SCATTERING OF HIGH FREQUENCY ELECTROMAGNETIC WAVES FROM AN OCEAN SURFACE: AN ALTERNATIVE APPROACH INCORPORATING A DIPOLE SOURCE

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By

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ABSTRACT

A theoretical analysis of the scattering of High-Frequency (HF) electromagnetic waves from a rough surface such as the ocean surface is proposed. This is required for an interpretation of the radar signature when using an HF radar for the remote sensing of ocean surface parameters. The analysis is based on Walsh's (1980b) formulation in the spatial Fourier transform domain for the scattering from a general time invariant surface. Initially, a two dimensional spatially periodic surface with a high refractive index is considered. For this surface, a series solution is derived for the surface electric field in the spatial transform domain maintaining the choice of any finite source. The source then considered is an elementary vertical dipole excited by a pulsed sinusoidal current. It is assumed that the surface slopes are small compared to unity. For this source, zero, first, and second order approximations of the vertical component of the surface field are inverse spatially transformed in an asymptotic sense. These solutions are in the form of ground waves with modified surface impedances. By assuming a narrow beam receiving antenna, the inverse transforms for the first and second order solutions that involve spatial convolution integrals, are evaluated asymptotically for the two orders of the backscattered surface field.

By modelling the ocean surface as a three dimensional periodic surface in space and time with random Fourier coefficients, the above two orders of the backscattered field are suitably modified to include the time dependency and the statistical variation of the surface. Based on this model an average first and second order backscattered Doppler spectra and consequently the two orders of the cross section are derived.

The first order cross section is the same as that derived by Barrick (1972b, 1977a) using the Rice perturbation technique. The second order result contains three parts. The first part is almost the same as that obtained by Barrick. The two additional parts result from surface and incident field interaction along the path from transmitter to the primary ocean scattering patch and off-patch double scatter respectively. The effect of the former may be very pronounced, particularly at the higher Doppler frequencies when the radar is situated in the middle of the ocean.

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LIST OF SYMBOLS

A_p	:	Area of the surface patch
A_r	:	Effective aperture of the receiving antenna
$\mathcal{C}_a$	:	Dipole constant in the time domain $= -j\omega_o\mu_o I_o dl$
C_d	:	Dipole constant in the temporal Fourier transform domain $= -j\omega\mu_o I dl$
c	:	Radio propagation velocity $\approx 3 \times 10^8$ m/sec
det A	:	Determinant of matrix A
dl	:	Length of the elementary dipole source
$\vec{E}_a$	:	Electric field intensity immediately above the surface (surface field) in the phasor form or in the temporal Fourier transform domain $= E_{ax}\hat{x} + E_{ay}\hat{y} + E_{az}\hat{z}$
$\underline{E_{az}}$	:	Spatial Fourier transform of E_{az}
E_{azo}, E_{az1}, E_{az2}	:	Zero, first and second orders of E_{az}
$\vec{E}_i$	:	Incident or source electric field intensity in the phasor form or in the temporal Fourier transform domain $= E_{ix}\hat{x} + E_{iy}\hat{y} + E_{iz}\hat{z}$
$\underline{\vec{E}_i}$	:	Spatial Fourier transform of $\vec{E}_i$ in a plane $z = z^- < f(\vec{x})$
$\vec{E}_s$	:	Scattered electric field intensity above the surface in the phasor form or in the temporal Fourier transform domain $= E_{sx}\hat{x} + E_{sy}\hat{y} + E_{sz}\hat{z}$
$\underline{\vec{E}_s}$	:	Spatial Fourier transform of $\vec{E}_s$ in a plane $z = z^+ > f(\vec{x})$
E_{zb1}, E_{zb2}	:	First and second orders of the vertical component of the backscattered surface field in the temporal Fourier transform domain
$\tilde{E}_{zb1}$, $\tilde{E}_{zb2}$	:	First and second orders of the vertical component of the backscattered surface field in the time domain.

$\tilde{E}_{zb21}$, $\tilde{E}_{zb22}$, $\tilde{E}_{zb23}$	:	Three parts of $\tilde{E}_{zb2}$
erf(z)	:	Error function
erfc(z)	:	Complementary error function = 1 - erf(z)
$F(\tilde{p}_a)$, F_a, F_b, F_c	:	Ground wave attenuation functions
F_p	:	One way ground wave attenuation function between the radar and the surface patch
$f(\vec{x})$	:	Non-time varying rough surface
$f(\vec{x},t)$	:	Time varying rough surface
f_x, f_y	:	x and y slopes of the surface
G(r)	:	Free space Green's function
$G_{\tau_o}(t)$	:	Gate function of width τ_o
g	:	Acceleration due to gravity
g_r, g_t	:	Free space gains of the receiving and transmitting antennas in the direction of surface patch
$h_e(x)$	:	Heaviside function
h_o	:	Height of the dipole source above the surface
I	:	Temporal Fourier transform of $\tilde{I}$
$\tilde{I}$	:	Dipole current in the time domain
I_o	:	Peak value of the dipole current
$\vec{J}_s$	:	Source current density
j	=	$(-1)^{\frac{1}{2}}$
$\vec{K}$	:	Spatial Fourier transform variable or wave number vector = $K_x\hat{x} + K_y\hat{y}$
K	:	Magnitude of $\vec{K}$
k_o	:	Incident radio wave number = $\omega_o(\mu_o\epsilon_o)^{\frac{1}{2}}$

L : Spatial period of the surface

N : Fundamental wave number of the surface = $2\pi/L$

$\vec{n}$: Normal vector to the surface in the direction of increasing z

n_o : Refractive index of the lower medium = $\left(\epsilon_r - j\frac{\sigma_o}{\omega\epsilon_o}\right)^{\frac{1}{2}}$

$P_{m,n}$: Fourier coefficient of the two dimensional periodic surface

$P_{m,n,l}$: Fourier coefficient of the three dimensional periodic surface (ocean surface)

${}_1P_{m,n,l}$, ${}_2P_{m,n,l}$: First and second orders of $P_{m,n,l}$

P_r : Average power density spectrum (Doppler spectrum) of the backscattered signal

P_r^o : Average power of the backscattered signal

P_{rp} : Average backscattered Doppler spectrum received from the surface patch

P_{rp}^o : Average backscattered power received from the surface patch

P_t : Average transmitted power

p_a : Numerical distance containing a modified surface impedance

p_{am} : Numerical distance containing an average modified surface impedance (average modified numerical distance)

$R(\vec{\tau}_x, \tau_t)$: Three dimensional autocorrelation function of the ocean surface

$R_r(\tau)$: Autocorrelation function of the backscattered signal

$S(\vec{K},\omega)$: Three dimensional wave height spectrum

$S_1(\vec{K},\omega)$: First order of $S(\vec{K}, \omega)$

$S_1^+(\vec{K})$, $S_1^-(\vec{K})$	:	Two dimensional (spatial) wave height spectrum of the first order gravity waves moving with +/- velocity components in the x direction
$S_1(\omega)$	:	One dimensional (frequency) wave height spectrum of the first order gravity waves
$Sa(x)$	:	Sampling function
$S_t(\omega,\theta)$	:	Directional-frequency wave height spectrum of the first order gravity waves
$sgn(x)$	:	Sign function
T	:	Temporal period of the time varying surface
t_o	:	Time delay between the transmitted and received signals
U	:	Wind speed measured at anemometer height
W	:	Fundamental frequency of the time varying surface $= 2\pi/T$
$\vec{x}$		$= x\hat{x} + y\hat{y}$
$\hat{x},\hat{y},\hat{z}$	:	Unit vectors along the positive x,y, and z axes
Δ	:	Surface impedance normalized to η
Δ_a, Δ_b, Δ_c, Δ_m	:	Modified surface impedances normalized to η
Δ_{am}	:	Average modified surface impedance normalized to η
$2\Delta_\theta$	:	Angular width of the surface patch
$2\Delta_\rho$	:	Radial width of the surface patch
$\delta(x)$	:	Dirac delta function
ε_o	:	Permittivity of free space
ε_r	:	Relative permittivity of the lower medium
η	:	Intrinsic impedance of free space
λ_o	:	Incident radio wavelength
μ_o	:	Permeability of free space

ν	:	Doppler frequency normalized to the Bragg frequency
ρ_o	:	Radial distance of the surface patch from the source
σ_o	:	Conductivity of the lower medium
σ_{ff}	:	First order backscattered Doppler frequency dependent cross section of the ocean surface patch normalized to A_p
σ_{s11}, σ_{s22}, σ_{s33}	:	Three parts of the second order backscattered Doppler frequency dependent cross section of the ocean surface patch normalized to A_p
ω	:	Temporal Fourier transform variable
ω_B	:	Bragg frequency = $(2gk_o)^{\frac{1}{2}}$
ω_d	:	Doppler frequency = $\omega - \omega_o$
ω_o	:	Incident radio frequency
$\vec{\nabla}$	:	Del operator
$\langle \cdot \rangle$	:	Ensemble average
$\sum_{(m,n)\neq(p,q)}$	:	Double summation over the integars m and n from $-\infty$ to ∞ with a restriction on the summations that both m and n can not be simultaneously equal to p and q respectively

Other symbols or functions are defined as they occur and should be interpreted within the proper context.

CHAPTER 1

INTRODUCTION

1.1 General

With the increasing activity in the ocean environment, the need to quickly measure ocean surface parameters over large areas has arisen. The relevant surface parameters include spatial or directional wave height spectrum, surface wind condition and water current. This information is required by a wide variety of users: in navigation, fishing, offshore petroleum operations, environmental control, and search and rescue operations. Conventional oceanographic techniques for making these measurements are available, such as by using buoys, pressure transducers, drifters, current meters etc. However these techniques are slow, have a small area of coverage, and are quite expensive. Moreover, poor sea conditions sometimes makes the measurements very difficult.

An alternative to oceanographic techniques is the remote sensing of ocean surface parameters by radar. Microwave radars may be used for these measurements but they have very limited detection range with exception to aircraft or satellite borne radars. Also, at microwave frequencies (> 1 GHz) ocean surface waves interacting with radio waves are of small wavelengths, mainly ultragravity and capillary waves (wavelengths < 15 cm). Whereas the importance lies in detection of long ocean waves (gravity waves) which is possible by using High Frequency, abbreviated as HF(3 - 30 MHz), radars. The detection range for HF radars is relatively much higher (beyond the horizon). In the last two decades significant advances have been made for HF radars as a new remote sensor for ocean surface parameters. This powerful technique offers an opportunity for making the necessary measurements over a large area in a shorter time. There are two types of HF radars depending upon the mode of radio propagation, namely ground wave and sky wave radars. But we will consider only ground wave radar.

As the ocean waves are moving, they induce different Doppler frequencies to the incident radio wave. As a result, the return from the ocean surface consists of a band of frequencies in the form of a power density spectrum or the Doppler spectrum. Thus the return spectrum carries the relevant information about the ocean surface conditions. A major task lies in the interpretation of the return so that the required information may be extracted. The return varies with changes in the radar frequency, antenna system (wide beam or narrow beam and polarization), mode of operation (monostatic (backscattered) or bistatic) for the same ocean conditions. These complications demand an understanding of the interaction process occuring in the scattering of radio waves from the ocean surface. This understanding would be simpler if the ocean surface was deterministic. Unfortunately, it falls under the category of a random rough surface.

The Rice (1951) perturbation technique has been used by many investigators for dealing with problems of scatterings from random and slightly rough surfaces. In an alternative approach, Walsh (1980b) has proposed a formulation for the scattering from a general time invariant rough surface. This formulation is open to any finite source unlike the perturbation where only plane wave incidence is used. The scattering analysis for an ocean surface carried out in this thesis is based on Walsh's formulation. Initially, a non-time varying two dimensional periodic surface is considered. For analytical convenience it is assumed that the x,y slopes of the surface are small compared to unity. The source is taken to be an elementary vertical dipole excited by a pulsed sinusoidal current. Assuming a narrow beam receiving antenna, expressions for the vertical component of first and second orders of the backscattered field are derived. The expressions appear in the form of ground waves [Norton (1937)] with modified surface impedances [see section 3.3].

The ocean surface ismodelled as a three dimensional periodic surface in space and time with random Fourier coefficients. The previous results for the backscattered field are then applied to this model after making a suitable modification to include the time dependency of the model. Based on this model an average first order and second order backscattered Doppler spectra and consequently the two orders of the Doppler frequency dependent cross section have been derived. The Doppler spectrum or the cross section relate to the spatial (directional) wave height spectrum of gravity waves in deep water. This relationship thus serves as a link between the radar return and ocean surface conditions, when using an HF radar in a monostatic configuration.

1.2 Literature Review

Electromagnetic scattering from rough surfaces has been treated by many investigators using classical methods. These methods are physical and geometrical optics (tangent plane method) and perturbation. These procedures may be applied to a random rough surface in a statistical average sense. An excellent review of these techniques as applied to the scattering from the ocean surface is presented by Valenzuela (1978). Proceeding in a different manner Walsh (1980b) has presented a general formulation for scattering from a rough surface. We shall now briefly discuss these techniques and mention some of the other approaches as well.

1.2.1 Physical and Geometrical Optics Technique

In this approach the scattered field is obtained from the Stratton-Chu [Stratton (1941, ch. 8)] integral equation by using a tangent plane approximation for the surface field. That is, the surface fields are approximated at each point on the surface by considering the surface to be an infinite tangent plane at that point. The result then obtained is known as the physical optics or the Kirchhoff solution. The method is described in detail by Beckmann and Spizzichino (1963). The incident field is usually taken to be plane wave. The tangent plane approximation is limited to those surfaces whose principal radii of curvature are much greater than the incident radio wavelength. In the case of an ocean surface the requirement is met at UHF or higher radio frequencies. Also, the method neglects multiple scattering between different points of the surface.

As the frequency increases or the wavelength decreases the approximation improves and, in the limit, the method approaches geometrical optics or ray theory. Accordingly, the scattering reduces to the specular reflection and so the backscattered field is received from those surface facets which are normal to the direction of the incident radiation. By using the physical optics technique and the stationary phase method for integration, Kodis (1966) derived the radar cross section for a time invariant perfectly conducting rough surface at very short radio wavelengths. The case of a finitely conducting surface was treated by Barrick (1968) [also, Barrick and Bahar (1981)] and cross section derived. Because of the curvature and multiple scattering limitations the technique is not suitable for analyzing HF scattering from a rough surface such as an ocean surface.

1.2.2 Perturbation Technique

In 1896 Lord Rayleigh (1945, ch. 11) introduced a perturbation technique to study the acoustic reflection from a sinusoidal surface. An application of the technique to electromagnetic field problems was first made by Rice (1951) in finding the scattered field from a non-time varying slightly rough surface. In this approach the surface is assumed to be periodic in two dimensions and the incident field is taken as a plane wave. The solution for the scattered electric field components are then written as a sum of discrete angular spectra, or modes, of plane waves with unknown coefficients. The three components of the scattered field and thus the corresponding coefficients for a fixed mode are related by the divergence relation $\vec{\nabla} \cdot \vec{E} = 0$.

The method proceeds by expanding the unknown coefficients in increasing perturbational orders. Assuming the order of perturbation is the same as the order of smallness of the surface parameters (height and slope), the first order coefficients are derived from the boundary condition. Hence, the first order solution for the scattered field is obtained. The property that the tangential electric field should vanish for a perfect conductor is the boundary condition assumed by Rice. By repeating this process and using the results of previous orders, higher order coefficients and consequently higher orders of the solution are obtained. Of course, the complexity increases along with the order. The general restrictions in the technique are that the surface height variation should be small compared to the radio wavelength and the x,y slopes should be small compared to unity.

Rice obtained all the three scattered field components for both horizontal and vertical polarizations up to the second order. Furthermore, he derived the first order solution for horizontal polarization when the surface was taken to be finitely conducting. The results were also extended to a random rough surface by treating the Fourier coefficients of the surface as random variables. Thus the first and second statistical moments of the scattered field were derived in the form of a "roughness spectrum". The roughness spectrum is similar to the spatial wave height spectrum of the ocean surface.

The above perturbation technique has been extended by other investigators for finitely conducting surfaces and vertical polarization. Moreover, the derivations are relatively simplified by using a surface impedance boundary condition. But then it is limited to the surfaces with low skin depths. This boundary condition has been successfully used before in other electromagnetic problems, e.g., ground wave propagation over flat earth [Wait (1964)]. By using this concept, Wait (1971) derived the scattering results as well as the modified surface impedance for specular reflection up to the second order for a vertically polarized plane wave with arbitrary angle of incidence. He assumed the surface to be varying in one dimension with a periodic variation. By using a surface wave formulation in the perturbation technique, Barrick (1971) derived an expression for the modified surface impedance for ground wave propagation across the rough sea. A general perturbation solution for the scattered field for all orders was later given by Rosich and Wait (1977) for the case of vertical polarization and a finitely conducting surface with periodic variations in one dimension.

Peake (1959) appears to be the first to use the Rice perturbation theory for radar applications by deriving an average first order cross section for a random rough surface such as a terrain. Valenzuela (1967), in a similar manner, obtained the second order cross section also. From Crombie's (1955) pioneering experiment on the radar return from the ocean at 13.56 MHz, it is evident that HF radars are suitable for the detection of long ocean waves (gravity waves). Based on measurements of the backscattered Doppler spectrum, he found that the two peaks in the spectrum were caused by two ocean waves, one moving toward and the other moving away from the radar, and each having a wavelength equal to one-half the radar wavelength. Further, at HF the restrictions of the perturbation analysis are met in the case of an ocean surface under moderate sea conditions. These positive results led Barrick (1970, 1972a,b) to extend the perturbation technique for finding the Doppler frequency dependent cross section of the ocean surface. For this he included the third dimensional variation in time in the rough surface model of Rice. Thus he derived the first and second orders of the cross section in terms of the spatial wave height spectrum of gravity waves and confirmed the experimental observation made by Crombie. Similar results were also derived by Johnstone (1975) using again the above perturbation technique.

1.2.3 Walsh's Technique

Walsh (1980b) has proposed a general formulation to deal with the problem of scattering from a non-time varying rough surface. Since his formulation forms the basis of the thesis, it is worthwhile to mention his procedure in a relatively detailed manner. A complete description of the method including mathematical details is given in chapter 4 of the above reference.

The method is based on the treatment of both the physical geometry of the space and the Maxwell equations in a generalized functional sense. This is an extension of the procedure previously developed for the analysis of linear antenna system [Walsh and Srivastava (1980a)]. The model assumes that the two homogeneous and isotropic media are separated by a surface $z = f(x,y)$ for all x,y in a Cartesian coordinate system with the z axis pointing upward. The medium above is taken to be free space with permeability μ_o and permittivity ε_o. The medium below has conductivity σ_o, permeability μ_o, and relative permittivity ε_r. The surface is assumed to be the upper boundary for the lower medium. The electrical properties of the entire physical space are described with the aid of a Heaviside function located at the surface. The displacement and conduction current densities are thus described in terms of the electric field for the entire space.

The first two Maxwell equations (in point form) are reduced to the time harmonic (phasor) form either by assuming a time dependency of exp $(j\omega_o t)$, where ω_o is the radio frequency, or better by Fourier transforming with respect to time, using ω as the transform variable. Using the two equations a partial differential equation is then derived for the complete space electric field. In this derivation no gauge conditions are invoked. However, the forcing function of the equation includes an arbitrary finite current source, located in the upper half space, as well as the field immediately above the surface. This field may be considered as a secondary source. The partial differential equation takes the form

$$\nabla^2 \vec{E} + \gamma_o^2 \vec{E} = \frac{n_o^2 - 1}{n_o^2} \vec{\nabla} [\hat{n} \cdot \vec{E}_a \, \delta(z - f)] - T_{sE}(\vec{J}_s) \quad (1.1)$$

where

$f = f(x,y) = f(\vec{x})$

$\gamma_o^2 = k^2 [n_o^2 (1 - h_e) + h_e]$

$k = \omega(\mu_o \varepsilon_o)^{\frac{1}{2}}$

$\vec{x} = \hat{x}x + \hat{y}y$

$\vec{\nabla}$: del operator $= \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$

n_o: refractive index of the lower medium

$$= (\varepsilon_r - j \frac{\sigma_o}{\omega \varepsilon_o})^{\frac{1}{2}} \text{ , with } j = \sqrt{-1} \quad (1.2)$$

h_e: Heaviside function $= \begin{cases} 0 \ , & \text{for } z \leq f \\ 1 \ , & \text{for } z > f \end{cases}$

$\vec{n}$: normal vector to the surface in the direction of increasing z

$$= - \frac{\partial f}{\partial x} \hat{x} - \frac{\partial f}{\partial y} \hat{y} + \hat{z}$$

$\delta(x)$: Dirac delta function of argument x

T_{sE} : electrical source operator $= \frac{1}{j\omega\varepsilon_o} [\vec{\nabla}(\vec{\nabla} \cdot \ \ ) + k^2]$

$\vec{J}_s$ is the source current density. $\vec{E}$ denotes the electric field for the complete space, whereas $\vec{E}_a$ means the field immediately above the surface, i.e.,

$$\vec{E}_a = \lim_{z \to f_+} \vec{E}(x,y,z) = \vec{E}_a(x,y) \quad (1.3)$$

The symbol ($\hat{}$) as a superscript means a unit vector.

The above equation (1.1) may be either interpreted in the phasor form with $\omega = \omega_o$ or in the temporal Fourier transform form with ω as the transform variable. For the latter, the transform pair to be taken is

$$\vec{E}(x,y,z,\omega) = \int_t \tilde{\vec{E}}(x,y,z,t) \exp(-j\omega t)dt$$
$$\tilde{\vec{E}}(x,y,z,t) = \frac{1}{2\pi} \int_\omega \vec{E}(x,y,z,\omega) \exp(j\omega t)d\omega, \quad (1.4)$$

where the superscript ($\tilde{}$) denotes the time varying form and the symbol $\int_x$ means the integration with respect to x from $-\infty$ to ∞.

To arrive at a solution of the partial differential equation the method chosen is to decompose $\vec{E}$ into two parts above and below the surface. This procedure has the effect of splitting the equation into three sub-equations. Two of these equations represent partial differential equations for the fields above and below the surface, while the third gives a set of boundary conditions which the field must satisfy at the surface. It is thus seen that this technique provides its own boundary conditions directly from the main equation (1.1). These do not have to be supplied externally. By using a Green's function approach the solutions for the fields above and below the surface are derived. These solutions are in the form of convolutions of $\vec{E}_a$, $\vec{E}_b$, and their derivatives with respect to normal to the surface with the proper Green's functions. $\vec{E}_a$ and $\vec{E}_b$ refer to the fields immediately above and immediately below the surface respectively.

The scattered field above the surface may be determined from a solution of $\vec{E}_a$ and its derivative with respect to the normal to the surface. To develop integral equations for these unknowns a horizontal plane above the surface is chosen in the previously obtained equation for the field below the surface. Similarly, a horizontal plane below the surface is chosen in the equation for the field above the surface. The process is facilitated by taking the spatial Fourier transform of the equations with respect to x and y with K_x and K_y as the transform variables. Thus a set of two vector integral equations is obtained which should be solved simultaneously to determine the required unknowns. To simplify the simultaneous solution, one of the two integral equations is solved approximately and the solution is inserted in the second. The approximation used is based on the assumption that the functions in that particular integral equation are spatially band limited and the bandwidth is small compared to $|kn_o|$.

The second integral equation now to be solved is given as follows:

$$\int_x \int_y [|\vec{n}|^2 \vec{E}_a(\vec{x}) - \frac{\vec{R}_a(\vec{x})}{u}] \exp\{-uf(\vec{x}) - j\vec{K}\cdot\vec{x}\}\, dxdy$$

$$= 2\,\underline{\vec{E}_i}(\vec{K})\, \exp(-z^- u) \qquad (1.5)$$

with

$$\vec{R}_a(\vec{x}) = -\frac{(n_o^2-1)}{n_o^2}\vec{\nabla}_{xy}(\vec{n}\cdot\vec{E}_a) - jkn_o[|\vec{n}|^2\vec{E}_a$$

$$-\frac{(n_o^2-1)}{n_o^2}(\vec{n}\cdot\vec{E}_a)\vec{n}]$$

In the above $\vec{E}_i$ is the free space incident or source field spatially Fourier transformed in a plane $z = z^- < f(\vec{x})$. The incident field is derived from the current source by the relation

$$\vec{E}_i = T_{sE}(\vec{J}_s) * G = \int_{x'}\int_{y'}\int_{z'}\{T_{sE}[\vec{J}_s(\vec{x}', z')]\}G(\vec{x}-\vec{x}', z-z')dx'dy'dz' \qquad (1.6)$$

where the asterisk (*) means a three dimensional convolution with respect to x,y, and z. G is the Green's function for free space given by $G = \exp(-jkr)/(4\pi r)$, $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ (1.7)

The other terms in (1.5) are

$$\vec{K} = K_x\hat{x} + K_y\hat{y}$$

$$\vec{\nabla}_{xy} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} \qquad (1.8)$$

$$u = \begin{cases}(K^2 - k^2)^{\frac{1}{2}} & \text{for real root}\\ j(k^2 - K^2)^{\frac{1}{2}} & \text{for imaginary root}\end{cases}$$

$$K = |\vec{K}| = (K_x^2 + K_y^2)^{\frac{1}{2}}$$

A bar under a function denotes its spatial Fourier transform with respect to x and y with K_x and K_y as the transform variables. The transform pair used is

$$\underline{g}(\vec{K}) = \int_x \int_y g(\vec{x}) \exp(-j\vec{K} \cdot \vec{x}) dxdy$$

$$g(\vec{x}) = \frac{1}{4\pi^2} \int_{K_x} \int_{K_y} \underline{g}(\vec{K}) \exp(j\vec{K} \cdot \vec{x}) dK_x dK_y \tag{1.9}$$

In order to solve for the scattered field above the surface an inversion of (1.5) is required for finding the unknowns. The unknowns are the surface field $(\vec{E}_a)$ and $\vec{\nabla}_{xy}(\vec{n} \cdot \vec{E}_a)$. The equation for the scattered field is given by the integral;

$$\underline{\vec{E}_s}(\vec{K}) = \frac{1}{2} \exp(-z^+ u) \int_x \int_y [|n|^2 \vec{E}_a(\vec{x}) + \frac{\vec{R}_a(\vec{x})}{u}]$$

$$\cdot \exp\{uf(\vec{x}) - j\vec{K} \cdot \vec{x}\} dxdy \tag{1.10}$$

where $\underline{\vec{E}_s}$ is the scattered field spatially Fourier transformed in a plane $z = z^+ > f(\vec{x})$. It may be added here that when the expression for $\vec{R}_a(\vec{x})$ is inserted in (1.5) and (1.10) and a suitable integration by parts is carried out, the two equations (1.5) and (1.10) reduce to the following form respectively. In doing so it is assumed that the surface field $(\vec{E}_a)$ vanishes at infinity. The unknown is now $\vec{E}_a$ only.

$$\int_x \int_y \Bigg\{ (u + jkn_o) \; |\vec{n}|^2 \vec{E}_a(\vec{x}) + \frac{(n_o^2 - 1)}{n_o^2} (\vec{n} \cdot \vec{E}_a(\vec{x}))$$

$$\cdot [j\vec{K} - jkn_o\vec{n} + u\, \vec{\nabla}_{xy} f(\vec{x})] \Bigg\} \exp\{-uf(\vec{x}) - j\vec{K} \cdot \vec{x}\} dxdy$$

$$= 2u\,\vec{E}_i(\vec{K})\exp(-z^- u) \quad (1.11)$$

and

$$\vec{E}_s(\vec{K}) = \frac{1}{2u}\exp(-z^+ u)$$

$$\int_x\int_y \left\{ (u - jkn_o)\,|\vec{n}|^2\,\vec{E}_a(\vec{x}) - \frac{(n_o^2 - 1)}{n_o^2}(\vec{n}\cdot\vec{E}_a(\vec{x}))\right.$$

$$\left.[j\vec{K} - jkn_o\vec{n} - u\,\vec{\nabla}_{xy}\,f(\vec{x})]\right\}$$

$$\cdot\exp\{uf(\vec{x}) - j\vec{K}\cdot\vec{x}\}dxdy \quad (1.12)$$

where the various terms have been defined previously by (1.2) and (1.8).

It may readily be seen that Walsh's formulation accepts the field from any finite source, e.g., a dipole source. On the contrary, the Rice (1951) perturbation technique is based on plane wave incidence. The choice for the surface is arbitrary, i.e., no limitations on the height and slope variations are there. Further, the lower medium may have a finite or infinite (perfect) conductivity. Of course, the problems of finding a solution for the surface field and the inverse spatial Fourier transforms still exist. These difficulties may be eased somewhat for a deterministic surface. However, for an ocean surface this is not the case. In spite of these problems this formulation is felt preferable and is used in the thesis.

1.2.4 Other Techniques

It is now worthwhile to mention some of the other approaches. Mitzner (1964) has developed a general perturbation formulation applied to surfaces with small irregularities (small height and slope) and with or without finite boundaries. The approach taken by him is that of the dyadic Green's function to calculate the scattered field. The surface irregularities are transfered to the surface current in a perturbational order through the boundary condition so that a proper dyadic Green's function may be used for the underlying smooth surface. The problem remains with finding this function and the integral evaluation. The choice for the source is kept arbitrary. He has given the formulation up to the second order and the dyadic Green's functions for some special cases including the half space problem with which we are dealing.

A common technique to deal with those surfaces, where a small scale roughness (small slope and height variations) is riding over a gently undulating (small slope) surface but with large height variation, is to use a composite model of the physical optics and perturbation. That is, the surface may be decomposed into two sub-surfaces so that a perturbation technique may be applied to the surface with small roughness and the physical optics may be applied to the other with large height variation [Brown (1978)]. However, in some situations this decomposition may not be possible in the sense that requirements of the individual technique may not be met. To this end an answer has been provided by Bahar and Barrick (1983), using a technique developed by Bahar (1980), where the above decomposition is not necessary. In Bahar's method the problem is treated in general with exact boundary conditions. The total field is written as the sum of three components namely radiation, lateral, and surface wave terms and thus the technique is called "full wave analysis". The source taken is an infinite line source and the concept of a generalized Fourier transform is used.

The formulation is in a form of coupled differential equations. One method of solution is by an iteration and under appropriate limits, the results are shown to agree with those found by the physical optics and perturbation.

The case of a finite source in an application to the rough surface scattering problem is treated by Wait (1966). By using a vertical dipole source he studied the backscatter of HF ground wave signal from sea waves. In his work an integral formulation is used to find the change in the self-impedance of the dipole due to a slightly rough surface from which the reflection coefficient is derived. It is shown there that this coefficient is maximum when the sea wave number is equal to twice the incident radio wave number as experimentally observed by Crombie (1955).

1.3 Scope of Thesis

The analysis of HF scattering from an ocean surface presented in thesis is based on Walsh's (1980b) formulation for the scattering from a general time invariant rough surface (see section 1.2.3). The formulation appears as a set of two vector integral equations in the two dimensional spatial Fourier transform domain. Initially, a non-time varying two dimensional (in x and y) periodic surface is considered. The mean level of the surface is taken to be the z = 0 plane. It is assumed that the surface has high refractive index, such as sea water at HF. The medium above the surface is taken to be free space. This periodicity of the surface reduces both the integral equations to summation equations. The first equation is then formally inverted in the form of a Neumann series to yield a solution for the electric field on the surface in the spatial Fourier transform domain. It is shown that when this series solution is used in the second equation, the result obtained is that of Rice (1951) perturbation. For this purpose, the surface is taken to be a perfect conductor and the incident field assumed is plane wave. The Neumann series solution is partially summed to form another series which is in a more tractable form and further, is more amenable to physical interpretation.

In order to study the above series solution and apply it to remote sensing of ocean surface parameters, attention is confined to the vertical component of the surface field. At HF with both transmitting and receiving antennas located close to the surface, the ground wave [Wait (1964), Norton (1937)] mode of propagation is dominant. Moreover, vertical polarization is most efficient in this mode for a good conducting surface such as sea water. For analytical simplicity it is assumed that surface slopes are much less than unity. The source taken is an elementary vertical electric dipole located close to the surface. The excitation of the source is assumed to be a pulsed sinusoid and, therefore, we have treated the previous equations in the temporal Fourier transform domain. For a non-pulsed sinusoidal excitation, these may be treated in the standard phasor form. For this dipole source, zero, first, and second order approximations of the vertical component of the surface field in the spatial and temporal transform domain are derived. In these three orders of the solution we have considered the terms up to second order in surface Fourier coefficients, i.e., up to product of two coefficients.

The zero order solution represents a ground wave propagating outward from the source with a modified surface impedance. The first order solution represents two ground waves, again with modified surface impedances, propagating in different directions due to scattering. Similarly, the second and higher orders of the solution may be interpreted. The modified surface impedances take into account surface roughness, which is to be expected. These interpretations of the ground waves with modified surface impedances are based on an asymptotic evaluation of integrals for the inverse spatial transform using first the stationary phase method and then the steepest descent method given by Wait (1970, ch. 2). It should be mentioned that the trapped surface wave phenomenon as discussed by Wait may also be present. The inverse transforms for the first and second order solutions are in the form of spatial convolutions.

Considering the monostatic case, the inverse temporal Fourier transforms for the first and second order surface field are approximately taken for returning to time domain. The above spatial convolution integrals are then evaluated, again asymptotically, by the stationary phase method for these two orders of the backscattered surface field, assuming a narrow beam receiving antenna. However, the results for each order may be extended for a wide beam or omnidirectional receiving antenna. This backscattered field is received from an area of the surface which corresponds to the time delay between the transmitted and received signals. In the case of the first order backscattered field this area defines a patch of the surface, whereas this is not so for the second order. In the latter case the fields are received from the patch as well as from other regions of the surface, all arriving at the same time.

A more realistic model is now considered. In this model, both the time dependence and the statistical variation of the surface are introduced into the previous first and second order solutions for the backscattered field from a time invariant surface, thus making them more applicable to the ocean. The ocean surface is modelled as a three dimensional periodic surface in space and time. The randomness of the surface variation is introduced in the surface Fourier coefficients by treating them as random variables [Rice (1951), Barrick (1972a)]. A second order hydrodynamic effect, derived by Weber and Barrick (1977), is also included in the second order solution for the backscattered field. For modelling a pulsed radar, we have extended the excitation of the source dipole from a single pulsed sinusoid to a periodic pulsed sinusoid. Using this time dependent result and the dispersion relationship for deep water ocean gravity waves, an average first and second order backscattered Doppler spectra (power density spectra) have been derived, again assuming a narrow beam receiving antenna. Consequently the two orders of the Doppler frequency dependent cross section have been derived.

The backscattered Doppler spectrum or the cross section relate to the spatial (directional) wave height spectrum of gravity waves. The result obtained for the first order Doppler spectrum is the same as derived by Barrick (1972a) using the Rice (1951) perturbation technique, whereas this is not so for the second order. Our second order result contains three terms. The first term is almost the same as that obtained by Barrick (1972b, 1977a), and it represents the case where double scattering occurs on the patch only [Srivastava and Walsh (1983)]. The effect of the second term is very pronounced at higher Doppler frequencies. This term may be viewed as a result of interaction between the surface and incident field along the path from source point to the patch. The third term results from off the patch double scattering [Walsh and Srivastava (1984)]. These differences may be traced to the use of a dipole source in the model. However, if the transmitting antenna is also a narrow beam one, the third term may be neglected.

CHAPTER 2

TIME INVARIANT TWO DIMENSIONAL PERIODIC SURFACE

2.1 General

Walsh (1980b) has formulated a pair of integral equations (1.5 and 1.10) in the two dimensional spatial Fourier transform domain for determining the scattered electric field from an arbitrary surface $z = f(x,y)$. The surface is assumed to be time invariant but not necessarily perfectly conducting. Further, the formulation is open to any finite source. These equations in the form (1.11) and (1.12) are taken as the starting point of this thesis. The subsequent work in this chapter deals with determining a solution for the surface field for the case of an assumed model of the rough surface.

The basic equations (1.11) and (1.12) are first simplified for a good conducting surface such as sea water at HF. The surface is assumed to be a two dimensional periodic surface in x and y. This approach is a standard one taken in modelling rough surfaces [see, e.g., Rice (1951)]. In this event both the equations are reduced to functional (summation) equations. The first equation (1.11) is then formally inverted in the form of a Neumann series to yield a solution for the surface field in the transform domain. It is shown in Appendix A that when this series solution is used in the other equation (1.12), the result obtained is that of Rice perturbation. Of course, to obtain his result, the surface is taken to be a perfect conductor and the incident field assumed to be plane wave, although we have violated the assumption of a finite source for this purpose. For a comparison with Rice's result we have carried out the derivation for the scattered field up to the second order of perturbation for the case of vertical polarization. In a similar way the result for horizontal polarization may also be derived.

It is shown in Appendix B that the Neumann series solution for the surface field may be partially summed to form another series which is in a more tractable form and further, is more amenable to physical interpretation. In order to study this series solution and to apply it to remote sensing of ocean surface parameters we have confined our attention to the vertical component of the surface field (derived in the following chapter). This is because in an HF radar system with both transmitting and receiving antennas located close to the surface, the ground wave [Norton (1937)] mode of propagation is predominant. Moreover, vertical polarization is most efficient in this mode for a good conducting surface, such as an ocean surface.

2.2 Basic Integral Equations and their Simplification

In order to seek a solution of (1.11) for the surface field, this equation is written in cartesian component form. The resulting set of three equations will be referred to as the "Source field equations".

Source field equations

$$\int_x\int_y [\{(u + jkn_o)(1 + \frac{f_x^2}{n_o^2} + f_y^2) - j\,(1 - \frac{1}{n_o^2})\,K_x f_x\}\,E_{ax}$$

$$- (1 - \frac{1}{n_o^2})\,f_y\,\{jK_x + (u + jkn_o)f_x\}\,E_{ay}$$

$$+ (1 - \frac{1}{n_o^2})\,\{jK_x + (u + jkn_o)f_x\}\,E_{az}]$$

$$\cdot \exp(-fu - j\vec{K}\cdot\vec{x})\,dxdy = 2u\,\underline{E_{ix}}\,(\vec{K})\,\exp(-z^- u) \qquad (2.1a)$$

$$\int_x\int_y [-(1 - \frac{1}{n_o^2})\,f_x\,\{jK_y + (u + jkn_o)f_y\}\,E_{ax}$$

$$+ \{(u + jkn_o)(1 + f_x^2 + \frac{f_y^2}{n_o^2}) - j(1 - \frac{1}{n_o^2})\,K_y f_y\}\,E_{ay}$$

$$+ (1 - \frac{1}{n_o^2})\,\{jK_y + (u + jkn_o)f_y\}\,E_{az}]$$

$$\cdot \exp(-fu - j\vec{K}\cdot\vec{x})\,dxdy = 2u\,\underline{E_{iy}}\,(\vec{K})\,\exp(-z^- u) \qquad (2.1b)$$

$$\int_x\int_y [jk(n_o - \frac{1}{n_o})(f_x E_{ax} + f_y E_{ay})$$

$$+ \{u + j\frac{k}{n_o} + (u + jkn_o)(f_x^2 + f_y^2)\} E_{az}]$$

$$\cdot \exp(-fu - j\vec{K}\cdot\vec{x})\, dxdy = 2u\, \underline{E_{iz}}\, (\vec{K})\, \exp(-z^- u) \qquad (2.1c)$$

In the above f_x and f_y are partial derivatives of $f(\vec{x})$ with respect to x and y respectively. E_{ax}, E_{ay}, and E_{az} are the cartesian components of $\vec{E_a}$. Similarly $\underline{E_{ix}}$, $\underline{E_{iy}}$, and $\underline{E_{iz}}$ are the cartesian components of $\underline{\vec{E_i}}$.

In a similar way, (1.12) may be written in component form. The resulting three equations will be referred to as the "Scattered field equations".

Scattered field equations

$$\underline{E_{sx}}\, (\vec{K}) = \frac{1}{2u} \exp(-z^+ u)$$

$$\cdot \int_x\int_y [\{(u - jkn_o)(1 + \frac{f_x^2}{n_o^2} + f_y^2) + j(1 - \frac{1}{n_o^2})K_x f_x\} E_{ax}$$

$$+ (1 - \frac{1}{n_o^2})f_y \{jK_x - (u - jkn_o)f_x\} E_{ay}$$

$$- (1 - \frac{1}{n_o^2}) \{jK_x - (u - jkn_o)f_x\} E_{az}]$$

$$\cdot \exp(fu - j\vec{K}\cdot\vec{x})\, dxdy \qquad (2.2a)$$

$$\underline{E_{sy}}\, (\vec{K}) = \frac{1}{2u} \exp(-z^+ u)$$

$$\cdot \int_x\int_y [(1 - \frac{1}{n_o^2})f_x \{jK_y - (u - jkn_o)f_y\} E_{ax}$$

$$+ \{(u - jkn_o)(1 + f_x^2 + \frac{f_y^2}{n_o^2}) + j(1 - \frac{1}{n_o^2})K_y f_y\} E_{ay}$$

$$- (1 - \frac{1}{n_o^2}) \{jK_y - (u - jkn_o)f_y\} E_{az}]$$

$$\cdot \exp (fu - j\vec{K} \cdot \vec{x}) \, dxdy \qquad (2.2b)$$

$$E_{sz} (\vec{K}) = \frac{1}{2u} \exp (-z^+ u)$$

$$\cdot \int_x \int_y [-jk(n_o - \frac{1}{n_o})(f_x E_{ax} + f_y E_{ay})$$

$$+ \{u - j \frac{k}{n_o} + (u - jkn_o)(f_x^2 + f_y^2)\} E_{az}]$$

$$\cdot \exp (fu - j\vec{K} \cdot \vec{x}) \, dxdy \qquad (2.2c)$$

The above equations (2.1) and (2.2) may be simplified by making the following assumptions:

1) the magnitude of the refractive index (n_o) is large compared to both unity and the slopes of the surface, i.e., $|n_o| >> 1$, $|f_x|$, $|f_y|$ for all x and y. This in turn implies that $(n_o^2 - 1) \cong n_o^2$, $(n_o^2 + f_x^2) \cong n_o^2$, and $(n_o^2 + f_y^2) \cong n_o^2$. At HF and lower radio frequencies this condition is easily met by the ocean surface. The typical r.m.s. value of x,y slopes of the ocean surface is 0.24 as measured by Cox and Munk (1954) using aerial photographs of the sun's glitter on the surface. Whereas the magnitude of the refractive index for sea water ($\epsilon_r = 81$, $\sigma_0 = 4$ mhos/m) lies in the range 155 to 49 for the HF band (3 - 30 MHz). The above value for the slope corresponds to a wind speed of 10 m/sec.

2). the important wave numbers (values of $|\vec{K}|$) are much less than $|kn_o|$. Combining this assumption with the previous one implies that $u \pm jkn_o \cong \pm jkn_o$.

Considering these assumptions with the following substitutions,

$$E_{tx} = jkn_o \; (E_{ax} + f_x E_{az}) \qquad\qquad (2.3)$$
$$E_{ty} = jkn_o \; (E_{ay} + f_y E_{az}) \; ,$$

and

$$\underline{E_{xi}}(\vec{K}) = 2u \; \underline{E_{ix}} \; (\vec{K}) \; \exp \; (-z^- u)$$
$$\underline{E_{yi}}(\vec{K}) = 2u \; \underline{E_{iy}} \; (\vec{K}) \; \exp \; (-z^- u) \qquad\qquad (2.4)$$
$$\underline{E_{zi}}(\vec{K}) = 2u \; \underline{E_{iz}} \; (\vec{K}) \; \exp \; (-z^- u),$$

both sets of equations (2.1) and (2.2) reduce to:

<u>Source field equations</u>

$$\int_x \int_y [(1 + f_y^2 - \frac{K_x f_x}{kn_o})E_{tx} - f_y \, (f_x + \frac{K_x}{kn_o}) \, E_{ty}$$
$$+ jK_x (1 + f_x^2 + f_y^2) \, E_{az}] \exp \, (-fu - j\vec{K} \cdot \vec{x}) \, dxdy = \underline{E_{xi}}(\vec{K}) \qquad (2.5a)$$

$$\int_x \int_y [-f_x (f_y + \frac{K_y}{kn_o}) \, E_{tx} + (1 + f_x^2 - \frac{K_y f_y}{kn_o})E_{ty}$$
$$+ jK_y (1 + f_x^2 + f_y^2) \, E_{az}] \exp \, (-fu - j\vec{K} \cdot \vec{x}) \, dxdy = \underline{E_{yi}}(\vec{K}) \qquad (2.5b)$$

$$\int_x\int_y [f_x E_{tx} + f_y E_{ty} + (u + j\frac{k}{n_o})E_{az}]$$

$$\cdot \exp(-fu - j\vec{K}\cdot\vec{x})\, dxdy = \underline{E_{zi}}(\vec{K}) , \qquad (2.5c)$$

Scattered field equations

$$\underline{E_{sx}}(\vec{K}) = \frac{1}{2u}\exp(-z^+u)\int_x\int_y [-(1 + f_y^2 - \frac{K_x f_x}{kn_o})E_{tx}$$

$$+ f_y(f_x + \frac{K_x}{kn_o})E_{ty} - jK_x(1 + f_x^2 + f_y^2)E_{az}]$$

$$\cdot \exp(fu - j\vec{K}\cdot\vec{x})\, dxdy \qquad (2.6a)$$

$$\underline{E_{sy}}(\vec{K}) = \frac{1}{2u}\exp(-z^+u)\int_x\int_y [f_x(f_y + \frac{K_y}{kn_o})E_{tx}$$

$$- (1 + f_x^2 - \frac{K_y f_y}{kn_o})E_{ty} - jK_y(1 + f_x^2 + f_y^2)E_{az}]$$

$$\cdot \exp(fu - j\vec{K}\cdot\vec{x})\, dxdy \qquad (2.6b)$$

$$\underline{E_{sz}}(\vec{K}) = \frac{1}{2u}\exp(-z^+u)\int_x\int_y [-f_x E_{tx} - f_y E_{ty} + (u - j\frac{k}{n_o})E_{az}]$$

$$\cdot \exp(fu - j\vec{K}\cdot\vec{x})\, dxdy . \qquad (2.6c)$$

2.3 Reduction to Summation or Functional Equations

It is now assumed that $f(\vec{x})$ is a two dimensional periodic surface in both x and y with spatial periods L (figure 2.1). Therefore, we have the two dimensional Fourier series expansion for $f(\vec{x})$ as

$$f(\vec{x}) = \sum_{m,n}' P_{m,n} \exp [jN(mx + ny)] \qquad (2.7)$$

where $P_{m,n}$'s are the Fourier coefficients corresponding to the wave numbers mN and nN of the surface. These are given as

$$P_{m,n} = \frac{1}{L^2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} f(\vec{x}) \exp [-jN(mx + ny)] \, dxdy \qquad (2.8)$$

$N(= 2\pi/L)$ is the fundamental wave number. The symbol $\sum_{m,n}$ means a double summation over the integers m and n from $-\infty$ to ∞. Since $f(\vec{x})$ is a real surface we have $P_{-m,-n} = P^*_{m,n}$. The asterisk (*) as superscript denotes the complex conjugate of the quantity. For convenience we have assumed the same period L in both x and y although two different periods may be taken. This choice, however, is not a restriction in the procedure as later in the analysis [Chapter 4] the limit of the periods will be extended to infinity. The mean level of the surface is taken to be the plane $z = 0$, which implies $P_{o,o} = 0$.

In view of (2.7) $\exp(-fu)$, $f_x \exp(-fu)$ etc. are also then periodic. Therefore we may write

$$[1, f_x, f_y, f_x f_y, f_x^2, f_y^2] \exp(-fu) =$$

$$\sum_{m,n} [A_{m,n}, B_{m,n}, C_{m,n}, D_{m,n}, I_{m,n}, J_{m,n}] \exp [jN(mx + ny)]. \qquad (2.9)$$

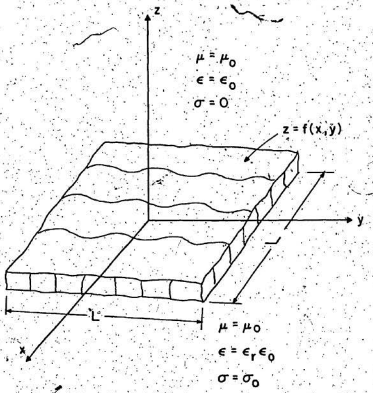


Figure 2.1 Geometry of the surface ($f(x,y)$ is periodic in x and y with spatial periods L. The medium above is free space with permeability μ_o and permittivity ε_o. The medium below has permeability μ_o, relative permittivity ε_r, and conductivity σ_o.)

In a similar way,

$[1,\ f_x,\ f_y,\ f_x f_y,\ f_x^2,\ f_y^2]\ \exp\ (fu) =$

$$\sum_{m,n} [\alpha_{m,n},\ \beta_{m,n},\ \gamma_{m,n},\ \theta_{m,n},\ \phi_{m,n},\ \mu_{m,n}]\ \exp\ [jN(mx + ny)] \qquad (2.10)$$

where $A_{m,n}$ to $J_{m,n}$ and $\alpha_{m,n}$ to $\mu_{m,n}$ are the Fourier coefficients for the respective functions in (2.9) and (2.10) for fixed u. This means this set of coefficients are functions of K_x and K_y in the form of u, for example $B_{m,n}$ is given as

$$B_{m,n} = \frac{1}{L^2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} f_x\ \exp\ [-fu - jN(mx + ny)]\ dxdy\ .$$

The Fourier coefficients given by (2.9) and (2.10) may be treated as auxiliary coefficients as they may be expanded in terms of the surface Fourier coefficients ($P_{m,n}$'s).

By using (2.9) in the source field equations (2.5) the integrals may be recognized as spatial Fourier transforms of E_{tx}, E_{ty} and E_{az} with shifts. These shifts are in the transform variables K_x and K_y by mN and nN respectively. We then have

<u>Source field equations</u>

$$\sum_{m,n} [(A_{m,n} + J_{m,n} - \frac{K_x}{kn_o} B_{m,n})\ \underline{E_{tx}}\ (K_x - mN,\ K_y - nN)$$
$$- (D_{m,n} + \frac{K_x}{kn_o} C_{m,n})\ \underline{E_{ty}}\ (K_x - mN,\ K_y - nN) + jK_x (A_{m,n}$$
$$+ I_{m,n} + J_{m,n})\ \underline{E_{az}}\ (K_x - mN,\ K_y - nN)] = \underline{E_{xi}}\ (\vec{K}) \qquad (2.11a)$$

$$\sum_{m,n}\left[-\left(D_{m,n} + \frac{K_y}{kn_o} B_{m,n}\right) \underline{E_{tx}} (K_x - mN, K_y - nN) + \left(A_{m,n} + I_{m,n} - \frac{K_y}{kn_o} C_{m,n}\right) \underline{E_{ty}} (K_x - mN, K_y - nN) + jK_y\left(A_{m,n} + I_{m,n} + J_{m,n}\right) \underline{E_{az}} (K_x - mN, K_y - nN)\right] = \underline{E_{yi}} (\vec{K}) \qquad (2.11b)$$

$$\sum_{m,n}\left[B_{m,n} \underline{E_{tx}} (K_x - mN, K_y - nN) + C_{m,n} \underline{E_{ty}} (K_x - mN, K_y - nN) + \left(u + j \frac{k}{n_o}\right) A_{m,n} \underline{E_{az}} (K_x - mN, K_y - nN)\right] = \underline{E_{zi}} (\vec{K}), \qquad (2.11c)$$

where a bar under E_{tx}, E_{ty}, and E_{az} denotes their spatial Fourier transform with respect to $\vec{x}$ [$= (x,y)$] with $\vec{K}$ [$= (K_x,K_y)$] as the transform variables. Similarly, by using (2.10) in the scattered field equations (2.6) we have

Scattered field equations

$$\underline{E_{sx}}(\vec{K}) = \frac{e^{-z^+ u}}{2u} \sum_{m,n} \Big[-\left(\alpha_{m,n} + \mu_{m,n} - \frac{K_x}{kn_o} \beta_{m,n}\right) \cdot \underline{E_{tx}} (K_x - mN, K_y - nN) + \left(\theta_{m,n} + \frac{K_x}{kn_o} \gamma_{m,n}\right) \underline{E_{ty}} (K_x - mN, K_y - nN) - jK_x \left(\alpha_{m,n} + \emptyset_{m,n} + \mu_{m,n}\right) \underline{E_{az}}(K_x - mN, K_y - nN)\Big] \qquad (2.12a)$$

$$\underline{E_{sy}}(\vec{K}) = \frac{e^{-z^+ u}}{2u} \sum_{m,n} \Big[\left(\theta_{m,n} + \frac{K_y}{kn_o} \beta_{m,n}\right) \underline{E_{tx}} (K_x - mN, K_y - nN) - \left(\alpha_{m,n} + \emptyset_{m,n} - \frac{K_y}{kn_o} \gamma_{m,n}\right) \underline{E_{ty}} (K_x - mN, K_y - nN) - jK_y \left(\alpha_{m,n} + \emptyset_{m,n} + \mu_{m,n}\right) \underline{E_{az}} (K_x - mN, K_y - nN)\Big] \qquad (2.12b)$$

$$\underline{E_{sz}}(\vec{K}) = \frac{e^{-z^{+}u}}{2u} \sum_{m,n} [-\beta_{m,n} \underline{E_{tx}}(K_x - mN, K_y - nN) - \gamma_{m,n}$$
$$\cdot \underline{E_{ty}}(K_x - mN, K_y - nN) + (u - j\frac{k}{n_o})\alpha_{m,n} \underline{E_{az}}(K_x - mN, K_y - nN)]. \quad (2.12c)$$

We now introduce a shift operator $S^{m,n}$ such that when this operates on a function it shifts K_x and K_y by mN and nN respectively. Mathematically, the operator is defined as

$$S^{m,n} F(K_x,K_y) = F(K_x - mN, K_y - nN) = F^{m,n}. \quad (2.13)$$

By using this shift operator (2.13), the source field equations (2.11) may conveniently be written in a matrix form as

$$\sum_{m,n} \begin{bmatrix} (A_{m,n} + J_{m,n} - \frac{K_x}{kn_o} B_{m,n}) & -(D_{m,n} + \frac{K_x}{kn_o} C_{m,n}) & jK_x(A_{m,n} + I_{m,n} + J_{m,n}) \\ -(D_{m,n} + \frac{K_y}{kn_o} B_{m,n}) & (A_{m,n} + I_{m,n} - \frac{K_y}{kn_o} C_{m,n}) & jK_y(A_{m,n} + I_{m,n} + J_{m,n}) \\ B_{m,n} & C_{m,n} & (u + j\frac{k}{n_o})A_{m,n} \end{bmatrix}$$

$$\begin{bmatrix} S^{m,n} \underline{E_{tx}}(\vec{K}) \\ S^{m,n} \underline{E_{ty}}(\vec{K}) \\ S^{m,n} \underline{E_{az}}(\vec{K}) \end{bmatrix} = \begin{bmatrix} \underline{E_{xi}}(\vec{K}) \\ \underline{E_{yi}}(\vec{K}) \\ \underline{E_{zi}}(\vec{K}) \end{bmatrix} \quad (2.14)$$

Symbolically, we may write the above matrix equation as

$$\sum_{m,n} T_{m,n}(\vec{K})\, S^{m,n}\, \overline{\underline{E_t}}(\vec{K}) = \overline{\underline{E_{oi}}}(\vec{K}), \quad (2.15)$$

where the matrix $T_{m,n}$ is given from (2.14) as

$$T_{m,n}(\vec{K}) =$$

$$\begin{bmatrix} (A_{m,n} + J_{m,n} - \frac{K_x}{kn_o} B_{m,n}) & -(D_{m,n} + \frac{K_x}{kn_o} C_{m,n}) & jK_x(A_{m,n} + I_{m,n} + J_{m,n}) \\ -(D_{m,n} + \frac{K_y}{kn_o} B_{m,n}) & (A_{m,n} + I_{m,n} - \frac{K_y}{kn_o} C_{m,n}) & jK_y(A_{m,n} + I_{m,n} + J_{m,n}) \\ B_{m,n} & C_{m,n} & -(u + j\frac{k}{n_o})A_{m,n} \end{bmatrix} \quad (2.16)$$

The two column vectors (denoted by an overhead bar) $\overline{E_t}(\vec{K})$ and $\overline{E_{oi}}(\vec{K})$ are

$$\overline{E_t}(\vec{K}) = \begin{bmatrix} E_{tx}(\vec{K}) \\ E_{ty}(\vec{K}) \\ E_{az}(\vec{K}) \end{bmatrix} , \quad \overline{E_{oi}}(\vec{K}) = \begin{bmatrix} E_{xi}(\vec{K}) \\ E_{yi}(\vec{K}) \\ E_{zi}(\vec{K}) \end{bmatrix} . \quad (2.17)$$

The equations (2.12) and (2.15) are the basic functional equations in the spatial transform domain. So that the scattered field may be found, equation (2.15) must be inverted to yield an expression for $\overline{E_t}$. Such an inversion will now be proposed.

2.4 A Series Solution for the Surface Field ($\overline{E_t}$).

The equation (2.15) may be written as

$$T_{o,o}\,\overline{E_t} + \sum_{(m,n)\neq(o,o)} T_{m,n}\, S^{m,n}\,\overline{E_t} = \overline{E_{oi}} \quad (2.18)$$

or

$$\overline{E_t} + \sum_{(m,n)\neq(o,o)} (T_{o,o})^{-1}\, T_{m,n}\, S^{m,n}\,\overline{E_t} = (T_{o,o})^{-1}\,\overline{E_{oi}} \quad (2.19)$$

where it is assumed that the inverse of matrix $T_{o,o}$, i.e., $(T_{o,o})^{-1}$, exists. The general expression $(m,n) \neq (p,q)$ means a restriction on the summations that both m and n can not be simultaneously equal to p and q respectively. For convenience let

$$L_{m,n} = (T_{o,o})^{-1}\, T_{m,n}\,,$$

$$\overline{E_{ci}} = (T_{o,o}^{t})^{-1}\, \overline{E_{oi}}\,. \tag{2.20}$$

Substituting these in (2.19) yields

$$[I + \sum_{(m,n)\neq(o,o)} L_{m,n}\, S^{m,n}]\, \overline{E_t} = \overline{E_{ci}} \tag{2.21}$$

where I is the identity operator. Or,

$$\overline{E_t} = [I + \sum_{(m,n)\neq(o,o)} L_{m,n}\, S^{m,n}]^{-1}\, \overline{E_{ci}}\,. \tag{2.22}$$

The above inverse of the linear operator may be formally expanded in the Neumann series, i.e.,

$$(I + A)^{-1} = \sum_{i=o}^{\infty} (-A)^i = I - A + A^2 - A^3 + \ldots \tag{2.23}$$

to obtain

$$\overline{E_t} = \sum_{i=o}^{\infty} [-\sum_{(m,n)\neq(o,o)} L_{m,n}\, S^{m,n}]^i\, \overline{E_{ci}} \tag{2.24}$$

$$= \overline{E_{ci}} - \underbrace{L_{m,n}\, S^{m,n}\, \overline{E_{ci}}}_{(m,n)\neq(o,o)}$$

$$+ \underbrace{L_{p,q}\, S^{p,q}}_{(p,q)\neq(o,o)}\, \underbrace{L_{m,n}\, S^{m,n}\, \overline{E_{ci}}}_{(m,n)\neq(o,o)}$$

$$- \underbrace{L_{r,s}\, S^{r,s}}_{(r,s)\neq(o,o)}\, \underbrace{L_{p,q}\, S^{p,q}}_{(p,q)\neq(o,o)}\, \underbrace{L_{m,n}\, S^{m,n}\, \overline{E_{ci}}}_{(m,n)\neq(o,o)} + \ldots \tag{2.25}$$

In (2.25) the summation symbols have been dropped for clarity but they are implied by subscripts on L: i.e., with respect to summation indices m,n, p,q, r, and s etc. with the proper restrictions placed on them.

By operating with shift operators and then collecting all the terms together in a general mode (m,n) of $\overline{E_{ci}}$ yields $\overline{E_t}$ as

$$\overline{E_t} = \overline{E_{ci}} - \sum_{(m,n)\neq(o,o)} L_{m,n} \overline{E^{m,n}_{ci}} + \sum_{(p,q)\neq \left\{\begin{matrix}(o,o)\\(m,n)\end{matrix}\right\}} L_{p,q} L^{p,q}_{m-p,n-q} \overline{E^{m,n}_{ci}}$$

$$- \sum_{(r,s)\neq \left\{\begin{matrix}(o,o)\\(p,q)\end{matrix}\right\}} \sum_{(p,q)\neq(m,n)} L_{r,s} L^{r,s}_{p-r,q-s} L^{p,q}_{p-r,q-s} L^{p,q}_{m-p,n-q} \overline{E^{m,n}_{ci}} + \cdots \quad (2.26)$$

In the above the superscripts mean respective shifts in K_x and K_y as defined by (2.13). The subscripts for a general term $L_{p-r,q-s}$ mean that p and q are the indices for double summation with any given restrictions on the summation. For example the third term

$$\sum_{(p,q)\neq \left\{\begin{matrix}(o,o)\\(m,n)\end{matrix}\right\}} L_{p,q} L^{p,q}_{m-p,n-q} \overline{E^{m,n}_{ci}} \text{ means}$$

$$\sum_{m,n} \sum_{(p,q)\neq \left\{\begin{matrix}(o,o)\\(m,n)\end{matrix}\right\}} L_{p,q} (K_x,K_y)\, L_{m-p,n-q} (K_x - pN, K_y - qN)$$

$$\cdot\ \overline{E_{ci}} (K_x - mN, K_y - nN) .$$

It is shown in Appendix A that when this solution for $\overline{E_t}$ (2.26) is used in the equation (2.12) for the scattered field, and the auxiliary Fourier coefficients (2.9 and 2.10) are expanded in terms of the surface Fourier coefficients (2.7), the result obtained is that of the Rice (1951) perturbation analysis. To obtain his result, the surface is taken to be a perfect conductor and the incident field assumed is plane wave. For comparison the derivation is carried out up to the second order of perturbation for vertical polarization. For horizontal polarization the result may similarly be derived.

2.5 Partial Summation of the Series Solution

It would be better if the series solution (2.26) could be summed to give an elegant closed form solution for $\overline{E_t}$. Nevertheless this series may be partially summed to form another series which is in a more tractable form. The procedure of the partial summation is summarized below whereas the mathematical details are given in Appendix B.

Initially the second term in (2.26) is considered. Those higher terms which are equivalent to the second term are collected as if they provide corrections to this term. For example the first correction comes from the fourth term by setting (p,q) = (o,o) there. The second correction comes from the fifth term and so on. These correction terms appear on the left side of the second term and may be formally summed in the form of the inverse of a series. This process is similarly repeated for higher terms. Then by letting (m,n) = (o,o) in the resultant series and adding it to the first term gives the correction to the first term. With this correction the first term is now known as the zero order term. No further correction is possible to this term.

The process is continued by collecting the correction terms but in this case on the right side of the second term matrix $L_{m,n}$. These correction terms may again be formally summed in the form of the inverse of another series. The second term is now referred to as the first order as no further correction is possible to this term. The procedure may be repeated for higher terms, however, the details have been carried out up to the second order and on that basis a generalization for higher orders has been achieved. The result of this partial summation is as follows:

$$\overline{E_t} = (I - R_1)^{-1}\,\overline{E_{ci}} - (I - R_1)^{-1} \underset{(m,n)\neq(o,o)}{L_{m,n}} (I - R_2)^{-1}\,\overline{E_{ci}^{m,n}}$$

$$+ (I - R_1)^{-1} \underset{(p,q)\neq\left\{\begin{matrix}(o,o)\\(m,n)\end{matrix}\right\}}{L_{p,q}} (I - R_3)^{-1} \underset{(m,n)\neq(o,o)}{L^{p,q}_{m-p,n-q}} (I - R_2)^{-1}\,\overline{E_{ci}^{m,n}}$$

$$- (I - R_1)^{-1} \underset{(r,s)\neq\left\{\begin{matrix}(o,o)\\(p,q)\\(m,n)\end{matrix}\right\}}{L_{r,s}} (I - R_4)^{-1} \underset{(p,q)\neq\left\{\begin{matrix}(o,o)\\(m,n)\end{matrix}\right\}}{L^{r,s}_{p-r,q-s}} (I - R_3)^{-1}$$

$$\underset{(m,n)\neq(o,o)}{L^{p,q}_{m-q,n-q}} (I - R_2)^{-1}\,\overline{E_{ci}^{m,n}} + \ldots \qquad (2.27)$$

where I is the identity matrix. Further, R_i's are given as

$$R_1 = \underset{(v,w)\neq(o,o)}{L_{v,w}} L^{v,w}_{-v,-w} - \underset{(k,l)=\left\{\begin{matrix}(o,o)\\(v,w)\end{matrix}\right\}}{L_{k,l}} L^{k,l}_{v-k,w-l} \underset{(v,w)\neq(o,o)}{L^{v,w}_{-v,-w}}$$

$$+ \underset{(i,j)\neq\left\{\begin{matrix}(o,o)\\(k,l)\end{matrix}\right\}}{L_{i,j}} \underset{(k,l)=\left\{\begin{matrix}(o,o)\\(v,w)\end{matrix}\right\}}{L^{i,j}_{k-i,l-j}} L^{k,l}_{v-k,w-l} \underset{(v,w)\neq(o,o)}{L^{v,w}_{-v,-w}} - \ldots, \qquad (2.28a)$$

$$R_2 = \underset{(v,w)\neq\left\{\begin{matrix}(o,o)\\(m,n)\end{matrix}\right\}}{L^{m,n}_{v-m,w-n}} L^{v,w}_{m-v,n-w} - \underset{(k,l)\neq\left\{\begin{matrix}(o,o)\\(v,w)\\(m,n)\end{matrix}\right\}}{L^{m,n}_{k-m,l-n}} L^{k,l}_{v-k,w-l} \underset{(v,w)\neq\left\{\begin{matrix}(o,o)\\(m,n)\end{matrix}\right\}}{L^{v,w}_{m-v,n-w}} + \ldots, \qquad (2.28b)$$

$$R_3 = \underset{(v,w)\neq\left\{\begin{matrix}(o,o)\\(p,q)\\(m,n)\end{matrix}\right\}}{L^{p,q}_{v-p,w-q}} L^{v,w}_{p-v,q-w} - \underset{(k,l)\neq\left\{\begin{matrix}(o,o)\\(v,w)\\(p,q)\\(m,n)\end{matrix}\right\}}{L^{p,q}_{k-p,l-q}} L^{k,l}_{v-k,w-l} \underset{(v,w)\neq\left\{\begin{matrix}(o,o)\\(p,q)\\(m,n)\end{matrix}\right\}}{L^{v,w}_{p-v,q-w}} + \ldots, \qquad (2.28c)$$

$$R_4 = \sum_{(v,w)\neq\begin{Bmatrix}(o,o)\\(r,s)\\(p,q)\\(m,n)\end{Bmatrix}} L^{r,s}_{v-r,w-s}\, L^{v,w}_{r-v,s-w} - \sum_{(k,l)\neq\begin{Bmatrix}(o,o)\\(v,w)\\(r,s)\\(p,q)\\(m,n)\end{Bmatrix}} \sum_{(v,w)\neq\begin{Bmatrix}(o,o)\\(r,s)\\(p,q)\\(m,n)\end{Bmatrix}} L^{r,s}_{k-r,l-s}\, L^{k,l}_{v-k,w-l}\, L^{v,w}_{r-v,s-w}$$

$$+ \sum_{(i,j)\neq\begin{Bmatrix}(o,o)\\(k,l)\\(r,s)\\(p,q)\\(m,n)\end{Bmatrix}} \sum_{(k,l)\neq\begin{Bmatrix}(o,o)\\(v,w)\\(r,s)\\(p,q)\\(m,n)\end{Bmatrix}} \sum_{(v,w)\neq\begin{Bmatrix}(o,o)\\(r,s)\\(p,q)\\(m,n)\end{Bmatrix}} L^{r,s}_{i-r,j-s}\, L^{i,j}_{k-i,l-j}\, L^{k,l}_{v-k,w-l}\, L^{v,w}_{r-v,s-w} - \dots \qquad (2.28d)$$

In (2.28) v, w, k, l, i, and j are dummy indices for local summations with given restrictions on summations, treating m, n, p, q, r, and s as constants. These constants refer to summation indices in (2.27). For example, taking only the first term in R_1, R_2, and R_3, the second order solution may be written as

$$[I - \sum_{(v,w)\neq(o,o)} L_{v,w}\, L^{v,w}_{-v,-w}]^{-1} \sum_{(m,n)\neq(o,o)} \; \sum_{(p,q)\neq\begin{Bmatrix}(o,o)\\(m,n)\end{Bmatrix}} L_{p,q}$$

$$\cdot [I - \sum_{(v,w)\neq\begin{Bmatrix}(o,o)\\(p,q)\\(m,n)\end{Bmatrix}} L^{p,q}_{v-p,w-q}\, L^{v,w}_{p-v,q-w}]^{-1} L_{m-p,n-q}$$

$$\cdot [I - \sum_{(v,w)\neq\begin{Bmatrix}(o,o)\\(m,n)\end{Bmatrix}} L^{m,n}_{v-m,w-n}\, L^{v,w}_{m-v,n-w}]^{-1} \overline{E^{m,n}_{ci}}$$

It may be mentioned here that the meaning of term "order", referred to above, is different than that used by Rice (1951). In his analysis the meaning of order is based on the form of appearance of the surface Fourier coefficients, i.e., $P_{m,n}$'s, in the perturbation series. For example, the second order means the term containing a product of two Fourier coefficients. Whereas in this analysis the ordering of terms is as they appear in the above series solution (2.27), starting with first term as the zero order solution. This ordering is based on the collection of correction terms and their formal summation in a manner described above. As a matter of fact each term in either of the series solutions, (2.26) or (2.27), contain many orders of perturbation.

The equation (2.27) for $\overline{E_t}$ may be treated as a general solution for the case of a two dimensional periodic and highly conducting surface. This solution is open to any finite source. This is evident from the column vector $\overline{E_{ci}}$ present in the solution which contains the incident field in the transform domain. From this solution the vertical component of the surface field in the spatial transform domain, i.e., $\underline{E_{az}}$, may be directly extracted, and may be inverse spatially transformed to obtain E_{az}. Whereas the two horizontal components E_{ax} and E_{ay} are not readily available because of the substitutions induced by (2.3). However, these two surface field components may be found from (2.3) once $\underline{E_{tx}}$, $\underline{E_{ty}}$, and $\underline{E_{az}}$ are determined from the above solution (2.27) and inverse spatially transformed. To find the scattered field above the surface, this result for $\overline{E_t}$ may be used directly in the scattered field equation (2.12) since the same surface field substitutions has been also used in (2.12).

In order to study the above series solution and to apply it to remote sensing of the ocean surface parameters by an HF radar system we will confine our attention to the vertical component of the surface field. This is so because at HF with transmitting and receiving antennas very close to the surface, the dominant mode of radio wave propagation is the ground wave mode [Norton (1937), Jordon and Balmain (1968, ch. 16)]. Moreover, the vertical component is most efficient in this mode for a good conducting surface such as an ocean surface.

2.6 Physical Interpretation of the Series Solution

Although the series solution (2.27) is in a complicated form in the spatial transform domain, a physical interpretation may still be attached to it. Initial consideration will be given to the case of a flat surface. In this situation all the auxiliary Fourier coefficients are zero with exception of $A_{o,o}$ which becomes unity. Hence, all the terms along with R_1, R_2, etc. are zero except $\underline{E_{ci}}$. The same point may also be reached from the previous solution (2.26) for $\underline{E_t}$. Under this flat surface condition we have

$$\underline{E_t} = \underline{E_{ci}} = (T_{o,o})^{-1} \underline{E_{oi}} \tag{2.29}$$

By inverting the matrix $T_{o,o}$ given by (2.16) and using (2.17) and (2.4) in the above equation, the vertical component of the surface field may be given as

$$\underline{E_{az}} = \frac{2u}{u + j\frac{k}{n_o}} \underline{E_{iz}}(\vec{K}) \exp(-z^- u)$$

Some investigators [Wait (1964), Barrick (1971)] have used the concept of "normalized surface impedance" when dealing with problems of radio wave propagation and scattering from a flat or rough surface. This normalized surface impedance helps in forming a boundary condition for such problems. For a flat surface this is denoted as Δ and is equal to $\frac{1}{n_o}(1 - \frac{1}{n_o^2})^{\frac{1}{2}}$. For a good conducting surface ($|n_o| >> 1$) Δ may be approximated as $\frac{1}{n_o}$. By using this approximation for Δ and taking the incident field as the far field from an elementary vertical dipole source located on z axis at a short height h_o above the surface, the above equation becomes,

$$E_{az}(\vec{K}) = \frac{C_d \exp(-h_o u)}{u + jk\Delta} = C_d\left[\frac{1}{2u} + \frac{u - jk\Delta}{2u(u + jk\Delta)}\right] \exp(-h_o u) \qquad (2.30)$$

where C_d is the dipole constant. This equation may be inverse spatially transformed to obtain E_{az} by a method used by Wait (1970, ch. 2). The technique for inversion is based on an asymptotic evaluation of integral using a steepest descent method. This inversion of (2.30) yields a ground wave [Wait (1970, ch. 2)] propagating away from the dipole source with Δ as the normalized surface impedance, as it should to agree with established theory.

Now in the case of a periodic surface the zero order term will be considered first. This term is $(I - R_1)^{-1}\,\overline{E_{ci}}$, and is obtained from a partial summation of the series (2.26). Once again, this represents an outward propagating ground wave, but this time with a modified surface impedance Δ_m, which is to be expected. This modification in the surface impedance may be effected by reducing the determinant of the matrix $(I - R_1)$ to the form $(u + jk\Delta_m)$ when taking the inverse of this matrix. Of course, this matrix is not in a simple form to achieve its inverse, since R_1 (or for that matter, R_2, R_3, . . .) is in the form of a series of matrices with summations. However, to facilitate the analysis and for comparing with the results of other investigators these R_i will be approximated by their first term only. The resulting modified surface impedance Δ_m, as will be seen later, consists of the surface

Fourier coefficients accounting for the surface variations. In this respect the solution in the form of (2.27) differs from (2.26). In the form of (2.26), i.e., before the series is partially summed, the surface impedance is $\frac{1}{n_0}$. Therefore (2.27) is to be preferred over (2.26).

Consideration will now be given to the first order term, which is given as $(I - R_1)^{-1} L_{m,n} (I - R_2)^{-1} \underline{\overline{E_{ci}^{m,n}}}$. This term accounts for a first order scattering of the incident field from the surface to everywhere else on the surface. Suppose now an observation point is fixed on the surface. At this point a ground wave is received, originating from the source and given by the zero order term discussed above. Additionally, a first order scattered field is received, which may be explained as follows. A ground wave travels from the source in a direction given by $(I - R_2)^{-1} \underline{\overline{E_{ci}^{m,n}}}$. In this direction, some surface patch acts to scatter this surface wave to everywhere else on the surface. A part of this scattered surface field is also received at the observation point. The mode of propagation of this scattered field is again a ground wave with a modified surface impedance and is given by $(I - R_1)^{-1} L_{m,n}$. If the observation point now moves back to the source point what is received is the first order backscattered surface field. In a similar manner the second and higher order terms may be explained.

At this point it is worthwhile to mention some of the differences between the Rice (1951) perturbation technique and that used here. In perturbation analysis the successive order of solution for the scattered field depends upon the solution for previous order. For example, the second order solution may only be derived when first order is known, whereas a complete solution in a symbolic form may be written here as evident from the series solution. In other words, e.g., our second order solution may be derived directly from the series after performing the necessary algebra and inverse Fourier transforms. For this purpose the first order result need not be evaluated. Further, this series solution is open to any source unlike the perturbation method where only plane wave incidence is used. This plane wave incidence causes breakdown of the perturbation solution at grazing observation in the case of perfect conductivity and vertical polarization. However, in this special case Rice has given a corrective modification using a surface wave formulation in the perturbation technique.

For the case of finite conductivity, the perturbation technique has been used by other investigators [Barrick (1971), Wait (1971)] and a modified surface impedance derived. However, in all these cases the surface field solution does not appear in the form of ground wave. This is due to the assumed plane wave incidence. Whereas in the present work the source assumed is an elementary vertical electric dipole which is clearly more realistic than plane wave. It may be mentioned here that in another treatment Wait (1959) has used a dipole source for studying the propagation across hemispherical bosses distributed uniformly over a perfectly conducting plane. The result obtained is in the form of a ground wave. The influence of surface curvature on the propagation has also been considered by him.

CHAPTER 3

SURFACE FIELD FOR A DIPOLE SOURCE

3.1 General

In the previous chapter we have derived a series solution (2.27) for the surface field for a two dimensional periodic surface. The solution is in the spatial Fourier transform domain and is open to any finite source. The source now assumed is an elementary vertical dipole located close to the surface and it is having a pulsed sinusoidal excitation. For analytical simplicity it is assumed that the slopes of the surface are much less than unity. For this source the series solution may be interpretated as: 1) a ground wave propagating out from the source with a modified surface impedance, 2) various ground waves, again with modified surface impedances, propagating in different directions due to scattering. These modified surface impedances take into account surface roughness, which is to be expected. This interpretation of ground waves is based on an asymptotic evaluation of integrals using the methods of stationary phase and steepest descent for the inverse spatial transform. By using this integration technique a zero, first, and second order approximation of the vertical component of the surface waves are inverse spatially transformed. The inverse transform for the first and second order appromimations are in the form of convolutions. These convolution integrals are again asymptotically evaluated using the method of stationary phase for these two orders of backscattered surface field, assuming a narrow beam receiving antenna. However, the result for each order may be extended for a wide beam or an omnidirectional receiving antenna.

3.2 Incident Field from a Dipole Source

We consider the source as an elementary vertical dipole of length dl. The dipole is located on the z axis at a small height h_o above the mean level of the surface, i.e., z = 0 plane, as shown in figure 3.1. The dipole is assumed to be carrying a sinusoidal pulsed current of the form

$$\tilde{I}(t) = I_o\, G_{\tau_o}(t)\, \exp(j\omega_o t) \tag{3.1}$$

where I_o is the peak current, ω_o is the radian frequency of the current and $G_{\tau_o}(t)$ is a gate function of width τ_o with the definition

$$G_{\tau_o}(t) = \begin{cases} 1 ; & \text{for } |t| \le \tau_o/2 \\ 0 ; & \text{for } |t| > 0 \end{cases} \tag{3.2}$$

The choice of a complex sinusoid in (3.1) is made for identifying the power spectrum of the scattered signal at positive and negative Doppler frequencies in the case of a time varying rough surface (see Ch. 4).

For this dipole the current density $\vec{J}_s$ may be given by [see e.g., Jordan and Balmain (1968, ch. 10)]

$$\vec{J}_s = I dl\, \delta(x)\, \delta(y)\, \delta(z - h_o)\, \hat{z} \tag{3.3}$$

where $I = I(\omega)$ is the temporal Fourier transform of $\tilde{I}(t)$ and $\delta(x)$ is the Dirac delta function. From (1.6) the incident field may be given as

$$\vec{E}_i = T_{sE}[\vec{J}_s] * G = \frac{1}{j\omega\varepsilon_o}\left[\vec{\nabla}(\vec{\nabla}\cdot\vec{J}_s) + k^2\vec{J}_s\right] * G \tag{3.4}$$

where the asterisk denotes a three dimensional convolution with respect to x, y and z. G is the free space Green's function given by

$$G = \frac{e^{-jkr}}{4\pi r} \quad ; \quad r = (x^2 + y^2 + z^2)^{\frac{1}{2}} \tag{3.5}$$

Now, we have

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{J}_s) = I dl\, [\delta_x(x)\, \delta(y)\, \delta_z(z - h_o)\hat{x} + \delta(x)\, \delta_y(y)\, \delta_z(z - h_o)\hat{y} + \delta(x)\, \delta(y)\, \delta_{zz}(z - h_o)\hat{z}] \tag{3.6}$$

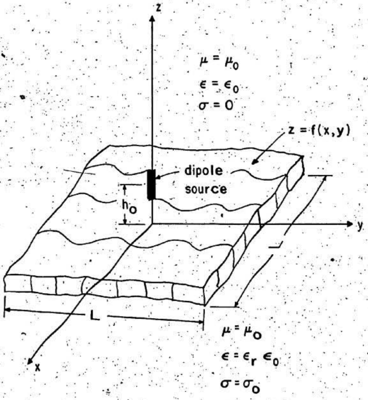


Figure 3.1 Surface geometry with an elementary vertical dipole source

where the subscripts x, y or z on delta functions refer to the partial derivative with respect x, y and z respectively. By performing the three dimensional convolution in (3.4), the incident field in the component form may be written as

$$E_{ix} = \frac{Idl}{j\omega\varepsilon_o}\left(\frac{3}{r_1^2} + \frac{3jk}{r_1} - k^2\right)\frac{x(z - h_o)}{r_1^2}\,\frac{\exp(-jkr_1)}{4\pi r_1}$$

$$E_{iy} = \frac{Idl}{j\omega\varepsilon_o}\left(\frac{3}{r_1^2} + \frac{3jk}{r_1} - k^2\right)\frac{y(z - h_o)}{r_1^2}\,\frac{\exp(-jkr_1)}{4\pi r_1}$$

$$E_{iz} = \frac{Idl}{j\omega\varepsilon_o}\left[\left(\frac{3}{r_1^2} + \frac{3jk}{r_1} - k^2\right)\frac{(z - h_o)^2}{r_1^2} - \left(\frac{1}{r_1^2} + \frac{jk}{r_1} - k^2\right)\right]\frac{\exp(-jkr_1)}{4\pi r_1} \quad (3.7)$$

where

$$r_1 = [x^2 + y^2 + (z - h_o)^2]^{\frac{1}{2}} . \quad (3.8)$$

Under the far field assumption, i.e., neglecting $\frac{1}{r_1^2}$ and $\frac{1}{r_1^3}$ dependency of the dipole field, and evaluating the above equation (3.7) in a plane $z = z^- < f(x,y)$ assuming $|z^- - h_o| << r_1$ the above incident field may be approximated as

$$E_{ix} \cong 0$$

$$E_{iy} \cong 0 \quad (3.9)$$

$$E_{iz} \cong \frac{Idl}{j\omega\varepsilon_o}\,k^2\,\frac{\exp(-jkr_d)}{4\pi r_d} = C_d\,\frac{\exp(-jkr_d)}{4\pi r_d}$$

where

$$C_d = \frac{I(\omega)\,dl\,k^2}{j\omega\epsilon_o} = -j\omega I(\omega)\,dl\,\mu_o, \tag{3.10}$$

$$r_d = [x^2 + y^2 + (z^- - h_o)^2]^{\frac{1}{2}}$$

By spatially Fourier transforming (3.9) we obtain

$$\underline{E_{ix}}(\vec{K}) = 0 \ ,$$

$$\underline{E_{iy}}(\vec{K}) = 0 \ , \tag{3.11}$$

$$\underline{E_{iz}}(\vec{K}) = C_d \frac{\exp\,(-|z^- - h_o|u)}{2u}$$

Since $z^- < h_o$,

$$\underline{E_{iz}}(\vec{K}) = C_d \frac{\exp\,[(z^- - h_o)u]}{2u} \tag{3.12}$$

The above transformed incident field may be used in (2.4) to obtain

$$\underline{E_{xi}}(\vec{K}) = 0$$

$$\underline{E_{yi}}(\vec{K}) = 0 \tag{3.13}$$

$$\underline{E_{zi}}(\vec{K}) = C_d \exp\,(-h_o u)$$

which may be written in the form of a column vector, defined by (2.17), as

$$\overline{\underline{E_{oi}}}(\vec{K}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} C_d \exp\,(-h_o u) \ . \tag{3.14}$$

3.3 Zero Order Surface Field

From the series solution (2.27) the zero order surface field, now denoted as $\overline{\underline{E_{to}}}$, may be written as

$$\overline{E_{to}}(\vec{K}) = (I - R_1)^{-1}\,\overline{E_{ci}} \tag{3.15}$$

where R_1 is given by another series (2.28a).

By using (2.20) in the above equation we obtain

$$\overline{E_{to}} = (I - R_1)^{-1}\,(T_{o,o})^{-1}\,\overline{E_{oi}}$$

$$= (T_{o,o} - T_{o,o}R_1)^{-1}\,\overline{E_{oi}} \tag{3.16}$$

To facilitate the analysis and for the comparing with the results of other investigators we approximate the series R_1 by its first term only. Therefore by using the first term of (2.28a) we have

$$\overline{E_{to}} = [T_{o,o} - T_{o,o}\sum_{(v,w)\neq(o,o)} L_{v,w}\,L^{v,w}_{-v,-w}]^{-1}\,\overline{E_{oi}} \tag{3.17}$$

where we have written a double summation symbol explicitly which was earlier implied in (2.28a). Use of (2.20) in the above yields

$$\overline{E_{to}} = [T_{o,o} - \sum_{(v,w)\neq(o,o)} T_{v,w}\,(T^{v,w}_{o,o})^{-1}\,T^{v,w}_{-v,-w}]^{-1}\,\overline{E_{oi}} \tag{3.18}$$

where the matrix $T_{v,w}$ is defined by (2.16).

For a simplification of the integral equations (2.1) and (2.2) in the form of (2.5) and (2.6) one of the assumptions made was that the slopes of the surface are much less than the magnitude of the refractive index. We will now narrow the range of surface to consider only those cases where squared slopes are much less than unity, i.e., f_x^2 and $f_y^2 \ll 1$. This additional restriction simplifies further analysis. Under this condition the source field equation (2.5) simplifies to the following:

$$\int_x \int_y [E_{tx} - f_y (f_x + \frac{K_x}{kn_o}) E_{ty} + jK_x E_{az}]$$

$$\cdot \exp(-fu - j\vec{K} \cdot \vec{x}) \, dxdy = \underline{E_{xi}}(\vec{K}) \tag{3.19a}$$

$$\int_x \int_y [-f_x (f_y + \frac{K_y}{kn_o}) E_{tx} + E_{ty} + jK_y E_{az}]$$

$$\cdot \exp(-fu - j\vec{K} \cdot \vec{x}) \, dxdy = \underline{E_{yi}}(\vec{K}) \tag{3.19b}$$

$$\int_x \int_y [f_x E_{tx} + f_y E_{ty} + (u + j\frac{k}{n_o}) E_{az}]$$

$$\cdot \exp(-fu - j\vec{K} \cdot \vec{x}) \, dxdy = \underline{E_{zi}}(\vec{K}) \tag{3.19c}$$

In view of the above equations the matrix $T_{v,w}$, earlier given by (2.16), for an arbitrary shifts mN and nN simplifies to the following:

$$T^{m,n}_{v,w} = \begin{bmatrix} A^{m,n}_{v,w} & -[D^{m,n}_{v,w} + \frac{(K_x - mN)}{kn_o} C^{m,n}_{v,w}] & j(K_x - mN)A^{m,n}_{v,w} \\ -[D^{m,n}_{v,w} + \frac{(K_y - nN)}{kn_o} B^{m,n}_{v,w}] & A^{m,n}_{v,w} & j(K_y - nN)A^{m,n}_{v,w} \\ B^{m,n}_{v,w} & C^{m,n}_{v,w} & (u^{m,n} + j\frac{k}{n_o})A^{m,n}_{v,w} \end{bmatrix} \tag{3.20}$$

For $v = w = 0$, (3.20) reduces to;

$$T^{m,n}_{o,o} = \begin{bmatrix} A^{m,n}_{o,o} & -D^{m,n}_{o,o} & j(K_x - mN) A^{m,n}_{o,o} \\ -D^{m,n}_{o,o} & A^{m,n}_{o,o} & j(K_y - nN) A^{m,n}_{o,o} \\ 0 & 0 & (u^{m,n} + j\frac{k}{n_o}) A^{m,n}_{o,o} \end{bmatrix} \tag{3.21}$$

where we have used the property $B^{m,n}_{o,o} = C^{m,n}_{o,o} = 0$, which may easily be verified from (2.9). The matrix $T^{m,n}_{o,o}$ may be inverted as

$$(T^{m,n}_{o,o})^{-1} = \frac{1}{(u^{m,n} + j\frac{k}{n_o})A^{m,n}_{o,o}\,[(A^{m,n}_{o,o})^2 - (D^{m,n}_{o,o})^2]}$$

$$\begin{bmatrix} (A^{m,n}_{o,o})^2(u^{m,n} + j\frac{k}{n_o}) & A^{m,n}_{o,o}D^{m,n}_{o,o}(u^{m,n} + j\frac{k}{n_o}) & -jA^{m,n}_{o,o}[(K_x - mN)A^{m,n}_{o,o} + (K_y - nN)D^{m,n}_{o,o}] \\ A^{m,n}_{o,o}D^{m,n}_{o,o}(u^{m,n} + j\frac{k}{n_o}) & (A^{m,n}_{o,o})^2(u^{m,n} + j\frac{k}{n_o}) & -jA^{m,n}_{o,o}[(K_x - mN)D^{m,n}_{o,o} + (K_y - nN)A^{m,n}_{o,o}] \\ 0 & 0 & (A^{m,n}_{o,o})^2 - (D^{m,n}_{o,o})^2 \end{bmatrix} \tag{3.22}$$

Also, under this small slope assumption, $(A^{m,n}_{o,o})^2 - (D^{m,n}_{o,o})^2 \cong (A^{m,n}_{o,o})^2$. With this approximation the above matrix inverse may be rewritten as

$$(T^{m,n}_{o,o})^{-1} \cong \frac{1}{A^{m,n}_{o,o}} \begin{bmatrix} 1 & \dfrac{D^{m,n}_{o,o}}{A^{m,n}_{o,o}} & \dfrac{-j[(K_x - mN)A^{m,n}_{o,o} + (K_y - nN)D^{m,n}_{o,o}]}{A^{m,n}_{o,o}(u^{m,n} + j\frac{k}{n_o})} \\ \dfrac{D^{m,n}_{o,o}}{A^{m,n}_{o,o}} & 1 & \dfrac{-j[(K_y - nN)A^{m,n}_{o,o} + (K_x - mN)D^{m,n}_{o,o}]}{A^{m,n}_{o,o}(u^{m,n} + j\frac{k}{n_o})} \\ 0 & 0 & \dfrac{1}{u^{m,n} + j\frac{k}{n_o}} \end{bmatrix} \tag{3.23}$$

Let us put

$$A = T_{o,o} - \sum_{(v,w)\neq(o,o)} T_{v,w}\,(T^{v,w}_{o,o})^{-1}\,T^{v,w}_{-v,-w} \tag{3.24}$$

Therefore, (3.18) may be written as

$$\overline{E_{to}} = A^{-1}\,\overline{E_{oi}} \tag{3.25}$$

The inverse of matrix A may symbolically be given as

$$A^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \frac{1}{\det A} \tag{3.26}$$

where a_{ij} are the cofactors of matrix A and det A is the determinant of A.

By using (2.17), (3.14) and (3.26) in (3.25) the vertical component of the surface field to zero order may be given as

$$\underline{E_{azo}}(K) = C_d \frac{a_{33}}{\det A} \exp(-h_o u). \tag{3.27}$$

Now by utilizing (3.20), (3.21), (3.23), and (3.24) and performing the required matrix algebra for the inverse of matrix A, the element a_{33} and det A may respectively be given as follows:

$$a_{33} = (A_{o,o})^2 \Big[1 - \frac{1}{A_{o,o}} \sum_{(p,q)\neq(o,o)} \frac{1}{A^{p,q}_{o,o}} [A_{p,q} \; \{2A^{p,q}_{-q,-q} + \frac{jN}{u^{p,q} + j\frac{k}{n_o}} (pB^{p,q}_{-p,-q} + qC^{p,q}_{-p,-q})\} + \frac{1}{kn_o} (\frac{1}{kn_o} + \frac{j}{u^{p,q} + j\frac{k}{n_o}}) \cdot \{K_y(K_x - pN)B_{p,q} \; C^{p,q}_{-p,-q} + K_x(K_y - qN)C_{p,q} \; B^{p,q}_{-p,-q}\}]\Big] \tag{3.28}$$

$$\det A = (A_{o,o})^2 [(u + j\frac{k}{n_o})A_{o,o} + \sum_{(v,w)\neq(o,o)} \frac{1}{A^{v,w}_{o,o}} [(u + j\frac{k}{n_o})A_{v,w} \cdot \{-3A^{v,w}_{-v,-w} + \frac{j}{u^{v,w} + j\frac{k}{n_o}} [(K_x - vN)B^{v,w}_{-v,-w} + (K_y - wN)C^{v,w}_{-v,-w}]\}$$

$$- \{K_y(K_x - vN)B_{v,w}C^{v,w}_{-v,-w} + K_x(K_y - wN)C_{v,w}B^{v,w}_{-v,-w}\}$$

$$\cdot \left(\frac{1}{kn_o} + \frac{j}{u^{v,w} + j\frac{k}{n_o}}\right)\left(j + \frac{u + j\frac{k}{n_o}}{kn_o}\right)$$

$$+ j A^{v,w}_{-v,-w}(K_x B_{v,w} + K_y C_{v,w}) + \frac{1}{u^{v,w} + j\frac{k}{n_o}}\{K_x(K_x - vN)B_{v,w}B^{v,w}_{-v,-w}$$

$$+ K_y(K_y - wN)C_{v,w}C^{v,w}_{-v,-w}\}]] \qquad (3.29)$$

In arriving at the above two equations we have not taken those terms which, when expanded in terms of the surface Fourier coefficients ($P_{m,n}$'s), contain products of three or more surface Fourier coefficients. That is, we have considered the terms only up to second order in $P_{m,n}$'s. Since we have assumed that $|n_o| >> 1$ and both f_x^2 and $f_y^2 << 1$, we may discard the bracketed term within the double summation in (3.28) which has $\frac{1}{kn_o}$ as the coefficient. By doing so and by using the expansion of auxiliary Fourier coefficients in terms of the surface Fourier coefficient, given by the equation (A.8), the vertical component of the surface field up to the second order in $P_{m,n}$'s may be given as follows:

$$\underline{E_{azo}}(\vec{K}) = \frac{C_d\, G(K_x,K_y)}{u + jk\Delta_a(K_x,K_y)} \exp(-h_o u) \qquad (3.30)$$

where

$$G(K_x,K_y) = 1 + \frac{u}{A_{o,o}} \sum_{(p,q)\neq(o,o)} \left\{-2u^{p,q} + \frac{N^2(p^2+q^2)}{(u^{p,q} + jk\Delta)}\right\} \frac{|P_{p,q}|^2}{A^{p,q}_{o,o}}, \qquad (3.31)$$

$$\Delta_a(K_x,K_y) = \Delta - \frac{j}{k}(u + jk\Delta)\frac{u^2h^2}{2}$$

$$- \frac{j}{k} \sum_{(v,w)\neq(o,o)} [-u(u + jk\Delta) \{3u^{v,w} + \frac{vN(K_x - vN) + wN(K_y - wN)}{(u^{v,w} + jk\Delta)}\}$$

$$- \{K_x(K_y - wN) + K_y(K_x - vN)\}\{j + \frac{\Delta}{k}(u + jk\ )\}\{\frac{\Delta}{k} + \frac{j}{u^{v,w} + jk\Delta}\}vwN^2$$

$$+ \frac{N^2}{u^{v,w} + jk\Delta} \{K_x(K_x - vN)v^2 + K_y(K_y - wN)w^2\}$$

$$+ u^{v,w}(K_x v + K_y w)N] \frac{|P_{v,w}|^2}{A^{v,w}_{o,o}} , \qquad (3.32)$$

$$A^{v,w}_{o,o} : \text{spatial average of } \exp(-fu^{v,w}) = \frac{1}{L^2} \int_{-L/2}^{L/2}\int_{-L/2}^{L/2} \exp(-fu^{v,w})dxdy, \qquad (3.33)$$

and

$$u^{v,w} = \begin{cases} [(K_x - vN)^2 + (K_y - wN)^2 - k^2]^{\frac{1}{2}} ; & \text{for real root} \\ j[k^2 - (K_x - vN)^2 - (K_y - wN)^2]^{\frac{1}{2}}; & \text{for imaginary root} . \end{cases} \qquad (3.34)$$

h^2 is the mean square height of the surface, i.e.,

$$h^2 = \frac{1}{L^2} \int_{-L/2}^{L/2}\int_{-L/2}^{L/2} f(x,y)dxdy = \sum_{p,q} |P_{p,q}|^2 . \qquad (3.35)$$

In deriving (3.31) and (3.32) we have approximated $\frac{1}{n_o}$ as Δ, where Δ is the normalized surface impedance for the flat surface [Wait (1964)]. It is given as

$$\Delta = \frac{(n_o^2 - 1)^{\frac{1}{2}}}{n_o^2} \cong \frac{1}{n_o} \text{ for } |n_o| >> 1. \qquad (3.36)$$

The nature of (3.30) when compared to the similar equation (2.30) for the flat surface suggests that Δ_a may be treated as a modified normalized surface impedance or simply to call it a modified surface impedance. It may easily be seen that this modified surface impedance takes into an account the surface roughness which is to be expected.

When dealing with any problem in a transform domain a return in the original domain itself becomes sometimes a problem. This hurdle for an inverse transform quite often occurs in the area of EM propagation and scattering. In the present case we are faced with the problem of finding the inverse spatial transform of (3.30) to return to the physical x,y domain, i.e., evaluation of the following double integral is required;

$$Q = \frac{1}{4\pi^2} \int_{K_x} \int_{K_y} \frac{G(K_x,K_y)}{u + jk\Delta_a(K_x,K_y)} \exp(-h_o u + jK_x x + jK_y y) dK_x dK_y \tag{3.37}$$

where G and Δ_a are given by (3.31) and (3.32) respectively.

It may not be possible to evaluate the above double integral exactly. However, some techniques such as stationary phase and steepest descent [Friedman (1969, ch. 3)] are available to evaluate such integrals asymptotically and serve as a good approximation for the integral. By using these two techniques, i.e., stationary phase and steepest descent, we have evaluated the above integral (3.37). The procedure is summarized here but the details of the evaluation are given in appendix C. The integration variables (K_x,K_y) and inverse transform variables (x,y) are changed to polar coordinate variables (λ,ψ) and (ρ,θ) such that λ

is allowed to take negative value also. By assuming the function $G/(u + jk\Delta_a)$ as slowly varying and $|\rho\lambda|$ as a large parameter, the ψ integral is evalued first using the stationary phase method. We have taken only the leading term of the resulting asymptotic series which is of the order of $(\rho|\lambda|)^{-\frac{1}{2}}$. The remaining integral with respect λ is then reduced to a sum of two integrals. The first integral is evaluated asymptotically using the standard steepest descent method. The form of the remaining second integral is similar to the one evaluated by Wait (1970, ch. 2) using a modified steepest descent method. Following his method and the assumption that $|\Delta_a| << 1$, the second integral has been evaluated. The large parameter taken for both the integrals is kR, where k is assumed positive and $R = [\rho^2 + h_o^2]^{\frac{1}{2}}$. Both integrations are performed to their leading terms which are of the order of $(kR)^{-\frac{1}{2}}$. The end result of these evaluations is as follows [equation (C.50) in Appendix C].

$$Q \cong [1 - (\frac{p_a}{w_a})^{\frac{1}{2}} \{1 - F(w_a)\}] \, G(-k \cos \theta, -k \sin \theta) \cdot \exp(-jkR)/(2\pi R) \qquad (3.38)$$

where

$$p_a = -jk\rho\frac{\Delta_a^2}{2}$$

$$w_a = p_a(1 + \frac{h_o}{R\Delta_a})^2$$

$$\Delta_a = \Delta_a(-k \cos \theta, -k \sin \theta) \quad \text{with } \theta = \tan^{-1}(\frac{y}{x}) \qquad (3.39)$$

$$R = (\rho^2 + h_o^2)^{\frac{1}{2}} \quad \text{with } \rho = (x^2 + y^2)^{\frac{1}{2}}$$

$$F(w_a) = 1 - j(\pi w_a)^{\frac{1}{2}} \exp(-w_a)\text{erfc}(j\sqrt{w_a})$$

$\text{erfc}(z_o)$: complementary error function $= 1 - \text{erf}(z_o)$

$\text{erf}(z_o)$: error function $= \frac{2}{\sqrt{\pi}} \int_0^{z_o} \exp(-z^2)dz$.

For convenience we take the limit as the dipole height (h_o) tends to zero. In this event $R = \rho$ and $w_a = p_a$ and thus we have

$$Q = G(-k \cos \theta, -k \sin \theta) F(p_a) \exp(-jk\rho)/(2\pi\rho) \quad (3.40)$$

By using the above equation and noting from (3.31) that $G(-k \cos \theta, -k \sin \theta) = 1$ we have an approximate inverse spatial transform of the vertical component of the zero order surface field as

$$E_{azo}(\vec{x}) \cong C_d F(p_a) \exp(-jk\rho)/(2\pi\rho) \quad . \quad (3.41)$$

The function $F(p_a)$ is commonly referred to as the ground wave attenuation function with p_a as the numerical distance [Wait (1970, ch. 2)]. The form of this function is the same as the attenuation coefficient of the ground wave propagating over a flat ground presented by Norton (1937). For the flat surface case Δ_a reduces to Δ as may be seen from (3.32), and, therefore, $P_a = -jk\rho\frac{\Delta^2}{2}$. For a periodic surface the numerical distance is dependent on a modified surface impedance Δ_a as evident from (3.39). Thus (3.41) represents a ground wave propagating out from the source with a modified surface impedance. This solution may be treated either in the phasor form for a continuously excited dipole source or in the temporal Fourier transform domain for the dipole having a pulsed excitation such as assumed in (3.1).

The surface impedance Δ_a, given by (3.32), remains to be evaluated at $K_x = -k \cos \theta$ and $K_y = -k \sin \theta$. This evaluation yields

$$\Delta_a(-k \cos \theta, -k \sin \theta) = \Delta + \sum_{v,w} [\frac{k(v \cos \theta + w \sin \theta)^2 N^2}{D(v,w) + k\Delta}$$

$$- \Delta \{\frac{k(v \cos \theta + w \sin \theta) N D(v,w)}{D(v,w) + k\Delta} + (\sin 2\theta + \frac{vN}{k} \sin \theta + \frac{wN}{k} \cos \theta)$$

$$\cdot (1 + \frac{k}{D(v,w) + k\Delta} + \Delta^2) vwN^2\}] \frac{|P_{v,w}|^2}{A(v,w)} \quad (3.42)$$

In (3.42) we have removed the restriction $(v,w) \neq (o,o)$ on the summation indices as $P_{o,o} = 0$. The variables D and A are

$$D(v,w) = \begin{cases} [k^2 - (vN + k\cos\theta)^2 - (wN + k\sin\theta)^2]^{\frac{1}{2}} \ ; \text{ for real root} \\ -j[(vN + k\cos\theta)^2 + (wN + k\sin\theta)^2 - k^2]^{\frac{1}{2}} \ ; \text{ for imaginary root} \end{cases} \tag{3.43}$$

$$A(v,w) = \frac{1}{L^2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \exp\,[-jD(v,w)f(x,y)]dxdy \ .$$

The above modified surface impedance is a function of θ, i.e., it is dependent on the direction of propagation. This dependency on θ is to be expected as surface roughness may not necessarily be the same in all directions. It may be shown that by changing $\cos\theta$ and $\sin\theta$ to $-\cos\theta$ and $-\sin\theta$ in the expression (3.42) for Δ_a the expression remains unaltered. That is to say

$$\Delta_a\,(-k\cos\theta,\ -k\sin\theta) = \Delta_a\,(k\cos\theta,\ k\sin\theta)\ . \tag{3.44}$$

It may be seen that this modified surface impedance does not have any contribution from the first order term, i.e., $P_{v,w}$. This behaviour has been observed by Barrick (1971) and Wait (1971) also. By using a surface wave formulation in the perturbation technique Barrick (1971) has derived an expression for the modified surface impedance to the second order for propagation along x axis ($\theta = 0$). The result derived here does not agree when compared directly with his expression, However, under the following approximations a comparison may be made.

If we assume small variations in the surface height such that

$$\exp\,[-jD(v,w)f(x,y)] \cong 1 - j\,D(v,w)f(x,y)$$

and neglect $O(\Delta^2)$ and $O(\Delta^3)$ terms in (3.42) as Δ is small, we arrive at the following expression for Δ_a in $\theta = 0$ direction, hence denoted as Δ_{ax}.

$$\Delta_{ax} = \Delta + \sum_{v,w} \left[\frac{k\ (vN)^2}{b + k\Delta} - \Delta \left\{\frac{k\ vNb}{b + k\Delta} + \frac{vN\ (wN)^2}{k}\right\}\right] |P_{v,w}|^2 \qquad (3.45)$$

where the variable b is defined as

$$b = \begin{cases} [k^2 - (vN + k)^2 - (wN)^2]^{\frac{1}{2}} ; \text{ for real root} \\ -j[(vN + k)^2 + (wN)^2 - k^2]^{\frac{1}{2}} ; \text{ for imaginary root} \end{cases} \qquad (3.46)$$

Since the last term $\frac{vN\ (wN)^2}{k}$ in the above double summation sums to zero, we have

$$\Delta_{ax} = \Delta + \sum_{v,w} \left[\frac{k\ (vn)^2}{b + k\Delta} - \Delta\ \frac{k\ vNb}{b + k\Delta}\right] |P_{v,w}|^2 \qquad (3.47)$$

If we now approximate Barrick's equation (20) as

$$D(k,1) = 1 + \frac{\Delta k_o}{b(k + h,1)} \quad \text{with } h = k_o a$$

and compare our result (3.47), when written in Barrick's notation, with his equation (21) we find that the two expressions agree in part. That is, his result contains a few terms more in addition to those given by (3.47). However, if the limit of Δ in the above double summation is taken to be zero we obtain

$$\Delta_{ax} = \Delta + k \sum_{v,w} \frac{(vN)^2}{b} |P_{v,w}|^2 \qquad (3.48)$$

which agrees with Feinberg's result [Barrick (1971)]. Barrick's result also agree with that of Feinberg when the above limit of Δ is taken there. The above equation (3.48) in the present form means that the modified surface impedance is equal to Δ, accounting for the finite conductivity of the flat surface, plus an effect of surface roughness but then treating the surface as perfectly conducting.

3.4 First Order Surface Field

The first order surface field in the spatial transform domain is given from (2.27) as

$$\overline{E_{t1}}(\vec{K}) = -(I - R_1)^{-1} \underset{(m,n)\neq(o,o)}{L_{m,n}} (I - R_2)^{-1} \overline{E_{ci}^{m,n}} \qquad (3.49)$$

where an additional subscript 1 on $\overline{E_t}$ means the first order field. I is the identity matrix and the series R_1 and R_2 are respectively given by (2.28a) and (2.28b). By using (2.20) and by approximating R_1 and R_2 by their first term only the above may be written as

$$\overline{E_{t1}}(\vec{K}) = -[T_{o,o} - \sum_{(v,w)\neq(o,o)} T_{v,w} (T_{o,o}^{v,w})^{-1} T_{-v,-w}^{v,w}]^{-1}$$

$$\cdot \sum_{(m,n)\neq(o,o)} T_{m,n} [T_{o,o}^{m,n} - \sum_{(v,w)\neq\left\{\begin{matrix}(o,o)\\(m,n)\end{matrix}\right\}} T_{v-m,w-n}^{m,n} (T_{o,o}^{v,w})^{-1} T_{m-v,n-w}^{v,w}]^{-1}$$

$$\cdot \overline{E_{oi}^{m,n}} \qquad (3.50)$$

The general matrix $T_{v,w}^{m,n}$ and the column vector $\overline{E_{oi}}$ are given by (3.20) and (3.14) respectively. Let

$$A = T_{o,o} - \sum_{(v,w)\neq(o,o)} T_{v,w} (T_{o,o}^{v,w})^{-1} T_{-v,-w}^{v,w},$$

$$B = T_{o,o}^{m,n} - \sum_{(v,w)\neq\left\{\begin{matrix}(o,o)\\(m,n)\end{matrix}\right\}} T_{v-m,w-n}^{m,n} (T_{o,o}^{v,w})^{-1} T_{m-v,n-w}^{v,w}, \qquad (3.51)$$

and assume their inverses may symbolically be given as

$$A^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \frac{1}{\det A} \quad (3.52)$$

$$B^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \frac{1}{\det B} \quad (3.53)$$

where $a_{i\,j}$ and $b_{i\,j}$ are the cofactors of the matrices A and B respectively and det A and det B are their determinants. The matrix A is the same as for the zero order surface field.

By employing (2.17), (3.14), (3.20), (3.52), and (3.53) in (3.50) the vertical component of the first order surface field may be derived as follows:

$$\underline{E_{az1}} (\vec{K}) = -C_d \sum_{(m,n)\neq(o,o)} \frac{1}{\det A} [a_{31} \{A_{m,n} b_{13} - (D_{m,n} + \frac{K_x}{kn_o} C_{m,n}) b_{23} + jK_x A_{m,n} b_{33}\} + a_{32} \{-(D_{m,n} + \frac{K_y}{kn_o} B_{m,n}) b_{13} + A_{m,n} b_{23} + jK_y A_{m,n} b_{33}\} + a_{33} \{B_{m,n} b_{13} + C_{m,n} b_{23} + (u + jkn_o) A_{m,n} b_{33}\}] \cdot \frac{\exp(-h_o u^{m,n})}{\det B} \quad (3.54)$$

where C_d is given by (3.10). The above coefactors of the matrices A and B may be found by using (3.20) and (3.23) and performing the required algebra in (3.51). Similarly, det A and det B may also be derived. By doing so and by discarding those terms which are of the third or higher orders in $P_{m,n}$'s (the surface Fourier coefficients) we arrive at

$$E_{az1}(\bar{K}) = -C_d \sum_{(m,n)\neq(o,o)} \frac{1}{u + jk\Delta_a(K_x,K_y)} \Bigg\{ \{mN(K_x - mN) + nN(K_y - nN)$$

$$- u(u + jk\Delta)\}P_{m,n} - u \sum_{p,q} \{pN(K_x - mN) + qN(K_y - nN) - \frac{u^2}{2}(u + jk\Delta)\}$$

$$\cdot P_{p,q} P_{m-p,n-q} \Bigg\} \frac{\exp(-h_o u^{m,n})}{u^{m,n} + jk\Delta_b(K_x - mN, K_y - nN)} \qquad (3.55)$$

where $\Delta = \frac{1}{n_o}$. Δ_a and Δ_b are two modified surface impedances. Δ_a is the same as for the zero order and is given by (3.32). The second surface impedance Δ_b turns out to be

$$\Delta_b(K_x - mN, K_y - nN) = \Delta - \frac{j}{k}(u^{m,n} + jk\Delta)\frac{(u^{m,n}h)^2}{2}$$

$$-\frac{j}{k} \sum_{(v,w)\neq \left\{\begin{matrix}(o,o)\\(m,n)\end{matrix}\right\}} \Bigg\{ -u^{m,n}(u^{m,n} + jk\Delta)\{3u^{v,w} + \frac{(v - m)(K_x - vN) + (w - n)(K_y - wN)}{u^{v,w} + jk\Delta} N\}$$

$$- \{(K_x - mN)(K_y - wN) + (K_y - nN)(K_x - vN)\}\{j + \frac{\Delta}{k}(u^{m,n} + jk\Delta)\}\{\frac{\Delta}{k} + \frac{j}{u^{v,w} + jk\Delta}\}$$

$$\cdot (v - m)(w - n)N^2 + u^{v,w}\{(K_x - mN)(v - m) + (K_y - nN)(w - n)\}N$$

$$+ \frac{N^2}{u^{v,w} + jk\Delta}\{(K_x - mN)(K_x - vN)(v - m)^2 + (K_y - nN)(K_y - wN)(w - n)^2\} \Bigg\}$$

$$\cdot \frac{|P_{v-m,w-n}|^2}{A^{v,w}_{o,o}} \qquad (3.56)$$

where $A^{v,w}_{o,o}$, $u^{v,w}$ and h^2 are defined by (3.33) to (3.35). In deriving the above two equations we have used (A.8) which gives an expansion of the auxiliary Fourier coefficients in terms of the surface Fourier coefficients.

In order to return (3.55) to the original (x,y) domain the inverse spatial Fourier transform is required. For this purpose we find it convenient to use the well known convolution and shifting theorems of Fourier transforms. These theorems enable us to write the first order field in (x,y) domain as

$$\underline{E_{az1}}(\vec{x}) = - C_d$$

$$\cdot \sum_{(m,n)\neq(o,o)} [\exp(jmNx + jnNy) F^{-1}_{spa} \{\frac{\exp(-h_o u)}{u + jk\Delta_b(K_x,K_y)}\}]$$

$$*[F^{-1}_{spa} \{\frac{G(K_x,K_y)}{u + jk\Delta_a(K_x,K_y)}\}] \qquad (3.57)$$

where F^{-1}_{spa} { } means the inverse spatial transform as per the definition (1.9) and the asterisk(*) here means a two dimensional convolution with respect to x and y. From (3.55) the function G may be given as

$$G(K_x,K_y) = [mN(K_x - mN) + nN(K_y - nN) - u(u + jk\Delta)]P_{m,n}$$

$$- u \sum_{p,q} [pN(K_x - mN) + qN(K_y - nN) - \frac{u^2}{2}(u + jk\Delta)]P_{p,q}P_{m-p,n-q} \qquad (3.58)$$

The method used in the zero order case for finding an approximate inverse spatial transform in the far field sense may again be utilized here. This method is based on an asymptotic evaluation of the double Fourier integral and is described in Appendix C. Since the factor $\exp(-h_o u)$ is not present in the second function in (3.57) we may artificially introduce this factor to comply with the form of the integral that has been evaluated in Appendix C. Then after evaluation the limit of h_o may be taken to zero to get the inverse transform of the second function in (3.57). This way we obtain

$$F^{-1}_{spa}\left\{\frac{G(K_x,K_y)}{u + jk\Delta_a(K_x,K_y)}\right\} \cong G(-k\cos\theta, -k\sin\theta)F(p_a)\exp(-jk\rho)/(2\pi\rho). \quad (3.59)$$

In a similar way an approximate inverse transform for the first function in (3.57) may also be derived with the result analogus to (3.38). Which in the limit of zero antenna height reduces to:

$$F^{-1}_{spa}\left\{\frac{\exp(-h_o u)}{u + jk\Delta_b(K_x,K_y)}\right\}\Bigg|_{h_o\to 0} \cong F(P_b)\exp(-jk\rho)/(2\pi\rho). \quad (3.60)$$

In the above two equations F is the attenuation function and is given by (3.39). p_a and p_b are two numerical distances with modified surface impedances Δ_a and Δ_b respectively. They may be given as

$$p_a = -jk\frac{\rho}{2}\Delta_a^2(-k\cos\theta, -k\sin\theta)$$
$$p_b = -jk\frac{\rho}{2}\Delta_b^2(-k\cos\theta, -k\sin\theta) \quad (3.61)$$

where Δ_a is given by (3.42). From (3.56) Δ_b $(-k\cos\theta, -k\sin\theta)$ may be evaluated as

$$\Delta_b(-k\cos\theta, -k\sin\theta) = \Delta_b(K_x - mN, K_y - nN)\Bigg|_{\substack{(K_x - mN) = -k\cos\theta \\ (K_y - nN) = -k\sin\theta}}$$

$$= \Delta + \sum_{(v,w)\neq(-m,-n)} \left[\frac{k(v\cos\theta + w\sin\theta)^2 N^2}{D(v,w) + k\Delta} - \Delta\left\{\frac{k(v\cos\theta + w\sin\theta)ND(v,w)}{D(v,w) + k\Delta}\right.\right.$$
$$\left.\left. + (\sin 2\theta + \frac{vN}{k}\sin\theta + \frac{wN}{k}\cos\theta)(1 + \frac{k\Delta}{D(v,w) + k\Delta} + \Delta^2)vwN^2\right\}\right]\frac{|P_{v,w}|^2}{A(v,w)} \quad (3.62)$$

where D(v,w) and A(v,w) are defined by (3.43). In a similar way $G(-k\cos\theta, -k\sin\theta)$ may be evaluated from (3.58) as

$$G\,(-k\cos\theta,\,-k\sin\theta) = [-mN(k\cos\theta + mN) - nN(k\sin\theta + nN)]P_{m,n} \tag{3.63}$$

In deriving (3.62) from (3.56) we have replaced the summation indices v and w by v + m and w + n respectively. Further, a restriction $(v,w) \neq (m,n)$ on the summations in (3.56) has been removed on the assumption $P_{o,o} = 0$. This restriction is equivalent to $(v,w) \neq (o,o)$ in (3.62). It may be seen that the expressions for the two surface impedances Δ_a (3.42) and Δ_b (3.62) are the same except that there is a restriction $(v,w) \neq (-m,-n)$ on the summations for Δ_b, whereas no restriction is there for Δ_a. These m and n refer to the summation indices in (3.57). Further, except for this restriction on Δ_b, the expressions for Δ_a or Δ_b are not functions of m and n.

Since the two numerical distances p_a and p_b are functions of ρ, cos θ, and sin θ, it is more convenient to denote the two attenuation functions as

$$F(p_a) = F_a\,\{\rho,\,\cos\theta,\,\sin\theta\} \tag{3.64a}$$

$$F(p_b) = F_b\,\{\rho,\,\cos\theta,\,\sin\theta,\,(v,w) \neq (-m,-n)\} \tag{3.64b}$$

where in (3.64b) we have explicitly shown the restriction on (v,w) for Δ_b through the argument of F_b.

By using (3.59), (3.60), (3.63), and (3.64) in (3.57) yields E_{az1} as

$$E_{az1}\,(\vec{x}) \approx \frac{C_d}{4\pi^2} \sum_{m,n} P_{m,n}\,[F_b\,\{\rho,\,\cos\theta,\,\sin\theta,(v,w) \neq (-m,-n)\}$$

$$\cdot\,\frac{1}{\rho}\exp\,(-jk\rho + jmNx + jnNy)*F_a\{\rho,\,\cos\,\theta,\,\sin\,\theta\}\,\frac{1}{\rho}$$

$$\cdot\quad \{mN(k\cos\theta + mN) + nN(k\sin\theta + nN)\}\,\exp(-jk\rho)] \tag{3.65}$$

where we have removed the restriction $(m,n) \neq (o,o)$ on the summations as $P_{o,o}$ is assumed to be zero. The above equation may also be written in the following form of a double convolution integral,

$$E_{az1}(\vec{x}) \cong \frac{C_d}{4\pi^2} \sum_{m,n} P_{m,n} \int_{x'} \int_{y'} F_b \{\rho', \cos\theta', \sin\theta', (v,w) \neq (-m,-n)\}$$

$$\cdot F_a \{\rho_x, \cos\theta_x, \sin\theta_x\} [mN(k \cos\theta_x + mN) + nN(k \sin\theta_x + nN)]$$

$$\cdot \exp [j(-k\rho' - k\rho_x + mNx' + nNy')]$$

$$\cdot \frac{1}{\rho'\rho_x} dx'dy' \qquad (3.66)$$

where

$$\rho' = (x'^2 + y'^2)^{\frac{1}{2}}$$

$$\rho_x = [(x - x')^2 + (y - y')^2]^{\frac{1}{2}}$$

$$\cos\theta' = \frac{x'}{\rho'} \; ; \; \sin\theta' = \frac{y'}{\rho'}$$

$$\cos\theta_x = \frac{x - x'}{\rho_x} \; ; \; \sin\theta_x = \frac{y - y'}{\rho_x} .$$

The above equation (3.65) or (3.66) thus represents the vertical component of a first order approximation of the surface field in (x,y) domain. For a continuously excited source this electric field may be understood in the phasor form, whereas for a pulsed source this may be treated in the temporal Fourier transform domain with ω as the transform variable. Atthis point it is better to understand this solution physically. We have assumed that the surface is periodic in both x and y. That is, it consists of a linear superposition of many two dimensional sinusoidal surfaces through its representation in the form of a two dimensional Fourier series. Consider one such sinusoidal surface with wave numbers mN and nN. For this sinusoidal surface we have a source field propagating in the ground wave mode in a direction θ'. This source field is identified by the attenuation function F_b with modified surface impedance Δ_b. Because of the variations in the surface elevation, this field is being scattered in all directions by an elementary area of the surface located at (ρ',θ'). A part of this scattered field, which is also travelling in ground wave mode, is being received at the observation point (x,y). This scattered surface wave has attenuation function F_a with modified surface impedance Δ_a. Therefore the first order scattered field being received at the observation point from the complete area of this sinusoial surface is given by the integration over all x' and y'. As a result, the scattered field from the original surface may be approximated by linearly summing the contributions to this field coming from each sinusoial surface, i.e., summing the contributions over all mN and nN wave numbers space. Further, this scattered surface field is in addition to what is being received directly from the source at the observation point. This directly received field is termed as the zero order surface field and has been discussed previously.

In an analogy with the circuit theory extended to two dimensions, the solution (3.65) represents the output of a linear system, the system being surface here. The term containing F_b, i.e., the source field term, may be considered as the input to the system. Whereas the term containing F_a may be taken as the impulse response of the system.

3.4.1 First Order Backscattered Surface Field

In (3.66) if the observation point (x,y) is moved back to the source point, i.e., x = y = 0, a field received then is termed the first order backscattered surface field. This backscattered field will be denoted as E_{zb1}. Under this condition we have

$$E_{zb1} = E_{az1}(0,0)$$

$$= \frac{C_d}{4\pi^2} \cdot \sum_{m,n} P_{m,n} \int_0^\infty \int_{-\pi}^{\pi} F_b \{\rho', \cos\theta', \sin\theta', (v,w) \neq (-m,-n)\}$$

$$\cdot F_a \{\rho', -\cos\theta', -\sin\theta'\} [mN(mN - k\cos\theta') + nN(nN - k\sin\theta')]$$

$$\frac{\exp(-2jk\rho')}{\rho'} \exp[j\rho' N(m\cos\theta' + n\sin\theta')] \, d\theta' d\rho' \quad . \qquad (3.67)$$

In deriving the above equation we have transformed the integration variables from cartesian (x,',y') to polar coordinate (ρ',θ').

Since we are interested in a pulsed exitation of the source dipole from an HF radar point of view, we will prefer to treat the above equation in the temporal Fourier transform domain instead of the phasor form. This means E_{zb1} is a function of ω, where ω is the transform variable. Let us rewrite the equation (3.1) for a pulsed exitation of the dipole source of the form

$$\tilde{I}(t) = I_o \, G_{\tau_o}(t) \exp(j\omega_o t) \qquad (3.68)$$

where I_o is the peak current and ω_o is the radian frequency of the current. $G_{\tau_o}(t)$ is a gate function and is defined by (3.2). Fourier transforming the above equation with respect to time t yields [Lathi (1968, ch. 1)]

$$I(\omega) = I_o \, \tau_o \, Sa\left[(\omega - \omega_o)\frac{\tau_o}{2}\right] \qquad (3.69)$$

where Sa represents the sampling function. It is defined as

$$Sa(\omega) = \frac{\sin\omega}{\omega} \qquad (3.70)$$

Therefore, from (3.10), C_d may be given as

$$C_d = -j\ \mu_o\ I_o\ dl\ \tau_o\ \omega\ Sa\ [(\omega - \omega_o)\ \frac{\tau_o}{2}]\ . \tag{3.71}$$

We now consider those terms in (3.67) in conjunction with (3.71) which are functions of ω or k $[k = \omega(\mu_o \epsilon_o)^{\frac{1}{2}}]$. By denoting these terms as $A_1(\omega)$ we have

$$A_1(\omega) = \omega\ Sa[(\omega - \omega_o)\frac{\tau_o}{2}]\ F_b\{\rho',\ \cos\theta',\ \sin\theta',\ \omega,\ (v,w) \neq (-m,-n)\}$$
$$\cdot F_a\ \{\rho',\ -\cos\theta',\ -\sin\theta',\ \omega\}$$
$$\cdot\ [mN(mN - k\cos\theta') + nN(nN - k\sin\theta')]\ \exp\ (-2jk\rho') \tag{3.72}$$

where for convenience we have written ω explicitly in the argument of F's as they are functions of ω also. Let us consider the sampling function. This is an oscillatory function with decaying peaks. It has an absolute maximum of unit amplitude and this peak occurs at the point where the argument of the function is zero, i.e., in this case at $\omega = \omega_o$. The function is symmetrical also about this peak point. The first zero crossing of the function occurs when $(\omega - \omega_o)\ \frac{\tau_o}{2} = \pm\pi$, or at $\omega = \omega_o \pm \frac{2\pi}{\tau_o}$. If we consider the range of ω between the first zero crossings only, i.e., $(\omega_o - \frac{2\pi}{\tau_o}) \leq \omega \leq (\omega_o + \frac{2\pi}{\tau_o})$, we find that this range is very small for HF radars. For example, for a typical radar operating at 25.4 MHz with a pulse width of 8 μsec, this range may be calculated to be $158.81 \leq \omega \leq 160.38$ M rad/sec or $25.275 \leq f \leq 25.525$ MHz. Further, the two attenuation functions F_a and F_b may be considered as slowly varying with respect to ω especially for a highly conducting surface. For example, for a flat surface with $\epsilon_r = 81$ and $\sigma_0 \approx 4$ mhos/m (sea water) the numerical distance

p_e ($p_e = -jk\rho\Delta^2/2$) varies from $0.93 \times 10^{-4}\rho \angle -1.63°$ to $0.95 \times 10^{-4}\rho \angle -1.64°$ for the above range of ω. Over this range of p_e both magnitude and phase of the attenuation function remain almost constant for $\rho \leq 100$ Km as may be seen from the graphs presented by Wait (1970, ch. 2). Therefore in the region between the first zero crossings, i.e., over the first cycle of the sampling function centered at $\omega = \omega_o$, we may have the following approximation for the slowly varying functions in (3.72),

$$\omega \, \text{Sa}\,[(\omega - \omega_o)\frac{\tau_o}{2}]\, F_b\{\rho', \cos\theta', \sin\theta', \omega, (v,w) \neq (-m, -n)\}$$

$$\cdot F_a\{\rho', -\cos\theta', -\sin\theta', \omega\}$$

$$\cdot [mN(mN - k\cos\theta') + nN(nN - k\sin\theta')]$$

$$\approx \omega_o \, \text{Sa}[(\omega - \omega_o)\frac{\tau_o}{2}]\, F_b\{\rho', \cos\theta', \sin\theta', \omega_o, (v,w) \neq (-m,-n)\}$$

$$\cdot F_a\{\rho', -\cos\theta', -\sin\theta', \omega_o\}[mN(mN - k_o\cos\theta') + nN(nN - k_o\sin\theta')] \qquad (3.73)$$

where

$$k_o = \omega_o(\mu_o\, \varepsilon_o)^{\frac{1}{2}} . \qquad (3.74)$$

The above approximation may be extended to all ω since the peaks in the subsequent cycles of the sampling function decay fast with increasing number of cycles. The second cycle has a maximum amplitude of 20% to that of the first cycle peak and it occurs when the argument of the function is $\pm 3\pi/2$. Therefore the main contribution of the sampling function comes from its first cycle. It may be seen that in this approximation we have not included the exponential factor of (3.72) which is $\exp[-2jk\rho']$ or $\exp[-2j\rho'\omega(\mu_o\, \varepsilon_o)^{\frac{1}{2}}]$. This is because this function being a purely oscillatory function may change rapidly even for a small change in ω when ρ' is large. Thus by using the above approximation (3.73) in

(3.72) and by extending it to all ω we obtain

$$A_1(\omega) \cong \omega_o \; F_b \{\rho', \cos\theta', \sin\theta', \omega_o, (v,w) \neq (-m,-n)\}$$
$$\cdot F_a \{\rho', -\cos\theta', -\sin\theta', \omega_o\}$$
$$\cdot [mN(mN - k_o\cos\theta') + nN(nN - k_o\sin\theta')]$$
$$\cdot Sa[(\omega - \omega_o)\frac{\tau_o}{2}] \; \exp[-2j\rho'\omega(\mu_o \varepsilon_o)^{\frac{1}{2}}] \; . \tag{3.75}$$

The above equation may easily be inverse Fourier transformed to give [Lathi (1968, ch. 1)] $A_1(t)$ as

$$A_1(t) = \omega_o \; F_b \{\rho', \cos\theta', \sin\theta', \omega_o, (v,w) \neq (-m,-n)\}$$
$$\cdot F_a \{\rho', -\cos\theta', -\sin\theta', \omega_o\}$$
$$\cdot [mN(mN - k_o\cos\theta') + nN(nN - k_o\sin\theta')] \exp(j\omega_o t - 2jk_o\rho')$$
$$\cdot \frac{1}{\tau_o} G_{\tau_o}(t - \frac{2\rho'}{c}) \tag{3.76}$$

where c is the velocity of propagation and is given by

$$c = (\varepsilon_o \mu_o)^{-\frac{1}{2}} \cong 3 \times 10^8 \text{ m/sec} \; . \tag{3.77}$$

By using (3.71), (3.72), and (3.76) in (3.67) we thus obtain the vertical component of the first order backscattered surface field in time domain which may be written as

$$E_{zb1}(t) = \frac{C_a}{4\pi^2} \exp(j\omega_o t) \sum_{m,n} P_{m,n} \int_0^{\infty} \frac{\exp(-2jk_o\rho')}{\rho'} G_{\tau_o}(t - \frac{2\rho'}{c})$$
$$\cdot [\int_{-\pi}^{\pi} F_b \{\rho', \cos\theta', \sin\theta', (v,w) \neq (-m,-n)\}$$
$$\cdot F_a \{\rho', -\cos\theta', -\sin\theta'\}$$
$$\cdot [mN(mN - k_o\cos\theta') + nN(nN - k_o\sin\theta')]$$
$$\cdot \exp[j\rho' N(m\cos\theta' + n\sin\theta')] d\theta'] d\rho' \; . \tag{3.78}$$

In the above

$$C_a = -j\omega_o \mu_o I_o dl$$
$$k_o = \omega_o/c \qquad (3.79)$$
$$c = (\mu_o \varepsilon_o)^{-\frac{1}{2}},$$

and it is meant that ω has been replaced by ω_o in F_a and F_b.

We have considered that the transmitted signal is a pulse of sinusoid centered at time $t = 0$. We now suppose that the observation time of the received signal is at $t = t_o$ where $t_o > \frac{\tau_o}{2}$. Therefore the received backscattered signal at time t_o may be given as

$$\tilde{E}_{zb1}(t_o) = \frac{C_a}{4\pi^2} \exp(j\omega_o t_o) \sum_{m,n} P_{m,n} \int_0^\infty \frac{\exp(-2jk_o\rho')}{\rho'} G_{\tau_o}(t_o - \frac{2\rho'}{c})$$
$$\cdot \left[\int_{-\pi}^{\pi} F_b \{\rho', \cos\theta', \sin\theta', (v,w) \neq (-m,-n)\} \right.$$
$$\cdot F_a \{\rho', -\cos\theta', -\sin\theta'\}$$
$$\cdot [mN(mN - k_o\cos\theta') + nN(nN - k_o\sin\theta')]$$
$$\cdot \exp[j\rho' N(m\cos\theta' + n\sin\theta')] d\theta'] d\rho' . \qquad (3.80)$$

Now consider the above gate function. From its definition (3.2)

$$G_{\tau_o}(t_o - \frac{2\rho'}{c}) = \begin{cases} 1 ; & \frac{c}{2}(t_o - \frac{\tau_o}{2}) \le \rho' \le \frac{c}{2}(t_o + \frac{\tau_o}{2}) \\ 0 ; & \text{otherwise} \end{cases}$$

The time t_o is a two way transmission time, i.e., the time taken for the transmitted signal to travel to a point on the surface and return back from that point to the receiving point because of the first order scattering at that point. The transmitting and receiving points are the same and are taken as the origin of the coordinate system. Thus $c\,t_o/2$ may be regarded as ρ_o, where ρ_o is the distance of one way travel. Similarly, since τ_o is the pulse width, $c\,\tau_o/2$ may be taken as the radial width of a radar resolution cell located at ρ_o [Skolnik (1970, ch. 1)]. The above gate function may then be rewritten as

$$G_{\tau_o}(t_o - \frac{2\rho'}{c}) = \begin{cases} 1 \ ; \ \rho_o - \Delta_\rho \le \rho' \le \rho_o + \Delta_\rho \\ 0 \ ; \ \text{otherwise} \end{cases} \tag{3.81}$$

where

$$\rho_o = \frac{c\,t_o}{2} \ ; \ \Delta_\rho = \frac{c\,\tau_o}{4} . \tag{3.82}$$

By using (3.81) in (3.80) it follows that

$$\tilde{E}_{zb1}(t_o) \doteq \frac{C_a}{4\pi^2} \exp(j\omega_o t_o) \sum_{m,n} P_{m,n} \int_{\rho_o - \Delta_\rho}^{\rho_o + \Delta_\rho} \frac{\exp(-2jk_o\rho')}{\rho'}$$

$$\cdot \Big[\int_{-\pi}^{\pi} F_b\,(\rho', \cos\theta', \sin\theta', (v,w) \neq (-m,-n))$$

$$\cdot F_a\,\{\rho', -\cos\theta', -\sin\theta'\}$$

$$\cdot [N^2(m^2 + n^2) - Nk_o(m\cos\theta' + n\sin\theta')]$$

$$\cdot \exp[j\rho' N(m\cos\theta' + n\sin\theta')]d\theta'\Big]\, d\rho' . \tag{3.83}$$

The above equation thus represents a first order backscattered surface field received after a elapsed time t_o form an annulus of the surface provided the receiving antenna also is omnidirectional. The mean radius of the annulus is ρ_o and the width is $2\Delta_\rho$. F_b and F_a are the attenuation functions for transmitted and received signals respectively. If the receiving antenna is a narrow beam one, the received backscattered field is from a patch of the surface instead from the annulus. We will now consider this case.

3.4.1.1 Narrow Beam Receiving Antenna

Suppose for receiving purposes we have a very narrow beam antenna of a beam width $2\Delta_\theta$ which is directed along the positive x axis, i.e., the beam is centered at $\theta = 0$. Under this situation the variation of θ' in (3.83) is limited from $-\Delta_\theta$ to Δ_θ and, therefore, the received first order field is from a patch of the surface located at a distance ρ_o. The radial and angular widths of the patch are $2\Delta_\rho$ and $2\Delta_\theta$ as shown in figure 3.2. We thus have

$$\tilde{E}_{zb1}(t_o) = \frac{C_a}{4\pi^2} \exp(j\omega_o t_o) \sum_{m,n} P_{m,n} \int_{\rho_o - \Delta_\rho}^{\rho_o + \Delta_\rho} \frac{\exp(-2jk_o\rho')}{\rho'}$$

$$\cdot \Big[\int_{-\Delta_\theta}^{\Delta_\theta} F_b \{\rho', \cos\theta', \sin\theta', (v,w) \neq (-m,-n)\}$$

$$\cdot F_a \{\rho', -\cos\theta', -\sin\theta'\}$$

$$\cdot [N^2(m^2 + n^2) - Nk_o(m\cos\theta' + n\sin\theta')]$$

$$\cdot \exp[j\rho' N(m\cos\theta' + n\sin\theta')]\, d\theta' \Big]\, d\rho' \quad (3.84)$$

The two integrals with respect to ρ' and θ' in either (3.83) or (3.84) need to be evaluated. Again it may not be possible to evaluate them exactly. However, they may be evaluated approximately. We will now perform such an approximate evaluation for the integrals in (3.84). Since the two attenuation functions F_a and F_b are slowly varying, their variations with respect to ρ' and θ' over the patch may be approximated by setting $\rho' = \rho_o$ and $\theta' = 0$ there.

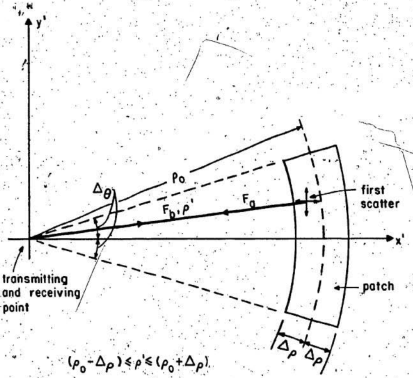


Figure 3.2 First order backscatter from a surface patch for omnidirectional transmission and narrow beam reception (ρ_o = distance of patch, $2\Delta_\rho$ = radial width of patch, $2\Delta_\theta$ = beam width of receiving antenna, F's = ground wave attenuation functions with modified surface impedances)

That is to say,

$$\tilde{E}_{ab1}(t_o) \cong \frac{C_a}{4\pi^2} \exp(j\omega_o t_o) F_a(\rho_o, 1, 0) \sum_{m,n} P_{m,n}$$

$$\cdot F_b\{\rho_o, 1, 0, (v,w) \neq (-m,-n)\} \int_{\rho_o - \Delta_\rho}^{\rho_o + \Delta_\rho} \frac{\exp(-2jk_o\rho')}{\rho'} I_\theta d\rho' \tag{3.85}$$

where we have used the symmetry

$$F_a(\rho_o, -1, 0) = F_a(\rho_o, 1, 0), \tag{3.86}$$

which holds because of the relation (3.44). The integral I_θ in (3.85) is given as

$$I_\theta = \int_{-\Delta_\theta}^{\Delta_\theta} [N^2(m^2 + n^2) - Nk_o(m \cos\theta' + n \sin\theta')]$$

$$\cdot \exp[j\rho' N(m \cos\theta' + \sin\theta')] \, d\theta' \tag{3.87}$$

The above integral I_θ may be evaluated asymptotically using the stationary phase method. The procedure for evaluation is the same as used before for evaluating the integral I_ϕ (C.10) in appendix C. ρ' may be taken as the large parameter for this purpose by assuming a far location of the patch. The result of this evaluation to the leading term may be shown to be as follows:

$$I_\theta \sim \left(\frac{2\pi K_m}{\rho'}\right)^{\frac{1}{2}} [K_m - k_o \, \mathrm{sgn}(m)] \exp[j\rho' K_m \, \mathrm{sgn}(m) - j\frac{\pi}{4} \mathrm{sgn}(m)]$$

$$-|m| \, \alpha \leq n \leq |m| \, \alpha \tag{3.88}$$

where

$$\vec{K}_m = mN\hat{x} + nN\hat{y} \quad ; \quad K_m = |\vec{K}_m|$$

$$\alpha = \tan \Delta_\theta \tag{3.89}$$

$$\mathrm{sgn}(m) = \begin{cases} 1 & , m > 0 \\ -1 & , m < 0 \end{cases}$$

To satisfy the restriction, the nearest integer may be taken for the summation indice n when $(m\alpha)$ is not an integer. By using (3.88) in (3.85) yields $\tilde{E}_{zb1}$ as

$$\tilde{E}_{zb1}(t_o) = \frac{C_a}{(2\pi)^{3/2}} \exp(j\omega_o t_o) F_a(\rho_o, 1, 0) \sum_{\substack{m,n \\ -|m|\alpha \le n \le |m|\alpha}} P_{m,n}$$

$$\cdot F_b\{\rho_o, 1, 0, \ (v,w) \neq (-m,-n)\}$$

$$\cdot \sqrt{K_m}\ [K_m - k_o\ \mathrm{sgn}(m)]\ \exp[-j \tfrac{\pi}{4}\ \mathrm{sgn}(m)]$$

$$\cdot \int_{\rho_o - \Delta_\rho}^{\rho_o + \Delta_\rho} \frac{1}{(\rho')^{3/2}} \exp\ [j\rho' \{\mathrm{sgn}(m) K_m - 2k_o\}]\ d\rho' \ . \tag{3.90}$$

The integration with respect to ρ' may be considered now. Since ρ' varies only from $(\rho_o - \Delta_\rho)$ to $(\rho_o + \Delta_\rho)$ and by assuming $\Delta_\rho \ll \rho_o$, which means that the radial width of the patch is much smaller than the distance of its location, $\frac{1}{(\rho')^{3/2}}$ may be approximated as $\frac{1}{(\rho_o)^{3/2}}$. The remaining integral then may be evaluated exactly to give a following result for $\tilde{E}_{zb1}(t_o)$,

$$\tilde{E}_{zb1}(t_o) \simeq \frac{2C_a \Delta_\rho}{(2\pi\rho_o)^{3/2}} F_a(\rho_o,1,0) \sum_{\substack{m,n \\ -|m|\alpha \le n \le |m|\alpha}} P_{m,n}$$

$$\cdot F_b\{\rho_o,1,0,(v,w) \ne (-m,-n)\}$$

$$\cdot \sqrt{K_m}\,[K_m - k_o\,\mathrm{sgn}(m)]\,\mathrm{Sa}[\Delta_\rho\{\mathrm{sgn}(m)K_m - 2k_o\}]$$

$$\cdot \exp[j\,\mathrm{sgn}(m)(K_m\rho_o - \frac{\pi}{4})] \qquad (3.91)$$

where for completeness the meaning of various symbols may again be given as follows:

$F_b\{\rho_o,1,0,(v,w) \ne -(m,n)\}$: attenuation function for the transmitted signal given by (3.39), (3.42), (3.61), and (3.64a)

$F_a(\rho_o,1,0)$: attenuation function for the received signal given by (3.39), (3.61), (3.62), and (3.64b)

C_a : dipole constant of the transmitting antenna $= -j\omega_o\mu_o I_o dl$

ρ_o : radial distance of the patch from the source $= \frac{ct_o}{2}$

$2\Delta_\theta$: angular width of the patch = beam width of the receiving antenna

c : velocity of propagation $= (\mu_o\varepsilon_o)^{-\frac{1}{2}}$

τ_o : pulse width of the transmitted signal

ω_o : frequency of the transmitted signal (3.92)

k_o : wave number of the transmitted signal $= \omega_o/c$

$\mathrm{Sa}(x)$: sampling function given by (3.70)

$\mathrm{sgn}(m)$: sign function given by (3.89)

$2\Delta_\rho$: radial width of the patch $= c\tau_o/2$

α = $\tan\Delta_\theta$

K_m = $N(m^2 + n^2)^{\frac{1}{2}}$

The above equation (3.91) thus represents an approximate solution for the vertical component of the first order backscattered surface field received at a time t_o from a patch of a two dimensional periodic surface. The transmitted signal is a pulsed sinusoid centered at time $t = 0$ from an elementary vertical electric dipole. For convenience the height of the dipole has been taken to be zero. The two attenuation functions are the same except that in the expression for the modified surface impedance Δ_b (3.62) for F_b there is a restriction on the summation indices (v,w) as shown through the argument of F_b, whereas this restriction is not there for Δ_a (3.42).

3.5 Second Order Surface Field

The procedure developed in sections (3.3) and (3.4) for deriving zero and first orders of the surface field will be closely followed here to derive the second order. From the equation (2.27) the second order surface field in the spatial transform domain may be given as

$$\underline{\overline{E_{t2}}}(\vec{K}) = (I - R_1)^{-1} \underset{(p,q)\neq \left\{\begin{matrix}(o,o)\\(m,n)\end{matrix}\right\}}{L_{p,q}} (I - R_3)^{-1} \underset{(m,n)\neq(o,o)}{L^{p,q}_{m-p,n-q}} (I - R_2)^{-1} \underline{\overline{E^{m,n}_{ci}}} \tag{3.93}$$

where an additional subscript 2 on E_t means the second order field. I is the identity matrix and the series R_1, R_2, and R_3 are given by (2.28a), (2.28b), and (2.28c). By approximating R_1, R_2, and R_3 by their first term only and using (2.20) we get

$$\underline{\overline{E_{t2}}}(\vec{K}) = A^{-1} \sum_{(m,n)\neq(o,o)} \sum_{(p,q)\neq \left\{\begin{matrix}(o,o)\\(m,n)\end{matrix}\right\}} T_{p,q}\, C^{-1}\, T^{p,q}_{m-p,n-q}\, B^{-1}\, \underline{\overline{E^{m,n}_{oi}}} \tag{3.94}$$

where $\overline{E_{oi}}$ is the incident field vector and is given by (3.14) for the dipole source. The matrices A and B with their inverses in a symbolic form are given by (3.51) to (3.53). The matrix C may be given as

$$C = T^{p,q}_{o,o} - \sum_{(v,w)\neq \begin{Bmatrix}(o,o)\\(m,n)\\(p,q)\end{Bmatrix}} T^{p,q}_{v-p,w-q} \left(T^{v,w}_{o,o}\right)^{-1} T^{v,w}_{p-v,q-w} \tag{3.95}$$

The general matrix $T^{m,n}_{v,w}$ is given by (3.20). Let us assume the inverse of C may symbolically be given as

$$C^{-1} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \frac{1}{\det C} \tag{3.96}$$

where c_{ij} are the cofactors of matrix C and det C is the determinant of C.

By employing (2.17), (3.14), (3.20), (3.52), (3.53), and (3.96) in (3.94) the vertical component of the second order surface field may be derived as

$$\begin{aligned} \underline{E_{az2}}(\vec{K}) = C_d \sum_{(m,n)\neq(o,o)} \sum_{(p,q)\neq \begin{Bmatrix}(o,o)\\(m,n)\end{Bmatrix}} \frac{1}{\det A} \\ \cdot \Big\{ [a_{31}\{A_{p,q}c_{11} - (D_{p,q} + \frac{K_x}{kn_o} C_{p,q})c_{21} + jK_x A_{p,q}c_{31}\} \\ + a_{32}\{-(D_{p,q} + \frac{K_y}{kn_o} B_{p,q})c_{11} + A_{p,q}c_{21} + jK_y A_{p,q}c_{31}\} \\ + a_{33}\{B_{p,q}c_{11} + C_{p,q}c_{21} + (u + j\frac{k}{n_o})A_{p,q}c_{31}\}] \\ \cdot [A^{p,q}_{m-p,n-q}b_{13} - (D^{p,q}_{m-p,n-q} + \frac{K_x - pN}{kn_o} C^{p,q}_{m-p,n-q})b_{23} \\ + j(K_x - pN)A^{p,q}_{m-p,n-q}b_{33}] \end{aligned}$$

$$+ [a_{31}\{A_{p,q}c_{12} - (D_{p,q} + \frac{K_x}{kn_o}C_{p,q})c_{22} + jK_xA_{p,q}c_{32}\}$$

$$+ a_{32}\{-(D_{p,q} + \frac{K_y}{kn_o}B_{p,q})c_{12} + A_{p,q}c_{22} + jK_yA_{p,q}c_{32}\}$$

$$+ a_{33}\{B_{p,q}c_{12} + C_{p,q}c_{22} + (u + j\frac{k}{n_o})A_{p,q}c_{32}\}]$$

$$[-(D^{p,q}_{m-p,n-q} + \frac{K_y - qN}{kn_o}B^{p,q}_{m-p,n-q})b_{13} + A^{p,q}_{m-p,n-q}b_{23} + j(K_y - qN)$$

$$A^{p,q}_{m-p,n-q}b_{33}].$$

$$+ [a_{31}\{A_{p,q}c_{13} - (D_{p,q} + \frac{K_x}{kn_o}C_{p,q})c_{23} + jK_xA_{p,q}c_{33}\}$$

$$+ a_{32}\{-(D_{p,q} + \frac{K_y}{kn_o}B_{p,q})c_{13} + A_{p,q}c_{23} + jK_yA_{p,q}c_{33}\}$$

$$+ a_{33}\{B_{p,q}c_{13} + C_{p,q}c_{23} + (u + j\frac{k}{no})A_{p,q}c_{33}\}]$$

$$[B^{p,q}_{m-p,n-q}b_{13} + C^{p,q}_{m-p,n-q}b_{23} + (u^{p,q} + j\frac{k}{n_o})A^{p,q}_{m-p,n-q}b_{33}]\Big\}$$

$$\cdot \frac{\exp(-h_o u^{m,n})}{\det C \det B}. \qquad (3.97)$$

The above cofactors of the matrices A, B, and C may again be found by using (3.20) and (3.23) in (3.51) and (3.95) as well as the three determinants. By doing so and by using the expansion of the auxiliary Fourier coefficients in terms of the surface Fourier coefficients given by (A.8) yields $\underline{E_{az2}}$ as

$$\underline{E_{az2}}(\vec{K}) = C_d \sum_{(m,n)\neq(o,o)} \sum_{(p,q)\neq\left\{\begin{matrix}(o,o)\\(m,n)\end{matrix}\right\}} P_{p,q} P_{m-p,n-q} \frac{G(K_x,K_y)}{u + jk\Delta_a(K_x,K_y)}$$

$$\cdot \frac{1}{u^{p,q} + jk\Delta_c(K_x - pN, K_y - qN)} \cdot \frac{\exp(-h_o u^{m,n})}{u^{m,n} + jk\Delta_b(K_x - mN, K_y - nN)} \qquad (3.98)$$

where we have discarded those terms which are of the third or higher orders in $P_{m,n}$'s. The function G may be given as

$$G(K_x,K_y) = u^{p,q}(u^{p,q} + jk\Delta)\{p(m-p) + q(n-q)\}N^2$$
$$- j\frac{\Delta}{k}(u^{p,q} + jk\Delta)\{p(m-p)(K_x - pN)(K_y - nN) + q(n-q)$$
$$\cdot (K_y - qN)(K_x - mN)\}N^2$$
$$+ \{pN(K_x - pN) + qN(K_y - qN) - u(u + jk\Delta)\}$$
$$\cdot \{(m-p)N(K_x - mN) + (n-q)N(K_y - nN) - u^{p,q}(u^{p,q} + jk\Delta)\}$$

(3.99)

where $\Delta \equiv \frac{1}{n_o}$. Δ_a, Δ_b, and Δ_c are modified surface impedances. Δ_a and Δ_b are the same as for the first order surface field and are given by (3.32) and (3.56) respectively. Δ_c may be given as

$$\Delta_c(K_x - pN, K_y - qN) = \Delta - \frac{j}{k}(u^{p,q} + jk\Delta)\frac{(u^{p,q}h)^2}{2}$$
$$- \frac{j}{k}\sum_{(v,w)\neq\begin{Bmatrix}(o,o)\\(m,n)\\(p,q)\end{Bmatrix}}\left\{-u^{p,q}(u^{p,q} + jk\Delta)\{3u^{v,w} + \frac{(v-p)(K_x - vN) + (w-q)(K_y - wN)}{u^{v,w} + jk\Delta}N\}\right.$$
$$- \{(K_x - pN)(K_y - wN) + (K_y - qN)(K_x - vN)\}\{j + \frac{\Delta}{k}(u^{p,q} + jk\Delta)\}$$
$$\cdot \{\frac{\Delta}{k} + \frac{j}{u^{v,w} + jk\Delta}\}(v-p)(w-q)N^2$$
$$+ u^{v,w}\{(K_x - pN)(v-p) + (K_y - qN)(w-q)\}N$$
$$\left. + \frac{N^2}{u^{v,w} + jk\Delta}\{(K_x - pN)(K_x - vN)(v-p)^2 + (K_y - qN)(K_y - wN)(w-q)^2\}\right\}$$
$$\cdot \frac{|P_{v-p,w-q}|^2}{A^{v,w}_{o,o}}$$

(3.100)

where $A_{o,o}^{v,w}$, $u^{v,w}$, and h^2 are defined by (3.33) to (3.35).

Let us consider the function G in (3.99). Since we have assumed $|n_o| >> 1$ or $|\Delta| << 1$, we may discard these terms in G which are of the order of Δ or Δ^2. In other words G may be approximated as

$$G(K_x,K_y) \cong (u^{p,q})^2 \{p(m - p) + q(n - q)\}N^2$$
$$+ \{pN(K_x - pN) + qN(K_y - qN) - u^2\}$$
$$\cdot \{(m - p)N(K_x - mN) + (n - q)N(K_y - nN) - (u^{p,q})^2\}. \quad (3.101)$$

The above equation (3.98) may be inverse spatially transformed to yield the vertical component of the second order surface field in the (x,y) domain. This inverse transform may be given as

$$E_{az2}(\vec{x}) = C_d \sum_{(m,n)\neq(o,o)} \sum_{p,q} P_{p,q} P_{m-p,n-q}$$
$$\cdot [\exp(jmNx + jnNy)F_{spa}^{-1}\{\frac{\exp(-h_o u)}{u + jk\Delta_b(K_x,K_y)}\}]$$
$$*[\exp(jpNx + jqNy)F_{spa}^{-1}\{\frac{1}{u + jk\Delta_c(K_x,K_y)}\}]$$
$$*[F_{spa}^{-1}\{\frac{G(K_x,K_y)}{u + jk\Delta_a(K_x,K_y)}\}]. \quad (3.102)$$

In the above $F_{spa}^{-1}\{\ \}$ means the inverse spatial transform as per the definition (1.9) and the asterisk means a two dimensional convolution with respect to x and y. Since $P_{o,o}$ is assumed to be zero, we have removed the restrictions $(p,q) \neq (o,o)$ and (m,n) on the summations with respect to p and q in (3.102). Again, as in the first order case, the following inverse spatial transforms may be approximately evaluated as

$$F^{-1}_{spa}\left\{\frac{\exp(-h_o u)}{u + jk\Delta_b(K_x,K_y)}\right\}\Bigg|_{h_o \to 0} \cong F(p_b)\exp(-jk\rho)/(2\pi\rho) \quad (3.103)$$

$$F^{-1}_{spa}\left\{\frac{1}{u + jk\Delta_c(K_x,K_y)}\right\} \cong F(p_c)\exp(-jk\rho)/(2\pi\rho) \quad (3.104)$$

$$F^{-1}_{spa}\left\{\frac{G(K_x,K_y)}{u + jk\Delta_a(K_x,K_y)}\right\} \cong G(-k\cos\theta, -k\sin\theta)F(p_a)\frac{\exp(-jk\rho)}{2\pi\rho}. \quad (3.105)$$

F is the attenuation function and is given by (3.39). p_a, p_b and p_c are three numerical distances with modified surface impedances Δ_a, Δ_b, and Δ_c respectively. These numerical distances are given as

$$p_a = -jk\frac{\rho}{2}\Delta_a^2(-k\cos\theta, -k\sin\theta)$$

$$p_b = -jk\frac{\rho}{2}\Delta_b^2(-k\cos\theta, -k\sin\theta) \quad (3.106)$$

$$p_c = -jk\frac{\rho}{2}\Delta_c^2(-k\cos\theta, -k\sin\theta).$$

The modified surface impedances Δ_a and Δ_b are given by (3.42) and (3.62) respectively. The third impedance Δ_c may be obtained from (3.100) by substituting $-k\cos\theta$ and $-k\sin\theta$ for $(K_x - pN)$ and $(K_y - qN)$ in that equation. It may be shown that the expression thus obtained for Δ_c is the same as for Δ_b (3.62) except that the restrictions on the summation indices v and w are different. For Δ_b the restriction is $(v,w) \neq (-m,-n)$, whereas for Δ_c the restrictions would be $(v,w) \neq (-p,-q)$ and $(m-p,n-q)$. These m, n, p, and q refer to the summation indices in (3.98). It has been mentioned already that the expression for Δ_a (3.42) is the same as for Δ_b except that the summation indices v and w for Δ_a have no restriction.

The function G in (3.105) needs to be evaluated at $K_x = -k \cos\theta$ and $K_y = -k \sin\theta$. This evaluation in (3.101) yields

$$G(-k\cos\theta, -k\sin\theta) = p(m+p)N^2k^2\cos^2\theta + q(n+q)N^2k^2\sin^2\theta$$
$$+ \{p(n+q) + q(m+p)\}N^2k^2\cos\theta\sin\theta + (m^2+n^2)(p^2+q^2)N^4$$
$$+ \{mp(m+p) + np(n+q) + m(p^2+q^2)\}N^3k\cos\theta$$
$$+ \{mq(m+p) + nq(n+q) + n(p^2+q^2)\}N^3k\sin\theta. \qquad (3.107)$$

The three attenuation functions may again be denoted in the following form similar to that of (3.64) used in the first order case.

$$F(p_a) = F_a\{\rho, \cos\theta, \sin\theta\} \qquad (3.108a)$$
$$F(p_a) \quad F_b\{\rho, \cos\theta, \sin\theta, (v,w) \neq (-m,-n)\} \qquad (3.108b)$$
$$F(p_c) = F_c\left\{\rho, \cos\theta, \sin\theta, (v,w) \neq \begin{Bmatrix}(m-p, n-q)\\(-p,-q)\end{Bmatrix}\right\} \qquad (3.108c)$$

Similarly, as G in (3.107) is a function of the surface wave numbers mN, nN, pN, and qN and the angle θ, it is convenient to denote this function as

$$G(-k\cos\theta, -k\sin\theta) = Q(m, n, p, q, \cos\theta, \sin\theta), \qquad (3.109)$$

that is, the function Q is equal to the right hand side of the equation (3.107).

The above three approximate inverse spatial transforms in conjunction with (3.108) and (3.109) may now be used in (3.102) to yield the following expression for E_{az2}.

$$E_{az2}(\vec{x}) \cong \frac{C_d}{(2\pi)^3} \sum_{(m,n)\neq(o,o)} \sum_{p,q} P_{p,q} P_{m-p,n-q}$$

$$\cdot\ [F_b\{\rho,\ \cos\theta,\ \sin\theta,\ (v,w) \neq (-m,-n)\} \frac{1}{\rho} \exp(-jk\rho + jmNx + jnNy)]$$

$$*[F_c\{\rho,\ \cos\theta,\ \sin\theta,\ (v,w) \neq \begin{Bmatrix}(m-p,n-q)\\(-p,-q)\end{Bmatrix}\} \frac{1}{\rho} \exp(-jk\rho + jpNx + jqNy)]$$

$$*\ [Q(m,\ n,\ p,\ q,\ \cos\theta,\ \sin\theta) F_a\{\rho,\ \cos\theta,\ \sin\theta\} \frac{1}{\rho} \exp(-jk\rho)] \tag{3.110}$$

This may also be written in a form of integrals as

$$E_{az2}(\vec{x}) \cong \frac{C_d}{(2\pi)^3} \sum_{(m,n)\neq(o,o)} \sum_{p,q} P_{p,q} P_{m-p,n-q} \int_{x'}\int_{y'}\int_{x''}\int_{y''} \{\frac{1}{\rho''}$$

$$\cdot\ F_b\{\rho'',\ \cos\theta'',\ \sin\theta'',\ (v,w) \neq (-m,-n)\} \exp(-jk\rho' + jmNx'' + jnNy'')$$

$$\cdot\ \frac{1}{\rho'_x} F_c\ \{\rho'_x,\ \cos\theta'_x,\ \sin\theta'_x,\ (v,w) \neq \begin{Bmatrix}(m-p,n-q)\\(-p,-q)\end{Bmatrix}\}$$

$$\cdot\ \exp[-jk\rho'_x + jpN(x' - x'') + jqN(y' - y'')]$$

$$\cdot\ \frac{1}{\rho_x} Q(m,\ n,\ p,\ q,\ \cos\theta_x,\ \sin\theta_x)\ F_a\ \{\rho_x,\ \cos\theta_x,\ \sin\theta_x\}$$

$$\cdot\ \exp(-jk\rho_x)\}\ dx''dy''dx'dy' \tag{3.111}$$

where

$$\rho' = (x''^2 + y''^2)^{\frac{1}{2}}$$

$$\rho'_x = [(x' - x'')^2 + (y' - y'')^2]^{\frac{1}{2}},$$

$$\rho_x = [(x - x')^2 + (y - y')^2]^{\frac{1}{2}}$$

$$\cos\theta'' = \frac{x''}{\rho''},\ \sin\theta'' = \frac{y''}{\rho''} \tag{3.112}$$

$$\cos\theta'_x = \frac{x' - x''}{\rho'_x},\ \sin\theta'_x = \frac{y' - y''}{\rho'_x}$$

$$\cos\theta_x = \frac{x - x'}{\rho_x},\ \sin\theta_x = \frac{y - y'}{\rho_x}.$$

The above equation (3.110) or (3.111) thus represents the vertical component of a second order approximation of the surface field. For a continuously excited source this result may be interpreted in the phasor form, whereas for a pulsed source (3.1) this may be taken in the temporal Fourier transform domain. A physical interpretation may be given to this solution also in a way similar to that given for the first order solution (3.65) or (3.66). F_b represents a ground wave propagating out from the source with a modified surface impedance Δ_b in a direction θ''. This wave is scattered in all directions by an elementary area of the surface located at (ρ'', θ''). A part of this scattered wave travels, again in a ground wave mode with the attenuation function F_c, in a direction θ'_x. This scattered wave is again rescattered in all directions by another elementary area of the surface located at (ρ'_x, θ'_x). A part of this rescattered wave travels in the ground wave mode with the attenuation function F_a in the direction θ_x and which is received at the observation point (x,y). Thus the total field received at the observation point due to this double scattering from the complete area of the surface may be given as the integration over (x",y") and (x',y') and summing over (mN, nN) and (pN, qN) in wave number space. Further, this scattered surface field is in addition to that which is received directly from the source as well as due to the first order scattering.

3.5.1 Second Order Backscattered Surface Field

The second order backscattered surface field may easily be obtained from (3.111) by moving the observation point back to the source point, i.e., by substituting x = y = 0 in that equation. This second order backscattered field will be denoted as E_{zb2}. Thus we have

$$E_{zb2} = E_{az2}(0,0) = \frac{C_d}{(2\pi)^3} \sum_{(m,n)\neq(o,o)} \sum_{p,q} P_{p,q} P_{m-p,n-q}$$

$$\cdot \int_{x'} \int_{y'} \int_{x''} \int_{y''} F_b \{\rho'', \cos\theta'', \sin\theta'', (v,w) \neq (-m,-n)\}$$

$$\cdot F_c \{\rho'_x, \cos\theta'_x, \sin\theta'_x, (v,w) \neq \begin{Bmatrix}(m-p,n-q)\\(-p,-q)\end{Bmatrix}\}$$

$$\cdot F_a \{\rho', -\cos\theta', -\sin\theta'\} Q(m,n,p,q, -\cos\theta', -\sin\theta')$$

$$\cdot \frac{1}{\rho'' \rho'_x \rho'} \exp[jN\{mx'' + ny'' + p(x' - x'') + q(y' - y'')\}]$$

$$\cdot \exp[-jk(\rho'' + \rho'_x + \rho')]dx''dy''dx'dy' \qquad (3.113)$$

where

$$\rho' = (x'^2 + y'^2)^{\frac{1}{2}}, \cos\theta' = \frac{x'}{\rho'}, \sin\theta' = \frac{y'}{\rho'}, \qquad (3.114)$$

and other distances and angles are given by (3.112).

From the point of view of the pulsed excitation of the source dipole (3.1) the above solution (3.113) may be treated in the temporal Fourier transform domain with ω as the transform variable. For returning to the time domain similar steps, i.e., from (3.68) to (3.77), may again be followed which approximately reduced the first order field from frequency (ω) domain (3.67) to time (t) domain (3.78). By doing so it may be shown that the second order backscattered surface field received at time t_o may be given in the form

$$\tilde{E}_{zb2}(t_o) = \frac{C_a}{(2\pi)^3} \sum_{p,q} \sum_{(m,n)\neq(-p,-q)} P_{p,q} P_{m,n}$$

$$\int_{\rho'=0}^{\infty} \int_{\theta'=-\pi}^{\pi} \int_{x''} \int_{y''} F_b \{\rho'', \cos\theta'', \sin\theta'', (v,w) \neq (-m-p,-n-q)\}$$

$$\cdot F_c \{\rho_x', \cos\theta_x', \sin\theta_x', (v,w) = \begin{Bmatrix}(m,n)\\(-p,-q)\end{Bmatrix}\} F_a \{\rho', -\cos\theta', -\sin\theta'\}$$

$$\cdot Q(m+p, n+q, p,q, -\cos\theta', -\sin\theta')$$

$$\cdot \exp [jN\{mx'' + ny'' + \rho' (p\cos\theta' + q\sin\theta')\}]$$

$$\cdot \exp [j\omega_o t_o - jk_o(\rho'' + \rho_x' + \rho')] G_{\tau_o}[t_o - \frac{1}{c}(\rho'' + \rho_x' + \rho')]$$

$$\cdot \frac{1}{\rho''\rho_x'} dx''dy''d\theta'd\rho' , \qquad (3.115)$$

where we have changed the integration variables (x',y') to the polar coordinate variables (ρ', θ'). Further, the summation indices m and n have been replaced by $(m+p)$ and $(n+q)$ respectively. G_{τ_o} is the gate function given by (3.2). The three constants C_a, k_o, c are given by (3.79). It is understood that the variables ω (or k) appearing in the expressions for the attenuation functions and Q is now replaced by ω_o (or k_o) as a result of the inverse temporal Fourier transformation. The four integrals in the above solution need to be evaluated. In the following section we will propose an approximate evaluation of these integrals for the case of a narrow beam receiving antenna.

3.5.1.1 Narrow Beam Receiving Antenna

We consider that the receiving antenna is a very narrow beam one with a beam width of $2\Delta_\theta$ and is directed along the positive x axis. However, we still maintain that the transmitting antenna has uniform radiation in the horizontal ($z = 0$) plane. Under this condition the variation of θ' in (3.115) is from $-\Delta_\theta$ to Δ_θ. Therefore we have

$$\tilde{E}_{zb2}(t_o) = \frac{C_a}{(2\pi)^3} \exp(j\omega_o t_o) \sum_{p,q} \sum_{(m,n)\neq(-p,-q)} P_{p,q} P_{m,n}$$

$$\cdot \int_{\rho'=0}^{\infty} \int_{\theta'=-\Delta_\theta}^{\Delta_\theta} \{F_a(\rho', -\cos\theta', -\sin\theta') Q(m+p, n+q, p, q, -\cos\theta', -\sin\theta')$$

$$\cdot \exp[-jk_o\rho' + j\rho'(p\cos\theta' + q\sin\theta')N]$$

$$\cdot \int_{x''} \int_{y''} F_b\{\rho'', \cos\theta'', \sin\theta'', (v,w) \neq (-m-p, -n-q)\}$$

$$\cdot F_c\left\{\rho'_x, \cos\theta'_x, \sin\theta'_x, (v,w) \neq \begin{Bmatrix}(m,n)\\(-p,-q)\end{Bmatrix}\right\}$$

$$\cdot \frac{1}{\rho''\rho'_x} \exp[-jk_o(\rho'' + \rho'_x) + jN(mx'' + ny'')]$$

$$\cdot G_{\tau_o}[t_o - \frac{1}{c}(\rho'' + \rho'_x + \rho')]dx''dy''\}d\theta' d\rho' . \tag{3.116}$$

Let us consider the gate function. From its definition (3.2), we have

$$G_{\tau_o}[t_o - \frac{1}{c}(\rho'' + \rho'_x + \rho')] = \begin{cases}1, & 2(\rho_o - \Delta_\rho) \le (\rho'' + \rho'_x + \rho') \le 2(\rho_o + \Delta_\rho)\\ 0, & \text{otherwise}\end{cases} \tag{3.117}$$

where $\rho_o = \frac{ct_o}{2}$ and $\Delta_\rho = \frac{c\tau_o}{4}$. By noting the triangular inequality

$\rho' \le (\rho'' + \rho'_x)$ or $2\rho' \le (\rho'' + \rho'_x + \rho')$,

and requiring $(\rho'' + \rho'_x + \rho') \le 2(\rho_o + \Delta_\rho)$ to hold for the gate function we have $\rho' \le (\rho_o + \Delta_\rho)$. In a similar way it may be shown that ρ'' and ρ'_x are also less than or equal to $(\rho_o + \Delta_\rho)$. In other words

$$\rho', \rho'_x, \rho'' \le (\rho_o + \Delta_\rho) \tag{3.118}$$

is a necessary condition for $G_{\tau_o} = 1$. This ρ_o and Δ_ρ along with Δ_θ define the same patch of the surface (figure 3.2) from which the first order backscattered signal is received at time t_o. It has been seen before that this first order signal is received from the patch only, whereas this does not totally follow in the case of second order. This characteristic of the second order solution may be explained as follows.

Consider the inner double integral of (3.116). For the integral the outer integration variables ρ' and θ' may be treated as fixed. Therefore, for a point (ρ', θ'), the inner integral may be taken to represent the field that is received at this point due to the first order scattering of the source field. This scattering is from that area of the surface which satisfies the inequality (3.117) dictated by the gate function. From the end equalities of (3.117) we have,

$$\text{upper equality:} \quad \rho'' + \rho_x' = 2(\rho_o - \Delta_\rho - \frac{\rho'}{2}) = 2c_1 \text{ (say)} \qquad (3.119a)$$

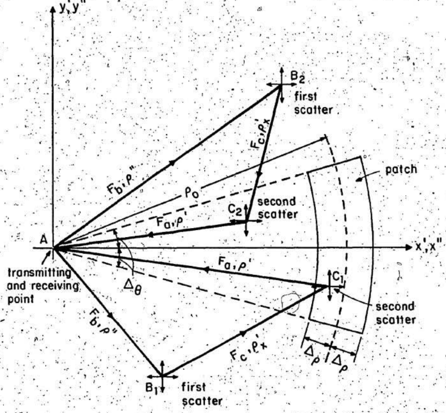
$$\text{lower equality:} \quad \rho'' + \rho_x' = 2(\rho_o + \Delta_\rho - \frac{\rho'}{2}) = 2c_2 \text{ (say)} \qquad (3.119b)$$

where c_1 and c_2 are positive constants for a fixed ρ' and $c_1 < c_2$. The above equations, therefore, represent two ellipses with common foci located at $(0,0)$ and (ρ', θ') in the x'', y'' plane. Also, the ellipse of (3.119a) is enclosed within the second one formed by (3.119b). Thus the surface area responsible for the first order scattering may be given as the region enclosed between the two ellipses.

The first order scattered signal thus received at (ρ', θ') gets scattered again and is then received by the receiving antenna as a part of the second order backscattered signal at time t_o. The surface area responsible for the second scattering may be given by the region enclosed between $0 \le \rho' \le (\rho_o + \Delta_\rho)$ and $-\Delta_\theta \le \theta' \le \Delta_\theta$. An integration over this region thus gives a total second order backscattered signal received at time t_o. It may be mentioned that these two regions of integrations for the first and second scattering are not present explicitly in the form of limits on the integrals in (3.116). But they are inherent through the gate function. Thus we see that the second order backscattered signal is not totally received from the patch only. However, a part of it does come from the patch when $\rho'' + \rho_x' = \rho'$, since under this condition $(\rho_o - \Delta_\rho) \leq \rho' \leq (\rho_o + \Delta_\rho)$. In other words the two second order backscattered signal, one from the patch and other from off the patch, received at any time t_o may not be distinguished from each other. The only requirement is on their path lengths which is dictated by the gate function such that both the signals arrive at the same time as shown in figure 3.3 .

Consider the equation (3.116) again from the point of view of evaluating the integrals. The inner double integral will be evaluated first asymptotically using the method of two dimensional stationary phase [Bleistein and Handelsman (1975, ch. 8), Friedman (1969, ch. 3)]. The outer integration variables ρ' and θ' will be treated as constants for this purpose. Since the integrand consists of a singularity factor $(\rho' \rho_x')^{-1}$, the integrand as such may not be considered slowly varying for applying the method of integration. Therefore a coordinate transformation for x" and y" is required such that this singularity factor may be eliminated. A suitable transformation for this purpose, which runs in two steps, may be given as follows. A rotation by θ' and a shifting by $\frac{\rho'}{2}$ for (x",y") yields

$$\begin{aligned} x'' &= (\frac{\rho'}{2} + \alpha) \cos\theta' - \beta \sin\theta' \\ y'' &= (\frac{\rho'}{2} + \alpha) \sin\theta' + \beta \cos\theta' \end{aligned} \qquad (3.120)$$



$$2(\rho_o - \Delta_\rho) \le (\rho'' + \rho_x' + \rho') \le 2(\rho_o + \Delta_\rho)$$

Figure 3.3 Second order backscatter from a surface patch and off the patch for omnidirectional transmission and narrow beam reception where α and β are labeled as the new axes in the cartesian form. Now by changing (α,β) to the elliptical coordinate variables (μ,δ) [Korn and Korn (1968, ch. 6)] as

$$\alpha = \frac{\rho'}{2} \cosh \mu \cos \delta$$

$$\beta = \frac{\rho'}{2} \sinh \mu \sin \delta, \tag{3.121}$$

the required transformation is

$$x'' = \frac{\rho'}{2} [(1 + \cosh \mu \cos \delta) \cos \theta' - \sinh \mu \sin \delta \sin \theta']$$

$$y'' = \frac{\rho'}{2} [(1 + \cosh \mu \cos \delta) \sin \theta' + \sinh \mu \sin \delta \cos \theta'] \tag{3.122}$$

The above elliptical coordinate transformation has been used by King (1968) for finding the electrical field from a dipole over a flat surface. The Jacobian of this transformation is $r_b r_c$, i.e.,

$$dx''dy'' = r_b r_c \, d\mu d\delta \tag{3.123}$$

where

$$r_b = \frac{\rho'}{2} (\cosh \mu + \cos \delta) = \rho'' = [(x'')^2 + (y'')^2]^{\frac{1}{2}}$$

$$r_c = \frac{\rho'}{2} (\cosh \mu - \cos \delta) = \rho'_x = [(x' - x'')^2 + (y' - y'')^2]^{\frac{1}{2}} \tag{3.124}$$

In the above equations $\sinh \mu$ and $\cosh \mu$ are the hyperbolic sine and cosine. By using (3.122) in (3.116) we get

$$\tilde{E}_{zb2}(t_o) = \frac{C_a}{(2\pi)^3} \exp(j\omega_o t_o) \sum_{p,q} \sum_{(m,n)\neq(-p,-q)} P_{p,q} P_{m,n}$$

$$\cdot \int_{\rho'=0}^{\infty} \int_{\theta'=-\Delta_\theta}^{\Delta_\theta} \cdot F_a(\rho', -\cos \theta', -\sin \theta') Q(m+p, n+q, p, q, -\cos \theta', -\sin \theta')$$

$$\cdot \exp[-jk_o\rho' + j \frac{\rho'}{2} \{(2p + m) \cos \theta' + (2q + n) \sin \theta'\} N]$$

$$\cdot I_1 \, d\theta' \, d\rho' \tag{3.125}$$

where the double integral I_1 is given as

$$I_1 = \int_{\mu=-\infty}^{\infty} \int_{\delta=0}^{\pi} F_b \{r_b,\ \cos(\psi_b + \theta'),\ \sin(\psi_b + \theta'),\ (v,w) \neq (-m-p,-n-q)\}$$

$$\cdot F_c^* \{r_c,\ -\cos(\psi_c + \theta'),\ -\sin(\psi_c + \theta'),\ (v,w) \neq \begin{Bmatrix}(m,n)\\(-p,-q)\end{Bmatrix}\}$$

$$\cdot G_{\tau_o}[t_o - \frac{\rho'}{c}(1 + \cosh\mu)]$$

$$\cdot \exp[j\frac{\rho'}{2} g(\mu,\delta)]\, d\mu d\delta \ . \qquad (3.126)$$

In the above integral angles ψ_b and ψ_c are defined as

$$\cos\psi_b = \frac{\cosh\mu\cos\delta + 1}{\cosh\mu + \cos\delta}\ , \quad \sin\psi_b = \frac{\sinh\mu\sin\delta}{\cosh\mu + \cos\delta}\ ,$$

$$\cos\psi_c = \frac{\cosh\mu\cos\delta - 1}{\cosh\mu - \cos\delta}\ , \quad \sin\psi_c = \frac{\sinh\mu\sin\delta}{\cosh\mu - \cos\delta}\ , \qquad (3.127)$$

and the function g is given as

$$g(\mu,\delta) = -2k_o\cosh\mu + N(m\cos\theta' + n\sin\theta')\cosh\mu\cos\delta$$

$$- N(m\sin\theta' - n\cos\theta')\sinh\mu\sin\delta\ . \qquad (3.128)$$

For convenience we change the surface wave numbers mN and nN from the present cartesian form to the following polar form.

$$mN = K_m\cos\phi_m\ ,\ nN = K_m\sin\phi_m\ . \qquad (3.129)$$

This transformation reduces (3.128) to:

$$g(\mu,\delta) = -2k_o\cosh\mu + K_m\cos(\phi_m - \theta')\cosh\mu\cos\delta$$

$$+ K_m\sin(\phi_m - \theta')\sinh\mu\sin\delta\ . \qquad (3.130)$$

In order to apply the two dimension stationary phase method to the double integral I_1 in (3.126) we consider ρ' as the large parameter. This means that the second scattering point C_1 or C_2 in figure (3.3) is at a far distance from the receiving point A. For C_1 this condition is easily met as this point lies on the patch. Whereas this may not be always true for C_2 which represents a case of off the patch scattering. However, we proceed with this assumption meaning that we are not considering those cases where the second scatterings occur near the receiving point. The stationary phase points of the exponential function $g(\mu,\delta)$ may be found by solving the two equations $\frac{\partial g}{\partial \mu} = 0$ and $\frac{\partial g}{\partial \delta} = 0$ simultaneously for μ and δ. These partial derivatives are

$$\frac{\partial g}{\partial \mu} = -2k_o \sinh \mu + K_m \cos(\emptyset_m - \theta')\sinh \mu \cos \delta$$

$$+ K_m \sin(\emptyset_m - \theta')\cosh \mu \sin \delta = 0 \; ; \tag{3.131a}$$

$$\frac{\partial g}{\partial \delta} = -K_m \cos(\emptyset_m - \theta')\cosh \mu \sin \delta$$

$$+ K_m \sin(\emptyset_m - \theta')\sinh \mu \cos \delta = 0 \; . \tag{3.131b}$$

By solving the above two transcendental equations we arrive at the following three stationary points:

i) $\mu_1 = 0 \; , \; \delta_1 = 0$ (3.132a)

ii) $\mu_2 = 0 \; , \; \delta_2 = 0$ (3.132b)

iii)
$$\mu_3 = \tanh^{-1} \left\{ \frac{[K_m^2 - 4k_o^2 \cos^2(\emptyset_m - \theta')]^{\frac{1}{2}}}{2k_o \sin(\emptyset_m - \theta')} \right\}$$

$$\delta_3 = \tan^{-1} \left\{ \frac{[K_m^2 - 4k_o^2 \cos^2(\emptyset_m - \theta')]^{\frac{1}{2}}}{2k_o \cos(\emptyset_m - \theta')} \right\} \tag{3.132c}$$

with the restriction $2k_o \left|\cos(\emptyset_m - \theta')\right| < K_m < 2k_o$.

It may be mentioned that the above restriction on K_m applies only to the third stationary point.

In view of the above three stationary points an asymptotic expansion of the integral I_1 to the leading term may be represented as

$$I_1 \sim I_{11} + I_{12} + I_{13} \quad (3.133)$$

where I_{11}, I_{12}, and I_{13} represent the respective contribution to the leading term from the above three points. We will now consider each stationary point separately to find its contribution.

First Stationary Point

This point is given as $\mu_1 = 0$, $\delta_1 = 0$. From (3.124) $r_b = \rho'$ and $r_c = 0$ for this point. This means that the point B_1 coincides with C_1 in figure 3.3, which implies that both first and second scatterings occur at the same point and this point lies on the patch. By following the procedure described by Bleistein and Handelsman (1975, ch. 8), the contribution to the integral I_1 in (3.126) from this stationary point may be found. This procedure for the two dimensional case is an extension of the one dimensional stationary phase method. A procedure for the one dimensional case has been described in Appendix C while evaluating the integral $I_\emptyset$ in equation (C.10). This contribution to the leading term is found to be

$$I_{11} = \frac{1}{2}\frac{4\pi}{\rho'} F_b\,\{\rho', \cos\theta', \sin\theta', (v,w) \neq (-m-p,-n-q)\}$$

$$\cdot \exp[j\frac{\rho'}{2}\{-2k_o + N(m\cos\theta' + n\sin\theta')\}\, G_{T_o}(t_o - \frac{2\rho'}{c})$$

$$\cdot \frac{1}{[(m^2+n^2)N^2 - 2k_o N(m\cos\theta' + n\sin\theta')]^{\frac{1}{2}}} \quad (3.134)$$

In the above the attenuation function F_c is not present. This is because r_c is zero for this stationary point and, therefore, F_c goes to unity. The factor of one-half in the above equation accounts for the occurrence of the stationary point at the boundary of the region of integration. For the square root term present above in the denominator, the root with imaginary part ≥ 0 should be taken.

Second Stationary Point

The second stationary point occurs at $\mu_2 = 0$, $\delta_2 = \pi$. This point also lies on the boundary of the integration region. For this point $r_b = 0$ and $r_c = \rho'$ as may be seen from (3.124). This means that the point B_1 coincides with A in figure 3.3. In other words the first scattering occurs at the source or transmitting point and the second scattering falls on the patch, therefore, $F_b = 1$. The contribution for this stationary point may also be similarly found. To the leading term this may be given as

$$I_{12}' = \frac{1}{2}\frac{4\pi}{\rho'} F_c \left\{\rho', \cos\theta', \sin\theta'; (v,w) \neq \begin{Bmatrix}(m,n)\\(-p,-q)\end{Bmatrix}\right\}$$

$$\cdot \exp[j \frac{\rho'}{2} \{-2k_o - N(m\cos\theta' + n\sin\theta')\}] G_{\tau_o}(t_o - \frac{2\rho'}{c})$$

$$\cdot \frac{1}{[(m^2+n^2)N^2 + 2k_o N(m\cos\theta' + n\sin\theta')]^{\frac{1}{2}}} \qquad (3.135)$$

Again, for the square root term, the root with imaginary part ≥ 0 should be taken.

It should be mentioned here that the above contributions (3.134) and (3.135) from the first and second stationary points are valid provided the respective denominators are not zero. If this condition is not met the cubic terms in the Taylor series expansion of the function $g(\mu,\delta)$ about each stationary point in (3.126) need to be considered. Accordingly, (3.134) and (3.135) would be modified. In this thesis this modification has not been carried out due to the added mathematical complexity. Nevertheless we shall proceed with the above with due regard to the regions of validity of the expressions.

Third Stationary Point

The third stationary point is denoted as μ_3, δ_3 and is given by (3.132c). From (3.124) and (3.127) for this point we have

$$r_b = \rho' k_o \left[\frac{|\sin(\phi_m - \theta')|}{(4k_o^2 - K_m^2)^{\frac{1}{2}}} + \frac{\cos(\phi_m - \theta')}{K_m}\right]$$

$$r_c = \rho' k_o \left[\frac{|\sin(\phi_m - \theta')|}{(4k_o^2 - K_m^2)^{\frac{1}{2}}} - \frac{\cos(\phi_m - \theta')}{K_m}\right]$$

$$\cos\psi_b = \frac{\rho'}{2r_b}\left[\frac{4k_o^2 \cos(\phi_m - \theta')\,|\sin(\phi_m - \theta')|}{K_m(4k_o^2 - K_m^2)^{\frac{1}{2}}} + 1\right] \qquad (3.136)$$

$$\sin\psi_b = \frac{\rho'}{2r_b}\left[\frac{\mathrm{sgn}\{\sin(\phi_m - \theta')\}\{K_m^2 - 4k_o^2\cos^2(\phi_m - \theta')\}}{K_m(4k_o^2 - K_m^2)^{\frac{1}{2}}}\right]$$

$$\cos\psi_c = \frac{\rho'}{2r_c}\left[\frac{4k_o^2 \cos(\phi_m - \theta')\,|\sin(\phi_m - \theta')|}{K_m(4k_o^2 - K_m^2)^{\frac{1}{2}}} - 1\right]$$

$$\sin\psi_c = \frac{\rho'}{2r_c}\left[\frac{\mathrm{sgn}\{\sin(\phi_m - \theta')\}\{K_m^2 - 4k_o^2\cos^2(\phi_m - \theta')\}}{K_m(4k_o^2 - K_m^2)^{\frac{1}{2}}}\right]$$

where K_m and ϕ_m are defined by (3.129). This point represents the case of off the patch scattering as shown in figure 3.3. For a given ρ', θ', and the surface wave number (mN, nN), the two points at which the first and the second scatterings occur may be found from the above equation. However, the wave numbers should lie within the limit mentioned in (3.132c). The contribution from this stationary point may also be similarly found and it turns out to be as follows.

$$I_{13} = -j \frac{4\pi}{\rho'} F_b \{r_b, \cos(\psi_b + \theta'), \sin(\psi_b + \theta'), (v,w) \neq (-m-p,-n-q)\}$$

$$\cdot F_c \{r_c, -\cos(\psi_c + \theta'), -\sin(\psi_c + \theta'), (v,w) \neq \begin{Bmatrix}(m,n)\\(-p,-q)\end{Bmatrix}\}$$

$$\cdot \exp[-j \frac{\rho'}{2} |\sin(\phi_m - \theta')| (4k_o^2 - K_m^2)^{\frac{1}{2}}]$$

$$\cdot \frac{1}{[K_m^2 - 4k_o^2 \cos^2(\phi_m - \theta')]^{\frac{1}{2}}} G_{\tau_o} [t_o - \frac{1}{c}(\rho' + r_b + r_c)]$$

with the restriction $2k_o |\cos(\phi_m - \theta')| < K_m < 2k_o$. (3.137)

The above three contributions may be summed together to give an asymptotic evaluation of the integral I_1 (3.126) to the leading term which is of the order of $\frac{1}{\rho'}$. The succeeding terms are of the order of $(\frac{1}{\rho'})^2$ or lower as shown by Bleistein and Handelsman (1975, ch. 8). The outer double integral with respect to θ' and ρ' for $\tilde{E}_{zb2}(t_o)$ in (3.125) still remains to be evaluated. We propose to evaluate the θ'-integral, again asymptotically, using one dimensional stationary phase method. Before we do this we will approximate I_{13} in (3.137) by setting $\theta' = 0$. The reason for this approximation is that it is difficult to find the stationary point for θ' after the exponent of I_{13} is combined with that already present in the outer integral in (3.125). Whereas this difficulty is not there when the exponent of either I_{11} in (3.134) or I_{12} in (3.135) is combined with that present in (3.125). Since θ' is varying over a small range, i.e., $-\Delta_\theta \leq \theta' \leq \Delta_\theta$, this approximation for I_{13} may be justified. Thus, under this approximation, I_{13} may be written as

$$I_{13} \cong -j \frac{4\pi}{\rho'} F_b \{r_b, \cos\psi_b, \sin\psi_b, (v,w) \neq (-m-p,-n-q)\}$$

$$\cdot F_c \{r_c, -\cos\psi_c, -\sin\psi_c, (v,w) \neq \begin{Bmatrix}(m,n) \\ (-p,-q)\end{Bmatrix}\}$$

$$\cdot \exp[-j \frac{\rho'|n|N}{2K_m} (4k_o^2 - K_m^2)^{\frac{1}{2}}]$$

$$\cdot \frac{K_m}{[K_m^4 - (2k_o mN)^2]^{\frac{1}{2}}} G_{\tau_o}(t_o - \frac{2\rho'}{c\beta_o})$$

with the restriction $2k_o|mN| < K_m^2 < 2k_o K_m$. (3.138)

In the above the different parameters, given earlier by (3.136), are now independent of θ' and may be given as follows:

$$r_b = \frac{\rho' k_o N}{K_m} (\frac{|n|}{K_{mo}} + \frac{m}{K_m})$$

$$r_c = \frac{\rho' k_o N}{K_m} (\frac{|n|}{K_{mo}} - \frac{m}{K_m})$$

$$\cos\psi_b = \frac{\rho'}{2r_b} (\frac{4k_o m|n|N^2}{K_{mo} K_m^3} + 1)$$

$$\sin\psi_b = \frac{\rho'}{2r_b} \operatorname{sgn}(n) \frac{(K_m^4 - 4k_o^2 m^2 N^2)}{K_{mo} K_m^3}$$

$$\cos\psi_c = \frac{\rho'}{2r_c} (\frac{4k_o^2 m|n|N^2}{K_{mo} K_m^3} - 1) \tag{3.139}$$

$$\sin\psi_c = \frac{\rho'}{2r_c} \operatorname{sgn}(n) \frac{(K_m^4 - 4k_o^2 m^2 N^2)}{K_{mo} K_m^3}$$

$$\beta_o = 2(1 + \frac{r_b + r_c}{\rho'})^{-1} = 2(1 + \frac{2k_o|n|N}{K_{mo} K_m})^{-1}$$

$$K_m = N(m^2 + n^2)^{\frac{1}{2}}$$

$$K_{mo} = (4k_o^2 - K_m^2)^{\frac{1}{2}} .$$

Thus using (3.133), (3.134), (3.135), and (3.138) in (3.125) yields the following expression for $\tilde{E}_{zb2}(t_o)$,

$$\tilde{E}_{zb2}(t_o) \cong \frac{C_a}{(2\pi)^2} \exp(j\omega_o t_o) \sum_{p,q} \sum_{(m,n)\neq(-p,-q)} P_{p,q} P_{m,n}$$

$$\cdot \left\{ \int_{\rho_o-\Delta_\rho}^{\rho_o+\Delta_\rho} \int_{-\Delta_\theta}^{\Delta_\theta} F_b\{\rho', \cos\theta', \sin\theta', (v,w)\neq(-m-p,-n-q)\} \right.$$

$$\cdot F_a\{\rho', -\cos\theta', -\sin\theta'\}\, Q(m+p, n+q, p, q, -\cos\theta', -\sin\theta')$$

$$\cdot \exp[-j2k_o\rho' + j\rho' N\{(p+m)\cos\theta' + (q+n)\sin\theta'\}]$$

$$\left. \cdot (\rho')^{-1}[K_m^2 - 2k_o N(m\cos\theta' + n\sin\theta')]^{-\frac{1}{2}}\, d\theta' d\rho' \right\}$$

$$+ \frac{C_a}{(2\pi)^2} \exp(j\omega_o t_o) \sum_{p,q} \sum_{(m,n)\neq(-p,-q)} P_{p,q} P_{m,n}$$

$$\cdot \left\{ \int_{\rho_o-\Delta_\rho}^{\rho_o+\Delta_\rho} \int_{-\Delta_\theta}^{\Delta_\theta} F_c\{\rho', \cos\theta', \sin\theta', (v,w)\neq \begin{Bmatrix}(m,n)\\(-p,-q)\end{Bmatrix}\} \right.$$

$$\cdot F_a\{\rho', -\cos\theta', -\sin\theta'\}\, Q(m+p, n+q, p, q, -\cos\theta', -\sin\theta')$$

$$\cdot \exp[-j\, 2k_o\rho' + j\rho' N(p\cos\theta' + q\sin\theta')]$$

$$\left. \cdot (\rho')^{-1}[K_m^2 + 2k_o N(m\cos\theta' + n\sin\theta')]^{-\frac{1}{2}}\, d\theta' d\rho' \right\}$$

$$- j\, \frac{2C_a}{(2\pi)^2} \exp(j\omega_o t_o) \sum_{p,q} \sum_{\substack{(m,n)\neq(-p,-q)\\ 2k_o|m|N < K_m^2 < 2k_o K_m}} P_{p,q} P_{m,n}$$

$$\cdot \left\{ \int_{\beta_o(\rho_o-\Delta_\rho)}^{\beta_o(\rho_o+\Delta_\rho)} \int_{-\Delta_\theta}^{\Delta_\theta} F_b \{r_b, \cos\psi_b, \sin\psi_b, (v,w) \neq (-m-p,-n-q)\} \right.$$

$$\cdot F_c \{r_c, -\cos\psi_c, -\sin\psi_c, (v,w) \neq \begin{Bmatrix}(m,n)\\(-p,-q)\end{Bmatrix}\}$$

$$\cdot F_a \{\rho', -\cos\theta', -\sin\theta'\} Q(m+p, n+q, p,q, -\cos\theta', -\sin\theta')$$

$$\cdot \exp\left[-j\frac{\rho'}{2}\{2k_o + \frac{|n|N}{K_m}(4k_o^2 - K_m^2)^{\frac{1}{2}}\}\right]$$

$$\cdot \exp\left[j\frac{\rho'}{2} N\{(2p+m)\cos\theta' + (2q+n)\sin\theta'\}\right]$$

$$\left. \cdot K_m(\rho')^{-1} [K_m^4 - (2k_o mN)^2]^{-\frac{1}{2}} d\theta' d\rho' \right\} \tag{3.140}$$

The above expression for $\tilde{E}_{zb2}$ consists of three parts. The first part corresponds to the case where both first and second scatterings occur on the patch. The second part corresponds to the case where the first scattering occurs at the source point and the second at the patch. The third part represents the scatterings occuring at other places on the surface and the second order field received at time t_o is the sum of all the three parts. In the above the gate function has been removed from each part by accordingly modifying the limits for the corresponding ρ' integral.

The above expression for $\tilde{E}_{zb2}$ may be made simpler by making a few minor approximations similar to those made in the first order case. The two attenuation functions in first two parts of (3.140) may be approximated at $\rho' = \rho_o$ and $\theta' = 0$ as they are slowly varying functions and their variations are over the patch only. In a similar way the three attenuation functions present in the third part may be approximated as $\rho' = \beta_o\rho_o$ since ρ' is varying only from $\beta_o(\rho_o - \Delta_\rho)$ to

$\beta_o(\rho_o + \Delta_\rho)$. The variation of β_o is between 0 and 1 as may be seen from (3.139). The θ' variation for the third attenuation function F_a may be approximated at $\theta' = 0$ as $-\Delta_\theta \leq \theta' \leq \Delta_\theta$. Thus under these approximations all attenuation functions in (3.140) are no longer functions of the integration variables ρ' and θ' and, therefore, they may be taken out from each double integral.

The θ' integral with the remaining integrand for each part may be evaluated asymptotically by using one dimensional stationary phase method. ρ' may again be taken as the large parameter for this purpose. This is valid for the first two parts in (3.140) where $(\rho_o - \Delta_\rho) \leq \rho' \leq (\rho_o + \Delta_\rho)$, and ρ_o is large. Whereas for the third part the variation of ρ' is $\beta_o(\rho_o - \Delta_\rho) \leq \rho' \leq \beta_o(\rho_o + \Delta_\rho)$ with β_o varying from 0 to 1. This means that ρ' may be small is some cases depending upon β_o. These cases are where second scatterings occur near to the receiving point. However, by maintaining a reasonable minimum for β_o we may proceed with the above assumption of large ρ'. By using the standard procedure the stationary phase point for θ' may easily be derived for each part. For the first part the stationary points occur at $\theta' = \gamma_1$ and $(\gamma_1 - \pi)$, where $\gamma_1 = \tan^{-1}(\frac{q + n}{p + m})$. For the second part they are at $\theta' = \gamma_2$ and $(\gamma_2 - \pi)$, where $\gamma_2 = \tan^{-1}(\frac{n}{m})$. In the third part these points are $\theta' = \gamma_3$ and $(\gamma_3 - \pi)$, where $\gamma_3 = \tan^{-1}(\frac{2q + n}{2p + m})$. After performing the required algebra for this method, the result of this evaluation to the leading term may be shown to be

$$\hat{E}_{zb2}(t_o) = \frac{C_a}{(2\pi)^{3/2}} \exp(j\omega_o t_o) F_a(\rho_o, -1, 0)$$

$$\cdot \sum_{p,q} \sum_{(m,n)\neq(-p,-q)} \left\{ P_{p,q} P_{m,n} F_b\{\rho_o, 1, 0, (v,w) \neq (-m-p, -n-q)\} \right.$$

$$-\alpha|p+m| \leq q+n \leq \alpha|p+m|$$

$$\cdot Q\{m+p, n+q, p, q, -\mathrm{sgn}(p+m) \frac{(p+m)}{K_{p+m}} N, -\mathrm{sgn}(p+m) \frac{(q+n)}{K_{p+m}} N\}$$

$$\cdot [K_m^2 K_{p+m} - 2k_o N^2 \mathrm{sgn}(p+m)\{m(p+m) + n(q+n)\}]^{-1/2}$$

$$\exp[-j \frac{\pi}{4} \mathrm{sgn}(p+m)]$$

$$\left. \cdot \int_{\rho_o-\Delta_\rho}^{\rho_o+\Delta_\rho} (\rho')^{-3/2} \exp[j\rho'\{\mathrm{sgn}(p+m) K_{p+m} - 2k_o] d\rho' \right\}$$

$$+ \frac{C_a}{(2\pi)^{3/2}} \exp(j\omega_o t_o) F_a(\rho_o -1, 0)$$

$$\cdot \sum_{p,q} \sum_{(m,n)\neq(-p,-q)} \left\{ P_{p,q} P_{m,n} F_c \left\{\rho_o, 1, 0, (v,w) \neq \begin{Bmatrix}(m,n)\\(-p,-q)\end{Bmatrix}\right\} \right.$$

$$-\alpha|p| \leq q \leq \alpha|p|$$

$$\cdot Q\{m+p, n+q, p, q, -\mathrm{sgn}(p) \frac{p}{K_p} N, -\mathrm{sgn}(p) \frac{q}{K_p} N\}$$

$$\cdot [K_m^2 K_p + 2k_o N^2 \mathrm{sgn}(p) \{mp + nq\}]^{-1/2} \exp[-j \frac{\pi}{4} \mathrm{sgn}(p)]$$

$$\left. \cdot \int_{\rho_o-\Delta_\rho}^{\rho_o+\Delta_\rho} (\rho')^{-3/2} \exp[j\rho'\{\mathrm{sgn}(p) K_p - 2k_o\}] d\rho' \right\}$$

$$- j \frac{C_a}{\pi^{3/2}} \exp(j\omega_o t_o) \sum_{p,q} \sum_{(m,n)\neq(-p,-q)} \left\{ P_{p,q} P_{m,n} \right.$$

$$2k_o|m|N < K_m^2 < 2k_o K_m$$

$$-\alpha|2p+m| \leq 2q+n \leq \alpha|2p+m|$$

$$\cdot F_b \{r_b, \cos\psi_b, \sin\psi_b, (v,w) \neq (-m-p, -n-q)\}$$

$$\cdot F_c \{r_c, -\cos\psi_c, -\sin\psi_c, (v,w) \neq \left\{\begin{matrix}(m,n)\\(-p,-q)\end{matrix}\right\}\}$$

$$\cdot F_a (\beta_o\rho_o, -1, 0)\, K_m \exp[j \frac{\pi}{4} \operatorname{sgn}(2p+m)]$$

$$\cdot Q \{m+p, n+q, p, q, -\operatorname{sgn}(2p+m) \frac{(2p+m)}{K_{2p+m}} N, -\operatorname{sgn}(2p+m) \frac{(2p+n)}{K_{2p+m}} N\}$$

$$\cdot (K_{2p+m})^{-\frac{1}{2}} [K_m^4 - (2k_o mN)^2]^{-\frac{1}{2}}$$

$$\cdot \int_{\beta_o(\rho_o-\Delta_\rho)}^{\beta_o(\rho_o+\Delta_\rho)} (\rho')^{-3/2} \exp[j \frac{\rho'}{2} \{\operatorname{sgn}(2p+m) K_{2p+m} - 2k_o - \frac{|n| N}{K_m} K_{mo}\}] d\rho' \Bigg\} \tag{3.141}$$

where

$$\begin{aligned}
\alpha &= \tan\Delta_\theta \\
K_m &= N[m^2 + n^2]^{\frac{1}{2}} \\
K_p &= N[p^2 + q^2]^{\frac{1}{2}} \\
K_{p+m} &= N[(p+m)^2 + (q+n)^2]^{\frac{1}{2}} \\
K_{2p+m} &= N[(2p+m)^2 + (2p+n)^2]^{\frac{1}{2}} \\
K_{mo} &= [4k_o^2 - K_m^2]^{\frac{1}{2}} .
\end{aligned} \tag{3.142}$$

The distance r_b, r_c and the angles ψ_b, ψ_c, required for the two attenuation functions F_b and F_c in the third part of (3.141), are given by (3.139) with ρ' replaced by $(\beta_o\rho_o)$. For clarity, they may again be written as follows:

$$r_b = \frac{\rho_o \beta_o k_o N}{K_m} \left(\frac{|n|}{K_{mo}} + \frac{m}{K_m}\right)$$

$$r_c = \frac{\rho_o \beta_o k_o N}{K_m} \left(\frac{|n|}{K_{mo}} - \frac{m}{K_m}\right)$$

$$\cos \psi_b = \frac{\rho_o \beta_o}{2r_b} \left(\frac{4k_o\, m|n|N^2}{K_{mo}\, K_m^3} + 1\right)$$

$$\sin \psi_b = \frac{\rho_o \beta_o}{2r_b} \operatorname{sgn}(n) \frac{(K_m^4 - 4\, k_o^2\, m^2 N^2)}{K_{mo}\, K_m^3} \tag{3.143}$$

$$\cos \psi_c = \frac{\rho_o \beta_o}{2r_c} \left(\frac{4k_o^2\, m|n|N^2}{K_{mo}\, K_m^3} - 1\right)$$

$$\sin \psi_c = \frac{\rho_o \beta_o}{2r_c} \operatorname{sgn}(n) \frac{(K_m^4 - 4k_o^2\, m^2 N^2)}{K_{mo}\, K_m^3}$$

$$\beta_o = 2 \left\{1 + \frac{2k_o |n| N}{K_{mo}\, K_m}\right\}^{-1}$$

With the help of (3.109) and (3.107) the function Q for each part in (3.141) may also be given here for convenience. They will be denoted as Q_1, Q_2, and Q_3 corresponding to each of the three parts,

$$Q\{m+p,\, n+q,\, p,q,\, -\operatorname{sgn}(p+m)\frac{(p+m)}{K_{p+m}} N,\, -\operatorname{sgn}(p+m)\frac{(q+n)}{K_{p+m}} N\}$$

$$= p(2p+m)(p+m)^2 k_o^2 \frac{N^4}{K_{p+m}^2} + q(2q+n)(q+n)^2 k_o^2 \frac{N^4}{K_{p+m}^2}$$

$$+ \{p(2q+n) + q(2p+m)\}\, (p+m)(q+n) k_o^2 \frac{N^4}{K_{p+m}^2} + K_p^2\, K_{m+p}^2$$

$$- [(p+m)\{p(p+m)(2p+m) + p(q+n)(2q+n) + (p+m)(p^2+q^2)\}$$

$$+ (q+n)\{q(p+m)(2p+m) + q(q+n)(2q+n) + (q+n)(p^2 + q^2)\}]$$

$$\cdot \operatorname{sgn}(p+m) \frac{k_o N^4}{K_{p+m}} = Q_1(m,n,p,q) \tag{3.144a}$$

$$Q\{m+p,\ n+q,\ p,q,\ -\mathrm{sgn}(p)\ \frac{p}{K_p}N,\ -\mathrm{sgn}(p)\ \frac{q}{K_p}N\}$$

$$= (2p+m)p^3 k_o^2 \frac{N^4}{K_p^2} + (2q+n)q^3 k_o^2 \frac{N^4}{K_p^2}$$

$$+ \{p(2q+n) + q(2p+m)\}pq\ k_o^2 \frac{N^4}{K_p^2} + K_p^2\ K_{p+m}^2$$

$$- [p\{p(p+m)(2p+m) + p(q+n)(2q+n) + (p+m)(p^2+q^2)\}$$

$$+ q\{q(p+m)(2p+m) + q(q+n)(2q+n) + (q+n)(p^2+q^2)\}]$$

$$\cdot\ \mathrm{sgn}(p)\ \frac{k_o N^4}{K_p}$$

$$= Q_2(m,\ n,\ p,\ q) \tag{3.144b}$$

$$Q\{m+p,\ n+q,\ p,q,\ -\mathrm{sgn}(2p+m)\ \frac{(2p+m)}{K_{2p+m}}N,\ -\mathrm{sgn}(2p+m)\ \frac{(2q+n)}{K_{2p+m}}N\}$$

$$= p(2p+m)^3\ k_o^2 \frac{N^4}{K_{2p+m}^2} + q(2q+n)^3\ k_o^2 \frac{N^4}{K_{2p+m}^2}$$

$$+ \{p(2q+n) + q(2p+m)\}(2p+m)(2q+n)k_o^2 \frac{N^4}{K_{2p+m}^2} + K_p^2\ K_{p+m}^2$$

$$- [(2p+m)\{p(p+m)(2p+m) + p(q+n)(2q+n) + (p+m)(p^2+q^2)\}$$

$$+ (2q+n)\{q(p+m)(2p+m) + q(q+n)(2q+n) + (q+n)(p^2+q^2)\}]$$

$$\cdot\ \mathrm{sgn}(2p+m)\ \frac{k_o N^4}{K_{2p+m}}$$

$$= Q_3(m,\ n,\ p,\ q) \tag{3.144c}$$

At this point it is worthwhile to comment on the stationary phase method applied above. In reducing (3.125) to the above form (3.141) we have performed stationary phase integrations to the leading terms in two stages. First the two dimensional method has been applied to the double integral with respect to μ and δ while treating θ' as a constant for this purpose. This application, along with an approximation for θ' in the contribution from the third stationary point, has reduced the equation (3.125) to (3.140). This has been further reduced to (3.141) by using the one dimensional method for evaluating the θ' integral in (3.140). Instead of these, the three dimensional stationary phase may be used directly to evaluate the three integrals with respect to μ, δ, and θ' together. But in using this direct method solutions of three simultaneous equations are required as opposed to solving two such equations in the two dimensional method. In the present situation these three equations are in a transcendental form and it is difficult to solve for the third stationary point equivalent to (3.132c). However, these three equations may be easily solved for the other two stationary points equivalent to (3.132a) and (3.132b). Further, it may be shown that the contributions to the leading term for these two stationary points are the same as obtained above, i.e., the first two parts of the equation (3.141). Because of this difficulty, we have evaluated the triple integral successively.

The remaining integral with respect to ρ' in each part in (3.141) may be considered now. Since ρ' is varying only from $(\rho_o - \Delta_\rho)$ to $(\rho_o + \Delta_\rho)$ in the first two parts, $(\rho')^{-3/2}$ may be approximated as $(\rho_o)^{-3/2}$. In a similar way, $(\rho')^{-3/2}$ may be approximated as $(\beta_o\rho_o)^{-3/2}$ in the third part where ρ' is varying from $\beta_o(\rho_o - \Delta_\rho)$ to $\beta_o(\rho_o + \Delta_\rho)$. This means that only the amplitude term and not the phase term (exponent term) is being approximated. This approximation is valid in the sense that the amplitude variations are not that significant compared to the phase variations when the distance is large and its variation is over a small range. The three integrals remaining after this approximation may be evaluated exactly to give the following solution for $\tilde{E}_{zb2}(t_o)$.

$$\tilde{E}_{zb2}(t_o) =$$

$$\frac{2C_a \Delta_\rho}{(2\pi\rho_o)^{3/2}} F_a(\rho_o, 1, 0) \sum_{p,q} \sum_{\substack{(m,n)\neq(-p,-q) \\ -\alpha|p+m| \leq q+n \leq \alpha|p+m|}} \Big\{ P_{p,q} P_{m,n}$$

$$\cdot F_b\{\rho_o, 1, 0, (v,w) \neq (-m-p, -n-q)\} Sa[\Delta_\rho\{sgn(p+m)K_{p+m} - 2k_o\}]$$

$$\cdot [K_m^2 K_{p+m} - 2k_o N^2 \, sgn(p+m)\{m(p+m) + n(q+n)\}]^{-\frac{1}{2}}$$

$$Q_1(m, n, p, q) \exp[j \, sgn(p+m)\{K_{p+m}\rho_o - \frac{\pi}{4}\}] \Big\}$$

$$+ \frac{2C_a \Delta_\rho}{(2\pi\rho_o)^{3/2}} F_a(\rho_o, 1, 0) \sum_{p,q} \sum_{\substack{(m,n)\neq(-p,-q) \\ -\alpha|p| \leq q \leq \alpha|p|}} \Big\{ P_{p,q} P_{m,n}$$

$$\cdot F_c\left\{\rho_o, 1, 0, (v,w) \neq \begin{Bmatrix}(m,n) \\ (-p,-q)\end{Bmatrix}\right\} Sa[\Delta_\rho\{sgn(p)K_p - 2k_o\}]$$

$$\cdot [K_m^2 K_p + 2k_o N^2 \, sgn(p)\{mp+nq\}]^{-\frac{1}{2}}$$

$$\cdot Q_2(m, n, p, q) \exp[j \, sgn(p)\{K_p\rho_o - \frac{\pi}{4}\}] \Big\}$$

$$- j \frac{2C_a \Delta_\rho}{(\pi\rho_o)^{3/2}} \sum_{p,q} \sum_{\substack{(m,n)\neq(-p,-q) \\ 2k_o|m|N < K_m^2 < 2k_o K_m \\ -\alpha|2p+m| \leq 2q+n \leq \alpha|2p+m|}} \Big\{ P_{p,q} P_{m,n} F_a(\beta_o\rho_o, 1, 0)$$

$$\cdot F_b\{r_b, \cos\psi_b, \sin\psi_b, (v,w) \neq (-m-p, -n-q)\}$$

$$\cdot F_c\left\{r_c, -\cos\psi_c, -\sin\psi_c, (v,w) \neq \begin{Bmatrix}(m,n) \\ (-p,-q)\end{Bmatrix}\right\}$$

$$\cdot \, Sa\left[\beta_o \frac{\Delta_\rho}{2}\left\{\operatorname{sgn}(2p+m)K_{2p+m} - 2k_o - \frac{|n|N}{K_m}K_{mo}\right\}\right]$$

$$\cdot \, K_m Q_3(m, n, p, q)\left[\beta_o K_{2p+m}\left\{K_m^4 - (2k_o mN)^2\right\}\right]^{-\frac{1}{2}}$$

$$\cdot \, \exp\left[j \operatorname{sgn}(2p+m)\left(K_{2p+m}\frac{\beta_o \rho_o}{2} - \frac{\pi}{4}\right)\right]$$

$$\left.\cdot \, \exp\left[j\rho_o\left\{2k_o - k_o\beta_o - \frac{|n|N}{2K_m}K_{mo}\beta_o\right\}\right]\right\} \qquad (3.145)$$

In the above solution we have used the relation (3.86) for F_a. F_a, F_b, and F_c are the attenuation functions with modified surface impedances Δ_a, Δ_b, and Δ_c respectively. They are given by (3.108) in conjunction with (3.106) and (3.39). The expressions for Δ_a, Δ_b, and Δ_c are the same except the differences in the restrictions on the summation indices (v,w) in their expressions. They may be given by (3.42) after replacing ω (or k) by ω_o (or k_o) and placing the appropriate restrictions on (v,w) there. These restrictions are indicated through the argument of the F's. The other symbols or functions present above are defined earlier by (3.92) and (3.142) to (3.144).

The above equation thus represents an approximate second order solution for the vertical component of the backscattered surface field received at time $t = t_o$. The transmitted signal is a pulsed sinusoid centered at time $t = 0$ from an elementary vertical electric dipole. The receiving antenna is assumed to be a narrow beam one and directed along the positive x axis. This solution consists of three parts. The first part represents the case where both first and second scatterings occur on the patch. The second part represents the case where the first scattering occurs at the source point and the second scattering occurs on the patch. Lastly, the third part represents the case where the two scatterings occur off the patch but within the surface area corresponding to t_o.

The assumption of narrow beam for the receiving antenna was made for the purpose of evaluating the integrals in (3.115). However, this result may be extended to the situation where the receiving antenna is wide beam or omnidirectional. This extension may be carried out by dividing the beam pattern into small angular segments. Then for each segment the backscattered field may be given by modifying the above result (3.145). This modification is via the azimuth rotation of the coordinate axes from the particular angular location of the beam segment to the reference position. Finally, by summing the results thus obtained for each angular segment with appropriate weighting factor for each segment depending upon the beam pattern, an approximate second order solution for the backscattered field may be derived.

It may be mentioned here that in the analysis we have assumed the rough surface without any curvature, whereas over large distances this assumption may not be justified. Since the scattering occuring at any point on the surface is a localized phenomenon the surface there may be treated with no curvature. On the other hand the propagation between any two points may be influenced by the earth's curvature depending upon the distance between the points. This effect may be accounted for by replacing the ground wave attenuation functions (F's) by the spherical earth attenuation functions with modified surface impendances in the previously derived results.

CHAPTER 4

APPLICATION TO THE OCEAN SURFACE

4.1 General

We have considered in the previous chapter the vertical component of the first and second order backscattered surface field from a rough surface. The surface assumed there is a non-time varying two dimensional periodic surface. The results derived for these two orders of the field may easily be applied to a class of time invariant and random rough surfaces such as terrain. The usual model assumed for a random rough surface is to describe it as a two dimensional periodic surface with Fourier coefficients taken as random variables [Rice (1951), Peake (1959)]. Hence, by assuming such a model, an average (statistical) "radar cross section" to the first and second order in terms of the "roughness spectrum" of the surface may be derived. The limit of the fundamental surface wave number (N) may then be taken to be zero to reduce the summations to integrals. Of course the results will be restricted to those random surfaces whose slopes and normalized surface impedances are much less than unity in magnitude. The cross section is the main parameter for identification of different targets in remote sensing by radar [Skolnik (1970, ch. 1)].

In this chapter a similar technique is applied to a model of the ocean surface. The ocean surface is modelled as a three dimensional periodic surface in space and time with the Fourier coefficients taken as random variables. The previously derived results are not applicable to this surface in a strict mathematical sense due to the fact that the ocean surface varies with time also. However, on a valid assumption that the time taken for any significant surface variation is very large compared to the propagation time of radio waves, the previous results are suitably modified to include the time dependency of the ocean surface. With this modification an average first and second order of the Doppler frequency dependent backscattered radar cross section are derived for a narrow beam receiving antenna. These two orders of the cross section are in terms of the first order ocean wave height spectrum of deep water gravity waves [Kinsman (1965, ch. 7)]. In the second order result the hydrodynamic effect, derived by Johnstone (1975) or Weber and Barrick (1977), is also included.

4.2 Model for the Ocean Surface

The ocean surface may be modelled as a three dimensional periodic surface in x, y, and t. This model is an extension of the Rice (1951) model for a time invariant rough surface and is used by other investigators, e.g., Barrick (1972a,b) and Johnstone (1975). The surface may then be described by the three dimensional Fourier series,

$$f(\vec{x},t) = \sum_{m,n,l} P_{m,n,l} \exp[j(mNx + nNy + lWt)]$$

$$= \sum_{m,n,l} P_{m,n,l} \exp[j(\vec{K}_m \cdot \vec{x} + lWt)] \qquad (4.1)$$

where

$$\vec{x} = x\hat{x} + y\hat{y} \text{ and } \vec{K}_m = mN\hat{x} + nN\hat{y} \qquad (4.2)$$

$P_{m,n,l}$ is the Fourier coefficient corresponding to the wave number $\vec{K}_m$ and temporal frequency lW of the surface and is given as

$$P_{m,n,l} = \frac{1}{L^2 T} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \int_{-T/2}^{T/2} f(\vec{x},t)$$

$$\cdot \exp[-j(\vec{K}_m \cdot \vec{x} + lWt)]dxdydt \qquad (4.3)$$

In the above equations $N = 2\pi/L$ and $W = 2\pi/T$. L is the fundamental wavelength (spatial period) in both x and y directions, and T is the fundamental temporal period. The randomness of the surface variations may be introduced in (4.1) by treating the Fourier coefficients as random variables [Rice (1951)]. Since $f(\vec{x},t)$ is a real surface, we have $P_{m,n,l} = P^*_{-m,-n,-l}$, where the asterisk(*) as a superscript denotes the complex conjugate. The mean level of the surface is taken to be the $z = 0$ plane, which implies $P_{o,o,o} = 0$.

A three dimensional autocorrelation function for the surface displacement or height from the mean level may be defined as [Papoulis (1965, ch. 10), Phillips (1977, ch. 4)]

$$R(\vec{x},t,\vec{\tau}_x,\tau_t) = \langle f_a f_b \rangle \qquad (4.4)$$

where

$$f_a = f(\vec{x} + \vec{\tau}_x, t + \tau_t)$$
$$f_b = f(\vec{x},t) \qquad (4.5)$$
$$\vec{\tau}_x = \tau_x \hat{x} + \tau_y \hat{y} .$$

The angular bracket <·> denotes the ensemble average. That is, the ensemble is formed by collecting the samples of surface displacements from many different oceans for a fixed $\vec{x}$, t, $\vec{\tau}_x$, and τ_t. Of course this collection is not physically realizable and it is only hypothetical. Mathematically, this ensemble average may be given as

$$\langle f_a f_b \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_a f_b \, p(f_a,f_b) df_a df_b \qquad (4.6)$$

where $p(f_a,f_b)$ is the joint probability density function of the surface displacements at $(\vec{x},t)$ and $(\vec{x} + \vec{\tau}_x, t + \tau_t)$. Since $f(\vec{x},t)$ is real and by using (4.1) in (4.4) yields R as

$$R(\vec{x},t,\vec{\tau}_x,\tau_t) = \sum_{m,n,l} \sum_{p,q,i} \langle P^*_{m,n,l} P_{p,q,i} \rangle$$
$$\cdot \exp[-jN(m-p)x - jN(n-q)y - jW(l-i)t]$$
$$\exp[jN(p\tau_x + q\tau_y) + jWi\tau_t] . \qquad (4.7)$$

We now assume that the ocean surface is, statistically, spatially homogeneous and temporally stationary as proposed by Phillips (1977, ch. 4) and Kinsman (1965, ch. 7). This means the above autocorrelation function is now only a function of spatial and temporal displacements $(\vec{\tau}_x,\tau_t)$ and not of positions $(\vec{x},t)$. In other words samples are still collected from many different oceans but only for a fixed $\vec{\tau}_x$ and τ_t. Again, this collection is not realizable. From the realizability point of view, one may assume that the random variation of the surface displacements belongs to an ergodic process. This means that the samples may be collected from the same ocean and any statistics derived from those samples applies to all such oceans which form a set of this ergodic process. Ergodicity always implies homogeneity and stationarity but the converse need not be true. Proceeding with the assumption of homogeneity and stationarity it implies that the right hand side of (4.7) is also independent of $\vec{x}$ and t, and which is possible if and only if

$$<P^*_{m,n,1} P_{p,q,i}> = \begin{cases} <|P_{m,n,1}|^2>, & \text{for } m = p,\ n = q,\ 1 = i \\ 0, & \text{otherwise.} \end{cases} \qquad (4.8)$$

Therefore, we have

$$R(\vec{\tau}_x, \tau_t) = \sum_{m,n,1} <|P_{m,n,1}|^2> \cdot \exp[j\,\vec{K}_m \cdot \vec{\tau}_x + j1W\tau_t] . \qquad (4.9)$$

As a special case, for $\tau_x = \tau_y = \tau_t = 0$, we get

$$R(\vec{0},0) = \sum_{m,n,1} <|P_{m,n,1}|^2> = <f^2(x,y,t)> = h_s^2 . \qquad (4.10)$$

where h_s^2 denotes the ensemble mean of the squared surface heights or displacements. We now define

$$<|P_{m,n,1}|^2> = \frac{N^2 W}{(2\pi)^3} S(\vec{K}_m, 1W) \qquad (4.11)$$

where $S(\vec{K}_m, 1W)$ is referred to as a three dimensional power spectrum of the surface displacement [Kinsman (1965, ch. 7)]. This is also known as a wave spectrum [Phillips (1977, ch. 4)] or a three dimensional wave height spectrum [Barrick (1972b)]. It has the following property of evenness:

$$S(\vec{K}_m, lW) = S(-\vec{K}_m, -lW) \tag{4.12}$$

By using (4.11) in (4.9) we get

$$R(\vec{\tau}_x, \tau_t) = \sum_{m,n,l} \frac{N^2 W}{(2\pi)^3} S(\vec{K}_m, lW) \cdot \exp[j\vec{K}_m \cdot \vec{\tau}_x + jlW\tau_t] \tag{4.13}$$

By setting $mN = K_x$, $nN = K_y$, and $lW = \omega$ and then extending the limits of both fundamental wave length (L) and temporal period (T) to infinity (i.e., N, W → 0), the above summations may be reduced to integrals as

$$R(\vec{\tau}_x, \tau_t) = (2\pi)^{-3} \int_{K_x}\int_{K_y}\int_{\omega} S(\vec{K},\omega) \cdot \exp(j\vec{K} \cdot \vec{\tau}_x + j\omega\tau_t)\, dK_x dK_y d\omega \tag{4.14}$$

where

$$\vec{K} = K_x\hat{x} + K_y\hat{y} \tag{4.15}$$

The above equation shows that the autocorrelation function and the wave height spectrum form a three dimensional Fourier transform pair. Further, from (4.10),

$$h_s^2 = (2\pi)^{-3} \int_{K_x}\int_{K_y}\int_{\omega} S(\vec{K},\omega)\, dK_x dK_y d\omega \tag{4.16}$$

Thus $(2\pi)^{-3} S(\vec{K},\omega)\, dK_x dK_y d\omega$ represents the contribution to the mean square surface height from those wave components whose wave numbers and frequencies lie within an infinitismal region formed by (K_x, K_y, ω) and $(K_x + dK_x, K_y + dK_y, \omega + d\omega)$.

4.2.1 First and Second Order Gravity Waves

Gravity waves are considered of main importance amongst the various surface waves of the ocean. These waves are always present in the ocean in almost any natural condition. They are generated by the wind blowing over the ocean and gravity acts as the chief restoring force to maintain the equilibrium for these waves. They are classified as those waves whose wavelengths are greater than $2\pi(\gamma/g)^{\frac{1}{2}}$, where g is the acceleration due to gravity and γ is the surface tension-to-water density ratio [Phillips (1977, ch. 3)]. The minimum wavelength of gravity waves is 1.7 cm. Their periods range from 0.1 seconds to 5 minutes [Kinsman (1965, ch. 1)]. Near the lower end of this range they are also referred to as ultragravity waves and as infragravity waves towards the upper end of the range. When the water is sufficiently deep such that the effect of the ocean bottom on the propagation of surface waves can be neglected, the waves are referred to as deep water waves. This approximation of deep water for a given wave is quite satisfactory if the water depth is greater than half the wavelength. For most practical purposes an ocean surface may be assumed to consist of mainly deep water gravity waves and, therefore, we will consider here only this type of wave.

Ocean waves are governed by a set of nonlinear hydrodynamic equations [Kinsman (1965, ch. 2)], namely, the continuity equation, the momentum equation, and the boundary conditions. In a linear or first order approximation of these equations an ocean surface may be described as a linear superposition of many propagating sinusoidal waves. These waves may have different amplitudes, wavelengths and may be travelling in different directions. However, for each wave the phase velocity is related to its wavelength and the relationship is commonly known as the dispersion relationship. For deep water gravity waves this relation may be given as [Kinsman (1965, ch. 3).]

$$v_1 = \pm (gL_1/2\pi)^{\frac{1}{2}} \tag{4.17}$$

where v_1 is the phase velocity and L_1 is wave length for a given wave. g is the acceleration due to gravity (=9.81 m/sec^2). Alternatively, the above relation for the corresponding frequency ω_1 may be written as

$$\omega_1 = \pm (2\pi g/L_1)^{\frac{1}{2}} = \pm (gK_1)^{\frac{1}{2}} \tag{4.18}$$

where K_1 is the corresponding wave number.

Since the above equation (4.17) is obtained by a first order approximation of the governing equations, the waves obeying this equation may be referred to as the first order gravity waves. When the nonlinearities of the hydrodynamic equations are taken into the account a solution for the ocean surface includes second and higher order waves also. The usual technique for solving these nonlinear equations is a perturbational approach [Kinsman (1965, ch. 13)]. Thus by considering up to the second order waves only, an ocean surface may be described as

$$\begin{aligned} f(\vec{x},t) &= f_1(\vec{x},t) + f_2(\vec{x},t) \\ &= \int \sum_{m,n,1} ({}_1P_{m,n,1} + {}_2P_{m,n,1}) \\ &\quad \cdot \exp [j\vec{K}_m \cdot \vec{x} + j1Wt] \end{aligned} \tag{4.19}$$

where f_1 and ${}_1P$'s correspond to the first order waves. Similarily, f_2 and ${}_2P$'s correspond to the second order waves.

By using a perturbation analysis Johnstone (1975) has derived a solution for the second order waves (${}_2P$) in terms of a sum of products of two first order waves (${}_1P$) for deep water gravity waves. The quantities which are perturbed in his method are the surface $f(x,y,t)$ and the velocity potential $\phi(x,y,z,t)$, i.e., perturbations of surface Fourier coefficients ($P_{m,n,l}$'s) and of the Fourier coefficients for the velocity potential. His method is an extension of Tick's (1959) method [Kinsman 1965, ch. 13)] by including the third dimensional (y) variations for the surface. Tick has considered only two dimensional variations (x,t) for the surface. In a more general perturbational approach Weber and Barrick (1977) [along with Barrick and Weber (1977b)] have also derived a solution for the above second order waves. In their method the frequency (ω) is also expanded in a perturbation series besides the above quantities. Thus a second order amplitude correction to the frequencies of the first order waves, i.e., a correction to the dispersion relationship (4.16), is also obtained there. By assuming small amplitudes for the waves these corrections to the frequencies may be ignored. Under this condition the second order result derived by Weber and Barrick is the same as that of Johnstone. From equations (22) and (23) of Weber and Barrick (1977), this second order result in our notation under this condition may be given as

$$ {}_2P_{m,n,l} = \sum_{p,q,i} H_2(m,n,l,p,q,i)\, {}_1P_{p,q,i}\, {}_1P_{m-p,n-q,l-i} \qquad (4.20) $$

where ${}_1P$'s correspond to the first order waves and

$$ H_2(m,n,l,p,q,i) = \begin{cases} 0, & \text{for } m = n = 0 \\ 0.5\left[K_p + K_{m-p} + (K_p K_{m-p} - \vec{K}_p \cdot \vec{K}_{m-p}) \cdot \left(\dfrac{gK_m + l^2W^2}{gK_m - l^2W^2}\right)\left(\dfrac{g}{i(l-i)W^2}\right)\right], & \text{otherwise} \end{cases} \qquad (4.21) $$

where

$$\vec{K}_p = pN\hat{x} + qN\hat{y}\ ,\ K_p = |\vec{K}_p|$$

$$\vec{K}_{m-p} = (m-p)N\hat{x} + (n-q)N\hat{y}\ ,\ K_{m-p} = |\vec{K}_{m-p}| \quad (4.22)$$

$$\vec{K}_m = mN\hat{x} + nN\hat{y}\ ,\ K_m = |\vec{K}_m|$$

g : acceleration due to gravity = 9.81 m/sec^2

In (4.19) $\vec{K}_p$, iW and $\vec{K}_{m-p}$, (1 - i)W are spatial wave numbers and frequencies for the two first order waves of respective amplitudes ${}_1P_{p,q,i}$ and ${}_1P_{m-p,n-q,l-i}$. These first order waves obey the dispersion relationship (4.18). On the other hand $\vec{K}_m$ and lW are wave number and frequency of a second order wave of amplitude ${}_2P_{m,n,l}$ and this does not obey the dispersion equation (4.18), i.e., $lW \neq (gK_m)^{\frac{1}{2}}$, as shown by Weber and Barrick. Therefore the function H is never singular. It may be mentioned that the authors have made the usual assumptions about the ocean in their analysis. These assumptions are that the ocean water is homogeneous, incompressible, inviscid, and without surface tension. Also, the ocean is considered infinitely deep, unbounded along its surface and has a free interface. By using (4.20) in (4.19) the ocean surface to the second order may thus be given as

$$f(\vec{x},t) \approx \sum_{m,n,l} [{}_1P_{m,n,l} + \sum_{p,q,i} H_2(m,n,l,p,q,i) \cdot {}_1P_{p,q,i}\ {}_1P_{m-p,n-q,l-i}] \exp[j\vec{K}_m \cdot \vec{x} + jlWt]. \quad (4.23)$$

By comparing the above with (4.1) we get

$$P_{m,n,l} \approx {}_1P_{m,n,l} + \sum_{p,q,i} H_2(m,n,l,p,q,i)\ {}_1P_{p,q,i}\ {}_1P_{m-p,n-q,l-i} \quad (4.24)$$

4.2.2 Wave Height Spectra for the First Order Gravity Waves

As mentioned before an ocean surface to a first order approximation may be described by a linear superposition of the first order gravity waves. This description from (4.19) may be given as

$$f_1(\vec{x},t) = \sum_{m,n,l} {}_1P_{m,n,l} \exp[j\vec{K}_m \cdot \vec{x} + jlWt] \tag{4.25}$$

with the corresponding three dimensional spectrum, given from (4.8) and (4.11), as

$$\langle {}_1P^*_{m,n,l}\, {}_1P_{p,q,i} \rangle = \begin{cases} \dfrac{N^2 W}{(2\pi)^3} S_1(\vec{K}_m, lW), \text{for } m = p, n = q, l = i \\ 0, \text{otherwise} . \end{cases} \tag{4.26}$$

Since the above first order waves obey the dispersion relation (4.18), we have

$$f_1(\vec{x},t) = \sum_{m,n} {}_1P^+_{m,n} \exp[j\vec{K}_m \cdot \vec{x} - j\,\mathrm{sgn}(m)(gK_m)^{\frac{1}{2}}t]$$
$$+ \sum_{m,n} {}_1P^-_{m,n} \exp[j\vec{K}_m \cdot \vec{x} + j\,\mathrm{sgn}(m)(gK_m)^{\frac{1}{2}}t], \tag{4.27}$$

where we have split the first order $P_{m,n}$ into two parts as ${}_1P_{m,n} = {}_1P^+_{m,n} + {}_1P^-_{m,n}$. This splitting, as suggested by Barrick (1972a), provides a distinction of the waves moving in the positive x half space from those moving in the negative half space. The superscript (+) on ${}_1P_{m,n}$ refers to those waves which have positive velocity components in the x direction. Similarly, (-) refers to those waves which have negative velocity components in the x direction. It is assumed that ${}_1P^+$ and ${}_1P^-$ are statistically independent of each other, i.e.,

$\langle {}_1P^+_{m,n}\, {}_1P^-_{p,q} \rangle = 0$

Since $f_1(x,t)$ is real, we have ${}_1P^{\pm}_{m,n} = ({}_1P^{\pm}_{-m,-n})^*$.

Following (4.8) and (4.11), a two dimensional spectrum may be defined as

$$<({}_1P^{\pm}_{m,n})^* \, {}_1P^{\pm}_{p,q}> = \begin{cases} (\frac{N}{2\pi})^2 \, S_1^{\pm}(\vec{K}_m), & \text{for } m = p, \ n = q \\ 0, & \text{otherwise} \end{cases} \tag{4.28}$$

where $S_1^+(\vec{K}_m)$ is a two dimensional wave height spectrum or a spatial spectrum for the first order gravity waves moving with positive velocity components in the x direction. Similarly, $S_1^-(\vec{K}_m)$ is a spatial spectrum for the same type of waves but moving with negative velocity components in the x direction. These two spatial spectra are even functions of $\vec{K}_m$.

By using (4.27) to (4.28) a three dimensional autocorrelation function to first order may be derived as

$$R_1(\vec{\tau}_x, \tau_t) = <f_1(\vec{x} + \vec{\tau}_x, t + \tau_t) f_1(\vec{x},t)>$$

$$= (2\pi)^{-2} \int_{K_x} \int_{K_y} [S_1^+(\vec{K}) \exp\{-j \, \mathrm{sgn}(K_x)(gK)^{\frac{1}{2}}\}\tau_t$$

$$+ S_1^-(\vec{K}) \exp\{j \, \mathrm{sgn}(K_x)(gK)^{\frac{1}{2}}\}\tau_t]$$

$$\cdot \exp(j\vec{K} \cdot \vec{\tau}_x) \, dK_x dK_y \tag{4.29}$$

where we have extended the limit of fundamental wavelength (L) to infinity (i.e., $N \to 0$) to reduce the summations to integrals. Now by taking the three dimensional Fourier transforms of (4.29) with respect to $\vec{\tau}_x$ and τ_t yields S_1 as

$$S_1(\vec{K},\omega) = 2\pi[S_1^+(\vec{K})\delta(\omega+\omega^o) + S_1^-(K)\delta(\omega-\omega^o)] \tag{4.30}$$

where

$$\omega^{o} = \mathrm{sgn}(K_x)(gK)^{\frac{1}{2}} \tag{4.31}$$

$\delta(\omega)$: Dirac delta function of argument ω .

In a similar way a one dimensional waveheight or frequency spectrum may easily be determined from (4.29) by setting $\tau_x = \tau_y = 0$ and then by Fourier transforming with respect to τ_t. This way we obtain

$$S_1(\omega) = (2\pi)^{-1} \int_{K_x} \int_{K_y} [S_1^+(\vec{K})\delta(\omega+\omega^o) + S_1^-(\vec{K})\delta(\omega-\omega^o)]\, dK_x dK_y \tag{4.32}$$

and, by using (4.30),

$$S_1(\omega) = (2\pi)^{-2} \int_{K_x} \int_{K_y} S_1(\vec{K},\omega)\, dK_x dK_y \tag{4.33}$$

Thus the above equations (4.30) to (4.33) provide an inter-relationship between the three dimensional wave height, two dimensional wave height (spatial) and one dimensional wave height (frequency) spectra for the first order gravity waves.

From the above inter-relationship and (4.16), the contribution to the mean square surface height due to the first order gravity waves may be given as

$$\begin{aligned} h_{s1}^2 &= (2\pi)^{-3} \int_{K_x} \int_{K_y} \int_{\omega} S_1(\vec{K},\omega)\, dK_x dK_y d\omega \\ &= (2\pi)^{-2} \int_{K_x} \int_{K_y} [S_1^+(\vec{K}) + S_1^-(\vec{K})]\, dK_x dK_y \\ &= (2\pi)^{-1} \int_{\omega} S_1(\omega) d\omega . \end{aligned} \tag{4.34}$$

It may be mentioned here that these one, two, and three dimensional spectra are double sided, i.e., they are defined for negative wave number and frequency space also. Further, the wave numbers for two and three dimensional spectra are in the cartesian form. In the case of a narrow beam receiving antenna the x axis is taken to be the direction of the beam for convenience. There are other forms available for these spectra in the literature. These various forms are based on the coordinate system chosen or on the single or double sidedness of the spectra. One such common form is $S_t(\omega,\theta)$ and it may be called a single sided directional-frequency spectrum. This spectrum is also a two dimensional spectrum and is dependent on the frequency (ω) and the direction (θ), where θ is measured with respect to the average direction of the wind [Longuet-Higgins et. al. (1963), Barrick (1977a)]. The mean square surface displacement from this spectrum may be given as

$$h_{s1}^2 = \int_0^{\infty}\int_{-\pi}^{\pi} S_t(\omega,\theta)\ d\omega d\theta \tag{4.35}$$

The above spectrum may be shown to be related with our spatial spectrum $S_1(\vec{K})$ as

$$S_1^+(\vec{K}) = \frac{(\pi g)^2}{\omega^3}\, S_t(\omega,\theta) \quad ; \quad \theta = \arg\{\mathrm{sgn}(K_x)\vec{K}\} - \theta_o \tag{4.36a}$$

$$S_1^-(\vec{K}) = \frac{(\pi g)^2}{\omega^3}\, S_t(\omega,\theta) \quad ; \quad \theta = \arg\{-\mathrm{sgn}(K_x)\vec{K}\} - \theta_o \tag{4.36b}$$

where

$\omega = (gK)^{1/2}$, $K = |\vec{K}|$

$\arg\{\pm\mathrm{sgn}(K_x)\vec{K}\}$: argument of the wave number vector $\{\pm\mathrm{sgn}(K_x)\vec{K}\}$

θ_o : average direction of the wind measured with respect to the x axis.

(4.37)

Clearly, θ lies between $-(\frac{\pi}{2} + \theta_o)$ and $(\frac{\pi}{2} - \theta_o)$ in (4.36a) and between $(\frac{\pi}{2} - \theta_o)$ and $(\frac{3\pi}{2} - \theta_o)$ in (4.36b).

Conventional oceanographic techniques are available for experimental determination of the directional-frequency spectrum $S_t(\omega,\theta)$, e.g., by using a pitch and roll buoy [Longuet-Higgins et. al. (1963)]. The frequency spectrum alone is relatively simpler to determine experimentally, e.g., by using an accelerometer buoy [Kinsman (1965, ch. 9)]. There are many semi-empirical models available for the frequency spectrum, whereas only a few models are available for the directional-frequency spectrum. The models available for the directional-frequency spectrum are assumed to be in the following separable form [Tyler et. al. (1974)]

$$S_t(\omega,\theta) = S_t(\omega)\, d(\omega,\theta) \tag{4.38}$$

where $S_t(\omega)$ is a frequency spectrum and is related to our frequency spectrum as

$$S_t(\omega) = \frac{1}{\pi}\, S_1(\omega) \tag{4.39}$$

$d(\omega,\theta)$ is the directional distribution factor and is normalized in such a way that

$$\int_{-\pi}^{\pi} d(\omega,\theta)\, d\theta = 1 \tag{4.40}$$

A few of these proposed models for the frequency spectrum and the directionality factor are as follows.

A. Frequency Spectrum

Neumann [Kinsman (1965, ch. 8)] has proposed that for a fully developed sea $S_t(\omega)$ may be given as

$$S_t(\omega) = C_n \frac{\pi}{2} \omega^{-6} \exp\{-2(\frac{g}{\omega U})^2\} . \qquad (4.41)$$

A sea is considered to be fully developed at a given wind speed U whose spectrum contains components of all frequencies, each with the maximum energy of which it is capable under the given wind. Fully developed seas are only encountered when the wind blows over a sufficiently long fetch and for a sufficiently long time. In this situation the phase speed of the dominant wave becomes equal to the wind speed, i.e., its frequency becomes equal to (g/U). In the above equation U is the wind speed measured at anemometer height (between 8 to 10 meters above the sea surface) and g is the acceleration due to gravity. C_n is a constant and is estimated from measurements as 3.04 m^2/sec^5. The above model is also referred to as the Neumann–Pierson spectrum.

By forming observed spectra as a basis Pierson and Moskowitz (1964) proposed the following form for the spectrum for fully developed seas

$$S_t(\omega) = 0.0081\, g^2\, \omega^{-5}\, \exp\{-0.74\, (\frac{g}{\omega U})^4\} \qquad (4.42)$$

By assuming that wave breaking is the main mechanism controlling the amplitude of ocean waves Phillips (1977, ch. 4) has proposed that, in the saturation range, the spectrum should adopt the following form

$$S_t(\omega) = C_p\, g^2\, \omega^{-5} \;;\; \text{for } \omega_m < \omega < 2g/U_* \qquad (4.43)$$

where C_p is a dimensionless constant and is determined experimentally. It varies from 0.008 to 0.0148. U_* is the friction velocity and is given by

$$U_* = (\tau_s/\rho_a)^{\frac{1}{2}} \qquad (4.44)$$

where τ_s is the mean tangential stress of the wind and ρ_a is the density of the air. ω_m is the frequency at which the spectral peak occurs and it may be taken as the frequency of dominant wave, i.e., $\omega_m = g/U$.

B. Directional Distribution Factor

The simplest form of the directional distribution is that of semi-isotropic, i.e., all waves moving in the half space with respect to the direction of wind are equally favoured by the wind. This form is independent of the frequency (ω) and may be given as

$$d(\theta,\omega) = d(\theta) = \begin{cases} \frac{1}{\pi}, & \text{for } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases} \tag{4.45}$$

The above form has been used by Barrick (1972a) and Johnstone (1975) for calculating the radar cross section for the ocean surface. A more preferable form is that proposed by Pierson, Neumann, and James [Kinsman (1965, ch. 8)]. In this form the directionality factor is taken to be a cosine square distribution instead of the above constant distribution in the half space with respect to the wind direction. This form is again independent of the frequency and may be given as

$$d(\theta,\omega) = d(\theta) = \begin{cases} \frac{2}{\pi}\cos^2\theta, & \text{for } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases} \tag{4.46}$$

In both of these forms waves are assumed to be propagating in the half space only.

From the directional-frequency spectra obtained from measurements Longuet-Higgins et. al. (1963) have proposed that the angular distribution is proportional to $\{\cos(\theta/2)\}^{2s}$, i.e.,

$$d(\theta,\omega) = \frac{1}{C(s)}\{\cos(\theta/2)\}^{2s}, \text{ for } -\pi \leq \theta \leq \pi \tag{4.47}$$

where the parameter s varies with frequency and wind speed. C(s) is a normalization constant such that

$$C(s) = \int_{-\pi}^{\pi} \{\cos(\theta/2)\}^{2s} \, d\theta$$

$$= 2\sqrt{\pi}\ \Gamma(s + \tfrac{1}{2})/\Gamma(s + 1) \qquad (4.48)$$

where Γ is the gamma function. The parameter s is found to vary with frequency. A graph showing the variation of s with $(\omega U_1/g)$ along with the definition of the reference wind speed U_1 is presented in their paper. For a fixed U_1, s decreases from near 4 at low frequencies to about 0.5 at high frequencies. The above distribution allows the propagation of waves in the other half space also except in the direction opposite to wind, i.e., in the downwind direction, where it is zero. By assuming s = 1 for all frequencies the above distribution reduces to a cardioid distribution as

$$d(\theta,\omega) = d(\omega) = \frac{1}{\pi}\cos^2(\theta/2)\ , \text{ for } -\pi \le \theta \le \pi \qquad (4.49)$$

Based on the observed radar backscattered Doppler spectra and in-situ measurements using a pitch and roll buoy Tyler et. al. (1974) have proposed the following modification to the above distribution (4.47) of Longuet-Higgins

$$d(\theta,\omega) = \frac{1}{H(s)}\left[\epsilon + (1 - \epsilon)\{\cos(\theta/2)\}^{2s}\right], \text{ for } -\pi \le \theta \le \pi \qquad (4.50)$$

where H(s) is a normalization constant and is equal to

$$H(s) = \epsilon + (1 - \epsilon)\ C(s) \qquad (4.51)$$

where C(s) is given by (4.48). To maintain a consistency with (4.47) we have used 2s for the power of the cosine function instead of s used by Tyler. This form allows a slight energy via ε to the waves travelling in the down wind direction. From the observed radar backscattered doppler spectra an upper limit of ε for an ocean wave of 7 sec period is found to be 0.2. The variation of s with $(\omega U_1/g)$, which was obtained from buoy measurements, is presented there and is in reasonable agreement with that found by Longuet-Higgins.

Now, for example, by combining the cardioid distribution factor (4.49) with the Pierson-Maskowitz frequency spectrum (4.42) the directional-frequency spectrum may be given as

$$S_t(\omega,\theta) = 0.0081 \frac{g^2}{\pi\omega^5} \exp\{-0.74 (\frac{g}{\omega U})^4\} \cos^2(\theta/2) \quad (4.52)$$

4.3 First and Second Orders of Backscattered Surface Field

In chapter 3 under sections 3.4.1 and 3.5.1 we have derived the vertical component of the first and second orders of the backscattered surface field received at time t_o from a time invariant two dimensional periodic surface. These two solutions are in the form of spatial convolution integrals and are given by the equations (3.80) and (3.115). The source assumed is an elementary vertical dipole which is excited by a pulse of sinusoidal current centered at time $t = 0$. Since the surface is a continuous object in the spatial dimension, the received backscattered signal is also continuous with respect to time even though the transmitted signal is a single pulse. In other words the backscattered field is continuously being received after the pulse is transmitted. As the observation time advances the propagation distance increases and, therefore, the transmission loss increases. This in turn reduces the strength of the received backscattered signal. Therefore, an upper limit on the observation time may be set, say t_m, such that the signal received then is just sensitive enough to be detected by the receiver. This t_m corresponds to the maximum range of detection. The range will depend upon the surface condition, external noise condition, and the radar system that is used.

If the observation time for the backscattered field is fixed as time t_o ($t_o \leq t_m$), then an area of the surface, from which the first order backscattered field is received, is also fixed. Clearly, from (3.80), this area is an annulus of radius $ct_o/2$ and width $c\tau_o/2$, where c is the propagation velocity ($\approx 3 \times 10^8$ m/sec) and τ_o is the transmitted pulse width. At the same time t_o the second order backscattered field is also received from the surface area bounded by the outer boundary of this annulus as is evident from the gate function in (3.115). For that matter higher orders of this field are also received at the same time but they are assumed to be relatively weaker and hence are not considered.

In a pulsed radar, however, the transmission is not restricted to a single sinusoidal pulse. Instead, it is a periodic transmission of the pulsed sinusoid. Of course, the pulse repetition period should be greater than the above time t_m so that the backscattered signal received due to a transmitted pulse is not mixed with that received due to the pulse transmitted subsequently. For example, for a typical ocean applications radar these specifications are: frequency = 25.4 MHz, pulse width = 8 μsec, and the pulse repetition period = 512 μsec. This means that for this radar the maximum theoretical range taken is 76.8 Km. In what follows we consider such a periodic transmission.

Now, for every transmitted pulse we have a continuous record of the backscattered field of t_m seconds long with starting time that of the pulse. Therefore, a set of many such records is collected by the periodic transmission of the pulsed sinusoid. The number of records in the set obviously can not exceed the number of pulses that are transmitted. Each of these records may be observed at a fixed time t_o (t_o measured from the starting time of each record), and thus a set of samples may be collected. The minimum interval between the two consecutive samples will be equal to the pulse repetition period. Clearly, these samples represent the observed backscattered field from the same area of the surface and this area corresponds to t_o as explained above. Further, all these samples are the same as the surface is not varying with time. Therefore, the two respective equations (3.80) and (3.115) for first and second orders of backscattered surface field hold here as well. Also, the sum of these two equations gives an approximation of these samples.

In the case of an ocean surface the situation is somewhat different, i.e., now we are dealing with a time varying surface. The above description for the case of a non-time varying surface is still valid in this case also except that these samples are not in general equal. Their values are now dependent upon the variability of the surface with time. Therefore, the above referred equations (3.80) and (3.115) are no more valid in a strict mathematical sense. However, on a valid assumption that the time taken for any significant surface variations is very large compared to the propagation time of radio waves, these two equations may be modified to include the time dependency of the ocean surface. By comparing the Fourier series description (2.7) for $f(\vec{x})$ with that of (4.1) for $f(\vec{x}, t)$, the proposed modification is to replace

$P_{m,n}$ by $\sum_{l} P_{m,n,l} \exp(jlWt)$

in (3.80) and (3.115), i.e., in the equations for the two orders of the backscattered surface field derived earlier for a time invariant surface.

By allowing the above modification the vertical component of the backscattered surface field approximated up to the second order may be given as

$$\tilde{E}_{zb}(t_o,t) \simeq \tilde{E}_{zb1}(t_o,t) + \tilde{E}_{zb2}(t_o,t) \tag{4.53}$$

where $\tilde{E}_{zb1}(t_o,t)$ is the first order backscattered field obtained by carrying the proposed modification in the corresponding equation (3.80). In a similar way, $\tilde{E}_{zb2}(t_o,t)$ is the second order field obtained by modifying (3.115). Let us consider now the arguments namely t_o and t in (4.53). As explained above, t_o is any selected time at which we are sampling each record of the backscattered field. Also, this t_o corresponds to the respective areas of the surface from which the two orders of the backscattered field are being received. Therefore, in this respect t_o is considered fixed. Whereas the variable t relates to the variability of the surface with time, hence relates to the variability of the backscattered field with time in each record at a fixed t_o. As mentioned above the information for the field with respect to t for a fixed t_o is available in the form of samples only. This means that this information is discrete and is available at $t = t_o \pm mT_p$, where T_p is the pulse repetition period and m = 0,1,2, ... However, this information may be theoretically extended to all t as if we have a radar system which can constantly receive the backscattered signal from a specific area of the surface corresponding to t_o. Therefore, the actual signal is nothing but the one obtained by sampling this theoretical signal at regular intervals.

Proceeding in this way and using (3.80) and (3.115) in (4.53) yields

$$\tilde{E}_{zb}(t_o,t) \simeq \frac{C_a}{(2\pi)^2} \exp(j\omega_o t_o) \sum_{m,n,l} \Big\{ P_{m,n,l} \exp(jlWt)$$

$$\cdot \left[\int_0^{\infty} \int_{-\pi}^{\pi} \frac{1}{\rho'} F_b \{\rho', \cos\theta', \sin\theta', (v,w) \neq (-m,-n)\} \right.$$

$$\cdot F_a \{\rho', -\cos\theta', -\sin\theta'\} G_{\tau_o}(t_o - \frac{2\rho'}{c})$$

$$\cdot [mN(mN - k_o \cos\theta') + nN(nN - k_o \sin\theta')]$$

$$\left. \cdot \exp[j\rho' N(m\cos\theta' + n\sin\theta') - 2jk_o\rho'] \, d\theta' d\rho' \right\}$$

$$+ \frac{C_a}{(2\pi)^3} \exp(j\omega_o t_o) \sum_{p,q,i} \sum_{\substack{m,n,l \\ (m,n)\neq(-p,-q)}} \left\{ P_{p,q,i} P_{m,n,l} \exp[jW(l+i)t] \right.$$

$$\cdot \left[\int_0^{\infty} \int_{-\pi}^{\pi} \int_{x''} \int_{y''} \frac{1}{\rho''\rho_x'} F_b \{\rho'', \cos\theta'', \sin\theta'', (v,w) \neq (-m-p, -n-q)\} \right.$$

$$\cdot F_c \{\rho_x', \cos\theta_x', \sin\theta_x', (v,w) \neq \begin{Bmatrix} (m,n) \\ (-p,-q) \end{Bmatrix} \}$$

$$\cdot F_a \{\rho', -\cos\theta', -\sin\theta'\} G_{\tau_o}[t_o - \frac{1}{c}(\rho'' + \rho_x' + \rho')]$$

$$\cdot Q(m+p, n+q, p, q, -\cos\theta', -\sin\theta')$$

$$\cdot \exp[jN\{mx'' + ny'' + \rho'(p\cos\theta' + q\sin\theta')\}]$$

$$\left. \cdot \exp[-jk_o(\rho'' + \rho_x' + \rho') \, dx'' dy'' d\theta' d\rho'] \right\} \qquad (4.54)$$

where we have carried out the proposed modification. The various terms and symbols appearing in the above equation have been defined earlier while deriving (3.80) and (3.115). For the non-time varying surface we have seen that the expressions for modified surface impedances contain the surface Fourier coefficients, i.e., $P_{v,w}$'s. Therefore, they also should have been similarly modified here to include the surface time dependency. However, in this work we have not performed this modification.

This means that we are assuming that the effect of instantaneous variations of the ocean surface on attenuation functions is negligible compared to the long term variations. The effect of long term variations is already accounted for through the $P_{v,w}$'s. To put it another way the ocean surface may be considered temporally frozen during any observation period as far as these attenuation functions are concerned.

The above equation for the backscattered field consists of two parts. The first part represents the first order field and the second part represents the second order field. As mentioned in section 4.2.1 the Fourier coefficients ($P_{m,n,l}$'s) may be expanded into a perturbational series representing the orders of ocean waves, e.g., first order, second order, and higher order waves. We will consider here only up to the second order. Therefore, the Fourier coefficients in the first part of (4.54) may be approximated as

$$P_{m,n,l} \simeq {}_1P_{m,n,l} + {}_2P_{m,n,l},$$

and in the second part as

$$P_{p,q,i}\, P_{m,n,l} \simeq {}_1P_{p,q,i}\; {}_1P_{m,n,l}\,.$$

A solution for the second order coefficients (${}_2P_{m,n,l}$'s) has been derived by Weber and Barrick (1977) in terms of a sum of products of two first order coefficients for deep water gravity waves. This solution is given by (4.21). By using the above approximation and (4.21) in (4.54) we arrive at the following solution for the backscattered field:

$$\tilde{E}_{zb}(t_o,t) \simeq \frac{C_a}{(2\pi)^2} \exp(j\omega_o t_o) \sum_{m,n,l} \left\{ {}_1P_{m,n,l} \exp(jlWt) \right.$$

$$\cdot \left[\int_0^{\infty} \int_{-\pi}^{\pi} \frac{1}{\rho'} F_b \{\rho', \cos\theta', \sin\theta', (v,w) \neq (-m,-n)\} \right.$$

$$\cdot F_a \{\rho', -\cos\theta', -\sin\theta'\}\ G_{\tau_o}(t_o - \frac{\rho'}{c})$$

$$\cdot [mN(mN - k_o \cos\theta') + nN(nN - k_o \sin\theta')]$$

$$\left.\cdot \exp[j\rho' N(m\cos\theta' + n\sin\theta') - 2jk_o\rho']\ d\theta' d\rho' \right\}$$

$$+ \frac{C_a}{(2\pi)^2} \exp(j\omega_o t_o) \sum_{p,q,i} \sum_{\substack{m,n,l \\ (m,n)\neq(-p,-q)}} \left\{ {}_1P_{p,q,i}\ {}_1P_{m,n,l} \right.$$

$$\cdot \exp[jW(l + i)t]\ H(m,n,l,p,q,i)$$

$$\cdot \left[\int_0^{\infty} \int_{-\pi}^{\pi} \frac{1}{\rho'} F_b \{\rho', \cos\theta', \sin\theta', (v,w) \neq (-m-p,-n-q)\}\right.$$

$$\cdot F_a \{\rho', -\cos\theta', -\sin\theta'\}\ G_{\tau_o}(t_o - \frac{\rho'}{c})$$

$$\cdot [(p + m)N\{(p + m)N - k_o \cos\theta'\}$$

$$- (q + n)N\{(q + n)N - k_o \sin\theta'\}]$$

$$\cdot \exp[j\rho' N\{(p + m)\cos\theta' + (q + n)\sin\theta'\}]$$

$$\left.\left.\cdot \exp(-2jk_o\rho')\ d\theta' d\rho' \right] \right\}$$

$$+ \frac{C_a}{(2\pi)^3} \exp(j\omega_o t_o) \sum_{p,q,i} \sum_{\substack{m,n,l \\ (m,n)\neq(-p,-q)}} \left\{ {}_1P_{p,q,i}\ {}_1P_{m,n,l} \right.$$

$$\cdot \exp[jW(l + i)t]$$

$$\cdot \int_0^{\infty} \int_{-\pi}^{\pi} \int_{x''} \int_{y''} \frac{1}{\rho''\rho_x'} F_b \{\rho'', \cos\theta'', \sin\theta'', (v,w) \neq (-m-p,-n-q)\}$$

$$\cdot F_c \{\rho_x', \cos\theta_x', \sin\theta_x', (v,w) \neq \begin{Bmatrix}(m,n) \\ (-p,-q)\end{Bmatrix}\}$$

$$\cdot F_a \{\rho', -\cos\theta', -\sin\theta'\}\ G_{\tau_o}[t_o - \frac{1}{c}(\rho'' + \rho_x' + \rho')]$$

$$\cdot\, Q(m+p,\ n+q,\ p,q,\ -\cos\theta',\ -\sin\theta')$$

$$\cdot\, \exp[jN\{mx'' + ny'' + \rho'(p\cos\theta' + q\sin\theta')\}]$$

$$\left.\cdot\, \exp[-jk_o(\rho'' + \rho_x' + \rho')\quad dx''dy''d\theta'd\rho']\right\} \tag{4.55}$$

where the function H is derived from H_2 by changing the summation indices (m,n,l) to $(m+p,\ n+q,\ l+i)$ in (4.21), i.e.,

$$H(m,n,l,p,q,i) = 0.5\,[K_p + K_m + (K_pK_m - \vec{K}_p \cdot \vec{K}_m)$$

$$\cdot\, \frac{g}{liW^2}\ \{\frac{gK_{m+p} + (l+i)^2W^2}{gK_{m+p} - (l+i)^2W^2}\}] \tag{4.56}$$

Further, the condition of H_2 being zero for $m = n = 0$ in (4.21) has been incorporated in (4.55) via the restriction $(m,n) \neq (-p,-q)$.

The above equation (4.55) thus gives a solution for the vertical component of the backscattered surface field up to the second order for the assumed model of deep water gravity waves. This solution consists of three parts. The first part represents a first order field corresponding to the first order scattering from the first order gravity waves. Therefore, this part may be referred to as the first order electromagnetic and hydrodynamic term. The second part represents a second order field corresponding to the first order scattering from the second order waves. This term is obtained from a second order solution of the hydrodynamic equations for gravity waves in deep water. Therefore, this term may be referred to as the second order hydrodynamic term. Finally, the third part represents a second order field corresponding to the second order scattering from the first order waves. Hence, this term may be referred to as the second order electromagnetic term. The above solution is in the form of a multi-dimensional integral and needs to be evaluated. These integrals may be evaluated asymptotically for the case of a narrow beam receiving antenna which we consider now.

4.3.1 Narrow Beam Receiving Antenna

Assume we have for receiving purposes a very narrow beam antenna with a beam width of $2\Delta_\theta$ directed along the positive x axis. The radiation uniformity in the horizontal plane for the transmitting antenna is still maintained. Therefore, the angle θ' in (4.55) varies only from $-\Delta_\theta$ to Δ_θ. The integrals in the first and third parts of (4.55) are the same as in the corresponding equations (3.80) and (3.115) for the time invariant surface. Under this narrow beam condition, they have been evaluated asymptotically by using the method of stationary phase in sections (3.4.1.1) and (3.5.1.1). The nature of the double integral in the second part of (4.55) is the same as in the first part. Therefore, this integral may be also evaluated similarly. Without repeating these evaluations the result may be shown to be

$$\tilde{E}_{zb}(t_o^{\frown}, t) \simeq$$

$$\frac{2C_a \Delta_\rho}{(2\pi\rho_o)^{3/2}} F_a(\rho_o, 1, 0) \sum_{\substack{m,n,1 \\ -|m|\alpha \leq n \leq |m|\alpha}} \Bigg\{ {}_1P_{m,n,1} \exp(j1Wt)$$

$$\cdot F_b\{\rho_o, 1, 0, (v,w) \neq (-m,-n)\}[K_m - k_o\, \mathrm{sgn}(m)](K_m)^{1/2}$$

$$\cdot \mathrm{Sa}[\Delta_\rho\{\mathrm{sgn}(m)K_m - 2k_o\}]\exp[j\, \mathrm{sgn}(m)(K_m\rho_o - \frac{\pi}{4})]\Bigg\}$$

$$+ \frac{2C_a \Delta_\rho}{(2\pi\rho_o)^{3/2}} F_a(\rho_o, 1, 0) \sum_{p,q,i} \sum_{\substack{m,n,1 \\ (m,n)\neq(-p,-q) \\ -\alpha|p+m| \leq q+n \leq \alpha|p+m|}} \Bigg\{ {}_1P_{p,q,i}\; {}_1P_{m,n,1}$$

$$\cdot \exp[j(1+i)Wt]F_b\{\rho_o, 1, 0, (v,w) \neq (-m-p, -n-q)\}$$

$$\cdot \mathrm{Sa}[\Delta_\rho\{\mathrm{sgn}(p+m)\, K_{p+m} - 2k_o\}]$$

$$\cdot \{(K_{p+m})^{\frac{1}{2}} [K_{p+m} - k_o\, \mathrm{sgn}(p+m)]\, H(m,n,l,p,q,i)$$

$$+ Q_1(m,n,p,q)[K_m^2\, K_{p+m} - 2k_o\, \mathrm{sgn}(p+m)\, \vec{K}_m \cdot \vec{K}_{p+m}]^{-\frac{1}{2}}\}$$

$$\left. \cdot \exp[j\, \mathrm{sgn}(p+m)(K_{p+m}\, \rho_o - \frac{\pi}{4})] \right\}$$

$$+ \frac{2C_a\, \Delta_\rho}{(2\pi\rho_o)^{3/2}} F_a(\rho_o, 1, 0) \sum_{\substack{p,q,i \\ -\alpha|p| \leq q \leq \alpha|p|}} \sum_{\substack{m,n,l \\ (m,n) \neq (-p,-q)}} \left\{ {}_1P_{p,q,i}\ {}_1P_{m,n,l} \right.$$

$$\cdot \exp[jW(I+i)t]F_c\left\{\rho_o, 1, 0, (v,w) \neq \begin{Bmatrix} (m,n) \\ (-p,-q) \end{Bmatrix}\right\}$$

$$\cdot \mathrm{Sa}[\Delta_\rho\{\mathrm{sgn}(p)\, K_p - 2k_o\}]$$

$$\cdot Q_2(m,n,p,q)[K_m^2\, K_p + 2k_o\, \mathrm{sgn}(p)\, \vec{K}_m \cdot \vec{K}_p]^{-\frac{1}{2}}$$

$$\left. \cdot \exp[j\, \mathrm{sgn}(p)\, (K_p\, \rho_o - \frac{\pi}{4})] \right\}$$

$$- j\, \frac{2C_a\, \Delta_\rho}{(\pi\rho_o)^{3/2}} \sum_{p,q,i} \sum_{\substack{m,n,l \\ (m,n) \neq (-p,-q) \\ 2k_o|m|N < K_m^2 < 2k_o K_m \\ -\alpha|2p+m| \leq 2q+n \leq \alpha|2p+m|}} \left\{ {}_1P_{p,q,i}\ {}_1P_{m,n,l}\ \exp[jW(l+i)t] \right.$$

$$\cdot F_a(\beta_o\rho_o, 1, 0)F_b\{r_b, \cos\psi_b, \sin\psi_b, (v,w) \neq (-m-p, -n-q)\}$$

$$\cdot F_c\left\{r_c, -\cos\psi_c, -\sin\psi_c, (v,w) \neq \begin{Bmatrix} (m,n) \\ (-p,-q) \end{Bmatrix}\right\}$$

$$\cdot \; Sa[\beta_o \frac{\Delta_\rho}{2} \{sgn(2p+m)\; K_{2p+m} - 2k_o - \frac{|n|N}{K_m} K_{mo}\}]$$

$$\cdot \; K_m Q_3(m,n,p,q)\; [\beta_o K_{2p+m} \{K_m^4 - (2k_o mN)^2\}]^{-\frac{1}{2}}$$

$$\cdot \; \exp[j\; sgn(2p+m)(K_{2p+m} \frac{\beta_o \rho_o}{2} - \frac{\pi}{4})]$$

$$\cdot \; \exp[j\rho_o (2k_o - k_o\beta_o - \frac{|n|N}{2K_m} K_{mo}\beta_o)]\Bigg\} \qquad (4.57)$$

In the above equation F_a, F_b, and F_c are attenuation functions with modified surface impedances. The three functions Q_1, Q_2, and Q_3 are defined by (3.144). The function H is the hydrodynamic term and is given by (4.56). The relevent equations which define the other symbols or functions are (3.92), (3.142), and (3.143).

A description of the above solution is almost the same as that already described for the corresponding first and second order solutions, (3.91) and (3.145), in the case of the time invariant surface. The first part represents the first order scattering occuring on the patch. The second part represents the case where both first and second scatterings occur on the patch. This part includes the second order hydrodynamic effect also. The third part represents the case where the first scattering occurs at the source point and the second scattering occurs on the patch. Finally, the fourth part represents the case where the two scatterings occur off the patch but within the surface area corresponding to t_o. A comparison of the above solution with that for the time invariant surface shows that this solution, apart from the inclusion of the hydrodynamic term, is time varying and this variation corresponds to the temporal variation of the ocean surface. Further, this variation appears in the form of a Doppler effect caused by moving ocean waves, which is to be expected. These Doppler frequencies are apparent in (4.57) through the time dependent exponential terms.

4.3.2 Narrow Beam Transmitting and Receiving Antennas

Suppose we have very narrow beam antennas for both transmitter and receiver. In this situation the contribution to the second order back-scattered field from off the patch scatterings may be neglected. This contribution comes from the third stationary point (3.132c) in the second order field and it corresponds to the last part in the solution (4.57). In this case the received backscattered field is mainly from the patch. Under this condition (4.57) reduces to the following equation

$$\tilde{E}_{zb}(t_o,t) \simeq$$

$$\frac{2C_a\,\Delta_\rho}{(2\pi\rho_o)^{3/2}}\,F_a(\rho_o,1,0)\sum_{\substack{m,n,l\\ -|m|\alpha\le n\le|m|\alpha}}\Big\{{}_1P_{m,n,l}\exp(jlWt)$$

$$\cdot F_b\{\rho_o,1,0,(v,w)\ne(-m,-n)\}[K_m - k_o\,\mathrm{sgn}(m)](K_m)^{\frac{1}{2}}$$

$$\cdot \mathrm{Sa}\,[\Delta_\rho\{\mathrm{sgn}(m)\,K_m - 2k_o\}]\ \exp[j\,\mathrm{sgn}(m)(K_m\rho_o - \tfrac{\pi}{4})]\Big\}$$

$$+\frac{2C_a\,\Delta_\rho}{(2\pi\rho_o)^{3/2}}\,F_a(\rho_o,1,0)\sum_{\substack{p,q,i\\(m,n)\ne(-p,-q)\\ -\alpha|p+m|\le q+n\le\alpha|p+m|}}\ \sum_{m,n,l}\Big\{{}_1P_{p,q,i}\ {}_1P_{m,n,l}$$

$$\cdot \exp\,[j(l+i)Wt]\ F_b\{\rho_o,1,0,(v,w)\ne(-m-p,-n-q)\}$$

$$\cdot \mathrm{Sa}\,[\Delta_\rho\{\mathrm{sgn}(p+m)\,K_{p+m} - 2k_o\}]$$

$$\cdot \{(K_{p+m})^{\frac{1}{2}}\,[K_{p+m} - k_o\,\mathrm{sgn}(p+m)]\,H(m,n,l,p,q,i)$$

$$+\,Q_I(m,n,p,q)[K_m^2\,K_{p+m} - 2k_o\,\mathrm{sgn}(p+m)\,\vec{K}_m\cdot\vec{K}_{p+m}]^{-\frac{1}{2}}\}$$

$$\cdot \exp\ [j\ \mathrm{sgn}(p+m)(K_{p+m}\rho_o - \tfrac{\pi}{4})]\Big\}$$

$$+ \frac{2C_a \Delta_\rho}{(2\pi\rho_o)^{3/2}} F_a(\rho_o, 1, 0) \sum_{\substack{p,q,i}} \sum_{\substack{m,n,1 \\ (m,n)\neq(-p,-q) \\ -\alpha|p|\leq q\leq\alpha|p|}} \Big\{ {}_1P_{p,q,i}\ {}_1P_{m,n,1}$$

$$\cdot \exp\ [j(1+i)Wt]\ F_c^{*}\{\rho_o, 1, 0, (v,w) \neq \begin{Bmatrix}(m,n)\\(-p,-q)\end{Bmatrix}\}$$

$$\cdot\ Sa\ [\Delta_\rho\{\mathrm{sgn}(p)\ K_p - 2k_o\}]$$

$$\cdot\ Q_2(m,n,p,q)\,[K_m^2 K_p + 2k_o\ \mathrm{sgn}(p)\,\vec{K}_m \cdot \vec{K}_p]^{-\frac{1}{2}}$$

$$\cdot \exp\ [j\ \mathrm{sgn}(p)(K_p\rho_o - \tfrac{\pi}{4})]\Big\} \qquad (4.58)$$

4.4 Average First and Second Orders of Backscattered Radar Cross Section

In the previous section we derived an approximate solution up to the second order for the vertical component of the backscattered surface field. In deriving this no statistics of the ocean surface have been used. The field may be calculated from this solution provided the Fourier coefficients of the first order gravity waves are known. But these coefficients are not deterministic as the ocean surface is random. However, some information is available in the form of the statistical averages of these coefficients as discussed in section 4.2. It is, therefore, more meaningful to talk in the statistical average sense for the field also and thereby obtain an average first and second order of the Doppler dependent backscattered radar cross section of the ocean surface. The cross section is a useful parameter in the remote sensing of ocean surface conditions. We will derive this cross section for two cases: 1) narrow beam reception, 2) narrow beam transmission and narrow beam reception.

4.4.1 Narrow Beam Receiving Antenna

Let us again consider the source. In chapter 3 we assumed that the source was an elementary vertical dipole excited by a pulsed sinusoidal current. To facilitate the analysis we later reduced the height of the dipole to zero. Further, in section 4.3 we extended the excitation from a single pulse to a periodic pulsed sinusoid. Consider now any one pulse of this periodic excitation. From (3.7), the incident field in the far field approximation may be given as

$$\vec{E}_i\ (r,\ \theta,\ t') = -\frac{C_a}{4\pi r} \sin\theta\ G_{\tau_o}(t' - \frac{r}{c})\ \exp(j\omega_o t' - jk_o r)\hat{\theta}. \quad (4.59)$$

The above equation is written in the spherical coordinates $(r,\ \theta,\ \phi)$ system. G_{τ_o} is the gate function defined by (3.2). The two constants C_a and c are defined by (3.79). The time t' here refers to the propagation time of the field and is measured with reference to the transmitting time of the pulse. We assume that the length of the pulse (τ_o) is long enough to include many cycles of the sinusoid of frequency ω_o so that phasor analysis may be applied in (4.59). Thus we may write the θ component of the incident field in phasor form as

$$E_{i\theta}(r,\theta) = \frac{C_a}{4\pi r} \sin\theta\ \exp(-jk_o r)\ . \quad (4.60)$$

Let g_t represents the free space gain of the transmitting antenna in the $\theta = \frac{\pi}{2}$ (or $z = 0$) plane. It may be defined as [Jordon and Balmain (1968, ch. 11)]

$$g_t = \frac{2\pi r^2}{\eta P_t} \left| E_{i\theta}\ (r, \ \frac{\pi}{2}) \right|^2 \qquad (4.61)$$

where P_t is the average power transmitted during the pulse. η is the intrinsic impedance for free space. Using (4.60) in (4.61) yields

$$|C_a|^2 = 8\pi\eta P_t g_t , \qquad (4.62)$$

or, from (3.79),

$$C_a = -j\omega_o \mu_o I_o dl = -j(8\pi\eta P_t g_t)^{\frac{1}{2}} \qquad (4.63)$$

In deriving an average Doppler frequency dependent backscattered cross section from (4.57) we will follow the procedure stated here. First derive an autocorrelation function (ensemble average of the product of fields) for the backscattered field with respect to time t. The variable t refers to that used in the equation (4.57). By Fourier transforming this function with respect to the time displacement (τ) we obtain the average power density spectrum of the backscattered field. Finally, by comparing the spectrum with the standard radar range equation, an average first and second order of the backscattered cross section may be obtained.

Consider now the averaging required for the autocorrelation function of the field. It is obvious from (4.57) that this averaging involves only the surface Fourier coefficients (${}_1P_{m,n,1}$'s) and the attenuation functions (F's). As per the model assumed for the ocean surface the Fourier coefficients are considered to be random variables. The attenuation functions are entering into this averaging process as they are also functions of these coefficients through their modified surface impedances. In order to get a simple but useful result we assume that the attenuation functions may be separated out from this averaging process. On the other hand each of the three modified surface impedances (Δ_a, Δ_b, and Δ_c) may be averaged separately to obtain average modified surface impedances [Barrick (1971)]. It may be again mentioned that the expressions for all three surface impedances are the same except for the differences in the restrictions on their summation indices v and w. These restrictions are shown in (4.57) through the argument of F's.

After the averaging (with F's separated) for the autocorrelation function is performed, it is intended to extend the limits of both fundamental wavelength (L) and fundamental period (T) to infinity. As a result the summations may be reduced to integrals. The same extension also applies to the expressions for the modified surface impedances. This in turn allows us to include the countable restrictive points for the summation indices v,w into integrations by continuity. Therefore, in the sense of these integrals the restrictions on indices v and w may be removed. In this event all the three modified surface impedances are the same. Thus we have

$$\begin{aligned}\Delta_a(-k_o \cos\theta, -k_o \sin\theta) &= \Delta_b(-k_o \cos\theta, -k_o \sin\theta)\\ &= \Delta_c(-k_o \cos\theta, -k_o \sin\theta)\\ &= \Delta_m(-k_o \cos\theta, -k_o \sin\theta) \text{ (say)}. \qquad (4.64)\end{aligned}$$

Therefore, from (3.106) and (3.108), it follows:

$$\begin{aligned}F_a(\rho, \cos\theta, \sin\theta) &= F_b(\rho, \cos\theta, \sin\theta)\\ &= F_c(\rho, \cos\theta, \sin\theta)\\ &= F(\rho, \cos\theta, \sin\theta) \text{ (say)}. \qquad (4.65)\end{aligned}$$

Also, from (3.44), we have

$$\Delta_m(-k_o \cos\theta, -k_o \sin\theta) = \Delta_m(k_o \cos\theta, k_o \sin\theta), \qquad (4.66)$$

and hence,

$$F(\rho,\ \cos\theta,\ \sin\theta) = F(\rho,\ -\cos\theta,\ -\sin\theta)\ . \tag{4.67}$$

Now by using (3.42) or (3.62) the average modified surface impedance may be given as

$$\langle \Delta_m(-k_o \cos\theta,\ -k_o \sin\theta) \rangle$$

$$= \Delta + \sum_{v,w} \Big[\frac{k_o\ (v\cos\theta + w\sin\theta)^2 N^2}{D(v,w) + k_o\Delta} - \Delta$$

$$\cdot \Big\{ \frac{k_o\ (v\cos\theta + w\sin\theta) N\, D(v,w)}{D(v,w) + k_o\Delta}$$

$$+ \left(\sin 2\theta + \frac{vN}{k_o}\sin\theta + \frac{wN}{k_o}\cos\theta\right)\left(1 + \frac{k_o}{D(v,w) + k_o\Delta} + \Delta^2\right)$$

$$\cdot\ vwN^2 \Big\} \Big] \left\langle \frac{|{}_1P_{v,w}|^2}{A(v,w)} \right\rangle$$

$$= \Delta_{am}(-k_o \cos\theta,\ -k_o \sin\theta)\ \text{(say)}. \tag{4.68}$$

In the above equation we have approximated the surface Fourier coefficients ($P_{v,w}$'s) by their first order (${}_1P_{v,w}$'s). The two functions $D(v,w)$ and $A(v,w)$ are given by (3.43) with k replaced by k_o. If we assume small variations in the surface height such that

$$\exp\,[-jD(v,w) f(\vec{x})] = 1 - j\,D(v,w) f(\vec{x})\ , \tag{4.69}$$

then from (3.43),

$$A(v,w) = 1\ . \tag{4.70}$$

By letting $vN = K'_x$ and $wN = K'_y$ and then extending L to infinity (i.e., $N \to 0$) reduces the summations in (4.68) into integrals as

$$\Delta_{am}(-k_o \cos\theta,\ -k_o \sin\theta)$$

$$= \Delta + \frac{1}{4\pi^2} \int_{K'_x} \int_{K'_y} \frac{k_o\ (K'_x \cos\theta + k'_y \sin\theta)^2}{D'(K'_x, K'_y) + k_o\Delta} - \Delta$$

$$\cdot \{\frac{k_o (K_x' \cos\theta + K_y' \sin\theta) D'(K_x', K_y')}{D'(K_x',K_y') + k_o\Delta}$$

$$+ (\sin 2\theta + \frac{K_x'}{k_o} \sin\theta + \frac{K_y'}{k_o}\cos\theta)$$

$$\cdot (1 + \frac{k_o\Delta}{D'(K_x',K_y') + k_o\Delta} + \Delta^2) K_x' K_y'\}] \; S_1(K_x',K_y')\, dK_x' dK_y' \qquad (4.71)$$

where

$$D'(K_x',K_y') = \begin{cases} [k_o^2 - (K_x' + k_o \cos\theta)^2 - (K_y' + k_o \sin\theta)^2]^{\frac{1}{2}}, & \text{for real root} \\ -j[(K_x' + k_o \cos\theta)^2 + (K_y' + k_o \sin\theta)^2 - k_o^2]^{\frac{1}{2}}, & \text{for imaginary root} \end{cases} \qquad (4.72)$$

and

$$S_1(K_x',K_y') = S_1^+(K_x',K_y') + S_1^-(K_x',K_y') \qquad (4.73)$$

In deriving (4.71) we have used (4.28) and (4.70). Further, from (3.39), (3.108), and (4.65), the attenuation function F may be given as

$$F(\rho, \cos\theta, \sin\theta) = F(p_{am})$$

$$= 1 - j(\pi p_{am})^{\frac{1}{2}} \exp(-p_{am})\, \mathrm{erfc}(j\sqrt{p_{am}}) \qquad (4.74)$$

with p_{am} as the average modified numerical distance,

$$p_{am} = -jk_o \frac{\rho}{2} [\Delta_{am}(-k_o \cos\theta, -k_o \sin\theta)]^2 . \qquad (4.75)$$

In view of the above discussion the solution (4.57) for the back-scattered field may be rewritten as

$$\tilde{E}_{zb}(t_o,t) = \tilde{E}_{zb1}(t_o,t) + \tilde{E}_{zb21}(t_o,t)$$

$$+ \tilde{E}_{zb22}(t_o,t) + \tilde{E}_{zb23}(t_o,t) . \qquad (4.76)$$

$\tilde{E}_{zb1}$ is the first order field and it corresponds to the first part of (4.57). In a similar way, the three parts of the second order field in (4.57) are represented by $\tilde{E}_{zb21}$, $\tilde{E}_{zb22}$, and $\tilde{E}_{zb23}$ respectively. They may be written as follows:

$$\tilde{E}_{zb1}(t_o,t) = -j(\eta P_t g_t)^{\frac{1}{2}} \frac{2\Delta_\rho}{\pi \rho_o^{3/2}} F^2(\rho_o,1,0)$$

$$\cdot \sum_{\substack{m,n,1 \\ -|m|\alpha \le n \le |m|\alpha}} \Big\{ {}_1P_{m,n,1} (K_m)^{\frac{1}{2}} [K_m - k_o \operatorname{sgn}(m)]$$

$$\cdot \operatorname{Sa}[\Delta_\rho\{\operatorname{sgn}(m)K_m - 2k_o\}]$$

$$\cdot \exp[j \operatorname{sgn}(m)(K_m \rho_o - \tfrac{\pi}{4}) + j1Wt] \Big\} \qquad (4.77)$$

$$\tilde{E}_{zb21}(t_o,t) = -j(\eta P_t g_t)^{\frac{1}{2}} \frac{2\Delta_\rho}{\pi \rho_o^{3/2}} F^2(\rho_o,1,0)$$

$$\cdot \sum_{p,q,i} \sum_{\substack{m,n,1 \\ (m,n)\ne(-p,-q) \\ -\alpha|p+m| \le q+n \le \alpha|p+m|}} \Big\{ {}_1P_{p,q,i}\ {}_1P_{m,n,1}$$

$$\cdot \operatorname{Sa}[\Delta_\rho\{\operatorname{sgn}(p+m)K_{p+m} - 2k_o\}]$$

$$\cdot \{(K_{p+m})^{\frac{1}{2}} [K_{p+m} - k_o \operatorname{sgn}(p+m)] H(m,n,1,p,q,i)$$

$$+ Q_1(m,n,p,q)[K_m^2 K_{p+m} - 2k_o \operatorname{sgn}(p+m) \vec{K}_m \cdot \vec{K}_{p+m}]^{-\frac{1}{2}}\}$$

$$\cdot \exp[j \operatorname{sgn}(p+m)(K_{p+m}\rho_o - \tfrac{\pi}{4}) + j(1+i)Wt] \Big\} \qquad (4.78)$$

$$\tilde{E}_{zb22}(t_o,t) = -j(\eta P_t g_t)^{1/2} \frac{2\Delta_\rho}{\pi \rho_o^{3/2}} F^2(\rho_o,1,0)$$

$$\cdot \sum_{\substack{p,q,i \\ -\alpha|p| \leq q \leq \alpha|p|}} \sum_{\substack{m,n,l \\ (m,n)\neq(-p,-q)}} \Big\{ {}_1P_{p,q,i}\ {}_1P_{m,n,l}\ Sa[\Delta_\rho\{sgn(p)K_p - 2k_o\}]$$

$$\cdot Q_2(m,n,p,q)[K_m^2 K_p + 2k_o\, sgn(p)\, \vec{K}_m \cdot \vec{K}_p]^{-1/2}$$

$$\cdot \exp\,[j\ sgn(p)(K_p\,\rho_o - \tfrac{\pi}{4}) + j(1+i)Wt]\Big\} \qquad (4.79)$$

$$\tilde{E}_{zb23}(t_o,t) = -(8\eta P_t g_t)^{1/2} \frac{2\Delta_\rho}{\pi \rho_o^{3/2}}$$

$$\cdot \sum_{\substack{p,q,i \\ 2k_o|m|N<K_m^2<2k_oK_m \\ -\alpha|2p+m| \leq 2q+n \leq \alpha|2p+m|}} \sum_{\substack{m,n,l \\ (m,n)\neq(-p,-q)}} \Big\{ {}_1P_{p,q,i}\ {}_1P_{m,n,l}\ F(\beta_o\rho_o,1,0)$$

$$\cdot F(r_b,\ \cos\psi_b,\ \sin\psi_b)\ F(r_c,\ -\cos\psi_c,\ -\sin\psi_c)$$

$$\cdot Sa\Big[\frac{\beta_o\Delta_\rho}{2}\ \{sgn(2p+m)K_{2p+m} - 2k_o - \frac{|n|N}{K_m}K_{mo}\}\Big]$$

$$\cdot K_m\, Q_3(m,n,p,q)[\beta_o\ K_{2p+m}\ \{K_m^4 - (2k_o mN)^2\}]^{-1/2}$$

$$\cdot \exp\,\Big[j\ sgn(2p+m)(K_{2p+m}\frac{\beta_o\rho_o}{2} - \frac{\pi}{4}) + j2\rho_o k_o\Big]$$

$$\cdot \exp\,\Big[-j\rho_o\beta_o(k_o + \frac{|n|N}{2K_m}K_{mo}) + j(1+i)Wt\Big]\Big\} \qquad (4.80)$$

In writing (4.77) to (4.80) from (4.57) we have used (4.63) and (4.65). Also, the attenuation functions (F's) in (4.77), (4.78), and (4.79) have been taken out from summations as they are no more functions of the summation indices. Whereas, in (4.80), they remain inside as they are still functions of the summation indices m and n through β_o, r_b, r_c, ψ_b, and ψ_c.

An autocorrelation function with respect to t for the backscattered field may be defined as [Papoulis (1965, ch. 10)]

$$R_r(\tau) = \frac{A_r}{2\eta} < \tilde{E}_{zb}(t_o, t + \tau) \tilde{E}^*_{zb}(t_o, t) > \qquad (4.81)$$

where η is the free space intrinsic impedance and A_r is the effective aperture of the receiving antenna. The effective aperture is given by [Jordan and Balmain (1968, ch. 11)]

$$A_r = \frac{\lambda_o^2}{4\pi} g_r \qquad (4.82)$$

where λ_o is the radar wavelength and g_r is the free space gain of the receiving antenna in the patch direction. The equation (4.81) is normalized by the factor $(\frac{A_r}{2\eta})$ so that $R_r(0)$ gives the average backscattered power (P_r^o) received from the surface area corresponding to t_o. That is,

$$P_r^o = R_r(0) = \frac{A_r}{2\eta} < |\tilde{E}_{zb}(t_o, t)|^2 >. \qquad (4.83)$$

In (4.81) or (4.83) there are two types of averaging involved. The first type is the time average of the received power during the pulse length (τ_o) in the sense of phasors. This average is accounted for in the two equations by the factor one-half. The second type is the ensemble (statistical) average of the time averaged power. The ensemble is a collection of samples of the backscattered signal obtained from different oceans for fixed t and t_o. For finding the autocorrelation function, the samples are required at both t and $(t + \tau)$. This average is denoted by the angular brackets and it implies the assumption of homogeneity and stationarity for the ocean surface. Because of this assumption and the other common assumption made later, it turns out from the solution (4.76) that the field $\tilde{E}_{zb}(t_o,t)$ is also statistically stationary. To put it another way the function $R_r(\tau)$ is independent of t as shown in appendix D. That is why the definition (4.81) for a stationary process has been taken here. However, both P_r and $R_r(\tau)$ are dependent on t_o in the sense that they are derived from the field which is received from the surface area corresponding to t_o. The other assumption for the ocean surface is that the probability distribution of the first order surface displacement is Gaussian with zero mean [Phillips (1977, ch. 4)].

By Fourier transforming the autocorrelation function with respect to τ we obtain

$$P_r(\omega_d) = \int_{\tau} R_r(\tau)\ \exp(-j\omega_d\tau)d\tau \qquad (4.84)$$

such that

$$P_r^o = (2\pi)^{-1} \int_{\omega_d} P_r\ (\omega_d)d\omega_d \qquad (4.85)$$

where $P_r(\omega_d)$ is the average power density spectrum of the backscattered signal with ω_d as the Doppler frequency. This may also be called the backscattered Doppler spectrum.

By using (4.76) in (4.81) yields $R_r(\tau)$ as

$$R_r(\tau) = R_{ff}(\tau) + R_{fs}(\tau) + R_{sf}(\tau)$$
$$+ R_{s11}(\tau) + R_{s22}(\tau) + R_{s33}(\tau)$$
$$+ R_{s12}(\tau) + R_{s21}(\tau) + R_{s13}(\tau)$$
$$+ R_{s31}(\tau) + R_{s23}(\tau) + R_{s32}(\tau) \qquad (4.86)$$

Because of the stationarity, the following relations hold for the cross-correlation terms appearing on the right hand side of the above equation [Papoulis (1965, ch. 10)].

$$R_{sf}(\tau) = R^*_{fs}(-\tau) \qquad R_{s31}(\tau) = R^*_{s13}(-\tau)$$
$$R_{s21}(\tau) = R^*_{s12}(-\tau) \qquad R_{s32}(\tau) = R^*_{s23}(-\tau) \qquad (4.87)$$

Therefore we have

$$R_r(\tau) = R_{ff}(\tau) + R_{fs}(\tau) + R^*_{fs}(-\tau)$$
$$+ R_{s11}(\tau) + R_{s22}(\tau) + R_{s33}(\tau)$$
$$+ R_{s12}(\tau) + R^*_{s12}(-\tau) + R_{s13}(\tau)$$
$$+ R^*_{s13}(-\tau) + R_{s23}(\tau) + R^*_{s23}(-\tau) \qquad (4.88)$$

where the various symbols have the following meaning.

$R_{ff}(\tau)$: autocorrelation of the first order field

$$= \frac{A_r}{2\eta} < \bar{E}_{zb1}(t_o, t + \tau) \bar{E}^*_{zb1}(t_o, t) > \qquad (4.89a)$$

$R_{fs}(\tau)$: cross correlation of the first order with the three parts of the second order field

$$= \frac{A_r}{2\eta} < \bar{E}_{zb1}(t_o, t + \tau)[\bar{E}^*_{zb21}(t_o, t)$$
$$+ \bar{E}^*_{zb22}(t_o, t) + \bar{E}^*_{zb23}(t_o, t)] > \qquad (4.89b)$$

$R_{s11}(\tau)$: autocorrelation of the first part of the second order field

$$= \frac{A_r}{2\eta} < \tilde{E}_{zb21}(t_o, t+\tau)\, \tilde{E}^*_{zb21}(t_o, t) > \quad (4.89c)$$

$R_{s22}(\tau)$: autocorrelation of the second part of the second order field

$$= \frac{A_r}{2\eta} < \tilde{E}_{zb22}(t_o, t+\tau)\, \tilde{E}^*_{zb22}(t_o, t) > \quad (4.89d)$$

$R_{s33}(\tau)$: autocorrelation of the third part of the second order field

$$= \frac{A_r}{2\eta} < \tilde{E}_{zb23}(t_o, t+\tau)\, \tilde{E}^*_{zb23}(t_o, t) > \quad (4.89e)$$

$R_{s12}(\tau)$: cross correlation of the first part with the second part of the second order field

$$= \frac{A_r}{2\eta} < \tilde{E}_{zb21}(t_o, t+\tau)\, \tilde{E}^*_{zb22}(t_o, t) > \quad (4.89f)$$

$R_{s13}(\tau)$: cross correlation of the first part with the third part of the second order field

$$= \frac{A_r}{2\eta} < \tilde{E}_{zb21}(t_o, t+\tau)\, \tilde{E}^*_{zb23}(t_o, t) > \quad (4.89g)$$

$R_{s23}(\tau)$: cross correlation of the second part with the third part of the second order field

$$= \frac{A_r}{2\eta} < \tilde{E}_{zb22}(t_o, t+\tau)\, \tilde{E}^*_{zb23}(t_o, t) > \quad (4.89h)$$

We may now introduce the symbols for the Doppler spectra corresponding to each of the auto and cross correlation functions mentioned above. By using (4.84) in (4.88) it translates to these spectra as

$$P_r(\omega_d) = P_{ff}(\omega_d) + 2\,\mathrm{Re}\,\{P_{fs}(\omega_d)\} + P_{s11}(\omega_d) + P_{s22}(\omega_d) + P_{s33}(\omega_d) + 2\,\mathrm{Re}\,\{P_{s12}(\omega_d)\} + 2\,\mathrm{Re}\,\{P_{s13}(\omega_d)\} + 2\,\mathrm{Re}\,\{P_{s23}(\omega_d)\} \quad (4.90)$$

where Re {·} means the real part. It may easily be verified that $P_r(\omega_d)$ is real and so is the right hand side of (4.90).

At this point we find it convenient to write the radar range equation for the average backscattered power received from a patch of the ocean surface [Barrick (1972a,b)].

$$P^o_{rp} = \frac{P_t g_t g_r \lambda_o^2}{(4\pi)^3 \rho_o^4} |F_p|^4 A_p \sigma_p^o, \tag{4.91}$$

or in terms of the backscattered Doppler spectrum as

$$P_{rp}(\omega_d) = \frac{P_t g_t g_r \lambda_o^2}{(4\pi)^3 \rho_o^4} |F_p|^4 A_p \sigma_p(\omega_d) \tag{4.92}$$

with

$$P^o_{rp} = (2\pi)^{-1} \int_{\omega_d} P_{rp}(\omega_d) d\omega_d . \tag{4.93}$$

In the above equations we have

P^o_{rp} : average backscattered power received from the patch (watts)

$P_{rp}(\omega_d)$: average backscattered Doppler spectrum received from the patch (watts/rad/sec)

P_t : average transmitted power (watts)

g_t, g_r : free space gains of transmitting and receiving antennas in the direction of the patch (dimensionless)

λ_o : radar wavelength (meter)

ρ_o : distance of the patch from the radar (meter)

F_p : one way attenuation function between the radar and the patch (dimensionless)

A_p : area of the patch (meter^2)

σ_p^o : average backscattered cross section of the patch normalized to the patch area (dimensionless)

$\sigma_p(\omega_d)$: average backscattered Doppler frequency dependent cross section of the patch normalized to the patch area (/rad/sec).

In view of (4.93) we may write

$$\sigma_p^o = (2\pi)^{-1} \int_{\omega_d} \sigma_p(\omega_d) d\omega_d . \tag{4.94}$$

We have described earlier that in the case of a narrow beam receiving antenna the first order of the backscattered field is received from the patch. Whereas in the second order, which consists of three parts, only the first two parts are received from the patch. In the first part the two scatterings occur on the patch. A second order hydrodynamic term has been included in this part as explained before. In the second part the first scattering occurs at the source point and the second scattering occurs on the patch. The third part arrises because of the two scatterings occuring elsewhere on the surface but within the surface area corresponding to t_o. For the first order and each of the three parts of the second order fields average auto and cross correlation functions, defined by (4.89a) to (4.89h), may be derived and hence the corresponding Doppler spectra may be obtained. The sum of these spectra thus gives an average backscattered doppler spectrum $[P_r(\omega_d)]$ up to the second order as represented by (4.90).

Although these auto and cross Doppler spectra are not all received from the patch, we may, for determining the respective cross sections, treat them as if they were. This treatment may be achieved by first deriving each auto and cross Doppler spectrum and then reducing it in the form of (4.92). Thus by comparing each of these spectra with (4.92) the respective cross sections may be derived. Therefore, in this sense we may say $P_r^o = P_{rp}^o$ or $P_r(\omega_d) = P_{rp}(\omega_d)$. This in turn implies from (4.90) that

$$\sigma_p(\omega_d) = \sigma_{ff}(\omega_d) + 2\,\mathrm{Re}\,\{\sigma_{fs}(\omega_d) + \sigma_{s11}(\omega_d)$$
$$+ \sigma_{s22}(\omega_d) + \sigma_{s33}(\omega_d) + 2\,\mathrm{Re}\,\{\sigma_{s12}(\omega_d)\}$$
$$+ 2\,\mathrm{Re}\,\{\sigma_{s13}(\omega_d)\} + 2\,\mathrm{Re}\,\{\sigma_{s23}(\omega_d)\} \,. \qquad (4.95)$$

The meaning of each cross section appearing on the right hand side of the above equation may easily be derived from the definition of the respective correlation functions given by (4.89).

In order to demonstrate the method used in deriving these cross sections, the first order case will be dealt with here. The other cross sections are derived in appendix D and only the results will be presented here.

4.4.1.1 First Order Cross Section

By using (4.77) in (4.89a), the autocorrelation function for the first order backscattered field may be given as

$$R_{ff}(\tau) = \frac{2P_t g_t \Delta_\rho^2}{\pi^2 \rho_o^3} A_r \, |F(\rho_o,1,0)|^4$$

$$\sum_{\substack{m,n,1 \\ -|m|\alpha \le n \le |m|\alpha}} \; \sum_{\substack{m',n',1' \\ -|m'|\alpha \le n' \le |m'|\alpha}} \left\{ < {}_1P^*_{m,n,1} \; {}_1P_{m',n',1'} > \right.$$

$$\cdot\ (K_m K_{m'})^{\frac{1}{2}}\,[K_m - k_o\,\mathrm{sgn}(m)][K_{m'} - k_o\,\mathrm{sgn}(m')]$$
$$\cdot\ Sa\,[\Delta_\rho\{\mathrm{sgn}(m)\,K_m - 2k_o\}]\ Sa\,[\Delta_\rho\{\mathrm{sgn}(m')\,K_{m'} - 2k_o\}]$$
$$\cdot\ \exp\,[-j\,\mathrm{sgn}(m)(K_m\,\rho_o - \tfrac{\pi}{4}) - jlWt]$$
$$\left.\cdot\ \exp\,[j\,\mathrm{sgn}(m')(K_{m'}\,\rho_o - \tfrac{\pi}{4}) + jl'W(t+\tau)]\right\} \qquad (4.96)$$

where $K_{m'}$ is given by K_m in (3.142) with m and n replaced by m' and n' respectively. The equation (4.26) may be used now for the ensemble average of the Fourier coefficients of the first order gravity waves to give

$$R_{ff}(\tau) = \frac{P_t g_t g_r \lambda_o^2 \Delta_\rho^2}{16\,\pi^6\,\rho_o^3}\,|F(\rho_o,1,0)|^4$$
$$\cdot \sum_{\substack{m,n,l \\ -|m|\alpha\le n\le |m|\alpha}} \left\{ N^2 W S_1\,(\vec{K}_m,\,lW) K_m\,[K_m - k_o\,\mathrm{sgn}(m)]^2 \right.$$
$$\left.\cdot\ Sa^2\,[\Delta_\rho\{\mathrm{sgn}(m)\,K_m - 2k_o\}]\,\exp(jlW\tau)\right\} \qquad (4.97)$$

where we have used (4.82) for the effective aperture (A_r) of the receiving antenna.

By substituting,

$mN = K_x$, $nN = K_y$, $lW = \omega$,

and then extending the limits of both fundamental wavelength (L) and temporal period (T) to infinity (i.e., N and $W \to 0$), the above summations may be reduced to integrals as

$$R_{ff}(\tau) = \frac{P_t g_t g_r \lambda_o^2 \Delta_\rho^2}{16\,\pi^6 \rho_o^3} |F(\rho_o,1,0)|^4$$

$$\cdot \int_{K_x} \int_{K_y = -\alpha|K_x|}^{\alpha|K_x|} \int_{\omega} S_1(\vec{K}, \omega) K\, [K - k_o \,\mathrm{sgn}(K_x)]^2$$

$$\cdot\, Sa^2\, [\Delta_\rho \{\mathrm{sgn}(K_x)\, K - 2k_o\}]\, \exp(j\omega\tau)\, dK_x dK_y d\omega \tag{4.98}$$

where

$$\vec{K} = K_x \hat{x} + K_y \hat{y}\ ,\ K = |\vec{K}|. \tag{4.99}$$

The Fourier transform with respect to τ yields the power density spectrum or the Doppler spectrum of the first order backscattered signal. Also, the area of the scattering patch may be given as

$$A_p = 4\,\rho_o\,\Delta_\rho\,\Delta_\theta\ . \tag{4.100}$$

Therefore we get

$$P_{ff}(\omega_d) = \frac{P_t g_t g_r \lambda_o^2}{32\,\pi^5\,\rho_o^4} |F(\rho_o,1,0)|^4 A_p \frac{\Delta_\rho}{\Delta_\theta}$$

$$\cdot \int_{K_x} \int_{K_y = -\alpha|K_x|}^{\alpha|K_x|} S_1(\vec{K}, \omega_d)\, K\, [K - k_o\, \mathrm{sgn}(K_x)]^2$$

$$\cdot\, Sa^2\, [\Delta_\rho \{\mathrm{sgn}(K_x)\, K - 2k_o\}]\, dK_x dK_y \tag{4.101}$$

With the help of (4.30) the three dimensional wave height spectrum of the gravity waves in the above integrand may be reduced to two dimensional spectra S_1^+ and S_1^-. The above equation may now be compared with the radar range equation (4.92). Since the attenuation function F in (4.101) is the same as F_p in (4.92), we obtain the following solution for an average first order Doppler frequency dependent backscattered cross section.

$$\sigma_{ff}(\omega_d) = \frac{4\Delta_\rho}{\pi\Delta_\theta} \int_{K_x} \int_{K_y = -\alpha|K_x|}^{\alpha|K_x|} K\,[K - k_o\,\mathrm{sgn}(K_x)]^2$$

$$\cdot\, Sa^2\,[\Delta_\rho\{\mathrm{sgn}(K_x)\,K - 2k_o\}]$$

$$\cdot\, \{S_1^+(\vec{K})\,\delta[\omega_d + \mathrm{sgn}(K_x)(gK)^{\frac{1}{2}}]$$

$$+ S_1^-(\vec{K})\,\delta[\omega_d - \mathrm{sgn}(K_x)(gK)^{\frac{1}{2}}]\}\,dK_x dK_y \qquad (4.102)$$

The above cross section is normalized to the patch area and is in form of a double integral. However, with a few minor approximations, the double integral may be evaluated and the solution simplified. The absolute maximum of the sampling squared function occurs in the region of positive K_x when $\mathrm{sgn}(K_x)K = 2k_o$. Also, this function is fastly decaying as Δ_ρ is usually large. For a 8 μsec pulse Δ_ρ is equal to 600 meter. Therefore, the contribution to the integral from the negative K_x region may be neglected. By doing so and by changing (K_x, K_y) to the polar form $(K, \emptyset)$ the above solution reduces to:

$$\delta_{ff}(\omega_d) \simeq \frac{4\Delta_\rho}{\pi\Delta_\theta} \int_{\emptyset = -\Delta_\theta}^{\Delta_\theta} \int_{K=0}^{\infty} K^2 (K - k_o)^2 Sa^2\,[\Delta_\rho(K - 2k_o)]$$

$$\cdot\, \{S_1^+(K\cos\emptyset,\, K\sin\emptyset)\,\delta[\omega_d + (gK)^{\frac{1}{2}}]$$

$$+ S_1^-(K\cos\emptyset,\, K\sin\emptyset)\,\delta[\omega_d - (gK)^{\frac{1}{2}}]\}\,dK d\emptyset\,. \qquad (4.103)$$

The integral with respect to K may be easily evaluated because of the delta function. The remaining integral with respect to $\emptyset$ may be approximated by assuming the integrand constant over the narrow range of ϕ. That is, the spatial spectra S_1^+ and S_1^- may be approximated at $\phi = 0$ in the range $-\Delta_\theta \leq \phi \leq \Delta_\theta$ to yield

$$\sigma_{ff}(\omega_d) \approx \frac{16}{\pi} \frac{|\omega_d|^5}{g^3} \left(\frac{\omega_d^2}{g} - k_o\right)^2 \cdot \Delta_\rho \, Sa^2 \left[\Delta_\rho\left(\frac{\omega_d^2}{g} - 2k_o\right)\right] \cdot \left[S_1^+\left(\frac{\omega_d^2}{g}, 0\right)_{\omega_d \leq 0} + S_1^-\left(\frac{\omega_d^2}{g}, 0\right)_{\omega_d \geq 0}\right] \quad (4.104)$$

The above algebraic equation relates the first order Doppler frequency dependent cross section with the spatial spectrum of the first order gravity waves. The positive Doppler frequencies are caused by those gravity waves which are moving toward the radar, i.e., in the negative x direction. In a similar way, the negative Doppler frequencies are caused by those waves which are moving away from the radar. The cross section peaks at two Doppler frequencies where the sampling squared function is maximum. These two frequencies are $\omega_d = \pm(2gk_o)^{\frac{1}{2}}$. From the dispersion relation (4.18), it may be shown that the two peaks are caused by those two ocean waves whose wavelengths are equal to one-half the radar wavelength and one is moving toward the radar while another is moving away from it. This scattering phenomenon of the first order peaks is commonly referred to as the "Bragg scattering" [Barrick (1972a,b)] because of the analogy with scattering from gratings in optics, and it confirms the experimental observation made first by Crombie (1955). Around these two peaks the cross section is a continuum but it is rapidly decaying because of the sampling squared function.

The above cross section in the present form does not agree with that derived by Barrick (1972a,b) using the Rice (1951) perturbation technique. However, when the limit of the sampling squared function is taken to be the Dirac delta function by assuming large Δ_ρ [Lathi (1968, ch. 1)], the result does go to that derived by Barrick. This may be shown as follows:

$$\lim_{\Delta_\rho \to \infty} \Delta_\rho \, Sa^2 \left[\Delta_\rho \left(\frac{\omega_d^2}{g} - 2k_o\right)\right] = \pi \, \delta \left(\frac{\omega_d^2}{g} - 2k_o\right) \tag{4.105}$$

Also, it may be shown that

$$\delta\left(\frac{\omega_d^2}{g} - 2k_o\right) = g \, \delta(\omega_d^2 - \omega_B^2)$$

$$= \frac{g}{2\omega_B} \left[\delta(\omega_d + \omega_B) + \delta(\omega_d - \omega_B)\right] \tag{4.106}$$

where

$$\omega_B : \text{Bragg frequency} = (2gk_o)^{1/2} \tag{4.107}$$

Therefore, (4.104) reduces to:

$$\sigma_{ff}(\omega_d) = 32k_o^4 \left[S_1^+(2k_o, 0) \, \delta(\omega_d + \omega_B) + S_1^-(2k_o, 0) \, \delta(\omega_d - \omega_B)\right] . \tag{4.108}$$

The result (4.108), after proper normalizations, is the same as given by Barrick (1972a,b) for vertical polarization at grazing. For comparison, equation (4.108) has to be normalized by constants since our definitions for spatial spectra (S_1^+ and S_1^-) for the ocean surface and the cross section (σ_p^o) differ in constants with those used by Barrick. Again by using the Rice perturbation method, Johnstone (1975) also has derived the expressions for first and second order cross sections. Although, the form for the spatial spectrum taken by Johnstone is somewhat different than used here or by Barrick, the result (4.108) agrees with that given by him when interpreted in his notation. A comparison of (4.108) with (4.104) shows that the solution (4.108) gives two spikes situated at plus and minus the Bragg frequency instead of the continuum given by (4.104).

4.4.1.2 Second Order Cross Section

The different parts of an average second order Doppler frequency dependent backscattered cross section have been derived in appendix D. The results are as follows.

1. Based on the cross correlation of the first order field with all three parts of the second order field [from (D.4)],

$$\sigma_{fs}(\omega_d) = 0 \tag{4.109}$$

2. Based on the autocorrelation of the first part of the second order field [from (D.19)],

$$\sigma_{s11}(\omega_d) \cong \frac{16\, k_o^4}{\pi^2} \sum_{+,-} \int_{\alpha_1} \int_{\beta_1} |C_e + C_h|^2\, S_1^{\pm}(\vec{K}_1) S_1^{\pm}(\vec{K}_2) \cdot \delta\{\omega_d \mp \mathrm{sgn}(K_{1x})(gK_1)^{\frac{1}{2}} \mp \mathrm{sgn}(K_{2x})(gK_2)^{\frac{1}{2}}\}\, d\alpha_1 d\beta_1 \tag{4.110}$$

where

$\sum_{+,-}$ = sum over superscripts (+) and (−), i.e., sum of the four combinations (+,+), (−,−), (+,−), and (−,+) of S_1's with corresponding arguments of delta function

$$\vec{K}_1 = K_{1x}\hat{x} + K_{1y}\hat{y} = (\alpha_1 - k_o)\hat{x} + \beta_1\hat{y}$$

$$\vec{K}_2 = K_{2x}\hat{x} + K_{2y}\hat{y} = -(\alpha_1 + k_o)\hat{x} - \beta_1\hat{y}$$

$$K_1 = |\vec{K}_1|, \quad K_2 = |\vec{K}_2| \tag{4.111}$$

$$C_e = \frac{1}{2}\frac{(K_{1x}K_{2x} - 2\vec{K}_1 \cdot \vec{K}_2)}{(\vec{K}_1 \cdot \vec{K}_2)^{\frac{1}{2}}}$$

$$C_h = -\frac{j}{2}\,[K_1 + K_2 + (K_1K_2 - \vec{K}_1 \cdot \vec{K}_2)\,\frac{(\omega_B^2 + \omega_d^2)}{(\omega_B^2 - \omega_d^2)}\;\frac{\mu}{\mathrm{sgn}(K_{1x}K_{2x})(K_1K_2)^{\frac{1}{2}}}]$$

$$\mu = \begin{cases} 1, & \text{for } (+,+) \text{ or } (-,-) \text{ combination} \\ -1, & \text{for } (+,-) \text{ or } (-,+) \text{ combination} \end{cases}$$

ω_B : Bragg frequency = $(2gk_o)^{\frac{1}{2}}$, $\delta\{x\}$: Dirac delta function.

The part containing the function C_e in the result (4.110) is valid provided C_e is not singular. This singularity corresponds to the denominator being zero in (3.134) (see section 3.5.1.1).

3. Based on the autocorrelation of the second part of the second order field [from (D.31)],

$$\sigma_{s22}(\omega_d) = \frac{144k_o^7\,\Delta_\theta}{\pi\Delta_\rho}\,[\{S_1^{+}(\vec{K}_1)\}^2\,\delta\{\omega_d + 2\omega_B\}$$

$$+ \{S_1^{-}(\vec{K}_1)\}^2\,\delta\{\omega_d - 2\omega_B\} + 2S_1^{+}(\vec{K}_1)S_1^{-}(\vec{K}_1)\delta\{\omega_d\}]$$

$$+ \frac{2k_o^4}{\pi^2}\sum_{+,-}\int_{\alpha_2}\int_{\beta_2}\frac{\{(\vec{K}_1 + 2\vec{K}_2)\cdot\vec{K}_2\}^2}{|(\vec{K}_1 + \vec{K}_2)\cdot\vec{K}_2|}\cdot S_1^{\pm}(\vec{K}_1)S_1^{\pm}(\vec{K}_2)$$

$$\cdot\,\delta\{\omega_d \pm (gK_1)^{\frac{1}{2}} \pm \mathrm{sgn}(K_{2x})(gK_2)^{\frac{1}{2}}\}\,d\alpha_2 d\beta_2 \qquad (4.112)$$

where

$$\vec{K}_1 = K_{1x}\hat{x} + K_{1y}\hat{y} = 2k_o\hat{x}\;,\quad \vec{K}_2 = K_{2x}\hat{x} + K_{2y}\hat{y} = \alpha_2\hat{x} + \beta_2\hat{y} \qquad (4.113)$$

$$K_1 = |\vec{K}_1| = 2k_o\;,\quad K_2 = |\vec{K}_2|\;.$$

Again, the part containing the double integral in the result (4.112) is valid provided the integrand is not singular. This singularity corresponds to the denominator being zero in (3.135) (see section 3.5.1.1).

4. Based on the autocorrelation of the third part of the second order field [from (D.46)],

$$\sigma_{s33}(\omega_d) \cong \frac{32}{\pi^2} |F_p|^{-4} \sum_{+,-} \int_{\alpha_2} \int_{\beta_2} |F(\xi_o \rho_o, 1, 0)|^2$$

$$2k_o |K_{2x}| < K_2^2 < 2k_o K_2$$

$$\cdot |F(R_b, \cos \phi_b, \sin \phi_b) F(R_c, -\cos \phi_c, -\sin \phi_c)|^2$$

$$\cdot [(k_o - K_{1x} - K_{2x}) \{k_o K_{1x} (2K_{1x} + K_{2x})$$

$$- K_1^2 (K_{1x} + K_{2x})\} + K_1^2 K_{1y}^2]^2$$

$$\cdot \frac{K_2^2}{\xi_o^2 [K_2^4 - (2k_o K_{2x})^2]} S_1^{\pm}(\vec{K}_1) S_1^{\pm}(\vec{K}_2)$$

$$\cdot \delta \{\omega_d \pm \mathrm{sgn}(K_{1x})(gK_1)^{\frac{1}{2}} \pm \mathrm{sgn}(K_{2x})(gK_2)^{\frac{1}{2}}\} \, d\alpha_2 d\beta_2 \qquad (4.114)$$

where

$$\vec{K}_2 = K_{2x}\hat{x} + K_{2y}\hat{y} = \alpha_2 \hat{x} + \beta_2 \hat{y}$$

$$\vec{K}_1 = K_{1x}\hat{x} + K_{1y}\hat{y}$$

$$K_{1x} = k_o + \frac{|K_{2y}|}{2K_2} K_{2s} - \frac{1}{2} K_{2x} \qquad (4.115)$$

$$K_{1y} = -\frac{1}{2} K_{2y}$$

$$K_1 = |\vec{K}_1| \quad , \quad K_2 = |\vec{K}_2|$$

and ξ_o, R_b, R_c, ϕ_b, ϕ_c, and K_{2s} are given by (D.41). These are functions of $\vec{K}_2$ only.

5. Based on the cross correlation of the first part with the second part of the second order field [from (D.57)],

$$\sigma_{s12}(\omega_d) \cong 0$$

6. Based on the cross correlation of the first part with the third part of the second order field [from (D.65)],

$\sigma_{s13}(\omega_d) = 0$ (4.117)

7. Based on the cross correlation of the second part with the third part of the second order field [from (D.75)],

$\sigma_{s23}(\omega_d) = 0$ (4.118)

The equation (4.109) means that the first and second orders of the backscattered field are uncorrelated. In a similar way, (4.116) to (4.118) state that three parts of the second order field are approximately uncorrelated to each other. Therefore, from (4.95), the cross section of the ocean surface patch up to the second order reduces to:

$$\sigma_p(\omega_d) \simeq \sigma_{ff}(\omega_d) + \sigma_{s11}(\omega_d) + \sigma_{s22}(\omega_d) + \sigma_{s33}(\omega_d) \qquad (4.119)$$

where σ_{ff} is the first order cross section and σ_{s11}, σ_{s22}, and σ_{s33} are three parts of the second order cross section. The backscattered power density spectrum may then be given from (4.90) and (4.92) as

$$P_r(\omega_d) = P_{rp}(\bar{\omega}_d) \simeq [P_{ff}(\omega_d) + P_{s11}(\omega_d) + P_{s22}(\omega_d) + P_{s33}(\omega_d)]. \qquad (4.120)$$

It may be mentioned that truly speaking equation (4.119) does not represent the actual cross section of the patch. The three parts of the second order cross section of the patch have been derived as if the respective three parts of the second order field are received from the patch only. The actual cross section of the patch may be given by the first two terms in (4.119), i.e., σ_{ff} and σ_{s11}, which represents the case where first and second order scatterings occur only on the patch. But these two terms can not be separated from the other two experimentally in the assumed antenna's configuration namely omnidirectional transmission and narrow beam reception. For this configuration the scattering area is not just the patch only. However, the cross section of this scattering area is equivalent to that of the patch, given by (4.119), in the sense $P_r(\omega_d) = P_{rp}(\omega_d)$.

By using the Rice (1951) perturbation method Barrick (1972b, 1977a) has derived an expression for the second order backscattered cross section of the ocean surface patch for vertical polarization at grazing. His expression is equivalent to σ_{s11} and it does not contain the additional second order terms namely σ_{s22} and σ_{s33} obtained above. A comparison of the expression (4.110) for σ_{s11} with that derived by Barrick shows that both expressions are the same except for two differences when his result is interpreted in our notation. The first difference is that the denominator for C_e in (4.111) is $[(\vec{K}_1 \cdot \vec{K}_2)^{\frac{1}{2}} + k_o\Delta^*]$ in his expression instead of $(\vec{K}_1 \cdot \vec{K}_2)^{\frac{1}{2}}$, where Δ^* is the complex conjugate of the normalized surface impedance. Since $|k_o\Delta|$ is very small for the ocean at HF, the effect of $(k_o\Delta^*)$ is negligible except when $(\vec{K}_1 \cdot \vec{K}_2)^{\frac{1}{2}}$ is near zero. The second difference is that in Barrick's expression for C_h in (4.111) $\mu = 1$, whereas we have $\mu = 1$ or -1 depending upon the combination of S_1's as given in (4.111). C_e and C_h are usually called the electromagnetic and hydrodynamic coupling coefficients respectively.

Again by using the perturbation method, Johnstone (1975) also has derived an expression for the second order cross section. Obviously, the nature of his expression is similar to that of Barrick. His result differs mainly in constants in the expressions for C_e and C_h when compared with Barrick's result. He has combined the electromagnetic and hydrodynamic terms of the second order field after the statistical averaging, whereas Barrick and the author have combined them before the averaging as it should be. Further, the form for the two dimensional ocean wave height spectrum taken by Johnstone is somewhat different than used here or by Barrick.

By using plane wave incidence Barrick (1970) has considered the scattering area of the surface to be the patch and thus he has derived the first and second orders of the cross section of the patch. Whereas in this work the source is assumed to be a pulsed dipole which does not limit the scattering area to the patch only as explained before. As a result the second order cross section derived above contains additional terms. The additional second order power corresponding to σ_{s22} may be viewed as a result of interaction between the surface and incident field along the path from source point to the patch. The other additional power corresponding to σ_{s33} results from off the patch double scattering.

4.4.2 Narrow Beam Transmitting and Receiving Antennas

We now consider the case where we have very narrow beam antennas for both transmitter and receiver and directed along x axis. For this case the received backscattered field is given by (4.58). This means that (4.76) reduces to:

$$\tilde{E}_{zb}(t_o,t) = \tilde{E}_{zb1}(t_o,t) + \tilde{E}_{zb21}(t_o,t) + \tilde{E}_{zb22}(t_o,t) \tag{4.121}$$

where $\tilde{E}_{zb1}$, $\tilde{E}_{zb21}$, and $\tilde{E}_{zb22}$ are given by (4.77) to (4.79) respectively. The only difference between omnidirectional transmission, narrow beam reception and narrow beam transmission, narrow beam reception is that off the patch double scatterings have been neglected in the latter. Therefore, the Doppler frequency dependent cross section of the patch from (4.119) may simply be given as

$$\sigma_p(\omega_d) \cong \sigma_{ff}(\omega_d) + \sigma_{s11}(\omega_d) + \sigma_{s22}(\omega_d) \,. \qquad (4.122)$$

In the above σ_{ff} is the first order cross section given by (4.104) or in a simplified form by (4.108). σ_{s11} and σ_{s22} are the second order cross sections given by (4.110) and (4.112) respectively. These cross sections are normalized to the area of the patch. In view of (4.122) the expression (4.120) for the backscattered power density spectrum becomes

$$P_r(\omega_d) = P_{rp}(\omega_d) \cong P_{ff}(\omega_d) + P_{s11}(\omega_d) + P_{s22}(\omega_d). \qquad (4.123)$$

4.4.3 Simplification of Second Order Cross Section

The three parts of the second order Doppler frequency dependent backscattered cross section, σ_{s11}, σ_{s22}, σ_{s33}, derived in appendix D and presented in section 4.4.1.2 are in the form of a double integral. They were left with double integrals as σ_{s11} is usually given in that form in literature. Since each double integral contains one Dirac delta function, one integral in each cross section term may be evaluated analytically. However, arguments of the delta functions not being simple, integral evaluations are algebraically somewhat tedius. We shall evaluate one integral in each term but the algebraic details of their evaluations will not be presented here. For an understanding of the method used, the main steps and transformations involved will be given. The integral evaluation for σ_{s33} in (4.114) requires solving transcendental equations which may be done numerically.

4.4.3.1 Integral Evaluation for $\sigma_{s11}(\omega_d)$

The expression for $\sigma_{s11}(\omega_d)$ is given by (4.110) along with (4.111). To evaluate one integral in this expression we use the following steps and transformations:

(a) We divide the limits of the α_1 integral into four intervals; $-\infty \leq \alpha_1 \leq -k_o$, $-k_o \leq \alpha_1 \leq 0$, $0 \leq \alpha_1 \leq k_o$, and $k_o \leq \alpha_1 \leq \infty$, so that the sign functions may be cleared from the argument of the delta function and in the expression for C_h.

(b) By changing (α_1, β_1) to $(-\alpha_1, -\beta_1)$ in the region with negative α_1 and then changing β_1 to $-\beta_1$ in the negative region of β_1, the total region of the double integral reduces to: $\{0 \leq \alpha_1 \leq k_o, \beta_1 \geq 0\}$ and $\{k_o \leq \alpha_1 \leq \infty, \beta_1 \geq 0\}$.

(c) The integration variables α_1 and β_1 are transformed to the variables K_1 and K_2 $(K_1 > 0, K_2 > 0)$, where K_1 and K_2 are defined by (4.111). There are now three different regions for the double integral in K_1, K_2 plane with K_1 along the x axis and K_2 along the y axis. The first region consists of the area bounded by the lines $K_2 = 2k_o - K_1$, $K_2 = 2k_o$, and $K_2 = K_1$. The second region is the area enclosed between $K_2 = 2k_o$, $K_2 = K_1$, and $K_2 = (K_1^2 + 4k_o^2)^{\frac{1}{2}}$. The third region is given by the area enclosed between $K_2 = K_1 + 2k_o$ and $K_2 = (K_1^2 + 4k_o^2)^{\frac{1}{2}}$. The argument of the delta function is now in the form of $\{\omega_d \mp (gK_1)^{\frac{1}{2}} \mp (gK_2)^{\frac{1}{2}}\}$.

(d) For convenience, we normalize the ocean wave number vectors by the radar wave number as $\vec{K}_1^o = \vec{K}_1/(2k_o)$ and $\vec{K}_2^o = \vec{K}_2/(2k_o)$. Similarly, the Doppler frequency is normalized by the Bragg frequency as $\nu = \omega_d/\omega_B$. Thus the argument of delta function reduces to the form $\{\nu \mp (\vec{K}_1^o)^{\frac{1}{2}} \mp (\vec{K}_2^o)^{\frac{1}{2}}\}$.

(e) Finally, by using the transformation $v = (\vec{K}_1^o)^{\frac{1}{2}}$ and $w = (\vec{K}_2^o)^{\frac{1}{2}}$, the integral with respect to either v or w may be evaluated keeping in view the regions of integrations with respect to K_1 and K_2 given above in (c). The result is as follows:

(i) For $0 \leq |v| \leq 1$

$$\sigma_{s11}(\omega_d) = \frac{8(2k_o)^8}{\pi^2 \omega_B}$$

$$\left\{ \int_{\frac{1}{2}\{a+(2-a^2)^{\frac{1}{2}}\}}^{1} \Gamma_m [S_1^{\mp}(\vec{K}_1)S_1^{\pm}(\vec{K}_2) + S_1^{\mp}(\vec{K}_1')S_1^{\pm}(\vec{K}_2')]dw \right.$$

$$+ \int_{1,\ |w-a| \geq (w^4-1)^{\frac{1}{2}}}^{\infty} \Gamma_m [S_1^{\mp}(\vec{K}_1)S_1^{\pm}(\vec{K}_2) + S_1^{\mp}(\vec{K}_1')S_1^{\pm}(\vec{K}_2')]dw$$

$$\left. + \int_{1,\ |w-a| \leq (w^4-1)^{\frac{1}{2}}}^{\frac{1+a^2}{2a}} \Gamma_m [S_1^{\pm}(\vec{K}_1)S_1^{\pm}(\vec{K}_2) + S_1^{\pm}(\vec{K}_1')S_1^{\pm}(\vec{K}_2')]dw \right\} \qquad (4.124a)$$

(ii) For $1 \leq |v| \leq (2)^{\frac{1}{2}}$

$$\sigma_{s11}(\omega_d) = \frac{8(2k_o)^8}{\pi^2 \omega_B}$$

$$\left\{ \int_{\frac{1}{2}\{a+(2-a^2)^{\frac{1}{2}}\}}^{1} \Gamma_p [S_1^{\pm}(\vec{K}_1)S_1^{\pm}(\vec{K}_2) + S_1^{\pm}(\vec{K}_1')S_1^{\pm}(\vec{K}_2')]dw \right.$$

$$+ \int_{1,\ |w-a| \geq (w^4-1)^{\frac{1}{2}}}^{a} \Gamma_p [S_1^{\pm}(\vec{K}_1)S_1^{\pm}(\vec{K}_2) + S_1^{\pm}(\vec{K}_1')S_1^{\pm}(\vec{K}_2')]dw$$

$$+\int_{1}^{\frac{1+a^2}{2a}} \Gamma_p[S_1^{\mp}(\vec{K}_1)S_1^{\pm}(\vec{K}_2) + S_1^{\mp}(\vec{K}_1')S_1^{\pm}(\vec{K}_2')]dw \Bigg\}$$
$$|w - a| \leq (w^4 - 1)^{\frac{1}{2}}$$

(4.124b)

(iii) For $(2)^{\frac{1}{2}} \leq |v| \leq 2$

$$\sigma_{s11}(\omega_d) = \frac{8(2k_o)^8}{\pi^2 \omega_B}$$

$$\cdot \Bigg\{ \int_{a/2}^{1} \Gamma_p[S_1^{\pm}(\vec{K}_1)S_1^{\pm}(\vec{K}_2) + S_1^{\pm}(\vec{K}_1')S_1^{\pm}(\vec{K}_2')]dw$$

$$+\int_{1}^{a} \Gamma_p[S_1^{\pm}(\vec{K}_1)S_1^{\pm}(\vec{K}_2) + S_1^{\pm}(\vec{K}_1')S_1^{\pm}(\vec{K}_2')]dw$$
$$|w - a| \geq (w^4 - 1)^{\frac{1}{2}}$$

$$+\int_{1}^{\frac{1+a^2}{2a}} \Gamma_p[S_1^{\mp}(\vec{K}_1)S_1^{\pm}(\vec{K}_2) + S_1^{\mp}(\vec{K}_1')S_1^{\pm}(\vec{K}_2')]dw \Bigg\}$$
$$|w - a| \leq (w^4 - 1)^{\frac{1}{2}}$$

(4.124c)

(iv) For $|v| \geq 2$

$$\sigma_{s11}(\omega_d) = \frac{8(2k_o)^8}{\pi^2 \omega_B}$$

$$\cdot \Bigg\{ \int_{a/2}^{a} \Gamma_p[S_1^{\pm}(\vec{K}_1)S_1^{\pm}(\vec{K}_2) + S_1^{\pm}(\vec{K}_1')S_1^{\pm}(\vec{K}_2')]dw$$
$$|w - a| \geq (w^4 - 1)^{\frac{1}{2}}$$

$$+\int_{\substack{1 \\ |w-a| \leq (w^4-1)^{\frac{1}{2}}}}^{\frac{1+a^2}{2a}} \Gamma_p [S_1^{\mp}(\vec{K}_1) S_1^{\pm}(\vec{K}_2) + S_1^{\mp}(\vec{K}_1') S_1^{\pm}(\vec{K}_2')] dw \Bigg\} \tag{4.124d}$$

In (4.124a) to (4.124d) the upper superscripts on S_1's should be taken for the negative Doppler frequencies, whereas for the positive frequencies lower superscripts should be taken. Also, the restrictions placed on w as a result of the integration have to be met. The different restrictions for the integrals are clearly mentioned in (4.124a) to (4.124d). The various symbols appearing in these equations are defined as

$$\nu = \omega_d/\omega_B \ , \quad a = |\nu|$$

$$\Gamma_m = \frac{(K_1^o K_2^o)^{3/2}}{K_y^o} \left| C_{en} + C_{hnm} \right|^2$$

$$\Gamma_p = \frac{(K_1^o K_2^o)^{3/2}}{K_y^o} \left| C_{en} + C_{hnp} \right|^2$$

$$C_{en} = \frac{1}{2} \frac{(K_{1x}^o K_{2x}^o - 2\vec{K}_1^o \cdot \vec{K}_2^o)}{(\vec{K}_1^o \cdot \vec{K}_2^o)^{\frac{1}{2}}}$$

$$C_{hnm} = -\frac{j}{2} \left[K_1^o + K_2^o - (K_1^o K_2^o - \vec{K}_1^o \cdot \vec{K}_2^o) \cdot \left(\frac{1+\nu^2}{1-\nu^2}\right) \frac{1}{(K_1^o K_2^o)^{\frac{1}{2}}}\right]$$

$$C_{hnp} = -\frac{j}{2} \left[K_1^o + K_2^o + (K_1^o K_2^o - \vec{K}_1^o \cdot \vec{K}_2^o) \cdot \left(\frac{1+\nu^2}{1-\nu^2}\right) \frac{1}{(K_1^o K_2^o)^{\frac{1}{2}}}\right] \tag{4.125}$$

$$\vec{K}_1^o = K_{1x}^o\hat{x} + K_y^o\hat{y} \quad , \quad K_1^o = |\vec{K}_1^o|$$

$$\vec{K}_2^o = K_{2x}^o\hat{x} - K_y^o\hat{y} \quad , \quad K_2^o = |\vec{K}_2^o|$$

$$\vec{K}_1 = 2k_o(K_{1x}^o\hat{x} + K_y^o\hat{y})$$

$$\vec{K}_2 = 2k_o(K_{2x}^o\hat{x} - K_y^o\hat{y})$$

$$\vec{K}_1' = 2k_o(K_{1x}^o\hat{x} - K_y^o\hat{y})$$

$$\vec{K}_2' = 2k_o(K_{2x}^o\hat{x} + K_y^o\hat{y})$$

$$K_{1x}^o = \frac{1}{2}(K_2^{o^2} - K_1^{o^2} - 1)$$

$$K_{2x}^o = \frac{1}{2}(K_1^{o^2} - K_2^{o^2} - 1)$$

$$K_y^o = \frac{1}{2}[4K_1^{o^2}K_2^{o^2} - (K_1^{o^2} + K_2^{o^2} - 1)^2]^{\frac{1}{2}}$$

$$K_1^o = (w - a)^2$$

$$K_2^o = w^2 \; .$$

It may easily be shown from (4.124a) or (4.124b) that $\sigma_{s11} = 0$ at $\omega_d = \omega_B$ or $-\omega_B$. This means that this part of the second order cross section is disjoint with the two first order peaks situated at plus and minus the Bragg frequency.

4.4.3.2 Integral Evaluation for $\sigma_{s22}(\omega_d)$

The expression for $\sigma_{s22}(\omega_d)$ is given by (4.112) in conjunction with (4.113). The method used to evaluate one integral in (4.112) is as follows;

(a) The limits of the α_2 integral are divided into two intervals; $-\infty \leq \alpha_2 \leq 0$ and $0 \leq \alpha_2 \leq \infty$, so that the sign functions are cleared.

(b) By changing (α_2, β_2) to $(-\alpha_2, -\beta_2)$ in the region with negative α_2 and then changing β_2 to $-\beta_2$ in the negative region of β_2, the total region of the double integral is reduced to: $\alpha_2 \geq 0$, $\beta_2 \geq 0$.

(c) The integration variables α_2 and β_2 are transformed to the polar coordinate variables K_2 and ϕ_2. The region of integrations may now be given as $K_2 \geq 0$, $0 \leq \phi_2 \leq \pi/2$.

(d) Again, the wave number vectors are normalized by the radar wave number and the Doppler frequency is normalized by the Bragg frequency. Thus the form of the argument of delta function reduces to $\{\nu \pm 1 \pm (K_2^o)^{\frac{1}{2}}\}$, where $K_2^o = K_2/2k_o$ and $\nu = \omega_d/\omega_B$.

(e) Finally, by using the transformation $v = (K_2^o)^{\frac{1}{2}}$, the integral with respect to v may be easily evaluated. The result of the evaluation is as follows:

$$\sigma_{s22}(\omega_d) = A(\omega_d) + B(\omega_d) \qquad (4.126)$$

where

$$A(\omega_d) = \frac{144\, k_o^7\, \Delta_\theta}{\pi\, \Delta_\rho\, \omega_B} [\{S_1^+(K_1)\}^2\, \delta(\nu + 2)$$

$$+ \{S_1^-(K_1)\}^2\, \delta(\nu - 2) + 2\, S_1^+(\vec{K}_1) S_1^-(\vec{K}_1) \delta(\nu)] \qquad (4.127)$$

and $B(\omega_d)$ for different regions of ν is as follows:

(i) <u>For $\nu \leq -1$</u>

$$B(\omega_d) = \frac{(2k_o)^8}{\pi^2 \omega_B} \int_0^{\pi/2} \left\{ S_1^+(\vec{K}_1)[C_1\{S_1^+(\vec{K}_2)\right.$$

$$+ S_1^+(\vec{K}_2')\} + C_2\{S_1^-(\vec{K}_2) + S_1^-(\vec{K}_2')\}]$$

$$+ S_1^-(\vec{K}_1)[C_3\{S_1^+(\vec{K}_3) + S_1^+(\vec{K}_3')\} + C_4\{S_1^-(\vec{K}_3)$$

$$+ S_1^-(\vec{K}_3')\}]\Bigg\} d\phi_2 \tag{4.128a}$$

(ii) For $|\nu| \leq 1$

$$B(\omega_d) = \frac{(2k_o)^8}{\pi^2\omega_B} \int_0^{\pi/2} \Bigg\{ S_1^+(\vec{K}_1)[C_1\{S_1^-(\vec{K}_2)$$

$$+ S_1^-(\vec{K}_2')\} + C_2\{S_1^+(\vec{K}_2) + S_1^+(\vec{K}_2')\}]$$

$$+ S_1^-(\vec{K}_1)[C_3\{S_1^+(\vec{K}_3) + S_1^+(\vec{K}_3')\} + C_4\{S_1^-(\vec{K}_3)$$

$$+ S_1^-(\vec{K}_3')\}]\Bigg\} d\phi_2 \tag{4.128b}$$

(iii) For $\nu \geq 1$

$$B(\omega_d) = \frac{(2k_o)^8}{\pi^2\omega_B} \int_0^{\pi/2} \Bigg\{ S_1^+(\vec{K}_1)[C_1\{S_1^-(\vec{K}_2)$$

$$+ S_1^-(\vec{K}_2')\} + C_2\{S_1^+(\vec{K}_2) + S_1^+(\vec{K}_2')\}]$$

$$+ S_1^-(\vec{K}_1)[C_3\{S_1^-(\vec{K}_3) + S_1^-(\vec{K}_3')\} + C_4\{S_1^+(\vec{K}_3)$$

$$+ S_1^+(\vec{K}_3')\}]\Bigg\} d\phi_2 \tag{4.128c}$$

In (4.127) to (4.128c) we have

$$\nu = \omega_d/\omega_B$$

$$C_1 = K_2^{o\,3/2} \frac{[(\vec{K}_2^o + \frac{1}{2}\vec{K}_1^o) \cdot \vec{K}_2^o]^2}{|(\vec{K}_2^o + \vec{K}_1^o) \cdot \vec{K}_2^o|}$$

$$C_2 = K_2^{o\,3/2} \frac{[(\vec{K}_2^o - \frac{1}{2}\vec{K}_1^o) \cdot \vec{K}_2^o]^2}{|(\vec{K}_2^o - \vec{K}_1^o) \cdot \vec{K}_2^o|}$$

$$C_3 = K_3^{o\,3/2} \frac{[(\vec{K}_3^o + \frac{1}{2}\vec{K}_1^o) \cdot \vec{K}_3^o]^2}{|(\vec{K}_3^o + \vec{K}_1^o) \cdot \vec{K}_3^o|}$$

$$C_4 = K_3^{o\,3/2} \frac{[(\vec{K}_3^o - \frac{1}{2}\vec{K}_1^o) \cdot \vec{K}_3^o]^2}{|(\vec{K}_3^o - \vec{K}_1^o) \cdot \vec{K}_3^o|}$$

$$\vec{K}_1^o = 1\hat{x} \quad , \quad K_1^o = |\vec{K}_1^o| = 1$$

$$\vec{K}_2^o = K_{2x}^o\hat{x} + K_{2y}^o\hat{y} \quad , \quad K_2^o = |\vec{K}_2^o|$$

$$\vec{K}_3^o = K_{3x}^o\hat{x} + K_{3y}^o\hat{y} \quad , \quad K_3^o = |\vec{K}_3^o|$$

$$\vec{K}_1 = 2k_o\vec{K}_1^o = 2k_o\hat{x}$$

$$\vec{K}_2 = 2k_o(K_{2x}^o\hat{x} + K_{2y}^o\hat{y}) \qquad (4.129)$$

$$\vec{K}_2' = 2k_o(K_{2x}^o\hat{x} - K_{2y}^o\hat{y})$$

$$\vec{K}_3 = 2k_o(K_{3x}^o\hat{x} + K_{3y}^o\hat{y})$$

$$\vec{K}_3' = 2k_o(K_{3x}^o\hat{x} - K_{3y}^o\hat{y})$$

$$K_{2x}^o = K_2^o \cos \phi_2$$

$$K_{2y}^o = K_2^o \sin \phi_2$$

$K^o_{3x} = K^o_3 \cos \phi_2$

$K^o_{3y} = K^o_3 \sin \phi_2$

$K^o_2 = (\nu + 1)^2$

$K^o_3 = (\nu - 1)^2$.

4.4.3.3 Integral Evaluation for $\sigma_{s33}(\omega_d)$

The expression for $\sigma_{s33}(\omega_d)$ is given by (4.114) along with (4.115) and (D.41). The method used to evaluate one integral in (4.114) is as follows. However, the evaluation requires solutions for eight transcendental equations which may be obtained numerically.

(a) By following the steps (a) and (b) mentioned in the case of $\sigma_{s22}(\omega_d)$ in section 4.4.3.2 the sign functions may be cleared and the region for integrations may be reduced to: $\alpha_2 \geq 0$, $\beta_2 \geq 0$, with the restriction $2k_o|\alpha_2| < (\alpha_2^2 + \beta_2^2) < 2k_o(\alpha_2^2 + \beta_2^2)^{\frac{1}{2}}$ as mentioned in (4.114).

(b) The integration variables are first transformed to polar coordinate variables (K_2, ϕ_2) as $\alpha_2 = K_2 \cos \phi_2$ and $\beta_2 = K_2 \sin \phi_2$. Since $K_2 < 2k_o$, it is then transformed as $K_2 = 2k_o \cos \theta$. The region of integration may now be given as $\epsilon_1 \leq \theta \leq (\frac{\pi}{2} - \epsilon_2)$, $(\theta + \epsilon_2) \leq \phi_2 \leq \frac{\pi}{2}$, where ϵ_1 and ϵ_2 are small positive constants. These constants arise while converting the end inequalities into equalities in the restriction mentioned in (a).

(c) The ocean wave number vectors $\vec{K}_1$ and $\vec{K}_2$ are normalized by $2k_o$ and the Doppler frequency ω_d is normalized by the Bragg frequency ω_B. Thus the arguments of the delta functions are of the forms $\{\nu \pm (K^o_1)^{\frac{1}{2}} \pm (\cos \theta)^{\frac{1}{2}}\}$

and $\{v \pm (K_3^o)^{\frac{1}{2}} \pm (\cos\theta)^{\frac{1}{2}}\}$ where both K_1^o and K_3^o are functions of θ and ϕ_2.

(d) Finally, by using $v = (K_1^o)^{\frac{1}{2}}$ in the first type of argument and $w = (K_3^o)^{\frac{1}{2}}$ in the second type, each integral with respect v or w may be evaluated. However, each evaluation requires a solution of a transcendental equation given by the argument of delta function. In other words, for a given v and θ, a solution for ϕ_2 is required so that the argument of delta function is zero as well as the solution is within the limits of ϕ_2. Obviously, if ϕ_2 has no solution or the solution does not lie within the limits, the integral is zero. The final expression for $\sigma_{s33}(\omega_d)$ may be given as follows:

$$\sigma_{s33}(\omega_d) = \frac{8(2k_o)^8}{\pi^2\omega_B}|F_p|^{-4}\int_{\theta=\varepsilon_1}^{\pi/2-\varepsilon_2}\frac{\sin 2\theta}{\xi_o^2(\cos^2\theta-\cos^2\phi_o)}$$

$$\left\{ B_1[A_1S_1^+(\vec{K}_1)S_1^+(\vec{K}_2) + A_2S_1^+(\vec{K}_1')S_1^+(\vec{K}_2')] \right. \quad \text{(first)}$$

$$+ B_1[A_1S_1^-(\vec{K}_1)S_1^-(\vec{K}_2) + A_2S_1^-(\vec{K}_1')S_1^-(\vec{K}_2')] \quad \text{(second)}$$

$$+ B_1[A_1S_1^-(\vec{K}_1)S_1^+(\vec{K}_2) + A_2S_1^-(\vec{K}_1')S_1^+(\vec{K}_2')] \quad \text{(third)}$$

$$+ B_1[A_1S_1^+(\vec{K}_1)S_1^-(\vec{K}_2) + A_2S_1^+(\vec{K}_1')S_1^-(\vec{K}_2')] \quad \text{(fourth)}$$

$$+ B_2[A_1S_1^+(\vec{K}_3)S_1^-(\vec{K}_2) + A_2S_1^+(\vec{K}_3')S_1^-(\vec{K}_2')] \quad \text{(fifth)}$$

$$+ B_2[A_1S_1^-(\vec{K}_3)S_1^+(\vec{K}_2) + A_2S_1^-(\vec{K}_3')S_1^+(\vec{K}_2')] \quad \text{(sixth)}$$

$$+ B_2[A_1S_1^-(\vec{K}_3)S_1^-(\vec{K}_2) + A_2S_1^-(\vec{K}_3')S_1^-(\vec{K}_2')] \quad \text{(seventh)}$$

$$+ B_2[A_1S_1^+(\vec{K}_3)S_1^+(\vec{K}_2) + A_2S_1^+(\vec{K}_3')S_1^+(\vec{K}_2')] \quad \text{(eighth)}$$

$$\Big\} \, d\theta \qquad (4.130)$$

In the above we have

$$\nu = \omega_d/\omega_B$$

$$A_1 = |F(\xi_o\rho_o, 1, 0)F(R_b, \cos\phi_b, \sin\phi_b)F(R_c, -\cos\phi_c, -\sin\phi_c)|^2$$

$$A_2 = |F(\xi_o\rho_o, 1, 0)F(R_b, \cos\phi_b, -\sin\phi_b)F(R_c, -\cos\phi_c, \sin\phi_c)|^2$$

$$B_1 = [(1 - 2K^o_{1x} - 2K^o_{2x})\{K^o_{1x}(2K^o_{1x} + K^o_{2x}) - 2(K^o_{1x} + K^o_{2x})K^{o^2}_1\} + (2K^o_1K^o_{1y})^2]^2 \cdot K^{o^{3/2}}_1 [K^o_{2x}K^o_{2y} + 2K^o_{1x}\sin(\phi_o + \theta)]^{-1}$$

$$B_2 = [(1 - 2K^o_{3x} + 2K^o_{2x})\{K^o_{3x}(2K^o_{3x} - K^o_{2x}) - 2(K^o_{3x} - K^o_{2x})K^{o^2}_3\} + (2K^o_3K^o_{3y})^2]^2 \cdot K^{o^{3/2}}_3 [K^o_{2x}K^o_{2y} - 2k^o_{3x}\sin(\phi_o - \theta)]^{-1}$$

$$\vec{K}^o_1 = K^o_{1x}\hat{x} + K^o_{1y}\hat{y} \quad , K^o_1 = |\vec{K}^o_1|$$

$$\vec{K}^o_2 = K^o_{2x}\hat{x} + K^o_{2y}\hat{y} \quad , K^o_2 = |\vec{K}^o_2|$$

$$\vec{K}^o_3 = K^o_{3x}\hat{x} + K^o_{3y}\hat{y} \quad ; K^o_3 = |\vec{K}^o_3|$$

$$\vec{K}_1 = 2k_o(K^o_{1x}\hat{x} + K^o_{1y}\hat{y})$$

$$\vec{K}'_1 = 2k_o(K^o_{1x}\hat{x} - K^o_{1y}\hat{y})$$

$$\vec{K}_2 = 2k_o(K^o_{2x}\hat{x} + K^o_{2y}\hat{y})$$

$$\vec{K}'_2 = 2k_o(K^o_{2x}\hat{x} - K^o_{2y}\hat{y})$$

$$\vec{K}_3 = 2k_o(K^o_{3x}\hat{x} + K^o_{3y}\hat{y})$$

$$\vec{K}'_3 = 2k_o(K^o_{3x}\hat{x} - K^o_{3y}\hat{y}) \tag{4.131}$$

$$K^o_{1x} = \frac{1}{2}[1 - \cos(\phi_o + \theta)]$$

$$K^o_{1y} = -\frac{1}{2}\sin\phi_o\cos\theta$$

$$K^o_{2x} = \cos\phi_o\cos\theta$$

$$K^o_{2y} = \sin\phi_o\cos\theta$$

$$K^o_{3x} = \frac{1}{2}[1 + \cos(\phi_o - \theta)]$$

$$K^o_{3y} = \frac{1}{2}\sin\phi_o\cos\theta$$

$$\xi_o = \frac{2\sin\theta}{\sin\phi_o + \sin\theta}$$

$$R_b = \rho_o\,\frac{\sin(\phi_o + \theta)}{\cos\theta(\sin\phi_o + \sin\theta)}$$

$$R_c = \rho_o\,\frac{\sin(\phi_o - \theta)}{\cos\theta(\sin\phi_o + \sin\theta)}$$

$$\cos\phi_b = \frac{\sin(2\phi_o) + \sin(2\theta)}{2\sin(\phi_o + \theta)}$$

$$\cos\phi_c = \frac{\sin(2\phi_o) - \sin(2\theta)}{2\sin(\phi_o - \theta)}$$

$$\sin\phi_b = \frac{\cos^2\theta - \cos^2\phi_o}{\sin(\phi_o + \theta)}$$

$$\sin\phi_c = \frac{\cos^2\theta - \cos^2\phi_o}{\sin(\phi_o - \theta)}$$

In (4.131) ϕ_o is a solution for ϕ_2 obtained by solving a transcendental equation for a given ν and θ such that $(\theta + \varepsilon_2) \leq \phi_o \leq \frac{\pi}{2}$. Since each part of (4.130) has a different equation to solve for ϕ_2, ϕ_o may be different for each part. These different equations corresponding to all eight parts are as follows:

$$\begin{aligned}
&\nu + g_1(\phi_2, \theta) + (\cos\theta)^{\frac{1}{2}} = 0 && \text{(first)}\\
&\nu - g_1(\phi_2, \theta) - (\cos\theta)^{\frac{1}{2}} = 0 && \text{(second)}\\
&\nu - g_1(\phi_2, \theta) + (\cos\theta)^{\frac{1}{2}} = 0 && \text{(third)}\\
&\nu + g_1(\phi_2, \theta) - (\cos\theta)^{\frac{1}{2}} = 0 && \text{(fourth)}\\
&\nu + g_2(\phi_2, \theta) + (\cos\theta)^{\frac{1}{2}} = 0 && \text{(fifth)}\\
&\nu - g_2(\phi_2, \theta) - (\cos\theta)^{\frac{1}{2}} = 0 && \text{(sixth)}\\
&\nu - g_2(\phi_2, \theta) + (\cos\theta)^{\frac{1}{2}} = 0 && \text{(seventh)}\\
&\nu + g_2(\phi_2, \theta) - (\cos\theta)^{\frac{1}{2}} = 0\ , && \text{(eighth)}
\end{aligned} \tag{4.132}$$

where the functions g_1 and g_2 are given as

$$\begin{aligned}
g_1(\phi_2, \theta) &= [\sin^4(\frac{\phi_2 + \theta}{2}) + \frac{1}{4}\sin^2\phi_2\cos^2\theta]^{\frac{1}{2}}\\
g_2(\phi_2, \theta) &= [\cos^4(\frac{\phi_2 - \theta}{2}) + \frac{1}{4}\sin^2\phi_2\cos^2\theta]^{\frac{1}{2}}.
\end{aligned} \tag{4.133}$$

It may be seen from (4.132) and (4.133) that no equations in (4.132) will be satisfied for $|\nu| > 2.06$. Hence $\sigma_{s33}(\omega_d)$ is zero for $|\omega_d| > 2.06\ \omega_B$.

4.5 Interpretation and Utilization of the Cross Section Solution

In the previous section we have developed a solution for the first and second order backscattered Doppler frequency dependent cross section of the ocean surface. The first order cross section, $\sigma_{ff}(\omega_d)$, is given by (4.104). The second order cross section consists of three parts; $\sigma_{s11}(\omega_d)$, $\sigma_{s22}(\omega_d)$, $\sigma_{s33}(\omega_d)$ which are available from (4.110) to (4.115) in the form of a double integral or from (4.124) to (4.133) in the form of a single integral. These expressions are normalized to the area of the surface patch. As mentioned in section 3.4.1.1, the first order backscattered field defines a patch of the surface depending upon the beam width of the receiving antenna, transmitted pulse duration, and the time delay between transmitted and received signals.

For an interpretation of these two orders of the cross section and their utilization in extracting relevant information about ocean surface conditions we shall now present graphs for a few typical cases. These graphs are in the form of the spectral density variation of the cross section with the Doppler frequency. The model for the wave height spectrum taken here is the Pierson-Moskowitz frequency spectrum with the cardioid directional distribution factor for a fully developed sea. This model is given by (4.52). The integrations required in different parts of the second order cross section are performed numerically.

Figure 4.1 shows the sum of the first and second order cross section at a radar frequency of 25.4 MHz. The wind speed is 30 knots and its direction is at 45° with respect to the x axis, where the latter is taken to be the direction of the patch or the radar look direction. The radial width of the patch required for the first order cross section is assumed to be 1.2 km, which corresponds to 8 μsec pulse duration.

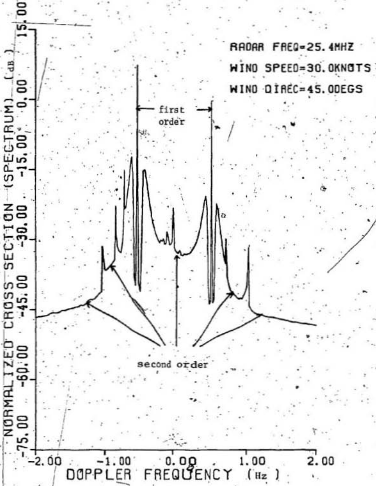


Figure 4.1 Backscattered cross section of first and second order of an ocean surface patch normalized to patch area for omnidirectional transmission and narrow beam reception at 25.4 MHz, wind velocity = 30 knots at 45° with respect to patch direction

The two large peaks in the figure show the first order effect, while the continuum with other peaks surrounding them is the second order effect. Let us first consider the first order peaks. As evident from (4.104), the two peaks are caused by those two ocean gravity waves whose wavelengths are equal to one-half the radar wavelength - one is moving toward the radar (+ve Doppler) while the other is moving away from it (-ve Doppler). Further, the two peaks occur at plus and minus the Bragg frequency $[\omega_B = (2gk_o)^{\frac{1}{2}}]$ which is about 0.51 Hz in this case.

In the model assumed for the ocean surface it has been considered that the surface itself is not moving. Only the waves are considered to be moving. However, if the surface is also physically moving in the form of surface current, the first order peaks are equally shifted from the Bragg position by a small amount in the same direction. The shift towards right or left and its amount depend upon the radial component (along x axis) of the current vector. The presence of current modifies the phase velocity of ocean waves. Thus by measuring the above shift one can obtain the radial component (V_{cr}) of the mean surface current vector from the relationship, $V_{cr} = \Delta_s\lambda_o/2$, where λ_o is the radar wavelength and and Δ_s is the Doppler shift in Hz [Barrick et al. (1974)].

The relative strength of the first order peaks depends upon the wave height spectra S_1^+ and S_1^-, which in turn depend upon the wind speed and its direction. In this particular case S_1^+ is higher than S_1^- as the wind is more effective for waves moving with positive velocity components along x axis. As a result the first order peak situated at the negative Bragg frequency is higher than at the plus frequency. By assuming a model for the wave height spectrum and measuring the relative strength of the two peaks, Long and Trizna (1973) have estimated the wind direction using a sky wave HF radar. The technique is equally applicable to ground wave radars also. It may be mentioned here that the two orders of the cross section have been derived assuming an infinitely long record length for the backscattered signal. Whereas in practice the record length is of a finite time duration which causes a broadening of the peaks and the other parts of the spectrum.

We now consider the second order cross section. To show the individual effects of the three parts of the second order cross section they are plotted separately in figure 4.2. Since the expression for the electromagnetic coupling coefficient (C_e), given by (4.111), for σ_{s11} is valid provided $\vec{K}_1 \cdot \vec{K}_2 \neq 0$, it is taken as zero in the neighborhood of $\vec{K}_1 \cdot \vec{K}_2 = 0$ in performing the numerical integration. The integrand for σ_{s22}, given by (4.112), is also taken as zero in the neighborhood of $(\vec{K}_1 + \vec{K}_2) \cdot \vec{K}_2 = 0$ for the same reason. The σ_{s11} part of the second order cross section corresponds to the case where the double scattering occurs on the patch only. The sum of any two ocean wave number vectors responsible for the scattering in this case is always equal to $-2k_o\hat{x}$, i.e., from (4.111), $\vec{K}_1 + \vec{K}_2 = -2k_o\hat{x}$.

The spectrum of σ_{s11} contains peaks at the Doppler frequencies of $\sqrt{2}$ and $2^{3/4}$ times the Bragg frequency. These peaks are marked in figure 4.2 as "a" and "b" respectively. The peak a occurs when the ocean wave numbers $\vec{K}_1$ and $\vec{K}_2$ producing the scatter are both equal to $-k_o\hat{x}$, which makes the coupling coefficients quite large at $\nu = \pm\sqrt{2}$ ($\nu = \omega_d/\omega_B$) [Johnstone (1975), Lipa and Barrick (1982)]. The peak b occurs when $\vec{K}_1 \cdot \vec{K}_2$ is nearly zero. Under this condition the electromagnetic coupling coefficient becomes quite large. Also, the integration contour dictated by the argument of the delta function in (4.110) touches tangentially to

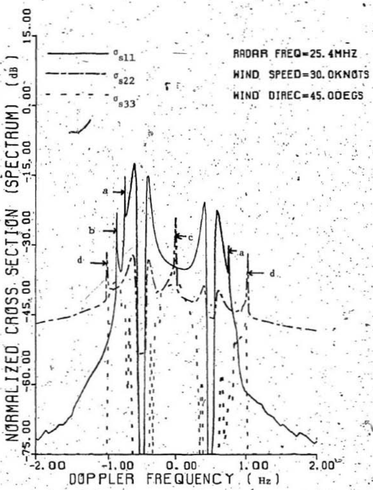


Figure 4.2 Different parts of second order cross section corresponding to figure 4.1

the circle described by $\vec{K}_1 \cdot \vec{K}_2 = 0$ for $\nu = \pm 2^{3/4}$ in the α_1, β_1 plane, where α_1 and β_1 are the integration variables in (4.110). It gives a significant contribution to the integral and thus the spectrum peaks [Johnstone (1975), Lipa and Barrick (1982)]. This peak is usually referred to as the corner reflector effect [Barrick (1972b)]. The occurrence and strength of these peaks, a and b, on either or both sides of the zero Doppler frequency depend upon the wind condition through the wave height spectrum (e.g., see figures 4.5 and 4.6).

An examination of equations (4.124a) to (4.124d) reveals that the spectrum of σ_{s11} at high Doppler frequencies (away from +/- Bragg frequency) are produced mainly by short ocean waves (large wave numbers). Since the wave height spectrum decreases rapidly as the wave number increases, proportional to K^{-4} from (4.36) and (4.52), σ_{s11} decreases quickly as the Doppler frequency increases. The long ocean waves contribute mainly around the two first order peaks. Again, near zero Doppler, the major contribution to the spectrum comes from relatively short waves. For example, the zero Doppler is caused by those ocean waves whose wave numbers are higher than $4k_o$, or at 25.4 MHz the required wavelengths are lower than 3 m.

Consideration is given to σ_{s22} now. This part of the second order cross section corresponds to the case where one scattering occurs at the source and another on the patch. The expression for σ_{s22}, given by (4.112) or (4.126), contains two types of terms. The first type consists of delta functions or impulses situated at the zero Doppler and at plus and minus twice the Bragg frequency. These impulses are caused by ocean waves having wavelengths equal to one-half the radar wavelength. It may be viewed as a repeated first order phenomenon; first at the source then on the patch or vice-versa. However, their weights are very small compared to the weights in the first order cross section, when the latter is taken in the similar form as given by (4.108). These impulses are not shown in the figures.

The second type of term for σ_{s22} is an integral form which gives a continuum as shown in figure 4.2. The continuum is produced by all ocean waves present at the source but by only two waves on the patch, whose wave numbers are equal to $2k_o$ where one is moving towards and another away from the radar. In a way similar to σ_{s11}, the spectrum of σ_{s22} also contains peaks but they are situated at $\nu = 0$ and $\pm 2(\nu = \omega_d/\omega_B)$. These peaks are marked as "c" and "d", respectively, in figure 4.2. They are in addition to those impulses already described which are also situated at the same three frequencies. Of course, the occurrence and strength of these peaks as well as the impulses are dependent on the wind condition. Referring to (4.112), these peaks are caused when $(2k_o\hat{x} + \vec{K}_2) \cdot \vec{K}_2$ is nearly zero. Under this condition the integrand becomes quite large. Also, the integration contour dictated by the argument of the delta function meets tangentially to the circle given by $(2k_o\hat{x} + \vec{K}_2) \cdot \vec{K}_2 = 0$. This happens at $\omega_d = 0$ and $\pm 2\omega_B$, and when the ocean wave number vector $(\vec{K}_2)$ responsible for the scattering at the source becomes equal to $-2k_o\hat{x}$.

It may be seen from (4.129) that, for a given ν, the magnitudes of the ocean wave number vectors responsible for the scattering at the source are constant. The integration is performed only on their directions. For example, at $\omega_d = 3\omega_B$, their magnitudes are $32k_o$ and $8k_o$. Thus the contribution to the Doppler spectrum may come from both long and short waves even at high Doppler frequencies depending upon the radar wavelength. Also, the wave numbers responsible for the scattering on the patch are fixed. They do not enter into the integration. Whereas, in the case of σ_{s11}, the integration is performed over the magnitude of the wave numbers. As a result, σ_{s22} decreases much slower than σ_{s11} as evident in figure 4.2. The contribution from long waves to the spectrum of σ_{s22} is effective near the zero Doppler frequency also.

We will now consider the spectrum of the third part (σ_{s33}) of the second order cross section. This corresponds to the case where the two scatterings occur off the patch. Although this term involves the attenuation functions with modified surface impedances, we have treated these functions with the normalized surface impedance for sea water for computational purposes. The distance of the patch required for them is taken to be 30 km. However, a variation of this distance is not found to affect the spectrum of σ_{s33} very significantly. The two small constants ε_1 and ε_2 appearing in the limits for the integral in (4.130) are respectively taken as 0.1 and 0.02. The value chosen for ε_1 is relatively high so that the second scattering point maintains a reasonable minimum distance from the receiving or source point. We made this assumption of a reasonable minimum distance while evaluating the integrals with respect to the third stationary phase point (3.132c) in section 3.5.1.1. This value of ε_1 along with 30 km patch distance correspond to the minimum distance of 6 km between the source and the second scattering point. It has been already mentioned in section 4.4.3.3 that $\sigma_{s33} = 0$ for $|\omega_d| > 2.06\ \omega_B$. At 25.4 MHz the cut off frequency is about 1.05 Hz. It is seen in figure 4.2 that σ_{s33} is about the order of σ_{s22} near the corner reflector point, and at the zero Doppler it is of the order of σ_{s11}. At other frequencies it is significantly small compared to σ_{s11} or σ_{s22}.

Figures 4.3 and 4.4 show the sum of the first and second order cross section for two different wind directions, 0° and 90°. The radar frequency and the wind speed are taken to be the same as in figure 4.1. The relative strength of different parts of the second order cross section at 0° wind is shown in figure 4.5, and in 4.6 for 90° wind condition. Since the assumed model for the wave height spectrum does not allow ocean waves to propagate in the direction opposite to wind, the first order spectrum at the positive Bragg frequency is zero in figure 4.3. In actual situations it is not zero as the ocean waves moving opposite to wind are invariably present, although they carry very little energy. Further, the spectrum of σ_{s11} is comparatively lower than that of 45° wind condition at positive Doppler frequencies. Whereas the spectrum of σ_{s22} is not changed that much. At 90° wind the Doppler spectrum is symmetrical about the zero Doppler frequency. This is because the wave height spectra S_1^+ and S_1^- are the same under the cross-wind condition. Thus all the second order peaks mentioned above are present in figures 4.5 and 4.6.

Another example is presented in figures 4.7, 4.8, and 4.9. The radar frequency taken is 10 MHz and wind speed of 30 knots. The wind

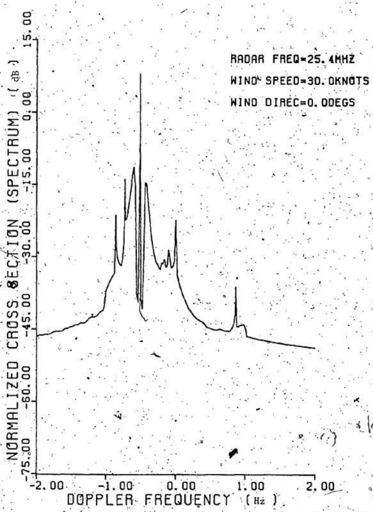


Figure 4.3 Backscattered cross section of first and second order of an ocean surface patch normalized to patch area for omnidirectional transmission and narrow beam reception at 25.4 MHz, wind velocity = 30 knots at 0° with respect to patch direction

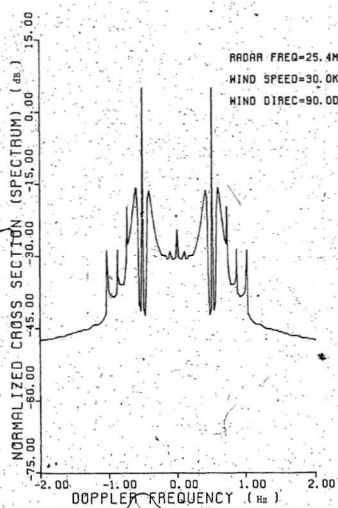


Figure 4.4 Backscattered cross section of first and second order of an ocean surface patch normalized to patch area for omni-directional transmission and narrow beam reception at 25.4 MHz, wind velocity = 30 knots at 90° with respect to patch direction

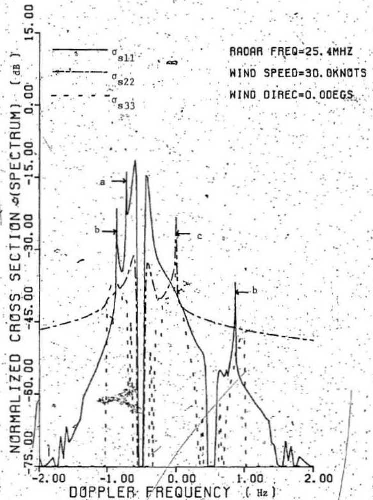


Figure 4.5 Different parts of second order cross section corresponding to figure 4.3

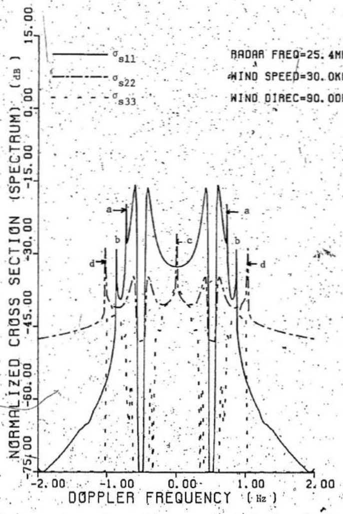


Figure 4.6 Different parts of second order cross section corresponding to figure 4.4

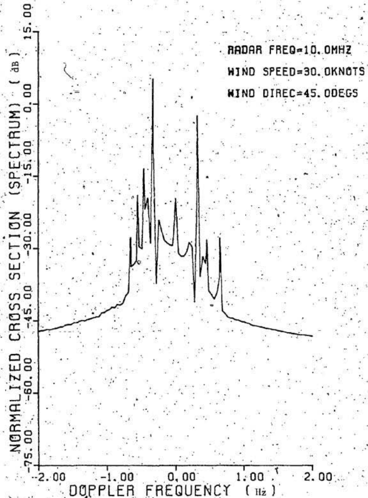


Figure 4.7 Backscattered cross section of first and second order of an ocean surface patch normalized to patch area for omni-directional transmission and narrow beam reception at 10 MHz, wind velocity = 30 knots and 45° with respect to patch direction

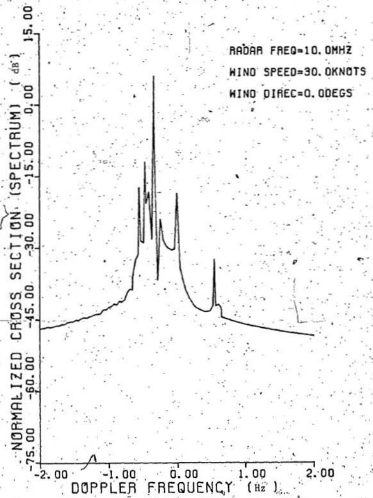


Figure 4.8 Backscattered cross section of first and second order of an ocean surface patch normalized to patch area for omnidirectional transmission and narrow beam reception at 10 MHz, wind velocity = 30 knots at 0° with respect to patch direction

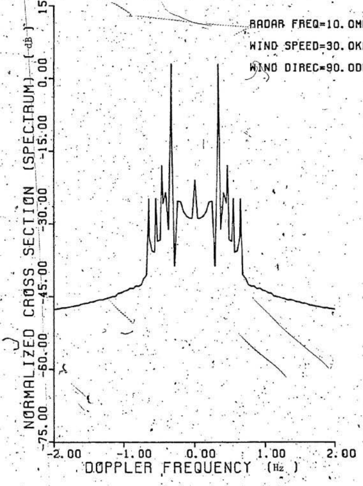


Figure 4.9 Backscattered cross section of first and second order of an ocean surface patch normalized to patch area for omni-directional transmission and narrow beam reception at 10 MHz, wind velocity = 30 knots at 90° with respect to patch direction

directions considered are again 45°, 0°, and 90°. The figures show the sum of the first and second order cross section for three different wind directions. The Bragg frequency in this case is about 0.32 Hz. Therefore, most of the second order features are concentrated in a relatively narrow Doppler band. The cut off of the spectrum is again slow due to the second part of the second order cross section.

Based on the above discussion and the graphs presented it may be inferred that the contribution to the second order cross section by σ_{s22} is significant around the zero Doppler and at frequencies greater than the corner reflector frequency ($|\omega_d| > 2^{3/4}\ \omega_B$). The contribution from σ_{s33} is effective only at a few isolated frequencies such as at the zero Doppler and near the corner reflector point. It may be mentioned here that the above results are based on the assumption that the sea is fully developed in the scattering region including the one surrounding the transmitting antenna. However, in most of the cases antennas are located on the beach or near the shore. The long ocean waves (depending upon the radio wave length) are very rarely present there and besides, they can not be considered fully developed. Therefore, under these common situations σ_{s22} may not be wholly present. Also, σ_{s33} may be neglected. Thus the second order cross section reduces to σ_{s11} which is equivalent to Barrick's second order result. His result is used widely to extract the wave height directional spectrum by inverting the integral equation.

On the other hand when the radar is based on a ship or on an off shore platform the situation is different. The sea may be fully developed there and thus the contribution by σ_{s22} to the second order cross section may be quite effective. Hence, it may modify the backscattered Doppler spectrum predicted by σ_{s11} alone. In order to estimate the wave height directional spectrum from the measured Doppler spectrum the total expression for the second order cross section has to be inverted. This of course is a problem involving the solution of the relevant expressions as integral equations, and should form the basis for further research.

It may be mentioned that while modifying the results for the back-scattered field (previously derived for a time invariant surface) to include the surface time dependency in section 4.3 we did not include the ground wave attenuation functions. These attenuation functions contain modified surface impedances which involve the surface Fourier coefficients. If they are also included in the modification they may give rise to additional Doppler. This phenomenon should be investigated so that a more complete model for the Doppler return from the ocean surface may be formed.

CHAPTER 5

CONCLUSIONS

A theoretical analysis of electromagnetic scattering from a rough surface is carried out. The analysis is based on Walsh's (1980b) formulation for the scattering from a general time invariant rough surface. The formulation appears in the form of two vector integral equations for the electric field in the two dimensional spatial Fourier transform domain. A non-time varying rough surface with a high refractive index is considered first. Following Rice (1951), it is modelled as a two dimensional periodic surface. The first equation is then formally inverted in a Neumann series to yield a solution for the electric field on the surface in the spatial transform domain. It is shown how the Rice perturbation result for the scattered field may be obtained from this series solution and the second equation when the surface is perfectly conducting. The Neumann series solution is partially summed, again formally, to form another series which is more amenable to physical interpretation. The choice of a finite source is kept arbitrary, thus maintaining the generality of the surface field solution in the spatial transform domain.

The source now assumed is an elementary vertical electric dipole located close to the surface and excited by a pulsed sinusoidal current. To accommodate this excitation, the previous equations are taken in the temporal Fourier transform domain. For a non-pulsed sinusoidal excitation, these may be considered in the standard phasor form. For analytical ease it is assumed that the surface slopes are small compared to unity. For this source, zero, first, and second order approximations of the vertical component of the surface field are derived. These three orders of the solution are then inverse spatially transformed, where we have used first the stationary phase method and then the steepest descent method given by Wait (1970, ch. 2) for evaluating the integrals asymptotically. The inverse transforms for the first and second order solutions are in the form of spatial convolutions.

The zero order solution represents a ground wave propagating outward from the source with a modified surface impedance. The first order solution represents two ground waves, again with modified surface impedances, propagating in different directions due to scattering. Similarly, the second and higher orders of the solution may be interpreted. The modified surface impedances take into account surface roughness, which is to be expected. It may be mentioned that the trapped surface wave phenomenon as discussed by Wait may also be present. Based on the average (statistical) modified surface impedance derived here, Dawe (1984) has made field calculations over the ocean surface. The results show trapped surface waves in some cases depending upon the sea state and the transmitted frequency. Thus it may be noticed that the analysis automatically provides the main features of HF radio wave propagation over a rough surface when using a ground based antenna. Although some investigators [e.g., Wait (1971), Barrick (1971)] have derived expressions for the modified surface impedance for the rough surface, their solutions for the field are not in the form of ground waves because of the assumed plane wave incident field.

Considering the backscattered surface field, the inverse temporal Fourier transforms for the first and second order surface fields are approximately obtained to return to the time domain. The spatial convolution integrals for these two orders of the backscattered field are then evaluated, again asymptotically, by the stationary phase method, assuming a narrow beam receiving antenna. However, the result for each order may be extended for a wide beam or omnidirectional receiving antenna. The time delay between the transmitted and received signals, transmitted pulse width, and the horizontal beam width of the receiving antenna define a patch of the surface from where the first order field is received. In the second order case the fields are received from the patch as well as from other regions of the surface, all arriving at the same time. Thus a pulsed excitation of the source clearly shows that the second order field is not received from the patch alone as opposed to some other analyses where considerations are given to the field received from the patch only.

With reference to the remote sensing of ocean surface conditions using an HF ground wave radar, the ocean surface is modelled as a three dimensional periodic surface in space and time with random Fourier coefficients [Barrick (1972a)]. For this model, the previously derived solutions for the first and second order backscattered surface fields from a time invariant surface are suitably modified to account for the time dependence and the statistical variation of the surface, thus making them more applicable to the ocean. A second order hydrodynamic effect, derived by Weber and Barrick (1977), is also included in the second order solution for the backscattered surface field. For modelling a pulsed radar, the excitation of the source dipole is extended from a single pulsed sinusoid to a periodic pulsed sinusoid. Using this time dependent result and the dispersion relationship for deep water ocean gravity waves, an average first and second order backscattered Doppler spectra (power density spectra) have been derived, again assuming a narrow beam receiving antenna. Consequently the two orders of the back-scattered Doppler frequency dependent cross section of the ocean surface have been derived. These solutions relate to the spatial (directional) spectrum of gravity waves.

The result obtained for the first order backscattered Doppler spectrum or the cross section is the same as derived by Barrick (1972a), whereas this is not so for the second order. The author's second order result consists of three parts. The first part is almost the same as derived by Barrick (1972b, 1977a), and it represents the case where double scattering occurs on the patch only [Srivastava and Walsh (1983)]. The second part corresponds to the case where one scattering occurs at the source and another on the patch. The third part represents the case of off the patch double scattering [Walsh and Srivastava (1984)].

Using an oceanographic model for the waveheight spectrum for a fully developed sea, graphs are presented for a few typical cases. These graphs are in the form of the spectral density variation of the first and second order backscattered cross sections with the Doppler frequency. The first order result shows two large peaks situated at plus and minus the Bragg frequency in accordance with Crombie's (1955) experimental observation. The Bragg frequency is given as $(2gk_o)^{\frac{1}{2}}$ rad/sec, where g is acceleration due to gravity and k_o is the incident radio wave number. These two first order peaks are surrounded by the second order continuum. A study of the individual effects of the three parts of the second order cross section is also made. The results show that, when compared with the first part, the contribution to the second order cross section by the second part is significant near the zero Doppler and at frequencies greater than the corner reflector frequency. The corner reflector effect is produced by the first part and the corresponding Doppler frequency is given by $2^{3/4}$ times the Bragg frequency [Barrick (1972b)]. Because of the second part the total Doppler spectrum decreases very slowly at higher Doppler frequencies. The contribution from the third part is effective only at a few isolated frequencies such as the zero Doppler and near the corner reflector frequency. Moreover, this part may be neglected if the transmitting antenna also is assumed to be a narrow beam one.

It may be mentioned that the above study is made on the assumption that the sea is fully developed in the scattering region including the region surrounding the transmitting antenna. However, in most cases antennas are located on the beach or near the shore. The long ocean waves (depending upon the radio wavelength), which are responsible for the above behaviour of the second part, are very rarely present near the antenna and besides, they can not be considered fully developed. Therefore, under these common situations the second part may not be wholly present. Also, the third part may be neglected. Thus the second order cross section reduces to the first part only which is equivalent to Barrick's second order result. On the other hand when the radar is based on a ship or on an offshore platform the situation is different. The sea may be fully developed there and thus the contribution by the second part may be quite effective. Hence, it may modify the second order backscattered Doppler spectrum predicted by the first part alone.

In conclusion it may be said that the rough surface scattering analysis presented in the thesis is quite general, although a series of approximations have been made to arrive at some useful engineering results. In order to orient the problem towards practical situations the use of a simple dipole source is an obvious choice. This choice provides some interesting phenomenon mentioned above, e.g., i) propagation and scattering solutions for the surface field in the form of ground waves with modified surface impedances, ii) the second order scattering from a surface patch and the other regions of the surface and consequently their effects on the second order cross section of the ocean surface. In these respects the author feels that this work forms an addition to the existing models for EM scattering from rough surfaces.

5.1 Proposed Future Work

The spatial convolution integrals for the first and second order scattered surface fields are evaluated for the backscattered case only. They should be evaluated for an arbitrary observation point also followed by a derivation of the bistatic cross section of the ocean surface in a way similar to that used in deriving the backscattered (monostatic) cross section. This would be useful for the bistatic radar configurations. Also, while modifying the results for the backscattered surface field (previously derived for a time invariant surface) to include the surface time dependency we have not included the attenuation functions. These attenuation functions contain the surface Fourier coefficients through their modified surface impedances. If they are also included in the modification they may give rise to additional Doppler. This phenomenon should be investigated so that a more complete model for the Doppler return from the ocean surface may be formed.

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APPENDIX A

RICE PERTURBATION RESULT

In this section a procedure to obtain the Rice (1951) perturbation result from the relevant equations is described. By using the series solution (2.26) for $\overline{E_t}$ in the scattered field equations (2.12) an expression equivalent to all orders of perturbation may be derived. The complexity however increases with the order. Rice has obtained all three scattered electric field components for both vertical and horizontal polarizations up to the second order. The order of perturbation is equivalent to the order of the product of surface Fourier coefficient $P_{m,n}$. e.g. second order implies terms containing the product of two $P_{m,n}$'s. For comparison with Rice's result, the derivation is carried out up to the second order for the case of vertical polarization. The results for horizontal polarization may similarly be derived. The incident field taken is a plane wave with exp $(j\omega_o t)$ time dependency. Therefore all equations are taken in standard phasor form with ω replaced by ω_o and and k replaced by k_o.

A.1 Simplified Basic Equations

Following Rice we will assume that 1) the surface is perfectly conducting i.e. the refractive index (n_o) tends to infinity; 2) both x and y slopes of the surface and $k_o f(x,y)$ are small compared to unity. We shall also maintain our earlier assumption that the mean level of the surface is zero (i.e. $P_{o,o} = 0$). Under the above assumptions the matrix $T_{m,n}$ given by (2.16) and the scattered field equations (2.12) may be simplified as

$$T_{m,n}(\vec{K}) = \begin{bmatrix} A_{m,n} & -D_{m,n} & jK_x A_{m,n} \\ -D_{m,n} & A_{m,n} & jK_y A_{m,n} \\ B_{m,n} & C_{m,n} & uA_{m,n} \end{bmatrix}. \qquad (A.1)$$

Scattered Field Equations

$$\underline{E_{sx}}(\vec{K}) = \frac{e^{-z^+u}}{2u} \sum_{m,n} [-\alpha_{m,n} \underline{E_{tx}^{m,n}} + \theta_{m,n} \underline{E_{ty}^{m,n}} - jK_x \alpha_{m,n} \underline{E_{az}^{m,n}}] \qquad (A.2a)$$

$$\underline{E_{sy}}(\vec{K}) = \frac{e^{-z^+u}}{2u} \sum_{m,n} [\theta_{m,n} \underline{E_{tx}^{m,n}} - \alpha_{m,n} \underline{E_{ty}^{m,n}} - jK_y \alpha_{m,n} \underline{E_{az}^{m,n}}] \qquad (A.2b)$$

$$\underline{E_{sz}}(\vec{K}) = \frac{e^{-z^+u}}{2u} \sum_{m,n} [-\beta_{m,n} \underline{E_{tx}^{m,n}} - \gamma_{m,n} \underline{E_{ty}^{m,n}} + u\alpha_{m,n} \underline{E_{az}^{m,n}}] \qquad (A.2c)$$

where $A_{m,n}$, $B_{m,n}$, $C_{m,n}$ and $D_{m,n}$ are defined by (2.9) and $\alpha_{m,n}$, $\beta_{m,n}$, $\gamma_{m,n}$, and $\theta_{m,n}$ by (2.10). The superscripts m and n identify negative shifts in K_x and K_y by mN and nN respectively as defined by (2.13).

For convenience we rewrite the series solution (2.26) for $\underline{\overline{E_t}}$ as

$$\underline{\overline{E_t}} = \underline{\overline{E_{ci}}} - L_{p,q} \underline{\overline{E_{ci}^{p,q}}} + L_{r,s} L_{p-r,q-s}^{r,s} \underline{\overline{E_{ci}^{p,q}}}$$

$$(p,q) \neq (0,0) \quad (r,s) \neq \begin{Bmatrix} (0,0) \\ (p,q) \end{Bmatrix}$$

$$- L_{v,w} L_{r-v,s-w}^{v,w} L_{p-r,q-s}^{r,s} \underline{\overline{E_{ci}^{p,q}}} + \ldots \qquad (A.3)$$

$$(v,w) \neq \begin{Bmatrix} (0,0) \\ (r,s) \end{Bmatrix} \quad (r,s) \neq (p,q)$$

where the summations with respect to indices p, q, r, s, v and w with proper restrictions are implied. With a little algebra the expression for $\overline{E_t}$ with a general shift mN and nN in K_x and K_y may be given as

$$\overline{E_t^{m,n}} = \overline{E_{ci}^{m,n}} - L_{p-m,q-n}^{m,n}\,\overline{E_{ci}^{p,q}} + L_{r-m,s-n}^{m,n}\,L_{p-r,q-s}^{r,s}\,\overline{E_{ci}^{p,q}}$$

$$(p,q) \neq (m,n) \qquad (r,s) \neq \left\{\begin{matrix}(m,n)\\(p,q)\end{matrix}\right\}$$

$$- L_{v-m,w-n}^{m,n}\,L_{r-v,s-w}^{v,w}\,L_{p-r,q-s}^{r,s}\,\overline{E_{ci}^{p,q}} + \ldots \tag{A.4}$$

$$(v,w) \neq \left\{\begin{matrix}(m,n)\\(r,s)\end{matrix}\right\} \quad (r,s) \neq (p,q)$$

A.2 <u>Expansion Of Auxiliary Fourier Coefficients</u>

From equation (2.9) we have

$$\exp(-fu) = \sum_{m,n} A_{m,n} \exp[jN(mx + ny)] \quad . \tag{A.5}$$

By expanding the function exp(−fu) as

$$\exp(-fu) \cong 1 - uf + \frac{u^2}{2} f^2,$$

where the terms up to the second order only have been kept, and by using the Fourier series expansion (2.7) for f we get

$$\exp(-fu) = 1 - u \sum_{m,n} P_{m,n} \exp[jN(mx + ny)]$$

$$+ \frac{u^2}{2} \sum_{m,n} \sum_{p,q} P_{p,q}\, P_{m-p,n-q} \exp[jN(mx + ny)] \quad .$$

Locating the coefficient of exp[jN(mx + ny)] yields

$$A_{m,n} = \delta_{m,n} - uP_{m,n} + \frac{u^2}{2} \sum_{p,q} P_{p,q}\, P_{m-p,n-q} \tag{A.6}$$

where $\delta_{m,n}$ is defined as

$$\delta_{m,n} = \begin{cases} 1 \text{ for } (m,n) = (0,0) \\ 0 \text{ for } (m,n) \neq (0,0) \end{cases} \qquad (A.7)$$

By following the above procedure the other auxiliary Fourier coefficients may also be expanded. By doing this expansion and making a general shift vN and wN in K_x and K_y respectively we have

$$A^{v,w}_{m,n} = \delta_{m,n} - u^{v,w} P_{m,n} + \frac{(u^{v,w})^2}{2} \sum_{p,q} P_{p,q} P_{m-p,n-q}$$

$$B^{v,w}_{m,n} = jmN\ P_{m,n} - ju^{v,w} \sum_{p,q} pN\ P_{p,q} P_{m-p,n-q}$$

$$C^{v,w}_{m,n} = jnN\ P_{m,n} - ju^{v,w} \sum_{p,q} qN\ P_{p,q} P_{m-p,n-q}$$

$$D^{v,w}_{m,n} = -\sum_{p,q} q(m-p)\ N^2 P_{p,q} P_{m-p,n-q} \qquad (A.8)$$

$$\alpha^{v,w}_{m,n} = \delta_{m,n} + u^{v,w} P_{m,n} + \frac{(u^{v,w})^2}{2} \sum_{p,q} P_{p,q} P_{m-p,n-q}$$

$$\beta^{v,w}_{m,n} = jmN\ P_{m,n} + ju^{v,w} \sum_{p,q} pN\ P_{p,q} P_{m-p,n-q}$$

$$\gamma^{v,w}_{m,n} = jnN\ P_{m,n} + ju^{v,w} \sum_{p,q} qN\ P_{p,q} P_{m-p,n-q}$$

$$\theta^{v,w}_{m,n} = -\sum_{p,q} q(m-p)\ N^2 P_{p,q} P_{m-p,n-q}$$

In (A.8) we have carried the expansion up to the second order only and $u^{v,w}$ is given as

$$u^{v,w} = \begin{cases} [(K_x - vN)^2 + (K_y - wN)^2 - k_o^2]^{\frac{1}{2}} & ; \text{ for real root} \\ j[k_o^2 - (K_x - vN)^2 - (K_y - wN)^2]^{\frac{1}{2}} & ; \text{ for imaginary root.} \end{cases} \qquad (A.9)$$

It may be seen than $B^{v,w}_{o,o}$, $C^{v,w}_{o,o}$, $\beta^{v,w}_{o,o}$, and $\gamma^{v,w}_{o,o}$ are identically zero due to the periodicity of the surface.

A.3 Incident Field and its Spatial Transform

We consider now that the incident field is a vertically polarized plane wave. The direction cosines are taken to be $\sin\theta, 0,$ and $-\cos\theta$ where θ is the angle of incidence measured from the z axis. Therefore componentwise the incident field may be given as

$$\begin{aligned} E_{ix} &= \cos\theta \exp(-jK_{xo}x + u_o z) \\ E_{iy} &= 0 \\ E_{iz} &= \sin\theta \exp(-jK_{xo}x + u_o z) \end{aligned} \tag{A.10}$$

where

$$\begin{aligned} K_{xo} &= k_o \sin\theta \\ u_o &= (K^2_{xo} - k^2_o)^{\frac{1}{2}} = jk_o \cos\theta \\ k_o &= \omega_o \sqrt{\mu_o \varepsilon_o} \end{aligned} \tag{A.11}$$

By using (1.9) the spatial Fourier transform of the above incident field in a plane $z = z^-$ may easily be evaluated to yield

$$\begin{aligned} \underline{E_{ix}} &= 4\pi^2 \cos\theta \exp(u_o z^-)\ \delta(K_x + K_{xo})\ \delta(K_y) \\ \underline{E_{iy}} &= 0 \\ \underline{E_{iz}} &= 4\pi^2 \sin\theta \exp(u_o z^-)\ \delta(K_x + K_{xo})\ \delta(K_y) \end{aligned} \tag{A.12}$$

where K_x and K_y are the spatial transform variables and $\delta(K_x)$ is the Dirac delta function of argument K_x. By using (A.12) in (2.4) we obtain

$$\underline{E_{xi}} = jk_o\, 8\pi^2 \cos^2\theta\, \delta(K_x + K_{xo})\, \delta(K_y)$$

$$\underline{E_{yi}} = 0 \qquad \text{(A.13)}$$

$$\underline{E_{zi}} = jk_o\, 8\pi^2 \sin\theta \cos\theta\, \delta(K_x + K_{xo})\, \delta(K_y)\ .$$

In obtaining the above equation we have used the following property of the delta function [Papoulis (1962, app I)]:

$$g(x,y)\, \delta(x - x_o)\, \delta(y - y_o) = g(x_o, y_o)\, \delta(x - x_o)\, \delta(y - y_o). \qquad \text{(A.14)}$$

For arbitrary shifts vN and wN in K_x and K_y respectively, the column vector $\overline{\underline{E_{oi}^{v,w}}}$ defined by (2.17) may then be written as

$$\overline{\underline{E_{oi}^{v,w}}} = jk_o\, 8\pi^2 \cos\theta \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \end{bmatrix} \delta(K_x - vN + K_{xo})\, \delta(K_y - wN)\ . \qquad \text{(A.15)}$$

A.4 Reduction of Series Solution (A.4) in Terms of $P_{m,n}$'s

From (A.1), the matrix $T_{m,n}$ with arbitrary shifts vN and wN may be given as

$$T_{m,n}(K_x - vN,\, K_y - wN) = T_{m,n}^{v,w}$$

$$= \begin{bmatrix} A_{m,n}^{v,w} & -D_{m,n}^{v,w} & j(K_x - vN)A_{m,n}^{v,w} \\ -D_{m,n}^{v,w} & A_{m,n}^{v,w} & j(K_y - wN)A_{m,n}^{v,w} \\ B_{m,n}^{v,w} & C_{m,n}^{v,w} & u^{v,w}\, A_{m,n}^{v,w} \end{bmatrix} \qquad \text{(A.16)}$$

For m = n = 0,

$$T^{v,w}_{o,o} = \begin{bmatrix} A^{v,w}_{o,o} & -D^{v,w}_{o,o} & j(K_x - vN)A^{v,w}_{o,o} \\ -D^{v,w}_{o,o} & A^{v,w}_{o,o} & j(K_y - wN)A^{v,w}_{o,o} \\ 0 & 0 & u^{v,w} A^{v,w}_{o,o} \end{bmatrix}$$

The matrix $T^{v,w}_{o,o}$ may be inverted as

$$(T^{v,w}_{o,o})^{-1} = \frac{1}{u^{v,w} A^{v,w}_{o,o} [(A^{v,w}_{o,o})^2 - (D^{v,w}_{o,o})^2]}$$

$$\begin{bmatrix} (A^{v,w}_{o,o})^2 u^{v,w} & A^{v,w}_{o,o} D^{v,w}_{o,o} u^{v,w} & -jA^{v,w}_{o,o}[(K_x - vN)A^{v,w}_{o,o} + (K_y - wN)D^{v,w}_{o,o}] \\ A^{v,w}_{o,o} D^{v,w}_{o,o} u^{v,w} & (A^{v,w}_{o,o})^2 u^{v,w} & -jA^{v,w}_{o,o}[(K_x - vN)D^{v,w}_{o,o} + (K_y - wN)A^{v,w}_{o,o}] \\ 0 & 0 & (A^{v,w}_{o,o})^2 - (D^{v,w}_{o,o})^2 \end{bmatrix} \tag{A.17}$$

Since we have assumed that the surface slopes are small, this implies

$$(A^{v,w}_{o,o})^2 - (D^{v,w}_{o,o})^2 \cong (A^{v,w}_{o,o})^2 \tag{A.18}$$

Further, under the small height condition, from (A.8)

$$(A^{v,w}_{o,o})^{-1} \cong 1 - \frac{(u^{v,w})^2}{2} \sum_{p,q} |P_{p,q}|^2 \tag{A.19}$$

Let us define

$$a^{v,w} = 1 - \frac{(u^{v,w})^2}{2} \sum_{p,q} |P_{p,q}|^2$$

$$d = D^{v,w}_{o,o} = \sum_{p,q} pq\, N_1^2\, |P_{p,q}|^2 . \tag{A.20}$$

By using (A.18) to (A.20) the inverse of the matrix $T^{v,w}_{o,o}$ (A.17) may be approximated as

$$(T_{o,o}^{v,w})^{-1} \cong \begin{bmatrix} a^{v,w} & d & -\frac{j}{u^{v,w}}[(K_x - vN)a^{v,w} + (K_y - wN)d] \\ d & a^{v,w} & -\frac{j}{u^{v,w}}[(K_x - vN)d + (K_y - wN)a^{v,w}] \\ 0 & 0 & \frac{a^{v,w}}{u^{v,w}} \end{bmatrix}. \quad (A.21)$$

In the above the elements have been written up to second order terms only.

For convenience let us rewrite the series solution (A.4) with summation symbols as

$$\overline{\underline{E_t^{m,n}}} = \overline{\underline{E_{ci}^{m,n}}} - \sum_{(p,q) \neq (m,n)} L_{p-m,q-n}^{m,n} \overline{\underline{E_{ci}^{p,q}}}$$

$$+ \sum_{p,q} \sum_{(r,s) \neq \left\{\begin{matrix}(p,q)\\(m,n)\end{matrix}\right\}} L_{r-m,s-n}^{m,n} L_{p-r,q-s}^{r,s} \overline{\underline{E_{ci}^{p,q}}} - \ldots \quad (A.22)$$

where from (2.21)

$$L_{p-m,q-n}^{m,n} = (T_{o,o}^{m,n})^{-1} T_{p-m,q-n}^{m,n}$$

$$\overline{\underline{E_{ci}^{m,n}}} = (T_{o,o}^{m,n})^{-1} \overline{\underline{E_{oi}^{m,n}}} \quad (A.23)$$

By inspection of (A.22) it is found that the first term of this series contains zero and higher order terms in the $P_{m,n}$'s. The second term contains first and higher orders. In a similar way the third term starts with second order and so on. As we are interested in terms up to the second order we will not go beyond the third term of the series. Further, in each of the first, second, and third terms of the series the various expressions will be written up to the second order only in the $P_{m,n}$'s.

For simplicity let us write $\overline{E_t^{m,n}}$ as

$$\overline{E_t^{m,n}} = \overline{E_{t1}^{m,n}} + \overline{E_{t2}^{m,n}} + \overline{E_{t3}^{m,n}} \tag{A.24}$$

where $\overline{E_{t1}^{m,n}}$, $\overline{E_{t2}^{m,n}}$, and $\overline{E_{t3}^{m,n}}$ are respectively the first three terms of the corresponding series (A.22). We will now expand each of the above three terms to the second order in $P_{m,n}$'s.

First Term

$$\overline{E_{t1}^{m,n}} = \overline{E_{ci}^{m,n}} = (T_{o,o}^{m,n})^{-1} \, \overline{E_{oi}^{m,n}} \tag{A.25}$$

By using (A.14), (A.15), and (A.21) the above equation may be written as

$$\overline{E_{t1}^{m,n}} = jk_o \, 8\pi^2 \begin{bmatrix} a^{m,n} \\ d \\ -\frac{j}{k_o} \sin\theta \, a^{m,n} \end{bmatrix} \delta(K_x - mN + K_{xo}) \, \delta(K_y - nN),$$

or by using (A.20),

$$\overline{E_{t1}^{m,n}} = jk_o \, 8\pi^2 \begin{bmatrix} 1 - \frac{(u^{m,n})^2}{2} \sum\limits_{p,q} |P_{p,q}|^2 \\ \sum\limits_{p,q} pq \, N^2 \, |P_{p,q}|^2 \\ -\frac{j}{k_o} \sin\theta + \frac{j}{k_o} \sin\theta \, \frac{(u^{m,n})^2}{2} \sum\limits_{p,q} |P_{p,q}|^2 \end{bmatrix}$$

$$\cdot \, \delta(K_x - mN + K_{xo}) \, \delta(K_y - nN) \tag{A.26}$$

Second Term

$$\overline{E^{m,n}_{t2}} = -\sum_{(p,q) \neq (m,n)} L^{m,n}_{p-m,q-n} \overline{E^{p,q}_{ci}} \tag{A.27}$$

By using (A.23), (A.21), and (A.16) the matrix $L^{m,n}_{p-m,q-n}$ may be evaluated to the second order as

$$L^{m,n}_{\substack{p-m,q-n \\ (p,q) \neq (m,n)}} =$$

$$\begin{bmatrix} A^{m,n}_{p-m,q-n} - \frac{j(K_x - mN)}{u^{m,n}} B^{m,n}_{p-m,q-n} & -D^{m,n}_{p-m,q-n} - \frac{j(K_x - mN)}{u^{m,n}} C^{m,n}_{p-m,q-n} & 0 \\ -D^{m,n}_{p-m,q-n} - \frac{j(K_y - nN)}{u^{m,n}} B^{m,n}_{p-m,q-n} & A^{m,n}_{p-m,q-n} - \frac{j(K_y - nN)}{u^{m,n}} C^{m,n}_{p-m,q-n} & 0 \\ \frac{B^{m,n}_{p-m,q-n}}{u^{m,n}} & \frac{C^{m,n}_{p-m,q-n}}{u^{m,n}} & A^{m,n}_{p-m,q-n} \end{bmatrix} \tag{A.28}$$

In the above the auxiliary Fourier coefficients are to be expanded up to the second order with the help of (A.8). However, in this expansion the zeroth order term should not be taken as $(p,q) \neq (m,n)$. By performing this operation and then multiplying the matrix by the column vector $\overline{E^{p,q}_{ci}}$ [given by (A.25) and (A.26)] we arrive at

$$\overline{E^{m,n}_{t2}} = jk_o 8\pi^2 \sum_{(p,q) \neq (m,n)} \begin{bmatrix} u^{m,n} - \frac{(K_x - mN)(p - m)N}{u^{m,n}} \\ -\frac{(K_y - nN)(p - m)N}{u^{m,n}} \\ -\frac{j}{k_o} \sin\theta \, u^{m,n} - j\frac{(p - m)N}{u^{m,n}} \end{bmatrix} \cdots$$

$$\cdot P_{p-m,q-n}\ \delta(K_x - pN + K_{xo})\ \delta(K_y - qN)$$

$$+ jk_o\ 8\pi^2 \sum_{(p,q) \neq (m,n)} \sum_{r,s} \left[\begin{array}{l} -\frac{(u^{m,n})^2}{2} + (K_x - mN)(r - m)N \\ - (s - n)(p - r)N^2 + (K_y - nN)(r - m)N \\ \frac{j}{k_o} \sin\theta \frac{(u^{m,n})^2}{2} + j(r - m)N \end{array} \right]$$

$$\cdot P_{r-m,s-n}\ P_{p-r,q-s}\ \delta(K_x - pN + K_{xo})\ \delta(K_y - qN) . \qquad \text{(A.29)}$$

The restriction on summation indices $(p,q) \neq (m,n)$ may easily be removed in the first part of the above equation i.e. the part with single double summation. This is because of the assumption $P_{o,o} = 0$. For the second part we may write

$$\sum_{(p,q) \neq (m,n)} \sum_{r,s} = \sum_{p,q} \sum_{r,s} - \sum_{r,s} \qquad \text{(A.30)}$$

$$[\text{with } (p,q) = (m,n)]$$

By using (A.30) and the identity

$$\sum_{r,s} (r - m)\ |P_{r-m,s-n}|^2 = \sum_{r,s} r\ |P_{r,s}|^2 = 0 , \qquad \text{(A.31)}$$

the equation for $\overline{E^{m,n}_{t2}}$ may be rewritten as

$$\overline{E^{m,n}_{t2}} = jk_o\ 8\pi^2 \sum_{p,q} \left[\begin{array}{l} u^{m,n} - \frac{(K_x - mN)(p - m)N}{u^{m,n}} \\ - (K_y - nN)(p - m) \frac{N}{u^{m,n}} \\ - \frac{j}{k_o} \sin\theta\ u^{m,n} - j \frac{(p - m)N}{u^{m,n}} \end{array} \right]$$

$$\cdot P_{p-m,q-n}\ \delta(K_x - pN + K_{xo})\ \delta(K_y - qN)$$

$$+ jk_o\ 8\pi^2 \sum_{p,q} \sum_{r,s} \begin{bmatrix} \frac{-(u^{m,n})^2}{2} + (K_x - mN)(r - m)N \\ - (s - n)(p - r)N^2 + (K_y - nN)(r - m)N \\ \frac{j}{k_o} \sin\theta \frac{(u^{m,n})^2}{2} + j(r - m)N \end{bmatrix}$$

$$\cdot P_{r-m,s-n} P_{p-r,q-s}\ \delta(K_x - pN + K_{xo})\ \ \delta(K_y - qN)$$

$$+ jk_o\ 8\pi^2 \sum_{r,s} \begin{bmatrix} \frac{(u^{m,n})^2}{2} \\ - rsN^2 \\ - \frac{j}{k_o} \sin\theta\ \frac{(u^{m,n})^2}{2} \end{bmatrix} |P_{r,s}|^2$$

$$\cdot\ \delta(K_x - mN + K_{xo})\ \ \delta(K_y - nN). \tag{A.32}$$

Third Term

$$\overline{E_{t3}^{m,n}} = \sum_{p,q} \sum_{(r,s) \neq \left\{ \begin{matrix} (p,q) \\ (m,n) \end{matrix} \right\}} L_{r-m,s-n}^{m,n}\ L_{p-r,q-s}^{r,s}\ \overline{E_{ci}^{p,q}} \tag{A.33}$$

A procedure similar to that used to derive $\overline{E_{t2}^{m,n}}$ (A.31) leads to the following expression for $\overline{E_{t3}^{m,n}}$. The relevant equations used are (A.8), (A.25), (A.26), and (A.28).

$$\overline{E^{m,n}_{t3}} = \sum_{p,q}\sum_{r,s} \begin{bmatrix} \{[u^{r,s} - \frac{(K_x - rN)(p-r)N}{u^{r,s}}][u^{m,n} - \frac{(K_x - mN)(r-m)N}{u^{m,n}}] \\ + (K_x - mN)(K_y - sN)(p-r)(s-n)\frac{N^2}{u^{m,n}\,u^{r,s}}\} \\ \{-[u^{r,s} - \frac{(K_x - rN)(p-r)N}{u^{r,s}}](K_y - nN)(r-m)\frac{N}{u^{m,n}} \\ -[u^{m,n} - \frac{(K_y - nN)(s-n)N}{u^{m,n}}](K_y - sN)\frac{(p-r)N}{u^{r,s}}\} \\ \{-j[u^{r,s} - \frac{(K_x - rN)(p-r)N}{u^{r,s}}]\frac{(r-m)N}{u^{m,n}} - \frac{j}{k_o}\sin\theta\, uu^{r,s} \\ + j(K_y - sN)(p-r)(s-n)\frac{N^2}{u^{m,n}\,u^{r,s}} - j(p-r)N\frac{u^{m,n}}{u^{r,s}}\} \end{bmatrix}$$

$$\cdot\, jk_o\, 8\pi^2\, P_{r-m,s-n}\, P_{p-r,q-s}\, \delta(K_x - pN + K_{xo})\, \delta(K_y - qN) \quad (A.34)$$

In the above the restrictions $(r,s) \neq (p,q)$ and (m,n) have been removed as $P_{o,o} = 0$.

The above three column vectors (A.26), (A.32), and (A.34) may be summed to give $\overline{E^{m,n}_{t}}$ to second order. By using (2.17) this may be written componentwise as

$$\underline{E^{m,n}_{tx}} = jk_o\, 8\pi^2\, \delta(K_x - mN + K_{xo})\, \delta(K_y - nN)$$

$$+ jk_o\, 8\pi^2 \sum_{p,q} [u^{m,n} - (K_x - mN)(p-m)N\frac{1}{u^{m,n}}]P_{p-m,q-n}\, \delta(K_x - pN + K_{xo})$$

$$\cdot\, \delta(K_y - qN)$$

$$+ jk_o\, 8\pi^2 \sum_{p,q}\sum_{r,s} [-\frac{(u^{m,n})^2}{2} + (K_x - mN)(r-m)N + \{u^{r,s} - (K_x - rN)$$

$$\cdot\, \frac{(p-r)N}{u^{r,s}}\}$$

$$\cdot \{u^{m,n} - (K_x - mN)(r - m)\frac{N}{u^{m,n}}\}$$

$$+ (K_x - mN)(K_y - sN)(p - r)(s - n)\frac{N^2}{u^{m,n}\,u^{r,s}}]$$

$$\cdot P_{r-m,s-n}\,P_{p-r,q-s}\,\delta(K_x - pN + K_{xo})\,\delta(K_y - qN) \qquad \text{(A.35a)}$$

$$\underline{E^{m,n}_{ty}} = -jk_o\,8\pi^2\sum_{p,q}(K_y - nN)(p - m)\frac{N}{u^{m,n}}P_{p-m,q-n}\,\delta(K_x - pN + K_{xo})\delta(K_y - qN)$$

$$+ jk_o\,8\pi^2\sum_{p,q}\sum_{r,s}[(K_y - nN)(r - m)N - (s - n)(p - r)N^2$$

$$-\{u^{r,s} - (K_x - rN)(p - r)\frac{N}{u^{r,s}}\}(K_y - nN)(r - m)\frac{N}{u^{m,n}}$$

$$-\{u^{m,n} - (K_y - nN)(s - n)\frac{N}{u^{m,n}}\}(K_y - sN)(p - r)\frac{N}{u^{r,s}}]$$

$$\cdot P_{r-m,s-n}\,P_{p-r,q-s}\,\delta(K_x - pN + K_{xo})\,\delta(K_y - qN)$$

$$\text{(A.35b)}$$

$$\underline{E^{m,n}_{az}} = 8\pi^2\sin\theta\,\delta(K_x - mN + K_{xo})\delta(K_y - nN)$$

$$+ k_o\,8\pi^2\sum_{p,q}[\sin\theta\,\frac{u^{m,n}}{k_o} + (p - m)\frac{N}{u^{m,n}}]P_{p-m,q-n}\,\delta(K_x - pN + K_{xo})\delta(K_y - qN)$$

$$+ k_o\,8\pi^2\sum_{p,q}\sum_{r,s}[-\sin\theta\,\frac{(u^{m,n})^2}{2k_o} - (r - m)N + \sin\theta\,\frac{uu^{r,s}}{k_o}$$

$$+ \{u^{r,s} - (K_x - rN)\frac{(p - r)N}{u^{r,s}}\}\,\frac{(r - m)N}{u^{m,n}}$$

$$- (K_y - sN)(p - r)(s - n)\frac{N^2}{u^{m,n}\,u^{r,s}} + (p - r)N\frac{u^{m,n}}{u^{r,s}}]$$

$$\cdot P_{r-m,s-n}\,P_{p-r,q-s}\,\delta(K_x - pN + K_{xo})\delta(K_y - qN)\,.$$

$$\text{(A.35c)}$$

A.5 Scattered Field

In the previous section we have obtained a series solution up to the second order for $E_{tx}^{m,n}$, $E_{ty}^{m,n}$, and $E_{az}^{m,n}$ for the case of a perfectly conducting surface. The incident electrical field taken is a vertically polarized plane wave. This solution may now be used in (A.2) to derive the scattered field in the spatial transform domain. By taking the inverse spatial transform the scattered field in (x,y,z) domain may thus be obtained. In this manner we now proceed for each component of the scattered field.

A.5.1 x Component

From (A.2a) the x component of the scattered field may be written as

$$E_{sx}(\vec{K}) = -\frac{e^{-z^{+}u}}{2u} \sum_{m,n} [\alpha_{m,n}(E_{tx}^{m,n} + jK_x E_{az}^{m,n}) - \theta_{m,n} E_{ty}^{m,n}] \tag{A.36}$$

For convenience we introduce another subscript "i" (i = 0,1,2) for E_{tx}, E_{ty}, and E_{az} as E_{txi}, E_{tyi}, and E_{azi}. This subscript indicates the order in the respective solutions given by (A.35). For example E_{txo} means the zeroth order solution for E_{tx} in (A.35a). Similarly E_{tx2} means the second order solution. By using this notation and expanding the auxiliary Fourier coefficient $\alpha_{m,n}$ and $\theta_{m,n}$ the following expression for E_{sx} up to the second order may be derived.

$$\underline{E_{sx}}(\vec{K}) = -\frac{e^{-z^+u}}{2u_0}[(\underline{E_{txo}} + jK_x \underline{E_{azo}})$$
$$+ (\underline{E_{tx1}} + jK_x \underline{E_{az1}}) + u \sum_{m,n} P_{m,n}(\underline{E^{m,n}_{txo}} + jK_x \underline{E^{m,n}_{azo}})$$
$$+ (\underline{E_{tx2}} + jK_x \underline{E_{az2}}) + u \sum_{m,n} P_{m,n}(\underline{E^{m,n}_{tx1}} + jK_x \underline{E^{m,n}_{az1}})$$
$$+ \frac{u^2}{2} \sum_{m,n} \sum_{p,q} P_{p,q} P_{m-p,n-q} (\underline{E^{m,n}_{txo}} + jK_x \underline{E^{m,n}_{azo}})] \qquad (A.37)$$

In the above we have neglected the contribution from $\theta_{m,n} \underline{E^{m,n}_{ty}}$ as the minimum order of this term is three. By using (A.35) and performing some algebra we arrive at

$$\underline{E_{sx}}(\vec{K}) = -4\pi^2 \cos\theta \exp(-z^+u_o)\, \delta(K_x + K_{xo})\, \delta(K_y)$$
$$-j8\pi^2 \sum_{m,n} \exp(-z^+u_o^{m,n})[\{k_o - \sin\theta\,(K_{xo} - mN)\}P_{m,n}$$
$$+ \sum_{p,q} [k_o p(m-p)N^2 - \{k_o - \sin\theta\,(K_{xo} - mN)\}(u_o^{p,q})^2]$$
$$\cdot \frac{P_{p,q} P_{m-p,n-q}}{u_o^{p,q}}]\, \delta(K_x - mN + K_{xo})\, \delta(K_y - nN) \qquad (A.38)$$

where

$$K_{xo} = k_o \sin\theta$$

$$u_o^{m,n} = \begin{cases} [(K_{xo} - mN)^2 + (nN)^2 - k_o^2]^{\frac{1}{2}} & ;\ \text{for real root} \\ j[k_o^2 - (K_{xo} - mN)^2 - (nN)^2]^{\frac{1}{2}} & ;\ \text{for imaginary root.} \end{cases} \qquad (A.39)$$

The above equation (A.38) may be inverse spatially transformed to give the x component of the scattered field in the (x,y,z) domain i.e. E_{sx}.

This inverse transformation may easily be achieved because the expression contains two Dirac delta functions for K_x and K_y. By using the definition of the inverse spatial transform (1.9) E_{sx} may be written as follows:

$$E_{sx} = -\cos\theta \exp[-jK_{xo}x - z^+u_o] - 2j \sum_{m,n} \exp[-j(K_{xo} - mN)x + jnNy - z^+u_o^{m,n}] \cdot [\{k_o - \sin\theta(K_{xo} - mN)\} P_{m,n} + \sum_{p,q} [k_o p(m-p)N^2 + \{k_o - \sin\theta\,(K_{xo} - mN)\}(u_o^{p,q})^2] \frac{P_{p,q} P_{m-p,n-q}}{u_o^{p,q}}] \quad (A.40)$$

A.5.2 y Component

From (A.2b) the y component of the scattered field may be written as

$$\underline{E_{sy}}(\vec{K}) = -\frac{e^{-z^+u}}{2u} \sum_{m,n} [\alpha_{m,n} (\underline{E^{m,n}_{ty}} + jK_y \underline{E^{m,n}_{az}}) - \theta_{m,n} \underline{E^{m,n}_{tx}}] . \quad (A.41)$$

A procedure similar to that used for x component leads to following expression for $\underline{E_{sy}}(\vec{K})$.

$$\underline{E_{sy}}(\vec{K}) = -\frac{e^{-z^+u}}{2u} [(\underline{E_{tyo}} + jK_y \underline{E_{azo}}) + (\underline{E_{ty1}} + jK_y \underline{E_{az1}}) + u \sum_{m,n} P_{m,n} (\underline{E^{m,n}_{tyo}} + jK_y \underline{E^{m,n}_{azo}}) + (\underline{E_{ty2}} + jK_y \underline{E_{az2}}) + u \sum_{m,n} P_{m,n} (\underline{E^{m,n}_{ty1}} + jK_y \underline{E^{m,n}_{az1}}) + \sum_{m,n} \sum_{p,q} P_{p,q} P_{m-p,n-q} (\frac{u^2}{2} \underline{E^{m,n}_{tyo}} + jK_y \frac{u^2}{2} \underline{E^{m,n}_{azo}} + q(m-p)N^2 \underline{E^{m,n}_{txo}})] \quad (A.42)$$

Again using (A.35) and carrying out the algebraic detail results in the following:

$$E_{sy}(\vec{K}) = -j\,8\pi^2 \sum_{m,n} \exp(-z^+ u_o^{m,n}) \, [nN \sin\theta \, P_{m,n} + \sum_{p,q} \{k_o p(n-q)N^2 + nN \sin\theta \, (u_o^{p,q})^2\} \cdot \frac{P_{p,q} \, P_{m-p,n-q}}{u_o^{p,q}}] \; \delta(K_x - mN + K_{xo}) \, \delta(K_y - nN) \tag{A.43}$$

The above may be inverse spatially transformed as

$$E_{sy} = -2j \sum_{m,n} \exp[-j(K_{xo} - mN)x + jnNy - z^+ u_o^{m,n}] \cdot [nN \sin\theta \, P_{m,n} + \sum_{p,q} \{k_o p(n-q)N^2 + nN \sin\theta \, (u_o^{p,q})^2\} \cdot \frac{P_{p,q} \, P_{m-p,n-q}}{u_o^{p,q}}] \tag{A.44}$$

A.5.3 z Component

From (A.2c) the z component of the scattered field may be given as

$$E_{sz}(\vec{K}) = \frac{e^{-z^+ u}}{2u} \sum_{m,n} [u\alpha_{m,n} E_{az}^{m,n} - \beta_{m,n} E_{tx}^{m,n} - \gamma_{m,n} E_{ty}^{m,n}] \tag{A.45}$$

By applying the same procedure as for the other two components the above may be rewritten as follows:

$$E_{sz}(\vec{K}) = \frac{e^{-z^+ u}}{2u} \, [uE_{azo} + \tilde{u}E_{az1} + \sum_{m,n} P_{m,n}(u^2 E_{azo}^{m,n} - jmN \, E_{txo}^{m,n} - jnN \, E_{tyo}^{m,n}) + uE_{az2} + \sum_{m,n} P_{m,n} \, (u^2 E_{az1}^{m,n} - jmN \, E_{tx1}^{m,n} - jnN \, E_{ty1}^{m,n})$$

$$+ u \sum_{m,n} \sum_{p,q} P_{p,q} P_{m-p,n-q} \left(\frac{u^2}{2} \underline{E}^{m,n}_{azo} - jpN \underline{E}^{m,n}_{txo} - jqN \underline{E}^{m,n}_{tyo}\right)] . \qquad (A.46)$$

Equation (A.35) may be used in the above and algebraic simplifications carried out. The final expression then is as follows;

$$\underline{E_{sz}}(\vec{K}) = 4\pi^2 \sin\theta \exp(-z^+u_o) \, \delta(K_x + K_{xo}) \, \delta(K_y)$$

$$+ 8\pi^2 \sum_{m,n} \frac{1}{u_o^{m,n}} \exp(-z^+u_o^{m,n}) \, [\{k_o mN + \sin\theta \, (u_o^{m,n})^2\} P_{m,n}$$

$$- \sum_{p,q} [k_o pN^2\{(m-p)(K_{xo} - mN) - n(n-q)N\}$$

$$- \{(K_{xo} - mN)^2 \sin\theta + n^2N^2 \sin\theta - k_o(K_{xo} - mN)\} \, (u_o^{p,q})^2]$$

$$\cdot \frac{P_{p,q} P_{m-p,n-q}}{u_o^{p,q}}] \, \delta(K_x - mN + K_{xo}) \, \delta(K_y - nN) \qquad (A.47)$$

Then taking the inverse spatial transform gives

$$E_{sz} = \sin\theta \exp[-jK_{xo}x - z^+u_o]$$

$$+ 2 \sum_{m,n} \frac{1}{u_o^{m,n}} \exp[-j(K_{xo} - mN)x + jnNy - z^+u_o^{m,n}]$$

$$\cdot [\{k_o mN + \sin\theta \, (u_o^{m,n})^2\} P_{m,n}$$

$$- \sum_{p,q} [k_o pN^2\{(m-p)(K_{xo} - mN) - n(n-q)N\}$$

$$- \{(K_{xo} - mN)^2 \sin\theta + n^2N^2 \sin\theta - k_o(K_{xo} - mN)\} \, (u_o^{p,q})^2]$$

$$\cdot \frac{P_{p,q} P_{m-p,n-q}}{u_o^{p,q}}] . \qquad (A.48)$$

Thus the cartesian components of the scattered field for the case of vertical polarization are given by (A.40), (A.44), and (A.48). It may easily be verified that this result is the same as derived by Rice (1951) when written in his notation. Clearly the result for the vertical component breaks down at grazing, i.e. when $u_o^{m,n} = 0$, as pointed out by Rice. However, in this special case he has given a correcting modification using a surface wave formulation in the perturbation technique.

APPENDIX B

PARTIAL SUMMATION OF THE SERIES (2.26)

In this appendix a procedure is described for partially summing the series (2.26). This series is given as

$$\underline{\overline{E_t}} = \underline{\overline{E_{ci}}} + [- \sum_{(m,n) \neq (o,o)} L_{m,n} + \sum_{(p,q) \neq \left\{\begin{matrix}(o,o)\\(m,n)\end{matrix}\right.} L_{p,q} L^{p,q}_{m-p,n-q}$$

$$- \sum_{(r,s) \neq \left\{\begin{matrix}(o,o)\\(p,q)\end{matrix}\right.} \sum_{(p,q) \neq (m,n)} L_{r,s} L^{r,s}_{p-r,q-s} L^{p,q}_{m-p,n-q} + \ldots] \underline{\overline{E^{m,n}_{ci}}} \qquad (B.1)$$

For convenience we change the summation indices m,n, p,q, r, and s to m_2, n_2, m_3, n_3, m_4, and n_4 respectively. Therefore we have

$$\underline{\overline{E_t}} = \underline{\overline{E_{ci}}} + [- \sum_{(m_2,n_2) \neq (o,o)} L_{m_2,n_2} + \sum_{(m_3,n_3) \neq \left\{\begin{matrix}(o,o)\\(m_2,n_2)\end{matrix}\right.} L_{m_3,n_3} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$- \sum_{(m_4,n_4) \neq \left\{\begin{matrix}(o,o)\\(m_3,n_3)\end{matrix}\right.} \sum_{(m_3,n_3) \neq (m_2,n_2)} L_{m_4,n_4} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$+ \sum_{(m_5,n_5) \neq \left\{\begin{matrix}(o,o)\\(m_4,n_4)\end{matrix}\right.} \sum_{(m_4,n_4) \neq (m_3,n_3)} \sum_{(m_3,n_3) \neq (m_2,n_2)} L_{m_5,n_5} L^{m_5,n_5}_{m_4-m_5,n_4-n_5} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$- \ldots] \underline{\overline{E^{m_2,n_2}_{ci}}} \qquad (B.2)$$

B.1 Left Side Partial Summation

We will first perform a partial summation on the left side of each term in the above series (B.2). For this we proceed by opening the sums present in the bracketed terms of the series. This opening may be performed by setting each double summation indices equal to (o,o) provided they do not have this restriction.

The case of setting $(m_2,n_2) = (o,o)$ will be considered later. Obviously, this opening starts from the fourth term of the series by setting $(m_3,n_3) = (o,o)$. In such a manner we formally perform the opening for each term.

$$\text{Fourth Term} = -\sum_{\substack{(m_4,n_4) \neq \left\{\begin{smallmatrix}(o,o)\\(m_3,n_3)\end{smallmatrix}\right\}}} L_{m_4,n_4} \sum_{(m_3,n_3) \neq (m_2,n_2)} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

By setting $(m_3,n_3) = (o,o)$ the sums in the above may be written as

$$= -\Big(\sum_{(m_4,n_4) \neq (o,o)} L_{m_4,n_4} L^{m_4,n_4}_{-m_4,-n_4}\Big) \sum_{(m_2,n_2) \neq (o,o)} L_{m_2,n_2} - \sum_{(m_4,n_4) \neq \left\{\begin{smallmatrix}(o,o)\\(m_3,n_3)\end{smallmatrix}\right\}} L_{m_4,n_4} \sum_{(m_3,n_3) \neq \left\{\begin{smallmatrix}(o,o)\\(m_2,n_2)\end{smallmatrix}\right\}} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$= -C_1 \sum_{(m_2,n_2) \neq (o,o)} L_{m_2,n_2} - \sum_{(m_4,n_4) \neq \left\{\begin{smallmatrix}(o,o)\\(m_3,n_3)\end{smallmatrix}\right\}} L_{m_4,n_4} \sum_{(m_3,n_3) \neq \left\{\begin{smallmatrix}(o,o)\\(m_2,n_2)\end{smallmatrix}\right\}} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

In a similar way the sums for the higher terms may be opened with the result as follows:

$$\text{Fifth Term} = C_2 \sum_{(m_2,n_2)\neq(o,o)} L_{m_2,n_2} + C_1 \sum_{(m_3,n_3)\neq\left\{\begin{matrix}(o,o)\\(m_2,n_2)\end{matrix}\right\}} L_{m_3,n_3} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$+ \sum_{(m_5,n_5)\neq\left\{\begin{matrix}(o,o)\\(m_4,n_4)\end{matrix}\right\}} \sum_{(m_4,n_4)\neq\left\{\begin{matrix}(o,o)\\(m_3,n_3)\end{matrix}\right\}} L_{m_5,n_5} L^{m_5,n_5}_{m_4-m_5,n_4-n_5} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$\text{Sixth Term} = -(C_1^2 + C_3) \sum_{(m_2,n_2)\neq(o,o)} L_{m_2,n_2} - C_2 \sum_{(m_3,n_3)\neq\left\{\begin{matrix}(o,o)\\(m_2,n_2)\end{matrix}\right\}} L_{m_3,n_3} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$- C_1 \sum_{(m_4,n_4)\neq\left\{\begin{matrix}(o,o)\\(m_3,n_3)\end{matrix}\right\}} \sum_{(m_3,n_3)\neq\left\{\begin{matrix}(o,o)\\(m_2,n_2)\end{matrix}\right\}} L_{m_4,n_4} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$- \sum_{(m_6,n_6)\neq\left\{\begin{matrix}(o,o)\\(m_5,n_5)\end{matrix}\right\}} \sum_{(m_5,n_5)\neq\left\{\begin{matrix}(o,o)\\(m_4,n_4)\end{matrix}\right\}} \sum_{(m_4,n_4)\neq\left\{\begin{matrix}(o,o)\\(m_3,n_3)\end{matrix}\right\}} L_{m_6,n_6} L^{m_6,n_6}_{m_5-m_6,n_5-n_6} L^{m_5,n_5}_{m_4-m_5,n_4-n_5}$$

$$\cdot \sum_{(m_3,n_3)\neq\begin{matrix}(o,o)\\(m_2,n_2)\end{matrix}} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$\text{Seventh Term} = (C_1C_2 + C_2C_1 + C_4) \sum_{(m_2,n_2)\neq(o,o)} L_{m_2,n_2} + (C_1^2 + C_3) \sum_{(m_3,n_3)\neq\left\{\begin{matrix}(o,o)\\(m_2,n_2)\end{matrix}\right\}} L_{m_3,n_3} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$+ C_2 L_{m_4,n_4} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_4,n_4) \neq \begin{Bmatrix}(0,0)\\(m_3,n_3)\end{Bmatrix} (m_3,n_3) \neq \begin{Bmatrix}(0,0)\\(m_2,n_2)\end{Bmatrix}$$

$$+ C_1 L_{m_5,n_5} L^{m_5,n_5}_{m_4-m_5,n_4-n_5} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_5,n_5) \neq \begin{Bmatrix}(0,0)\\(m_4,n_4)\end{Bmatrix} (m_4,n_4) \neq \begin{Bmatrix}(0,0)\\(m_3,n_3)\end{Bmatrix} (m_3,n_3) \neq \begin{Bmatrix}(0,0)\\(m_2,n_2)\end{Bmatrix}$$

$$+ L_{m_7,n_7} L^{m_7,n_7}_{m_6-m_7,n_6-n_7} L^{m_6,n_6}_{m_5-m_6,n_5-n_6} L^{m_5,n_5}_{m_4-m_5,n_4-n_5}$$

$$(m_7,n_7) \neq \begin{Bmatrix}(0,0)\\(m_6,n_6)\end{Bmatrix} (m_6,n_6) \neq \begin{Bmatrix}(0,0)\\(m_5,n_5)\end{Bmatrix} (m_5,n_5) \neq \begin{Bmatrix}(0,0)\\(m_4,n_4)\end{Bmatrix}$$

$$(m_4,n_4) \neq \begin{Bmatrix}(0,0)\\(m_3,n_3)\end{Bmatrix}$$

$$\cdot L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_3,n_3) \neq \begin{Bmatrix}(0,0)\\(m_2,n_2)\end{Bmatrix}$$

Eighth Term $= -(C_1^3 + C_1C_3 + C_3C_1 + C_2^2 + C_5) L_{m_2,n_2}$

$$(m_2,n_2) \neq (0,0)$$

$$-(C_1C_2 + C_2C_1 + C_4) L_{m_3,n_3} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_3,n_3) \neq \begin{Bmatrix}(0,0)\\(m_2,n_2)\end{Bmatrix}$$

$$- (C_1^2 + C_3) L_{m_4,n_4} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_4,n_4) \neq \begin{Bmatrix}(0,0)\\(m_3,n_3)\end{Bmatrix} \quad (m_3,n_3) \neq \begin{Bmatrix}(0,0)\\(m_2,n_2)\end{Bmatrix}$$

$$- C_2 L_{m_5,n_5} L^{m_5,n_5}_{m_4-m_5,n_4-n_5} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_5,n_5) \neq \begin{Bmatrix}(0,0)\\(m_4,n_4)\end{Bmatrix} \quad (m_4,n_4) \neq \begin{Bmatrix}(0,0)\\(m_3,n_3)\end{Bmatrix} \quad (m_3,n_3) \neq \begin{Bmatrix}(0,0)\\(m_2,n_2)\end{Bmatrix}$$

$$- C_1 L_{m_6,n_6} L^{m_6,n_6}_{m_5-m_6,n_5-n_6} L^{m_5,n_5}_{m_4-m_5,n_4-n_5} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_6,n_6) \neq \begin{Bmatrix}(0,0)\\(m_5,n_5)\end{Bmatrix} \quad (m_5,n_5) \neq \begin{Bmatrix}(0,0)\\(m_4,n_4)\end{Bmatrix} \quad (m_4,n_4) \neq \begin{Bmatrix}(0,0)\\(m_3,n_3)\end{Bmatrix}$$

$$(m_3,n_3) \neq \begin{Bmatrix}(0,0)\\(m_2,n_2)\end{Bmatrix}$$

$$- L_{m_8,n_8} L^{m_8,n_8}_{m_7-m_8,n_7-n_8} L^{m_7,n_7}_{m_6-m_7,n_6-n_7} L^{m_6,n_6}_{m_5-m_6,n_5-n_6}$$

$$(m_8,n_8) \neq \begin{Bmatrix}(0,0)\\(m_7,n_7)\end{Bmatrix} \quad (m_7,n_7) \neq \begin{Bmatrix}(0,0)\\(m_6,n_6)\end{Bmatrix} \quad (m_6,n_6) \neq \begin{Bmatrix}(0,0)\\(m_5,n_5)\end{Bmatrix}$$

$$(m_5,n_5) \neq \begin{Bmatrix}(0,0)\\(m_4,n_4)\end{Bmatrix}$$

$$\cdot L^{m_5,n_5}_{m_4-m_5,n_4-n_5} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_4,n_4) \neq \begin{Bmatrix}(0,0)\\(m_3,n_3)\end{Bmatrix} \quad (m_3,n_3) \neq \begin{Bmatrix}(0,0)\\(m_2,n_2)\end{Bmatrix}$$

In the above the matrices C_1, C_2, C_3, C_4, and C_5 are given as

$$C_1 = L_{p_2,q_2} L^{p_2,q_2}_{-p_2,-q_2}$$

$$(p_2,q_2) \neq (o,o)$$

$$C_2 = L_{p_3,q_3} L^{p_3,q_3}_{p_2-p_3,q_2-q_3} L^{p_2,q_2}_{-p_2,-q_2}$$

$$(p_3,q_3) \neq \begin{Bmatrix} (o,o) \\ (p_2,q_2) \end{Bmatrix} \quad (p_2,q_2) \neq (o,o)$$

$$C_3 = L_{p_4,q_4} L^{p_4,q_4}_{p_3-p_4,q_3-q_4} L^{p_3,q_3}_{p_2-p_3,q_2-q_3} L^{p_2,q_2}_{-p_2,-q_2} \tag{B.3}$$

$$(p_4,q_4) = \begin{Bmatrix} (o,o) \\ (p_3,q_3) \end{Bmatrix} \quad (p_3,q_3) = \begin{Bmatrix} (o,o) \\ (p_2,q_2) \end{Bmatrix} \quad (p_2,q_2) \neq (o,o)$$

$$C_4 = L_{p_5,q_5} L^{p_5,q_5}_{p_4-p_5,q_4-q_5} L^{p_4,q_4}_{p_3-p_4,q_3-q_4} L^{p_3,q_3}_{p_2-p_3,q_2-q_3} L^{p_2,q_2}_{-p_2,-q_2}$$

$$(p_5,q_5) \neq \begin{Bmatrix} (o,o) \\ (p_4,q_4) \end{Bmatrix} \quad (p_4,q_4) \neq \begin{Bmatrix} (o,o) \\ (p_3,q_3) \end{Bmatrix} \quad (p_3,q_3) \neq \begin{Bmatrix} (o,o) \\ (p_2,q_2) \end{Bmatrix}$$

$$(p_2,q_2) \neq (o,o)$$

$$C_5 = L_{p_6,q_6} L^{p_6,q_6}_{p_5-p_6,q_5-q_6} L^{p_5,q_5}_{p_4-p_5,q_4-q_5} L^{p_4,q_4}_{p_3-p_4,q_3-q_4} L^{p_3,q_3}_{p_2-p_3,q_2-q_3} L^{p_2,q_2}_{-p_2,-q_2}$$

$$(p_6,q_6) = \begin{Bmatrix} (o,o) \\ (p_5,q_5) \end{Bmatrix} \quad (p_5,q_5) \neq \begin{Bmatrix} (o,o) \\ (p_4,q_4) \end{Bmatrix} \quad (p_4,q_4) \neq \begin{Bmatrix} (o,o) \\ (p_3,q_3) \end{Bmatrix}$$

$$(p_3,q_3) \neq \begin{Bmatrix} (o,o) \\ (p_2,q_2) \end{Bmatrix} \quad (p_2,q_2) \neq (o,o)$$

where (p_2,q_2), (p_3,q_3), (p_4,q_4), (p_5,q_5) and (p_6,q_6) are the indices for local summations with proper restrictions given on the summations.

Thus we have carried out the openings of the sums up to eighth term. This procedure may be extended for higher terms. It may be seen that the opening of the sums in each term in (B.2), starting from the fourth term, provides two types of the terms. The first type consists of those terms which are the same as the lower terms but modified by coefficient matrices. These matrices are C_1, C_2, etc. The second type consists of new terms. These new terms are the same as the term which is opened but with additional restrictions on the summations. The first type of terms may be thought of as correction terms. For example, the fourth term of the series (B.2) provides a correction term to the second term of the series and a new fourth term. This new fourth term is the same as the previous fourth term but with additional restrictions. Similarly, the sixth term provides corrections to the second, third and new fourth terms and generates a new sixth term. These correction terms may be collected together for each term and an expression for say the second term may be written as

$$- [I + C_1 - C_2 + C_1^2 + C_3 - C_1C_2 - C_2C_1 - C_4 + C_1^3 + C_1C_3 + C_3C_1 + C_2^2 + C_5 - . . .] L_{m_2,n_2}$$

$$(m_2,n_2) \neq (o,o)$$

It is obvious that these coefficient matrices form a series which may be generalized as

$$- [I + (C_1 - C_2 + C_3 - C_4 + \cdot \cdot \cdot) + (C_1 - C_2 + C_3 - C_4 + \cdot \cdot \cdot)^2 + (C_1 - C_2 + C_3 - C_4 + \cdot \cdot \cdot)^3 + \cdot \cdot \cdot] L_{m_2,n_2} , \quad (m_2,n_2) \neq (o,o)$$

or, $$- [I + R_1 + R_1^2 + R_1^3 + \cdot \cdot \cdot] L_{m_2,n_2} \quad (m_2,n_2) = (o,o)$$

The above series may formally be summed as

$$-(I - R_1)^{-1} L_{m_2,n_2} \quad (m_2,n_2) \neq (o,o)$$

where

$$R_1 = C_1 - C_2 + C_3 - C_4 + \cdot \cdot \cdot \quad . \tag{B.4}$$

Since the same coefficient matrices as for the second term, are also generated for the third, new fourth, and the new fifth and higher terms, they may also be summed formally as above. Thus by carrying out these partial summations the original series (B.2) may be rewritten as follows;

$$\overline{E_t} = \overline{E_{ci}} + (I - R_1)^{-1} [-L_{m_2,n_2} + L_{m_3,n_3} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_2,n_2) \neq (o,o) \quad (m_3,n_3) \neq \begin{Bmatrix} (o,o) \\ (m_2,n_2) \end{Bmatrix}$$

$$- L_{m_4,n_4} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_4,n_4) \neq \begin{Bmatrix} (o,o) \\ (m_3,n_3) \end{Bmatrix} (m_3,n_3) \neq \begin{Bmatrix} (o,o) \\ (m_2,n_2) \end{Bmatrix}$$

$$+ L_{m_5,n_5} L^{m_5,n_5}_{m_4-m_5,n_4-n_5} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_5,n_5) \neq \begin{Bmatrix} (o,o) \\ (m_4,n_4) \end{Bmatrix} (m_4,n_4) \neq \begin{Bmatrix} (o,o) \\ (m_3,n_3) \end{Bmatrix} (m_3,n_3) \neq \begin{Bmatrix} (o,o) \\ (m_2,n_2) \end{Bmatrix}$$

$$- \ . \ . \ . \ ] \ \overline{E^{m_2,n_2}_{ci}} \ . \qquad (B.5)$$

In the above equation R_1 is given by (B.4) and (B.3).

The sums in the above series may again be opened by setting the summation indices (m_2,n_2) to (o,o). This operation results in the following:

$$\overline{E_t} = \overline{E_{ci}} + (I - R_1)^{-1} R_1 \overline{E_{ci}}$$

$$+ (I - R_1)^{-1} [-L_{m_2,n_2} + L_{m_3,n_3} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_2,n_2) \neq (o,o) \quad (m_3,n_3) \neq \begin{Bmatrix} (o,o) \\ (m_2,n_2) \end{Bmatrix} (m_2,n_2) \neq (o,o) \ .$$

$$- L_{m_4,n_4} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3} + \ . \ . \ ] \ \overline{E^{m_2,n_2}_{ci}} \ . \qquad (B.6)$$

$$(m_4,n_4) \neq \begin{Bmatrix} (o,o) \\ (m_3,n_3) \end{Bmatrix} (m_3,n_3) \neq \begin{Bmatrix} (o,o) \\ (m_2,n_2) \end{Bmatrix} (m_2,n_2) \neq (o,o)$$

The first two terms in the above series may be formally combined as $(I - R_1)^{-1}\,\overline{\underline{E_{ci}}}$. Therefore we have

$$\overline{\underline{E_t}} = (I - R_1)^{-1}\,\overline{\underline{E_{ci}}} + (I - R_1)^{-1}\Big[-\sum_{(m_2,n_2)\neq(o,o)} L_{m_2,n_2}$$

$$+ \sum_{(m_3,n_3)\neq\left\{\begin{matrix}(o,o)\\(m_2,n_2)\end{matrix}\right\}}\;\sum_{(m_2,n_2)\neq(o,o)} L_{m_3,n_3}\, L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$- \sum_{(m_4,n_4)\neq\left\{\begin{matrix}(o,o)\\(m_3,n_3)\end{matrix}\right\}}\;\sum_{(m_3,n_3)\neq\left\{\begin{matrix}(o,o)\\(m_2,n_2)\end{matrix}\right\}}\;\sum_{(m_2,n_2)\neq(o,o)} L_{m_4,n_4}\, L^{m_4,n_4}_{m_3-m_4,n_3-n_4}\, L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$+ \sum_{(m_5,n_5)\neq\left\{\begin{matrix}(o,o)\\(m_4,n_4)\end{matrix}\right\}}\;\sum_{(m_4,n_4)\neq\left\{\begin{matrix}(o,o)\\(m_3,n_3)\end{matrix}\right\}}\;\sum_{(m_3,n_3)\neq\left\{\begin{matrix}(o,o)\\(m_2,n_2)\end{matrix}\right\}}\;\sum_{(m_2,n_2)\neq(o,o)} L_{m_5,n_5}\, L^{m_5,n_5}_{m_4-m_5,n_4-n_5}\, L^{m_4,n_4}_{m_3-m_4,n_3-n_4}\, L^{m_3,n_3}_{m_2-m_3,n_2-n_3} \tag{B.7}$$

$$- \ldots\Big]\;\overline{\underline{E^{m_2,n_2}_{ci}}}\,.$$

In the above (m_2, n_2) may be regarded as a general mode of propagation. Therefore the first term represents the propagation in the (o,o) mode and the rest represent, as a sum, the propagation in the non (o,o) modes. This interpretation is based on the shifts by m_2N and n_2N in the spatial transform variables of the incident field. These shifts are analogus to the modes of plane wave propagation.

B.2 Right Side Partial Summation

It is possible to further partially sum the above series (B.7). We will perform this partial summation but now on the right side of each term of the series except the first term. For the first term the partial summation is over and is performed only on the left side of that term. We again open the sums present in the bracketed terms of (B.7) by setting each of the double summation indices equal to (m_2,n_2) provided the indices are free from this restriction. This opening starts from the fourth term of the series by setting $(m_4,n_4) = (m_2,n_2)$. That is the fourth term may be rewritten as follows;

$$\text{Fourth term} = -\sum_{(m_4,n_4)\neq\left\{\begin{smallmatrix}(o,o)\\(m_3,n_3)\end{smallmatrix}\right\}}\ \sum_{(m_3,n_3)\neq\left\{\begin{smallmatrix}(o,o)\\(m_2,n_2)\end{smallmatrix}\right\}}\ \sum_{(m_2,n_2)\neq(o,o)} L_{m_4,n_4} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$= -\sum_{(m_2,n_2)\neq(o,o)} L_{m_2,n_2} D_1 - \sum_{(m_4,n_4)\neq\left\{\begin{smallmatrix}(o,o)\\(m_3,n_3)\\(m_2,n_2)\end{smallmatrix}\right\}}\ \sum_{(m_3,n_3)\neq\left\{\begin{smallmatrix}(o,o)\\(m_2,n_2)\end{smallmatrix}\right\}}\ \sum_{(m_2,n_2)\neq(o,o)} L_{m_4,n_4} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

In a similar way the sums for the higher terms may be opened with the following result.

$$\text{Fifth term} = \sum_{(m_2,n_2)\neq(o,o)} L_{m_2,n_2} D_2 + \sum_{(m_3,n_3)\neq\left\{\begin{smallmatrix}(o,o)\\(m_2,n_2)\end{smallmatrix}\right\}}\ \sum_{(m_2,n_2)\neq(o,o)} L_{m_3,n_3} L^{m_3,n_3}_{m_2-m_3,n_2-n_3} D_1$$

$$+ L_{m_5,n_5} L^{m_5,n_4}_{m_4-m_5,n_4-n_5} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_5,n_5) \neq \begin{Bmatrix} (o,o) \\ (m_4,n_4) \\ (m_2,n_2) \end{Bmatrix} (m_4,n_4) \neq \begin{Bmatrix} (o,o) \\ (m_3,n_3) \\ (m_2,n_2) \end{Bmatrix} (m_3,n_3) \neq \begin{Bmatrix} (o,o) \\ (m_2,n_2) \end{Bmatrix}$$

$$(m_2,n_2) \neq (o,o)$$

$$\text{Sixth Term} = - L_{m_2,n_2} (D_1^2 + D_3) - L_{m_3,n_3} L^{m_3,n_3}_{m_2-m_3,n_2-m_3} D_2$$

$$(m_2,n_2) \neq (o,o) \; (m_3,n_3) \neq \begin{Bmatrix} (o,o) \\ (m_2,n_2) \end{Bmatrix} (m_2,n_2) \neq (o,o)$$

$$- L_{m_4,n_4} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3} D_1$$

$$(m_4,n_4) \neq \begin{Bmatrix} (o,o) \\ (m_3,n_3) \\ (m_2,n_2) \end{Bmatrix} (m_3,n_3) \neq \begin{Bmatrix} (o,o) \\ (m_2,n_2) \end{Bmatrix} (m_2,n_2) \neq (o,o)$$

$$- L_{m_6,n_6} L^{m_6,n_6}_{m_5-m_6,n_5-n_6} L^{m_5,n_5}_{m_4-m_5,n_4-n_5}$$

$$(m_6,n_6) \neq \begin{Bmatrix} (o,o) \\ (m_5,n_5) \\ (m_2,n_2) \end{Bmatrix} (m_5,n_5) \neq \begin{Bmatrix} (o,o) \\ (m_4,n_4) \\ (m_2,n_2) \end{Bmatrix} (m_4,n_4) \neq \begin{Bmatrix} (o,o) \\ (m_3,n_3) \\ (m_2,n_2) \end{Bmatrix}$$

$$\cdot L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_3,n_3) \neq \begin{Bmatrix} (o,o) \\ (m_2,n_2) \end{Bmatrix} (m_2,n_2) \neq (o,o)$$

$$\text{Seventh Term} = L_{m_2,n_2}\,(D_1D_2 + D_2D_1 + D_4)$$

$$(m_2,n_2) \neq (o,o)$$

$$+ L_{m_3,n_3} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}\,(D_1^2 + D_3)$$

$$(m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix} (m_2,n_2) \neq (o,o)$$

$$+ L_{m_4,n_4} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}\, D_2$$

$$(m_4,n_4) \neq \begin{Bmatrix}(o,o)\\(m_3,n_3)\\(m_2,n_2)\end{Bmatrix} (m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix} (m_2,n_2) \neq (o,o)$$

$$+ L_{m_5,n_5} L^{m_5,n_5}_{m_4-m_5,n_4-n_5} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}\, D_1$$

$$(m_5,n_5) \neq \begin{Bmatrix}(o,o)\\(m_4,n_4)\\(m_2,n_2)\end{Bmatrix} (m_4,n_4) \neq \begin{Bmatrix}(o,o)\\(m_3,n_3)\\(m_2,n_2)\end{Bmatrix} (m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix}$$

$$(m_2,n_2) \neq (o,o)$$

$$+ L_{m_7,n_7} L^{m_7,n_7}_{m_6-m_7,n_6-n_7} L^{m_6,n_6}_{m_5-m_6,n_5-n_6} L^{m_5,n_5}_{m_4-m_5,n_4-n_5}$$

$$(m_7,n_7) \neq \begin{Bmatrix}(o,o)\\(m_6,n_6)\\(m_2,n_2)\end{Bmatrix} (m_6,n_6) \neq \begin{Bmatrix}(o,o)\\(m_5,n_5)\\(m_2,n_2)\end{Bmatrix} (m_5,n_5) \neq \begin{Bmatrix}(o,o)\\(m_4,n_4)\\(m_2,n_2)\end{Bmatrix}$$

$$(m_4,n_4) \neq \begin{Bmatrix}(o,o)\\(m_3,n_3)\\(m_2,n_2)\end{Bmatrix}$$

$$L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix} (m_2,n_2) \neq (o,o)$$

In the above D_1, D_2, D_3 and D_4 are given as

$$D_1 = L^{m_2,n_2}_{p_2-m_2,q_2-n_2} L^{p_2,q_2}_{m_2-p_2,n_2-q_2} \qquad (p_2,q_2) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix}$$

$$D_2 = L^{m_2,n_2}_{p_3-m_2,q_3-n_2} L^{p_3,q_3}_{p_2-p_3,q_2-q_3} L^{p_2,q_2}_{m_2-p_2,n_2-q_2} \qquad (p_3,q_3) \neq \begin{Bmatrix}(o,o)\\(p_2,q_2)\\(m_2,n_2)\end{Bmatrix} (p_2,q_2) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix} \tag{B.8}$$

$$D_3 = L^{m_2,n_2}_{p_4-m_2,q_4-n_2} L^{p_4,q_4}_{p_3-p_4,q_3-q_4} L^{p_3,q_3}_{p_2-p_3,q_2-q_3} L^{p_2,q_2}_{m_2-p_2,n_2-q_2} \qquad (p_4,q_4) \neq \begin{Bmatrix}(o,o)\\(p_3,q_3)\\(m_2,n_2)\end{Bmatrix} (p_3,q_3) \neq \begin{Bmatrix}(o,o)\\(p_2,q_2)\\(m_2,n_2)\end{Bmatrix} (p_2,q_2) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix}$$

$$D_4 = L^{m_2,n_2}_{p_5-m_2,q_5-n_2} L^{p_5,q_5}_{p_4-p_5,q_4-q_5} L^{p_4,q_4}_{p_3-p_4,q_3-q_4} L^{p_3,q_3}_{p_2-p_3,q_2-q_3} L^{p_2,q_2}_{m_2-p_2,n_2-q_2}$$

$$(p_5,q_5) \neq \begin{Bmatrix}(o,o)\\(p_4,q_4)\\(m_2,n_2)\end{Bmatrix} (p_4,q_4) \neq \begin{Bmatrix}(o,o)\\(p_3,q_3)\\(m_2,n_2)\end{Bmatrix} (p_3,q_3) \neq \begin{Bmatrix}(o,o)\\(p_2,q_2)\\(m_2,n_2)\end{Bmatrix}$$

$$(p_2,q_2) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix}$$

where (p_2,q_2), (p_3,q_3), (p_4,q_4), and (p_5,q_5) are the indices for local summations with the given restrictions, treating (m_2,n_2) as constant. This constant refer to the double summation indices given in the series (B.7).

The above procedure of opening sums may be extended for higher terms of (B.7). But the complexity of the algebra also increases. However, by collecting the newly generated correction terms and summing them formally, as in the previous section, the series (B.7) may be rewritten as follows:

$$\underline{\overline{E_t}} = (I - R_1)^{-1}\,\underline{\overline{E_{ci}}} + (I - R_1)^{-1}\Big[- \sum_{(m_2,n_2)\neq(o,o)} L_{m_2,n_2}$$

$$+ \sum_{(m_3,n_3)\neq\left\{\begin{matrix}(o,o)\\(m_2,n_2)\end{matrix}\right\}}\ \sum_{(m_2,n_2)\neq(o,o)} L_{m_3,n_3} L^{m_3,n_3}_{m_2-m_3,n_2-m_3}$$

$$- \sum_{(m_4,n_4)\neq\left\{\begin{matrix}(o,o)\\(m_3,n_3)\\(m_2,n_2)\end{matrix}\right\}}\ \sum_{(m_3,n_3)\neq\left\{\begin{matrix}(o,o)\\(m_2,n_2)\end{matrix}\right\}}\ \sum_{(m_2,n_2)\neq(o,o)} L_{m_4,n_4} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$+ \sum_{(m_5,n_5)\neq\left\{\begin{matrix}(o,o)\\(m_4,n_4)\\(m_2,n_2)\end{matrix}\right\}}\ \sum_{(m_4,n_4)\neq\left\{\begin{matrix}(o,o)\\(m_3,n_3)\\(m_2,n_2)\end{matrix}\right\}}\ \sum_{(m_3,n_3)\neq\left\{\begin{matrix}(o,o)\\(m_2,n_2)\end{matrix}\right\}}\ \sum_{(m_2,n_2)\neq(o,o)} L_{m_5,n_5} L^{m_5,n_5}_{m_4-m_5,n_4-n_5} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-m_3}$$

$$- \ldots\Big]\,(I - R_2)^{-1}\,\underline{\overline{E_{ci}^{m_2,n_2}}} \qquad \text{(B.9)}$$

where R_1 is given by (B.4) and (B.3) and R_2 is given as

$$R_2 = D_1 - D_2 + D_3 - D_4 + \ldots \qquad \text{(B.10)}$$

with D_1, D_2 etc. given by (B.8).

B.3 Intermediate Partial Summation and Generalization

In the previous two sections we have carried out the partial summations, first on the left side of each term of the series (B.2) and then on the right side of (B.7). The result obtained after these two summations is given by (B.9). These partial summations for the first two terms of (B.9) are completed as no further correction term is available for them. We, therefore, call these two terms as zero and first order terms respectively. However, for third and higher terms, further partial summations may be performed. These summations will now be within each term. We call these further partial summations intermediate partial summations. We will carry out one such intermediate summation and on that basis generalize the result for all intermediate summations.

To carry out an intermediate summation we open the sums present in the fifth and higher terms of (B.9). For the opening we set each of the double summation indices equal to (m_3,n_3) which does not violate this restriction. This opening starts from the fifth term by setting $(m_5,n_5) = (m_3,n_3)$. Thus we may write the fifth term as

$$\text{Fifth Term} = L_{m_5,n_5} L^{m_5,n_5}_{m_4-m_5,n_4-n_5} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_5,n_5) \neq \begin{Bmatrix} (o,o) \\ (m_4,n_4) \\ (m_2,n_2) \end{Bmatrix} \quad (m_4,n_4) \neq \begin{Bmatrix} (o,o) \\ (m_3,n_3) \\ (m_2,n_2) \end{Bmatrix} \quad (m_3,n_3) \neq \begin{Bmatrix} (o,o) \\ (m_2,n_2) \end{Bmatrix}$$

$$(m_2,n_2) \neq (o,o)$$

$$= L_{m_3,n_3} F_1 L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix} \quad (m_2,n_2) \neq (o,o)$$

$$+ L_{m_5,n_5} L^{m_5,n_5}_{m_4-m_5,n_4-n_5} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_5,n_5) \neq \begin{Bmatrix}(o,o)\\(m_4,n_4)\\(m_2,n_2)\\(m_3,n_3)\end{Bmatrix} \quad (m_4,n_4) \neq \begin{Bmatrix}(o,o)\\(m_3,n_3)\\(m_2,n_2)\end{Bmatrix} \quad (m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix}$$

$$(m_2,n_2) \neq (o,o)$$

In a similar way,

$$\text{Sixth Term} = - L_{m_3,n_3} F_2 L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix} \quad (m_2,n_2) \neq (o,o)$$

$$- L_{m_4,n_4} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} F_1 L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_4,n_4) \neq \begin{Bmatrix}(o,o)\\(m_3,n_3)\\(m_2,n_2)\end{Bmatrix} \quad (m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix} \quad (m_2,n_2) \neq (o,o)$$

$$- L_{m_6,n_6} L^{m_6,n_6}_{m_5-m_6,n_5-n_6} L^{m_5,n_5}_{m_4-m_5,n_4-n_5} L^{m_4,n_4}_{m_3-m_4,n_3-n_4} L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_6,n_6) \neq \begin{Bmatrix}(o,o)\\(m_5,n_5)\\(m_2,n_2)\\(m_3,n_3)\end{Bmatrix} \quad (m_5,n_5) \neq \begin{Bmatrix}(o,o)\\(m_4,n_4)\\(m_2,n_2)\\(m_3,n_3)\end{Bmatrix} \quad (m_4,n_4) \neq \begin{Bmatrix}(o,o)\\(m_3,n_3)\\(m_2,n_2)\end{Bmatrix}$$

$$(m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix} \quad (m_2,n_2) \neq (o,o)$$

$$\text{Seventh Term} = L_{m_3,n_3}(F_1^2 + F_3)\, L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix} \quad (m_2,n_2) \neq (o,o)$$

$$+ L_{m_4,n_4}\, L^{m_3,n_3}_{m_3-m_4,n_3-n_4}\, F_2\, L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_4,n_4) \neq \begin{Bmatrix}(o,o)\\(m_3,n_3)\\(m_2,n_2)\end{Bmatrix} \quad (m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix} \quad (m_2,n_2) \neq (o,o)$$

$$+ L_{m_5,n_5}\, L^{m_5,n_5}_{m_4-m_5,n_4-n_5}\, L^{m_4,n_4}_{m_3-m_4,n_3-n_4}\, F_1\, L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_5,n_5) \neq \begin{Bmatrix}(o,o)\\(m_4,n_4)\\(m_2,n_2)\\(m_3,n_3)\end{Bmatrix} \quad (m_4,n_4) \neq \begin{Bmatrix}(o,o)\\(m_3,n_3)\\(m_2,n_2)\end{Bmatrix} \quad (m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix}$$

$$(m_2,n_2) \neq (o,o)$$

$$+ L_{m_7,n_7}\, L^{m_7,n_7}_{m_6-m_7,n_6-n_7}\, L^{m_6,n_6}_{m_5-m_6,n_5-n_6}\, L^{m_5,n_5}_{m_4-m_5,n_4-n_5}$$

$$(m_7,n_7) \neq \begin{Bmatrix}(o,o)\\(m_6,n_6)\\(m_2,n_2)\\(m_3,n_3)\end{Bmatrix} \quad (m_6,n_6) \neq \begin{Bmatrix}(o,o)\\(m_5,n_5)\\(m_2,n_2)\\(m_3,n_3)\end{Bmatrix} \quad (m_5,n_5) \neq \begin{Bmatrix}(o,o)\\(m_4,n_4)\\(m_2,n_2)\\(m_3,n_3)\end{Bmatrix}$$

$$(m_4,n_4) \neq \begin{Bmatrix}(o,o)\\(m_3,n_3)\\(m_2,n_2)\end{Bmatrix}$$

$$\cdot\, L^{m_4,n_4}_{m_3-m_4,n_3-n_4}\, L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix} \quad (m_2,n_2) \neq (o,o)$$

In the above F_1, F_2, and F_3 are given as

$$F_1 = L^{m_3,n_3}_{p_2-m_3,q_2-n_3} \; L^{p_2,q_2}_{m_3-p_2,n_3-q_2}$$

$$(p_2,q_2) \neq \begin{Bmatrix} (o,o) \\ (m_2,n_2) \\ (m_3,n_3) \end{Bmatrix}$$

$$F_2 = L^{m_3,n_3}_{p_3-m_3,q_3-n_3} \; L^{p_3,q_3}_{p_2-p_3,q_2-q_3} \; L^{p_2,q_2}_{m_3-p_2,n_3-q_2} \tag{B.11}$$

$$(p_3,q_3) \neq \begin{Bmatrix} (o,o) \\ (p_2,q_2) \\ (m_2,n_2) \\ (m_3,n_3) \end{Bmatrix} \quad (p_2,q_2) \neq \begin{Bmatrix} (o,o) \\ (m_2,n_2) \\ (m_3,n_3) \end{Bmatrix}$$

$$F_3 = L^{m_3,n_3}_{p_4-m_3,q_4-n_3} \; L^{p_4,q_4}_{p_3-p_4,q_3-q_4} \; L^{p_3,q_3}_{p_2-p_3,q_2-q_3} \; L^{p_2,q_2}_{m_3-p_2,n_3-q_2}$$

$$(p_4,q_4) \neq \begin{Bmatrix} (o,o) \\ (p_3,q_3) \\ (m_2,n_2) \\ (m_3,n_3) \end{Bmatrix} \quad (p_3,q_3) \neq \begin{Bmatrix} (o,o) \\ (p_2,q_2) \\ (m_2,n_2) \\ (m_3,n_3) \end{Bmatrix} \quad (p_2,q_2) \neq \begin{Bmatrix} (o,o) \\ (m_2,n_2) \\ (m_3,n_3) \end{Bmatrix}$$

where (p_2,q_2), (p_3,q_3), and (p_4,q_4) are the indices for the local summations with given restrictions. The summation indices (m_2,n_2) and (m_3,n_3), referred to in (B.9), may be treated as constants in the above equations (B.11).

It may be seen that the nature of the correction terms is similar to those obtained previously in the left and right partial summations. Therefore we may collect these correction terms and sum them formally. By doing so, the series (B.9) may be rewritten as follows:

$$\underline{\overline{E_t}} = (I - R_1)^{-1}\,\underline{\overline{E_{ci}}} + (I - R_1)^{-1}\,[-\,L_{m_2,n_2}$$

$$(m_2,n_2) \neq (o,o)$$

$$+\,L_{m_3,n_3}\,(I - R_3)^{-1}\,L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix}\;(m_2,n_2) \neq (o,o)$$

$$-\,L_{m_4,n_4}\,L^{m_4,n_4}_{m_3-m_4,n_3-n_4}\,(I - R_3)^{-1}\,L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_4,n_4) \neq \begin{Bmatrix}(o,o)\\(m_3,n_3)\\(m_2,n_2)\end{Bmatrix}\;(m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix}\;(m_2,n_2) \neq (o,o)$$

$$+\,L_{m_5,n_5}\,L^{m_5,n_5}_{m_4-m_5,n_4-n_5}\,L^{m_4,n_4}_{m_3-m_4,n_3-n_4}\,(I - R_3)^{-1}\,L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_5,n_5) \neq \begin{Bmatrix}(o,o)\\(m_4,n_4)\\(m_2,n_2)\\(m_3,n_3)\end{Bmatrix}\;(m_4,n_4) \neq \begin{Bmatrix}(o,o)\\(m_3,n_3)\\(m_2,n_2)\end{Bmatrix}\;(m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix}$$

$$-\;.\;.\;.\;]\;(I - R_2)^{-1}\,\underline{\overline{E^{m_2,n_2}_{ci}}} \tag{B.12}$$

where the new series R_3 is given by

$$R_3 = F_1 - F_2 + F_3 - \ldots \tag{B.13}$$

with F_1, F_2 etc. given by (B.11).

Thus we have carried an intermediate partial summation for the third and higher terms of the series (B.9). The result obtained is given by (B.12). The summation for the third term are now completed and therefore this term may be called the second order term. For the fourth and higher terms in (B.12) the process of intermediate summations may still be carried out. The procedure to be followed is the same as used before i.e., opening the sums, collecting the correction terms, and their formal summation. For opening the sums all the double summation indices are to be set now equal to (m_4,n_4). The process may be repeated again by setting the double indices equal to (m_5,n_5) and so on. We may thus generalize this procedure to a final result as follows:

$$\overline{E_t} = (I - R_1)^{-1}\,\overline{E_{ci}} + (I - R_1)^{-1}\,[-\,L_{m_2,n_2}$$

$$(m_2,n_2) \neq (o,o)$$

$$+\,L_{m_3,n_3}\,(I - R_3)^{-1}\,L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix}\;(m_2,n_2) \neq (o,o)$$

$$-\,L_{m_4,n_4}\,(I - R_4)^{-1}\,L^{m_4,n_4}_{m_3-m_4,n_3-n_4}\,(I - R_3)^{-1}\,L^{m_3,n_3}_{m_2-m_3,n_2-n_3}$$

$$(m_4,n_4) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\\(m_3,n_3)\end{Bmatrix}\;(m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix}\;(m_2,n_2) \neq (o,o)$$

$$+\,L_{m_5,n_5}\,(I - R_5)^{-1}\,L^{m_5,n_5}_{m_4-m_5,n_4-n_5}\,(I - R_4)^{-1}\,L^{m_4,n_4}_{m_3-m_4,n_3-n_4}$$

$$(m_5,n_5) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\\(m_3,n_3)\\(m_4,n_4)\end{Bmatrix}\;(m_4,n_4) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\\(m_3,n_3)\end{Bmatrix}\;(m_3,n_3) \neq \begin{Bmatrix}(o,o)\\(m_2,n_2)\end{Bmatrix}.$$

$$\cdot\ (I - R_3)^{-1}\ L^{m_3,n_3}_{m_2-m_3,n_2-n_3} - \ldots]\ (I - R_2)^{-1}\ \overline{\underline{E^{m_2,n_2}_{ci}}} \qquad (B.14)$$

$$(m_2,n_2) \neq (o,o)$$

where R_1, R_2, and R_3 are given earlier in a series form. R_4, R_5, etc. may also be written in a similar form. For example R_4 may be given as

$$R_4 = L^{m_4,n_4}_{p_2-m_4,q_2-n_4}\ L^{p_2,q_2}_{m_4-p_2,n_4-q_2}$$

$$(p_2,q_2) \neq \begin{Bmatrix} (o,o) \\ (m_2,n_2) \\ (m_3,n_3) \\ (m_4,n_4) \end{Bmatrix}$$

$$- L^{m_4,n_4}_{p_3-m_4,q_3-n_4}\ L^{p_3,q_3}_{p_2-p_3,q_2-q_3}\ L^{p_2,q_2}_{m_4-p_2,n_4-q_2}$$

$$(p_3,q_3) \neq \begin{Bmatrix} (o,o) \\ (p_2,q_2) \\ (m_2,n_2) \\ (m_3,n_3) \\ (m_4,n_4) \end{Bmatrix} \quad (p_2,q_2) \neq \begin{Bmatrix} (o,o) \\ (m_2,n_2) \\ (m_3,n_3) \\ (m_4,n_4) \end{Bmatrix}$$

$$+ L^{m_4,n_4}_{p_4-m_4,q_4-n_4}\ L^{p_4,q_4}_{p_3-p_4,q_3-q_4}\ L^{p_3,q_3}_{p_2-p_3,q_2-q_3}\ L^{p_2,q_2}_{m_4-p_2,n_4-q_2} - \ldots \qquad (B.15)$$

$$(p_4,q_4) \neq \begin{Bmatrix} (o,o) \\ (p_3,q_3) \\ (m_2,n_2) \\ (m_3,n_3) \\ (m_4,n_4) \end{Bmatrix} \quad (p_3,q_3) \neq \begin{Bmatrix} (o,o) \\ (p_2,q_2) \\ (m_2,n_2) \\ (m_3,n_3) \\ (m_4,n_4) \end{Bmatrix} \quad (p_2,q_2) \neq \begin{Bmatrix} (o,o) \\ (m_2,n_2) \\ (m_3,n_3) \\ (m_4,n_4) \end{Bmatrix}$$

In the above (p_2,q_2), (p_3,q_3), and (p_4,q_4) are again indices for local double summations treating (m_2,n_2), (m_3,n_3), and (m_4,n_4) as constants. These constants refer to summation indices in (B.14).

Thus the above series (B.14) represents the original series (B.2) after the latter is partially summed.

APPENDIX C

ASYMPTOTIC EVALUATION OF THE INTEGRAL Q(3.37)

In order to find the inverse spatial transform of the zero order surface field, as discussed in section 3.3, an evaluation of the following integral, given by (3.37), is required.

$$Q = \frac{1}{4\pi^2} \int_{K_x} \int_{K_y} \frac{G(K_x, K_y)}{u + jk\Delta_a(K_x, K_y)} \exp(-h_o u + jK_x x + jK_y y) dK_x \, dK_y \tag{C.1}$$

In the above h_o is the height of the dipole above the mean level of surface, i.e., $z = 0$ plane. h_o is assumed to be small. Δ_a is a modified surface impedance given by (3.32) and G is given by (3.31). u is defined as

$$u = \begin{cases} (K_x^2 + K_y^2 - k^2)^{\frac{1}{2}} ; & \text{for real root} \\ j(k^2 - K_x^2 - K_y^2)^{\frac{1}{2}} ; & \text{for imaginary root} \end{cases} \tag{C.2}$$

with K_x and K_y as transform variables. It is likely that the above double integral may not be evaluated exactly. However, an asymptotic evaluation is possible and which may serve as a good approximation for the integral. We, therefore, proceed in the following manner.

Let the transform variables K_x and K_y and the inverse transform variables x and y be changed to respective polar coordinates as per the following substitutions:

$$K_x = \lambda \cos\psi, \; K_y = \lambda \sin\psi, \; x = \rho \cos\theta, \; y = \rho \sin\theta . \tag{C.3}$$

We then have

$$Q = \frac{1}{4\pi^2} \int_{\lambda=0}^{\infty} \int_{\psi=0}^{2\pi} \frac{\lambda G(\lambda \cos \psi,\ \lambda \sin \psi)}{u + jk\Delta_a(\lambda \cos \psi, \lambda \sin \psi)}$$

$$\cdot \exp\,[-h_o u + j\rho\lambda \cos(\psi - \theta)]\, d\psi d\lambda. \tag{C.4}$$

By substituting $\psi - \theta = \emptyset$ it becomes

$$Q = \frac{1}{4\pi^2} \int_0^{\infty} \int_{-\theta}^{2\pi - \theta} \frac{\lambda G\,\{\lambda\cos(\emptyset + \theta),\ \lambda\sin(\emptyset + \theta)\}}{u + jk\Delta_a\{\lambda\cos(\emptyset + \theta),\ \lambda \sin(\emptyset + \theta)\}}$$

$$\exp(-h_o u + j\rho\lambda \cos \emptyset)\, d\emptyset d\lambda. \tag{C.5}$$

Since the integrand is a periodic function of $\emptyset$ and the range of integration with respect to $\emptyset$ is 2π, we may write Q as

$$Q = \frac{1}{4\pi^2}\,(Q_1 + Q_2) \tag{C.6}$$

where Q_1 and Q_2 are given as

$$Q_1 = \int_0^{\infty} \int_{-\pi/2}^{\pi/2} \frac{\lambda G\,\{\lambda \cos(\emptyset + \theta),\ \lambda \sin(\emptyset + \theta)\}}{u + jk\Delta_a\{\lambda \cos(\emptyset + \theta),\ \lambda \sin(\emptyset + \theta)\}}$$

$$\cdot \exp\,(-h_o u + j\rho\lambda \cos \emptyset)\, d\emptyset\, d\lambda \tag{C.7}$$

$$Q_2 = \int_0^{\infty} \int_{\pi/2}^{3\pi/2} \frac{\lambda G\,\{\lambda \cos(\emptyset + \theta),\ \lambda \sin(\emptyset + \theta)\}}{u + jk\Delta_a\{\lambda \cos(\emptyset + \theta),\ \lambda \sin(\emptyset + \theta)\}}$$

$$\cdot \exp\,(-h_o u + j\rho\lambda \cos \emptyset)\, d\emptyset\, d\lambda \tag{C.8}$$

By changing $\emptyset$ to $\emptyset - \pi$ and λ to $-\lambda$ in Q_2 (C.8) and then adding Q_2 to to Q_1 yields Q as

$$Q = \frac{1}{4\pi^2} \int_{\lambda=-\infty}^{\infty} |\lambda| \exp(-h_o u)$$

$$\cdot \left[\int_{\emptyset=-\pi/2}^{\pi/2} \frac{G\{\lambda \cos(\emptyset + \theta), \lambda \sin(\emptyset + \theta)\} \exp(j\rho\lambda \cos \emptyset)}{u + jk\Delta_a \{\lambda \cos(\emptyset + \theta), \lambda \sin(\emptyset + \theta)\}} d\emptyset \right] d\lambda. \quad \text{(C.9)}$$

We will evaluate asymptotically the inner integral with respect to $\emptyset$ first. This inner integral may be written as

$$I_\emptyset = \int_{-\pi/2}^{\pi/2} b(\emptyset) \exp[j\rho\lambda g(\emptyset)] d\emptyset \quad \text{(C.10)}$$

where

$$b(\emptyset) = \frac{G\{\lambda \cos(\emptyset + \theta), \lambda \sin(\emptyset + \theta)\}}{u + jk\Delta_a \{\lambda \cos(\emptyset + \theta), \lambda \sin(\emptyset + \theta)\}} \quad \text{(C.11)}$$

$$g(\emptyset) = \cos \emptyset .$$

It is now assumed that the function $b(\emptyset)$ is slowly varying and $|\rho\lambda|$ is large. Since the argument of the exponent in (C.10) is purely imaginary we would prefer to evaluate this integral by the method of stationary phase [Friedman (1969, Ch. 3)].

Now from (C.11) the stationary points $\emptyset_o$ for $g(\emptyset)$ may be derived as

$$g'(\emptyset) = -\sin \emptyset = 0$$

where a prime denotes the partial derivative with respect to $\emptyset$.

or

$$\emptyset_o = \pm m\pi; \quad m = 0, 1, 2, \ldots \quad \text{(C.12)}$$

Since the range of $\emptyset$ is from $-\pi/2$ to $\pi/2$, the stationary point to be taken is $\emptyset_o = 0$. The functions $b(\emptyset)$ and $g(\emptyset)$ in (C.10) may be expanded in a Taylor series about the stationary point $\emptyset_o = 0$ to obtain

$$I_\emptyset = \int_{-\pi/2}^{\pi/2} [b(0) + \emptyset\, b'(0) + \frac{\emptyset^2}{2} b''(0) + \ldots]$$

$$\cdot \exp [j\rho\lambda \{g(0) + \frac{\emptyset^2}{2} g''(0) + \ldots\}]\, d\emptyset .$$

Substituting $\emptyset = \frac{\tau}{\sqrt{|\rho\lambda|}}$,

$$I_\emptyset = \frac{1}{\sqrt{|\rho\lambda|}} \exp [j\rho\lambda\, g(0)] \int_{-(\pi/2)\sqrt{|\rho\lambda|}}^{(\pi/2)\sqrt{|\rho\lambda|}} [b(0) + \frac{\tau}{\sqrt{|\rho\lambda|}} b'(0)$$

$$+ \frac{\tau^2}{2|\rho\lambda|} b''(0) \ldots] \exp [j \operatorname{sgn} (\rho\lambda)\; g''(0) \frac{\tau^2}{2}]\, d\tau \qquad \text{(C.13)}$$

where we have taken up to the g" term in the exponent. The first term in the above series is $O(1/\sqrt{|\rho\lambda|})$. The second term is $O(1/|\rho\lambda|)$ and all succeeding terms are even smaller. To the leading term,

$$I_\emptyset \sim \frac{2b(0)}{\sqrt{|\rho\lambda|}} \exp (j\rho\lambda) \int_0^{(\pi/2)\sqrt{|\rho\lambda|}} \exp [-j \operatorname{sgn} (\lambda) \frac{\tau^2}{2}]\, d\tau \qquad \text{(C.14)}$$

where we have used (C.11) to derive $g(0)$ and $g''(0)$. $\operatorname{sgn}(x)$ is the sign function and is defined as

$$\mathrm{sgn}(x) = \begin{cases} 1 & ; \quad x > 0 \\ -1 & ; \quad x < 0 \quad . \end{cases} \tag{C.15}$$

Further, $\mathrm{sgn}(\rho\lambda) = \mathrm{sgn}(\lambda)$ as ρ is positive. If the limit of $|\rho\lambda|$ is extended to infinity for the integral in (C.14) we obtain

$$I_{\emptyset} \cong \frac{2b(0)}{\sqrt{|\rho\lambda|}} \exp\,(j\rho\lambda) \int_0^{\infty} \exp\,[-j\ \mathrm{sgn}(\lambda)\ \frac{\tau^2}{2}]\ d\tau. \tag{C.16}$$

The above integral may easily be evaluated by using the values for the Fresnel integrals C(x) and S(x) at infinity [Abramowitz and Stegun (1972, ch. 7)] with the result

$$I_{\emptyset} \cong \left(\frac{2\pi}{\rho|\lambda|}\right)^{\frac{1}{2}} b(0)\ \exp\,[j\rho\lambda - j\ \pi/4\ \mathrm{sgn}(\lambda)] \quad . \tag{C.17}$$

This completes an asymptotic evaluation of the integral $I_{\emptyset}$ to the leading term. By using the above result and (C.11) in (C.9) and by noting

$\sqrt{|\lambda|}\ \exp\,[-j\ \pi/4\ \mathrm{sgn}(\lambda)] = \sqrt{\lambda}\ \exp\,(-j\ \pi/4)$

we obtain

$$Q = \frac{\exp\,(-j\ \pi/4)}{2\pi\ \sqrt{2\pi\rho}} \int_{-\infty}^{\infty} \frac{\sqrt{\lambda}\ \ G\ (\lambda\cos\,\theta,\ \lambda\sin\,\theta)}{u + jk\Delta_a(\lambda\cos\,\theta,\ \lambda\sin\,\theta)}$$

$$\exp\,(-h_o u + j\rho\lambda)\ d\lambda\ . \tag{C.18}$$

The above integral also may be evaluated asymptotically to the leading term but using the method of steepest descent [Friedman (1969, Ch. 3), Wait (1970, Ch. 2)]. The method of stationary phase can not be used here as the exponent is no longer pure imaginary for all λ. The parameter considered to be large is now Rk, where k is assumed positive and $R = (h_o^2 + \rho^2)^{\frac{1}{2}}$. Let λ be substituted as

$$\lambda = k \cos \mu \qquad \text{(C.19)}$$

where μ is complex with

$$\mu = \alpha + j\beta . \qquad \text{(C.20)}$$

We, therefore, have

$$u = \sqrt{\lambda^2 - k^2} = jk \sin \mu \qquad \text{(C.21)}$$

and

$$Q = -\frac{\exp(j\ \pi/4)}{2\pi} \sqrt{\frac{k}{2\pi\rho}} \int_{C_1} \frac{\sqrt{\cos \mu}\ G\ (k \cos \mu \cos \theta,\ k \cos \mu \sin \theta)}{\sin \mu + \Delta_a\ (k \cos \mu \cos \theta,\ k \cos \mu \sin \theta)}$$

$$\cdot \sin \mu \exp\left[-jk\ (h_o \sin \mu - \rho \cos \mu)\right] d\mu \qquad \text{(C.22)}$$

where the contour C_1 is shown in the figure C.1. By substituting

$$h_o = R \sin a \ ; \quad \rho = -R \cos a \qquad \text{(C.23)}$$

we obtain

$$Q = -\frac{\exp(j\ \pi/4)}{2\pi} \sqrt{\frac{k}{2\pi\rho}} \int_{C_1} \frac{\sqrt{\cos \mu}\ G\ (k \cos \mu \cos \theta,\ k \cos \mu \sin \theta)}{\sin \mu + \Delta_a\ (k \cos \mu \cos \theta,\ k \cos \mu \sin \theta)}$$

$$\cdot \sin \mu \exp\left[kRg(\mu)\right] d\mu \qquad \text{(C.24)}$$

where $g(\mu)$ is given by

$$g(\mu) = -j \cos (a - \mu) . \qquad \text{(C.25)}$$

We follow the standard procedure of deforming the contour C_1 to the path of steepest descent passing through the saddle point. However, in this deformation care must be taken to include the contribution from poles of the integrand if they are captured in the deformation. The contribution from such poles may be found separately by the residue theorem. By differentiating (C.25) with respect to μ and then setting it equal to zero, the saddle points may be found as $\mu_o = a + m\pi$ ($m = 0, \pm1, \pm2 \ldots$). Since we consider $h_o << \rho$ or R, it follows from (C.23) that $a \cong \pi$ or $a = \pi - \varepsilon$ where ε is a very small real positive constant. Further, since Re [μ] lies between 0 and π the saddle point may be taken as

$$\mu_o = a = \pi - \varepsilon \cong \pi \quad . \tag{C.26}$$

The path of steepest descent may be given by setting Im [$g(\mu)$] = constant. That is

$$\cos (a - \alpha) \cosh \beta = \text{constant}. \tag{C.27}$$

Since this path passes through $\mu_o = a$, the constant may be found as equal to unity. Therefore the equation for the steepest descent path is

$$\cos (a - \alpha) \cosh \beta = 1 \ . \tag{C.28}$$

By deforming the contour C_1 to the path of steepest descent C_2, where C_2 is given by (C.28) and shown in figure C.1, the integral Q may be rewritten as

$$Q = I_a + I_b \ . \tag{C.29}$$

In the above the two integrals I_a and I_b are

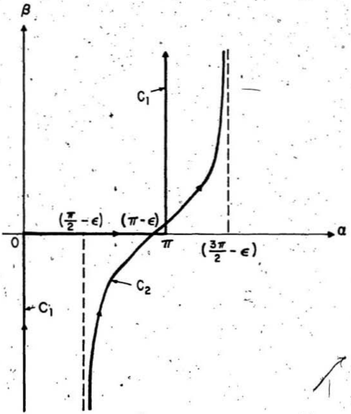

Figure C.1 Integration contours in the complex μ plane for equations (C.22), (C.30), and (C.31)

$$I_a = -\frac{\exp(j\ \pi/4)}{2\pi}\sqrt{\frac{k}{2\pi\rho}}\int_{C_2}\sqrt{\cos\mu}\ G\ (k\cos\mu\cos\theta,\ k\cos\mu\sin\theta)$$

$$\cdot\ \exp\ [kRg(\mu)]\ d\mu\ , \tag{C.30}$$

$$I_b = \frac{\exp(j\ \pi/4)}{2\pi}\sqrt{\frac{k}{2\pi\rho}}\int_{C_2}\frac{\Delta_a'\sqrt{\cos\mu}\ G\ (k\cos\mu\cos\theta,\ k\cos\mu\sin\theta)}{\sin\mu + \Delta_a\ (k\cos\mu\cos\theta,\ k\cos\mu\sin\theta)}$$

$$\cdot\ \exp\ [kRg(\mu)]\ d\mu\ . \tag{C.31}$$

The integral I_a may be considered first. Assuming G is a nice function and is slowly varying at the saddle point we find that the integrand is free from poles. This allows us to use the simple technique for an evaluation of (C.30) asymptotically. Since $[g(\mu_o) - g(\mu)]$ is purely real and positive along C_2 we may make the following substitution:

$$g(\mu_o) - g(\mu) = \tau^2 \tag{C.32}$$

where τ is real. Or

$$\cos\ (a - \mu) = 1 - j\tau^2 \quad ; \quad \frac{d\mu}{d\tau} = \frac{-\sqrt{2j}}{\sqrt{1 - j\frac{\tau^2}{2}}}\ . \tag{C.33}$$

By using (C.33) in (C.30), I_a may be given as

$$I_a = \frac{1}{2\pi}\sqrt{\frac{-k}{\pi\rho}}\ \exp\ (-jkR)\int_{-\infty}^{\infty}\frac{\sqrt{\cos\mu}\ G}{\sqrt{1 - j\frac{\tau^2}{2}}}\ \exp\ (-kR\tau^2)\ d\tau\ . \tag{C.34}$$

By expanding the slowly varying function in the above integral about the saddle point $\mu_o = a\ (\cong\pi)$, i.e., $\tau = 0$, and taking the first term of the expansion, I_a may asymptotically be given to the leading term as follows:

$$I_a \sim \frac{1}{\pi} \sqrt{\frac{k}{\pi\rho}} \exp(-jkR)\, G(-k\cos\theta, -k\sin\theta) \cdot \int_0^{\infty} \exp(-kR\tau^2)\, d\tau . \tag{C.35}$$

The succeeding terms in the asymptotic expansion may be shown to be $O(\frac{1}{kR})$ or smaller. The remaining integral in the above is the error function and is equal to $(\frac{\pi}{4kR})^{\frac{1}{2}}$ [Abramowitz and Stegun (1972, Ch. 7)]. Thus we have

$$I_a \simeq \frac{1}{2\pi} \sqrt{\frac{1}{\rho R}}\, G(-k\cos\theta, -k\sin\theta) \exp(-jkR). \tag{C.36}$$

We now consider the integral I_b given by (C.31). The form of this integral is similar to the one evaluated by Wait (1970, Ch. 2). Following his procedure and the assumptions: i) treating Δ_a as a constant by evaluating it at the saddle point $\mu_o = \pi$, ii) $|\Delta_a| << 1$, and iii) $0 < \arg \Delta_a < \frac{\pi}{4}$, it may be shown that

$$I_b \simeq \frac{\exp(j\pi/4)}{4\pi} \sqrt{\frac{k}{2\pi\rho}}\, \Delta_a \int_{C_2} \frac{\sqrt{\cos\mu}\, G(k\cos\mu\cos\theta, k\cos\mu\sin\theta)}{\sin(\frac{\mu+\nu}{2})\cos(\frac{\mu-\nu}{2})} \cdot \exp[kRg(\mu)]\, d\mu \tag{C.37}$$

where the following substitution has been made,

$$\sin\nu = \Delta_a(-k\cos\theta, -k\sin\theta) , \tag{C.38}$$

We note here that the integrand has a pole at $\mu = \pi + \nu$ because of the cosine function in the denominator. This pole is near the saddle point as ν (or Δ_a) is assumed to be small. However, because of the assumption $0 < \arg \Delta_a < \frac{\pi}{4}$, this pole is not crossed over when deforming the integration contour from C_1 to C_2. This singularity however makes the integrand not slowly varying near the saddle point. The slowly varying part, i.e., $\frac{\sqrt{\cos \mu}\, G}{\sin (\frac{\mu + \nu}{2})}$, may be taken out from the integral by evaluating it at the saddle point. We are then left with the following integral for I_b:

$$I_b \cong \frac{1}{4\pi} \left\{ \frac{-jk\Delta_a^2}{2\pi\rho} \right\}^{\frac{1}{2}} G\,(-k \cos \theta, -k \sin \theta) I_c \tag{C.39}$$

where we have approximated $\cos (\frac{\nu}{2})$ as unity, and

$$\Delta_a = \Delta_a\,(-k \cos \theta, -k \sin \theta) \tag{C.40}$$

$$I_c = \int_{C_2} \frac{\exp [kRg(\mu)]}{\cos (\frac{\mu - \nu}{2})}\, d\mu \tag{C.41}$$

By making the same substitution (C.32) or (C.33) as for the integral I_a yields

$$I_c = -2\sqrt{-2j}\, \exp\, (-jkR)\, \cos (\frac{a - \nu}{2}) \int_{\infty}^{\infty} \frac{\exp\, (-kR\tau^2)}{\tau^2 + 2j \cos^2 (\frac{a - \nu}{2})}\, d\tau . \tag{C.42}$$

The above integral has been evaluated by Wait (1970, Ch. 2) and the result is as follows:

$$I_c = -j2\pi \exp(-jkR) \exp(-w_a) \operatorname{erfc}(j\sqrt{w_a}) \quad (C.43)$$

where

$$w_a = -2jkR \cos^2 \left(\frac{a - \nu}{2}\right) . \quad (C.44)$$

The function erfc $(j\sqrt{w_a})$ is the complementary error function and is defined as

$$\operatorname{erfc}(z_o) = 1 - \operatorname{erf}(z_o) \quad (C.45)$$

where erf (z_o) is the error function. It is defined as

$$\operatorname{erf}(z_o) = \frac{2}{\sqrt{\pi}} \int_0^{z_o} \exp(-z^2)\, dz \quad . \quad (C.46)$$

Use of (C.43) in (C.39) yields

$$I_b \cong -\frac{j}{2} \left\{\frac{-jk\Delta_a^2}{2\pi\rho}\right\}^{\frac{1}{2}} G(-k \cos\theta, -k \sin\theta) \exp(-jkR)$$

$$\cdot \exp(-w_a) \operatorname{erfc}(j\sqrt{w_a}) \quad . \quad (C.47)$$

The above may also be written as

$$I_b \cong -\left(\frac{p_a}{w_a}\right)^{\frac{1}{2}} [1 - F(w_a)]\, G(-k \cos\theta, -k \sin\theta) \exp(-jkR)/(2\pi\rho) \quad (C.48)$$

where p_a and $F(w_a)$ are given as

$$p_a = -jk \frac{\rho}{2} \Delta_a^2 (-k \cos\theta, -k \sin\theta) \quad (C.49)$$

$$F(w_a) = 1 - j(\pi w_a)^{\frac{1}{2}} \exp(-w_a) \operatorname{erfc}(j\sqrt{w_a}) \quad .$$

In deriving the above result for the integral I_b it was initially assumed that $0 < \arg \Delta_a < \frac{\pi}{4}$. However, according to Wait, the result may be extended to include all arguments of Δ_a by analytic continuation.

By using the respective results (C.36) and (C.48) for I_a and I_b the original integral Q may be given as

$$Q \cong G\ (-k \cos \theta, -k \sin \theta)\ \left[1 - \left(\frac{p_a}{w_a} \right)^{\frac{1}{2}} \{1 - F\ (w_a)\} \right] \cdot \exp\ (-jkR)/(2\pi R) \quad \text{(C.50)}$$

In the above we have approximated $\rho = (R^2 - h_o^2)^{\frac{1}{2}} \cong R$ since the dipole height (h_o) is assumed to be much smaller compared to the distance (ρ).

Now from (C.44) w_a is given as

$$w_a = -2jkR \cos^2 (\frac{a - \nu}{2}) = -jk \frac{R}{2} [\frac{\sin \nu}{\sin (\frac{a + \nu}{2})}]^2 [1 + \frac{\sin a}{\sin \nu}]^2 ,$$

or by using (C.23) and (C.38)

$$w_a = -jk \frac{R}{2} [\frac{\Delta_a}{\sin (\frac{a + \nu}{2})}]^2 [1 + \frac{h_o}{R\Delta_a}]^2 .$$

Since $a \cong \pi$ and ν is very small we have $\sin (\frac{a + \nu}{2}) \cong 1$ and $R \cong \rho$. Therefore w_a may be related approximately to p_a as

$$w_a \cong p_a \left\{ 1 + \frac{h_o}{R\Delta_a} \right\}^2 . \quad \text{(C.51)}$$

It may easily be seen that in the limit as h_o tends to zero we get $w_a \cong p_a$ and, therefore, Q reduces to,

$$Q \cong G\ (-k \cos \theta, -k \sin \theta)\ F\ (p_a) \exp\ (-jk\rho)/(2\pi\rho) . \quad \text{(C.52)}$$

The function $F(p_a)$ and the argument p_a are given by (C.49).

APPENDIX D

DERIVATION OF SECOND ORDER DOPPLER SPECTRA AND ASSOCIATED CROSS SECTIONS

In this appendix we will carry out the derivation of the second order auto and cross correlation functions defined by ((4.89b) to (4.89h). By Fourier transforming these functions the corresponding backscattered Doppler spectra and hence the associated cross sections will be obtained.

D.1 $R_{fs}(\tau)$, $P_{fs}(\omega_d)$, and $\sigma_{fs}(\omega_d)$

The cross correlation of the first order backscattered field with the second order field is defined by (4.89b) as

$$R_{fs}(\tau) = \frac{A_r}{2\eta} \cdot < \tilde{E}_{zb1}(t_o, t+\tau) \, [\tilde{E}^*_{zb21}(t_o,t) + \tilde{E}^*_{zb22}(t_o,t) + \tilde{E}^*_{zb23}(t_o,t)] > \qquad (D.1)$$

where $\tilde{E}_{zb1}$ is the first order field and it is given by (4.77). $\tilde{E}_{zb21}$, $\tilde{E}_{zb22}$, and $\tilde{E}_{zb23}$ are the respective three parts of the second order field given by (4.78) to (4.80). From the expressions for these fields it is obvious that the above correlation function involves the averaging of the product of three first order surface Fourier coefficients. The form of this average may be written as

$$< {}_1P_{m,n,l} \; {}_1P^*_{p',q',i'} \; {}_1P^*_{m',n',l'} >$$

From oceanographic measurements it is found that the probability distribution of surface displacement is almost Gaussian with zero mean. Therefore, to the first order displacement, i.e., for $f_1(\vec{x},t)$, the distribution may be assumed to be Gaussian with zero mean [Phillips (1977, ch. 4)]. Since any linear operation on a Gaussian process is also

Gaussian, it implies that the first order surface Fourier coefficients ($_1P$'s) are also Gaussian random variables with zero mean [Thomas (1969, ch. 4)]. Further, the average of the product of three Gaussian variables, each with zero mean, is zero as shown by Thomas (1969, ch. 2). Therefore,

$$< {}_1P_{m,n,l} \; {}_1P^*_{p',q',i'} \; {}_1P^*_{m',n',l'} > = 0 \; . \tag{D.2}$$

This in turn gives

$$R_{fs}(\tau) = 0 \; , \tag{D.3}$$

hence

$$P_{fs}(\omega_d) = \sigma_{fs}(\omega_d) = 0 \; . \tag{D.4}$$

The equation (D.3) means that first and second orders of backscattered fields are uncorrelated.

D.2 $R_{s11}(\tau)$, $P_{s11}(\omega_d)$, and $\sigma_{s11}(\omega_d)$

The autocorrelation function of the first part of the second order field is defined by (4.89c) as

$$R_{s11}(\tau) = \frac{A_r}{2\eta} < \tilde{E}_{zb21}(t_o, t+\tau) \; \tilde{E}^*_{zb21}(t_o, t) > \; . \tag{D.5}$$

By using (4.78) for $\tilde{E}_{zb21}$ and (4.82) for A_r it becomes

$$R_{s11}(\tau) = \frac{P_t g_t g_r \lambda_o^2}{2(\pi\rho_o)^3} \, |F(\rho_o,1,0)|^4 \, \Delta_\rho^2$$

$$\cdot \sum_{p,q,i} \; \sum_{\substack{m,n,l \\ (m,n)\neq(-p,-q) \\ (m',n')\neq(-p',-q') \\ -\alpha|p+m|\leq q+n\leq\alpha|p+m| \\ -\alpha|p'+m'|\leq q'+n'\leq\alpha|p'+m'|}} \; \sum_{p',q',i'} \; \sum_{m',n',l'} \Big\{ < {}_1P_{p,q,i} \; {}_1P_{m,n,l} \; {}_1P^*_{p',q',i'} \; {}_1P^*_{m',n',l'} >$$

$$\cdot \, Sa[\Delta_\rho\{\mathrm{sgn}(p+m)\, K_{p+m} - 2k_o\}] \; Sa\,[\Delta_\rho\{\mathrm{sgn}(p'+m')K_{p'+m'} - 2k_o\}]$$

$$\cdot \{(K_{p+m})^{\frac{1}{2}} [K_{p+m} - k_o \ \mathrm{sgn}(p+m)] \ H(m,n,l,p,q,i)$$

$$+ Q_1(m,n,p,q)[K_m^2 K_{p+m} - 2k_o \ \mathrm{sgn}(p+m) \ \vec{K}_m \cdot \vec{K}_{p+m}]^{-\frac{1}{2}}\}$$

$$\cdot \{(K_{p'+m'})^{\frac{1}{2}} [K_{p'+m'} - k_o \ \mathrm{sgn}(p'+m')] \ H(m',n',l',p',q',i')$$

$$+ Q_1(m',n',p',q') \ [K_{m'}^2 \ K_{p'+m'} - 2k_o \ \mathrm{sgn}(p'+m') \ \vec{K}_{m'} \cdot \vec{K}_{p'+m'}]^{-\frac{1}{2}}\}^*$$

$$\cdot \exp \{j \ \mathrm{sgn}(p+m)(K_{p+m} \ \rho_o - \tfrac{\pi}{4}) + j(i+l)W(t+\tau)\}$$

$$\left. \cdot \exp \{-j \ \mathrm{sgn}(p'+m')(K_{p'+m'} \ \rho_o - \tfrac{\pi}{4}) - j(i'+l')Wt\} \right\} . \qquad (D.6)$$

In (D.6) the asterisk (*) as a superscript denotes complex conjugate. The two functions Q_1 and H are respectively given by (3.144a) and (4.56). $K_{m'}$ and $K_{p'+m'}$ are respectively given by K_m and K_{p+m} in (3.142) with (m,n,p,q) replaced by (m',n',p',q'). Now consider the average of the product of four first order Fourier coefficients in (D.6). Since these coefficients are Gaussian with zero mean, it follows then [Thomas (1969, ch. 2)],

$$< {}_1P_{p,q,i} \ {}_1P_{m,n,l} \ {}_1P^*_{p',q',i'} \ {}_1P^*_{m',n',l'} >$$

$$= < {}_1P_{p,q,i} \ {}_1P_{m,n,l} > < {}_1P^*_{p',q',i'} \ {}_1P^*_{m',n',l'} >$$

$$+ < {}_1P_{p,q,i} \ {}_1P^*_{p',q',i'} > < {}_1P_{m,n,l} \ {}_1P^*_{m',n',l'} >$$

$$+ < {}_1P_{p,q,i} \ {}_1P^*_{m',n',l'} > < {}_1P_{m,n,l} \ {}_1P^*_{p',q',i'} > . \qquad (D.7)$$

Since $f_1(\vec{x},t)$ is real surface, the first part of the three averages appearing on the right hand side in (D.7) may be rewritten as

$$< {}_1P^*_{-p,-q,-i} \ {}_1P_{m,n,l} > \ < {}_1P_{-p',-q',-i'} \ {}_1P^*_{m',n',l'} > .$$

From (4.26) this average is non-zero only when (m,n,l) = (-p,-q,-i) and (m',n',l') = (-p',-q',-i'). But these two equalities can not be satisfied in (D.6) because the equation has restrictions namely $(m,n) \neq (-p,-q)$ and $(m',n') \neq (-p',-q')$. Therefore this average may be treated as zero for (D.6). This is true also for the other auto and cross correlation functions generated by the three parts of the second order backscattered field as each part of the field has the restriction $(m,n) \neq (-p,-q)$. Thus, as far as the second order correlation functions are concerned, equation (D.7) reduces to:

$$< {}_1P_{p,q,i}\ {}_1P_{m,n,l}\ {}_1P^*_{p',q',i'}\ {}_1P^*_{m',n',l'} >$$

$$= <{}_1P_{p,q,i}\ {}_1P^*_{p',q',i'}> <{}_1P_{m,n,l}\ {}_1P^*_{m',n',l'}>$$

$$+ <{}_1P_{p,q,i}\ {}_1P^*_{m',n',l'}> <{}_1P_{m,n,l}\ {}_1P^*_{p',q',i'}> ,$$

or, from (4.26),

$$= \frac{N^4W^2}{(2\pi)^6} S_1(\vec{K}_p, iW)\, S_1(\vec{K}_m, lW)$$

with $(p',q',i',m',n',l')=(p,q,i,m,n,l)$

$$+ \frac{N^4W^2}{(2\pi)^6} S_1(\vec{K}_p, iW)\, S_1(\vec{K}_m, lW)$$

with $(p',q',i',m',n',l')=(m,n,l,p,q,i)$. (D.8)

By using (D.8) for the average in (D.6) it becomes

$$R_{s11}(\tau) = \frac{P_t g_t g_r \lambda_o^2}{(2\pi)^6 (\pi\rho_o)^3} |F(\rho_o,1,0)|^4 \Delta_\rho^2$$

$$\sum_{p,q,i} \sum_{\substack{m,n,l \\ (m,n)\neq(-p,-q) \\ -\alpha|p+m|\leq q+n\leq\alpha|p+m|}} \left\{ N^4 W^2 S_1(\vec{K}_p, iW)\, S_1(\vec{K}_m, lW) \right.$$

$$\cdot\ Sa^2 [\Delta_\rho \{sgn(p+m) K_{p+m} - 2k_o\}]$$

$$\cdot\ |(K_{p+m})^{\frac{1}{2}} [K_{p+m} - k_o\ sgn(p+m)] H(m,n,l,p,q,i)$$

$$+ \frac{1}{2} Q_1(m,n,p,q) [K_m^2\ K_{p+m} - 2k_o\ sgn(p+m)\ \vec{K}_m \cdot \vec{K}_{p+m}]^{-\frac{1}{2}}$$

$$+ \frac{1}{2} Q_1 (p,q,m,n)[K_p^2 \; K_{p+m} - 2k_o \; sgn(p+m) \; \vec{K}_p \cdot \vec{K}_{p+m}]^{-\frac{1}{2}}\Big|^2$$

$$\left. \cdot \; \exp \; [j(i+1)W\tau] \right\} \qquad (D.9)$$

where we have used the following symmetry for the function H given by (4.56),

$$H \; (m,n,1,p,q,i) = H \; (p,q,i,m,n,1) \; . \qquad (D.10)$$

By substituting

$$\begin{aligned} \alpha_1 &= (m - p)N/2 \;, & \beta_1 &= (n - q)N/2 \\ \alpha_2 &= (m + p)N \;, & \beta_2 &= (n + q)N \\ \omega_1 &= -1W \;, & \omega_2 &= -1W \;, \end{aligned} \qquad (D.11)$$

and extending the limits of L and T to infinity (i.e., $N,W \to 0$), the summations in (D.9) may be reduced to integrals as

$$R_{s11}(\tau) = \frac{P_t g_t g_r \lambda_o^2}{(2\pi)^6 (\pi\rho_o)^3} \; |F\;(\rho_o,1,0)|^4 \; \Delta_\rho^2$$

$$\cdot \int_{\alpha_1}\int_{\beta_1}\int_{\alpha_2}\int_{\beta_2=-\alpha|\alpha_2|}^{\alpha|\alpha_2|}\int_{\omega_1}\int_{\omega_2} S_1\;(\vec{K}_1,\omega_1)\; S_1\;(\vec{K}_2,\omega_2)$$

$$\cdot \; Sa^2\;[\Delta_\rho\{sgn(K_{3x})\;K_3 + 2k_o\}]$$

$$\cdot \; \Big|(K_3)^{\frac{1}{2}}\;[K_3 + sgn(K_{3x})k_o]\;H_c\;(\vec{K}_1,\omega_1,\vec{K}_2,\omega_2)$$

$$+ \frac{1}{2} Q_{1c}(\vec{K}_1,\vec{K}_3)[K_3K_2^2 + 2k_o \; sgn(K_{3x}) \; \vec{K}_2 \cdot \vec{K}_3]^{-\frac{1}{2}}$$

$$+ \frac{1}{2} Q_{1c}(\vec{K}_2,\vec{K}_3)[K_3K_1^2 + 2k_o \; sgn(K_{3x}) \; \vec{K}_1 \cdot \vec{K}_3]^{-\frac{1}{2}}\Big|^2$$

$$\cdot \; \exp\;[-j\;(\omega_1 + \omega_2)\tau]\; d\alpha_1\, d\beta_1\, d\alpha_2\, d\beta_2\, d\omega_1\, d\omega_2 \qquad (D.12)$$

where we have included the point $(\alpha_2,\beta_2) = (0,0)$ into the integrations by continuity. This point corresponds to the restriction $(m,n) \neq (-p,-q)$ on summations in (D.9). Also, the evenness property (4.12) for S_1 has been used in deriving (D.12). In the above equation we have

$$\vec{K}_1 = (\alpha_1 - \frac{\alpha_2}{2})\hat{x} + (\beta_1 - \frac{\beta_2}{2})\hat{y} \quad , \quad K_1 = |\vec{K}_1|$$

$$\vec{K}_2 = -(\alpha_1 + \frac{\alpha_2}{2})\hat{x} - (\beta_1 + \frac{\beta_2}{2})\hat{y} \quad , \quad K_2 = |\vec{K}_2| \qquad (D.13)$$

$$\vec{K}_3 = \vec{K}_1 + \vec{K}_2 = -\alpha_2\hat{x} - \beta_2\hat{y} \quad , \quad K_3 = |\vec{K}_3|$$

The two functions H_c and Q_{1c} may be given from the expressions (4.56) and (3.144a) of the corresponding functions H and Q_1 after substituting (D.11) in the two expressions. The result of this substitution may be given as follows:

$$H_c(\vec{K}_1,\omega_1,\vec{K}_2,\omega_2) = \frac{1}{2}[K_1 + K_2 + (K_1K_2 - \vec{K}_1 \cdot \vec{K}_2)\{\frac{gK_3 + (\omega_1+\omega_2)^2}{gK_3 - (\omega_1+\omega_2)^2}\}\frac{g}{\omega_1\omega_2}] \qquad (D.14)$$

where g is acceleration due to gravity.

$$Q_{1c}(\vec{K}_1,\vec{K}_3) = K_{1x}(K_{1x}+K_{3x})(\frac{K_{3x}}{K_3})^2 k_o^2$$
$$+ K_{1y}(K_{1y}+K_{3y})(\frac{K_{3y}}{K_3})^2 k_o^2 + K_1^2K_3^2$$
$$+ [K_{1x}(K_{1y}+K_{3y}) + K_{1y}(K_{1x}+K_{3x})]K_{3x}K_{3y}(\frac{k_o}{K_3})^2$$
$$+ [K_{3x}\{K_{1x}K_{3x}(K_{1x}+K_{3x}) + K_{1x}K_{3y}(K_{1y}+K_{3y}) + K_{3x}K_1^2\}$$
$$+ K_{3y}\{K_{1y}K_{3x}(K_{1x}+K_{3x}) + K_{1y}K_{3y}(K_{1y}+K_{3y}) + K_{3y}K_1^2\}]$$
$$\mathrm{sgn}(K_{3x})\frac{k_o}{K_3} \qquad (D.15)$$

The three dimensional wave height spectra present in (D.12) may be converted to two dimension or spatial spectra with the help of (4.30). The two integrals with respect to ω_1 and ω_2 in (D.12) may be then evaluated easily because of the associated delta functions in the conversion. The resulting expression may be Fourier transformed with respect to τ to give the following solution for Doppler spectrum.

$$P_{s11}(\omega_d) = \frac{P_t g_t g_r \lambda_o^2}{(4\pi)^3 \rho_o^4} |F(\rho_\theta, 1, 0)|^4 A_p$$

$$\cdot \frac{2\Delta_\rho}{\pi^3 \Delta_\theta} \sum_{+,-} \int_{\alpha_1} \int_{\beta_1} \int_{\alpha_2} \int_{\beta_2=-\alpha|\alpha_2|}^{\alpha|\alpha_2|} Sa^2[\Delta_\rho\{sgn(K_{3x})K_\beta + 2k_o\}]$$

$$\cdot \Big| (K_3)^{\frac{1}{2}}[K_3 + sgn(K_{3x})k_o] \, H_d(\vec{K}_1, \vec{K}_2, \omega_d)$$

$$+ \frac{1}{2} Q_{1c}(\vec{K}_1, \vec{K}_3)[K_3 K_2^2 + 2k_o \, sgn(K_{3x})\vec{K}_2 \cdot \vec{K}_3]^{-\frac{1}{2}}$$

$$+ \frac{1}{2} Q_{1c}(\vec{K}_2, \vec{K}_3)[K_3 K_1^2 + 2k_o \, sgn(K_{3x})\vec{K}_1 \cdot \vec{K}_3]^{-\frac{1}{2}} \Big|^2$$

$$\cdot S_1^{\pm}(\vec{K}_1) \, S_1^{\pm}(\vec{K}_2)$$

$$\cdot \delta\{\omega_d \mp sgn(K_{1x})(gK_1)^{\frac{1}{2}} \mp sgn(K_{2x})(gK_2)^{\frac{1}{2}}\}$$

$$\cdot d\alpha_1 \, d\beta_1 \, d\alpha_2 \, d\beta_2 \qquad (D.16)$$

where

$\sum_{+,-}$ = sum over superscripts (+) and (-), i.e., sum of the four combinations (+,+), (-,-), (+,-), and (-,+) of S_1's with corresponding argument of delta function

$$\mu = \begin{cases} 1, & \text{for } (+,+) \text{ or } (-,-) \text{ combination} \\ -1, & \text{for } (+,-) \text{ or } (-,+) \text{ combination} \end{cases} \qquad (D.17)$$

A_p: area of the patch = $4\rho_o \Delta_\rho \Delta_\theta$

$$H_d(\vec{K}_1,\vec{K}_2,\omega_d) = \frac{1}{2}[K_1 + K_2 + (K_1K_2 - \vec{K}_1 \cdot \vec{K}_2)$$

$$\cdot \frac{(gK_3 + \omega_d^2)}{(gK_3 - \omega_d^2)} \frac{\mu}{\mathrm{sgn}(K_{1x}K_{2x})(K_1K_2)^{1/2}}].$$

The above equation (D.16) may now be compared with the radar range equation (4.92). Since the attenuation function F in (D.16) is the same as F_p in (4.92), we obtain the following solution for the Doppler frequency dependent cross section.

$$\sigma_{s11}(\omega_d) = \frac{2\Delta_\rho}{\pi \Delta_\theta}$$

$$\cdot \sum_{+,-} \int_{\alpha_1} \int_{\beta_1} \int_{\alpha_2} \int_{\beta_2=-\alpha|\alpha_2|}^{\alpha|\alpha_2|} Sa^2 [\Delta_\rho\{\mathrm{sgn}(K_{3x})K_3 + 2k_o\}]$$

$$\cdot |(K_3)^{1/2} [K_3 + \mathrm{sgn}(K_{3x})k_o] H_d(\vec{K}_1,\vec{K}_2,\omega_d)$$

$$+ \frac{1}{2} Q_{1c} (\vec{K}_1,\vec{K}_3)[K_3K_2^2 + 2k_o \,\mathrm{sgn}(K_{3x})\vec{K}_2 \cdot \vec{K}_3]^{-1/2}$$

$$+ \frac{1}{2} Q_{1c} (\vec{K}_2,\vec{K}_3)[K_3K_1^2 + 2k_o \,\mathrm{sgn}(K_{3x})\vec{K}_1 \cdot \vec{K}_3]^{-1/2}|^2$$

$$\cdot S_1^{\pm}(\vec{K}_1) S_1^{\pm}(\vec{K}_2)$$

$$\cdot \{\omega_d \mp \mathrm{sgn}(K_{1x})(gK_1)^{1/2} \mp \mathrm{sgn}(K_{2x})(gK_2)^{1/2}\}$$

$$\cdot d\alpha_1 \, d\beta_1 \, d\alpha_2 \, d\beta_2 \qquad (D.18)$$

The equation (D.16) thus represents an average backscattered Doppler spectrum of the first part of the second order field. The corresponding cross section is given by (D.18). This part of the field is received from the patch with both first and second scatterings occuring on the patch. In either (D.16) or (D.18) the solution is in the form of four integrals. However, the two integrals with respect to α_2 and β_2 may be evaluated approximately. Since Δ_ρ is usually large, the limit of the sampling squared function may be taken to be the delta function like (4.108) in the first order case. By changing (α_2,β_2) to the polar form (λ_2,ϕ_2), the integral with respect to λ_2 may be evaluated easily because of the delta function obtained from this limiting process. The remaining integral with respect to ϕ_2 may be approximated by assuming the integrand constant over the narrow range of ϕ_2. That is, the integrand may be approximated at $\phi_2 = 0$ in the range $-\Delta_\theta \leq \phi_2 \leq -\Delta_\theta$. The result of these evaluations is then as follows:

$$\sigma_{s11}(\omega_d) \simeq \frac{16k_o^4}{\pi^2} \sum_{+,-} \int_{\alpha_1}\int_{\beta_1}^{\infty} |C_e + C_h|^2 \, S_1^{\pm}(\vec{K}_1)\, S_1^{\pm}(\vec{K}_2)$$

$$\cdot\, \delta\{\omega_d \mp \mathrm{sgn}(K_{1x})(gK_1)^{\frac{1}{2}} \mp \mathrm{sgn}(K_{2x})(gK_2)^{\frac{1}{2}}\}\, d\alpha_1\, d\beta_1 \qquad \text{(D.19)}$$

where now we have

$$\vec{K}_1 = (\alpha_1 - k_o)\hat{x} + \beta_1\hat{y} \quad , \; K_1 = |\vec{K}_1|$$

$$\vec{K}_2 = -(\alpha_1 + k_o)\hat{x} - \beta_1\hat{y} \; , \quad K_2 = |\vec{K}_2|$$

$$C_e = \frac{1}{2}\,\frac{(K_{1x}K_{2x} - 2\vec{K}_1 \cdot \vec{K}_2)}{(\vec{K}_1 \cdot \vec{K}_2)^{\frac{1}{2}}}$$

$$C_h = -\tfrac{j}{2}[K_1 + K_2 + (K_1K_2 - \vec{K}_1 \cdot \vec{K}_2)\frac{(\omega_B^2 + \omega_d^2)}{(\omega_B^2 - \omega_d^2)} \qquad \text{(D.20)}$$

$$\cdot\, \frac{\mu}{\mathrm{sgn}(K_{1x}K_{2x})(K_1K_2)^{\frac{1}{2}}}\;]$$

ω_B: Bragg frequency $= (2gk_o)^{\frac{1}{2}}$.

For the square root term present in the denominator in the expression for C_e, the root with imaginary part ≥ 0 should be taken.

D.3 $R_{s22}(\tau)$, $P_{s22}(\omega_d)$, and $\sigma_{s22}(\omega_d)$

The autocorrelation function of the second part of the second order field is defined by (4.89d) as

$$R_{s22}(\tau) = \frac{A_r}{2\eta} < \tilde{E}_{zb22}(t_o, t + \tau)\ \tilde{E}^*_{zb22}(t_o, t) > . \qquad (D.21)$$

By using (4.79) and (4.82) in (D.21) we obtain

$$R_{s22}(\tau) = \frac{P_t g_t g_r \lambda_o^2}{2(\pi\rho_o)^3} |F(\rho_o, 1, 0)|^4 \Delta_\rho^2$$

$$\cdot \sum_{p,q,i} \sum_{\substack{m,n,l \\ (m,n)\neq(-p,-q) \\ (m',n')\neq(-p',-q') \\ -\alpha|p|\leq q\leq\alpha|p| \\ -\alpha|p'|\leq q'\leq\alpha|p'|}} \sum_{p',q',i'} \sum_{m',n',l'} \left\{ < {}_1P_{p,q,i}\ {}_1P_{m,n,l}\ {}_1P^*_{p',q',i'}\ {}_1P^*_{m',n',l'} > \right.$$

$$\cdot \text{Sa}\,[\Delta_\rho\{\text{sgn}(p)K_p - 2k_o\}]\ \text{Sa}\,[\Delta_\rho\{\text{sgn}(p')K_{p'} - 2k_o\}]$$

$$\cdot Q_2(m,n,p,q)[K_m^2\ K_p + 2k_o\ \text{sgn}(p)\ \vec{K}_m \cdot \vec{K}_p\ ]^{-\frac{1}{2}}$$

$$\cdot Q_2(m',n',p',q')\{[K_{m'}^2\ K_{p'} + 2k_o\ \text{sgn}(p')\ \vec{K}_{m'} \cdot \vec{K}_{p'}]^{-\frac{1}{2}}\}^*$$

$$\cdot \exp\{j\ \text{sgn}(p)(K_p\ \rho_o - \frac{\pi}{4}) + j(i + 1)W(t + \tau)\}$$

$$\left. \cdot \exp\{-j\ \text{sgn}(p')(K_{p'}\ \rho_o - \frac{\pi}{4}) - j(i'+l')Wt\} \right\} . \qquad (D.22)$$

By using (D.8) for the average in (D.22) it becomes

$$R_{s22}(\tau) = \frac{P_t g_t g_r \lambda_o^2}{2(2\pi)^6(\pi\rho_o)^3} |F(\rho_o, 1, 0)|^4 \Delta_\rho^2$$

$$\cdot \left[\sum_{\substack{p,q,i \\ (m,n)\neq(-p,-q) \\ -\alpha|p|\leq q\leq\alpha|p|}} \sum_{m,n,l} \left\{ N^4 W^2 S_1(\vec{K}_p, iW)\, S_1(\vec{K}_m, lW) \right.\right.$$

$$\cdot\ Q_2^2(m,n,p,q)\ \left| K_m^2 K_p + 2k_o \operatorname{sgn}(p)\, \vec{K}_m \cdot \vec{K}_p \right|^{-1}$$

$$\left. \cdot\ Sa^2 \left[\Delta_\rho \{\operatorname{sgn}(p)\, K_p - 2k_o\}\right] \exp\left[j(i+l)W\tau\right] \right\}$$

$$+ \sum_{\substack{p,q,i \\ (m,n)\neq(-p,-q) \\ -\alpha|p|\leq q\leq\alpha|p| \\ \alpha|m|\leq n\leq\alpha|m|}} \sum_{m,n,l} \left\{ N^4 W^2 S_1(\vec{K}_p, i\overline{W})\, S_1(\vec{K}_m, lW) \right.$$

$$\cdot\ Q_2(m,n,p,q)\left[K_m^2 K_p + 2k_o \operatorname{sgn}(p)\, \vec{K}_m \cdot \vec{K}_p\right]^{-\frac{1}{2}}$$

$$\cdot\ Q_2(p,q,m,n)\{[K_p^2 K_m + 2k_o \operatorname{sgn}(m)\, \vec{K}_p \cdot \vec{K}_m]^{-\frac{1}{2}}\}^*$$

$$\cdot\ Sa\left[\Delta_\rho\{\operatorname{sgn}(p)\, K_p - 2k_o\}\right] Sa\left[\Delta_\rho\{\operatorname{sgn}(m)\, K_m - 2k_o\}\right]$$

$$\cdot\ \exp\left[j \operatorname{sgn}(p)\left(K_p \rho_o - \frac{\pi}{4}\right) - j \operatorname{sgn}(m)\left(K_m \rho_o - \frac{\pi}{4}\right)\right]$$

$$\left.\left. \cdot\ \exp\left[j(i+l)W\tau\right] \right\} \right] . \qquad (D.23)$$

By substituting

$$\alpha_1 = pN\ ;\quad \beta_1 = qN$$
$$\alpha_2 = mN\ ,\quad \beta_2 = nN \qquad (D.24)$$
$$\omega_1 = iW\ ,\quad \omega_2 = lW$$

and then taking the limits of both N and W to zero, the summations in (D.23) may be reduced to integrals as

$$R_{s22}(\tau) = \frac{P_t g_t g_r \lambda_o^2}{2(2\pi)^6 (\pi\rho_o)^3} \, |F\ (\rho_o,1,0)|^4 \, \Delta_\rho^2$$

$$\cdot \left\{ \int_{\alpha_1} \int_{\beta_1=-\alpha|\alpha_1|}^{\alpha|\alpha_1|} \int_{\alpha_2} \int_{\beta_2} \cdot \int_{\omega_1} \int_{\omega_2} S_1(\vec{K}_1,\omega_1)\ S_1(\vec{K}_2,\omega_2) \right.$$

$$\cdot\ Q_{2c}^2(\vec{K}_1,\vec{K}_2)\ |K_1K_2^2 + 2k_o\ \mathrm{sgn}(K_{1x})\vec{K}_1 \cdot \vec{K}_2|^{-1}$$

$$\cdot\ Sa^2\ [\Delta_\rho\{\mathrm{sgn}(K_{1x})K_1 - 2k_o\}]$$

$$\cdot\ \exp\ [j(\omega_1 + \omega_2)\tau]\ \ d\alpha_1\ d\beta_1\ d\alpha_2\ d\beta_2\ d\omega_1\ d\omega_2$$

$$+ \int_{\alpha_1} \int_{\beta_1=-\alpha|\alpha_1|}^{\alpha|\alpha_1|} \int_{\alpha_2} \int_{\beta_2=-\alpha|\alpha_2|}^{\alpha|\alpha_2|} \int_{\omega_1} \int_{\omega_2} S_1(\vec{K}_1,\omega_1) S_1(\vec{K}_2,\omega_2)$$

$$\cdot\ Q_{2c}(\vec{K}_1,\vec{K}_2)[K_1K_2^2 + 2k_o\ \mathrm{sgn}(K_{1x})\vec{K}_1 \cdot \vec{K}_2]^{-\frac{1}{2}}$$

$$\cdot\ Q_{2c}(\vec{K}_2,\vec{K}_1)\{[K_2K_1^2 + 2k_o\ \mathrm{sgn}(K_{2x})\vec{K}_2 \cdot \vec{K}_1]^{-\frac{1}{2}}\}^*$$

$$\cdot\ Sa\ [\Delta_\rho\{\mathrm{sgn}(K_{1x})K_1 - 2k_o\}]\ Sa\ [\Delta_\rho\{\mathrm{sgn}(K_{2x})K_2 - 2k_o\}]$$

$$\cdot\ \exp\ [j\ \mathrm{sgn}(K_{1x})(K_1\rho_o - \tfrac{\pi}{4}) - j\ \mathrm{sgn}(K_{2x})(K_2\rho_o - \tfrac{\pi}{4})]$$

$$\left. \cdot\ \exp\ [j\ (\omega_1 + \omega_2)\tau]\ d\alpha_1\ d\beta_1\ d\alpha_2\ d\beta_2\ d\omega_1\ d\omega_2 \right\} \tag{D.25}$$

where we have included the point $(\alpha_2,\beta_2) = (-\alpha_1,-\beta_1)$ into integrations by continuity. This point corresponds to the restriction $(m,n) = (-p,-q)$ on summations in (D.23). In the above equation we have

$$\begin{aligned} \vec{K}_1 &= \alpha_1\hat{x} + \beta_1\hat{y}\ , \quad K_1 = |\vec{K}_1| \\ \vec{K}_2 &= \alpha_2\hat{x} + \beta_2\hat{y}\ , \quad K_2 = |\vec{K}_2|\ , \end{aligned} \tag{D.26}$$

and the function Q_{2c} may be given from the expression of Q_2 in (3.144b) after substituting (D.24) in (3.144b). This substitution yields

$$Q_{2c}(\vec{K}_1,\vec{K}_2) = (2K_{1x} + K_{2x})K_{1x}^3 \left(\frac{k_o}{K_1}\right)^2$$

$$+ (2K_{1y} + K_{2y})K_{1y}^3 \left(\frac{k_o}{K_1}\right)^2 + K_1^2 |\vec{K}_1 + \vec{K}_2|^2$$

$$+ \{K_{1x}(2K_{1y} + K_{2y}) + K_{1y}(2K_{1x} + K_{2x})\} K_{1x}K_{1y} \left(\frac{k_o}{K_1}\right)^2$$

$$- \{(K_{1x} + K_{2x})(3K_{1x} + K_{2x}) + (K_{1y} + K_{2y})(3K_{1y} + K_{2y})\}$$

$$\cdot \operatorname{sgn}(K_{1x})K_1 k_o \qquad \text{(D.27)}$$

The three dimensional wave height spectra in (D.25) may be reduced to spatial spectra with the help of (4.30). The two integrations with respect to ω_1 and ω_2 may be easily performed because of the associated delta functions in this reduction. By doing so and then taking the Fourier transform of (D.25) with respect to τ yields the following expression for the backscattered Doppler spectrum,

$$P_{s22}(\omega_d) = \frac{P_t g_t g_r \lambda_o^2}{(4\pi)^3 \rho_o^4} |F_p|^4 A_p$$

$$\cdot \frac{\Delta_\rho}{\pi^3 \Delta_\theta} \left\{ \sum_{+,-} \int_{\alpha_1} \int_{\beta_1=-\alpha|\alpha_1|}^{\alpha|\alpha_1|} \int_{\alpha_2} \int_{\beta_2} Sa^2 [\Delta_\rho \{\operatorname{sgn}(K_{1x})K_1 - 2k_o\}] \right.$$

$$\cdot Q_{2c}^2 (\vec{K}_1,\vec{K}_2) \; |K_1 K_2^2 + 2k_o \operatorname{sgn}(K_{1x}) \vec{K}_1 \cdot \vec{K}_2|^{-1}$$

$$\cdot S_1^{\pm}(\vec{K}_1) \; S_1^{\pm}(\vec{K}_2)$$

$$\cdot \delta\{\omega_d \pm \operatorname{sgn}(K_{1x})(gK_1)^{1/2} \pm \operatorname{sgn}(K_{2x})(gK_2)^{1/2}\}$$

$$\cdot d\alpha_1 \, d\beta_1 \, d\alpha_2 \, d\beta_2$$

$$+ \sum_{+,-} \int_{\alpha_1} \int_{\beta_1=-\alpha|\alpha_1|}^{\alpha|\alpha_1|} \int_{\alpha_2} \int_{\beta_2=-\alpha|\alpha_2|}^{\alpha|\alpha_2|} \cdot \text{Sa}\,[\Delta_\rho\{\text{sgn}(K_{1x})K_1 - 2k_o\}]$$

$$\cdot \text{Sa}\,[\Delta_\rho\{\text{sgn}(K_{2x})K_2 - 2k_o\}]$$

$$\cdot Q_{2c}(\vec{K}_1,\vec{K}_2)\,[K_1K_2^2 + 2k_o\,\text{sgn}(K_{1x})\,\vec{K}_1 \cdot \vec{K}_2]^{-\frac{1}{2}}$$

$$\cdot Q_{2c}(\vec{K}_2,\vec{K}_1)\,\{[K_2K_1^2 + 2k_o\,\text{sgn}(K_{2x})\,\vec{K}_2 \cdot \vec{K}_1]^{-\frac{1}{2}}\}^*$$

$$\cdot \exp\,[j\,\text{sgn}(K_{1x})(K_1\rho_o - \tfrac{\pi}{4}) - j\,\text{sgn}(K_{2x})(K_2\rho_o - \tfrac{\pi}{4})]$$

$$\cdot S_1^{\pm}(\vec{K}_1)\,S_1^{\pm}(\vec{K}_2)$$

$$\cdot \delta\,\{\omega_d \pm \text{sgn}(K_{1x})(gK_1)^{\frac{1}{2}} \pm \text{sgn}(K_{2x})(gK_2)^{\frac{1}{2}}\}$$

$$\left. \cdot\, d\alpha_1\, d\beta_1\, d\alpha_2\, d\beta_2 \right\} \qquad (D.28)$$

where

$$F_p = F(\rho_o,1,0)$$
$$A_p = 4\,\rho_o\,\Delta_\rho\,\Delta_\theta \qquad (D.29)$$

and $\sum_{+,-}$ is defined by (D.17).

The above equation (D.28) thus represents an average backscattered Doppler spectrum of the second part of the second order field. This part of the field is received from the patch where the first scattering occurs at the source point while the second scattering occurs on the patch. By comparing this equation with the radar range equation (4.92) we obtain the corresponding Doppler frequency dependent cross section.

$$\sigma_{s22}(\omega_d) = \frac{\Delta_\rho}{\pi^3 \Delta_\theta} \sum_{+,-} \int_{\alpha_1} \int_{\beta_1=-\alpha|\alpha_1|}^{\alpha|\alpha_1|} \int_{\alpha_2} \int_{\beta_2}$$

$$\cdot \; Sa^2 \, [\Delta_\rho \{sgn(K_{1x}) \, K_1 - 2k_o\}]$$

$$\cdot \; Q_{2c}^2 \, (\vec{K}_1, \vec{K}_2) \, |K_1 K_2^2 + 2k_o \, sgn(K_{1x}) \, \vec{K}_1 \cdot \vec{K}_2|^{-1}$$

$$\cdot \; S_1^{\pm} \, (\vec{K}_1) \, S_1^{\pm} \, (\vec{K}_2)$$

$$\cdot \; \delta\{\omega_d \pm sgn(K_{1x})(gK_1)^{1/2} \pm sgn(K_{2x})(gK_2)^{1/2}\}$$

$$\cdot \; d\alpha_1 \, d\beta_1 \, d\alpha_2 \, d\beta_2$$

$$+ \frac{\Delta_\rho}{\pi^3 \Delta_\theta} \sum_{+,-} \int_{\alpha_1} \int_{\beta_1=-\alpha|\alpha_1|}^{\alpha|\alpha_1|} \int_{\alpha_2} \int_{\beta_2=-\alpha|\alpha_2|}^{\alpha|\alpha_2|}$$

$$\cdot \; Sa \, [\Delta_\rho \{sgn(K_{1x}) K_1 - 2k_o\}] \; Sa \, [\Delta_\rho \{sgn(K_{2x}) K_{2x} - 2k_o\}]$$

$$\cdot \; Q_{2c}(\vec{K}_1, \vec{K}_2) \, [K_1 K_2^2 + 2k_o \, sgn(K_{1x}) \, \vec{K}_1 \cdot \vec{K}_2]^{-1/2}$$

$$\cdot \; Q_{2c}(\vec{K}_2, \vec{K}_1) \, \{[K_2 K_1^2 + 2k_o \, sgn(K_{2x}) \vec{K}_2 \cdot \vec{K}_1]^{-1/2}\}^*$$

$$\cdot \; \exp \, [j \, sgn(K_{1x})(K_1 \rho_o - \tfrac{\pi}{4}) - j \, sgn(K_{2x})(K_2 \rho_o - \tfrac{\pi}{4})]$$

$$\cdot \; S_1^{\pm} \, (\vec{K}_1) \, S_1^{\pm} \, (\vec{K}_2)$$

$$\cdot \; \delta \, \{\omega_d \pm sgn(K_{1x})(gK_1)^{1/2} \pm sgn(K_{2x})(gK_2)^{1/2}\}$$

$$\cdot \; d\alpha_1 \, d\beta_1 \, d\alpha_2 \, d\beta_2 \qquad \text{(D.30)}$$

In order to simplify the above solution we may use the same approximations for evaluating the integrals which have been used previously in simplifying (D.18). The procedure for these approximate evaluations may be stated as follows. In the first part of (D.30) the integration variables (α_1, β_1) may be changed to polar form (λ_1, ϕ_1). By approximating the sampling squared function as the delta function, assuming large Δ_ρ, the integration with respect to λ_1 may be easily performed. The integration with respect to ϕ_1 may be approximated by treating the integrand constant over the small range of ϕ_1. The variable ϕ_1 ranges from $-\Delta_\theta$ to Δ_θ. In the second part of (D.30) both (α_1, β_1) and (α_2, β_2) may be changed to polar form (λ_1, ϕ_1) and (λ_2, ϕ_2). In this case both ϕ_1 and ϕ_2 range from $-\Delta_\theta$ to Δ_θ. Further, there are two sampling functions in this part. However, both of these functions may be approximated to respective delta functions. In this event the integrals with respect to λ_1 and λ_2 may be evaluated. The remaining integrals with respect to ϕ_1 and ϕ_2 may be approximated by treating the integrand constant over the narrow range of ϕ_1 and ϕ_2. The result of these evaluations is as follows:

$$\sigma_{s22}(\omega_d) = \frac{2k_o^4}{\pi^2} \sum_{+,-} \int_{\alpha_2} \int_{\beta_2} \frac{[(\vec{K}_1 + 2\vec{K}_2) \cdot \vec{K}_2]^2}{|(\vec{K}_1 + \vec{K}_2) \cdot \vec{K}_2|}$$

$$\cdot\ S_1^{\pm}(\vec{K}_1)\ S_1^{\pm}(\vec{K}_2)$$

$$\cdot\ \delta\{\omega_d \pm (gK_1)^{1/2} \pm \mathrm{sgn}(K_{2x})(gK_2)^{1/2}\}\ d\alpha_2\, d\beta_2$$

$$+ \frac{144\, k_o^7 \Delta_\theta}{\pi \Delta_\rho}[\{S_1^{+}(\vec{K}_1)\}^2\ \delta(\omega_d + 2\omega_B)$$

$$+ \{S_1^{-}(\vec{K}_1)\}^2\ \delta(\omega_d - 2\omega_B) + 2S_1^{+}(\vec{K}_1)\ S_1^{-}(\vec{K}_2)\ \delta(\omega_d)] \qquad (D.31)$$

where

$$\vec{K}_1 = 2k_o\hat{x} \quad , \quad K_1 = |\vec{K}_1| = 2k_o$$

$$\vec{K}_2 = \alpha_2\hat{x} + \beta_2\hat{y} \quad , \quad K_2 = |\vec{K}_2| \qquad (D.32)$$

$$\omega_B = (2gk_o)^{1/2}$$

D.4 $R_{s33}(\tau)$, $P_{s33}(\omega_d)$, and $\sigma_{s33}(\omega_d)$.

The autocorrelation function of the third part of the second order field is defined by (4.89e) as

$$R_{s33}(\tau) = \frac{A_r}{2\eta} < \tilde{E}_{zb23}(t_o, t+\tau)\, \tilde{E}^*_{zb23}(t_o, t)> \quad . \qquad (D.33)$$

By using (4.80) for $\tilde{E}_{zb23}$, (4.82) for A_r, and (D.8) for the average in (D.33) it becomes

$$R_{s33}(\tau) = \frac{4P_t g_t g_r \lambda_o^2 \Delta_\rho^2}{(2\pi)^6 \ (\pi\rho_o)^3}$$

$$\cdot \Bigg[\sum_{\substack{p,q,i \\ (m,n)\neq(-p,-q) \\ -\alpha|2p+m|\leq 2q+n\leq\alpha|2p+m| \\ 2k_o|m|N<K_m^2<2k_oK_m}} \sum_{m,n,l} \Bigg\{ N^4W^2 S_1(\vec{K}_p, iW) S_1(\vec{K}_m, lW)$$

$$\cdot \left| F(\beta_o\rho_o, 1, 0) F(r_b, \cos\psi_b, \sin\psi_b) F(r_c, -\cos\psi_c, -\sin\psi_c) \right|^2$$

$$\cdot Sa^2 \left[\frac{\beta_o\Delta_\rho}{2} \{ \mathrm{sgn}(2p+m) K_{2p+m} - 2k_o - \frac{|n|N}{K_m} K_{mo} \} \right]$$

$$\cdot K_m^2 Q_3^2(m,n,p,q) [\beta_o K_{2p+m} \{K_m^4 - (2k_o mN)^2\}]^{-1}$$

$$\cdot \exp[j(i+l)W\tau] \Bigg\}$$

$$+ \sum_{\substack{p,q,i \\ (m,n)\neq(-p,-q) \\ -\alpha|2p+m|\leq 2q+n\leq\alpha|2p+m| \\ -\alpha|2m+p|\leq 2n+q\leq\alpha|2m+p| \\ 2k_o|m|N<K_m^2<2k_oK_m \\ 2k_o|p|N<K_p^2<2k_oK_p}} \sum_{m,n,l} \Bigg\{ N^4W^2 S_1(\vec{K}_p, iW) S_1(\vec{K}_m, lW)$$

$$\cdot F(\beta_o\rho_o,1,0)\,F(r_b,\cos\psi_b,\sin\psi_b)F(r_c,-\cos\psi_c,-\sin\psi_c)$$

$$\cdot F^*(\beta_o'\rho_o,1,0)F^*(r_b',\cos\psi_b',\sin\psi_b')F^*(r_c',-\cos\psi_c',-\sin\psi_c')$$

$$\cdot \mathrm{Sa}\left[\frac{\beta_o\Delta_\rho}{2}\{\mathrm{sgn}(2p+m)K_{2p+m}-2k_o-\frac{|n|N}{K_m}K_{mo}\}\right]$$

$$\cdot \mathrm{Sa}\left[\frac{\beta_o\Delta_\rho}{2}\{\mathrm{sgn}(2m+p)K_{2m+p}-2k_o-\frac{|q|N}{K_p}K_{po}\}\right]$$

$$\cdot K_m\,Q_3(m,n,p,q)[\beta_o K_{2p+m}\{K_m^4-(2k_o\,mN)^2\}]^{-\frac{1}{2}}$$

$$\cdot K_p\,Q_3(p,q,m,n)[\beta_o' K_{2m+p}\{K_p^4-(2k_o\,pN)^2\}]^{-\frac{1}{2}}$$

$$\cdot \exp\left[j\,\mathrm{sgn}(2p+m)\left(K_{2p+m}\frac{\beta_o\rho_o}{2}-\frac{\pi}{4}\right)\right]$$

$$\cdot \exp\left[-j\,\mathrm{sgn}(2m+p)\left(K_{2m+p}\frac{\beta_o'\rho_o}{2}-\frac{\pi}{4}\right)\right]$$

$$\cdot \exp\left[-j\rho_o\left\{\beta_o\left(k_o+\frac{|n|N}{2K_m}K_{mo}\right)-\beta_o'\left(k_o+\frac{|q|N}{2K_p}K_{po}\right)\right\}\right]$$

$$\left.\cdot \exp[j(i+1)W\tau]\right]\Bigg] \qquad (D.34)$$

In (D.34) β_o, r_b, r_c, ψ_b, ψ_c, and K_{mo} are given by (3.142) and (3.143). They are functions of mN and nN only. By replacing (m,n) by (p,q) in their expressions the corresponding quantities β_o', r_b', r_c', ψ_b', ψ_c', and K_{po} may be given.

We shall first consider the second part of the right hand side of (D.34). Because of tighter restrictions on summation indices m,n,p, and q, the contribution to $R_{s33}(\tau)$ from this part may be approximated to zero. This may be shown here in the sense of integrals. By substituting

$$\alpha_1 = (2p + m)N \quad , \quad \beta_1 = (2q + n)N$$
$$\alpha_2 = (2m + p)N \quad , \quad \beta_2 = (2n + q)N \qquad (D.35)$$
$$\omega_1 = iW \quad , \quad \omega_2 = iW$$

and then extending the limits of L and T to infinity (i.e., $N,W \to 0$), the summations may be reduced to integrals. Symbolically, it may be written as

$$\int_{\alpha_1}\int_{\beta_1}\int_{\alpha_2}\int_{\beta_2}\int_{\omega_1}\int_{\omega_2} ---- \, d\alpha_1 d\beta_1 d\alpha_2 \, d\beta_2 d\omega_1 d\omega_2, \qquad (D.36)$$

where the integration variables α_1, β_1, α_2, and β_2 must comply with the set of four restrictions corresponding to the restrictions on summation indices mentioned in the second part of (D.34). These restrictions are

Res. 1: $-\alpha|\alpha_1| \leq \beta_1 \leq \alpha|\alpha_1|$

Res. 2: $-\alpha|\alpha_2| \leq \beta_2 \leq \alpha|\alpha_2|$

Res. 3: $6k_o \, |2\alpha_2 - \alpha_1| < \{(2\alpha_2 - \alpha_1)^2 + (2\beta_2 - \beta_1)^2\}$

and $\{(2\alpha_2 - \alpha_1)^2 + (2\beta_2 - \beta_1)^2\}^{\frac{1}{2}} < 6k_o$

Res. 4: $6k_o \, |2\alpha_1 - \alpha_2| < \{(2\alpha_1 - \alpha_2)^2 + (2\beta_1 - \beta_2)^2\}$

and $\{(2\alpha_1 - \alpha_2)^2 + (2\beta_1 - \beta_2)^2\}^{\frac{1}{2}} < 6k_o$

The restriction $(m,n) \neq (-p,-q)$ is not included in the above set as the integrations may be extended to this point by continuity.

By transforming the integration variables from cartesian form to polar form, i.e., changing (α_1, β_1) to $(\lambda_1, \emptyset_1)$ and (α_2,β_2) to $(\lambda_2, \emptyset_2)$, (D.36) may be written as

$$\int_{\emptyset_1}\int_{\emptyset_2}\int_{\lambda_1}\int_{\lambda_2}\int_{\omega_1}\int_{\omega_2} ---- \, d\lambda_1 d\lambda_2 d\emptyset_1 d\emptyset_2 d\omega_1 d\omega_2 \quad . \qquad (D.37)$$

The set of restrictions in terms of new variables are

Res. 1: $-\Delta_\theta \leq \emptyset_1 \leq \Delta_\theta$ and $(\pi - \Delta_\theta) \leq \emptyset_1 \leq (\pi + \Delta_\theta)$

Res. 2: $-\Delta_\theta \leq \emptyset_2 \leq \Delta_\theta$ and $(\pi - \Delta_\theta) \leq \emptyset_2 \leq (\pi + \Delta_\theta)$

Res. 3: $6k_o \left|2\lambda_2 \cos \emptyset_2 - \lambda_1 \cos \emptyset_1\right| < \{4\lambda_2^2 + \lambda_1^2 - 4\lambda_1\lambda_2 \cos (\emptyset_1 - \emptyset_2)\}$

and $\{4\lambda_2^2 + \lambda_1^2 - 4\lambda_1\lambda_2 \cos (\emptyset_1 - \emptyset_2)\}^{\frac{1}{2}} < 6k_o$

Res. 4: $6k_o \left|2\lambda_1 \cos \emptyset_1 - \lambda_2 \cos \emptyset_2\right| < \{4\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2 \cos (\emptyset_1 - \emptyset_2)\}$

and $\{4\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2 \cos (\emptyset_1 - \emptyset_2)\}^{\frac{1}{2}} < 6k_o$.

The integrand along with integrals with respect to λ_1, λ_2, ω_1, and ω_2 may be treated as a function of $\emptyset_1$ and $\emptyset_2$. Since Δ_θ, the one-half beam width of the receiving antenna, is assumed to be very small, the two integrals with respect to $\emptyset_1$ and $\emptyset_2$ may be approximated by treating this function constant over the narrow ranges $\emptyset_1$ and $\emptyset_2$. That is, the function may be evaluated at $\emptyset_1 = 0$ and π and $\emptyset_2 = 0$ and π. In this event the restrictions 3 and 4 reduce to:

$$\text{Res. 3:} \begin{cases} 6k_o < |2\lambda_2 - \lambda_1| < 6k_o, \text{ for } (\emptyset_1, \emptyset_2) = (0,0) \text{ or } (\pi, \pi) \\ 6k_o < |2\lambda_2 + \lambda_1| < 6k_o, \text{ for } (\emptyset_1, \emptyset_2) = (0,\pi) \text{ or } (\pi, 0) \end{cases}$$

$$\text{Res. 4:} \begin{cases} 6k_o < |2\lambda_1 - \lambda_2| < 6k_o, \text{ for } (\emptyset_1, \emptyset_2) = (0,0) \text{ or } (\pi, \pi) \\ 6k_o < |2\lambda_1 + \lambda_2| < 6k_o, \text{ for } (\emptyset_1, \emptyset_2) = (0,\pi) \text{ or } (\pi, 0) \end{cases}$$

Clearly, the above restrictions for λ_1 and λ_2 can not be met. However, again by continuity, the inequalities in the above restrictions may be allowed to include the end points as well. Thus the restrictions are met. But then both λ_1 and λ_2 reduce to a set of finite number of points and they do not span continuous regions any more. For $(\phi_1, \phi_2) = (0,0)$ or (π, π), $\lambda_1 = \lambda_2 = 6k_o$. Similarly, $\lambda_1 = \lambda_2 = 2k_o$ for $(\phi_1, \phi_2) = (0,\pi)$ or $(\pi, 0)$. Therefore, the integrals with respect to λ_1 and λ_2 in (D.37) may be taken to be zero. This means that the second part of (D.34) may be taken as zero under the above approximation for the ϕ_1 and ϕ_2 integrals.

The first part of (D.34) may be considered now. Let, for this part,

$$\begin{aligned} \alpha_1 &= (2p + m)n \quad ; \quad \beta_1 = (2q + n)N \\ \alpha_2 &= mN \quad , \quad \beta_2 = nN \\ \omega_1 &= iW \quad ; \quad \omega_2 = lW \end{aligned} \tag{D.38}$$

By taking the limits of N and W to zero, $R_{s33}(\tau)$ becomes

$$\begin{aligned} R_{s33}(\tau) = {} & \frac{4P_t g_t g_r \lambda_o^2 \Delta_\rho^2}{(2\pi)^6 \, (\pi\rho_o)^3} \\ & \cdot \int_{\alpha_1} \int_{\beta_1=-\alpha|\alpha_1|}^{\alpha|\alpha_1|} \int_{\alpha_2} \int_{\beta_2} \int_{\omega_1} \int_{\omega_2} S_1(\vec{K}_1, \omega_1) S_1(\vec{K}_2, \omega_2) \\ & \qquad 2k_o|K_{2x}| < K_2^2 < 2k_o K_2 \\ & \cdot \left|F(\xi_o\rho_o, 1, 0) F(R_b, \cos\phi_b, \sin\phi_b) F(R_c, -\cos\phi_c, -\sin\phi_c)\right|^2 \\ & \cdot Sa^2 \left[\frac{\xi_o\Delta_\rho}{2} \left\{\mathrm{sgn}(K_{3x})K_3 - 2k_o - \frac{|K_{2y}|}{K_2} K_{2s}\right\}\right] \\ & \cdot K_2^2 \, Q_{3c}^2 (\vec{K}_1, \vec{K}_3) [\xi_o K_3 \{K_2^4 - (2k_o K_{2x})^2\}]^{-1} \\ & \cdot \exp[j(\omega_1 + \omega_2)\tau] d\alpha_1 d\beta_1 d\alpha_2 d\beta_2 d\omega_1 d\omega_2 \end{aligned} \tag{D.39}$$

In the above we have

$$\vec{K}_1 = \frac{1}{2}(\alpha_1 - \alpha_2)\hat{x} + \frac{1}{2}(\beta_1 - \beta_2)\hat{y} \quad , \quad K_1 = |\vec{K}_1|$$

$$\vec{K}_2 = \alpha_2\hat{x} + \beta_2\hat{y} \quad , \quad K_2 = |\vec{K}_2| \tag{D.40}$$

$$\vec{K}_3 = 2\vec{K}_1 + \vec{K}_2 = \alpha_1\hat{x} + \beta_1\hat{y} \quad , \quad K_3 = |\vec{K}_3|$$

and ξ_o, R_b, R_c, ϕ_b, ϕ_c, and K_{2s} are given by substituting (D.38) in the corresponding expressions for β_o, r_b, r_c, ψ_b, ψ_c, and K_{mo} which are given by (3.142) and (3.143). By making this set of substitutions in (3.142) and (3.143) we get

$$\xi_o = \xi_o(\vec{K}_2) = 2\left\{1 + \frac{2k_o|K_{2y}|}{K_2K_{2s}}\right\}^{-1}$$

$$R_b = R_b(\vec{K}_2) = \frac{\rho_o k_o \xi_o}{K_2}\left\{\frac{|K_{2y}|}{K_{2s}} + \frac{K_{2x}}{K_2}\right\}$$

$$R_c = R_c(\vec{K}_2) = \frac{\rho_o k_o \xi_o}{K_2}\left\{\frac{|K_{2y}|}{K_{2s}} - \frac{K_{2x}}{K_2}\right\} \tag{D.41}$$

$$\cos\phi_b = \cos\{\phi_b(\vec{K}_2)\} = \frac{\rho_o\xi_o}{2R_b}\left\{\frac{4k_o^2K_{2x}|K_{2y}|}{K_{2s}K_2^3} + 1\right\}$$

$$\sin\phi_b = \sin\{\phi_b(\vec{K}_2)\} = \frac{\rho_o\xi_o}{2R_b}\,\mathrm{sgn}(K_{2y})\,\frac{\{K_2^4 - 4k_o^2K_{2x}^2\}}{K_{2s}K_2^3}$$

$$\cos\phi_c = \cos\{\phi_c(\vec{K}_2)\} = \frac{\rho_o\xi_o}{2R_c}\left\{\frac{4k_o^2K_{2x}|K_{2y}|}{K_{2s}K_2^3} - 1\right\}$$

$$\sin\phi_c = \sin\{\phi_c(\vec{K}_2)\} = \frac{\rho_o\xi_o}{2R_c}\,\mathrm{sgn}(K_{2y})\,\frac{\{K_2^4 - 4k_o^2K_{2x}^2\}}{K_{2s}K_2^3}$$

$$K_{2s} = (4k_o^2 - K_2^2)^{\frac{1}{2}}.$$

In a similar way, Q_{3c} may be given from the equation (3.144c) for Q_3 as

$$Q_{3c}(\vec{K}_1,\vec{K}_3) = K_{1x}K_{3x}^3 \left(\frac{k_o}{K_3}\right)^2 + K_{1y}K_{3y}^3\left(\frac{k_o}{K_3}\right)^2$$

$$+ K_1^2 |\vec{K}_3 - \vec{K}_1|^2 + \{K_{1x}K_{3y} + K_{1y}K_{3x}\} K_{3x}K_{3y} \left(\frac{k_o}{K_3}\right)^2$$

$$- [K_{3x}\{(K_{3x} - K_{1x})(K_{1x}K_{3x} + K_1^2) + K_{1x}K_{3y}(K_{3y} - K_{1y})\}$$

$$+ K_{3y}\{(K_{3y} - K_{1y})(K_{1y}K_{3y} + K_1^2) + K_{1y}K_{3x}(K_{3x} - K_{1x})\}]$$

$$\cdot \operatorname{sgn}(K_{3x}) \frac{k_o}{K_3} . \qquad \text{(D.42)}$$

By using (4.30) in (D.39) and then taking the Fourier transform with respect to τ we obtain

$$P_{s33}(\omega_d) = \frac{4P_t g_t g_r \lambda_o^2 \Delta_\rho^2}{(2\pi)^3 (\pi\rho_o)^3}$$

$$\cdot \sum_{+,-} \int_{\alpha_1} \int_{\beta_1=-\alpha|\alpha_1|}^{\alpha|\alpha_1|} \int_{\alpha_2} \int_{\beta_2} |F(\xi_o\rho_o, 1, 0)|^2$$

$$2k_o|K_{2x}| < K_2^2 < 2k_oK_2$$

$$\cdot |F(R_b, \cos\phi_b, \sin\phi_b) F(R_c, -\cos\phi_c, -\sin\phi_c)|^2$$

$$\cdot Sa^2\left[\frac{\xi_o\Delta_\rho}{2} \{\operatorname{sgn}(K_{3x})K_3 - 2k_o - \frac{|K_{2y}|}{K_2} K_{2s}\}\right]$$

$$\cdot K_2^2 Q_{3c}^2 (\vec{K}_1,\vec{K}_3)[\xi_o K_3 \{K_2^4 - (2k_oK_{2x})^2\}]^{-1}$$

$$\cdot S_1^{\pm}(\vec{K}_1) S_1^{\pm}(\vec{K}_2)$$

$$\cdot \delta\{\omega_d \pm \operatorname{sgn}(K_{1x})(gK_1)^{\frac{1}{2}} \pm \operatorname{sgn}(K_{2x})(gK_2)^{\frac{1}{2}}\}$$

$$\cdot d\alpha_1 d\beta_1 d\alpha_2\, d\beta_2 . \qquad \text{(D.43)}$$

The above solution thus gives an average backscattered Doppler spectrum of the third part of the second order field. This part of the field represents the case where the two scatterings occur elsewhere on the surface other than the patch. However, for deriving the average cross section corresponding to the above Doppler spectrum we may treat this solution as if the third part of the field also is received from the patch. Then by rearranging and comparing (D.43) with the radar range equation (4.92) we get the following cross section.

$$\sigma_{s33}(\omega_d) = |F_p|^{-4} \frac{8\Delta_\rho}{\pi^3 \Delta_\theta}$$

$$\cdot \sum_{+,-} \int_{\alpha_1} \int_{\beta_1=-\alpha|\alpha_1|}^{\alpha|\alpha_1|} \int_{\alpha_2} \int_{\beta_2} |F(\xi_o\rho_o, 1, 0)|^2$$

$$2k_o|K_{2x}|<K_2^2<2k_oK_2$$

$$\cdot |F(R_b, \cos \phi_b, \sin \phi_b)F(R_c, -\cos \phi_c, -\sin \phi_c)|^2$$

$$\cdot Sa^2[\frac{\xi_o\Delta_\rho}{2}\{\mathrm{sgn}\ (K_{3x})K_3 - 2k_o - \frac{|K_{2y}|}{K_2} K_{2s}\}]$$

$$\cdot K_2^2\ Q_{3c}^2(\vec{K}_1,\vec{K}_3)[\xi_o K_3\{K_2^4 - (2k_oK_{2x})^2\}]^{-1}$$

$$\cdot S_1^{\pm}(\vec{K}_1)\ S_1^{\pm}(\vec{K}_2)$$

$$\cdot \delta\ \{\omega_d \pm \mathrm{sgn}\ (K_{1x})(gK_1)^{\frac{1}{2}} \pm \mathrm{sgn}\ (K_{2x})(gK_2)^{\frac{1}{2}}\}$$

$$\cdot d\alpha_1 d\beta_1 d\alpha_2 d\beta_2 \qquad \text{(D.44)}$$

where F_p is the one way attenuation function between transmitting point and the patch, i.e.,

$$F_p = F(\rho_o, 1, 0)\ . \qquad \text{(D.45)}$$

The two integrals with respect to α_1 and β_1 in (D.45) may be evaluated approximately by following the procedure used for similar evaluations in (D.18) or (D.30). The result is as follows:

$$\sigma_{s33}(\omega_d) \cong \frac{32}{\pi^2} |F_p|^{-4} \sum_{+,-} \int_{\alpha_2} \int_{\beta_2} |F(\xi_o \rho_o, 1, 0)|^2 \cdot$$

$$2k_o|K_{2x}| < K_2^2 < 2k_o K_2$$

$$\cdot \; |F(R_b, \cos \phi_b, \sin \phi_b) F(R_c, -\cos \phi_c, -\sin \phi_c)|^2$$

$$\cdot \; [(k_o - K_{1x} - K_{2x})\{k_o K_{1x}(2K_{1x} + K_{2x})$$

$$- K_1^2 (K_{1x} + K_{2x})\} + K_1^2 K_{1y}^2]^2$$

$$\cdot \; \frac{K_2^2}{\xi_o^2 [K_2^4 - (2k_o K_{2x})^2]} S_1^{\pm}(\vec{K}_1) S_1^{\pm}(\vec{K}_2)$$

$$\cdot \; \delta\{\omega_d \pm \text{sgn}(K_{1x})(gK_1)^{\frac{1}{2}} \pm \text{sgn}(K_{2x})(gK_2)^{\frac{1}{2}}\} \, d\alpha_2 d\beta_2 \qquad \text{(D.46)}$$

where now we have

$$\vec{K}_2 = \alpha_2 \hat{x} + \beta_2 \hat{y} \qquad , \quad K_2 = |\vec{K}_2|$$

$$\vec{K}_1 = (k_o + \frac{|K_{2y}|}{2K_2} K_{2s} - \frac{1}{2} K_{2x})\hat{x} - \frac{1}{2} K_{2y}\hat{y} \quad , \quad K_1 = |\vec{K}_1| \qquad \text{(D.47)}$$

The other quantities ξ_o; R_b, R_c, ϕ_b, ϕ_c, and K_{2s} are given by (D.41).

D.5 $R_{s12}(\tau)$, $\mathcal{P}_{s12}(\omega_d)$, and $\sigma_{s12}(\omega_d)$

The cross correlation function of the first part with the second part of the second order field is defined by (4.89f) as

$$R_{s12}(\tau) = \frac{A_r}{2\eta} < \tilde{E}_{zb21}(t_o, t + \tau) \tilde{E}^*_{zb22}(t_o, t) > \qquad \text{(D.48)}$$

where $\tilde{E}_{zb21}$ and $\tilde{E}_{zb22}$ are given by (4.78) and (4.79) respectively. A_r is given by (4.82). Using these equations in (D.48) yields

$$R_{s12}(\tau) = \frac{P_t g_t g_r \lambda_o^2}{2(\pi\rho_o)^3} |F(\rho_o, 1, 0)|^2 \Delta_\rho^2$$

$$\cdot \sum_{p,q,i} \sum_{\substack{m,n,1 \\ (m,n)\neq(-p,-q) \\ (m',n')\neq(-p',-q') \\ -\alpha|p+m|\leq q+n\leq\alpha|p+m| \\ \alpha|p'|\leq q'\leq\alpha|p'|}} \sum_{p',q',i'} \sum_{m',n',1'} \Bigg\{ \langle {}_1P_{p,q,i}\ {}_1P_{m,n,1}\ {}_1P^*_{p',q',i'}\ {}_1P^*_{m',n',1'} \rangle$$

$$\cdot \text{Sa}\,[\Delta_\rho\{\text{sgn}(p+m)K_{p+m} - 2k_o\}]\ \text{Sa}\,[\Delta_\rho\,\{\text{sgn}(p')K_{p'} - 2k_o\}]$$

$$\cdot \{(K_{p+m})^{\frac{1}{2}}[K_{p+m} - k_o\ \text{sgn}(p+m)]\ H(m,n,1,p,q,i)$$

$$+ Q_1(m,n,p,q)\ [K_m^2 K_{p+m} - 2k_o\ \text{sgn}(p+m)\vec{K}_m \cdot \vec{K}_{p+m}]^{-\frac{1}{2}}\}$$

$$\cdot Q_2(m',n',p',q')\{[K_{m'}^2 K_{p'} + 2k_o\ \text{sgn}(p')\vec{K}_{m'} \cdot \vec{K}_{p'}]^{-\frac{1}{2}}\}^*$$

$$\cdot \exp\,[j\ \text{sgn}(p+m)(K_{p+m}\rho_o - \tfrac{\pi}{4}) + j(i+1)W(t+\tau)]$$

$$\cdot \exp\,[-j\ \text{sgn}(p')(K_{p'}\rho_o - \tfrac{\pi}{4}) - j(i'+1')Wt] \Bigg\} \tag{D.49}$$

where $\vec{K}_{p'}$ and $\vec{K}_{m'}$ are respectively given by $\vec{K}_p$ and $\vec{K}_m$ in (4.22) with (m,n,p,q) replaced by (m',n',p',q'). By using (D.8) for the average and (D.10) it becomes

$$R_{s12}(\tau) = \frac{P_t g_t g_r \lambda_o^2}{(2\pi)^6(\pi\rho_o)^3} |F(\rho_o, 1, 0)|^4 \Delta_\rho^2$$

$$\cdot \sum_{\substack{p,q,i \\ -\alpha|p|\le q\le\alpha|p|}} \sum_{\substack{m,n,1 \\ (m,n)\neq(-p,-q) \\ -\alpha|p+m|\le q+n\le\alpha|p+m|}} \Big\{ N^4 W^2 S_1(\vec{K}_p, iW) S_1(\vec{K}_m, 1W)$$

$$\cdot \, Sa[\Delta_\rho\{sgn(p+m)K_{p+m} - 2k_o\}] Sa[\Delta_\rho\{sgn(p)K_p - 2k_o\}]$$

$$\cdot \, \{(K_{p+m})^{\frac{1}{2}}[K_{p+m} - k_o\, sgn(p+m)] H(m,n,1,p,q,i)$$

$$+ \frac{1}{2} Q_1(m,n,p,q)[K_m^2 K_{p+m} - 2k_o\, sgn(p+m)\vec{K}_m \cdot \vec{K}_{p+m}]^{-\frac{1}{2}}$$

$$+ \frac{1}{2} Q_1(p,q,m,n)[K_p^2 K_{p+m} - 2k_o\, sgn(p+m)\vec{K}_p \cdot \vec{K}_{p+m}]^{-\frac{1}{2}}\}$$

$$\cdot \, Q_2(m,n,p,q)\{[K_m^2 K_p + 2k_o\, sgn(p)\vec{K}_m \cdot \vec{K}_p]^{-\frac{1}{2}}\}^*$$

$$\cdot \, \exp\,[j\, sgn(p+m)(K_{p+m}\rho_o - \tfrac{\pi}{4}) - j\, sgn(p)(K_p\rho_o - \tfrac{\pi}{4})]$$

$$\cdot \, \exp\,[j(i+1)W\tau] \Big\} \; . \tag{D.50}$$

By substituting

$$\begin{aligned} \alpha_1 &= pN , & \beta_1 &= qN \\ \alpha_2 &= (p+m)N , & \beta_2 &= (q+n)N \\ \omega_1 &= iW , & \omega_2 &= 1W \end{aligned} \tag{D.51}$$

and then by taking the limits of N and W to zero we get

$$R_{s12}(\tau) = \frac{P_t g_t g_r \lambda_o^2}{(2\pi)^6 (\pi\rho_o)^3} |F(\rho_o, 1, 0)|^4 \Delta_\rho^2$$

$$\cdot \int_{\alpha_1} \int_{\beta_1=-\alpha|\alpha_1|}^{\alpha|\alpha_1|} \int_{\alpha_2} \int_{\beta_2=-\alpha|\alpha_2|}^{\alpha|\alpha_2|} \int_{\omega_1} \int_{\omega_2} S_1(\vec{K}_1,\omega_1) S_1(\vec{K}_2,\omega_2)$$

$$\cdot \, Sa[\Delta_\rho\{sgn(K_{3x})K_3 - 2k_o\}]Sa[\Delta_\rho\{sgn(K_{1x})K_1 - 2k_o\}]$$

$$\cdot \, \{(K_3)^{\frac{1}{2}}[K_3 - k_o \; sgn(K_{3x})]H_c(\vec{K}_1,\omega_1,\vec{K}_2,\omega_2)$$

$$+ \frac{1}{2} Q_{1d}(\vec{K}_1,\vec{K}_3)[K_3K_2^2 - 2k_o \; sgn(K_{3x})\vec{K}_2 \cdot \vec{K}_3]^{-\frac{1}{2}}$$

$$+ \frac{1}{2} Q_{1d}(\vec{K}_2,\vec{K}_3)[K_3K_1^2 - 2k_o \; sgn(K_{3x})\vec{K}_1 \cdot \vec{K}_3]^{-\frac{1}{2}}\}$$

$$\cdot \, Q_{2c}(\vec{K}_1,\vec{K}_2)\{[K_1K_2^2 + 2k_o \; sgn(K_{1x})\vec{K}_1 \cdot \vec{K}_2]\}^*$$

$$\cdot \, \exp\,[j \; sgn(K_{3x})(K_3\rho_o - \frac{\pi}{4}) - j \; sgn(K_{1x})(K_1\rho_o - \frac{\pi}{4})$$

$$\cdot \, \exp\,[j\,(\omega_1 + \omega_2)\tau]d\alpha_1 d\beta_1 d\alpha_2 d\beta_2 d\omega_1 d\omega_2 \qquad (D.52)$$

where

$$\vec{K}_1 = \alpha_1\hat{x} + \beta_1\hat{y} \qquad , \qquad K_1 = |\vec{K}_1|$$

$$\vec{K}_2 = (\alpha_2 - \alpha_1)\hat{x} + (\beta_2 - \beta_1)\hat{y} \qquad , \qquad K_2 = |\vec{K}_2|$$

$$\vec{K}_3 = \vec{K}_1 + \vec{K}_2 = \alpha_2\hat{x} + \beta_2\hat{y} \qquad , \qquad K_3 = |\vec{K}_3| \qquad (D.53)$$

The functions H_c and Q_{2c} are given by (D.14) and (D.27) respectively. The function Q_{1d} may be given from the expression (3.144a) of the corresponding function Q_1 after substituting (D.51) and (D.53) in that expression. The result of this substitution is as follows:

$$Q_{1d}(K_1, K_3) = K_{1x}(K_{1x} + K_{3x})(\frac{K_{3x}}{K_3})^2 k_o^2$$

$$+ K_{1y}(K_{1y} + K_{3y})(\frac{K_{3y}}{K_3})^2 k_o^2 + K_1^2K_3^2$$

$$+ [K_{1x}(K_{1y} + K_{3y}) + K_{1y}(K_{1x} + K_{3x})]K_{3x}K_{3y}(\frac{k_o}{K_3})^2$$

$$- [K_{3x}\{K_{1x}K_{3x}(K_{1x} + K_{3x}) + K_{1x}K_{3y}(K_{1y} + K_{3y}) + K_{3x}K_1^2\}$$

$$+ K_{3y}\{K_{1y}K_{3x}(K_{1x} + K_{3x}) + K_{1y}K_{3y}(K_{1y} + K_{3y}) + K_{3y}K_1^2\}]$$

$$\cdot \operatorname{sgn}(K_{3x})\frac{k_o}{K_3} \quad \text{(D.54)}$$

The three dimensional wave height spectra appearing in (D.52) may be reduced to two dimensional or spatial spectra with the help of (4.30). Then by taking the Fourier transform of (D.52) with respect to τ we get the following expression for the Doppler spectrum.

$$P_{s12}(\omega_d) = \frac{P_t g_t g_r \lambda_o^2}{(4\pi)^3 \rho_o^4} |F(\rho_o, 1, 0)|^4 A_p \frac{2\Delta_\rho}{\pi^3 \Delta_\theta}$$

$$\cdot \sum_{+,-} \int_{\alpha_1} \int_{\beta_1=-\alpha|\alpha_1|}^{\alpha|\alpha_1|} \int_{\alpha_2} \int_{\beta_2=-\alpha|\alpha_2|}^{\alpha|\alpha_2|} \operatorname{Sa}[\Delta_\rho\{\operatorname{sgn}(K_{3x})K_3 - 2k_o\}]$$

$$\cdot \operatorname{Sa}[\Delta_\rho\{\operatorname{sgn}(K_{1x})K_1 - 2k_o\}]$$

$$\cdot \{(K_3)^{\frac{1}{2}}[K_3 - k_o \operatorname{sgn}(K_{3x})]H_d(\vec{K}_1, \vec{K}_2, \omega_d)$$

$$+ \frac{1}{2} Q_{1d}(\vec{K}_1, \vec{K}_3)[K_3K_2^2 - 2k_o \operatorname{sgn}(K_{3x})\vec{K}_2 \cdot \vec{K}_3]^{-\frac{1}{2}}$$

$$+ \frac{1}{2} Q_{1d}(\vec{K}_2, \vec{K}_3)[K_3K_1^2 - 2k_o \operatorname{sgn}(K_{3x})\vec{K}_1 \cdot \vec{K}_3]^{-\frac{1}{2}}\}$$

$$\cdot Q_{2c}(\vec{K}_1, \vec{K}_2)\{[K_1K_2^2 + 2k_o \operatorname{sgn}(K_{1x})\vec{K}_1 \cdot \vec{K}_2]\}^*$$

$$\cdot \exp\,[j \operatorname{sgn}(K_{3x})(K_3\rho_o - \frac{\pi}{4}) - j \operatorname{sgn}(K_{1x})(K_1\rho_o - \frac{\pi}{4})]$$

$$\cdot S_1^{\pm}(\vec{K}_1)\, S_1^{\pm}(\vec{K}_2)$$

$$\cdot \delta\{\omega_d \pm \operatorname{sgn}(K_{1x})(gK_1)^{\frac{1}{2}} \pm \operatorname{sgn}(K_{2x})(gK_2)^{\frac{1}{2}}\}$$

$$\cdot d\alpha_1 d\beta_1 d\alpha_2 d\beta_2 \quad \text{(D.55)}$$

where H_d and A_p are given by (D.17). The above solution thus gives an average backscattered Doppler spectrum due to cross correlation of the first part with the second part of the second order field. By comparing this solution with the radar range equation (4.92) we get the corresponding Doppler frequency dependent cross section.

$$\sigma_{s12}(\omega_d) = \frac{2\Delta_\rho}{\pi^3 \Delta_\theta}$$

$$\cdot \sum_{+,-} \int_{\alpha_1} \int_{\beta_1=-\alpha|\alpha_1|}^{\alpha|\alpha_1|} \int_{\alpha_2} \int_{\beta_2=-\alpha|\alpha_2|}^{\alpha|\alpha_2|} \mathrm{Sa}[\Delta_\rho\{\mathrm{sgn}(K_{3x})K_3 - 2k_o\}]$$

$$\cdot \mathrm{Sa}[\Delta_\rho\{\mathrm{sgn}(K_{1x})K_1 - 2k_o\}]$$

$$\cdot \{(K_3)^{\frac{1}{2}}[K_3 - k_o\ \mathrm{sgn}(K_{3x})]H_d(\vec{K}_1,\vec{K}_2,\omega_d)$$

$$+ \frac{1}{2} Q_{1d}(\vec{K}_1,\vec{K}_3)[K_3K_2^2 - 2k_o\ \mathrm{sgn}(K_{3x})\vec{K}_2 \cdot \vec{K}_3]^{-\frac{1}{2}}$$

$$+ \frac{1}{2} Q_{1d}(\vec{K}_2,\vec{K}_3)[K_3K_1^2 - 2k_o\ \mathrm{sgn}(K_{3x})\vec{K}_1 \cdot \vec{K}_3]^{-\frac{1}{2}}\}$$

$$\cdot Q_{2c}(\vec{K}_1,\vec{K}_2)\{[K_1K_2^2 + 2k_o\ \mathrm{sgn}(K_{1x})\vec{K}_1 \cdot \vec{K}_2]^{-\frac{1}{2}}\}^*$$

$$\cdot \exp\ [j\ \mathrm{sgn}(K_{3x})(K_3\rho_o - \tfrac{\pi}{4}) - j\ \mathrm{sgn}(K_{1x})(K_1\rho_o - \tfrac{\pi}{4})]$$

$$\cdot S_1^{\pm}(\vec{K}_1)\ S_1^{\pm}(\vec{K}_2)$$

$$\cdot \delta\{\omega_d \pm \mathrm{sgn}(K_{1x})(gK_1)^{\frac{1}{2}} \pm \mathrm{sgn}(K_{2x})(gK_2)^{\frac{1}{2}}\}$$

$$\cdot d\alpha_1 d\beta_1 d\alpha_2 d\beta_2 \qquad \text{(D.56)}$$

The four integrals present in (D.56) may be evaluated approximately as per the similar evaluation in the second part in (D.30). By doing so the surface wave number vectors $\vec{K}_1$ and $\vec{K}_3$ become

$\vec{K}_1 = \vec{K}_3 = 2k_o\hat{x}$

and which implies from (D.53)

$\vec{K}_2 = \vec{0}$.

Since the mean level of the surface is assumed to be zero,

$S_1^{\pm}(\vec{0}) = 0$. Therefore,

$$\sigma_{s12}(\omega_d) \cong 0 \qquad \text{(D.57)}$$

and thus $P_{s12}(\omega_d) \cong 0$ and $R_{s12}(\omega_d) \cong 0$. This means that $\tilde{E}_{zb21}$ and $\tilde{E}_{zb22}$ are approximately uncorrelated.

D.6 $R_{s13}(\tau)$, $P_{s13}(\omega_d)$, and $\sigma_{s13}(\omega_d)$

The cross correlation function of the first part with the third part of the second order field is defined by (4.89g) as

$$R_{s13}(\tau) = \frac{A_r}{2\eta} < \tilde{E}_{zb21}(t_o, t + \tau)\tilde{E}^*_{zb23}(t_o, t) > \qquad \text{(D.58)}$$

where $\tilde{E}_{zb21}$ and $\tilde{E}_{zb23}$ are respectively given by (4.78) and (4.80). Using these equations and (D.8) for the average in (D.58) it becomes

$$R_{s13}(\tau) = j \frac{\sqrt{8}\, P_t g_t g_r \lambda_o^2}{(2\pi)^6 (\pi\rho_o)^3} \Delta_\rho^2 F^2(\rho_o, 1, 0)$$

$$\cdot \sum_{p,q,i} \sum_{\substack{m,n,1 \\ (m,n)\neq(-p,-q) \\ -\alpha|p+m|\leq q+n\leq\alpha|p+m| \\ -\alpha|2p+m|\leq 2q+n\leq\alpha|2p+m| \\ 2k_o|m|N<K_m^2<2k_oK_m}} \left\{ N^4 W^2 S_1(\vec{K}_p, iW) S_1(\vec{K}_m, 1W) \right.$$

$$[F(\beta_o\rho_o, 1, 0)F(r_b, \cos\psi_b, \underline{\sin}\,\psi_b)F(r_c, -\cos\psi_c, -\sin\psi_c)]^*$$

$$\cdot\ \mathrm{Sa}\ [\Delta_\rho\{\mathrm{sgn}(p+m)K_{p+m} - 2k_o\}]$$

$$\cdot\ \mathrm{Sa}\ [\frac{\beta_o\Delta_\rho}{2}\{\mathrm{sgn}(2p+m)K_{2p+m} - 2k_o - \frac{|n|N}{K_m}K_{mo}\}]$$

$$\cdot\ \{(K_{p+m})^{\frac{1}{2}}[K_{p+m} - k_o\ \mathrm{sgn}(p+m)]H(m,n,1,p,q,i)$$

$$+ \frac{1}{2}Q_1(m,n,p,q)[K_m^2K_{p+m} - 2k_o\ \mathrm{sgn}(p+m)\vec{K}_m \cdot \vec{K}_{p+m}]^{-\frac{1}{2}}$$

$$+ \frac{1}{2}Q_1(p,q,m,n)[K_p^2K_{p+m} - 2k_o\ \mathrm{sgn}(p+m)\vec{K}_p \cdot \vec{K}_{p+m}]^{-\frac{1}{2}}\}$$

$$\cdot\ K_m\ Q_3(m,n,p,q)[\beta_oK_{2p+m}\{K_m^4 - (2k_o mN)^2\}]^{-\frac{1}{2}}$$

$$\cdot\ \exp\ [j\ \mathrm{sgn}(p+m)(K_{p+m}\rho_o - \frac{\pi}{4}) - j\ \mathrm{sgn}(2p+m)(K_{2p+m}\frac{\beta_o\rho_o}{2} - \frac{\pi}{4})]$$

$$\cdot\ \exp\ [-j\ \rho_o(2k_o - k_o\beta_o - \frac{|n|N}{2K_m}K_{mo}\beta_o)]$$

$$\cdot\ \exp\ [j(i+1)W\tau]\Bigg\} \qquad (D.59)$$

Because of tighter restrictions on the summation indices m,n,p, and q, $R_{s13}(\tau)$ may be approximated to zero in the sense of integrals. The approximation is similar to that used in the second part of (D.34) and which may be shown here as follows:

By substituting

$$\alpha_1 = (p + m)N \quad , \quad \beta_1 = (q + n)N$$
$$\alpha_2 = (2p + m)N \quad , \quad \beta_2 = (2q + n)N \qquad (D.60)$$
$$\omega_1 = iW \quad , \quad \omega_2 = 1W$$

and then taking the limits of N and W to zero in (D.59), the summations may be reduced to integrals. The form of the integrals may be written as

$$R_{s13}(\tau) = \int_{\alpha_1}\int_{\beta_1}\int_{\alpha_2}\int_{\beta_2}\int_{\omega_1}\int_{\omega_2} - - - \; d\alpha_1 d\beta_1 d\alpha_2 d\beta_2 d\omega_1 d\omega_2 \; , \tag{D.61}$$

where the integration variables α_1, β_1, α_2, and β_2 must comply with the set of three restrictions corresponding to the restrictions on summation indices in (D.59). These restrictions are

Res. 1: $-\alpha|\alpha_1| \leq \beta_1 \leq \alpha|\alpha_1|$

Res. 2: $-\alpha|\alpha_2| \leq \beta_2 \leq \alpha|\alpha_2|$

Res. 3: $2k_o|2\alpha_1 - \alpha_2| < \{(2\alpha_1 - \alpha_2)^2 + (2\beta_1 - \beta_2)^2\}$

and $\{(2\alpha_1 - \alpha_2)^2 + (2\beta_1 - \beta_2)^2\}^{\frac{1}{2}} < 2k_o$.

The restriction $(m,n) \neq (-p,-q)$ is not included in the above set as the integrations may be extended to this point by continuity.

By transforming (α_1, β_1) and (α_2, β_2) to polar variables (λ_1, ϕ_1) and (λ_2, ϕ_2), (D.61) may be written as

$$R_{s13}(\tau) = \int_{\phi_1}\int_{\phi_2}\int_{\lambda_1}\int_{\lambda_2}\int_{\omega_1}\int_{\omega_2} - - - \; d\lambda_1 d\lambda_2 d\phi_1 d\phi_2 d\omega_1 d\omega_2 \; . \tag{D.62}$$

The set of restrictions in terms of the new variables are

Res. 1: $-\Delta_\theta \leq \phi_1 \leq \Delta_\theta$ and $(\pi - \Delta_\theta) \leq \phi_1 \leq (\pi + \Delta_\theta)$

Res. 2: $-\Delta_\theta \leq \phi_2 \leq \Delta_\theta$ and $(\pi - \Delta_\theta) \leq \phi_2 \leq (\pi + \Delta_\theta)$

Res. 3: $2k_o|2\lambda_1 \cos \phi_1 - \lambda_2 \cos \phi_2| < \{4\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2 \cos (\phi_1 - \phi_2)\}$

and $\{4\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2 \cos (\phi_1 - \phi_2)\}^{\frac{1}{2}} < 2k_o$.

The integrand along with the integrals with respect to λ_1, λ_2, ω_1, and

ω_2 in (D.62) may be treated as a function of $\emptyset_1$ and $\emptyset_2$. The two integrals with respect to $\emptyset_1$ and $\emptyset_2$ may be evaluated approximately by treating this function constant over the narrow ranges of $\emptyset_1$ and $\emptyset_2$. That is, the function may be evaluated at $\emptyset_1 = 0$ and π and $\emptyset_2 = 0$ and π. By doing so the restriction 3 becomes

$$\text{Res. 3:} \begin{cases} 2k_o < |2\lambda_1 - \lambda_2| < 2k_o \text{ for } (\emptyset_1,\emptyset_2) = (0,0) \text{ or } (\pi,\pi) \\ 2k_o < |2\lambda_1 + \lambda_2| < 2k_o \text{ for } (\emptyset_1,\emptyset_2) = (0,\pi) \text{ or } (\pi,0) \end{cases} .$$

Clearly, the integration variables λ_1 and λ_2 can not meet the above restriction. However, again by continuity, the inequalities in the restriction may be allowed to include the end points as well. Thus the restriction is met. But then for any fixed λ_1, the region of integration with respect to λ_2 reduces to a set of a finite number of points and it does not span any continuous region. Therefore, the λ_2 integral is zero. Alternatively, the variables λ_1 reduces to a set of a finite number of points for any fixed λ_2, which makes the λ_1 integral zero. Thus we have

$$R_{s13}(\tau) \cong 0 , \qquad (D.63)$$

which means that the fields $\tilde{E}_{zb21}$ and $\tilde{E}_{zb23}$ are approximately uncorrelated. This also implies

$$P_{s13}(\omega_d) \cong 0, \qquad (D.64)$$

or

$$\sigma_{s13}(\omega_d) \cong 0 . \qquad (D.65)$$

D.7 $R_{s23}(\tau)$, $P_{s23}(\omega_d)$, and $\sigma_{s23}(\omega_d)$

The cross correlation function of the second part with the third part of the second order field is defined by (4.89h) as

$$R_{s23}(\tau) = \frac{A_r}{2\eta} < \tilde{E}_{zb22}(t_o, t+\tau)\tilde{E}^*_{zb23}(t_o, t) > \quad (D.66)$$

where $\tilde{E}_{zb22}$ and $\tilde{E}_{zb23}$ are given by (4.79) and (4.80) respectively. A_r is given by (4.82). By using these equations and (D.8) for the average in (D.66), $R_{s23}(\tau)$ may be written as

$$R_{s23}(\tau) = j\,\frac{\sqrt{2}\,P_t g_t g_r \lambda_o^2}{(2\pi)^6 (\pi\rho_o)^3}\,\Delta_\rho^2\,F^2(\rho_o, 1, 0)\,\{A + B\} \quad (D.67)$$

where

$$A = \sum_{\substack{p,q,i\\ (m,n)\neq(-p,-q)\\ -\alpha|p|\leq q\leq\alpha|p|\\ -\alpha|2p+m|\leq 2q+n\leq\alpha|2p+m|\\ 2k_o|m|N<K_m^2<2k_oK_m}} \sum_{m,n,l} \Big\{ N^4W^2 S_1(\vec{K}_p, iW) S_1(\vec{K}_m, lW)$$

$$\cdot [F(\beta_o\rho_o, 1, 0)F(r_b, \cos\psi_b, \sin\psi_b)F(r_c, -\cos\psi_c, -\sin\psi_c)]^*$$

$$\cdot \mathrm{Sa}\,[\Delta_\rho\{\mathrm{sgn}(p)K_p - 2k_o\}]$$

$$\cdot \mathrm{Sa}\,[\frac{\beta_o\Delta_\rho}{2}\{\mathrm{sgn}(2p+m)K_{2p+m} - 2k_o - \frac{|n|N}{K_m}K_{mo}\}]$$

$$\cdot Q_2(m,n,p,q)[K_m^2K_p + 2k_o\,\mathrm{sgn}(p)\,\vec{K}_m\cdot\vec{K}_p]^{-\frac{1}{2}}$$

$$\cdot K_m\,Q_3(m,n,p,q)[\beta_oK_{2p+m}\{K_m^4 - (2k_omN)^2\}]^{-\frac{1}{2}}$$

$$\cdot \exp\,[j\,\mathrm{sgn}(p)(K_p\rho_o - \frac{\pi}{4}) - j\,\mathrm{sgn}(2p+m)(K_{2p+m}\frac{\beta_o\rho_o}{2} - \frac{\pi}{4})]$$

$$\cdot \exp\,[-j\rho_o(2k_o - k_o\beta_o - \frac{|n|N}{2K_m}K_{mo}\beta_o)]$$

$$\cdot \exp\,[j(i+1)W\tau]\Big\}, \quad (D.68)$$

and

$$B = \sum_{\substack{p,q,i \\ (m,n)\neq(-p,-q) \\ -\alpha|m|\leq n\leq \alpha|m| \\ -\alpha|2p+m|\leq 2q+n\leq q|2p+m| \\ 2k_o|m|N<K_m^2<2k_oK_m}} \sum_{m,n,l} \Big\{ N^4W^2 S_1(\vec{K}_p, iW) S_1(\vec{K}_m, lW)$$

$$\cdot\, [F(\beta_o\rho_o, 1, 0)F(r_b, \cos\psi_b, \sin\psi_b)F(r_c, -\cos\psi_c, -\sin\psi_c)]^*$$

$$\cdot\, Sa\,[\Delta_\rho\{\mathrm{sgn}(m)\,K_m - 2k_o\}]$$

$$\cdot\, Sa\,[\frac{\beta_o\Delta_\rho}{2}\{\mathrm{sgn}(2p+m)K_{2p+m} - 2k_o - \frac{|n|N}{K_m}K_{mo}\}]$$

$$\cdot\, Q_2(p,q,m,n)[K_p^2K_m + 2k_o\,\mathrm{sgn}(m)\vec{K}_p\cdot\vec{K}_m]^{-\frac{1}{2}}$$

$$\cdot\, K_m\,Q_3(m,n,p,q)[\,{}_oK_{2p+m}\{K_m^4 - (2k_omN)^2\}]^{-\frac{1}{2}}$$

$$\cdot\, \exp\,[j\ \mathrm{sgn}(m)(K_m\rho_o - \frac{\pi}{4}) - j\ \mathrm{sgn}(2p+m)(K_{2p+m}\frac{\beta_o\rho_o}{2} - \frac{\pi}{4})]$$

$$\cdot\, \exp\,[-j\rho_o(2k_o - k_o\beta_o - \frac{|n|N}{2K_m}K_{mo}\beta_o)]$$

$$\cdot\, \exp\,[j(i+l)W\tau]\Big\}\,. \qquad (D.69)$$

Let in (D.68)

$$\alpha_1 = pN \quad , \quad \beta_1 = qN$$
$$\alpha_2 = (2p+m)N \quad , \quad \beta_2 = (2q+n)N \qquad (D.70a)$$
$$\omega_1 = iW \quad , \quad \omega_2 = lW\ ,$$

and in (D,69) .

$$\alpha_1' = mN \quad , \qquad \beta_1' = nN$$

$$\alpha_2' = (2p + m)N \quad , \qquad \beta_2' = (2q + n) \qquad \text{(D.70b)}$$

$$\omega_1' = iW \quad , \qquad \omega_2' = lW \, .$$

By using above substitutions and taking the limits of N and W to zero, the summations in (D.68) and (D.69) may be reduced to integrals. Symbolically, A and B in the form of integrals may then be written as

$$A = \int_{\alpha_1} \int_{\beta_1} \int_{\alpha_2} \int_{\beta_2} \int_{\omega_1} \int_{\omega_2} - - - - d\alpha_1 d\beta_1 d\alpha_2 d\beta_2 d\omega_1 d\omega_2 \, , \qquad \text{(D.71)}$$

and

$$B = \int_{\alpha_1'} \int_{\beta_1'} \int_{\alpha_2'} \int_{\beta_2'} \int_{\omega_1'} \int_{\omega_2'} - - - - d\alpha_1' d\beta_1' d\alpha_2' d\beta_2' d\omega_1' d\omega_2' \, , \qquad \text{(D.72)}$$

where the integration variables α_1, β_1, α_2, and β_2 have to satisfy the restrictions corresponding to the restrictions on the summation indices in (D.68). Similarly, α_1', β_1', α_2', and β_2' have to satisfy the restrictions corresponding to those in (D.69). These restrictions for (D.71) are

Res. 1: $-\alpha|\alpha_1| \le \beta_1 \le \alpha|\alpha_1|$

Res. 2: $-\alpha|\alpha_2| \le \beta_2 \le \alpha|\alpha_2|$

Res. 3: $2k_o|\alpha_2 - 2\alpha_1| < \{(\alpha_2 - 2\alpha_1)^2 + (\beta_2 - 2\beta_1)^2\}$

and $\{(\alpha_2 - 2\alpha_1)^2 + (\beta_2 - 2\beta_1)^2\}^{\frac{1}{2}} < 2k_o$.

The restriction $(m,n) \neq (-p,-q)$ is not included in the above set as the integrations may be extended to this point by continuity. In a similar way, the restrictions for (D.72) are

Res. 1: $-\alpha|\alpha_1'| \leq \beta_1' \leq \alpha|\alpha_1'|$

Res. 2: $-\alpha|\alpha_2'| \leq \beta_2' \leq \alpha|\alpha_2'|$

Res. 3: $2k_o|\alpha_1'| < \{\alpha_1'^2 + \beta_1'^2\} < 2k_o \{\alpha_1'^2 + \beta_1'^2\}^{\frac{1}{2}}$.

By following the same procedure as in section D.6, it may be shown that because of the above restrictions, A and B are approximately zero. Therefore, from (D.67),

$$R_{s23}(\tau) \cong 0 \ . \tag{D.73}$$

This means that $\tilde{E}_{zb22}$ and $\tilde{E}_{zb23}$ also are approximately uncorrelated. As a result

$$P_{s23}(\omega_d) \cong 0 \ , \tag{D.74}$$

or

$$\sigma_{s23}(\omega_d) \cong 0 \ . \tag{D.75}$$