FIXED POINT THEORY OF FINITE POLYEDRA

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by

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ABSTRACT

In this thesis the behavior of the fixed point property (f.p.p.) in the category of finite polyhedra is studied. The f.p.p. behaves very badly with respect to many geometric constructions, for example, it is not invariant under topological products, products with the unit interval, suspensions, smash products, join products, nor is it a homotopy invariant. The only construction under which the f.p.p. trivially behaves nicely is the wedge (i.e. one point union) of two spaces. Even if one restricts attention to very nice spaces (e.g. simply connected polyhedra) the f.p.p. is not preserved under topological products and many other geometric constructions. This can be seen from the classical counter-examples due to Fadell-Lopez and Bredon which, together with many of their consequences, are described and discussed in detail. In a more restrictive setting, for example for simply connected polyhedra satisfying the so-called Shi condition, the f.p.p. behaves more nicely, and its invariance under topological products can also be proved under specific assumptions on the cohomology of the spaces involved.

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INTRODUCTION

Fixed point theory is a certain collection of topics, some of which come from analysis and some from topology. All these topics are concerned in one way or the other with the question which, in its simplest form, can be stated as follows: Given a function $f: X \longrightarrow X$, what is the nature and number of points $x \in X$, such that $f(x) = x$. The assumptions on f and X vary from practically none (e.g. X is a set, f a function) to quite strict assumptions (e.g. X is a differentiable manifold, f a diffeomorphism). We will turn our attention to the case where X is a fairly reasonable space (namely a finite polyhedron) and f a map (i.e. a continuous function).

In 1912 Brouwer [4] proved his classical theorem which states that the closed n-ball E^n has the fixed point property (f.p.p.) for continuous mappings, i.e. for every continuous $f: E^n \longrightarrow E^n$ there exists a point $x_0 \in E^n$ such that $f(x_0) = x_0$. This result was extended to compact convex subsets of certain function spaces, Banach spaces, locally convex topological spaces, etc., by various authors.

One of the most useful and important topological results in fixed point theory is the Lefschetz fixed point theorem which is a statement of the following form (see Chapter I for a detailed discussion): If X is a reasonably nice space, one associates to each continuous map $f: X \longrightarrow X$ an integer $L(f)$, called the Lefschetz number of f (defined in Chapter I) such that, whenever $L(f) \neq 0$, then f has at least one fixed point. The converse of the Lefschetz fixed point theorem is obviously not true in general, which can be easily seen by taking, for instance, X to be any polyhedron of Euler characteristic zero and f the identity map, then the Lefschetz number L(f) is equal to the Euler characteristic, hence, 0, but f clearly has lots of fixed points. However, if one puts sufficient restrictions on the type of the polyhedron (e.g. a simply connected finite polyhedron satisfying the so-called Shi condition (see Chapter III for definition)), it turns out that some kind of converse is also true.

A topological space X is said to have the f.p.p. if every self-map $f: X \longrightarrow X$ has at least one fixed point. The Lefschetz theorem is one of the most important tools for studying the f.p.p. in general, and, in particular, the central topic of this thesis, namely the question whether the f.p.p. is preserved under various constructions, such as topological products, suspensions, smash products, join products etc. From the classical counter-examples due to Fadell-Lopez and Bredon, a detailed description of which is included in Chapters IV and V, it follows that the answer is usually negative. However, under additional specific assumptions on the spaces some positive results can be obtained. Most of these results are due to E. Fadell. This material has been compiled in this thesis, and many fine points in the proofs (e.g. in [9] and [10]) and some gaps in the original presentation are filled and included here in detail.

We first state and prove the Lefschetz fixed point theorem itself, together with some consequences, in Chapter I. Then several geometric constructions (e.g. wedges, suspensions, smash products, join products, mapping cones etc.) and a description of the f.p.p. in general constitutes the content of Chapter II. (In studying the f.p.p. we are interested only in the existence of at least one fixed point. This ignores the very interesting question about the precise number of fixed points or about lower bounds for this number. This is a very difficult question which is discussed in [18]).

It is known that the f.p.p. behaves very badly with respect to many geometric constructions, for example, it is not invariant under topological products, smash products, join products, not even if one restricts attention to certain classes of very nice spaces, such as (finite) polyhedra, or even simply connected polyhedra. Even the product of manifolds with f.p.p. need not have f.p.p. as Husseini [14] has shown. However, in the following we will be considering only the case of finite polyhedra. Chapter III deals with the question whether the f.p.p. is preserved under these constructions. (See also [1], [2], [5], [6], [9] for the history and a general exposition of this old problem).

In Chapter IV we discuss the two classical examples due to Fadell-Lopez and Bredon in detail. They provide the counter-examples of non-preservation of the f.p.p. under the constructions mentioned above.

In the final chapter we consider the behavior of the f.p.p. in two more restrictive categories of spaces, namely S (= polyhedra satisfying the Shi condition) and S_0 (= simply connected polyhedra in S). In the category S the f.p.p. is a homotopy type invariant; i.e. if X has f.p.p. then any space of the same homotopy type as X also has the f.p.p. In this chapter we also discuss examples due to Fadell to show that the f.p.p. is not invariant under suspensions and join products in the category S_0. We also exhibit a simply connected polyhedron X such that the smash product $X \wedge X$ fails to have f.p.p. if one choice of base point is used to form $X \wedge X$, while $X \wedge X$ retains f.p.p. if one chooses another base point. In the last part we also prove that the f.p.p. is preserved by topological products under special additional assumptions on the spaces involved.

Although most terms used are defined within the thesis, some undefined terms and notations can be found in the references [5], [19], [22].

CHAPTER I

THE LEFSCHETZ FIXED POINT THEOREM

One of the most celebrated and useful theorems of fixed point theory is the Lefschetz fixed point theorem which associates with each self-map $f : X \longrightarrow X$ of a topological space an integer $L(f)$ (called the Lefschetz number of f). Given a space X we denote its singular homology by $H_*(X) = \{H_q(X) \mid q \geq 0\}$ where the coefficients are taken in a field F. If $H_q(X)$ is finite dimensional for every q and all but a finite number of the $H_q(X)$ are trivial then we say that $H_*(X)$ is finitary. A map $f : X \longrightarrow X$ induces a linear transformation $f_{*q} : H_q(X) \longrightarrow H_q(X)$. If $H_*(X)$ is finitary, we can define $L(f)$, the Lefschetz number of f by

$$L(f) = \sum_{q=0}^{\infty}(-1)^q \ \mathrm{tr}(f_{*q}),$$

where $\mathrm{tr}(f_{*q}) = 0$ if $H_q(X)$ is trivial. The Lefschetz fixed point theorem is a statement of the form: If $f : X \longrightarrow X$ is a map such that $L(f) \neq 0$, then f has a fixed point.

Of course, such a statement is not going to be true without some hypotheses on X. First of all $H_*(X)$ should be finitary in order that $L(f)$ is defined. This assumption above is, however, not sufficient because the fixed point free map $f : R \longrightarrow R$ defined by $f(x) = x + 1$ has a non-zero Lefschetz number. The assumption of X being compact will also not suffice even if $H_*(X)$ is finitary as can be seen from Borsuk's example [2].

The first one to prove such a general theorem for all finite polyhedra was H. Hopf [13]. This was later generalized by S. Lefschetz to compact ANR's (absolute neighborhood retracts, a generalization of polyhedra) [15]. In the following we will be interested in the case of finite polyhedra and we therefore state and prove the theorem for this case only. From now on instead of writing finite polyhedra we will write polyhedra.

1.1. THEOREM (LEFSCHETZ FIXED POINT THEOREM). Let X be a polyhedron and $f : X \longrightarrow X$ a map (continuous function). If f is without fixed point, then $L(f) = 0$.

PROOF: Without loss of generality, we may assume $X = |L|$ for some finite simplicial complex L. Since $|L|$ is a compact metric space, if f has no fixed points, there is a number $\varepsilon > 0$ such that $d(x, f(x)) \geq \varepsilon$ for all $x \in |L|$. Replacing L by a barycentric subdivision K, if necessary, we represent $X = |K|$ by a triangulation with mesh $K < \varepsilon/3$. According to the simplicial approximation theorem $f : |K| \longrightarrow |K|$ can be approximated simplicialy by a simplicial map $g : |K'| \longrightarrow |K|$ from a subdivision K' of K; i.e.

(1) $g \simeq f$ (g is homotopic to f)

(2) For each $x \in X$, if $f(x)$ and $g(x)$ lie on a common simplex of K then $d(f(x), g(x)) \leq \varepsilon/3$.

Suppose, some simplex S of K contains a point y such that $g(y)$ is in S, then one has

$$d(y, f(y)) \leq d(y, g(y)) + d(g(y), f(y)) < \varepsilon/3 + \varepsilon/3 = 2\varepsilon/3$$

which contradicts the choice of ε. Therefore, $g(S)$ is disjoint from S, for all $S \in K$.

Consider the (iterated barycentric) subdivision K' of K and the chain map $g_q : C_q(K') \longrightarrow C_q(K)$ associated with the simplicial map g. Also consider the chain map made up by the subdivision homomorphism

$$\tau_q : C_q(K) \longrightarrow C_q(K').$$

Since S and $g(S)$ are disjoint for each S in K, the chains $\tau_q(S) \in C_q(K')$ and $g_q\tau_q(S) \in C_q(K)$ are such that none of their simplices with non-zero coefficients intersect S. Consequently $g_q\tau_q(S)$ has coefficients zero on S (as a linear combination of generators of $C_q(K)$). Since this is so for each S in K, all coefficients occurring in the $tr(g_q\tau_q)$ are zero, for each q. Hence the alternating sum

$$L(g\tau) = \sum_q (-1)^q \, tr(g_q\tau_q) = 0.$$

By Hopf trace formula, we get

$$L(g\tau) = \sum_q (-1)^q \, tr(g_q\tau_q) = \sum_q (-1)^q \, tr(g_{*q}\tau_{*q}) = 0.$$

Let $h : |K'| \longrightarrow |K|$ be a simplicial approximation to the identity map

$$I_{|K|} : |K| \longrightarrow |K|,$$

then $fh \simeq f \simeq g$ implies that $g_* = f_*h_*$. It is well known that

$$\tau_* : H_*(K') \longrightarrow H_*(K)$$

is an isomorphism (invariance of homomorphism with respect to subdivision).

Therefore one has, $tr(g_q\tau_q) = tr(f_q h_q \tau_q) = tr(f_q)$ which implies

$$\sum_q (-1)^q \, tr(g_q\tau_q) = \sum_q (-1)^q \, tr(f_q) = L(f).$$

Hence $L(f) = 0$. □

REMARK: If $L(f) \neq 0$ then any map homotopic to f also has a fixed point. Let g be any map from X to X homotopic to f then $f_* = g_*$ which implies $L(f) = L(g)$. From Theorem 1.1 one has $L(f) = L(g) \neq 0$, hence g has a fixed point.

Consider $1_X : X \longrightarrow X$. Clearly $1_{*q} : H_q(X) \longrightarrow H_q(X)$ is an identity automorphism in each dimension q. Then one has

$$\begin{aligned} tr(1_*) &= \text{dimension of vector space } H_q(X) \\ &= \text{rank of } H_q(X) \text{ (denoted by } \rho_q(X)) \\ &= q^{th} \text{ Betti number of } X. \end{aligned}$$

$$\text{and } \sum_q (-1)^q \, tr(1_*) = \sum_q (-1)^q \rho_q(X) = \chi(X)$$

$$= \text{Euler-Poincare characteristic of } X.$$

1.2. COROLLARY. If $X = |K|$ is a path-connected polyhedron with $H_q(X; Q) = 0$, for $q \neq 0$ (i.e. X is Q-acyclic) then every map $f : X \longrightarrow X$ has Lefschetz number $L(f) = 1 \neq 0$ hence has a fixed point.

NOTE: $L(f) = 1$ because f_* is the identity map on $H_0(X) \approx Z$.

1.3. SPECIAL CASE. Let X be a contractible (compact) polyhedron. So we have $H_q(X) = 0$, for all $q \neq 0$. Hence every map $f : X \longrightarrow X$ has a fixed point.

1.4. SPECIAL CASE (of 1.3). Let $X = E^n$ (closed n-ball). In this case Brouwer's fixed point theorem is a special consequence of 1.3.

NOTE: The result 1.3 is false for noncompact polyhedra. In fact, R is a contractible polyhedron and any translation different from 1_R fails to have a fixed point.

NOTE: In the literature the Lefschetz number is also defined and computed as $L(f) = \sum_q (-1)^q \operatorname{tr}(f_*^q)$ which is, in fact, equivalent to our definition because the coefficients are taken in a field.

CHAPTER II

THE FIXED POINT PROPERTY AND SOME GEOMETRIC CONSTRUCTIONS

In this chapter we will discuss the fixed point property (f.p.p.) in general and, in particular, some aspects of its invariance under various constructions. It is well known that the f.p.p. behaves very badly under topological products even if one restricts attention to very nice spaces such as polyhedra or even simply connected polyhedra. On the other hand the f.p.p. behaves very nicely with respect to wedges (one point unions) of spaces : If X and Y are topological spaces then $X \vee Y$ has the f.p.p. if and only if X and Y both have the f.p.p. In the following we also want to consider geometric constructions, such as suspensions, smash products and join products, from this point of view.

2.1 : DEFINITION. A space X has the f.p.p. if every map $f : X \to X$ has a fixed point (a map $f : X \to X$ has a fixed point if $f(x) = x$, for some $x \in X$).

2.2 : THEOREM. The f.p.p. is a topological property (i.e. homeomorphisms preserve the f.p.p.).

PROOF: Let X be a space with the f.p.p. and let $h : X \to Y$ be an arbitrary homeomorphism. It suffices to prove that Y has the f.p.p. Let $f : Y \to Y$ denote an arbitrary given map. Consider the composed map

$$g = h^{-1} f h : X \to X .$$

Since X has the f.p.p., there exists a point p of X with $g(p) = p$. Then we have

$$f[h(p)] = h[g(p)] = h(p)$$

since $h g = f h$. This proves that $h(p) \in Y$ is a fixed point of f and hence Y has the f.p.p. □

2.3. THEOREM. If $X \times Y$ has the f.p.p. then X and Y both have the f.p.p.

PROOF: Let $f : X \to X$ be any map. Since

$$f \times 1_Y : X \times Y \to X \times Y$$

is a map and $X \times Y$ has the f.p.p. therefore $f \times 1_Y(x , y) = (x , y)$, for some $(x, y) \in X \times Y$. So $(f(x), y) = (x, y)$, hence $f(x) = x$ which implies X has the f.p.p. Similarly it can be shown that Y has the f.p.p. Hence $X \times Y$ has the f.p.p. implies that X and Y both have the f.p.p. □

2.4. PROPOSITION. If X has the f.p.p. and A is a retract (i.e. if there exists a map $r : X \to A$, called a retraction, such that $r i = 1_A$) of X, then A has the f.p.p.

PROOF: Assume that X is a space with the f.p.p. and that $r : X \to A$ is an arbitrary retraction of X onto a subspace A of X. It suffices to prove that A has the f.p.p. Let $f : A \to A$ denote an arbitrary given map. Consider the composed map

$$g = i f r : X \to X$$

where $i : A \to X$ is the inclusion map. Since X has the f.p.p., there exists a point p of X with $g(p) = p$. Since

$$g(p) = f[r(p)] \in A,$$

this implies that the fixed point p of g must be in A. Hence we obtain $r(p) = p$ and

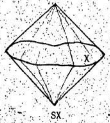
$$p = g(p) = f[r(p)] = f(p).$$

This proves that p is a fixed point of f and hence A has the f.p.p. □

We will define some geometric constructions and will study the behaviour of f.p.p. under some of them.

2.5. DEFINITION. Let X, Y be two spaces with base points $x_0 \in X$ and $y_0 \in Y$. The one point union (or wedge) $X \vee Y$ is defined to be the quotient space $X \cup Y / x_0 \sim y_0$, where $X \cup Y$ is the disjoint union of the spaces X and Y. The base point of $X \vee Y$ corresponds to the point $x_0 = y_0$. In other words, $X \vee Y$ is the space obtained from $X \cup Y$ (disjoint) identifying together the base points x_0, y_0. $X \vee Y$ can also be viewed as the subspace $(X \times y_0) \cup (x_0 \times Y)$ of $X \times Y$.

2.6. DEFINITION. Let X be a topological space. In the space $X \times I$, identify the closed subspace $X \times 0$ to one point and $X \times 1$ to another. The quotient space SX under these identifications is called the (unreduced) suspension of X. For example



the suspension SS^n of n-sphere S^n is homeomorphic to S^{n+1}.

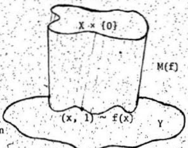
2.7. DEFINITION. Given (based) spaces X and Y, the reduced (or smashed) product $X \wedge Y$ is defined to be the quotient space $X \times Y / X \vee Y$, where $X \vee Y$ is regarded as a subspace of $X \times Y$. The base point of $X \wedge Y$ is of course the point corresponding to $X \vee Y$. Points of $X \wedge Y$ are written in the form $x \wedge y$; this denotes the equivalence class of (x, y) in $X \times Y$.

2.8. EXAMPLES. (i) The smash product $S^m \wedge S^n$ of two spheres of dimensions m and n is homeomorphic to S^{m+n}. (ii) For the special case $Y = S^1$, the smash product $X \wedge S^1 = SX$ is the (reduced) suspension of X.

2.9. DEFINITION. The join product $X * Y$ of two topological spaces X and Y is defined as a quotient space of $X \times Y \times I$ under the following identifications : $(x, y, 0) \sim (x, y', 0)$ and $(x, y, 1) \sim (x', y, 1)$, for all $x, x' \in X$ and all $y, y' \in Y$. For any specific $x \in X$, $y \in Y$, the "line segment" from x to y in $X * Y$ is the subset $[x, y] = \{(x, y, t) \mid 0 \leq t \leq 1\}$, obviously each point of $X * Y$ with $t \neq 0, 1$ lies on a unique such line segment.

2.10. EXAMPLES. (i) For Y a single point y_0, the join $X * \{y_0\}$ is the cone CX over X, which clearly is always contractible. (ii) The join $E^m * E^n$ of two closed balls of dimensios m and n is homeomorphic to E^{m+n+1}. (iii) The join $S^m * S^n$ of two spheres of dimensions m and n is homeomorphic to a sphere S^{m+n+1}. (iv) For the special case $Y = S^0$ = 0-sphere $\{S_0, S_1\}$, the join product $X * S^0 = SX$ is the suspension of X.

2.11. DEFINITION. Let $f : X \to Y$ be a map. The mapping cylinder $M(f)$ is the quotient space obtained from the disjoint union $X \times I \cup Y$ by identifying $(x, 1) \in X \times I$ with $f(x) \in Y$. We use $[x, t]$ to denote the points of $M(f)$ corresponding to $(x, t) \in X \times I$ under the identification map and $[y]$ to denote the point of $M(f)$ corresponding to $y \in Y$ (thus $[x, 1] = [f(x)]$ for $x \in X$).



There is an inclusion map $i : X \to M(f)$ with $i(x) = [x, 0]$ and an inclusion $j : Y \to M(f)$ with $j(y) = [y]$. X and Y are regarded as subspaces of $M(f)$ by means of these imbeddings. A retraction $r : M(f) \to Y$ is defined by $r[x, t] = [f(x)]$ for $x \in X$ and $t \in I$ and $r[y] = [y]$ for $y \in Y$. (The mapping cylinder can also be defined as the subspace of $X * Y$ that includes all line segments $[x, f(x)]$ for $x \in X$, together with the points of Y. When the top X in $M(f)$ is identified to a point, the resulting quotient space $C(f)$ is the mapping cone of f.

2.12. THEOREM. If SX has the f.p.p., then X also has the f.p.p.

PROOF: Let $g : X \to X$ be any map, then $Sg : SX \to SX$, defined by

$$Sg(x, t) = (g(x), t), \text{ where } 0 \leq t \leq 1,$$

is also a map. Since SX has the f.p.p., one has

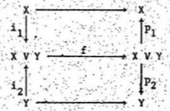
$Sg(x, t) = (x, t) \Longrightarrow (g(x, t) = (x, t) \Longrightarrow g(x, t)$, where $t \neq 0, 1$. Therefore g has a fixed point. If $t = 0$ or 1, define a reflection map p by $p(x, t) = (x, 1-t)$. Since $Sg\,p(x, t)$ is a map, therefore

$$Sg\,p(x, t) = Sg(x, 1-t) = (g(x), 1-t) = (x, t) \text{ which implies } t = 1/2.$$

Hence in this case $g(x) = x$, i.e. g has a fixed point. □

2.13. THEOREM. Let X and Y be spaces with the f.p.p. Then $X \vee Y$ also has the f.p.p.

PROOF: Let $f : X \vee Y \to X \vee Y$ be any map. Let $i_1 : X \to X \vee Y$ and $i_2 : Y \to X \vee Y$ be the inclusions and $p_1 : X \vee Y \to X$ and $p_2 : X \vee Y \to Y$ be the projections (i.e. $i_1(x) = (x, y_0)$; $p_1(x, y) = x$ and similarly for i_2 and p_2). Consider the following diagram,



Since i_1, i_2, p_1 and p_2 are continuous, the compositions $g_1 = p_1 f i_1 : X \to X$ and $g_2 = p_2 f i_2 : Y \to Y$ are also continuous. Hence there exists $x \in X$, $y \in Y$ such that $g_1(x) = x$, $g_2(y) = y$ (since X and Y both have the f.p.p.). Now,

$x = g_1(x) = p_1 f i_1(x) = p_1 f(x, y_0) = p_1(x_1, y_0) = x_1$ [case 1_x]; or is equal to $p_1 f(x, y_0) = p_1(x_0, y_1) = x_0$ [case 2_x].

$y =$ [illegible]$(y) = p_2 f\, i_2(y) = p_2 f(x_0, y) = p_2(x_0, y_2) = y_2$ [case 1_y] or is equal to $p_2 f(x_0, y_2) = p_2(x_2, y_0) = y_0$ [case 2_y].

Given case 1_x or 1_y, we have,

$$x = x_1 \text{ implies } f(x, y_0) = (x, y_0), \text{ or}$$

$$y = y_2 \text{ implies } f(x_0, y) = (x_0, y) \text{ respectively.}$$

Hence (x, y_0) or (x_0, y) is a fixed point of f. Assume cases 1_x and 1_y both do not hold. Then cases 2_x and 2_y both hold. Thus

$$x = x_0 \text{ implies } f(x_0, y_0) = (x_0, y_1) \text{ and}$$

$$y = y_0 \text{ implies } f(x_0, y_0) = (x_2, y_0).$$

Since $f(x_0, y_0) = f(x_0, y_0)$, we have,

$(x_0, y_1) = (x_2, y_0)$ implies $x_2 = x_0$, $y_1 = y_0$ implies $f(x_0, y_0) = (x_0, y_0)$ □

In fact, this theorem could have been formulated as an if and only if statement i.e. X and Y have the f.p.p. if and only if $X \vee Y$ has the f.p.p. This is actually even true for the wedge of any number of spaces (i.e. if $Z = \vee X_\lambda$ be the one-point union of the topological spaces X_λ, $\lambda \in \Lambda$(= arbitrary index set) then Z has the f.p.p. if and only if each X_λ has the f.p.p, for more details see in [20].

The join product does not preserve the f.p.p. , but one has the following

2.14. PROPOSITION. If the join product $X * X$ possesses the f.p.p., then at least one of X, Y possesses the f.p.p.

PROOF: See in [20].

From Proposition 2.14 it can be easily seen that if $SX = X * S^0$ has the f.p.p., then X also has the f.p.p. (since S^0 does not have the f.p.p.).

CHAPTER III

MORE ON THE FIXED POINT PROPERTY

This chapter deals with the question whether the f.p.p. of a space is retained under the various constructions discussed in Chapter II. The spaces under consideration will again be simply connected finite polyhedra. Examples of particular importance will be the real, complex and quaternionic projective spaces. The usual method of showing that a space X has the f.p.p. is to show that the Lefschetz number $L(f; F) \neq 0$ for every self-map $f : X \to X$, where coefficients are taken in any field. We therefore, first compile the information on the homology and cohomology of these projective spaces in order to determine which of them possess the f.p.p.

3.1. THEOREM. The real projective spaces RP^n have the f.p.p. when n is even.

PROOF: The Z-homology of RP^n is

$$H_q(RP^n; Z) = \begin{cases} Z & \text{if } q = 0 \\ Z_2 & \text{if } q \text{ is even and } 0 < q < n \\ Z & \text{if } q = n \text{ and } n \text{ is odd} \\ 0 & \text{otherwise,} \end{cases}$$

where Z_2 is cyclic group of order 2. Since

$$H_*(RP^n) \neq H_*(x_0; Z),$$

then RP^n is not contractible. By the universal coefficient theorem it follows that if $H_q(X; Z)$ is finite then $H_q(X; Q) = 0$ if $q > 0$ and $H_0(X; Q) = Q$, therefore RP^n is Q-acyclic when n is even. Let $f : RP^n \to RP^n$ be a map. f induces trivial homomorphisms $f_{*q} : H_q(RP^n; Q) \to H_q(RP^n; Q)$, for all $q \neq 0$. Since $H_0(RP^n; Q) \approx Q$ and $f_{*0} : H_0(RP^n; Q) \to H_0(RP^n; Q)$ is the identity isomorphism, therefore it follows that $L(f; Q) = 1$. Hence RP^n has the f.p.p. when n is even.

By the way this result could have also been obtained from Corollary 1.2, since RP^n is Q-acyclic for n even. That this result need not be true if n is odd can be seen, for example, by noticing that RP^1 is homeomorphic with S^1, and taking the antipodal map of S^1.

3.2. THEOREM. Complex projective spaces CP^n and quaternionic projective spaces HP^n both have the f.p.p. if n is even.

PROOF: Let ΓP^n be the complex projective space or the quaternionic projective space where Γ denotes either the complex numbers C or the quaternions H. It is known that

$$H^q(\Gamma P^n; Z_2) = \begin{cases} Z_2 & \text{if } p = dk;\ k = 0, 1, \ldots, n \\ 0 & \text{otherwise,} \end{cases}$$

where $d = 2$ if $\Gamma = C$ and $d = 4$ if $\Gamma = H$. Let $u \in H^d(\Gamma P^n; Z_2)$ be the non-zero element; then $u^k = u.u.\ldots.u$ (k-fold cup product) is the non-zero element in $H^{dk}(\Gamma P^n; Z_2)$ for $k = 1, \ldots, n$. Let $f : \Gamma P^n \to \Gamma P^n$ be any map. Since ΓP^n is connected, $f^*(1) = 1 \in H^0(\Gamma P^n; Z_2)$. Let $f^*(u) = au$, where $a \in Z_2$. Since f^* preserves cup products, $f^*(u^k) = a^k u^k = au^k$. Thus $L(f; Z_2) = 1 + na$, so if n is even then

$L(f; Z_2)$ is odd and, by the Lefschetz fixed point theorem, f has a fixed point. Therefore complex projective spaces CP^n and quaternionic projective spaces HP^n have the f.p.p. when n is even.

So we see that in all these cases choosing coefficients in Z_2 is sufficient to conclude that the even dimensional projective spaces have the f.p.p. It was Fadell who asked the question whether coefficients in Z_2 are always sufficient for such a conclusion.

3.3. QUESTION A. Does there exist a polyhedron X with the f.p.p. which admits a self-map f such that $L(f)$ is an even integer?

We will now investigate the implications of an affirmative answer to the Question A. We will consider the following category F. The objects of F are based maps $f : (X, x_0) \to (X, x_0)$ where X is a compact, simply connected, triangulable space with the f.p.p. A morphism in F, say $\phi : f \to f'$, is a map where

$$\begin{array}{ccc} (X, x_0) & \xrightarrow{\ f\ } & (X, x_0) \\ \phi \downarrow & & \downarrow \phi \\ (Y, y_0) & \xrightarrow{\ f'\ } & (Y, y_0) \end{array}$$

is a commutative diagram. Notice that if ϕ is an equivalence in F (i.e. ϕ has an inverse) then ϕ is a homeomorphism such that $\phi f = f'\phi$. Using the wedge operation

$$f \vee g : (X \vee Y, (x_0, y_0)) \longrightarrow (X \vee Y, (x_0, y_0))$$

where $f : (X, x_0) \longrightarrow (X, x_0)$, $g : (Y, y_0) \longrightarrow (Y, y_0)$, the category $\mathcal{F}$ admits a "sum" operation. Here we make use of the Theorem 2.13 that the wedge of two spaces with the f.p.p. also has the f.p.p. Define a relation in $\mathcal{F}$ as follows

$$f \sim g \text{ if and only if } \phi f = g\phi, \text{ for some equivalence } \phi \text{ in } \mathcal{F}.$$

It can easly be verified that this is an equivalence relation in the category $\mathcal{F}$. Let $[\mathcal{F}] = \mathcal{F}/\sim$ denote the set of equivalence classes of $\mathcal{F}$ under this relation.

3.4. THEOREM. The set $[\mathcal{F}]$ of equivalence classes forms an abelian semigroup with zero under the sum operation V.

PROOF: Clearly $[f]$ and $[g] \in [\mathcal{F}]$ implies that $[f \vee g] \in [\mathcal{F}]$. V is associative. By considering the map $\phi = 1_{X \vee Y \vee Z}$ (= identity map map on $X \vee Y \vee Z$), one has $1.(f \vee g) \vee h = f \vee (g \vee h).1$ (since $1_{(X \vee Y) \vee Z} = 1_{X \vee (Y \vee Z)}$ implies that $(f \vee g) \vee h \sim f \vee (g \vee h)$ hence $([f] \vee [g]) \vee [h] = [f] \vee ([g] \vee [h])$. For commutativity one has to show that $[f \vee g] = [g \vee f]$ i.e. $\phi.(f \vee g) = (g \vee f).\phi$ for some ϕ. Define ϕ such that $\phi(x, y_0) = (y_0, x)$ or $\phi(x_0, y) = (y, x_0)$, then

$$\phi.(f \vee g)(x, y_0) = \phi(f(x), g(y_0)) = (g(y_0), f(x)) \text{ and}$$

$$(g \vee f).\phi(x, y_0) = (g \vee f)(y_0, x) = (g(y_0), f(x))$$

implies that $\phi(f \vee g) = (g \vee f).\phi$ and similarly for the other case.

The zero element corresponds to a point map $x_0 \longrightarrow x_0$. Hence, indeed, $[F]$ is an abelian semigroup with zero. □

If $f \in F$, we let $\widetilde{L}(f)$ denote the reduced Lefschetz number of f, i.e.

$$\widetilde{L}(f) = \sum_{k\geq 1}(-1)^k \, tr(f_{*k}).$$

Since we are dealing only with connected spaces therefore in dimension zero we have $H_0(X) \approx Z$. Also the isomorphism induced by f on $H_0(X)$ is an identity automorphism which implies $tr(f_{*0}) = 1$. Therefore, $\widetilde{L}(f) = L(f) - 1$.

3.5. THEOREM. $\widetilde{L}(f \vee g) = \widetilde{L}(f) + \widetilde{L}(g)$.

PROOF: It is known that

$$\widetilde{H}_q(X \vee Y) = \widetilde{H}_q(X) \oplus \widetilde{H}_q(Y), \quad \text{for every } q.$$

Let $X \xrightarrow{i} X \vee Y \xrightarrow{r} X$ be the natural inclusion and retraction. Let the generators of $\widetilde{H}_q(X)$ be $x_1, \ldots, x_m$ and of $\widetilde{H}_q(Y)$ be $y_1, \ldots, y_n$. Let $(f \vee g)_*(x_i)$ be the sum of $\sum_{j=1}^{m} a_{ij}x_j$ and a linear combination of y's and $(f \vee g)_*(y_i)$ be the sum of the linear combination of x's and $\sum_{j=1}^{n} b_{ij}y_j$.

Therefore, $tr(f \vee g)_* = \sum_{i=1}^{m} a_{ii} + \sum_{i=1}^{n} b_{ii}$. $\quad (*)$

By naturality of the diagram,

$$\begin{array}{ccc}
\widetilde{H}_q(X \vee Y) = \widetilde{H}_q(X) \oplus \widetilde{H}_q(Y) & \xrightarrow{(f \vee g)_*} & \widetilde{H}_q(X) \oplus \widetilde{H}_q(Y) \\
\uparrow i_* & & \downarrow r_* \\
\widetilde{H}_q(X) & \xrightarrow{\widetilde{f}_*} & \widetilde{H}_q(X)
\end{array}$$

one has $f_*(x_i) = r_* \cdot (f \vee g)_* \cdot i_*(x_i) = \sum_{j=1}^{m} a_{ij} x_j$ which implies that $tr(f_*) = \sum_{i=1}^{m} a_{ii}$. Similarly $tr(g_*) = \sum_{i=1}^{n} b_{ii}$. Therefore by equation $(*)$, one has $tr(f \vee g)_* = tr(f_*) + tr(g_*)$, and hence, taking alternating sums one gets

$$\tilde{L}(f \vee g) = \tilde{L}(f) + \tilde{L}(g). \square$$

3.6. THEOREM. If f and g are equivalent in F, then $\tilde{L}(f) = \tilde{L}(g)$.

PROOF: $f \sim g$ implies that there exists a ϕ such that $\phi f = g\phi$ which implies that $f = \phi^{-1} g\phi$ which in turn implies that

$$f_* = (\phi^{-1} g\phi)_* = \phi_*^{-1} g_* \phi_*.$$

Since ϕ_* is an isomorphism its matrix is non-singular. From Linear Algebra one knows that for non-singular matrices P one has $tr(A) = tr(P^{-1}AP)$. Therefore, $tr(f_*) = tr(g_*)$, hence, by taking alternating sums one gets the result

$$\tilde{L}(f) = \tilde{L}(g). \square$$

Therefore from Theorems 3.5 and 3.6 it is clear that $\tilde{L}$ induces a homomorphism

$$\tilde{L} : [F] \longrightarrow Z \qquad (**)$$

We want to investigate the consequences of the following hypothesis.

3.7. HYPOTHESIS B. There is a simply connected polyhedron X with f.p.p. which admits a self-map $f : X \to X$ such that $L(f)$ is even.

3.8. THEOREM. Hypothesis B implies that $\tilde{L}$ in $(**)$ is surjective.

PROOF: If we let $\tilde{\chi}$ denote the reduced Euler characteristic, then

$$\tilde{\chi}(CP^n) = n, \quad \tilde{\chi}(SCP^n) = -n.$$

From homology one knows

$$H_q(CP^n; F) = \begin{cases} F & \text{if } q = 0 \text{ or even} \\ 0 & \text{otherwise,} \end{cases}$$

and $\chi(CP^n) := \sum_{q=0}^{n} (-1)^q \rho_q$ (= rank of $H_q(CP^n)$) $= n + 1$. Therefore, $\tilde{\chi}(CP^n) = \chi(CP^n) - 1 = n$. One also has

$$H_q(X) = \begin{cases} H_{q+1}(SX) & \text{if } q > 0 \\ Z & \text{if } q = 0. \end{cases}$$

Therefore,

$$H_q(SCP^n) = \begin{cases} Z & \text{if } q \text{ is odd, or } q = 0 \\ 0 & \text{if } q \text{ is even,} \end{cases}$$

and $\chi(SCP^n) = \sum (-1)^q \rho_q$ (= rank of $H_q(SCP^n)$) $= -n + 1$. Hence $\tilde{\chi}(SCP^n) = -n$. Since CP^n has the f.p.p. and its suspension SCP^n will be shown in Chapter 4 to have the f.p.p. too (see Proposition 4.1), for n even, it follows that the image of $\tilde{L}$ contains the even integers (since $\tilde{L}(1_*) = \tilde{\chi}$ = even). Hypothesis B asserts that the image of $\tilde{L}$ contains some odd integer (since $L(f)$ is even hence $\tilde{L}(f)$ is odd). This implies that $\tilde{L}$ is surjective.□

3.9. COROLLARY. Hypothesis B implies there exists a simply connected polyhedron X with f.p.p. which admits a self-map f such that $L(f) = 0$.

PROOF: Since $\widetilde{L}$ is surjective from Theorem (3.8), therefore there exists a map $f \in [F]$ such that $\widetilde{L}(f) = -1$ i.e. $\widetilde{L}(f) + 1 = 0$ which implies that $L(f) = 0$ for some simply connected polyhedron X with the f.p.p.

The following lemma implies additional interesting consequences of the Hypothesis B.

3.10. LEMMA. Given maps $f : X \to X$, $g : Y \to Y$ one has $L(f \times g) = L(f)L(g)$, $\widetilde{L}(Sf) = -\widetilde{L}(f)$, $\widetilde{L}(f \wedge g) = \widetilde{L}(f)\widetilde{L}(g)$ and $\widetilde{L}(f * g) = -\widetilde{L}(f)\widetilde{L}(g)$, where S stands for suspension, $\wedge$ for smash product, and $*$ for join.

PROOF: We first need a few simple facts from Linear Algebra. Suppose V and W are finite dimensional vector spaces with linear transformations $\alpha : V \to V$, $\beta : W \to W$ and let $\phi = \alpha \otimes \beta : V \otimes W \to V \otimes W$. If $\dim V = m$, and $\dim W = n$, then $\dim V \otimes W = mn$. Let $(v_1, \ldots, v_m)$ and $(w_1, \ldots, w_n)$ be bases for V and W respectively. The basis for $V \otimes W$ is given by

$$\begin{array}{l} v_1 \otimes w_1, \ldots, v_1 \otimes w_n; \\ v_2 \otimes w_1, \ldots, v_2 \otimes w_n; \\ \ldots \quad \ldots \quad \ldots \quad \ldots; \\ v_m \otimes w_1, \ldots, v_m \otimes w_n. \end{array}$$

Hence one has the following

$$\phi(v_1 \otimes w_1) = \alpha(v_1) \otimes \beta(w_1) = \sum_i a_{1i} v_i \otimes \sum_j b_{1j} w_j = \sum_{i,j} a_{1i} b_{1j} (v_i \otimes w_j).$$

The element in 1st row and 1st column is $a_{11} b_{11}$ and in the 2nd row and 2nd column $a_{22}b_{11}$ etc. By taking the sum on the diagonal elements we get

$$(a_{11}b_{11}+a_{22}b_{11}+\dots+a_{mm}b_{11})+(a_{11}b_{22}+a_{22}b_{22}+\dots+a_{mm}b_{22})+\dots\dots+$$
$$(a_{11}b_{nn}+a_{22}b_{nn}+\dots+a_{mm}b_{nn})=(a_{11}+a_{22}+\dots+a_{mm})\times(b_{11}+b_{22}+\dots+b_{nn}).$$

Therefore, $tr(\phi)=tr(\alpha)\,tr(\beta)$.

If the coefficients are taken in a PID(principal ideal domain) R then the Künneth formula states that

$$H_n(X\times Y)=\bigoplus_{p=0}^{n}H_p(X)\otimes H_{n-p}(Y)\oplus\bigoplus_{p=0}^{n}Tor(H_p(X),H_{n-p-1}(Y)).$$

Since in our case the coefficients are in the field F, so we have

$$H_n(X\times Y)=\bigoplus_{p=0}^{n}H_p(X)\otimes H_{n-p}(Y).$$

From the naturality of the following diagram,

$$\begin{array}{ccc}
H_n(X\times Y;\ F) & \xrightarrow{(f\times g)_*} & H_n(X\times Y;\ F)\\
\downarrow\alpha & & \downarrow\alpha\\
\bigoplus H_p(X;\ F)\otimes H_{n-p}(Y;\ F) & \xrightarrow{f_*\otimes g_*} & \bigoplus H_p(X;\ F)\otimes H_{n-p}(Y;\ F)
\end{array}$$

one has

$$tr(f\times g)_{*n}=tr(\bigoplus_{p=0}^{n}f_{*p}\otimes g_{*n-p})$$
$$=\sum_{p=0}^{n}tr\,f_{*p}\,tr\,g_{*n-p}.$$

Therefore,

$$L(f \times g) = \sum(-1)^n \operatorname{tr}(f \times g)_{*n}$$

$$= \sum(-1)^n \operatorname{tr}(f_{*0}) \operatorname{tr}(g_{*n}) + \ldots + \sum(-1)^n \operatorname{tr}(f_{*n}) \operatorname{tr}(g_{*0}).$$

Also,

$L(f) = \sum(-1)^n \operatorname{tr}(f_{*n}) = \operatorname{tr}(f_{*0}) - \operatorname{tr}(f_{*1}) + \ldots + (-1)^n \operatorname{tr}(f_{*n})$; and

$L(g) = \sum(-1)^n \operatorname{tr}(g_{*n}) = \operatorname{tr}(g_{*0}) - \operatorname{tr}(g_{*1}) + \ldots + (-1)^n \operatorname{tr}(g_{*n})$.

Therefore,

$$L(f)\, L(g) = \operatorname{tr}(f_{*0}) \sum(-1)^n \operatorname{tr}(g_{*n}) + \ldots + \operatorname{tr}(g_{*0}) \sum(-1)^n \operatorname{tr}(f_{*n}).$$

Hence,

$$L(f \times g) = L(f)\, L(g).$$

Similarly in the case of join products one has

$$\tilde{H}_n(X * Y) = \bigoplus_{p+q=n-1} \tilde{H}_p(X) \otimes \tilde{H}_q(Y) \;\oplus\; \bigoplus_{p+q=n-2} \operatorname{Tor}(\tilde{H}_p(X), \tilde{H}_q(Y)),$$

(compare for example [8], p 126ff). Since in our case coefficients are from a field, therefore

$$\tilde{H}_n(X * Y) \cong \bigoplus_{p+q=n-1} \tilde{H}_p(X) \otimes \tilde{H}_q(Y)$$

This is completely analogous to the Künneth formula result for the case of topological products, except for a shift in dimension by 1. Therefore we get

$$\tilde{L}(f * g) = - \tilde{L}(f)\, \tilde{L}(g).$$

To prove $\tilde{L}(f) = -\tilde{L}(Sf)$, we start by noticing that

$$\tilde{H}_q(X) = \tilde{H}_{q+1}(SX), \text{ for every } q.$$

Therefore,

$$\mathrm{tr}(\tilde{f}_{*0}) = \mathrm{tr}(S\tilde{f}_{*1})$$

$$\mathrm{tr}(\tilde{f}_{*1}) = \mathrm{tr}(S\tilde{f}_{*2})$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\mathrm{tr}(\tilde{f}_{*n}) = \mathrm{tr}(S\tilde{f}_{*n+1}).$$

Hence,

$$\begin{aligned}
\tilde{L}(f) &= -\mathrm{tr}(\tilde{f}_{*1}) + \mathrm{tr}(\tilde{f}_{*2}) - \dots + (-1)^n \mathrm{tr}(\tilde{f}_{*n}) \\
&= -\mathrm{tr}(S\tilde{f}_{*2}) + \mathrm{tr}(S\tilde{f}_{*3}) - \dots + (-1)^n \mathrm{tr}(S\tilde{f}_{*n+1}) \\
&= -[\mathrm{tr}(S\tilde{f}_{*2}) - \mathrm{tr}(S\tilde{f}_{*3}) + \dots + (-1)^n \mathrm{tr}(S\tilde{f}_{*n+1})] \\
&= -\sum_{n\geq 1} (-1)^{n+1} \mathrm{tr}(S\tilde{f}_{*n+1}) \\
&= -\tilde{L}(Sf).
\end{aligned}$$

Using the fact that $S(X \wedge Y) = (X * Y)$ (i.e. are of the same homotopy type), one has

$\tilde{L}(f \wedge g) = -\tilde{L}(S\ f{\wedge}g) = -\tilde{L}(f * g) = -\tilde{L}(f)\ \tilde{L}(g)$. □

3.11. COROLLARY. Hypothesis B implies that for each of the following constructions C, there exists a simply connected polyhedron X with f.p.p. such that C(X) admits a self-map f with L(f) = 0.

(a) $C(X) = X \times I$,

(b) $C(X) = X \times X$,

(c) $C(X) = SX$,

(d) $C(X) = X \wedge X$,

(e) $C(X) = X * X$.

PROOF: For (a) and (b), choose $g \in F$ such that $L(g) = 0$. Let $f = g \times 1_I$. $L(f) = L(g \times 1_I) = L(g)\, L(1_I) = 0$, since $L(g) = 0$. Similarly for (b), let $f = g \times 1_X$. Therefore $L(f) = L(g)\, L(1_X) = 0$. For (e), choose the same g i.e. $L(g) = 0$ so $\tilde{L}(g) = -1$. Let $f = g * g$. Hence $L(g * g) = \tilde{L}(g * g) + 1 = -\tilde{L}(g)\, \tilde{L}(g) + 1 = -1 + 1 = 0$. For (c), choose $g \in F$ such that $L(g) = 2$. Then by taking $f = Sg$, one has,

$L(f) = L(Sg) = \tilde{L}(Sg) + 1 = -\tilde{L}(g) + 1 = -L(g) + 2 = 0$.

To prove (d), we will need a space X which admits two self-maps g_1 and g_2 such that $L(g_1) = 0$, and $L(g_2) = 2$. Therefore,

$\tilde{L}(g_1 \wedge g_2) = \tilde{L}(g_1)\, \tilde{L}(g_2) = (L(g_1) - 1)(L(g_2) - 1) = -1.1 = -1$.

Hence, $L(g_1 \wedge g_2) = \tilde{L}(g_1 \wedge g_2) + 1 = 0$. Thus $f = g_1 \wedge g_2$ works in this case. This can be easily accomplished by choosing $X = A \vee B$, where $g_1' : A \to A$, $g_2' : B \to B$ are such that $L(g_1') = 0$, $L(g_2') = 2$. ☐

In the following an additional condition (the Shi-condition) on polyhedra will sometimes be required, which is defined as follows.

3.12. DEFINITION. A polyhedron X is said to satisfy the Shi-condition if (i) $\dim X \geq 3$ and (ii) X does not possess any local cut points.

It is well known (see [9], Lemma 4.7) and geometrically plausible that, if X is a simply connected polyhedron with $\dim X \geq 2$ then for each of the constructions C in Corollary 3.11, C(X) is simply connected and satisfies the Shi-condition. The smash product requires taking as base point (x_0, y_0), where x_0 and y_0 are not separating points.

As a consequence of Corollary 3.11 we have the following Corollary.

3.13. COROLLARY. Hypothesis B implies that for each of the following constructions C in Corollary 3.11, there exists a simply connected polyhedron X with f.p.p. such that C(X) fails to have f.p.p.

Before proving the Corollary we need the following theorem.

3.14. THEOREM. Let X be a simply connected polyhedron satisfying the Shi-condition. Then X has the f.p.p. if and only if $L(f) \neq 0$ for every self-map $f : X \to X$.

This follows immediately from Theorem 2 in [5 ; p.146] because the fundamental group $\pi_1(X) = 0$ for simply connected spaces.

PROOF (of Corollary 3.13). Since C(X) is simply connected and satisfies the Shi-condition, from Corollary 3.11, C(X) admits a self-map f such that $L(f) = 0$. Hence from Theorem 3.14, C(X) fails to have f.p.p. □

Summarizing, Hypothesis B implies

3.15. THEOREM. In the category of simply connected polyhedra, the fixed point property is neither preserved under cartesian products, nor under cartesian products with the unit interval I, nor under suspensions, smash products, and joins.

Since $X \times I$, the cartesian product with I, is of the same homotopy type as X, it also implies that the f.p.p. is not a homotopy property in general.

CHAPTER IV

TWO CLASSICAL EXAMPLES

In this chapter it will be shown, among other things, that Hypothesis B is in fact true. This will be done by first considering an example $Z = X \vee Y$ of a simply connected polyhedron such that Z has the f.p.p. but admits a self-map $f : Z \to Z$ with $L(f) = 0$. This example will also settle several other questions in the fixed point theory of polyhedra, for example the following:

(i) The union along an edge of two polyhedra with the f.p.p. need not have the f.p.p.

(ii) The f.p.p. is not a homotopy type invariant in the category of finite polyhedra.

(iii) The f.p.p. is not a product invariant in the category of finite polyhedra.

(iv) The f.p.p. is not a suspension invariant in the category of finite polyhedra.

We first consider the suspension of even dimensional complex projective spaces.

4.1. THEOREM. The suspension of even dimensional complex projective spaces (i.e. SCP^{2n}) has the f.p.p.

For the proof we need to use the Steenrod squares operators Sq^i, whose basic properties are therefore stated first:

(1) For all integers $i \geq 0$ and $q \geq 0$, there is a natural transformation of functors which is a homomorphism

$$Sq^i : H^n(X, A) \longrightarrow H^{n+i}(X, A), \quad n \geq 0.$$

(2) $Sq^0 = 1$.

(3) If $\dim x = n$, $Sq^n x = x^2$.

(4) If $i > \dim x$, $Sq^i x = 0$.

(5) Cartan formula: $Sq^k(xy) = \sum_{i=0}^{k} Sq^i x \cdot Sq^{k-i} y$.

PROOF (of the theorem) : We know that the cohomology ring of CP^{2n} over Z_2 is a truncated polynomial ring in one generator $\alpha \in H^2(CP^{2n}; Z_2)$; $\alpha^{2n+1} = 0$. If we let S denote the suspension homomorphism, then $\beta_i = S\alpha^i \in H^{2i+1}(SCP^{2n}; Z_2)$, $1 \leq i \leq 2n$, are the generators of $H^*(SCP^{2n}; Z_2)$ (since $H^q(CP^{2n}; Z_2) \cong H^{q+1}(SCP^{2n}; Z_2)$ hence $\beta_i = S\alpha^i \in H^{2i+1}(SCP^{2n}; Z_2)$). Furthermore, if Sq denotes the Steenrod squaring operations, we have

$Sq^2(\beta_1) = Sq^2(S\alpha) = S\, Sq^2\alpha = S\alpha^2 = \beta_2$ (since Steenrod squares operators Sq^i commute with suspension i.e. $S\, Sq^i = Sq^i S$) and $\alpha \in H^2(CP^{2n}; Z_2)$, therefore $Sq^2\alpha = \alpha^2$,

$$Sq^2(\beta_3) = Sq^2(S\alpha^3) = S\ Sq^2(\alpha^3),$$

$$Sq^2(\alpha^3) = Sq^2(\alpha^2.\alpha) = Sq^2(\alpha^2)Sq^0(\alpha) + Sq^1(\alpha^2)\, Sq^1(\alpha) + Sq^0(\alpha^2)Sq^2(\alpha)$$

$$= \alpha Sq^2(\alpha^2) + Sq^1(\alpha)\, Sq^1(\alpha) + \alpha^2.\alpha^2,$$

$$Sq^2(\alpha^2) = Sq^2(\alpha.\alpha) = Sq^2(\alpha)Sq^0(\alpha) + Sq^1(\alpha)Sq^1(\alpha) + Sq^0(\alpha)Sq^2(\alpha)$$

$$= 2\alpha^3 = 0 \text{ (since the coefficients are from } Z_2).$$

$Sq^1(\alpha) = 0$, because $Sq^1 : H^2(CP^{2n}; Z_2) \longrightarrow H^3(CP^{2n}; Z_2) = 0$ is the zero-homomorphism. Thus $Sq^2(\alpha^3) = \alpha^4$.

Similarly, $Sq^2(\beta_3) = Sq^2 S\alpha^3 = S\, Sq^2\alpha^3 = S\alpha^4 = \beta_4$.

$$\cdots \quad \cdots \quad \cdots$$

$$Sq^2(\beta_{2n-1}) = Sq^2 S\alpha^{2n-1} = S\, Sq^2\alpha^{2n-1} = S\alpha^{2n} = \beta_{2n}.$$

Now if $f : SCP^{2n} \rightarrow SCP^{2n}$ is a map and $f^*(\beta_k) = b_k\beta_k$, then the Lefschetz number of f over Z_2 is given by

$$L(f; Z_2) = 1 + \sum_{i=1}^{n}(b_{2i-1} + b_{2i}).$$

From the above discussion we have

$f^*(\beta_{2i}) = f^*(Sq^2\beta_{2i-1}) = Sq^2 f^*(\beta_{2i-1}) = Sq^2 b_{2i-1}\beta_{2i-1} = b_{2i-1}\beta_{2i}$ (by the naturality of the Steenrod squares). Therefore $b_{2i-1} = b_{2i}$, where $1 \leq i \leq 2n$. $L(f; Z_2) = 1 + 2(b_1 + b_3 + \dots + b_{2n-1}) = 1 \neq 0$. Hence SCP^{2n} has the f.p.p.

We now focus our attention to the specific example already announced in the chapter's introductory paragraph, which is due to W. Lopez, and which will turn out to have very interesting consequences.

Consider the disjoint union of $CP^2 \cup S_1 \times S_2 \cup CP^4$ where CP^2 and CP^4 are complex projective spaces and S_1 and S_2 are 2-spheres. Identify all $x \in CP^1 \subset CP^2$ with $(x, x_2) \in S_1 \times \{x_2\}$ and all $x' \in CP^1 \subset CP^4$ with $(x_1, x') \in \{x_1\} \times S_2$. Denote the resulting quotient space by X.

4.2. PROPOSITION. X has the f.p.p. and $\chi(X) = 8$.

PROOF: We first obtain the cohomology ring structure of X over the rationals Q. We notice that $H^q(X, S_1 \vee S_2) \approx H^q(X/ S_1 \vee S_2)$ and $X/ S_1 \vee S_2 \approx CP^2/ CP^1 \vee S_1 \times S_2/ S_1 \vee S_2 \vee CP^4/ CP^1$. Therefore

$$H^q(X/ S_1 \vee S_2) \approx H^q(CP^2/ CP^1) \oplus H^q(S_1 \times S_2/ S_1 \vee S_2) \oplus H^q(CP^4/ CP^1)$$

$$\approx H^q(CP^2, CP^1) \oplus H^q(S^4) \oplus H^q(CP^4, CP^1).$$

Since $H^2(CP^2, CP^1) \approx 0 \approx H^2(CP^4, CP^1)$ and $H^2(S^4) \approx 0$, therefore $H^2(X, S_1 \vee S_2) = 0$. Similarly it can be seen that $H^3(X, S_1 \vee S_2) = 0$. By considering the cohomology sequence of the pair $(X, S_1 \vee S_1)$, one has

$$0 = H^2(X, S_1 \vee S_2) \longrightarrow H^2(X) \longrightarrow H^2(S_1 \vee S_2) \longrightarrow H^3(X, S_1 \vee S_2) = 0.$$

Hence $H^2(X) \approx H^2(S_1 \vee S_2) \approx H^2(S_1) \oplus H^2(S_2) \approx Q \oplus Q$. Therefore there are two generators α and β of $H^2(X)$. Consider $X = X_1 \cup X_2$, where $X_1 = CP^2 \vee CP^4$, $X_2 = S_1 \times S_2$ and therefore $X_1 \cap X_2 = S_1 \vee S_2$. From the following Mayer-Vietoris sequence

$$0 \longrightarrow H^2(X) \longrightarrow H^2(CP^2 \vee CP^4) \oplus H^2(S_1 \times S_2) \longrightarrow H^2(S_1 \vee S_2) \longrightarrow 0,$$

and for $q \geq 2$ we have

$$0 \longrightarrow H^q(X) \longrightarrow H^q(CP^2 \vee CP^4) \oplus H^q(S_1 \times S_2) \longrightarrow 0.$$

Therefore $H^q(X) \approx H^q(CP^2 \vee CP^4) \oplus H^q(S_1 \times S_2)$

$$\approx H^q(CP^2) \oplus H^q(CP^4) \oplus H^q(S_1 \times S_2), \text{ for } q > 2.$$

Clearly $H^q(X) = 0$ when q is odd. For $q = 4$, we get

$$H^4(X) \approx H^4(CP^2) \oplus H^4(CP^4) \oplus H^4(S_1 \times S_2)$$

$$\approx H^4(CP^2) \oplus H^4(CP^4) \oplus H^2(S_1) \oplus H^2(S_2)$$

$$\approx Q \oplus Q \oplus Q \,.$$

For $q = 6$, one has

$$H^6(X) \approx H^6(CP^2) \oplus H^6(CP^4) \oplus H^6(S_1 \times S_2)$$

$$\approx H^6(CP^4) \approx Q.$$

For $q = 8$, we get

$$H^8(X) \approx H^8(CP^2) \oplus H^8(CP^4) \oplus H^8(S_1 \times S_2)$$

$$\approx H^8(CP^4) \approx Q.$$

From the above discussion and by the naturality of the cup products, the ring structure of X is given as follows

$H^0(X;\, Q)$	$= Q$	with generator 1
$H^2(X;\, Q)$	$= Q \oplus Q$	with generators α and β
$H^4(X;\, Q)$	$= Q \oplus Q \oplus Q$	with generators α^2, $\alpha\beta$, β^2
$H^6(X;\, Q)$	$= Q$	with generator β^3
$H^8(X;\, Q)$	$= Q$	with generator β^4.

Therefore the Betti numbers of X are $\rho_0(X) = 1$, $\rho_2(X) = 2$, $\rho_4(X) = 3$, $\rho_6(X) = 1$ and $\rho_8(X) = 1$. Hence the Euler characteristic of X is

$$\chi(X) = \sum_q (-1)^q \rho_q(X) = \rho_0(X) + \rho_2(X) + \rho_4(X) + \rho_6(X) + \rho_8(X) = 8$$

and it remains to show that X has the f.p.p.

Let $i : CP^4 \longrightarrow X$ be the inclusion and consider $i^* : H^2(X; Q) \longrightarrow H^2(CP^4; Q) = Q$. There is a generator $\beta' \in H^2(CP^4; Q)$ such that $i^*(\beta) = \beta'$ and $i^*(\alpha) = 0$. There is a retraction $r : X \longrightarrow CP^2$ defined by sending $CP^1 \times CP^1$ onto $CP^1 \times (x_2)$ and CP^4 onto (x_1, x_2), and a generator $\alpha' \in H^2(CP^2; Q)$ such that $r^*(\alpha') = \alpha$. Let $f : X \longrightarrow X$ be any map. We want to show that $L(f) \neq 0$. Consider $f^* : H^2(X; Q) \longrightarrow H^2(X; Q)$ and assume $f^*(\alpha) = a\alpha + b\beta$ and $f^*(\beta) = c\alpha + d\beta$, then $tr(f^{*2}) = a + d$. Next, we compute $i^*f^*r^*$. To do this we have

$$H^2(CP^2; Q) \xrightarrow{r^*} H^2(X; Q) \xrightarrow{f^*} H^2(X; Q) \xrightarrow{i^*} H^2(CP^4; Q).$$

Therefore $i^*f^*r^*(\alpha') = i^*f^*(\alpha) = i^*(a\alpha + b\beta) = bi^*(\beta) = b\beta'$. Since induced homomorphisms preserve cup products, we therefore know that $i^*f^*r^*(\alpha'^3) = b^3\beta'^3$. But β' generates $H^2(CP^4; Q)$, therefore β'^3 generates $H^6(CP^4; Q)$. So b must be zero.

In order to compute $L(f)$ we wish to determine the traces of f^* on $H^p(X; Q)$ for $p = 4, 6, 8$. For $H^4(X; Q)$ we use the fact that $b = 0$ to discover that $f^*(\alpha^2) = (a\alpha + b\beta)^2 = a^2\alpha^2$

$$\begin{aligned} f^*(\alpha \cup \beta) &= (a\alpha + b\beta) \cup (c\alpha + d\beta) \\ &= a\alpha \cup (c\alpha + d\beta) \\ &= ac\alpha + ad(\alpha \cup \beta), \quad \text{since } \alpha^2 = 0. \end{aligned}$$

$$\begin{aligned} f^*(\beta^2) &= (c\alpha + d\beta)^2 \\ &= (c\alpha + d\beta) \cup (c\alpha + d\beta) \end{aligned}$$

$$= c^2\alpha^2 + 2cd(\alpha \cup \beta) + d^2\beta^2.$$

The matrix of $f^* : H^4(X; Q) \longrightarrow H^4(X; Q)$ with respect to the basis α^2, $\alpha \cup \beta$, β^2 is therefore

$$\begin{bmatrix} a^2 & 0 & 0 \\ ac & ad & 0 \\ c^2 & 2cd & d^2 \end{bmatrix}$$

Since $\alpha^3 = \alpha^4 = \alpha \cup \beta^2 = \alpha^2 \cup \beta = 0$ we have $f^*(\beta^3) = (c\alpha + d\beta)^3 = d^3\beta^3$ and $f^*(\beta^4) = (c\alpha + d\beta)^4 = d^4\beta^4$, therefore $tr(f^{*6}) = d^3$; $tr(f^{*8}) = d^4$.

Thus $L(f) = 1 + (a + d) + (a^2 + ad + d^2) + d^3 + d^4$

$$= (a + 1/2 + d/2)^2 + 1/4(4d^4 + 4d^3 + 3d^2 + 2d + 3).$$

If $p(d) = 4d^4 + 4d^3 + 3d^2 + 2d + 3$, then $p'(d) = 2(2d + 1)(4d^2 + d + 1)$ and $p(d) \geq p(-1/2) = 5/2$. So $L(f) = (a + 1/2 + d/2)^2 + p(d)/4 \geq 5/8$ and f has a fixed point. Therefore X has f.p.p. □

This example verifies the Hypothesis B because X is a simply connected polyhedron with f.p.p. and the Euler characteristic, which is the Lefschetz number of the identity map is $\chi(X) = 8$, which is even. Let $Y = SCP^8$. Then as proved in 3.8, $\chi(Y) = -8 + 1 = -7$. Therefore $\chi(X \vee Y) = \chi(X) + \chi(Y) - 1 = 8 - 7 - 1 = 0$. Being the wedge of two spaces having f.p.p., $X \vee Y$ also has f.p.p. This is a specific example verifying Corollary 3.9.

4.3. THEOREM. There exist finite polyhedra X and Y with f.p.p. such that their union along an edge fails to have the f.p.p.

We now use a result originally due to Wecken [21], in the form proved by Shi Gen-Hua [11], p.238 , (see also [9], [5]):

4.4. THEOREM. Let K be a finite polyhedron such that no finite number of points separates K. Then if $\chi(K) = 0$, there is a fixed point free map homotopic to the identity.

PROOF (of the theorem 4.3). From the Mayer-Vietoris sequence we have the following relationship between the Euler characteristic of the union of the two spaces along an edge and the Euler characteristic of those spaces and their intersection:

$$\chi(X \cup_I Y) = \chi(X) + \chi(Y) - \chi(X \cap Y)$$
$$= \chi(X) + \chi(Y) - \chi(I) = 8 - 7 - 1 = 0.$$

Since $X \cup_I Y$ obviously satisfies the hypothesis of Theorem 4.4 it follows therefore that it does not have the f.p.p.

Since $X \cup_I Y$ is of the same homotopy type as $X \vee Y$ (because I is contractible) we conclude:

4.5. THEOREM. The f.p.p. is not a homotopy type invariant in the category of finite polyhedra.

This wedge $Z = X \vee Y$ also provides an example, showing that the f.p.p. is not invariant under cartesian products and suspensions.

4.6. THEOREM. $Z = X \vee Y$ is a finite polyhedron with the f.p.p. such that $Z \times I$ and $Z \times Z$ fail to have the f.p.p.

PROOF: Since $\chi(Z \times I) = \chi(Z)\,\chi(I) = 0$ and $\chi(Z \times Z) = \chi(Z)\,\chi(Z) = 0$

hence by Theorem 4.4 they fail to have the f.p.p. □

4.7. THEOREM. There is a finite polyhedron K with the f.p.p. whose suspension SK fails to have the f.p.p.

PROOF: Let $Z' = X \vee SCP^6$. By Theorem 2.13 Z' has f.p.p. since X and SCP^6 both have the f.p.p. However, $\chi(Z') = \chi(X) + \chi(SCP^6) - 1 = 8 - 5 - 1 = 2$, so for SZ' we have $\chi(SZ') = \tilde{\chi}(SZ') + 1 = -\tilde{\chi}(Z') + 1 = -\chi(Z') + 2 = 0$. Hence by Theorem 4.4 SZ' does not have f.p.p. The Z' is the K of our theorem. □

Next we want to discuss another example in some detail which is originally due to G. Bredon.

Consider $X = HP^3 \vee SCP^4$. We have already seen that SCP^4 has the f.p.p. To prove that HP^3 also has the f.p.p., let $f : HP^3 \longrightarrow HP^3$ be a self-map. We know that f induces a homomorphism $f^* : H^q(HP^3; Z_3) \longrightarrow H^q(HP^3; Z_3)$. Assume $f^*(\alpha) = a\alpha$, where α is a generator of $H^4(HP^3; Z_3)$ and $a \in Z_3$. Then we have $f^*(\alpha^n) = a^n\alpha^n$. Therefore $L(f) = \sum_j (-1)^j \operatorname{tr}(f_j^*) = 1 + a + a^2 + a^3$. If $a = 0$ or 1, then clearly $L(f) = 1 \neq 0$, hence HP^3 has f.p.p. Consider the case when $a = -1$. Let P^i be the Steenrod reduced power operator whose properties are similar to those of the Steenrod squares. We also need the following Adem relation. If $a < pb$ then

$$P^a P^b = \sum_{t=0}^{[a/p]} (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt} P^{a+b-t} P^t.$$

Since α is a generator of $H^4(HP^3; Z_3)$ therefore $P^2(\alpha) = \alpha^3$. Also from the Adem relation we get $P^1P^1 = 2P^2$. Therefore $P^1P^1(\alpha) = 2P^2(\alpha) = 2\alpha^3 \neq 0$ which implies that $P^1(\alpha) \neq 0$. This in turn implies that $P^1(\alpha)$ is a generator of $H^8(HP^3; Z_3) = Z_3$ since P^1 is a homomorphism and α is a generator of $H^4(HP^3; Z_3)$. Therefore $P^1(\alpha) = \pm\alpha^2$. Assume $P^1(\alpha) = \alpha^2$. Since P^1 commutes with f^* one has $f^*(P^1(\alpha)) = f^*(\alpha^2) = a^2\alpha^2$; $P^1(f^*(\alpha)) = P^1(a\alpha) = a\alpha^2$. Therefore $a^2\alpha^2 = a\alpha^2$ hence $a^2 = a$ i.e. $a^2 \equiv a \pmod 3$. This implies that $a \neq -1$, contradicting the choice of a. Hence HP^3 has f.p.p. (this is already mentioned in [9] without proof). We further conclude therefore that $HP^3 \vee SCP^4$ has f.p.p., since both HP^3 and SCP^4 have the f.p.p.

REMARK: In [10] it is left open whether $P^1(\alpha) = +\alpha^2$ or $-\alpha^2$, however, it should be noted that $P^1(\alpha)$ is actually not $-\alpha^2$ but $+\alpha^2$, as the following argument shows. From the Adem relation one has

$$P^1P^1 = \sum_{t=0}^{[1/3]} (-1)^{1+t} \binom{2(1-t)-1}{1-3t} P^{2-t} P^t$$
$$= -P^2 P^0 = -P^2 = 2P^2.$$

Suppose $P^1(\alpha) = -\alpha^2$. Therefore $P^1P^1(\alpha) = P^1(-\alpha^2) = -P^1(\alpha^2) = -2\alpha^3$. But $2P^2(\alpha) = 2\alpha^3$. Thus $P^1P^1(\alpha) \neq 2P^2(\alpha)$ which contradicts the relationship between P^1P^1 and P^2 (i.e. $P^1P^1 = P^2$).

REMARK: According to Fadell [9], G. Bredon was the first to observe that HP^3 has the f.p.p.

In more generality the following can be shown:

4.8. THEOREM. HP^n has the f.p.p. for all $n \geq 2$.

PROOF: The proof is similar to the above. Let $\phi : HP^n \longrightarrow HP^n$ be a map. We know that ϕ induces a homomorphism $\phi^* : H^q(HP^n; Z_3) \longrightarrow H^q(HP^n; Z_3)$. Assume $\phi^*(\alpha) = a\alpha$, where α is a generator in $H^4(HP^n)$. Therefore $\phi^*(\alpha^n) = a^n\alpha^n$. Hence $L(\phi) = 1 + a + a^2 + a^3 + \ldots + a^n$. If n is even we have already seen by Theorem 3.2 that HP^n has the f.p.p. If n is odd, then for $a = 0$ or 1 clearly $L(\phi) = 1 \neq 0$. Consider the case when $a = -1$. As before $P^1(\alpha) = \alpha^2 \in H^8(HP^n; Z_3)$. This forces $a^2 \equiv a \pmod 3$. Therefore $a \neq -1$ which is not possible. Hence HP^n has the f.p.p. for $n \geq 2$.

Next we want to show that, although $X = HP^3 \vee SCP^4$ has the f.p.p. as we have seen, $X \times I$ does not have the f.p.p. Since X is a finite polyhedron, $X \times I$ clearly is a finite polyhedron, too. However, although the wedge X does have a local cut point, $X \times I$ does not have one. The proof depends on the fact that X is a finite polyhedron and therefore is locally pathwise-connected. (The space X is locally pathwise-connected if for any $x \in X$ and neighborhood U of x there is a pathwise-connected neighborhood V of x in U. That is, for $x' \in V$ there is a path in V from x' to x). Figures 1 and 2 indicate why taking the cartesian product with I eliminates local cut points. Given $(x^*, t^*) \in X \times I$ and any neighborhood W of it, we can find a neighborhood U of (x^*, t^*) in W such that $U - \{(x^*, t^*)\}$ is pathwise-connected and therefore connected. To do so, we use the local pathwise-connectedness of X to find a pathwise-connected neighborhood V of x^* in X such that $U = V \times I_0$ is

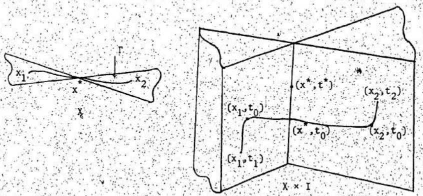


Figure 1.

contained in W, where I_0 is an open interval in I containing t^*. Let (x_1, t_1) and (x_2, t_2) be two points in $U - \{(x^*, t^*)\}$. First

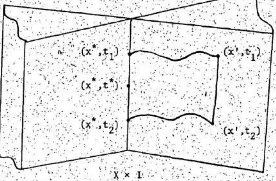


Figure 2.

consider the case where at least one of x_1 and x_2 is different from x^*. Choose $t_0 \neq t^*$ in I_0 and define a path between them as in Figure 1. The path Γ in V from x_1 to x_2 exists because V is pathwise-connected. For the second case $x_1 = x_2 = x^*$, one chooses some $x' \in V$ and considers a path Γ in V from x^* to x' and proceeds as indicated in Figure 2. Figures 1 and 2 illustrate the case where x^* is a local cut point, but the argument applies as well to any point of x.

We have already seen in Chapter II that $\chi(SCP^n) = -n + 1$ and similarly $\chi(HP^n) = n + 1$, for all n. The Euler characteristics being a special case of Lefschetz numbers (namely the Lefschetz number of the identity map) obviously also satisfy the following relationships for wedges and cartesian products.

$$\chi(X \vee Y) = L(1_X \vee 1_Y) = L(1_X) + L(1_Y) - 1$$
$$= \chi(X) + \chi(Y) - 1,$$

and similarly

$$\chi(X \times Y) = \chi(X)\,\chi(Y).$$

Then we have

$$\chi(X \times I) = \chi(X)\chi(I) = \chi(HP^3 \vee SCP^4)\chi(I)$$
$$= [\chi(HP^3) + \chi(SCP^4) - 1]\chi(I)$$
$$= [(4) + (-3) - 1]\chi(I) = 0.$$

Hence we can apply Theorem 4.4 (since for finite polyhedra the non-existence of a finite number of seperating points is equivalent to the non-existence of local cut points), and it thus follows that $X \times I$ does not have the f.p.p.

CHAPTER V

SMASH PRODUCTS AND THE FIXED POINT PROPERTY

In the previous chapters we have seen that the fixed point property in the category of simply connected polyhedra is not invariant under cartesian products, smash products, suspensions, joins, nor is it a homotopy type invariant. In all cases the counterexamples are based upon polyhedra which fail to satisfy the Shi condition. It is therefore natural to consider the behavior of the f.p.p. in more restrictive settings. As suggested by Fadell [10], we consider the following two categories:

S : Polyhedra satisfying the Shi condition.

S_0 : Simply connected polyhedra in S.

In the category S the f.p.p. is a homotopy type invariant (see [5]). (In fact, if X is any compact ANR dominated by Y, where Y is in S such that Y has f.p.p. then X has f.p.p. also). Thus the result that Y having f.p.p. implies $Y \times I$ having f.p.p., is valid in the category S even though it is false for (even simply connected) polyhedra in general (see, for example, [5], p.147). The question whether the implication

$$X \text{ f.p.p. and } Y \text{ f.p.p.} \Longrightarrow X \times Y \text{ f.p.p.?}$$

holds, in the category S in general still seems to remain open.

However, in the latter part of this chapter we shall give an example of such a product theorem under very special additional assumptions on the factor spaces (see Theorem 5.22). In the earlier parts of this chapter we discuss some further examples, due to Fadell [10], to show that the f.p.p. is not even invariant under suspensions and join products in the category S_0, and also an example of a simply connected polyhedron X such that the smash product $X \wedge X$ does or does not have f.p.p. depending on which point is chosen as the base point. Many fine points and proofs relating to this material, but not given in [10] are included in this chapter in detail.

In order to deal with the f.p.p. of certain wedges, suspensions, joins and smash products, we first need the following Lemma:

5.1. LEMMA. Suppose F is a field of characteristic $p \neq 2$ and X and Y are spaces with the property that for every self-map $f : X \longrightarrow X$, $\tilde{L}(f; F) = 0$ or 1 and every self-map $g : Y \longrightarrow Y$, $\tilde{L}(g; F) = 0$. Then any space $W = X \vee Y$ has f.p.p.

PROOF: Let

$$X \xrightarrow{i_1} X \vee Y \xrightarrow{r_1} X$$

$$Y \xrightarrow{i_2} X \vee Y \xrightarrow{r_2} Y$$

denote the natural inclusions and retractions. Then, if $\phi : X \vee Y \longrightarrow X \vee Y$ is any map, let $f = r_1 \phi i_1$ and $g = r_2 \phi i_2$. Let the generators of $\tilde{H}_k(X)$ be $x_1, \ldots, x_m$ and $\tilde{H}_k(Y)$ be $y_1, \ldots, y_n$. Let $\tilde{\phi}_*(x_i)$ be the sum of $\sum_{j=1}^{m} a_{ij} x_j$ and a linear combination of y's and $\tilde{\phi}_*(y_i)$ be the sum of the linear combination of x's and $\sum_{j=1}^{n} b_{ij}y_j$. Then

$$(\tilde{\phi}_*)_{ii} = a_{ii}, \text{ where } 1 \le i \le m$$

$$(\tilde{\phi}_*)_{i+m,\, i+m} = b_{ii}, \text{ where } 1 \le i \le n$$

(where the basis for $H_k(X \vee Y)$ is $\{z_1, \ldots, z_{m+n}\}$ such that $z_i = i_{1*}x_i$, $z_{i+m} = i_{2*}y_i$). The contribution in dimension k to $\tilde{L}(\phi; F)$ is $(-1)^k \operatorname{tr}(\phi_{*k})$, therefore $\tilde{L}(\phi; F) = \sum (-1)^k \operatorname{tr}(\phi_{*k}) = \sum_{k=1}^{m} (-1)^k a_{kk} + \sum_{k=1}^{n} (-1)^k b_{kk}$. By commutativity of the following diagram,

$$\begin{array}{ccc}
\tilde{H}_n(X \vee Y) = \tilde{H}_n(X) \oplus \tilde{H}_n(Y) & \xrightarrow{\ \tilde{\phi}_*\ } & \tilde{H}_n(X) \oplus \tilde{H}_n(Y) \\
\uparrow{\scriptstyle i_{1*}} & & \downarrow{\scriptstyle r_{1*}} \\
\tilde{H}_n(X) & \xrightarrow{\ \tilde{f}_*\ } & \tilde{H}_n(X)
\end{array}$$

one has $\tilde{f}_*(x_i) = r_{1*}\tilde{\phi}_* i_{1*}(x_i) = r_{1*}\tilde{\phi}_*(z_i)$

$$= r_{1*}\Big(\sum_{j=1}^{m} a_{ij}z_j\Big) + r_{1*}(\text{ linear combination of } z_j\text{'s}, j > m)$$

$$= \sum_{j=1}^{m} a_{ij}x_j .$$

Therefore $\operatorname{tr}(\tilde{f}_{*k}) = \sum_k (-1)^k a_{kk}$ and similarly $\operatorname{tr}(\tilde{g}_{*k}) = \sum_k (-1)^k b_{kk}$. Hence $\tilde{L}(\phi; F) = \tilde{L}(f; F) + \tilde{L}(g; F)$ is 0 or 1. Thus, $L(\phi; F) \neq 0$. So $X \vee Y$ has the property that every self-map ϕ has a nonzero Lefschetz number over F. Since this property is a homotopy type invariant, it follows that if $W \simeq X \vee Y$, then W has f.p.p. □

5.2. LEMMA. If HP^4 is quaternionic projective 4-space, then for every self-map $f : HP^4 \longrightarrow HP^4$, $\tilde{L}(f; Z_3)$ is 0 or 1.

PROOF: Let u denote a generator in $H^4(HP^4; Z_3)$. Assume $f^*(u) = au$. Therefore $f^*(u^n) = a^n u^n$. Hence $\tilde{L}(f) = a + a^2 + a^3 + a^4$ is 0 or 1. □

5.3. LEMMA. If SHP^3 is the suspension of quaternionic projective 3-space, then for every self-map $g : SHP^3 \longrightarrow SHP^3$, $\tilde{L}(g; Z_3) = 0$.

PROOF: Choose a generator $v \in H^5(SHP^3; Z_3)$ such that $P^1(v)$ and $P^2(v)$ generate the Z_3-cohomology in dimensions 9 and 13, respectively. P^i is the mod-3 Steenrod reduced power operator. Now, if $g : SHP^3 \longrightarrow SHP^3$ and $g^*(v) = bv$ then $g^*(P^1 v) = P^1 g^*(v) = P^1 bv = bP^1 v$. Similarly, $g^* P^2(v) = P^2 g^*(v) = P^2 bv = bP^2(v)$. Therefore, $\tilde{L}(g; Z_3) = b + b + b = 0$. □

5.4. PROPOSITION. Any space $W = HP^4 \vee SHP^3$ has the f.p.p.

PROOF: From Lemma 5.1 we will get the result. □

Let $K = HP^4 \cup_I SHP^3$ denote the union of HP^4 and SHP^3 along an edge. Since both constituent parts are simply connected polyhedra satisfying the Shi condition and have the f.p.p. the same is true for K. Moreover, one has

5.5. PROPOSITION. $\chi(K) = 2$.

PROOF: Since $\chi(HP^4) = 4 + 1 = 5$, $\chi(SHP^3) = -3 + 1 = -2$ and also from the result in the proof of Theorem 4.4 (i.e. $\chi(X \cup_I Y) =$

$\chi(X) + \chi(Y) - \chi(X \cap Y))$ we have

$$\chi(HP^4 \cup_I SHP^3) = \chi(HP^4) + \chi(SHP^3) - \chi(I)$$
$$= 5 - 2 - 1 = 2. \square$$

REMARK: $K' = (HP^4 \vee SHP^3) \times I$ has the same above mentioned properties as K.

5.6. PROPOSITION. The suspension SK and the join $K * K$ fail to have the f.p.p.

PROOF: Since $\tilde{\chi}(SK) = \tilde{L}(S1_K) = -\tilde{L}(1_K) = -\tilde{\chi}(K)$ and similarly $\tilde{\chi}(K * K) = -\tilde{\chi}(K)\tilde{\chi}(K)$, therefore $\chi(SK) = -\chi(K) + 2 = 0$ and $\chi(K * K) = \tilde{\chi}(K * K) + 1 = 0$. Since SK and $K * K$ satisfy the Shi condition, hence by Theorem 4.4 both admit maps homotopic to the identity map which are fixed point free. $\square$

We therefore conclude

5.7. THEOREM. The f.p.p. is not invariant under suspensions and joins in the category S_0.

Now we want to show that there is a simply connected polyhedron X with the f.p.p. such that the smash product $X \wedge X = X \times X/X \vee X$ has f.p.p. with one choice of base point $x_0 \in X$ while it fails to have f.p.p. if one chooses another base point $x_1 \in X$. We will make use of the polyhedron $K = HP^4 \cup_I SHP^3$ discussed previously. If $N = SHP^2$ and

$$X = K \vee N = (HP^4 \cup_I SHP^3) \vee SHP^2$$

we will show that $X \wedge X$ fails to have f.p.p. if the base point $x_0 \in X$ is chosen distinct from the wedge point $v \in X$. On the other hand, if the wedge point v is employed to form $X \wedge X$, then $X \wedge X$ retains f.p.p.

5.8. THEOREM. If $x_0 \neq v$, then $X \wedge X = X \times X / x_0 \times X \cup X \times x_0$ fails to have f.p.p.

PROOF: First we have

$$\begin{aligned} \chi(X) &= \chi(K \vee N) = \chi(K) + \chi(N) - 1 \\ &= \chi(HP^4 \cup_I SHP^3) + \chi(SHP^2) - 1 \\ &= 2 - 1 - 1 = 0. \end{aligned}$$

Therefore $\tilde{L}(1_X) = -1$. Since $\tilde{\chi}(K) = 1$ it follows that X admits a map g such that $\tilde{L}(g) = 1$. Thus, $\tilde{L}(1 \wedge g) = \tilde{L}(1)\tilde{L}(g) = -1$, and we see that $f = 1 \wedge g$ is a self-map of $X \wedge X$ with $L(f) = 0$. $X \wedge X$ is simply connected and satisfies the Shi condition (using the fact that $x_0 \times X \cup X \times x_0$ fails to separate $X \times X$). By Theorem 3.14, it follows that there is a map $g \sim f$ such that g has no fixed points. Thus, $X \wedge X$ fails to have f.p.p. □

We now show that using the wedge point v, the smash product

$$X \wedge X = X \times X / v \times X \cup X \times v$$

has f.p.p. Using v as base point in the formation of $X \wedge X$ gives the result

$$X \wedge X = (K \wedge K) \vee (K \wedge N) \vee (N \wedge K) \vee (N \wedge N)$$

where the four-fold wedge on the right is understood to have a single wedge point v' corresponding to $v \times X \cup X \times v$. Now, since f.p.p. is invariant under wedge operation, it suffices to show that the four individual spaces $K \wedge K$, $K \wedge N$, $N \wedge K$, $N \wedge N$ all have the f.p.p.

5.9. LEMMA. $HP^4 \wedge HP^4$ has f.p.p. Specifically, for any self-map ϕ, $\tilde{L}(\phi; Z_3)$ is 0 or 1, and thus $L(\phi; Z_3) \neq 0$.

PROOF: We will identify $H^*(A \wedge B)$ with $H^*(A \times B, A \vee B) = H^*(A, a_0) \otimes H^*(B, b_0)$ where the coefficients are taken from a field. Then, if the coefficients are from Z_3, $H^*(HP^4)$ has a basis of the form $1, \alpha, P^1\alpha, P^2\alpha, \alpha^4$ where P^i is the Steenrod reduced power operator. Then the basis for $H^*(HP^4 \wedge HP^4)$ in positive dimensions can be arranged as follows:

$\alpha \times \alpha$	$\alpha \times P^1\alpha + P^1\alpha \times \alpha$	$\alpha \times P^2\alpha + P^1\alpha \times P^1\alpha + P^2\alpha \times \alpha$
$P^1\alpha \times \alpha$	$P^1\alpha \times P^1\alpha - P^2\alpha \times \alpha$	$-P^2\alpha \times P^1\alpha + P^1\alpha \times P^2\alpha$
$\alpha \times P^2\alpha$	$P^1\alpha \times P^2\alpha$	$P^2\alpha \times P^2\alpha$
$\alpha^4 \times \alpha$	$\alpha^4 \times P^1\alpha$	$\alpha^4 \times P^2\alpha$
$\alpha \times \alpha^4$	$P^1\alpha \times \alpha^4$	$P^2\alpha \times \alpha^4$
$\alpha^4 \times \alpha^4$		

It can be easily verified (for the first five rows) that applying P^1 and P^2 to the first column yields the second and third columns. If $\phi : HP^4 \wedge HP^4$ is a self-map then $\phi^* : H^*(HP^4 \wedge HP^4) \longrightarrow H^*(HP^4 \wedge HP^4)$ is a homomorphism. Assume $\phi^*(\alpha \times \alpha) = \lambda(\alpha \times \alpha)$ where $\lambda \in Z_3$. Then, one has $\phi^*(\alpha \times P^1\alpha + P^1\alpha \times \alpha) = \phi^* P^1(\alpha \times \alpha) = P^1\phi^*(\alpha \times \alpha)$ (since P^1 commutes with ϕ^*) $= \lambda P^1(\alpha \times \alpha) = \lambda(\alpha \times P^1\alpha + P^1\alpha \times \alpha)$. Similarly

$\phi^*(\alpha \times P^2\alpha + P^1\alpha \times P^1\alpha + P^2\alpha \times \alpha) = \lambda(\alpha \times P^2\alpha + P^1\alpha \times P^1\alpha + P^2\alpha \times \alpha)$.

By similar arguments it can be seen that if $\phi^*(P^1\alpha \times \alpha) = \lambda_1(P^1\alpha \times \alpha)$, then $\phi^*(P^1\alpha \times P^1\alpha - P^2\alpha \times \alpha) = \lambda_1(P^1\alpha \times P^1\alpha - P^2\alpha \times \alpha)$, and $\phi^*(-P^2\alpha \times P^1\alpha + P^1\alpha \times P^2\alpha) = \lambda_1(-P^2\alpha \times P^1\alpha + P^1\alpha \times P^2\alpha)$. If $\phi^*(\alpha \times P^2\alpha) = \lambda_2(\alpha \times P^2\alpha)$, then $\phi^*(P^1\alpha \times P^2\alpha) = \lambda_2(P^1\alpha \times P^2\alpha)$, and $\phi^*(P^2\alpha \times P^2\alpha) = \lambda_2(P^2\alpha \times P^2\alpha)$. If $\phi^*(\alpha^4 \times \alpha) = \lambda_3(\alpha^4 \times \alpha)$, then $\phi^*(\alpha^4 \times P^1\alpha) = \lambda_3(\alpha^4 \times P^1\alpha)$, and $\phi^*(\alpha^4 \times P^2\alpha) = \lambda_3(\alpha^4 \times P^2\alpha)$. If $\phi^*(\alpha \times \alpha^4) = \lambda_4(\alpha \times \alpha^4)$, then $\phi^*(P^1\alpha \times \alpha^4) = \lambda_4(P^1\alpha \times \alpha^4)$, and $\phi^*(P^2\alpha \times \alpha^4) = \lambda_4(P^2\alpha \times \alpha^4)$. Finally, since $(\alpha \times \alpha) \cup (\alpha \times \alpha) = \alpha^2 \times \alpha^2$, therefore $(\alpha \times \alpha) \cup (\alpha \times \alpha) \cup (\alpha \times \alpha) \cup (\alpha \times \alpha) = (\alpha^2 \times \alpha^2) \cup (\alpha^2 \times \alpha^2) = \alpha^4 \times \alpha^4$. Hence $\phi^*(\alpha^4 \times \alpha^4) = \phi^*(\alpha \times \alpha) \cup \phi^*(\alpha \times \alpha) \cup \phi^*(\alpha \times \alpha) \cup \phi^*(\alpha \times \alpha) = \lambda^4(\alpha \times \alpha) \cup (\alpha \times \alpha) \cup (\alpha \times \alpha) \cup (\alpha \times \alpha) = \lambda^4(\alpha^4 \times \alpha^4)$. Therefore it follows that

$$\tilde{L}(\phi; Z_3) = 3\lambda + 3\lambda_1 + 3\lambda_2 + 3\lambda_3 + 3\lambda_4 + \lambda^4 = \lambda^4$$

is either 0 or 1 (since the coefficients are from Z_3).□

5.10. LEMMA. $HP^4 \wedge SHP^3$ has f.p.p. Specifically, for any self-map ϕ, $\tilde{L}(\phi; Z_3) = 0$.

PROOF: The generators of $H^*(HP^4)$ are given by $1, \alpha, P^1\alpha, P^2\alpha, \alpha^4$ and those of $\tilde{H}^*(SHP^3)$ by $S\alpha, SP^1\alpha, SP^2\alpha$. The basis for $H^*(HP^4 \wedge SHP^3)$ can be arranged in positive dimensions as follows:

$$\begin{array}{lll}
\alpha \times S\alpha & P^1\alpha \times S\alpha + \alpha \times SP^1\alpha & P^2\alpha \times S\alpha + P^1\alpha \times SP^1\alpha + \alpha \times SP^2\alpha \\
\alpha \times SP^1\alpha & P^1\alpha \times SP^1\alpha - \alpha \times SP^2\alpha & P^2\alpha \times SP^1\alpha - P^1\alpha \times SP^2\alpha \\
\alpha \times SP^2\alpha & P^1\alpha \times SP^2\alpha & P^2\alpha \times SP^2\alpha \\
\alpha^4 \times S\alpha & \alpha \times SP^1\alpha & \alpha^4 \times SP^2\alpha
\end{array}$$

It can be easily seen that applying P^1 and P^2 to the first column yields the second and third columns. By an argument similar to that in the proof of the preceding lemma it follows that for any self-map ϕ of $HP^4 \wedge SHP^3$ one has $\tilde{L}(\phi;\, Z_3) = 0$ and hence ϕ has a fixed point. □

5.11. LEMMA. $SHP^3 \wedge SHP^3$ has f.p.p. Specifically, for any self-map ϕ, $\tilde{L}(\phi;\, Z_3) = 0$, and thus $L(\phi;\, Z_3) \neq 0$.

PROOF: The basis for Z_3-cohomology of $H^*(SHP^3 \wedge SHP^3)$ can be arranged as follows

$$\begin{array}{lll}
S\alpha \times S\alpha & S\alpha \times SP^1\alpha + SP^1\alpha \times S\alpha & S\alpha \times SP^2\alpha + SP^1\alpha \times SP^1\alpha + S\alpha \times SP^2\alpha \\
S\alpha \times SP^1\alpha & SP^1\alpha \times SP^1\alpha - S\alpha \times SP^2\alpha & SP^2\alpha \times SP^1\alpha - SP^1\alpha \times SP^2\alpha \\
S\alpha \times SP^2\alpha & SP^1\alpha \times SP^2\alpha & SP^2\alpha \times SP^2\alpha
\end{array}$$

We notice that applying P^1 and P^2 to the first column yields the second and third columns. Using the same method as before it follows that if ϕ is a self-map of $SHP^3 \wedge SHP^3$ then $\tilde{L}(\phi;\, Z_3) = 0$ and thus $L(\phi;\, Z_3) \neq 0$. Hence ϕ has a fixed point.

5.12. PROPOSITION. $K \wedge K$ has f.p.p.

PROOF: Let $K' = HP^4 \vee SHP^3$. Using the above lemmas and the same method as in the proof of the Lemma 5.1, one first observes that every self-map ϕ' of

$$K' \wedge K' = (HP^4 \wedge HP^4) \vee (HP^4 \wedge SHP^3) \vee (SHP^3 \wedge HP^4) \vee (SHP^3 \wedge SHP^3)$$

has the property that $\tilde{L}(\phi'; Z_3)$ is 0 or 1 (considering the first of these four spaces as X and the wedge of the other three as Y). Since $K \simeq K'$ and hence $K \wedge K \simeq K' \wedge K'$, every self-map ϕ of $K \wedge K$ has $L(\phi; Z_3) \neq 0$. Thus $K \wedge K$ has the f.p.p.□

5.13. LEMMA. $HP^4 \wedge SHP^2$ has f.p.p. Specifically, for every self-map ϕ, $\tilde{L}(\phi; Z_2) = 0$.

PROOF: We may choose a basis for the Z_2-cohomology of HP^4 and SHP^2, respectively, as follows

$$1, \alpha, Sq^4\alpha, \beta, Sq^4\beta \quad \text{for } H^*(HP^4; Z_2)$$

$$1, u, Sq^4u, \quad \text{for } H^*(SHP^2; Z_2).$$

Then, the basis for the Z_2-cohomology of $HP^4 \wedge SHP^2$ can be arranged as follows

$$\begin{array}{ll} \alpha \times u & \alpha \times Sq^4u + Sq^4\alpha \times u \\ Sq^4\alpha \times u & Sq^4\alpha \times Sq^4u \\ \beta \times u & \beta \times Sq^4u + Sq^4\beta \times u \\ Sq^4\beta \times u & Sq^4\beta \times Sq^4u \end{array},$$

where Sq^4 applied to the first column yields the second column. This suffices to show that for every self-map ϕ, $\tilde{L}(\phi; Z_2) = 0$ (using the same argument as before) and hence $L(\phi;\ Z_2) = 1$. Thus $HP^4 \wedge SHP^2$ has f.p.p. □

5.14. LEMMA. $SHP^3 \wedge SHP^2$ has f.p.p. Specifically, for every self-map ϕ, $\tilde{L}(\phi;\ Z_2) = 0$ and thus $L(\phi;\ Z_2) \neq 0$.

PROOF: We may choose a basis for the Z_2-cohomology of SHP^3 and SHP^2, respectively, as follows

$$\alpha,\ Sq^4\alpha,\ \beta \quad \text{for} \quad H^*(SHP^3;\ Z_2) \quad \text{and}$$
$$u,\ Sq^4 u \quad \text{for} \quad H^*(SHP^2;\ Z_2).$$

Then, we may arrange a basis for the Z_2-cohomology of $SHP^3 \wedge SHP^2$ as follows

$$\begin{array}{ll} \alpha \times u & \alpha \times Sq^4 u + Sq^4\alpha \times u \\ Sq^4\alpha \times u & Sq^4\alpha \times Sq^4 u \\ \beta \times u & \beta \times Sq^4 u, \end{array}$$

where Sq^4 applied to the first column yields the second column. If ϕ is any self-map of $SHP^3 \wedge SHP^2$ then it can again be shown by the same method that $\tilde{L}(\phi;\ Z_2) = 0$ and thus $L(\phi;\ Z_2) = 1$. Hence $SHP^3 \wedge SHP^2$ has the f.p.p. □

5.15. PROPOSITION. $K \wedge N$ has f.p.p.

PROOF: $K \wedge N$ is of the same homotopy type as

$$W = (HP^4 \vee SHP^3) \wedge SHP^2 = (HP^4 \wedge SHP^2) \vee (SHP^3 \wedge SHP^2).$$

By the previous lemmas and the same method as in the proof of Lemma 5.1, it follows that every self-map ϕ' of W has the property that $\tilde{L}(\phi'; Z_2) = 0$ and hence every self-map of $K \wedge N$ has Lefschetz number 1 (over Z_2). Thus, $K \wedge N$ has f.p.p. □

5.16. PROPOSITION. $N \wedge N$ has f.p.p.

PROOF: We know that the basis for the Z_2-cohomology of $N = SHP^2$ has the form $1, u, Sq^4u$. Therefore the basis for the cohomology of $N \wedge N$ can be written as follows

$$\begin{array}{ll} u \times u & Sq^4u \times u + u \times Sq^4u \\ Sq^4u \times u & Sq^4u \times Sq^4u, \end{array}$$

where Sq^4 applied to the first column yields the second column. By the same argument as before it follows that given any self-map ϕ of $N \wedge N$, $\tilde{L}(\phi; Z_2) = 0$ and therefore $L(\phi; Z_2) = 1$. Hence $N \wedge N$ has f.p.p.□

5.17. THEOREM. Using the wedge point v of X, the smash product

$$X \wedge X = X \times X / v \times X \cup X \times v ,$$

has f.p.p.

PROOF: Since we have seen that $K \wedge K$, $K \wedge N$, $N \wedge N$ (and similarly $N \wedge K$) has the f.p.p., this follows from the above mentioned fact that

$$X \wedge X = (K \wedge K) \vee (K \wedge N) \vee (N \wedge K) \vee (N \wedge N)$$

is a wedge of four spaces with f.p.p. and hence has f.p.p., too. □

REMARK: There is a misprint in the proof of Lemma 3.6 in [10] on page 96 line 4. The second basis element should read $\alpha \times Sq^4 u + Sq^4 \alpha \times u$.

We now turn to a special case of a product theorem. Consider the following property:

5.18. PROPERTY F. X is said to have property F if, and only if, $L(f) \neq 0$ for every self-map $f : X \longrightarrow X$.

In terms of this property we recall the following theorem (see [9], [10]).

5.19. THEOREM. If X belongs to S_0, then X has f.p.p. if, and only if, X has property F.

Thus for spaces in S_0, the question of the invariance of f.p.p. under cartesian products is equivalent to the question

(1) X and Y have property F $\overset{?}{\Longrightarrow}$ $X \times Y$ has property F?

Theorem 5.20 answers (1) in the affirmative under quite special conditions on the spaces X and Y. In the following we will use singular cohomology with coefficients in the field of rational numbers.

5.20. THEOREM. Suppose X and Y are spaces having property F. Suppose further that X has trivial cup products and X and Y have disjoint cohomology, i.e., $H^p(X) \neq 0$, $H^q(Y) \neq 0$, $p, q \geq 1$, implies $p \neq q$. Then $X \times Y$ has property F.

Before proving the theorem we need the following lemma which states

5.21. LEMMA. Suppose $\psi : X \times Y \longrightarrow X \times Y$ is a map and $\psi_0 : Y \longrightarrow Y$ is defined by the diagram

$$\begin{array}{ccc} X \times Y & \xrightarrow{\ \psi\ } & X \times Y \\ {\scriptstyle i_2}\uparrow & & \downarrow{\scriptstyle p_2} \\ Y & \xrightarrow{\ \psi_0\ } & Y \end{array}$$

where i_2 is the inclusion given by $i_2(y) = (x_0, y)$, $x_0 \in X$ and p_2 is a projection on the second factor Y. Then, for $v \in H^n(Y)$

$$\psi^*(1 \times v) = 1 \times \psi_0^*(v) + E(v)$$

where $E(v)$ is a linear combination of terms of the form $a \times b$ where $\dim a \geq 1$.

PROOF: Let $p_1 : X \times Y \longrightarrow X$ and $p_2 : X \times Y \longrightarrow Y$ be the projections. By Corollary 14 [19], p.253, one has $1 \times v = (P_1^* 1) \cup (P_2^* v)$. Therefore $\psi^*(1 \times v) = (\psi^* P_1^* 1) \cup (\psi^* P_2^* v) = \psi^* P_2^* v$. If $i_1 : X \longrightarrow X \times Y$ and $i_2 : Y \longrightarrow X \times Y$ are inclusions then $i_2^* \psi^*(1 \times v) = i_2^* \psi^* P_2^* v = \psi_0^* v$ (since $i_2^* \psi^* P_2^* = \psi_0^*$). But $i_2^* : H^n(X \times Y) \longrightarrow H^n(Y) = H^0(X) \otimes H^n(Y)$ is the projection on to the direct summand (see [12], Exercise 29.27, p.208). Hence $\psi^*(1 \times v) = 1 \times \psi_0^* v + E(v)$, where $E(v)$ is a linear combination of terms of the form $a \times b$ where $\dim a \geq 1$. □

PROOF (of 5.20): Let $\phi : X \times Y \longrightarrow X \times Y$ be an arbitrary map and let f and g be defined by the diagrams

$$\begin{array}{ccc} X \times Y & \xrightarrow{\phi} & X \times Y \\ {\scriptstyle i_1}\uparrow & & \downarrow{\scriptstyle p_1} \\ X & \xrightarrow{f} & X \end{array}$$

$$\begin{array}{ccc} X \times Y & \xrightarrow{\phi} & X \times Y \\ {\scriptstyle i_2}\uparrow & & \downarrow{\scriptstyle p_2} \\ Y & \xrightarrow{g} & Y \end{array}$$

where i_1 and i_2 are inclusions and p_1 and p_2 are projections (as in Lemma 5.21). Let $1 = u_1, \ldots, u_k$ and $1 = v_1, \ldots, v_h$ be the bases for the cohomology of X and Y, respectively, where the coefficients are taken from the field of rational numbers. Therefore, elements of the form $u_i \times v_j$ form a basis for the cohomology of $X \times Y$. If u and v are typical basis elements, then from Lemma 5.21, it follows that

$$\phi^*(u \times 1) = f^*(u) \times 1 + E(u)$$

$$\phi^*(1 \times v) = 1 \times g^*(v) + E(v)$$

where $E(u)$ is a linear combination of terms of the form $a \times b$ with $\dim b \geq 1$ and $E(v)$ is a linear combination of terms of the form $a' \times b'$, $\dim a' \geq 1$. Suppose $\dim u = m$ and $\dim v = n$. Then

$$\phi^*(u \times v) = \phi^*(u \times 1 \cup 1 \times v) = \phi^*(u \times 1)\phi^*(1 \times v)$$

$$= (f^*(u) \times 1 + E(u))(1 \times g^*(v) + E(v))$$

$= (f^*(u) \times 1)(1 \times g^*(v)) + E(u)(1 \times g^*(v)) + (f^*(u) \times 1)E(v) + E(u)E(v)$

$= f^*(u)g^*(v) + E(u)(1 \times g^*(v)) + (f^*(u) \times 1)E(v) + E(u)E(v)$

(since $f^*(u) \times 1 \cup 1 \times g^*(v) = f^*(u) g^*(v)$). Now $E(u)$ is a linear combination of terms of the form $a \times b$ where $\dim a \le m-1$ so that $u \times v$ cannot appear in the term $E(u)(1 \times g^*(v))$. Similarly $u \times v$, cannot appear in the term $(f^*(u) \times 1)E(v)$. In $E(u)E(v)$ a typical term has the form

$$(a \times b)(a' \times b') = \pm aa' \times bb' \qquad (2)$$

where $\dim a \le m-1$, $\dim b \ge 1$, $\dim a' \ge 1$ and $\dim b' \le n-1$. If $\dim a \ge 1$, $aa' = 0$ (since X has trivial cup products) so that (2) is zero. On the other hand if $\dim a = 0$ then $\dim b = m$. Since $\dim u = m$ therefore $H^m(X) \neq 0$ implies $H^m(Y) = 0$ implies $b = 0$ and hence (2) is zero in this case, too. Thus $E(u)E(v) = 0$. Hence it follows that $\phi^*(u \times v)$ and $(f \times g)^*(u \times v)$ have the same coefficients of $u \times v$. Therefore, $L(f \times g) = L(f)L(g) = L(\phi) \neq 0$. Thus, $X \times Y$ has property F.□

Therefore we have the following consequence:

5.22. THEOREM. Suppose X and Y belong to S_0 and have f.p.p. Then $X \times Y$ has f.p.p. if either X or Y has trivial rational cup products and X and Y have disjoint rational cohomology.

As an specific example of this Theorem 5.22 consider the case where $X = CP^i$, $Y = SCP^j$ for i and j even, $i, j \ge 2$. Because the hypotheses of the Theorem 5.22 are satisfied, and since X and Y have the f.p.p., It does follow here that $X \times Y = CP^i \times SCP^j$ has the f.p.p., too.

In this context Fadell raises the following question:

5.23. QUESTION. If $SX \times Y$ has f.p.p., does this imply that $X \times Y$ has f.p.p.?

He observes that an affirmative answer to this question would also settle the following conjecture.

5.24. CONJECTURE. Suppose X and Y belong to S_0 and X and all its suspensions have f.p.p. Then if Y has f.p.p., so does $X \times Y$.

Although this conjecture still seems to be open in general, there is the following somewhat related result due to Thomeier [20]:

5.24. THEOREM. Suppose X and Y belong to class S (i.e. are polyhedra satisfying the Shi condition), and that the suspensions SX and SY and also the join product $X * Y$ all have the f.p.p. Then also $X \times Y$ and its suspension $S(X \times Y)$ have the f.p.p.

It is interesting to observe that this theorem holds in the more general setting of the category S (i.e. there are no assumptions on the simply-connectedness of the polyhedra).

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