimorph } (X,Y;Z) \cong \hom_M(X,\hom_M(Y,Z))$$

To do this, consider an M-bimorphism $f:X \times Y \longrightarrow Z$.
Write $f(x,y) = F_x(y)$. Therefore $F_x:Y \longrightarrow Z$ is a "partial function" for f. Since f is an M-bimorphism, it follows easily that F_x is a morphism of M-sets.

Define a function $F:X \longrightarrow \hom_M(Y,Z)$ by the assignment $x \longmapsto F_x$.
Since for all $x \in X$, $y \in Y$, $m \in M$

$$\begin{aligned}F(mx)(y) &= F_{mx}(y) \\ &= f(mx,y) \\ &= mf(x,y) \\ &= m\,F_x(y) \\ &= m(F(x))(y)\ ,\end{aligned}$$

F is a morphism of M-sets.

Proposition 3.6

$$\begin{aligned}\varphi = \varphi_{XYZ}: \text{Bimorph } (X,Y;Z) &\longrightarrow \hom_M(X,\hom_M(Y,Z)) \\ f &\longmapsto F\end{aligned}$$

is a natural isomorphism of M-sets.

<u>Proof:</u> First we show that the assignment $f \longmapsto F$ is a bijection by constructing an inverse. Given any $F: X \longrightarrow \hom_M(X, \hom(Y,Z))$ define f by $f(x,y) = F_x(y)$; then since F is a morphism of M-sets

$$f(mx,y) = F_{mx}(y) = mF_x(y) = mf(x,y)$$

also since $F_x: Y \longrightarrow Z$ is a morphism of M-sets

$$f(x,my) = F_x(my) = mF_x(y) = mf(x,y)\,.$$

Therefore f is an M-bimorphism, and $F \longmapsto f$ is the desired inverse.

The assignment $f \longmapsto F$ is a morphism of M-sets (and hence an isomorphism) because the "actions" on both f and F are defined pointwise.

Naturality follows by considering a configuration of three squares; one for each of X,Y,Z varying.

<u>Proposition 3.7</u>

For M-sets X,Y,Z

$$\rho = \rho_{XYZ}: \hom_M(X \otimes Y, Z) \longrightarrow \hom_M(X, \hom_M(Y,Z))$$

defined by $\rho(f)(x)(y) = f(x \otimes y)$ is a natural isomorphism.

<u>Proof:</u> The universality of the tensor product states that every M-bimorphism $h: X \times Y \longrightarrow Z$ has the form $h(x,y) = f(x \otimes y)$ for a unique morphism of M-sets $f: X \otimes Y \longrightarrow Z$. That is $f \longmapsto h$ is a bijection. Therefore consider

$$\alpha: \hom_M(X \otimes Y, Z) \longrightarrow \text{Bimorph}\ (X,Y;Z)$$

defined by $\alpha(f) = h$. For all $x \in X$, $y \in Y$, $m \in M$

$$\begin{aligned}\alpha(mf)(x,y) &= (mf)(x \otimes y)\\ &= m.f(x \otimes y)\\ &= m.h(x,y)\\ &= m\alpha(f)(x,y)\end{aligned}$$

Hence α is an isomorphism of M-sets. Again by considering a configuration of three squares; one for each of X,Y and Z varying, we can show that α is natural.

This isomorphism followed by that of Proposition 3.6 is the desired isomorphism ρ.

Proposition 3.8

For M-sets X and Y

$X \otimes Y \cong Y \otimes X$

Proof: Consider the following diagram

$$\begin{tikzcd} X \times Y \arrow[r, "t_1"] \arrow[d, "f"'] & X \otimes Y \arrow[d, dashed, shift right, "C_{XY}"'] \\ Y \times X \arrow[r, "t_2"'] & Y \otimes X \arrow[u, dashed, shift right, "C_{YX}"'] \end{tikzcd}$$

where the function $f: X \times Y \longrightarrow Y \times X$ is defined by $f(x,y) = (y,x)$ and t_1, t_2 are the previously described universal M-bimorphisms, $t_1(x,y) = x \otimes y$ and $t_2(y,z) = y \otimes x$.

Clearly f is a bijective function and therefore has an inverse f^{-1} such that $f^{-1}(y,x) = (x,y)$. The compositions $t_1 \circ f^{-1}$ and $t_2 \circ f$ are easily verified to be an M-bimorphisms. Therefore by the universality of t_1 there exists a unique morphism of M-sets $C_{XY}: X \otimes Y \longrightarrow Y \otimes X$ such that

$$C_{XY}(x \otimes y) = (t_1 \circ f)(x,y) = t_2(y,x) = y \otimes x$$

Similarly there exists a unique morphism of M-sets

$C_{YX} : Y \otimes X \longrightarrow X \otimes Y$ such that

$$C_{YX}(y \otimes x) = (t_1 \circ f^{-1})(y,x) = t_1(x,y) = x \otimes y$$

Therefore $C = C_{XY}: X \otimes Y \longrightarrow Y \otimes X$ is an isomorphism which can be shown to be natural in X and Y by a routine method.

4. The Category S_M of M-sets

We are now in a position to show that for a fixed commutative monoid M, the category S_M of M-sets is a symmetric monoidal closed category.

To do this we first of all show that S_M is a closed monoidal category by showing that the seven data of Proposition 2.6 are provided .

(i) $C_o = S_M$

(ii) The tensor product defined in section 3 is clearly a functor

$\otimes : S_M \times S_M \longrightarrow S_M$

(iii) $H = \hom_M : S_M^* \times S_M \longrightarrow S_M$

(iv) Take $F: S_M \longrightarrow S$ to be the 'forgetful functor'. Clearly F is faithful.

(v) K = M, the commutative monoid.

(vi) Proposition 3.1 provides the natural isomorphism

$\pi = \pi_X : X \longrightarrow \hom_M(M,X)$

(vii) The natural isomorphism

$\rho = \rho_{XYZ} : \hom_M(X \otimes Y, Z) \longrightarrow \hom_M(X, \hom_M(Y,Z))$

is provided by Proposition 3.7.

Since the basic functor F is faithful, it follows from Proposition 2.6 that S_M is a monoidal closed category.

Note that for S_M data (vi) and (vii) of Definition 2.1 are $\theta_X : M \longrightarrow \hom_M(X,X)$ defined by $\theta_X = \pi_{\hom(X,X)}(1_X)$ and

$L = L^X_{YZ}: \hom_M(Y,Z) \longrightarrow \hom_M(X,X), \hom_M(X,Z))$

defined by $L(f)(g) = f \circ g$ where $f:Y \longrightarrow Z$, $g:X \longrightarrow Y$.

Since the basic functor F is faithful, the natural isomorphism $C_{XY}:X \otimes Y \longrightarrow Y \otimes X$ defined in Proposition 3.8 is a unique symmetry for $\mathcal{S}_M$ if it satisfies MC6 and MC7. Clearly MC6 is satisfied. It is a routine exercise to show that if $f:X \longrightarrow X'$ and $g:Y \longrightarrow Y'$ are morphisms of M-sets, then $f \otimes g:X \otimes Y \longrightarrow X' \otimes Y'$, defined by $(f \otimes g)(x \otimes y) = f(x) \otimes g(y$ is a morphism of M-sets. With this definition the commutativity of the diagram in MC7 follows trivially for the category $\mathcal{S}_M$. Therefore $\mathcal{S}_M$ is a symmetric monoidal closed category with a unique symmetry.

CHAPTER FOUR

MODULOIDS

Many books on modern algebra provide detailed accounts of algebraic structures called modules. In this chapter definitions and basic results will be given for slightly more general structures, which we shall call moduloids.

1. Definitions and Examples

For the purposes of this thesis we formulate the following modified definition of a semiring.

Definition 4.1

A system $\langle R;+,\cdot\rangle$ is called a semiring if

(i) $\langle R,+\rangle$ is a commutative monoid;

(ii) $\langle R,\cdot\rangle$ is a monoid such that $\kappa 0 = 0\kappa = 0$ for all $\kappa \in R$, where 0 is the identity element for $\langle R,+\rangle$;

(iii) $\cdot$ is distributive (on both sides) over $+$.

Note that in terms of the usual definition of a semiring the above structure is a semiring with a multiplicative identity 1 and an additive identity 0, which is an absorbent element. The set of nonnegative integers is an obvious example of such a structure. Also every ring is clearly a semiring of the type defined above if the definition of a ring is, as given by MacLane and Birkoff [17], a "ring with identity".

A commutative semiring K is one in which the multiplication is commutative.

Let $\langle R;+,\cdot\rangle$ be any semiring,

<u>Definition 4.2</u>

An R-moduloid A consists of

(i) a commutative monoid A ;

(ii) a function $R \times A \longrightarrow A$ such that

$$(\kappa, a) \longmapsto \kappa a$$

(a) $\kappa(a + b) = \kappa a + \kappa b$ for all $a, b \in A$, $\kappa \in R$;

(b) $(\kappa + \lambda)a = \kappa a + \lambda a$ for all $a \in A$, $\kappa, \lambda \in R$;

(c) $(\kappa\lambda)a = \kappa(\lambda a)$ for all $a \in A$, $\kappa, \lambda \in R$;

(d) $1a = a$ for all $a \in A$; $1 \in R$;

(e) $0a = \underline{0}$ for all $a \in A$; $\underline{0} \in A$, $0 \in R$.

<u>Remark:</u> (i) The term moduloid has been used with a different meaning by some authors (e.g. Rosenfeld [21])

(ii) Note that because there are no additive inverses in the above structures it is necessary to assume that $0 \in R$ is an absorbent for the semiring R and to axiomatize a similar condition for R-moduloids. (axiom (e)).

<u>Lemma 4.1</u>

$\kappa\underline{0} = \underline{0}$ for all $\kappa \in R$; $\underline{0} \in A$

<u>Proof:</u>

$\kappa\underline{0}$	$= \kappa(0a)$	axiom (e)
	$= (\kappa 0)a$	axiom (c)
	$= 0a$	Definition of semiring
	$= \underline{0}$	axiom (e)

Examples

(1) Take R be the semiring Z^+ of all nonnegative integers. Any commutative monoid A can be considered as a Z^+-moduloid. Also every Z^+-moduloid is a commutative monoid.

(2) Every R-module, where R is a ring, is clearly an R-moduloid.

(3) Given a semiring R and a positive integer n, the set R^n is an R-moduloid under the termwise operations defined by
$(r_1,r_2, \ldots , r_n) + (s_1,s_2, \ldots , s_n) = (r_1 + s_1,r_2 + s_2, \ldots, r_n + s_n)$
and $\kappa(r_1,r_2, \ldots , r_n) = (\kappa r_1,\kappa r_2, \ldots ,\kappa r_n)$.

(4) By a subsemiring of the semiring X we mean a nonempty subset R of X which is itself a semiring under the binary operations defined in X. Any semiring X with a subsemiring $R \subset X$ is an R-moduloid. The operations are the addition in X, and a restriction of the multiplication in X; namely the function

$$(\kappa,a) \longmapsto \kappa a$$

which takes the product of an element κ in the subsemiring with any a in the whole semiring X. The case $X = R$ gives the moduloid $R' = R$. Thus it is a special case of examples three and four that every semiring is a moduloid over itself.

(5) Example number three may be generalized in the following way. If X is any set, the function moduloid A^X is the set of all functions $f:X \longrightarrow A$ from the set X to the R-moduloid A with the usual "pointwise" moduloid operations.

$$(f + g)(x) = f(x) + g(x)$$
$$(\kappa f)(x) = \kappa f(x) \qquad \text{for all } x \in X \ \ \kappa \in R$$

The moduloid axioms for these operations follow at once.

(6) A Z-moduloid is an abelian group.

(Define $-a = (-1)a$)

(7) In a similar way for a given ring R, an R-moduloid is an R-module. In particular, if $R = F$, a field, then the R-moduloid is a vector space over F.

More explicitly the moduloids defined and described above are left moduloids. Right moduloids can be defined in a similar way. If $R = K$, a commutative semiring, then it follows that right and left moduloids are essentially the same.

2. Submoduloids

Let X be an arbitrary R-moduloid. By a submoduloid of X we mean a nonempty subset A of X which is itself a moduloid over R relative to addition and scalar multiplication of the moduloid X.

Among the submoduloids of X are X itself and the set $\{0\}$ consisting of the zero element alone. Any submoduloid of X different from these two is said to be a proper submoduloid. Clearly every submodule is a submoduloid.

Let S be an arbitrary nonempty subset of an R-moduloid X. Then S is contained in at least one submoduloid of X, namely X itself. It can be easily shown that the intersection A of all submoduloids of X containing S is a submoduloid of X. In fact A is the smallest submoduloid of X that contains the given subset S. This submoduloid of X is called the submoduloid generated by S. In case $A = X$, we say that S is a set of generators of X and that S is generated by S.

An element a of an R-moduloid X is said to be a linear combination of elements in a subset S of X iff there exists a finite number of elements

$x_1, x_2, \ldots, x_n \in S$ such that $a = \sum_{i=1}^{n} \lambda_i x_i$ holds with coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ in R.

3. Congruences and Quotient Moduloids

As already indicated in the chapter on M-sets, a relation $\sim$ on an R-moduloid A is said to be compatible with the action on A if $a \sim b$ (a,b in A) implies that $\lambda a \sim \lambda b$ for every λ in R. Similarly, a relation $\sim$ on an R-moduloid A is said to be compatible with the addition on A if $a \sim b$ (a,b in A) implies that $a + x \sim b + x$ for all x in A.

Note that if the relation $\sim$ on A is an equivalence relation, then the above condition for compatibility with addition is equivalent to the following condition:

$$a \sim b \text{ and } c \sim d \text{ (a,b,c,d in A) imply that } a + c \sim b + d$$

Definition 4.3

By a congruence E on an R-moduloid A we mean an equivalence relation which is compatible with both scalar multiplication and addition. That is $a \, E \, b$ (a,b in A) implies that $\lambda a + \lambda c \; E \; \lambda b + \lambda c$ for all c in A and all λ in R.

Examples

(1) The equivalence relations $A \times A$ and $I_A = \{(a,a) \mid a \in A\}$ are congruences on any R-moduloid A.

(2) If E is a congruence on an R-moduloid A and X is a submoduloid of A, then the restriction of E to X is a congruence on X.

Since a congruence is simply a special kind of equivalence relation, it is meaningful to say that one congruence is finer than another. Recall that E is finer than E' iff $a E b$ implies that $a E' b$, that is $E \subset E'$. Clearly I_A, the equality relation, is the finest congruence on any R-moduloid A.

Definition 4.4

The ordered set S is called a lattice if every nonempty finite subset of S has a sup and an inf. S is called a complete lattice if every subset of S has a sup and an inf.

Note that a complete lattice must have a least element $0 = \sup \emptyset = \inf S$ and a greatest element $1 = \inf \emptyset = \sup S$.

Proposition 4.2

Let X be any set, and S any set of subsets of X which contains X itself and is "closed under intersection" - that is, for all nonempty $\mathcal{T} \subseteq S$ we have $\bigcap_{\mathcal{T}} Y \in S$. Then S, ordered by $\subseteq$, is a complete lattice.

Proof: For any nonempty $\mathcal{T} \subseteq S$ we have $\inf \mathcal{T} = \bigcap_{\mathcal{T}} Y$. On the other hand, let $\mathcal{T}^*$ be the set of upper bounds of $\mathcal{T}$ in S - that is, the set of $Y^* \in S$ such that $Y \subseteq Y^*$ for all $Y \in \mathcal{T}$. Then $\mathcal{T}^* \neq \emptyset$ since $X \in \mathcal{T}^*$ and $\bigcap_{\mathcal{T}^*} Y^* = \sup \mathcal{T}$.

Such an S is called an I-lattice on X ([21]). The subgroups of a group, the subrings of a ring and the subspaces of a vector space are nontrivial examples of I-lattices.

Proposition 4.3

The set C_A of all congruences on an R-moduloid A, ordered by inclusion, is a complete lattice with least element I_A and greatest element $A \times A$.

<u>Proof:</u> We need only show that any intersection of congruences on the R-moduloid A is a congruence. Consider $\mathcal{T}$ a nonempty set of congruences on A. We will show that $\bigcap_{\mathcal{T}} E \in C_A$. It is a standard result that $\bigcap_{\mathcal{T}} E$ is an equivalence relation. For $(a,b) \in \bigcap_{\mathcal{T}} E$ we have $(a,b) \in E_f$, where E_f is the finest congruence on A such that $E_f \in \mathcal{T}$; clearly $I_A \subseteq E_f$. Therefore,

$$(a + x,\ b + x) \in E_f \subseteq \bigcap_{\mathcal{T}} E \quad \text{for all } x \text{ in } A$$

which shows that $\bigcap_{\mathcal{T}} E$ is compatible with addition on A. Similarly $\bigcap_{\mathcal{T}} E$ is compatible with scalar multiplication and hence is a congruence for the R-moduloid A.

<u>Proposition 4.4</u>

Let E be a congruence on the R-moduloid A. Then the E-class containing 0 is a submoduloid of A which we shall call a normal submoduloid of A.

<u>Proof:</u> Let $E_o = \{a \mid (a,0) \in E\}$ where $E \subseteq A \times A$.

E_o is closed under addition since

$$\begin{aligned} a,b \in E_o &\Rightarrow (a,0),\ (b,0) \in E \\ &\Rightarrow (a + b,\ 0) \in E \\ &\Rightarrow a + b \in E_o \end{aligned}$$

Also E_o is closed under scalar multiplication from R since

$$\begin{aligned} a \in E_o &\Rightarrow (a,0) \in E \\ &\Rightarrow (\lambda a,0) \in E \quad \text{for all } \lambda \in R \\ &\Rightarrow \lambda a \in E_o \quad \text{for all } \lambda \in R \end{aligned}$$

The moduloid axioms are immediate.

Remark: Clearly the $A \times A$-class containing 0 is the R-moduloid A itself, and the I_A-class containing 0 is the trivial submoduloid of A.

Using this concept of congruence we will now construct a quotient moduloid of the R-moduloid A. Given a congruence E on the R-moduloid A, we first show that $A/_E$ with an appropriate binary operation is a commutative monoid.

For $a \in A$, $[a] = \{x \mid (x,a) \in E\}$ is an element of $A/_E$. We define $[a] \oplus [b] = [a + b]$. The operation $\oplus$ is well defined since if $[a] = [a']$ and $[b] = [b']$, then $(a,a') \in E$ and $(b,b') \in E$ imply that $(a + b, a' + b') \in E$; that is $[a + b] = [a' + b']$.

It follows easily that $\langle A/_E, \oplus \rangle$ is a commutative monoid.

Define $R \times A/_E \xrightarrow{\cdot} A/_E$ by

$$(\lambda,[a]) \longmapsto [\lambda a]$$

This scalar multiplication is well defined since if $[a] = [a']$, then $a\ E\ a'$, which implies that $\lambda a\ E\ \lambda a'$ for all $\lambda \in R$. That is $[\lambda a] = [\lambda a']$

A routine check shows that $\langle A/_E; \oplus, \cdot \rangle$ is an R-moduloid which we shall call a quotient moduloid of A.

4. Morphism of Moduloids

Given two R-moduloids X and Y, a morphism of R-moduloids is a function $f:X \longrightarrow Y$ such that

$$f(a + b) = f(a) + f(b)$$

$$\text{and,} \quad f(\lambda a) = \lambda f(a)$$

$$a,b \text{ in } X \text{ and all } \lambda \text{ in } R$$

Note that condition (e) of Definition 4.2, combined with the second part of the above definition ensures that a morphism of R-moduloids preserves additive identities.

It can be easily shown that the composite of two morphisms of R-moduloids, when defined, is a morphism of R-moduloids.

A morphism of R-moduloids $f:X \longrightarrow Y$ is said to be a monomorphism iff it is injective; f is said to be an epimorphism iff it is surjective. A bijective morphism of R-moduloids is called an isomorphism.

Define the projection $p:A \longrightarrow A/_E$ from any R-moduloid A to a quotient moduloid of A by $p(a) = [a]$. It follows readily from the definitions of addition and scalar multiplication on $A/_E$ that p is a morphism of R-moduloids.

Proposition 4.5 (The universal property of p)

Let E be a congruence on the R-moduloid A. To each morphism of R-moduloids $t:A \longrightarrow B$ such that $a_1, a_2 \in A$ with $(a_1, a_2) \in E$ implies $t(a_1) = t(a_2)$ there is a unique morphism of R-moduloids

$$s:A/_E \longrightarrow B \quad \text{with} \quad s \circ p = t$$

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, "t"'] & A/_E \arrow[d, dashed, "s"] \\ & B \end{tikzcd}$$

Proof: Define $s:A/_E \longrightarrow B$ by $s([a]) = t(a)$. s is well defined since if $[a] = [a']$, that is $(a,a') \in E$, then $t(a) = t(a')$. Because t is a morphism of R-moduloids, it follows that s is a morphism of R-moduloids. This morphism s has the required property since

$$(s \circ p)(a) = s(p(a)) = s([a]) = t(a) \quad \text{for all } a \text{ in } A.$$

Moreover s is uniquely determined. For suppose $s':A/_E \longrightarrow B$ is such that $s' \circ p = t$, then

$$s([a]) = t(a) = (s' \circ p)(a) = s'([a]) \quad \text{for all } [a] \text{ in } A/_E.$$

5. Free Moduloids

Let S be an arbitrarily given set. By a <u>free moduloid</u> over R on the set S we mean a moduloid F over R together with a function $f:S \longrightarrow F$ such that for every function $g:S \longrightarrow X$ from the set S into a moduloid X over R, there is a unique morphism of moduloids $h:F \longrightarrow X$ such that the commutativity relation $h \circ f = g$ holds in the following diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "g"'] & F \arrow[d, dashed, "h"] \\ & X \end{tikzcd}$$

The following two theorems can be easily proved in the usual way.

Theorem 4.6

If an R-moduloid F together with a function $f:S \longrightarrow F$ is a free R-moduloid on the set S, then f is injective and its image $f(S)$ generates F.

Theorem 4.7 (Uniqueness Theorem)

If (F,f) and (F',f') are free R-moduloids on the same set S, then there exists a unique isomorphism $j:F \longrightarrow F'$ such that $j \circ f = f'$

We now establish the following theorem.

Theorem 4.8 (Existence Theorem)

For any set S, there always exists a free R-moduloid on S.

<u>Proof:</u> Let R denote the given semiring and consider the set of all functions $f:S \longrightarrow R$ satisfying $f(x) = 0$ for all except at most a finite number of elements $x \in S$. This set is closed under pointwise addition and scalar multiplication. It is a submoduloid of the function moduloid R^S and is denoted by $R^{(S)}$.

Next let us define a function $E:S \longrightarrow R^{(S)}$ by assigning to each element $x \in S$ the function

$$E_x:S \longrightarrow R \quad \text{defined by} \quad E_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \qquad x,y \in S$$

If $f \in R^{(S)}$ then f has nonzero values for at most n elements of S, say $\{x\ ,x\ , \ldots, x_n\}$. Then f is determined by its n values $f(x_i)$ and indeed

$$f = \sum_{i=1}^{n} f(x_i)E_{x_i}$$

That is the submoduloid $R^{(S)}$ is spanned by all the elements E_x. Let $h:S \longrightarrow A$ be an arbitrary function from the set S to an R-moduloid A. We now show that there is exactly one morphism of moduloids $t:R^{(S)} \longrightarrow A$ with $t \circ E = h$ as in the diagram.

$$\begin{tikzcd} S \arrow[r, "E"] \arrow[dr, "h"'] & R^{(S)} \arrow[d, dashed, "t"] \\ & A \end{tikzcd}$$

Now $t \circ E = h$ states that $t(E_x) = h(x)$ for all x; so any such morphism t must have

$$t(f) = \sum_{i=1}^{n} f(x_i)h(x_i)$$

for each $f \in R^{(S)}$. This shows that t is unique if it exists; conversely one may verify that the function $t:R^{(S)} \longrightarrow A$ defined by this formula is indeed a morphism of R-moduloids.

Thus every set S of elements determines an essentially unique free R-moduloid. Since the function

$$E:S \longrightarrow R^{(S)}$$

is injective, we may identify S with its image $E(S)$ in $R^{(S)}$. This having been done, the set S becomes a subset of $R^{(S)}$ which generates $R^{(S)}$. This R-moduloid $R^{(S)}$ will be referred to as the free R-moduloid generated by the given set S.

6. Biproducts

Consider the Cartesian product $A \times B$ of the R-moduloids A and B. Under the usual pair addition $A \times B$ is clearly a commutative monoid since A and B are commutative monoids. Define scalar multiplication by $\lambda(a,b) = (\lambda a, \lambda b)$ for all $\lambda \in R$, $a \in A$, $b \in B$. It follows readily that the set $A \times B$ is an R-moduloid under the above operations.

We now define the following functions

$$A \underset{i_1}{\overset{p_1}{\rightleftarrows}} A \times B \underset{i_2}{\overset{p_2}{\rightleftarrows}} B$$

by $p_1(a,b) = a$; $p_2(a,b) = b$; $i_1(a) = (a,0)$ and $i_2(b) = (0,b)$ for all $a \in A$, $b \in B$. Clearly p_1, p_2, i_1 and i_2 are morphisms of R-moduloids, and the following lemma can be established in exactly the same way as for modules.

Lemma 4.9

$$p_1 i_1 = 1_A; \quad p_2 i_2 = 1_B; \quad p_1 i_2 = 0;$$

$$p_2 i_1 = 0; \quad \text{and} \quad i_1 p_1 + i_2 p_2 = 1_{A \times B}$$

The moduloid $A \times B$ will now be shown to be both the product and sum of the R-moduloids A and B.

Theorem 4.10

If C is any R-moduloid and $f:C \longrightarrow A$, $g:C \longrightarrow B$ are two morphisms of moduloids, there is a unique morphism of moduloids

$$t:C \longrightarrow A \times B$$

such that $p_1 \circ t = f$ and $p_2 \circ t = g$. That is, $A \times B$ is a product object.

Proof: We must show that the following diagram

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, dashed, "t"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

can be filled in with a unique morphism t so as to be commutative. Now this commutativity $p_1 t = f$ and $p_2 t = g$ implies

$$i_1 f + i_2 g = i_1 p_1 t + i_2 p_2 t = (i_1 p_1 + i_2 p_2)t = t$$

Hence t, if it exists, must be $t = i_1 f + i_2 g$. Conversely, this sum $i_1 f + i_2 g$ is a morphism of R-moduloids $C \longrightarrow A \times B$ such that

$$p_1(i_1 f + i_2 g) = p_1 i_1 f + p_1 i_2 g = f + 0g = f$$

and $$p_2(i_1 f + i_2 g) = p_2 i_1 f + p_2 i_2 g = 0f \pm g = g$$

Hence $t = i_1 f + i_2 g$ is the morphism required.

Theorem 4.11

If C is any R-moduloid and $f:A \longrightarrow C$, $g:B \longrightarrow C$ are two morphisms of R-moduloids, there is a unique morphism $s:A \times B \longrightarrow C$ of moduloids such that $s \circ i_1 = f$ and $s \circ i_2 = g$. That is, $A \times B$ is a sum (or coproduct) object.

Proof: (Similar to the proof of Theorem 4.10).

Since $A \times B$ is both a sum and a product, we write it $A \oplus B$. $A \oplus B$ will therefore be called the biproduct of the R-moduloids A and B.

7. Moduloids of Morphisms

If A and B are moduloids over the commutative semiring K, the set hom(A,B) of morphisms from A to B, under the usual pointwise addition of morphisms will be associative, have an identity, viz the zero morphisms and will be commutative since B is commutative. However, in general, there will be no inverse elements under pointwise addition in hom(A,B) since B is a monoid. Thus hom(A,B) can be given the structure of a commutative monoid under pointwise addition.

Next for any $\kappa \in K$ and any $f \in \hom(A,B)$, consider the function $\kappa f : A \longrightarrow B$ defined by $(\kappa f)(a) = \kappa(f(a))$ for every $a \in A$. Since K is a commutative semiring, it can be easily verified that κf is a morphism of the moduloid A into the moduloid B. The assignment $(\kappa, f) \longmapsto \kappa f$ defines a scalar multiplication in hom(A,B) and gives hom(A,B) the structure of a K-moduloid called the moduloid of all morphisms of the moduloid A into the moduloid B. When given this extra structure hom(A,B) is written as Hom(A,B).

Now let $f : A' \longrightarrow A$ and $g : B \longrightarrow B'$ denote arbitrarily given morphisms of K-moduloids and consider the moduloids Hom(A,B) and Hom(A',B').

Define a function

$$\phi : \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(A',B')$$

by taking

$$\phi(h) = g \circ h \circ f$$

for every h in Hom(A,B). Clearly ϕ is a morphism of K-moduloids. Denote ϕ by Hom(f,g).

Proposition 4.12

For any moduloid A over a commutative semiring K

$\pi = \pi_A : A \longrightarrow \mathrm{Hom}(K,A)$ defined by $\pi(a) = f_a$ such that $f_a(k) = ka$ for all $a \in A$, $k \in K$, is a natural isomorphism of K-moduloids.

Proof: π is well defined since

$$f_a(k_1 + k_2) = (k_1 + k_2)a = k_1 a + k_2 a = f_a(k_1) + f_a(k_2)$$

$$f_a(\lambda k) = (\lambda k)a = \lambda(ka) = \lambda f_a(k)$$

To prove that π is a morphism of moduloids consider for all $k \in K$

$$\begin{aligned}\pi(a_1 + a_2)(k) = f_{a_1 + a_2}(k) &= k(a_1 + a_2)\\ &= ka_1 + ka_2\\ &= f_{a_1}(k) + f_{a_2}(k)\\ &= (f_{a_1} + f_{a_2})(k)\\ &= (\pi(a_1) + \pi(a_2))(k)\end{aligned}$$

Thus $\pi(a_1 + a_2) = \pi(a_1) + \pi(a_2)$ for all a_1, a_2 in A.

For all $k, \lambda \in K$

$$\begin{aligned}\pi(\lambda a)(k) = f_{\lambda a}(k) &= k(\lambda a)\\ &= (k\lambda)a\\ &= (\lambda k)a\\ &= \lambda(ka)\\ &= \lambda f_a(k)\\ &= \lambda\pi(a)(k)\end{aligned}$$

Thus $\pi(\lambda a) = \lambda\pi(a)$ for all $a \in A$, $\lambda \in K$

To prove that π is an isomorphism we first show that π is a monomorphism. Consider $a_1, a_2 \in A$ such that $a_1 \neq a_2$

$$\pi(a_1)(k) = f_{a_1}(k) = ka_1$$

and $\pi(a_2)(k) = f_{a_2}(k) = ka_2$ for all $k \in K$

Since $f_{a_1}(1) = a_1 \neq a_2 = f_{a_2}(1)$, $f_{a_1} \neq f_{a_2}$

To show that π is an epimorphism of K-moduloids, consider $f \in \mathrm{Hom}(K,A)$. For all $k \in K$

$$f(k) = f(k.1) = kf(1) = ka' \quad \text{where } a' = f(1)$$

Therefore $\pi(a') = f$.

To see that the above isomorphism is natural in A, consider the following diagram

$$\begin{tikzcd} A \arrow[r, "\pi_A"] \arrow[d, "g"] & \mathrm{Hom}(K,A) \arrow[d, "\mathrm{Hom}(1,g)"] \\ B \arrow[r, "\pi_B"'] & \mathrm{Hom}(K,B) \end{tikzcd}$$

For $a \in A$, $k \in K$

$$\begin{aligned} \mathrm{Hom}(1,g)\ \pi_A(a)(k) &= \mathrm{Hom}(1,g)(f_a(k)) \\ &= g(f_a(k)) \\ &= g(ka) \end{aligned}$$

and

$$\begin{aligned} \pi_B(g(a))(k) &= f_{g(a)}(k) \\ &= kg(a) \end{aligned}$$

Since $g(ka) = kg(a)$ for all a in A, k in K, the diagram commutes.

8. Bilinear Functions

Let A and B denote arbitrarily given moduloids over a commutative semiring K and consider the cartesian product $A \times B$ of the sets A and B.

A function $g: A \times B \longrightarrow X$ from $A \times B$ to a K-moduloid X is said to be bilinear (or K-bilinear) iff

$$g(\alpha_1 a_1 + \alpha_2 a_2, b) = \alpha_1 g(a_1, b) + \alpha_2 g(a_2, b)$$
$$g(a, \beta_1 b_1 + \beta_2 b_2) = \beta_1 g(a, b_1) + \beta_2 g(a, b_2)$$

holds for all elements a_1, a_2, a in A; b_1, b_2, b in B and $\alpha_1, \alpha_2, \beta_1, \beta_2$ in K.

Let Bilin $(A,B;X)$ denote the set of all bilinear functions $h: A \times B \longrightarrow X$. Then if $t: X \longrightarrow Y$ is a morphism of K-moduloids (i.e. is linear), the composite $t \circ h: A \times B \longrightarrow Y$ is bilinear. For fixed K-moduloids A and B the formulas

$$F(X) = \textbf{Bilin}\ (A,B;X) \qquad F(t)(h) = t \circ h, \quad h \in F(X)$$

define a functor F from K-moduloids to sets. A universal element h_o for this functor F is called a "universal bilinear function" on $A \times B$. That is, $h_o: A \times B \longrightarrow X$ is universal if and only if for every bilinear function $h: A \times B \longrightarrow Y$ there exists a unique morphism of K-moduloids $t: X \longrightarrow Y$ such that the following diagram commutes:

$$\begin{tikzcd} A \times B \arrow[r, "h_o"] \arrow[dr, "h"'] & X \arrow[d, dashed, "t"] \\ & Y \end{tikzcd}$$

9. Tensor Product of K-Moduloids

For any two moduloids A and B over a commutative semiring K we shall construct a universal K-bilinear function on $A \times B$. That is, we simultaneously construct a new K-moduloid $A \otimes B$ and a K-bilinear map $A \times B \longrightarrow A \otimes B$ which is universal among bilinear functions from $A \times B$ to a K-moduloid.

Consider the K-moduloid $F = K^{(A \times B)}$. By theorem 4.8 this moduloid is free on the set $A \dot{\times} B \subset F$ of free generators (a,b). This means that the inclusion $i: A \times B \longrightarrow F$ is universal among functions from the set $A \times B$ to a K-moduloid, however, i is by no means bilinear; for example, if $b \neq 0$ the element $(a_1,b) + (a_2,b)$ in F is never the element $(a_1 + a_2,b)$ of F.

What we now do is to take the "biggest possible" quotient moduloid $F/_E$ so that the composite

$$A \times B \longrightarrow F \longrightarrow F/_E \quad \text{will be bilinear.}$$

To do this, let E be the finest congruence relation on the K-moduloid F for which the following relation R on F is contained in E.

$$(\lambda_1 a_1 + \lambda_2 a_2, b) R \ \lambda_1 (a_1,b) + \lambda_2 (a_2,b) \quad \text{and}$$
$$(a, \mu_1 b_1 + \mu_2 b_2) R \ \mu_1 (a,b_1) + \mu_2 (a,b_2) \quad \text{for all}$$
$$a_1, a_2, a \text{ in } A, \quad b_1, b_2, b \text{ in } B \quad \text{and} \quad \lambda_1, \lambda_2, \mu_1 \mu_2 \text{ in } K.$$

This procedure is certainly possible since the set C_F of all congruences on F is a complete lattice with least element I_F and greatest element $F \times F$. The intersection of all congruences satisfying the above conditions would be the required congruence E. In the language of Clifford and Preston [3], E is the congruence on F generated by the relation R.

As was the case with M-sets, it is useful to describe more fully what is meant by the "finest" congruence on F containing. This is done by giving a fairly general construction similar to that used by Clifford and Preston [3] for determining the congruence on a semigroup generated by a given relation.

Let $T = R \cup R^{-1}$. By putting $\lambda_1 = 1$ and $\lambda_2 = 0$, we see that R is reflexive. Therefore T is clearly reflexive and symmetric.

For x,y in F (To simplify notation we use single letters to denote elements of F), define

$x \rho y$ to mean that

$x = z + \lambda c$, $y = z + \lambda d$ and cTd for some c,d,z in F and λ in K.

Proposition 4.13

The relation ρ on F is compatible with the operations of addition and scalar multiplication.

Proof: Consider x,y in F such that $x\rho y$. Then $x = z + \lambda c$, $y = z + \lambda d$ and cTd for some z,c,d in F and λ in K. For all $s \in F$ $x + s = (z + \lambda c) + s$, $y + s = (z + \lambda d) + s$. By the commutativity and associativity of F

$$x + s = (z + s) + \lambda c \quad \text{and} \quad y + s = (z + s) + \lambda d.$$

Therefore $(x + s)\rho(y + s)$ for all s in F and ρ is compatible with addition on F. Also for all $k \in K$

$$kx = k(z + \lambda c) \quad \text{and} \quad ky = k(z + \lambda d).$$

That is $kx = kz + (k\lambda)c$ and $ky = kz + (k\lambda)d$.

Therefore ρ is compatible with scalar multiplication on F.

We are now able to establish the required congruence E.

Definition 4.6

For x,y in F

xEy iff there exists a finite sequence $c_1, c_2, \ldots, c_n$ in F such that $x\rho c_1$, $c_1\rho c_2$, $c_2\rho c_3$, $\ldots$, $c_n\rho y$

Proposition 4.14

The relation E defined above is the finest congruence on F containing R.

Proof: Clearly $R \subset T \subset \rho \subset E$. The argument already given in the proof of Proposition 3.3 establishes that E is the finest equivalence relation containing R. Also since ρ is compatible with addition we have

$$xEy \Rightarrow x\rho c_1,\ c_1\rho c_2,\ \ldots,\ c_n\rho y$$
$$\Rightarrow (x+s)\rho(c_1+s),\ (c_1+s)\rho(c_2+s),\ \ldots,\ (c_n+s)\rho(y+s).$$
for all s in F.

Therefore $(x+s)E(y+s)$ for all s in F whenever xEy. Similarly the compatibility of ρ with scalar multiplication implies that E is compatible with scalar multiplication. Thus E is a congruence containing R.

Definition 4.7

The tensor product $A \otimes B$ of the K-moduloids A and B is the quotient moduloid $F/_E$.

For each $a \in A$, $b \in B$, the element $[i(a,b)]$ of $A \otimes B$ will be denoted by $a \otimes b$ and called the tensor product of the elements a and b. Every element t of $A \otimes B$ can be written in the form $t = \sum_{i=1}^{n} \lambda_i (a_i \otimes b_i)$ where $a_i \in A$, $b_i \in B$ and $\lambda_i \in K$ for every $i = 1,2,3, \ldots n$. As usual these expressions of the elements of $A \otimes B$ are by no means unique.

In fact $(\lambda_1 a_1 + \lambda_2 a_2) \otimes b = \lambda_1 (a_1 \otimes b) + \lambda_2 (a_2 \otimes b)$ and $a \otimes (\mu_1 b_1 + \mu_2 b_2) = \mu_1 (a \otimes b_1) + \mu_2 (a \otimes b_2)$. In particular, one can easily deduce that

$(\lambda a) \otimes b = \lambda(a \otimes b) = a \otimes (\lambda b)$ for all λ in K $a \in A$, $b \in B$.
It follows that every element t of $A \otimes B$ can be written in the form

$$t = \sum_{i=1}^{n} (a_i \otimes b_i) \quad \text{where } a_i \in A \quad b_i \in B.$$

We now show that this function $A \times B \longrightarrow A \otimes B$ is universal among K-bilinear functions from $A \times B$ to a K-moduloid.

Theorem 4.15

To each K-bilinear functio $h: A \times B \longrightarrow C$ there is exactly one morphism of K-moduloids $f: A \otimes B \longrightarrow C$ such that $f(a \otimes b) = h(a,b)$.

Proof: We are given the solid arrows in the diagram

$$\begin{tikzcd}
A \times B \arrow[r, "i"] \arrow[dr, "h"'] & F = K^{(A \times B)} \arrow[r, "p"] \arrow[d, dashed, "s"] & F_{/E} = A \otimes B \arrow[dl, dashed, "f"] \\
& C &
\end{tikzcd}$$

Since F is free on $A \times B$ the function h determines a unique morphism of K-moduloids $s: F \longrightarrow C$ with $s \circ i = h$. That is s is a unique morphism which sends each generator (a,b) into $h(a,b) \in C$.

We now show that for x,y in F with xEy we have $s(x) = s(y)$. If xEy then $x = (\lambda_1 a_1 + \lambda_2 a_2, b)$ and $y = \lambda_1(a_1,b) + \lambda_2(a_2,b)$ for some a_1, a_2 in A, $b \in B$ λ_1, λ_2 in K,
or $x = (a, \kappa_1 b_1 + \kappa_2 b_2)$ and $y = \kappa_1(a,b_1) + \kappa_2(a,b_2)$ for some $a \in A$, b_1, b_2 in B, κ_1, κ_2 in K. In either case, the bilinearity of h and the fact that $s \circ i = h$ imply that $s(x) = s(y)$.

Since $T = R \cup R^{-1}$, xTy implies $s(x) = s(y)$. If xEy then there exists a finite sequence $x_1, x_2, \ldots, x_n$ in F such that

$$x\rho x_1, \; x_1\rho x_2, \; x_2\rho x_3, \; \ldots, \; x_n\rho y$$

But $x\rho x_1$ implies that $x = u + \lambda c$, $x_1 = u + \lambda d$ and cTd for some u, c, d in F, λ in K.

$$s(x) = s(u + \lambda c) = s(u) + \lambda s(c)$$

$$s(x_1) = s(u + \lambda d) = s(u) + \lambda s(d)$$

However cTd implies $s(c) = s(d)$.

Therefore $s(x) = s(x_1)$.

Similarly $s(x_1) = s(x_2) = s(x_3) = \ldots = s(x_n) = s(y)$

Thus $s(x) = s(y)$ whenever xEy and by the universal property of the projection $p: F \longrightarrow F/_E$ there exists a unique morphism of K-moduloids

$$f: F/_E \longrightarrow C \quad \text{with} \quad f \circ p = s$$

Consequently, $(f \circ p) \circ i = s \circ i = h$.

But $(p \circ i)(a,b) = a \otimes b$.

Therefore, $f(a \otimes b) = h(a,b)$.

For K-moduloids A, B and C consider the set Bilin $(A,B;C)$ of all K-bilinear functions $A \times B \longrightarrow C$. The pointwise sum $f + g$ of two such bilinear functions is bilinear. Also since K is commutative, the pointwise scalar multiple κf of a bilinear map by a scalar is also bilinear. It follows readily that the set Bilin $(A,B;C)$ is a K-moduloid under these pointwise operations. Our immediate aim is to show that

$$\text{Bilin}\,(A,B;C) \cong \text{Hom}(A, \text{Hom}(B,C))$$

The details of setting up this isomorphism are almost entirely parallel to the details of the corresponding result for modules. First we write the values of any bilinear function $h:A \times B \longrightarrow C$ as $h(a,b) = F_a(b)$ so that $F_a:B \longrightarrow C$ is a "partial function" for h. Since h is bilinear, it follows that F_a is a morphism of K-modules. That is

$$F_a(\kappa_1 b_1 + \kappa_2 b_2) = h(a,\kappa_1 b_1 + \kappa_2 b_2) = \kappa_1 h(a,b_1) + \kappa_2 h(a,b_2)$$
$$= \kappa_1 F_a(b\ ) + \kappa_2 F_a(b_2)$$

for all b_1, b_2 in B, κ_1, κ_2 in K.

Therefore $F_a \in \mathrm{Hom}(B,C)$

Define a function $F:A \longrightarrow \mathrm{Hom}(B,C)$ by the assignment $a \longmapsto F_a$. We now show that F is a morphism of K-moduloids.

$$\begin{aligned} F(\lambda_1 a_1 + \lambda_2 a_2)(b) &= F_{\lambda_1 a_1 + \lambda_2 a_2}(b) \\ &= h(\lambda_1 a_1 + \lambda_2 a_2, b) \\ &= \lambda_1 h(a\ ,b) + \lambda_2 h(a_2,b) \\ &= \lambda_1 F_a\ (b) + \lambda_2 F_{a_2}(b) \\ &= (\lambda_1 F_{a_1} + \lambda_2 F_{a_2})(b) \\ &= (\lambda_1 F(a_1) + \lambda_2 F(a_2))(b) \end{aligned}$$

for all b in B; a_1, a_2 in A and λ_1, λ_2 in K.

Therefore $F \in \mathrm{Hom}(A,\mathrm{Hom}(B,C))$

Theorem 4.16

For K-moduloids A, B and C there is an isomorphism

$$\mathrm{Bilin}\ (A,B;C) \cong \mathrm{Hom}(A,\mathrm{Hom}(B,C))$$

of K-moduloids given by assigning to each K-bilinear function $h:A \times B \longrightarrow C$ the morphism of K-moduloids $F:A \longrightarrow \mathrm{Hom}(B,C)$.

Proof: The proof is parallel to that given for modules [17] p. 330.

Theorem 4.17

For K-moduloids A,B and C there is a natural isomorphism

$$\rho = \rho_{ABC} : \mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B,C))$$

where $f \longmapsto \rho(f)$

such that $\rho(f)(a)(b) = f(a \otimes b)$ for all $a \in A$, $b \in B$.

Proof: The universality of the tensor product states that every K-bilinear function $h: A \times B \longrightarrow C$ has the form $h(a,b) = f(a \otimes b)$ for a unique morphism of K-moduloids $f: A \otimes B \longrightarrow C$. That is,

$f \longmapsto h$ is a bijection

Let $\alpha: \mathrm{Hom}(A \otimes B, C) \longrightarrow \mathrm{Bilin}(A,B;C)$ be defined by $\alpha(f) = h$.

Consider $f,g \in \mathrm{Hom}(A \otimes B, C)$ such that $f \longmapsto h$; $g \longmapsto k$

For all (a,b) in $A \times B$; $\lambda, \mu \in K$

$$\begin{aligned} \alpha(\lambda f + \mu g)(a,b) &= (\lambda f + \mu g)(a \otimes b) \\ &= (\lambda f)(a \otimes b) + (\mu g)(a \otimes b) \\ &= \lambda(f(a \otimes b)) + \mu(g(a \otimes b)) \\ &= \lambda(h(a,b)) + \mu(k(a,b)) \\ &= \lambda(\alpha(f)(a,b)) + \mu(\alpha(g)(a,b)) \\ &= (\lambda\alpha(f) + \mu\alpha(g))(a,b) \end{aligned}$$

Therefore $\alpha(\lambda f + \mu g) = \lambda\alpha(f) + \mu\alpha(g)$ and α is an isomorphism of K-moduloids. This isomorphism followed by that of Theorem 4.16 is the desired isomorphism ρ

Proposition 4.18

For K-moduloids A and B, $A \otimes B \cong B \otimes A$

Proof: (Similar to the proof of Proposition 3.8)

10. The Category M_K of K-Moduloids

As was the case with M-sets, our main objective is to show that for a fixed commutative semiring K, the category M_K is a symmetric monoidal closed category.

To do this we first of all show that M_K is a closed monoidal category by showing that the seven data of Proposition 2.6 are provided.

(i) $C = M_K$

(ii) The tensor product defined in section 9 is clearly a functor $\otimes: M_K \times M_K \longrightarrow M_K$.

(iii) $H = \mathrm{Hom}: M_K^* \times M_K \longrightarrow M_K$.

(iv) $F: M_K \longrightarrow S$ is "the forgetful functor". Clearly F is faithful.

(v) K is the commutative semiring

(vi) Proposition 4.12 provides the natural isomorphism
$\pi = \pi_A : A \longrightarrow \mathrm{Hom}(K,A)$

(vii) The natural isomorphism
$$\rho = \rho_{ABC} : \mathrm{Hom}(A \otimes B, C) \longrightarrow \mathrm{Hom}(A, \mathrm{Hom}(B,C))$$
is provided by Theorem 4.17

Since the basic functor F is faithful, it follows from Proposition 2.6 that M_K is a monoidal closed category.

Note that data (vi) and (vii) of Definition 2.1, that is

$$\theta_A : K \longrightarrow \mathrm{Hom}(A,A) \quad \text{and}$$

$$L = L^A_{BC} : \mathrm{Hom}(B,C) \longrightarrow \mathrm{Hom}(\mathrm{Hom}(A,B), \mathrm{Hom}(A,C)),$$

are defined by

$$\theta_A = \pi_{\mathrm{Hom}(A,A)}(1_A) \qquad \text{and}$$

$$L(f)(g) = f \circ g \quad \text{where} \quad f:B \longrightarrow C, \quad g:A \longrightarrow B.$$

The natural isomorphism $C_{AB}:A \otimes B \longrightarrow B \otimes A$, defined in Proposition 4.18, can be shown to be a unique symmetry in the same way as already indicated for M-sets. Hence $\mathcal{M}_K$ is a symmetric monoidal closed category.

In particular, we have shown that the category Mod_K of modules over a commutative ring and the category Ab of abelian groups are symmetric monoidal closed categories.

CHAPTER FIVE

EMBEDDING THEOREMS

In this chapter we consider again closed categories. It will be shown that any such category satisfying certain conditions can be embedded into one or more of the following five symmetric monoidal closed categories:

(i) the category S_M of M-sets, where M is a commutative monoid;

(ii) the category S_G of G-sets, where G is an abelian group;

(iii) the category M_K of moduloids over a commutative semiring K;

(iv) the category Mod_K of modules over a commutative ring K;

(v) the category V_F of vector spaces over a field F.

Consequently, five embedding theorems, which form the focal point of this thesis, are established.

1. Embedding Theorems for M-Sets and G-Sets.

Let C be any closed category with a faithful basic functor F. We will now show that the natural isomorphism $A \cong H(K,A)$ provides a means of defining, for each $A \in \mathrm{Ob}\, C$, a function $FK \times FA \longrightarrow FA$. The inherent properties of such a function will be the basis for putting extra structure on the set FA.

Since $FA \cong \hom(K,A)$, $F\pi(a): K \longrightarrow A$ and $F(F\pi(a)): FK \longrightarrow FA$ for all $a \in FA$.

Definition 5.1

The function $FK \times FA \xrightarrow{\cdot} FA$ written $(\kappa, a) \longmapsto \kappa \cdot a$ is defined by $\kappa \cdot a = F(F\pi(a))(\kappa)$ for all $a \in FA$ and all $\kappa \in FK$

Proposition 5.1

If $f \in FH(B,C)$, then $\pi_C \circ f = H(1,f) \circ \pi_B$

Proof: Axiom CC4 and the fact that F is a functor imply that the following diagram commutes:

$$\begin{tikzcd} FH(B,C) \arrow[r, "FL^K"] \arrow[dr, "FH(1{,}\pi_C)"'] & FH(H(K,B),H(K,C)) \arrow[d, "FH(\pi_B{,}1)"] \\ & FH(B,H(K,C)) \end{tikzcd}$$

$$FH(\pi_B,1)(FL^K(f)) = FH(\pi_B,1)(H(1,f))$$
$$= H(1,f) \circ \pi_B$$

and $FH(1,\pi_C)(f) = \pi_C \circ f$

Therefore $H(1,f) \circ \pi_B = \pi_C \circ f$

Corollary:

$F\pi(\kappa \cdot a) = F\pi(a) \circ F\pi(\kappa)$ for all a in FA and all κ in FK.

Proof: Since F is a functor, it follows from

$H(1,f) \circ \pi_B = \pi_C \circ f$ that

$FH(1,f) \circ F\pi_B = F\pi_C \circ Ff : FB \longrightarrow FH(K,C)$

Put $B = K$, $C = A$ and $f = F\pi(a)$ where $a \in FA$. Then for all a in FA and all κ in FK

$$F\pi(\kappa\cdot a) = F\pi(F(F\pi(a))(\kappa))$$
$$= (F\pi \circ Ff)(\kappa)$$
$$= (FH(1,f) \circ F\pi)(\kappa)$$
$$= FH(1,f)(F\pi(\kappa))$$
$$= f \circ F\pi(\kappa)$$
$$= F\pi(a) \circ F\pi(\kappa).$$

Considering the natural isomorphism $\pi: K \longrightarrow H(K,K)$, we obtain the bijective function $F\pi: FK \longrightarrow \hom(K,K)$.

Definition 5.2

Define $1 \in FK$ to be $F\pi^{-1}(1_K)$; that is $F\pi(1) = 1_K$

Proposition 5.2

For all a in FA and all λ, μ in FK the function $FK \times FA \xrightarrow{\cdot} FA$ satisfies the following properties:

(i) $\lambda\cdot(\mu\cdot a) = (\lambda\cdot\mu)\cdot a$

(ii) $1\cdot a = a$

Proof: (i) $\lambda\cdot(\mu\cdot a) = F(F\pi(\mu\cdot a))(\lambda)$

$$= F(F\pi(a) \circ F\pi(\mu))(\lambda) \qquad \text{(Corollary to Proposition 5.1)}$$
$$= F(F\pi(a))(F(F\pi(\mu))(\lambda))$$
$$= F(F\pi(a))(\lambda\cdot\mu)$$
$$= (\lambda\cdot\mu)\cdot a$$

(ii) $F\pi(1\cdot a) = F\pi(a) \circ F\pi(1)$ (Corollary to Proposition 5.1)

$$= F\pi(a) \circ 1_K$$
$$= F\pi(a)$$

But $F\pi$ is a bijection. Hence $1\cdot a = a$ for all a in FA.

Corollary 1:

The set FK together with the binary operation $FK \times FK \xrightarrow{\cdot} FK$ is a commutative monoid.

Proof: Put $A = K$ in Proposition 5.2. Associativity and the existence of a left identity are immediate. For all λ, μ in FK

$$F\pi(\lambda\cdot\mu) = F\pi(\mu) \circ F\pi(\lambda)$$
$$= F\pi(\lambda) \circ F\pi(\mu) \qquad \text{(Corollary 3 of Proposition 2.5)}$$
$$= F\pi(\mu\cdot\lambda)$$

But $F\pi$ is a bijection. Hence $\lambda\cdot\mu = \mu\cdot\lambda$ for all λ, μ in FK.

Corollary 2

For each $A \in \mathrm{Ob}\, C$ the pair $\langle FA, FK \times FA \xrightarrow{\cdot} FA\rangle$ is an FK-set where $\langle FK, \cdot\rangle$ is a commutative monoid.

We now establish the first of the five embedding theorems indicated at the beginning of this chapter.

Theorem 5.3 (Embedding Theorem 1)

Let C be any closed category with a faithful basic functor. Then there is a canonical faithful functor

$$G: C \longrightarrow S_{FK}$$

from C to the category S_{FK} of sets under the action of the commutative monoid $\langle FK, \cdot\rangle \cong$ end (K) Furthermore, if $\bar{F}: S_{FK} \longrightarrow S$ is the underlying set functor, then $\bar{F}G: C \longrightarrow S$ is the basic functor for C; that is $\bar{F}G = F$.

Proof: Define $G: C \longrightarrow S_{FK}$ such that for each $A \in \mathrm{Ob}\, C$, $G(A)$ is the FK-set $(FA, FK \times FA \xrightarrow{\cdot} FA)$ and for each morphism $f: A \longrightarrow B$ in C,

$G(f) = Ff : FA \longrightarrow FB$ is a morphism of FK-sets.

$G(1_A) = F1_A = 1_{FA} = 1_{G(A)}$ for each $A \in \mathrm{Ob}\,\mathcal{C}$ and if $f \circ g$ is defined, then

$$G(f \circ g) = F(f \circ g) = Ff \circ Fg = G(f) \circ G(g)$$

Hence the above assignments do define a functor.

Actually G is such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "G"] \arrow[dr, "F"'] & & \mathcal{S}_{FK} \arrow[dl, "\bar{F}"] \\
& \mathcal{S} &
\end{tikzcd}$$

where $\bar{F}$, the forgetful functor, is the basic functor for $\mathcal{S}_{FK}$. In other words, G is a lifting of the basic functor $\bar{F}$ for $\mathcal{C}$ over the basic functor $\bar{F}$ for $\mathcal{S}_{FK}$.

We now show that the functor G is faithful; that is, it is an embedding of $\mathcal{C}$ into $\mathcal{S}_{FK}$.

Consider $f, g : A \longrightarrow B$ in $\mathcal{C}$. Then $G(f), G(g) : FA \longrightarrow FB$ are morphisms of FK-sets. If $G(f) = G(g)$, then $\bar{F}G(f) = \bar{F}G(g)$; that is $F(f) = F(g)$. But F is faithful implies that $f = g$. Hence G is faithful.

Proposition 5.4

If in a closed category $\mathcal{C}$ with faithful basic functor every morphism $\bar{x} \in \hom(K,K)$ is invertible, then $\langle FK, \cdot \rangle$ is an abelian group.

Proof: Each $\bar{x} \in \hom(K,K)$ being invertible gives rise to the following bijection:

$$FK \cong \hom(K,K) \longrightarrow \hom(K,K) \cong FK.$$

$$x \longmapsto \bar{x} \longmapsto \bar{x}^{-1} \longmapsto x^{-1}$$

where $x^{-1} \in FK$ is defined by $x^{-1} = F\pi^{-1}(\overline{x}^{-1})$

It has already been shown that $\langle FK, \cdot \rangle$ is a commutative monoid.

For each $x \in FK$.

$$
\begin{aligned}
F\pi(x \cdot x^{-1}) &= F\pi(x^{-1}) \circ F\pi(x) \\
&= \overline{x}^{-1} \circ \overline{x} \\
&= 1_K
\end{aligned}
$$

Therefore $x \cdot x^{-1} = F^{-1}(1_K) = 1 \in FK$.

Theorem 5.5 (Embedding Theorem 2)

If for a closed category C with faithful basic functor every morphism in $\hom(K,K)$ is invertible, then there is a canonical faithful functor $F:C \longrightarrow S_{FK}$ from C to the category S_{FK} of sets under the action of the abelian group $\langle FK, \cdot \rangle$.

Furthermore, if $U:S_{FK} \longrightarrow S$ is the underlying set functor, then $UF:C \longrightarrow S$ is the basic functor for C; that is $UF = F$.

Proof: The proof is immediate from Proposition 5.3 and 5.4.

2. Embedding Theorems for Moduloids, Modules and Vector Spaces

Consider again any closed category C with faithful basic functor but with the extra assumption that C has biproducts.

Proposition 5.6

If $A \in Ob\ C$, then FA can be given the structure of a commutative monoid (under $+$)

Proof: By Proposition 1.8 it was shown that whenever A and B are objects of a category with biproducts, $\hom(A,B)$ has the structure of a commutative monoid under +.

For x,y in FA define

$$x + y = F\pi^{-1}(F\pi(x) + F\pi(y))$$

where $F\pi : FA \longrightarrow \hom(K,A)$

Clearly $x + y \in FA$.

Define $\mathbf{0} \in FA$ by $\mathbf{0} = F^{-1}(0)$

It follows easily that $\langle FA, +\rangle$ is a commutative monoid.

Proposition 5.7

Let C be any closed category with faithful basic functor and biproducts. The function $FK \times FA \xrightarrow{\cdot} FA$ defined by $\lambda\cdot a = F(F\pi(a))(\lambda)$ satisfies the following properties for all a,b in FA and all λ,μ in FK:

(i) $\lambda\cdot(a + b) = \lambda\cdot a + \lambda\cdot b$

(ii) $(\lambda + \mu)\cdot a = \lambda\cdot a + \lambda\cdot a$

(iii) $0\cdot a = \underline{0}$ where $0 \in FK$ and $\underline{0} \in FA$

Proof: (i) $F\pi(\lambda\cdot(a + b) = F\pi(a + b) \circ F\pi(\lambda)$

$= (F\pi(a) + F\pi(b)) \circ F\pi(\lambda)$

$= F\pi(a) \circ F\pi(\lambda) + F\pi(b) \circ F\pi(\lambda)$

$= F\pi(\lambda\cdot a) + F\pi(\lambda\cdot b)$

$= F\pi(\lambda\cdot a + \lambda\cdot b)$

But $F\pi$ is a bijection. Therefore $\lambda\cdot(a + b) = \lambda\cdot a + \lambda\cdot b$ for all a,b in FA and all λ in FK.

(ii) $F\pi((\lambda + \mu)\cdot a) = F\pi(a) \circ F\pi(\lambda + \mu)$

$= F\pi(a) \circ (F\pi(\lambda) + F\pi(\mu))$

$= F\pi(a) \circ F\pi(\lambda) + F\pi(a) \circ F\pi(\mu)$

$= F\pi(\lambda\cdot a) + F\pi(\mu\cdot a)$

$= F\pi(\lambda\cdot a + \mu\cdot a)$

Hence $(\lambda + \mu)\cdot a = \lambda\cdot a + \mu\cdot a$ for all a in FA and all λ,μ in FK.

(iii) $F\pi(0\cdot a) = F\pi(a) \circ F\pi(0)$

$= F\pi(a) \circ \mathbf{0}$ where $\mathbf{0}: K \longrightarrow A$

$= \mathbf{0}$

Therefore $0\cdot a = F\pi^{-1}(\mathbf{0}) = \underline{0} \in FA$.

Corollary 1:

$\langle FK: +, \cdot\rangle$ is a commutative semiring.

Proof: The proof is immediate from Corollary 2 of Proposition 5.2, Proposition 5.6 and Proposition 5.7

Corollary 2:

For each $A \in \mathcal{O}b\, C$, $\langle FA; +, \cdot\rangle$ is a moduloid over the commutative semiring FK.

Proof: The proof is immediate from Proposition 5.6, Proposition 5.2 and Proposition 5.7.

We are now able to present the third embedding theorem indicated at the beginning of this chapter.

Theorem 5.8 (Embedding Theorem 3)

If C is any closed category with a faithful basic functor and biproducts, then there is a canonical faithful functor

$$\alpha: C \longrightarrow M_{FK}$$

from C to the category M_{FK} of moduloids over the commutative semiring FK. Furthermore, if $U: M_{FK} \longrightarrow S$ is the underlying set functor, then

$U\alpha$ is the basic functor for C; that is $U\alpha = F$.

Proof: This follows from Theorem 5.3 and Corollary 2 of Proposition 5.7

Since the categories that are replacements for C in the above theorem have zero objects and biproducts, it is of some interest to investigate how the embedding α affects this special structure.

Lemma 5.9

The image $\alpha(0)$ of the zero object of C is the trivial FK-moduloid F0.

Proof: By definition of α, $\alpha(0) = F0$ where F0 is a FK-moduloid. All that reamins is to show that F0 is the trivial moduloid. Since $F0 \cong \hom(K,0)$ and $\hom(K,0)$ consists of a single element, the zero morphism $0:K \longrightarrow 0$, F0 is a singleton set. Specifically $F0 = \{0\}$ where $0 = F\pi^{-1}(0)$. Clearly when the set is given the previously indicated moduloid structure, it is (up to isomorphism) the trivial FK-moduloid, which is the zero object of the category M_{FK}.

Before considering how α affects biproducts, we require the following definition.

Definition 5.4

Let C and C' be two categories with biproducts and $F:C \longrightarrow C'$ a functor. F is said to be additive if for any pair of morphisms $f,g \in \hom_C(A,B)$, we have

$$F(f+g) = F(f) + F(g).$$

Proposition 5.10

The embedding $\alpha : \mathcal{C} \longrightarrow \mathcal{M}_{FK}$ of theorem 5.8 preserves biproduct iff it is an additive functor. In other words, $\alpha(A) \underset{\alpha(p)}{\overset{\alpha(i)}{\rightleftarrows}} \alpha(A \oplus B) \underset{\alpha(q)}{\overset{\alpha(j)}{\leftrightarrows}} \alpha(B)$ is a biproduct in $\mathcal{M}_{FK}$ whenever $A \underset{p}{\overset{i}{\rightleftarrows}} A \oplus B \underset{q}{\overset{j}{\leftrightarrows}} B$ is a biproduct in $\mathcal{C}$ iff α is an additive functor.

Proof: ($\Rightarrow$)

Since $A \underset{p}{\overset{i}{\rightleftarrows}} A \oplus B \underset{q}{\overset{j}{\leftrightarrows}} B$ is a biproduct in $\mathcal{C}$,

$$pi = 1_A \qquad qj = 1_B$$

$$qi = 0 \qquad pj = 0$$

Therefore $\alpha(pi) = \alpha(1_A)$

That is $\alpha(p)\alpha(i) = 1_{\alpha(A)}$

Similarly $\alpha(q)\alpha(j) = 1_{\alpha(B)}$

Also $\alpha(q)\alpha(i) = \alpha(0)$ and $\alpha(p)\alpha(j) = \alpha(0)$

Since α sends the zero object of $\mathcal{C}$ to the zero object of $\mathcal{M}_{FK}$ (Lemma 5.9), $\alpha(0)$ is a zero morphism in $\mathcal{M}_{FK}$; that is $\alpha(0) = 0$.

Therefore $\alpha(q)\alpha(i) = 0$ and $\alpha(p)\alpha(j) = 0$.

Since $A \oplus B$ is a biproduct in $\mathcal{C}$, Proposition 1.9 states that $ip + jq = 1_{A \oplus B}$

Consequently $\alpha(ip + jq) = \alpha(1_{A \oplus B}) = 1_{\alpha(A \oplus B)}$

But since α is an additive functor, this gives

$$\alpha(ip) + \alpha(jq) = 1_{\alpha(A \oplus B)}$$

$$\alpha(i)\alpha(p) + \alpha(j)\alpha(q) = 1_{\alpha(A \oplus B)}$$

All that remains is to show that $(\alpha(p), \alpha(q))$ is a product in M_{FK} and $(\alpha(i), \alpha(j))$ is a sum in M_{FK}. This follows in exactly the same way as the arguments already given for moduloids in chapter IV.

($\Leftarrow$) To show that α is an additive functor, consider $f,g \in \hom_{\mathcal{C}}(A,B)$.

f + g is :~

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ A \arrow[ur, "1"] \arrow[r, "\Delta"] \arrow[dr, "1"'] & A \oplus A \arrow[u, "p"] \arrow[d, "q"] \arrow[r, "{[f,g]}"] & B \\ & A \arrow[ur, "g"'] & \end{tikzcd}$$

Since α preserves biproducts, the object $\alpha(A \oplus A)$ in the following diagram is the biproduct of $\alpha(A)$ and $\alpha(A)$.

$$\begin{tikzcd} & \alpha(A) \arrow[dr, "\alpha(f)"] & \\ \alpha(A) \arrow[ur, "1\alpha(A)"] \arrow[r, "\alpha(\Delta)"] \arrow[dr, "1\alpha(A)"'] & \alpha(A \oplus A) \arrow[u, "\alpha(p)"] \arrow[d, "\alpha(q)"] \arrow[r, "{\alpha[f,g]}"] & \alpha(B) \\ & \alpha(A) \arrow[ur, "\alpha(g)"'] & \end{tikzcd}$$

Therefore $\alpha[f,g] = [\alpha(f), \alpha(g)]$

That is $\alpha(f + g) = \alpha(f) + \alpha(g)$.

It has been shown that for any closed category C with faithful basic functor and biproducts the set FA can be given the structure of a FK-moduloid for each $A \in \mathrm{Ob}\ C$. Considering that a module is essentially a moduloid with additive inverses, the following question naturally arises: What extra property must the closed category C possess in order that FA can be given the structure of a FK-module for each $A \in \mathrm{Ob}\ C$, where FK is a commutative ring?

The following proposition answers the above question by providing the details of how FA is enriched with an abelian group structure.

Proposition 5.11

If a closed category $\mathcal{C}$ is such that

(i) the basic functor F is faithful,

(ii) $\mathcal{C}$ has biproducts,

and (iii) there exists a morphism $\bar{u}:K \longrightarrow K$ such that the following diagram commutes:

$$\begin{tikzcd}
K \arrow[r, "\Delta"] \arrow[d] & K \oplus K \arrow[d, "{[1,\bar{u}]}"] \\
0 \arrow[r] & K
\end{tikzcd}$$

that is $\bar{u} + 1_K = 0_K$, then FA can be given, in a canonical way for each $A \in \mathrm{Ob}\,\mathcal{C}$, the structure of an abelian group (under +).

Proof: Since it has already been shown that conditions (i) and (ii) ensure that FA can be given the structure of a commutative monoid, all that remains is to show that additive inverses exist. We have $FA \cong \hom(K,A)$. For each $x \in FA$ define $-x \in FA$ by $-\bar{x} = F\pi^{-1}(-\bar{\bar{x}})$ where $\bar{\bar{x}} = F\pi(x)$ and $-\bar{\bar{x}} = \bar{\bar{x}} \circ \bar{u}$. That is

$$FA \cong \hom(K,A) \longrightarrow \hom(K,A) \cong FA$$

$$x \longmapsto \bar{\bar{x}} \longmapsto -\bar{\bar{x}} \longmapsto -x$$

Then
$$\begin{tikzcd}
 & K \arrow[d] \arrow[dr, "1"] & & \\
K \arrow[ur, "1"] \arrow[r, "\Delta"] \arrow[dr, "1"'] & K \oplus K \arrow[r, "{[1,\bar{u}]}"'] & K \arrow[r, "\bar{x}"'] & A \\
 & K \arrow[u] \arrow[ur, "\bar{u}"'] & &
\end{tikzcd}$$

is equal to zero.

That is

$$\begin{tikzcd} & K \arrow[d] \arrow[dr, "\bar{x}"] & \\ K \arrow[r, "\Delta"'] & K \oplus K \arrow[r, "{[\bar{x}, \bar{x} \circ \bar{u}]}"'] & A \\ & K \arrow[u] \arrow[ur, "\bar{x} \circ \bar{u}"'] & \end{tikzcd}$$

is equal to zero.

By definition of $+$

$$\bar{x} + (\bar{x} \circ \bar{u}) = \bar{0}$$

But $\bar{x} \circ \bar{u} = -\bar{x}$, therefore $\bar{x} + (-\bar{x}) = \bar{0}$

For all $x \in FA$

$$\begin{aligned} x + (-x) &= F\pi^{-1}(F\pi(x) + F\pi(-x)) \\ &= F\pi^{-1}(\bar{x} + (-\bar{x})) \\ &= F\pi^{-1}(\bar{0}) \\ &= 0 \in FA \end{aligned}$$

Two obvious consequences of the above proposition are:

Corollary 1 $\langle FK; +, \cdot \rangle$ is a commutative ring.

Corollary 2. $\langle FA; +, \cdot \rangle$ is a module over FK.

We are now able to present the fourth embedding theorem indicated at the beginning of this chapter.

Theorem 5.12 (Embedding Theorem 4)

If $\mathcal{C}$ is any closed category such that

(i) the basic functor F is faithful ,

(ii) $\mathcal{C}$ has biproducts ,

(iii) there exists a morphism $\bar{u}: K \longrightarrow K$ such that $1_K + \bar{u} = 0_K$,

then there is a canonical faithful functor

$$\gamma : \mathcal{C} \longrightarrow Mod_{FK}$$

from C to the category Mod_{FK} of modules over the commutative ring FK. Furthermore, if $U:\mathrm{Mod}_K \longrightarrow S$ is the underlying set functor, then $U\gamma: C \longrightarrow S$ is the basic functor for C; that is $U\gamma = F$.

Proof: The proof follows from Proposition 5.11 (Corollaries 1 and 2) and Theorem 5.3.

Considering the possibility of obtaining a field structure on FK and consequently giving FA the structure of a vector space for each $A \in \mathrm{Ob}\ C$ brings us to the fifth embedding theorem.

Theorem 5.13 (Embedding Theorem 5)

If C is any closed category such that

(i) the basic functor F is faithful,

(ii) C has biproducts,

(iii) there exists a morphism $\bar{u}: K \longrightarrow K$ such that $1_K + \bar{u} = 0_K$,

(iv) every nonzero morphism $K \longrightarrow K$ is invertible, then there is a canonical faithful functor

$$\Phi: \tilde{C} \longrightarrow V_{FK}$$

from C to the category V_{FK} of vector spaces over the field FK. Furthermore, if $\bar{F}: V_{FK} \longrightarrow S$ is the underlying set functor, then $\bar{F}\Phi: C \longrightarrow S$ is the basic functor for C; that is $\bar{F}\Phi = F$.

Proof: We have to prove that the commutative ring FK is a field; the result then follows from 5.12. It follows from condition (iv) and a slight modification of the proof of Proposition 5.4 that each nonzero element of FK has a "multiplicative" inverse. Hence

$\langle FK; +, \cdot \rangle$ is a field and the proof is complete.

3. Examples

In chapter III it was shown that any category $\mathcal{S}_M$ of sets under the action of a commutative monoid M is a symmetric monoidal closed category. In particular, any category $\mathcal{S}_G$, where G is an abelian group, is a symmetric monoidal closed category. Since Theorems 5.3 and 5.5 involve the embedding of closed categories into these two categories, it is of interest to investigate what these embeddings are in the case of $\mathcal{S}$, $\mathcal{S}_*$, $\mathcal{S}_M$ and $\mathcal{S}_G$. Note that if categories of more structured algebraic objects are considered, these embeddings will be less interesting since in many cases the basic structure is not preserved. For example, if $\mathcal{C} = \text{Mod}_K$ in Theorem 5.3, the embedding obviously does not preserve the abelian group structure.

(1) The category $\mathcal{S}$ of sets admits, through the following data, the structure of a symmetric monoidal closed category:

(a) $F = 1: \mathcal{S} \longrightarrow \mathcal{S}$ (F is clearly faithful);

(b) $H(X,Y) = \hom(X,Y)$; $X,Y \in \text{Ob } \mathcal{S}$;

(c) $K = \{s\}$, a singleton set;

(d) $\pi_X : X \cong \hom(\{s\}, X)$ such that $\pi_X(x)(s) = x$ for all $x \in X$;

(e) $X \otimes Y = X \times Y$;

(f) $\rho_{XYZ} : \hom(X \times Y, Z) \cong \hom(X, \hom(Y,Z))$ such that $\rho(f)(x)(y) = f(x,y)$ where $f: X \times Y \longrightarrow Z$.

Under the operation $\cdot$ (Definition 5.1 with $A = K$) the set $FK = 1K = K$ has the following monoidal structure:

$\cdot$	s
s	s

It was stated in chapter III that any set could be regarded as an M-set where $M = \{1\}$ is the trivial monoid. Clearly the above structure $\langle FK, \cdot \rangle$ is (isomorphic to) the trivial monoid M.

If in Theorem 5.3 we put $\mathcal{C} = \mathcal{S}$, the embedding is simply a confirmation of the above observation that the category of sets is (isomorphic to) the category of $\{1\}$-sets. This result is also obtained from Theorem 5.5 by putting $\mathcal{C} = \mathcal{S}$.

(2) The category $\mathcal{S}_*$ of pointed sets admits the structure of a symmetric monoidal closed category through the following data:

(a) $F: \mathcal{S}_* \longrightarrow \mathcal{S}$ such that $F(X, *) = X$;

(b) $H(X_*\ Y_*) = \hom_*(X_*, Y_*)$, the set of base point preserving functions;

(c) $K = \{*, 1\}$, a set with two points, one of them distinguished;

(d) $\pi_X : X_* \cong \hom_*(K, X_*)$ such that $\pi(x)(*) = *$ and $\pi(x)(1) = x$;

(e) $X_* \otimes Y_* = X_* \# Y_*$, the "smash" product, consisting of the cartesian product $X \times Y$ with $X \times \{*\} \cup \{*\} \times Y$ shrunk to a single point;

(f) $\rho_{XYZ} : \hom_*(X_* \otimes Y_*, Z_*) \cong \hom_*(X_*, \hom_*(Y_*, Z_*))$ such that $\rho(f)(x)(y) = f(x \otimes y)$, $* \otimes y = y \otimes * = *$.

Under the operation $\bullet$ (Definition 5.1) the set $FK = \{1, *\}$ has the following structure:

$\bullet$	$*$	1
$*$	$*$	$*$
1	$*$	1

Clearly $\langle FK, \cdot \rangle$ is a commutative monoid, but not an abelian group.

The embedding of Theorem 5.3 shows that the category S_* of pointed sets may be regarded as a full subcategory of the category of M-sets, where $M \approx FK$ and the action is defined as follows:

For all $x \in X$ $\quad 1x = x$

$0x = *$ $\quad$ where $* \in X$.

Note that the tensor product of M-sets, as defined in section 5 of Chapter III, does turn out to be the smash product in the special case of $\{0,1\}$-sets with the above action.

(3) In general, any category S_M of M-sets has a faithful basic functor, namely the forgetful functor of underlying-set functor. If we put $C = S_M$ in Theorem 5.3, the embedding may be regarded as an "inclusion". Details will be given in example 5 to show that the given commutative monoid $\langle M, \cdot \rangle$ is essentially the same as the commutative monoid $\langle FK, \odot \rangle$. Note that if S_M is the category of all M-sets, then the embedding is actually the identity functor.

(4) If we consider S_G, a category of sets acted on by an abelian group G, then $K = G$. The set $\hom(G,G)$ of morphisms of the G-set G has the property that each morphism is invertible. This is easily seen by observing that each morphism $G \longrightarrow G$ is uniquely determined by the image of the identity $1 \in G$. That is, if $f(1) = a$, then $f^{-1}(1) = a^{-1}$. Hence FG can be given the structure of an abelian group (under $\bullet$). By putting $C = S_G$ in Theorem 5.5 we again have an "inclusion".

(5) If in Theorem 5.10 we put $C = M_K$, a category of moduloids over the commutative semiring K, then

$$\alpha: \mathcal{M}_K \longrightarrow \mathcal{M}_{FK}$$

is an embedding of $\mathcal{M}_K$ into the category of all moduloids over the commutative semiring FK.

We now show that the addition and scalar multiplication defined for the moduloid FA over FK are the same as the addition and scalar multiplication of the moduloid A over K. To distinguish the operations in $\mathcal{M}_K$ from those in $\mathcal{M}_{FK}$ we use the following notation

$$A = \langle \bar{A}; +, \cdot \rangle \qquad FA = \langle \bar{A}; \oplus, \odot \rangle$$

where $\bar{A}$ denotes the underlying set, that is $\bar{A}$ is FA without the added structure. Obviously both structures have the same underlying set.

The operations $\oplus$ and $\odot$ have been defined as follows:

$$k \odot x = F(F\pi(x))(k) \quad \text{for all } x \in \bar{A} \text{ and } k \in \bar{K}$$

and $$x \oplus y = F\pi^{-1}(F\pi(x) + F\pi(y)) \quad \text{for all } x,y \text{ in } \bar{A}.$$

In Proposition 4.12 the natural isomorphism

$\pi : A \longrightarrow \mathrm{Hom}(K,A)$ was defined by $\pi(x) = f_x$ such that $f_x(k) = k \cdot x$ for all $x \in \bar{A}$, $k \in \bar{K}$. Therefore $F\pi(x) : K \longrightarrow A$ and $F(F\pi(x)) : \bar{K} \longrightarrow \bar{A}$ such that $F(F\pi(x))(k) = k \cdot x$ for all $x \in \bar{A}$, $k \in \bar{K}$. That is $k \odot x = k \cdot x$ for all $x \in \bar{A}$, $k \in \bar{K}$

Consider also $x + y \in A$.

$F\pi(x + y) : K \longrightarrow A$ such that for all $k \in \bar{K}$,

$$\begin{aligned} F\pi(x + y)(k) &= k \cdot (x + y) \\ &= k \cdot x + k \cdot y \\ &= F\pi(x)(k) + F\pi(y)(k) \\ &= (F\pi(x) + F\pi(y))(k) \end{aligned}$$

Therefore $F\pi(x + y) = F\pi(x) + F\pi(y)$.

That is $x + y = F\pi^{-1}(F\pi(x) + F\pi(y)) = x \oplus y$

In particular if $A = K$, the above discussion shows that the commutative semiring FK is the commutative semiring K. Therefore α is the "inclusion" of a category of K-moduloids into the category of all moduloids over the same commutative semiring K.

That is

$$\mathcal{M}_K = \alpha(\mathcal{M}_K) \subset \mathcal{M}_{FK}$$

If $\mathcal{M}_K$ is the category of all moduloids over K, then $\alpha = 1: \mathcal{M}_K \longrightarrow \mathcal{M}_K$.

(6) In the category Mod_K of all modules over a commutative ring condition (iii) of Theorem 5.12 is satisfied since

$\overline{u}: K \longrightarrow K$ defined by $\overline{u}(k) = -k$ for all $k \in K$ is a morphism of modules which has the desired property $1_K + \overline{u} = 0_K$. Therefore if $\mathcal{C} = \mathrm{Mod}_K$ in Theorem 5.12, then γ is the identity functor.

(7) Consider a category V_F of vector spaces over a field F. The set $\hom(F,F)$ of all linear transformations of F has the property that each nonzero linear transformation is invertible. Since $K = F$ for V_F, condition (iv) of Theorem 5.13 is satisfied.

Putting $\mathcal{C} = V_F$ in Theorem 5.13 we see that $\Phi = 1_{V_F}$

Appendix

An obvious question pertaining to each of the five embedding theorems is the following: "Is the embedding full?" The aim of this appendix is to provide an example involving topological modules which shows that, in general, the "module embedding" is not full. Similar arguments using topological M-sets, topological G-sets, topological moduloids and topological vector spaces prove that the other embeddings are not full.

The word topological used above is used in the sense of k-space [22] (also called "compactly generated" Hausdorff space). Assuming that all spaces are Hausdorff, the relevant properties of k-spaces are as follows:

(a) $k:\mathrm{Top} \longrightarrow \mathrm{Top}$ is a functor;

(b) If X is a space, X is a k-space means that $kX = X$;

(c) If Y is any space, $kkY = kY$; so kY is a k-space;

(d) If X and Y are k-spaces, then their "product" in the category of k-spaces is $X \times_k Y = k(X \times Y)$. [$X \times Y$ denotes the cartesian product of X and Y]

This statement involves the continuity of the projections

$X \times_k Y \longrightarrow X$, $(x,y) \longmapsto x$

$X \times_k Y \longrightarrow Y$, $(x,y) \longmapsto y$

and the fact that if A is a k-space, $f:A \longrightarrow X$ and $g:A \longrightarrow Y$ are maps (continuous functions) then

$(f,g):A \longrightarrow X \times_k Y$, $a \longmapsto (f(a),g(a))$ is a map;

(e) If X,Y,Z are k-spaces

$$(X \times_k Y)\times_k Z = X \times_k (Y \times_k Z)$$

(f) If X and Y are k-spaces then Y^X will denote the space of maps $X \longrightarrow Y$ topologized as $k\mathcal{C}(X,Y)$, where $\mathcal{C}(X,Y)$ is the same underlying set with the compact open topology;

(g) If A is a subset of the k-space X, then we make A into a k-subspace of X by topologizing it as $k(A')$, where $A' = \{A$ with the usual subspace topology$\}$. k-spaces have the following universal property: If Y is any k-space and $f:Y \longrightarrow X$ is a map such that $f(Y) \subset$ the set A, then $g:Y \longrightarrow A$, where A has the k-subspace topology, defined by $g(y) = f(y)$, $y \in Y$ is continuous;

(h) Exponential Law. If X,Y,Z are k-spaces then there is a bijective correspondence between: (1) the set of maps $f:X \times_k Y \longrightarrow Z$ and (ii) the set of maps $g:X \longrightarrow Z^Y$ defined by $g(x)(y) = f(x,y)$; $x \in X$, $y \in Y$.

Remark: Whenever we mention space, topological space, topology, etc. below these terms should be understood in the k-space sense.

Definition A.1

By a topological ring $\tilde{R} = \langle R, \mathcal{T}_R, +, \cdot\rangle$ we mean that $\langle R; +, \cdot\rangle$ is a commutative ring with identity, $\langle R, \mathcal{T}_R\rangle$ is a topological space and $+:R \times R \longrightarrow R$, $\cdot:R \times R \longrightarrow R$ are continuous.

Definition A.2

An $\tilde{R}$-module $\tilde{A} = \langle A, \mathcal{T}_A, +, \mu:R \times A \longrightarrow A\rangle$ consists of an $\langle R, +, \cdot\rangle$ -module $\langle A, +, \mu\rangle$ and a topology $\mathcal{T}_A$ on A such that $+$ and μ are continuous.

Definition A.3

Let $\tilde{A}$ and $\tilde{B}$ be $\tilde{R}$-modules. A morphism $f:\tilde{A} \longrightarrow \tilde{B}$ is a continuous function $f:A \longrightarrow B$ that preserves both scalar multiplication and addition.

The category of $\tilde{R}$-modules, denoted by V_o, is clearly a well defined category. We now make V_o into a closed category. We define

(i) V_o as above;

(ii) the functor $F:V_o \longrightarrow S$ by $F(\tilde{A}) = A$ and $F(f) = f$;

(iii) the functor $\hom:V_o^* \times V_o \longrightarrow V_o$ by $(\tilde{A},\tilde{B}) \longmapsto \hom(\tilde{A},\tilde{B})$, where $\hom(\tilde{A},\tilde{B})$ consists of the usual $\hom(A,B)$ module, topologized as a k-subspace of B^A;

(iv) the object $\tilde{R} = \langle R, T_R, +, \cdot \rangle$ to be the "ground object" in V_o;

(v) the natural isomorphism

$$\pi_{\tilde{A}}:\tilde{A} \longrightarrow \hom(\tilde{R},\tilde{A})$$

by $$a \longmapsto f_a, \text{ where}$$

$$f_a:R \longrightarrow A \text{ is defined by } f_a(\lambda) = \lambda\cdot a \qquad \lambda \in R;$$

(vi) $\theta_A:\tilde{R} \longrightarrow \hom(\tilde{A},\tilde{A})$ by $\theta_A(\lambda) = f_\lambda:A \longrightarrow A$, $f_\lambda(a) = \lambda\cdot a$, $a \in A$

(vii) $L = L^{\tilde{A}}_{\tilde{B}\tilde{C}}:\hom(\tilde{B},\tilde{C}) \longrightarrow \hom(\hom(\tilde{A},\tilde{B}), \hom(\tilde{A},\tilde{C}))$

by $$f \longmapsto (g \longmapsto f \circ g)$$

<u>Remark:</u> (i) It is a standard result for R-modules that $\pi_{\tilde{A}}$ is a bijective linear transformation. We now show that $\pi_{\tilde{A}}$ is continuous. The function $A \times R \longrightarrow A$, $(a,\lambda) \longmapsto a$ is continuous; hence by the exponential law $\alpha:A \longrightarrow A^R$, $a \longmapsto f_a$ is continuous. Now $f_a \in \hom(R,A)$ for all $a \in A$; hence we can define the continuous function $\pi_{\tilde{A}}:A \longrightarrow \hom(\tilde{R},\tilde{A})$ defined by $\pi_{\tilde{A}}(a) = \alpha(\tilde{A})$, $a \in \tilde{A}$. Let η denote the inverse of $\pi_{\tilde{A}}$; we prove that η is continuous as follows:

The composite $\hom(R,A) \xrightarrow[1]{} \hom(R,A) \subset A^R$ is continuous; hence the corresponding function $\phi:\hom(R,A) \times R \longrightarrow A$ is also continuous. Consider the composite

$$\hom(R,A) \longrightarrow \hom(R,A) \times R \xrightarrow{\rho} A$$
$$f \longmapsto (f,\ 1)\ .$$

This composite is the required η and is clearly continuous.

(ii) θ_A is continuous because it corresponds under the exponential law to

$$R \times A \longrightarrow A; \quad (\lambda,a) \longmapsto a, \quad \lambda \in R, \quad a \in A.$$

(iii) The continuity of L follows from the following argument: The inclusion $\hom(\tilde{B},\tilde{C}) \longrightarrow C^B$ is continuous; hence so is the "evaluation function"

$$e : \hom(\tilde{B},\tilde{C}) \times B \longrightarrow C, \quad (f,b) \longmapsto f(b).$$

$L^{\tilde{A}}_{\tilde{B}\tilde{C}}$ is continuous iff the function

$$\hom(\tilde{B},\tilde{C}) \longrightarrow \hom(\tilde{A},\tilde{C})^{\hom(\tilde{A},\tilde{B})}$$
$$f \longmapsto (g \longmapsto f \circ g)$$

is continuous; this function is continuous iff the function

$$\hom(\tilde{B},\tilde{C}) \times_k \hom(\tilde{A},\tilde{B}) \longrightarrow \hom(\tilde{A},\tilde{C})$$
$$(f,\ g) \longmapsto f \circ g$$

is continuous; this function is continuous iff the function

$$(\hom(\tilde{B},\tilde{C}) \times_k \hom(\tilde{A},\tilde{B})) \times_k \tilde{A} \longrightarrow \tilde{C}$$
$$(f,\ g,\ a) \longmapsto fg(a)$$

is continuous; because $\times_k$ is associative the above function is continuous iff the function

$$\hom(\tilde{B},\tilde{C}) \times_k (\hom(\tilde{A},\tilde{B}) \times_k \tilde{A}) \longrightarrow \tilde{C}$$
$$(f,\ g,\ a) \longmapsto fg(a)$$

is continuous.

The last function is the composite

$$\hom(\tilde{B},\tilde{C}) \times_k (\hom(\tilde{A},\tilde{B}) \times_k \tilde{A}) \xrightarrow{1 \times e} \hom(\tilde{B},\tilde{C}) \times_k \tilde{B} \xrightarrow{e} \tilde{C}$$

Hence it is continuous and consequently $L^{\tilde{A}}_{\tilde{B}\tilde{C}}$ is continuous.

Proposition A.1

$V = (V_o, F, \hom, \tilde{R}, \pi, \theta, L)$ is a closed category.

Proof: The difficult parts have been established above; the rest is routine.

The closed category V_o has biproducts; viz. the usual biproduct of moduloids with the cartesian product topology and the obvious actions. The details are routine.

Applying the "Module Embedding Theorem" to V we obtain an embedding $\alpha : V_o \longrightarrow \mathrm{Mod}_R$; this is simply the underlying module functor.

We will now give an example to show that α is not, in general, a full embedding.

Let $\tilde{A} = \langle A, +, \mu, T_A \rangle$, $\tilde{A}' = \langle A, +, \mu, T'_A \rangle$ be R-modules such that $T_A \supsetneq T'_A$ (two R-modules with the same underlying set, the same +, the same μ, but different topologies). For example, take $\tilde{R} = \mathbb{R}$ with the usual topology, $A = \mathbb{R}$, + and μ as the usual addition and multiplication, T_A as the discrete topology, and T'_A as the usual topology.

Then the identity $1 : \tilde{A} \longrightarrow \tilde{A}'$ is not a morphism of $\tilde{R}$-modules, but $\alpha(1) : \alpha(\tilde{A}) \longrightarrow \alpha(\tilde{A}')$, that is the identity on A, is a morphism of R-modules. Hence

$$\hom(\tilde{A}, \tilde{A}') \longrightarrow \hom(\alpha(\tilde{A}), \alpha(\tilde{A}'))$$

is not surjective, and therefore α is not a full embedding.

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