THE ELECTROMAGNETIC SCATTERING FROM A VERTICAL DISCONTINUITY WITH APPLICATIONS TO ICE HAZARD DETECTION -AN OPERATOR EXPANSION APPROACH

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by

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ABSTRACT

The long range detection of ice hazards such as multi-year ice, pressure ridges and icebergs will allow for more efficient planning of Arctic navigation routes and exploration in ice infested waters. An analysis of the electromagnetic scattering from a vertical discontinuity representing the transition from sea water or first-year ice to a multi- year ice sheet has been carried out. The analysis is based on a method of Space/Field decomposition where two Heaviside functions are used to decompose a three dimensional space into three regions each having different electrical properties. Maxwell's equations are used to derive a partial differential field equation for the complete space. Making use of a field decomposition, this differential equation may be decomposed into three field equations, one for each region, and a boundary equation. A spherical Green's function is taken as the fundamental solution and the spatial Fourier transform is used to simplify the equations to a single integral equation. Selecting a vertical electric dipole as the source field the solution for the vertical component of the surface field is obtained by writing this resultant integral equation in an operator form and expanding the inverse operator in a Neumann series. Using the Laplace transform and stationary phase integration this series solution may be summed to provide expressions for both the backscattered field and the field propagated past the boundary separating the two media. The solution for the propagated field agrees with that of both Bremmer and Wait. The technique differs from that of previous investigators in that it is possible to obtain an expression for the backscattered field and thereby the radar cross-section of the vertical discontinuity. The results of this analysis indicate that radar operating in the High Frequency range (3 –

30 MHz) should provide a significant improvement over present methods for the detection of this type of hazard.

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CHAPTER 1

INTRODUCTION

1.1 General

With increasing exploration and development in the Arctic regions and the growing interest in offshore resources more emphasis is being placed on solving the problems associated with navigation and exploration in ice infested waters. Major problems include icebergs in the southern Grand Banks region, icebergs and sea ice off Labrador and multi-year ice and pressure ridges in the eastern Arctic and Beaufort Sea areas. While some groups are necessarily taking the precaution of building ships and drilling platforms able to withstand large scale ice forces, the early detection of advancing ice hazards will allow better planning of navigation routes and drilling schedules, thereby avoiding costly delays and possible disaster.

Many methods presently in use for ice hazard detection rely heavily on the use of standard microwave marine radar. These radars can provide a useful detection service when the energy reflected back to the radar from the ice target is large in comparison to the background noise or clutter level. For the case of detection in an interference free enviornment the noise level of the radar will limit detection. However, for the situation of ice targets at sea, the background or clutter signal can originate from scattering by waves, surrounding ice or precipitation. Unfortunately these clutter signals often overpower the desired signal and make detection very difficult. For example, the radar return from a small iceberg may be easily obscured by prevailing sea conditions. Similarily, a large multiyear ice flow may appear no different from first-year ice at grazing incidence.

As most of these situations may be essentially represented by a transition from one type of medium to another the general model chosen here for analysis is based on a flat earth consisting of two media of semi-infinite extent each having different electrical properties.

The analysis of this general model provides an insight into many of the special cases which occur in nature. The two media of the model could represent land/sea, sea/ice or first-year ice/multi-year ice situations depending on the the electrical properties chosen. The emphasis of the analysis is placed on the interaction of the surface wave mode of propagation with a vertical discontinuity. As the surface wave mode is predominant for short ranges with ground mounted antennas operating in the radio frequency range from three to thirty Megahertz the following analysis is directly applicable to these types of radar.

The problem of propagation over an inhomogeneous earth has been the subject of much research and many similar solutions have been derived for the field propagated past a boundary separating two homogeneous media. However, an explicit representation of the backscattered field from such a boundary is not available.

Herein a new method is used for treating the problem in the sense of generalized functions. The forward propagated field is derived for comparison with previous work, and in addition an expression for the backscattered field is also found.

1.2 Literature Review

Historically, the emphasis on solving the so-called mixed path problem has been directed towards deriving the field propagated past a boundary separating two semi-infinite homogeneous media.

The semi-empirical work of Millington (1944) presented an accurate method for calculating the field propagated past a boundary. His work predicted a "recovery effect" in the field when passing from a medium of low conductivity to one of higher conductivity. His derivation was based on satisfying the reciprocity requirement regarding the interchangeability of transmitter and receiver. Clemmow (1953) and Bremmer (1954) have both formulated general methods for treating the the problem of propagation over mixed paths. Their methods differ greatly. However, Bremmer demonstrates that Clemmow's result may be derived from his for large numerical distances.

Initially Clemmow considers a flat earth consisting of a semi-infinite half-space containing a homogeneous medium. The second medium is represented by an infinitely thin perfectly conducting half-plane lying on this homogeneous medium. He proceeds by deriving a spectrum of plane waves representation of the scattered field due to the induced surface currents in the perfectly conducting sheet. This method leads to dual integral equations which he solves by contour integration. Subsequently Clemmow relaxes the perfectly conducting requirement on the second medium and adopts a more appropriate boundary condition, namely, that the modulus of the complex permittivity of each section of earth is large. Solving this alternate problem he obtains results that are in agreement with the work of Millington.

Bremmer (1954) on the other hand takes an entirely different approach. Essentially he derives the solution in terms of an integral equation based on an application of Green's theorem and a homogeneous boundary condition at the air/earth interface. His formulation applies to all types of continuous distributions of the conductivity and dielectric constant of the earth and the resultant integral equation solution is similar to the integral equation considered by Hufford (1952) for the propagation over irregular terrain. Bremmer then treats as a special case the problem of two adjacent medium with homogeneous electrical properties and, with the aid of two-sided operational calculus, he derives a solution for the field both near and far from the boundary. Bremmer has also derived an expression for the field propagated over multi-section paths. The derivations of both Bremmer and Hufford are based on a scalar wave assumption which precludes the derivation of a reflected field component.

Later Wait (1956b) showed that the integral equation for a curved two-section path could be derived by an application of the compensation theorem. (Monteath, 1951). His integral equation is essentially the same as that considered by Hufford (1952), except that Wait retains the field in a vector form. In a more recent treatment of the problem Wait (1970) reduces the problem to dual integral equations which he solves by the Weiner-Hopf technique. His solution is in terms of the mode conversion matrices for the reflection and transmission coefficients and agreement with his previous work is demonstrated. Perhaps the latest work on this problem has been presented by Furutsu (1982). Furutsu has derived an explicit expression for the attenuation of radio waves over a curved earth path with sections having different electrical properties and different heights. His method is based on an application of Green's theorem and on the derivation of pairs of integral equations. Furutsu uses an iterative procedure to derive an infinite series solution to these integral equations.

Most of the reviewed work is in agreement with respect the field propagated past the boundary separating two semi-infinite media; however, an explicit expression for the backscattered field far from the boundary has not been presented. Wait (1963) has derived an expression for the field close to and on either side of the boundary of separation.

1.3 Scope of Thesis

The present analysis of the problem of electromagnetic scattering from a vertical discontinuity is based on a method of Space/Field decomposition developed by Walsh (1980b). Two Heaviside functions are utilized to decompose a three dimensional space into three regions. The region above z=0 represents free space and the region below represents two semi-infinite homogeneous media. Maxwell's equations are used to derive a differential field equation for the complete space. In a manner similar to the space decomposition this differential field equation is decomposed into three field equations, one for each region, and a boundary equation. This boundary equation represents the conditions which the electric field must satisfy at each of the interfaces. In this manner this technique provides its own boundary conditions. The electric field in each region is given in terms of the field and its normal derivative at the bounding interfaces. Using the appropriate spherical Green's function the three field equations are reduced to the form of convolution-type integral equations. The boundary equation may be utilized to eliminate half of the unknowns from the field equations, and further simplification is achieved by taking the two-dimensional (spatial) Fourier transform. Assuming the refractive indices of the media below are large compared to free space, and taking the source field as the far field of an elementary vertical electric dipole the three field equations may be reduced to a single algebraic equation. This equation may be inverse Fourier transformed by an asymptotic evaluation of the integrals using the saddle point method, (Wait , 1964). The resultant convolution integral equation is written in operator notation and the operator is formally inverted in the form of a Neumann series. Utilizing stationary phase integration and the Laplace transform the series may be summed to give either the propagated field or the backscattered field. The propagated field agrees with the results derived by both Bremmer (1954) and Wait (1964), and the backscattered field is utilized to derive an expression for the radar cross-section of the vertical discontinuity.

Subsequently, several types of ice hazard are chosen for analysis and detection ranges calculated from the radar equation.

Chapter 2 contains the complete analysis and derivation of the propagated and backscattered fields. Chapter 3 contains the derivation of a radar equation and radar cross-section of the vertical discontinuity. Chapter 5 presents numerical results for different combinations of sea/ice interfaces and Chapter 6 contains the conclusions.

CHAPTER 2

FORMULATION OF THE PROBLEM

2.1 Space Decomposition

The formulation of this problem is based on a method of Space/Field decomposition demonstrated by Walsh (1980b). The three-dimensional space is decomposed into three regions utilizing two Heaviside functions. The plane $z=0$ represents the earth's surface. The half-space below is decomposed into two regions, $x<d$: medium 1, $x>d$: medium 2, each having different electrical properties. This decomposition is illustrated in Figure 1 :

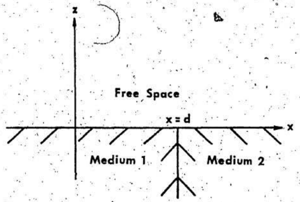


Figure 1 Space Decomposition by Electrical Properties

The electrical properties consist of the permeability, μ, the permittivity, ϵ, and the conductivity, σ. A subscript is used to denote the appropriate region (ie: medium 1 : ϵ_1 μ_1 σ_1). The two Heaviside functions are given as

$$h_1 = h(z) = \begin{cases} 0 & ; \quad z<0 \\ 1 & ; \quad z>0 \end{cases} \tag{2.1}$$

$$h_2 = h(x) = \begin{cases} 0 & ; \quad x<d \\ 1 & ; \quad x>d \end{cases} \tag{2.2}$$

These Heaviside functions are used to define the electrical properties of the complete space as

$$\mu = \mu_0 \tag{2.3}$$

$$\epsilon = \epsilon_0 h_1 + \epsilon_1 (1-h_1)(1-h_2) + \epsilon_2 (1-h_1) h_2 \tag{2.4}$$

$$\sigma = \sigma_1 (1-h_1)(1-h_2) + \sigma_2 (1-h_1) h_2 \tag{2.5}$$

The equations for the electrical properties of the complete space may be utilized with Maxwell's equations to derive a differential equation for the electric field, $\vec{E}$.

2.2 Maxwell's Equations

Maxwell's equations for the complete space are given in their point form as

$$\nabla\times\vec{E} = -j\omega\vec{B} \tag{2.6}$$

$$\nabla\times\vec{H} = j\omega\vec{D} + \vec{J} \tag{2.7}$$

$$\nabla\cdot\vec{B} = 0 \tag{2.8}$$

$$\nabla\cdot\vec{D} = \rho \tag{2.9}$$

where

$\vec{E}$= electric field intensity

H= magnetic field intensity

$\vec{D}$= electric flux density

$\vec{B}$= magnetic flux density

$\vec{J} = \vec{J}_c + \vec{J}_s$: conduction and source current densities respectively

ρ= charge density

These equations are in their 'time harmonic" form, which may be interpreted as the Fourier Transform, with respect to the time variable, of the time dependent equations. A positive time dependency of the fields is assumed (ie: the time dependent electric field $\vec{e}(t)$ = Re($\vec{E}$ exp($j\omega t$))where $j = \sqrt{-1}$).

In addition we have the constitutive relationships

$$\vec{D} = \epsilon\vec{E} \tag{2.10}$$

$$\vec{B} = \mu_0 \vec{H} \tag{2.11}$$

$$\vec{J}_c = \sigma \vec{E} \tag{2.12}$$

where ϵ and σ are given in (2.4) and (2.5).

We will interpret these Maxwell equations and the constitutive relationships in the sense of generalized functions. Further, it will be seen that the method of dealing with these equations implies certain restrictions on the fields and their spatial derivatives on the bounding interfaces. The Dirac delta function belongs to this class of functions. The electrical engineer is familiar with this function as it is used to derive the impulse response of a system.

2.3 The Equation for the Electric Field, $\vec{E}$

We may now proceed to derive a differential equation for the electric or magnetic fields. As the electric field is of prime importance to this problem the magnetic field equation will not be derived, however the same procedure may be applied to its derivation. For the electric field we may proceed by taking the curl of Maxwell's first equation, (2.6), and using the relationship of (2.11) we obtain

$$\nabla \times \nabla \times \vec{E} = -\nabla \times j\omega \vec{B} \tag{2.13}$$

$$= -j\omega\mu_0 \nabla \times \vec{H} \tag{2.14}$$

Inserting Maxwell's second equation, (2.7), into (2.13) yields

$$\nabla \times \nabla \times \vec{E} = -j\omega\mu_0 (j\omega \vec{D} + \vec{J}_c + \vec{J}_s) \tag{2.15}$$

Consider the vector identity.

$$\nabla^2\vec{E} + \nabla\times\nabla\times\vec{E} = \nabla(\nabla\cdot\vec{E}) \qquad (2.16)$$

Inserting (2.15) into this identity we obtain

$$\nabla^2\vec{E} - j\omega\mu_0\vec{D} - j\omega\mu_0\vec{J}_c = \nabla(\nabla\cdot\vec{E}) + j\omega\mu_0\vec{J}_s$$

and using the relationships of (2.10) and (2.12)

$$\nabla^2\vec{E} - j\omega\mu_0\epsilon\vec{E} - j\omega\mu_0\sigma\vec{E} = \nabla(\nabla\cdot\vec{E}) + j\omega\mu_0\vec{J}_s$$

This equation may be written as

$$\nabla^2\vec{E} + k^2n^2\vec{E} = \nabla(\nabla\cdot\vec{E}) + j\omega\mu_0\vec{J}_s \qquad (2.17)$$

where

$$k = \omega\sqrt{\mu_0\epsilon_0} \text{ ; freespace wavenumber} \qquad (2.18)$$

$$n^2 = h_1 + n_1^2(1-h_1)(1-h_2) + n_2^2(1-h_1)h_2 \qquad (2.19a)$$

$$n_1^2 = \frac{\epsilon_1'}{\epsilon_0} \text{ ; } n_2^2 = \frac{\epsilon_2'}{\epsilon_0} \qquad (2.19b)$$

$$\epsilon_1' = \epsilon_1 + \frac{\sigma_1}{j\omega} \text{ ; } \epsilon_2' = \epsilon_2 + \frac{\sigma_2}{j\omega} \qquad (2.19c)$$

where n_1 and n_2 are the refractive indices of medium 1 and medium 2 respectively and ϵ_1' and ϵ_2' are the complex permittivities of medium 1 and medium 2 respectively. Summarizing.

$$n^2 = \frac{\epsilon'}{\epsilon_0}$$

$$\epsilon' = \epsilon_0 h_1 + \epsilon_1'(1-h_1)(1-h_2) + \epsilon_2'(1-h_1)h_2 \qquad (2.20)$$

Consider again (2.7)

$$\nabla \times H = j\omega\vec{D} + \vec{J}_c + \vec{J}_s$$

$$= j\omega(\epsilon + \frac{\sigma}{j\omega})\vec{E} + \vec{J}_s$$

$$= j\omega\vec{D}_c + \vec{J}_s \qquad (2.21)$$

where

$$\vec{D}_c = (\epsilon + \frac{\sigma}{j\omega})\vec{E} = \epsilon'\vec{E} \qquad (2.22)$$

The divergence of (2.21) is

$$\nabla \cdot (\nabla \times \vec{H}) = 0 = \nabla \cdot (j\omega\vec{D}_c + \vec{J}_s) \qquad (2.23)$$

Now (2.23) may be rearranged to yield

$$\nabla \cdot \vec{D}_c = \frac{-1}{j\omega}\nabla \cdot \vec{J}_s \qquad (2.24)$$

It should be noted that the support of $\vec{J}_s$ is entirely in the half space $z > 0$ and therefore

$$\nabla \cdot \vec{D}_c = 0 \quad \text{for} \quad z < 0$$

Rewriting (2.22) and using (2.20) for ϵ'

$$\vec{D}_c = \epsilon'\vec{E}$$

$$= (\epsilon_0 h_1 + \epsilon_1'(1-h_1)(1-h_2) + \epsilon_2'(1-h_1)h_2)\vec{E}$$

This equation may be inverted to obtain an expression for $\vec{E}$.

$$\vec{E} = (\frac{h_1}{\epsilon_0} + \frac{(1-h_1)(1-h_2)}{\epsilon_1'} + \frac{(1-h_1)h_2}{\epsilon_2'})\vec{D}_c \qquad (2.25)$$

Equation (2.25) may be written in four equivalent forms as

$$\vec{E} = (\frac{1}{\epsilon_1} + h_1\epsilon_{10} - h_2\epsilon_{21} + h_1h_2\epsilon_{21})\vec{D}_c \qquad (2.26)$$

$$\vec{E} = (\frac{1}{\epsilon_0} - (1-h_1)\epsilon_{10} - h_2(1-h_1)\epsilon_{21})\vec{D}_c \qquad (2.27)$$

$$\vec{E} = (\frac{1}{\epsilon_2} - h_1(1-h_2)\epsilon_{21} - (1-h_2)\epsilon_{21} + h_1\epsilon_{20})\vec{D}_c \qquad (2.28)$$

$$\vec{E} = (\frac{1}{\epsilon_0} - (1-h_1)\epsilon_{20} + (1-h_1)(1-h_2)\epsilon_{21})\vec{D}_c \qquad (2.29)$$

where

$$\epsilon_{10} = \left[\frac{\epsilon'_1 - \epsilon_0}{\epsilon'_1\,\epsilon_0}\right] ; \quad \epsilon_{21} = \left[\frac{\epsilon'_2 - \epsilon'_1}{\epsilon'_2\,\epsilon'_1}\right] ; \quad \epsilon_{20} = \left[\frac{\epsilon'_2 - \epsilon_0}{\epsilon'_2\,\epsilon_0}\right]$$

As it is necessary to find an expression for the divergence of $\vec{E}$ we must interpret terms like $\nabla\cdot(1-h_1)\vec{D}_c$ and $\nabla\cdot h_1\vec{D}_c$. We have

$$\nabla\cdot(1-h_1)\vec{D}_c = (1-h_1)\nabla\cdot\vec{D}_c - \hat{z}\cdot\vec{D}_c^{-}\delta(z)$$

$$\nabla\cdot h_1\vec{D}_c = h_1\nabla\cdot\vec{D}_c + \hat{z}\cdot\vec{D}_c^{+}\delta(z)$$

$$\nabla\cdot(1-h_2)\vec{D}_c = (1-h_2)\nabla\cdot\vec{D}_c - \hat{x}\cdot\vec{D}_c^{L}\delta(x-d)$$

$$\nabla\cdot h_2\vec{D}_c = h_2\nabla\cdot\vec{D}_c + \hat{x}\cdot\vec{D}_c^{R}\delta(x-d)$$

where $\hat{x}$ and $\hat{z}$ are unit vectors in their respective directions and δ is the Dirac delta function.

The generalized derivatives, (Papoulis, 1962), of the Heaviside functions have been taken and

$$\vec{D}_c^{-}(x,y) = \lim_{z\to 0^-} \vec{D}_c(x,y,z)$$

$$\vec{D}_c^{+}(x,y) = \lim_{z\to 0^+} \vec{D}_c(x,y,z)$$

These quantities represent the value of the electric flux density, $\vec{D}_c$, just above, $\vec{D}_c^+$, and just below, $\vec{D}_c^-$, the horizontal interface and just to the right, $\vec{D}_c^R$, and just to the left, $\vec{D}_c^L$, of the vertical interface. The divergence of $\vec{E}$ may be derived using (2.26)–(2.29) and noting that $\nabla\cdot\vec{E}$ should be unique we find that the following relationships must hold, (Walsh, 1980b),

$$\hat{x}\cdot\vec{D}_c^R = \hat{x}\cdot\vec{D}_c^L \tag{2.30}$$

$$\hat{z}\cdot\vec{D}_c^+ = \hat{z}\cdot\vec{D}_c^- \tag{2.31}$$

These relationships represent the classical boundary conditions which the electric field must satisfy at the interfaces. That is the normal component of $\vec{D}_c$ must be continuous across an interface.

We may now proceed to derive an expression for $\nabla(\nabla\cdot\vec{E})$ using (2.26) by first forming the divergence of $\vec{E}$.

$$\nabla\cdot\vec{E} = \frac{-1}{j\omega\epsilon_0}\nabla\cdot\vec{J}_s + \epsilon_{10}\hat{z}\cdot\vec{D}_c^+\delta(z)$$

$$-\epsilon_{21}\hat{x}\cdot\vec{D}_c^R\delta(x-d) + \epsilon_{21}(h_1\hat{x}\cdot\vec{D}_c^R\delta(x-d) + h_2\hat{z}\cdot\vec{D}_c^+\delta(z)) \tag{2.32}$$

From (2.22) the following relationships hold on the interfaces.

$$\vec{D}_c^+ = \epsilon_0\vec{E}^+ \qquad (1-h_1)\vec{D}_c^- = (1-h_2)\epsilon_1'\vec{E}^-$$

$$h_2\vec{D}_c^- = h_2\epsilon_2'\vec{E}^- \qquad (1-h_1)\vec{D}_c^R = (1-h_1)\epsilon_2'\vec{E}^R \tag{2.33}$$

$$(1-h_1)\vec{D}_c^L = (1-h_1)\epsilon_1'\vec{E}^L$$

Rearranging (2.32)

$$\nabla\cdot\vec{E} = \frac{-1}{j\omega\epsilon_0}\nabla\cdot\vec{J}_s + \epsilon_{10}(1-h_2)\hat{z}\cdot\vec{D}_c^+\delta(z) + \epsilon_{20}h_2\hat{z}\cdot\vec{D}_c^+\delta(z) -$$

$$-\epsilon_{21}(1-h_1)\hat{x}\cdot\vec{D}_c^R\delta(x-d) \qquad (2.34)$$

Now forming $\nabla(\nabla\cdot\vec{E})$ by taking the gradient of (2.34) and making use of the relationships of (2.33) yields

$$\nabla(\nabla\cdot\vec{E}) = \frac{-1}{j\omega\epsilon_0}\nabla(\nabla\cdot\vec{J}_s) + \epsilon_{10}\epsilon_0\nabla\left\{(1-h_2)\hat{z}\cdot\vec{E}^+\delta(z)\right\} +$$

$$\epsilon_{20}\epsilon_0\nabla\left\{h_2\hat{z}\cdot\vec{E}^+\delta(z)\right\} + \epsilon_{21}\epsilon_2\nabla\left\{(1-h_1)\hat{x}\cdot\vec{E}^R\delta(x-d)\right\} \qquad (2.35)$$

Using this equation, (2.35), in (2.17) a partial differential equation for the electric field may be formed.

$$\nabla^2\vec{E} + \gamma^2\vec{E} = -T_{se}(\vec{J}_s) + \left[\frac{n_1^2-1}{n_1^2}\right]\nabla\left\{(1-h_2)\hat{z}\cdot\vec{E}^+\delta(z)\right\} +$$

$$\left[\frac{n_2^2-1}{n_2^2}\right]\nabla\left\{h_2\hat{z}\cdot\vec{E}^+\delta(z)\right\} +$$

$$\left[\frac{n_2^2-n_1^2}{n_1^2}\right]\nabla\left\{(1-h_1)\hat{x}\cdot\vec{E}^+\delta(x-d)\right\} \qquad (2.36)$$

where

$$\gamma^2 = k^2n^2 \qquad (2.37)$$

$$= k^2\left(h_1 + (1-h_1)(1-h_2)n_1^2 + (1-h_1)h_2n_2^2\right)$$

$$T_{se}(\vec{J}_s) = \frac{1}{j\omega\epsilon_0}(\nabla(\nabla\cdot\vec{J}_s) + k^2\vec{J}_s) \tag{2.38}$$

T_{se} will be referred to as the "electrical source operator". Equation (2.36) is the basic differential field equation which must be satisfied by the electric field, $\vec{E}$.

2.4 Field Decomposition

It can be seen in the field equation (2.36) that the right-hand side consists of terms like $(1-h_2)\vec{E}^{+}$, $h_2\vec{E}^{+}$ and $(1-h_1)\vec{E}^{+}$ which represent the value of the electric field on a particular interface. This suggests that we may seek a solution to (2.36) by decomposing the left-hand side of this equation into a similar form and equating terms with like support (ie: terms multiplied by $(1-h_2)$ have support in the region $x < d$).

To this end $\vec{E}$ may be decomposed as

$$\vec{E} = h_1\vec{E} + (1-h_1)(1-h_2)\vec{E} + (1-h_1)h_2\vec{E} \tag{2.39}$$

Applying this decomposition to the left-hand side of (2.36) and equating terms having like support results in four equations. Three of the equations are the partial differential equations that must be satisfied by the electric field in each region while the fourth represents the set of boundary equations that the field must satisfy at the interfaces. Thus the solution for $\vec{E}$ which satisfies these four equations will necessarily satisfy the required boundary conditions. In this manner this technique supplies its own boundary conditions.

For use in (2.36) we form the Laplacian of the electric field as given by equation (2.39), giving

$$\nabla^2\vec{E} = \nabla^2 h_1\vec{E} + \nabla^2(1-h_1)(1-h_2)\vec{E} + \nabla^2(1-h_1)h_2\vec{E} \qquad (2.40)$$

We consider equation (2.40) term by term. The first term will be

$$\nabla^2 h_1\vec{E} = \nabla^2 h_1 E_x\hat{x} + \nabla^2 h_1 E_y\hat{y} + \nabla^2 h_1 E_z\hat{z} \qquad (2.41)$$

Now

$$\nabla^2 h_1 E_x = \nabla\cdot\nabla h_1 E_x; \qquad \textit{for cartesian coordinates}$$

$$= \nabla\cdot\left\{h_1\frac{\partial E_x}{\partial x}\hat{x} + h_1\frac{\partial E_x}{\partial y}\hat{y} + \left[E_x\frac{\partial h_1}{\partial z} + h_1\frac{\partial E_x}{\partial z}\right]\hat{z}\right\}$$

$$= \nabla\cdot\left\{h_1\frac{\partial E_x}{\partial x}\hat{x} + h_1\frac{\partial E_x}{\partial y}\hat{y} + E_x^+\delta(z)\hat{z} + h_1\frac{\partial E_x}{\partial z}\hat{z}\right\}$$

$$= h_1\frac{\partial^2 E_x}{\partial x^2} + h_1\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial}{\partial z}\left[E_x^+\delta(z)\right] + \frac{\partial}{\partial z}\left[h_1\frac{\partial E_x}{\partial z}\right]$$

Noting that $E_x^+ = E_x^+(x,y)$ this equation simplifies further to

$$\nabla^2 h_1 E_x = h_1\frac{\partial^2 E_x}{\partial x^2} + h_1\frac{\partial^2 E_x}{\partial y^2} + E_x^+\delta'(z) + \left[\frac{\partial E_x}{\partial z}\right]^+\delta(z) + h_1\frac{\partial^2 E_x}{\partial z^2} \qquad (2.42)$$

Similarly,

$$\nabla^2 h_1 E_y = h_1\frac{\partial^2 E_y}{\partial x^2} + h_1\frac{\partial^2 E_y}{\partial y^2} + E_y^+\delta'(z) + \left[\frac{\partial E_y}{\partial z}\right]^+\delta(z) + h_1\frac{\partial^2 E_y}{\partial z^2} \qquad (2.43)$$

$$\nabla^2 h_1 E_z = h_1\frac{\partial^2 E_z}{\partial x^2} + h_1\frac{\partial^2 E_z}{\partial y^2} + E_z^+\delta'(z) + \left[\frac{\partial E_z}{\partial z}\right]^+\delta(z) + h_1\frac{\partial^2 E_z}{\partial z^2} \qquad (2.44)$$

where $\delta'(z)$ is the derivative of $\delta(z)$ with respect to z and

$$\left[\frac{\partial E_x}{\partial z}\right]^{+} = \lim_{z \to 0^{+}} \left[\frac{\partial E_x}{\partial z}\right]$$

Equations (2.42)–(2.44) may be combined in a single vector equation as

$$\nabla^2 h_1 \vec{E} = h_1 \nabla^2 \vec{E} + \vec{E}^{+} \delta'(z) + \left[\frac{\partial \vec{E}}{\partial z}\right]^{+} \delta(z) \qquad (2.45)$$

The third term of equation (2.40) will be

$$\nabla^2 (1-h_1) h_2 \vec{E} = \nabla^2 (1-h_1) h_2 E_x \, \hat{x} + \nabla^2 (1-h_1) h_2 E_y \, \hat{y} + \nabla^2 (1-h_1) h_2 E_z \, \hat{z}$$

Now for cartesian coordinates we have

$$\nabla^2 (1-h_1) h_2 E_x = \nabla \cdot \nabla (1-h_1) h_2 E_x$$

$$= \nabla \cdot \left\{ (1-h_1) \frac{\partial h_2 E_x}{\partial x} \hat{x} + (1-h_1) h_2 \frac{\partial E_x}{\partial y} \hat{y} + h_2 \frac{\partial}{\partial z}\left[(1-h_1) E_x \right] \hat{z} \right\}$$

$$= \nabla \cdot \left\{ (1-h_1) \left[E_x^R \delta(x-d) + h_2 \frac{\partial E_x}{\partial x} \right] \hat{x} + (1-h_1) h_2 \frac{\partial E_x}{\partial y} \hat{y} + \right.$$

$$\left. h_2 \left[(1-h_1) \frac{\partial E_x}{\partial z} - E_x^{-} \delta(z) \right] \hat{z} \right\}$$

$$= (1-h_1) \left[E_x^R \delta'(x-d) + \left[\frac{\partial E_x}{\partial x}\right]^R \delta(x-d) + h_2 \frac{\partial^2 E_x}{\partial x^2} \right] +$$

$$(1-h_1) h_2 \frac{\partial^2 E_x}{\partial y^2} +$$

$$+ h_2 \left[(1-h_1) \frac{\partial^2 E_x}{\partial z^2} - \left[\frac{\partial E_x}{\partial z}\right]^- \delta(z) - E_x^- \delta'(z) \right]$$

$$= (1-h_1) h_2 \left[\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \right] +$$

$$(1-h_1) \left[E_x^R \delta'(x-d) + \left[\frac{\partial E_x}{\partial x}\right]^R \delta(x-d) \right] -$$

$$h_2 \left[E_x^- \delta'(z) + \left[\frac{\partial E_x}{\partial z}\right]^- \delta(z) \right] \qquad (2.46)$$

Similarly,

$$\nabla^2 (1-h_1) h_2 E_y = (1-h_1) h_2 \left[\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} \right] +$$

$$(1-h_1) \left[E_y^R \delta'(x-d) + \left[\frac{\partial E_y}{\partial x}\right]^R \delta(x-d) \right] -$$

$$h_2 \left[E_y^- \delta'(z) + \left[\frac{\partial E_y}{\partial z}\right]^- \delta(z) \right] \qquad (2.47)$$

and

$$\nabla^2 (1-h_1) h_2 E_z = (1-h_1) h_2 \left[\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \right] +$$

$$+ (1-h_1) \left[E_z^R \delta'(x-d) + \left[\frac{\partial E_z}{\partial x}\right]^R \delta(x-d) \right] -$$

$$h_2 \left[E_z^- \delta'(z) + \left[\frac{\partial E_z}{\partial z}\right]^- \delta(z) \right] \quad (2.48)$$

Again these three equations may be combined into a single vector equation as

$$\nabla^2 (1-h_1) h_2 \vec{E} = (1-h_1) h_2 \nabla^2 \vec{E} + (1-h_1) \left[\vec{E}^R \delta'(x-d) + \left[\frac{\partial \vec{E}}{\partial x}\right]^R \delta(x-d) \right] -$$

$$h_2 \left[\vec{E}^- \delta'(z) + \left[\frac{\partial \vec{E}}{\partial z}\right]^- \delta(z) \right] \quad (2.49)$$

where

$$\vec{E}^R = \lim_{x \to d^+} \vec{E} \quad \text{and} \quad \left[\frac{\partial \vec{E}}{\partial x}\right]^R = \lim_{x \to d^+} \left[\frac{\partial \vec{E}}{\partial x}\right]$$

These quantities represent the electric field and its derivative immediately to the right(R) of the vertical interface.

Similarly the second term of equation (2.40) will be

$$\nabla^2 (1-h_1)(1-h_2) \vec{E} = (1-h_1)(1-h_2) \nabla^2 \vec{E} - (1-h_1) \left[\vec{E}^L \delta'(x-d) + \left[\frac{\partial \vec{E}}{\partial x}\right]^L \delta(x-d) \right] -$$

$$(1-h_2) \left[\vec{E}^- \delta'(z) + \left[\frac{\partial \vec{E}}{\partial z}\right]^- \delta(z) \right] \quad (2.50)$$

where,

$$\vec{E}^{L} = \lim_{x \to d^-} \vec{E} \quad \text{and} \quad \left[\frac{\partial \vec{E}}{\partial x}\right]^{L} = \lim_{x \to d^-} \left[\frac{\partial \vec{E}}{\partial x}\right]$$

These quantities represent the electric field and its derivative immediately to the left(L) of the vertical interface.

Also,

$$\vec{E}^{-} = \lim_{z \to 0^-} \vec{E} \quad \text{and} \quad \left[\frac{\partial \vec{E}}{\partial z}\right]^{-} = \lim_{z \to 0^-} \left[\frac{\partial \vec{E}}{\partial z}\right]$$

$$\vec{E}^{+} = \lim_{z \to 0^+} \vec{E} \quad \text{and} \quad \left[\frac{\partial \vec{E}}{\partial z}\right]^{+} = \lim_{z \to 0^+} \left[\frac{\partial \vec{E}}{\partial z}\right]$$

These quantities represent the electric field and its derivative immediately above(+) and immediately below(-) the horizontal interface.

Summarizing, we have obtained expressions for each of the terms of the field decomposition equation, (2.40), which may now be used in the differential field equation (2.36). Writing the left-hand side of (2.36) using (2.45), (2.49) and (2.50) we obtain

$$\nabla^2\vec{E} + \gamma^2\vec{E} = h_1(\nabla^2\vec{E} + k^2\vec{E}) + (1-h_1)h_2(\nabla^2\vec{E} + \gamma_2^2\vec{E}) +$$

$$(1-h_1)(1-h_2)(\nabla^2\vec{E} + \gamma_1^2\vec{E}) + \left[\frac{\partial \vec{E}}{\partial z}\right]^{+}\delta(z) + \vec{E}^{+}\delta'(z) +$$

$$(1-h_1)\left[\left[\frac{\partial \vec{E}}{\partial x}\right]^{R}\delta(x-d) + \vec{E}^{R}\delta'(x-d)\right] -$$

$$- h_2 \left[\left[\frac{\partial \vec{E}}{\partial z} \right]^- \delta(z) + \vec{E}^- \delta'(z) \right] -$$

$$(1-h_2) \left[\left[\frac{\partial \vec{E}}{\partial z} \right]^- \delta(z) + \vec{E}^- \delta'(z) \right] -$$

$$(1-h_1) \left[\left[\frac{\partial \vec{E}}{\partial x} \right]^L \delta(x-d) + \vec{E}^L \delta'(x-d) \right] \qquad (2.51)$$

Now inserting (2.51) in the left-hand side of (2.36) and equating terms with like support, we obtain four equations. Since the support of $\vec{J}_s$ is entirely in the region $z > 0$ the first equation will be

$$h_1 (\nabla^2 \vec{E} + k^2 \vec{E}) = - T_{se} (\vec{J}_s) \qquad (2.52)$$

We also have

$$(1-h_1) h_2 (\nabla^2 \vec{E} + \gamma_2^2 \vec{E}) = 0 \qquad (2.53)$$

$$(1-h_1)(1-h_2)(\nabla^2 \vec{E} + \gamma_1^2 \vec{E}) = 0 \qquad (2.54)$$

where

$$k^2 = \omega^2 \mu_0 \epsilon_0 \qquad (2.55)$$

$$\gamma_1^2 = n_1^2 k^2 \qquad (2.55)$$

$$\gamma_2^2 = n_2^2 k^2$$

The fourth equation will be

$$\left[\frac{\partial \vec{E}}{\partial z}\right]^{+} \delta(z) + \vec{E}^{+} \delta'(z) - \left[\left[\frac{\partial \vec{E}}{\partial z}\right]^{-} + \vec{E}^{-} \delta'(z)\right] +$$

$$(1-h_1)\left[\left[\frac{\partial \vec{E}}{\partial x}\right]^{R} \delta(x-d) + \vec{E}^{R} \delta'(x-d)\right] -$$

$$(1-h_1)\left[\left[\frac{\partial \vec{E}}{\partial x}\right]^{L} \delta(x-d) + \vec{E}^{L} \delta'(x-d)\right]$$

$$= \left[\frac{n_1^2-1}{n_1^2}\right]\nabla(1-h_1)\hat{z}\cdot\vec{E}^{+}\delta(z) + \left[\frac{n_2^2-1}{n_2^2}\right]\nabla h_2\hat{z}\cdot\vec{E}^{+}\delta(z) +$$

$$\left[\frac{n_2^2-n_1^2}{n_2^2}\right]\nabla(1-h_1)\hat{x}\cdot\vec{E}^{R}\delta(x-d) \qquad (2.56)$$

This last equation represents the boundary conditions which the electric field must satisfy at the vertical and horizontal interfaces and as such it will be refered to as the boundary equation.

2.5 Reduction to Integral Equations

In this section the problem as represented by equations (2.52)–(2.56) will be reduced to one involving convolution type integral equations. To accomplish this we make use of the fundamental solution subject to the Sommerfeld radiation condition. We have

$$K_0(x,y,z) = \frac{e^{-jkr}}{4\pi r} \qquad (2.57)$$

$$K_1(x,y,z) = \frac{e^{-j\gamma_1 r}}{4\pi r} \qquad (2.58)$$

$$K_2(x,y,z) = \frac{e^{-j\gamma_2 r}}{4\pi r} \tag{2.59}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and k, γ_1 and γ_2 are given by equation, (2.55).

These functions satisfy

$$\nabla^2 K_0 + k^2 K_0 = -\delta(x)\delta(y)\delta(z) \tag{2.60}$$

$$\nabla^2 K_1 + \gamma_1^2 K_1 = -\delta(x)\delta(y)\delta(z) \tag{2.61}$$

$$\nabla^2 K_2 + \gamma_2^2 K_2 = -\delta(x)\delta(y)\delta(z) \tag{2.62}$$

In order to make use of these functions we introduce the following identities,

$$\nabla^2 h_1 \vec{E} * K_0 = h_1 \vec{E} * \nabla^2 K_0 \tag{2.63}$$

$$\nabla^2 (1-h_1) h_2 \vec{E} * K_2 = (1-h_1) h_2 \vec{E} * \nabla^2 K_2 \tag{2.64}$$

$$\nabla^2 (1-h_1)(1-h_2) \vec{E} * K_1 = (1-h_1)(1-h_2) \vec{E} * \nabla^2 K_1 \tag{2.65}$$

where * denotes a three-dimensional convolution with respect to the x, y and z coordinates. It is assumed that these convolutions exist.

We may proceed by considering the first identity, (2.63), and the expression for $\nabla^2 h_1 \vec{E}$, (2.45). The left-hand side of (2.63) may be written as

$$\nabla^2 h_1\vec{E} * K_0 = h_1\nabla^2\vec{E} * K_0 + \left[\frac{\partial\vec{E}}{\partial z}\right]^+ \delta(z) * K_0$$

$$+ \vec{E}^+ \delta'(z) * K_0 \qquad (2.66)$$

and using equation (2.60) the right-hand side of (2.63) becomes

$$h_1\vec{E} * \nabla^2 K_0 = h_1\vec{E} * (-\delta(x)\delta(y)\delta(z) - k^2 K_0) \qquad (2.67)$$

Equating (2.66) and (2.67) we have

$$h_1\nabla^2\vec{E} * K_0 + \left[\frac{\partial\vec{E}}{\partial z}\right]^+ \delta(z) * K_0 + \vec{E}^+ \delta'(z) * K_0$$

$$= - h_1\vec{E} - h_1 k^2\vec{E} * K_0 \qquad (2.68)$$

Rearranging (2.68) to

$$h_1\vec{E} = -h_1(\nabla^2\vec{E} + k^2\vec{E}) * K_0 - \left[\frac{\partial\vec{E}}{\partial z}\right]^+ \delta(z) * K_0 - \vec{E}^+ \delta'(z) * K_0$$

and using (2.52) in this equation, we have

$$h_1\vec{E} = T_{se}(\vec{J}_s) * K_0 - \left[\left[\frac{\partial\vec{E}}{\partial z}\right]^+ \delta(z) + \vec{E}^+ \delta'(z)\right] * K_0 \qquad (2.69)$$

which is the equation for the electric field above the horizontal interface.

Following the same procedure for the second identity, (2.64), and using the expression for $\nabla^2 (1-h_1)h_2\vec{E}$, (2.49), the left-hand side of (2.64) will be

$$\nabla^2 (1-h_1) h_2 \vec{E} * K_2 = \left[(1-h_1) h_2 \nabla^2 \vec{E} + (1-h_1) \left[\frac{\partial \vec{E}}{\partial x}\right]^R \delta(x-d) + \right.$$

$$\left. + (1-h_1) \vec{E}^R \delta'(x-d) - h_2 \left[\left[\frac{\partial \vec{E}}{\partial z}\right]^- \delta(z) + \vec{E}^- \delta'(z) \right] \right] * K_2 \quad (2.70)$$

The right-hand side of (2.64) using (2.61) will be

$$(1-h_1) h_2 \vec{E} * \nabla^2 K_2 = (1-h_1) h_2 \vec{E} * (-\delta(x)\delta(y)\delta(z) - \gamma_2^2 K_2) \quad (2.71)$$

Equating (2.70) and (2.71) and utilizing (2.53) we obtain

$$(1-h_1) h_2 \vec{E} = \left[h_2 \left[\left[\frac{\partial \vec{E}}{\partial z}\right]^- \delta(z) + \vec{E}^- \delta'(z) \right] + \right.$$

$$\left. + (1-h_1) \left[\left[\frac{\partial \vec{E}}{\partial x}\right]^R \delta(x-d) + \vec{E}^R \delta'(x-d) \right] \right] * K_1 \quad (2.72)$$

which is the equation for the electric field below the horizontal interface and to the right of the vertical interface (ie: the field in medium 2). Following again the same procedure for equation (2.65), utilizing (2.50), (2.62) and (2.54) we may similarly obtain an equation for the electric field below the horizontal interface and to the left of the vertical interface (ie: the field in medium 1).

$$(1-h_1)(1-h_2) \vec{E} = \left[(1-h_2) \left[\left[\frac{\partial \vec{E}}{\partial z}\right]^- \delta(z) + \vec{E}^- \delta'(z) \right] + \right.$$

$$+ (1-h_1) \left[\left[\frac{\partial \vec{E}}{\partial x}\right]^{L} \delta(x-d) + \vec{E}^{L} \delta'(x-d) \right] \cdot K_1 \quad (2.73)$$

The boundary equation may be written as

$$\left[\frac{\partial \vec{E}}{\partial z}\right]^{+} \delta(z) + \vec{E}^{+} \delta'(z) - \left[\frac{\partial \vec{E}}{\partial z}\right]^{-} \delta(z) + \vec{E}^{-} \delta'(z) +$$

$$(1-h_1) \left[\left[\frac{\partial \vec{E}}{\partial x}\right]^{R} \delta(x-d) + \vec{E}^{R} \delta'(x-d) \right] - (1-h_1) \left[\left[\frac{\partial \vec{E}}{\partial x}\right]^{L} \delta(x-d) + \vec{E}^{L} \delta'(x-d) \right]$$

$$= N_1 ((1-h_1) \nabla_{xy} E_z^{+} \delta(z) - E_z^{L+} \delta(x-d) \hat{x} + (1-h_1) E_z^{+} \delta'(z) \hat{z}) +$$

$$+ N_2 (h_2 \nabla_{xy} E_z^{+} \delta(z) + E_z^{R+} \delta(x-d) \delta(z) \hat{x} + h_2 E_z^{+} \delta'(z) \hat{z})$$

$$+ N_{21} ((1-h_1) \nabla_{yz} E_x^{R} \delta(x-d) - E_x^{R-} \delta(x-d) \delta(z) \hat{z} +$$

$$+ (1-h_1) E_x^{R} \delta'(x-d) \hat{x}) \quad (2.74)$$

where the gradient operator has been expanded such that no Heaviside functions remain inside a differential operator and,

$$N_1 = \frac{n_1^2 - 1}{n_1^2} \quad ; \quad N_{21} = \frac{n_2^2 - n_1^2}{n_2^2} \quad ; \quad N_2 = \frac{n_2^2 - 1}{n_2^2}$$

and,

$$E_z^{L+} = E_z^{L+}(y) = \lim_{x \to d^-} E_z^{+}(x, y)$$

$$E_z^{R+} = E_z^{R+}(y) = \lim_{x \to d^+} E_z^{+}(x, y)$$

$$E_x^{R-} = E_x^{R-}(y) = \lim_{z\to 0^-} E_x^R(x,y)$$

This boundary equation may be separated into four sub-equations by considering terms having common support. Making use of the relationship

$$\left[\frac{\partial \vec{E}}{\partial z}\right]^+ \delta(z) + \vec{E}^+\delta'(z) = (1-h_1)\left\{\left[\frac{\partial \vec{E}}{\partial z}\right]\delta(z) + \vec{E}\delta'(z)\right\} + h_2\left\{\left[\frac{\partial \vec{E}}{\partial z}\right]\delta(z) + \vec{E}\delta'(z)\right\}$$

Equation (2.74) may be separated into

$$(1-h_2)\left\{\left[\frac{\partial \vec{E}}{\partial z}\right]^+ \delta(z) + \vec{E}^+\delta'(z)\right\} - (1-h_2)\left\{\left[\frac{\partial \vec{E}}{\partial z}\right]^- \delta(z) + \vec{E}^-\delta'(z)\right\}$$

$$= N_1(1-h_2)\left\{\nabla_{xy}E_z^+\delta(z) + E_z^+\delta'(z)\hat{z}\right\} \qquad (2.75)$$

$$h_2\left\{\left[\frac{\partial \vec{E}}{\partial z}\right]^+ \delta(z) + \vec{E}^+\delta'(z)\right\} - h_2\left\{\left[\frac{\partial \vec{E}}{\partial z}\right]^- \delta(z) + \vec{E}^-\delta'(z)\right\}$$

$$= N_2 h_2\left\{\nabla_{xy}E_z^+\delta(z) + E_z^+\delta'(z)\hat{z}\right\} \qquad (2.76)$$

$$(1-h_1)\left\{\left[\frac{\partial \vec{E}}{\partial x}\right]^R \delta(x-d) + \vec{E}^R\delta'(x-d)\right\} - (1-h_1)\left\{\left[\frac{\partial \vec{E}}{\partial x}\right]^L \delta(x-d) + \vec{E}^L\delta'(x-d)\right\}$$

$$= N_{21}(1-h_1)\left\{\nabla_{yz}E_x^R\delta(x-d) + E_x^R\delta'(x-d)\hat{x}\right\} \qquad (2.77)$$

and,

$$-N_1\left[E_z^{L+}\delta(x-d)\delta(z)\right]\hat{x} + N_2\left[E_z^{R+}\delta(x-d)\delta(z)\right]\hat{x} -$$

$$-N_{21}\left[E_x^{R-}\delta(x-d)\delta(z)\right]\hat{z} = 0 \qquad (2.78)$$

It is interesting to note that this last equation, (2.78), represents the boundary conditions which the electric field must satisfy on the line intersecting the two interfaces. As $N_1 \neq N_2$ then (2.78) will be satisfied only if

$$E_z^{R+} = E_z^{L+} = E_x^{R-} = 0$$

This agrees with the interpretation of the classical boundary conditions in this situation.

In this section, equations for the electric field in each region have been derived (ie: (2.69),(2.72),(2.73)) in terms of the field and its spatial derivative on the bounding interfaces. If the quantities

$$\left[\frac{\partial\vec{E}}{\partial z}\right]^{+},\ \vec{E}^{+},\ \left[\frac{\partial\vec{E}}{\partial z}\right]^{-},\ \vec{E}^{-},\ \left[\frac{\partial\vec{E}}{\partial x}\right]^{R}\ \vec{E}^{R},\ \left[\frac{\partial\vec{E}}{\partial x}\right]^{L} \text{ and } \vec{E}^{L}$$

are known then the electric field can be found for all regions. This is a statement of Green's theorem. The problem is thereby reduced to one of deriving expressions for these quantities.

2.6 Simplification of the Integral Equations

In the previous section we have derived three field equations and a set of boundary equations. In this section we simplify the field equations by using the boundary equations to eliminate half of the unknowns. To this end the boundary equations, (2.75)-(2.77), may be rewritten as

$$(1-h_2)\left[\left[\frac{\partial \vec{E}}{\partial z}\right]^{-}\delta(z)+\vec{E}^{-}\delta'(z)\right]=(1-h_2)\left[\left[\frac{\partial \vec{E}}{\partial z}\right]^{+}\delta(z)+\vec{E}^{+}\delta'(z)\right]-$$

$$-\;N_1(1-h_2)\left[\nabla_{xy}E_z^{+}\delta(z)+E_z^{+}\delta'(z)\hat{z}\right] \qquad (2.79)$$

$$h_2\left[\left[\frac{\partial \vec{E}}{\partial z}\right]^{-}\delta(z)+\vec{E}^{-}\delta'(z)\right]=h_2\left[\left[\frac{\partial \vec{E}}{\partial z}\right]^{+}\delta(z)+\vec{E}^{+}\delta'(z)\right]-$$

$$-\;N_2h_2\left[\nabla_{xy}E_z^{+}\delta(z)+E_z^{+}\delta'(z)\hat{z}\right] \qquad (2.80)$$

$$(1-h_1)\left[\left[\frac{\partial \vec{E}}{\partial x}\right]^{L}\delta(x-d)+\vec{E}^{L}\delta'(x-d)\right]=(1-h_1)\left[\left[\frac{\partial \vec{E}}{\partial x}\right]^{R}\delta(x-d)+\vec{E}^{R}\delta'(x-d)\right]-$$

$$-\;N_{21}(1-h_1)\left[\nabla_{yz}E_x^{R}\delta(x-d)+E_x^{R}\delta'(x-d)\hat{x}\right] \qquad (2.81)$$

The three field equations (2.69), (2.72) and (2.73) may be rewritten as

$$h_1\vec{E}=\vec{E}_s-\left[\left[\frac{\partial \vec{E}}{\partial z}\right]^{+}\delta(z)+\vec{E}^{+}\delta'(z)\right]*K_0 \qquad (2.82)$$

where $\vec{E}_s = T_{se}(\vec{J}_s) * K_0$; the source field

$$(1-h_1)h_2\vec{E} = \left[h_2\left[\left[\frac{\partial\vec{E}}{\partial z}\right]^{-}\delta(z) + \vec{E}^{-}\delta'(z)\right] - \right.$$

$$\left. -(1-h_1)\left[\left[\frac{\partial\vec{E}}{\partial x}\right]^{R}\delta(x-d) + \vec{E}^{R}\delta'(x-d)\right]\right] * K_2 \qquad (2.83)$$

$$(1-h_1)(1-h_2)\vec{E} = \left[(1-h_2)\left[\left[\frac{\partial\vec{E}}{\partial z}\right]^{-}\delta(z) + \vec{E}^{-}\delta'(z)\right] + \right.$$

$$\left. + (1-h_1)\left[\left[\frac{\partial\vec{E}}{\partial x}\right]^{L}\delta(x-d) + \vec{E}^{L}\delta'(x-d)\right]\right] * K_1 \qquad (2.84)$$

We have three equations, (2.82)-(2.84), in eight unknowns. Using the boundary equations, (2.79)-(2.81), four of the unknowns may be eliminated. Substituting (2.80) into (2.83) the field equation for medium 2 becomes

$$(1-h_1)h_2\vec{E} = \left[h_2\left[\left[\frac{\partial\vec{E}}{\partial z}\right]^{+}\delta(z) + \vec{E}^{+}\delta'(z)\right] - N_2 h_2\left[\nabla_{xy}E_z^{+}\delta(z) + \right.\right.$$

$$\left.\left. + E_z^{+}\delta'(z)\hat{z}\right] - (1-h_1)\left[\left[\frac{\partial\vec{E}}{\partial x}\right]^{R}\delta(x-d) + \vec{E}^{R}\delta'(x-d)\right]\right] * K_2 \qquad (2.85)$$

Substituting (2.79) and (2.81) into (2.84) the field equation for medium

1 becomes

$$(1-h_1)(1-h_2)\vec{E} = (1-h_2)\left\{\left[\left\{\frac{\partial \vec{E}}{\partial z}\right\}^{+}\delta(z) + \vec{E}^{+}\delta'(z)\right] - N_1(1-h_2)\left[\nabla_{xy}E_z^{+}\delta(z) + \right.\right.$$

$$\left.\left. + E_z^{+}\delta'(z)\hat{z}\right]\right\} + (1-h_1')\left[\left\{\frac{\partial \vec{E}}{\partial x}\right\}^{R}\delta(x-d) + \vec{E}^{R}\delta'(x-d)\right] -$$

$$- N_{21}(1-h_1)\left[\nabla_{yz}E_x^{R}\delta(x-d) + E_x^{R}\delta'(x-d)\hat{x}\right]\Big\} * K_1 \qquad (2.86)$$

The problem is thereby reduced to one of solving three equations, (2.82), (2.85) and (2.86), in four unknowns

$$\left\{\frac{\partial \vec{E}}{\partial z}\right\}^{+} \qquad \left\{\frac{\partial \vec{E}}{\partial x}\right\}^{R} \qquad \vec{E}^{+} \quad \text{and} \quad \vec{E}^{R}$$

If we could take planes of constant x and z such that the left-hand sides of the equations are zero, then we could generate six equations in four unknowns. However, in order to do this the spatial convolutions in the x and z directions must be carried out first. This approach has been used sucessfully by Walsh (1982) for the problem of propagation over layered media. Walsh also uses the spatial x-y Fourier transform to reduce the remaining convolutions to multiplications. In the present problem the presence of Dirac delta functions in both x and z prohibits a direct application of this method. However, if certain assumptions are made concerning the fields on the horizontal and vertical interfaces, the equations are reduced to a more favorable form.

We will assume that the refractive indices of medium 1 and medium 2 are large compared to that of freespace and as a consequence the fields below the horizontal interface may be neglected. In addition, the analysis will be confined to the vertical polarization of the surface field, since for a High Frequency radar system with both transmitting and receiving antennas located close to the earth's surface a Norton surface wave mode of propagation is predominant and the vertical polarization is most efficient in this mode.

As a consequence of these two assumptions, equations (2.85) and (2.86) will become

$$(1-h_1)(1-h_2)E_z = h_2\left[\left[\frac{\partial E_z}{\partial z}\right]^+\delta(z) + E_z^+\delta'(z) - N_2E_z^+\delta'(z)\right] * K_2$$

$$= h_2\left[\left[\frac{\partial E_z}{\partial z}\right]^+\delta(z) + \frac{1}{n_2^2}E_z^+\delta'(z)\right] * K_2 \qquad (2.87)$$

and

$$(1-h_1)(1-h_2)E_z = (1-h_2)\left[\left[\frac{\partial E_z}{\partial z}\right]^+\delta(z) + E_z^+\delta'(z) - N_1E_z^+\delta'(z)\right] * K_1$$

$$= (1-h_2)\left[\left[\frac{\partial E_z}{\partial z}\right]^+\delta(z) + \frac{1}{n_1^2}E_z^+\delta'(z)\right] * K_1 \qquad (2.88)$$

These equations may be reduced to an algebraic form by taking the two dimensional (spatial) Fourier transform with respect to the x and y variables. The Fourier transform is given as

$$\underline{f}(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j(k_xx+k_yy)}\,dx\,dy$$

where k_x and k_y are transform variables and the bar under the quantity indicates the Fourier transform.

The Fourier transform of (2.87) and (2.88) may be written as

$$\frac{\underline{(1-h_1)h_2E_z}}{4} = \underline{h_2\left[\frac{\partial\vec{E}}{\partial z}\right]^+}\,\underline{K_2} + \frac{1}{n_2^2}\,\underline{h_2E_z^+}\,\frac{\partial}{\partial z}\,\underline{K_2} \tag{2.89}$$

$$\underline{(1-h_1)(1-h_2)E_z} = \underline{(1-h_2)\left[\frac{\partial\vec{E}}{\partial z}\right]^+}\,\underline{K_1} + \frac{1}{n_1^2}\,\underline{(1-h_2)E_z^+}\,\frac{\partial}{\partial z}\,\underline{K_1} \tag{2.90}$$

Similarly the equation for the field above the horizontal interface, (2.82), is transformed to give

$$\underline{h_1E_z} = \underline{E_{zs}} - \left[\underline{\left[\frac{\partial E_z}{\partial z}\right]^+} + \underline{E_z^+}\,\frac{\partial}{\partial z}\right]\underline{K_0} \tag{2.91}$$

where $\underline{E_{zs}}$ is the z component of the source field.

The transforms of the fundamental solutions, (2.57)-(2.59), are given as, (Walsh 1980b),

$$\underline{K_0} = \underline{K_0}(k_x,k_y) = \frac{e^{-|z|u_0}}{2u_0} \tag{2.92}$$

$$\underline{K_1} = \underline{K_1}(k_x,k_y) = \frac{e^{-|z|u_1}}{2u_1} \tag{2.93}$$

$$\underline{K_2} = \underline{K_2}(k_x, k_y) = \frac{e^{-|z|u_2}}{2u_2} \qquad (2.94)$$

where

$$u_0 = \sqrt{\lambda^2 - k^2}$$

$$u_1 = \sqrt{\lambda^2 - n_1^2 k^2}$$

$$u_2 = \sqrt{\lambda^2 - n_2^2 k^2}$$

and $$\lambda^2 = k_x{}^2 + k_y{}^2$$

We may now choose planes of constant z above and below the horizontal interface which make the left-hand sides of (2.89)-(2.91) zero. For equations (2.89) and (2.90) choose a plane $z = z^+ : z^+ > 0$. Equation (2.89) will be

$$0 = \underline{h_2 \left[\frac{\partial E_z}{\partial z}\right]^+} \frac{e^{-|z^+|u_2}}{2u_2} + \frac{1}{n_2^2} \underline{h_2 E_z^+} \frac{\partial}{\partial z} \frac{e^{-|z^+|u_2}}{2u_2}$$

Noting that

$$\frac{e^{-|z^+|u_2}}{2u_2} = \frac{e^{-z^+u_2}}{2u_2}$$

we have

$$0 = \underline{h_2 \left[\frac{\partial E_z}{\partial z}\right]^+} \frac{e^{-z^+u_2}}{2u_2} - \frac{1}{n_2^2} \underline{h_2 E_z^+} \frac{e^{-z^+u_2}}{2}$$

Multiply by. $e^{z^+u_2} \; 2u_2$

$$0 = \underline{h_2 \left[\frac{\partial E_z}{\partial z}\right]^+} - \frac{u_2}{n_2^2}\, \underline{h_2 E_z^+} \tag{2.95}$$

Similarly for (2.90) we may obtain

$$0 = \underline{(1-h_2)\left[\frac{\partial E_z}{\partial z}\right]^+} - \frac{u_1}{n_1^2}\, \underline{(1-h_2)E_z^+} \tag{2.96}$$

Combining (2.95) and (2.96) we may derive an expression for $\left[\frac{\partial E_z}{\partial z}\right]^+$ in terms of E_z^+.

$$0 = \underline{\left[\frac{\partial E_z}{\partial z}\right]^+} - \frac{u_2}{n_2^2}\, \underline{h_2 E_z^+} - \frac{u_1}{n_1^2}\, \underline{(1-h_2)E_z^+}$$

$$\underline{\left[\frac{\partial E_z}{\partial z}\right]^+} = -\frac{u_1}{n_1^2} E_z^+ + \left[\frac{u_1}{n_1^2} - \frac{u_2}{n_2^2}\right] \underline{h_2 E_z^+} \tag{2.97}$$

noting that $\underline{(1-h_1)E_z^+} = \underline{E_z^+} - \underline{h_2 E_z^+}$.

For equation (2.91) choose a plane $z = z^-$; $z^- < 0$ which yields

$$0 = \underline{E_{zs}^{(z^-)}} - \left[\underline{\left[\frac{\partial E_z}{\partial z}\right]^+} + \underline{E_z^+}\frac{\partial}{\partial z}\right] \frac{e^{-|z^-|u_0}}{2u_0}$$

and with $\dfrac{e^{-|z^-|u_0}}{2u_0} = \dfrac{e^{z^-u_0}}{2u_0}$ we have

$$0 = \underline{E_{zs}^{(z^-)}} - \underline{\left[\frac{\partial E_z}{\partial z}\right]^+} \frac{e^{z^- u_0}}{2u_0} - \underline{E_z^+} \frac{e^{z^- u_0}}{2} \tag{2.98}$$

Multiplying equation (2.98) by $e^{-z^- u_0}\ 2u_0$ we have

$$0 = \underline{E_{zs}^{(z^-)}}\ e^{-z^- u_0}\ 2u_0 - \underline{\left[\frac{\partial E_z}{\partial z}\right]^+} - u_0 \underline{E_z^+} \tag{2.99}$$

Substituting (2.97) into (2.99) yields

$$-\underline{E_{zs}^{(z^-)}}\ e^{-z^- u_0}\ 2u_0 = -\frac{u_1}{n_1^2}\underline{E_z^+} - u_0 \underline{E_z^+} + \left[\frac{u_1}{n_1^2} - \frac{u_2}{n_2^2}\right] \underline{h_2 E_z^+}$$

or rearranging,

$$\underline{E_z^+} - \frac{\left[\frac{u_1}{n_1^2} - \frac{u_2}{n_2^2}\right]}{u_0 + \frac{u_1}{n_1^2}} \underline{h_2 E_z^+} = \underline{E_{zs}^{(z^-)}} \frac{e^{-z^- u_0}\ 2u_0}{u_0 + \frac{u_1}{n_1^2}} \tag{2.100}$$

The vertical electric field on the surface $z = 0$, E_z^+, will be given as the solution of (2.100). Similarly we may form an integral equation for $\left[\frac{\partial E_z}{\partial z}\right]^+$; however, we will confine our analysis to E_z^+ as we seek the effect of the vertical discontinuity on the vertical component of the surface field. In keeping with the assumption that the refractive indices of the media below are large compared to free space we may also state that $n_1 k$

and $n_2 k$ are large in comparison to the spatial wavenumbers of the electric field in the media below and we may therefore simplify the quantity $\frac{u_1}{n_1^2}$ to

$$\frac{u_1}{n_1^2} = \frac{(k_x^2 + k_y^2 - n_1^2 k^2)^{1/2}}{n_1^2}$$

$$\frac{u_1}{n_1^2} \approx \frac{(-n_1^2 k^2)^{1/2}}{n_1^2} \tag{2.101}$$

$$\approx \frac{jk}{n_1}$$

Now,

$$\Delta_1 = \frac{(n_1^2 - 1)^{1/2}}{n_1^2}$$

which is the standard definition of "surface impedence" and for $n_1 \gg 1$ we have,

$$\Delta_1 \approx \frac{1}{n_1}$$

and,

$$\frac{u_2}{n_2^2} \approx jk\Delta_2 \tag{2.102}$$

This assumption is equivalent to the homogeneous boundary condition of other investigators and is acceptable for a vertically polarized field (Hufford 1952, Bremmer 1954). Utilizing this approximation in the surface field equation , (2.100), we obtain

$$E_z^+ - \frac{jk(\Delta_1 - \Delta_2)}{u_0 + jk\Delta_1} h_2 E_z^+ = E_{zs}(z^-) \frac{e^{-z^- u_0} \, 2u_0}{u_0 + jk\Delta_1} \quad (2.103)$$

or alternatively taking the inverse spatial Fourier transform , F^{-1} , of (2.103) yields

$$E_z^+ - g(x,y) * h_2 E_z^+ = f(x,y) \quad (2.104)$$

where

$$g(x,y) = F^{-1}\left\{\frac{jk(\Delta_1 - \Delta_2)}{u_0 + jk\Delta_1}\right\} \quad (2.105)$$

and

$$f(x,y) = F^{-1}\left[E_{zs}(z^-) \frac{e^{-z^- u_0} \, 2u_0}{u_0 + jk\Delta_1}\right] \quad (2.106)$$

The solution of equation (2.104) will yield the vertical component of the electric field at any point on the surface of the earth when the earth is comprised of two homogeneous media. In the next section a method of solution is chosen and an elementary vertical electric dipole is selected for analysis.

2.7 Solution for the Surface field. E_z^+

The method chosen for solution of equation (2.104) is an operator expansion approach. We may proceed by writing the left-hand side of (2.104) in operator notation and formally inverting this equation to provide a solution for E_z^+ in terms of the Neumann series expansion of the inverse operator. We may define a linear operator T as

$$\mathbf{T} = g(x,y) * h_2(\cdot) \tag{2.107}$$

such that when T operates on f(x, y) we obtain

$$\mathbf{T}\, f = g(x,y) * h_2(f(x,y))$$

where g(x, y) and f(x, y) are given by (2.105) and (2.106) respectively. Equation (2.104) may now be written as

$$[\, \mathbf{I} - \mathbf{T}\,]\, E_z^+ = f(x,y) \tag{2.108}$$

where I is the identity operator. Inverting this equation we obtain

$$E_z^+ = [\, \mathbf{I} - \mathbf{T}\,]^{-1}\, f(x,y) \tag{2.109}$$

we obtain an expression for the solution of the problem. Formally, the inverse operator is given by the Neumann series, i.e.,

$$[\, \mathbf{I} - \mathbf{T}\,]^{-1} = \mathbf{I} + \mathbf{T} + \mathbf{T}^2 + \mathbf{T}^3 + \ldots\ldots. \tag{2.110}$$

$$= \sum_{n=0}^{\infty} \mathbf{T}^n$$

We shall show later that under certain assumptions this series may be summed. The solution for E_z^+ may now be formed by taking successive terms of (2.109) using the expansion of (2.110). Before proceeding further, the source field is taken to be the far field of an elementary vertical electric dipole.

2.7.1 The Field of a Dipole Source

It will be convenient at this point to introduce the source field as the far field of an elementary vertical electric dipole located at coordinates (0,0,h) carrying a current I with elementary length dL as given in Figure 2.

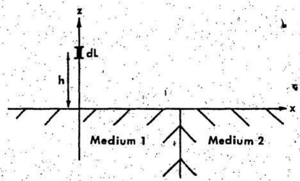


Figure 2 Geometry for a Dipole Source

The source current density, $\vec{J}_s$, is given by

$$\vec{J}_s = I\ \delta(x)\delta(y)\left[u\left(z-\left(h-\frac{dL}{2}\right)\right)-u\left(z-\left(h+\frac{dL}{2}\right)\right)\right]\hat{z} \quad (2.111)$$

where u is Heaviside's function.

Now,

$$\lim_{dL\to 0}\left\{ u\left(z-\left(h-\frac{dL}{2}\right)\right) - u\left(z-\left(h+\frac{dL}{2}\right)\right)\right\}$$

$$= \lim_{dL\to 0}\left[dL\frac{\left\{ u\left(z-\left(h-\frac{dL}{2}\right)\right) - u\left(z-\left(h+\frac{dL}{2}\right)\right)\right\}}{dL}\right]$$

$$= dL\frac{\partial}{\partial z}(u(z-h))$$

$$= dL\,\delta(z-h)$$ for an infinitesimal dipole length dL

Accordingly $\vec{J}_s$ is given by

$$\vec{J}_s = I\ dL\ \delta(x)\delta(y)\delta(z-h)\ \hat{z} \tag{2.112}$$

and the total field due to this source will be

$$\vec{E}_s = T_{se}(\vec{J}_s) * K_0$$

or equivalently

$$\vec{E}_s = T_{se}(\vec{J}_s * K_0) \tag{2.113}$$

Utilizing (2.112) for $\vec{J}_s$ and (2.57) for K_0, equation 2.113 becomes

$$\vec{E}_s = T_{se}\left[I dL\ \frac{e^{-jkr_1}}{4\pi r_1}\ \hat{z}\right] \tag{2.114}$$

where $r_1 = (x^2 + y^2 + (z-h)^2)^{1/2}$

Expanding the T_{se} operator we obtain

$$\vec{E}_s = \frac{1}{j\omega\epsilon_0}\left[\nabla\left[\nabla\cdot IdL\ \frac{e^{-jkr_1}}{4\pi r_1}\right] + k^2 IdL\ \frac{e^{-jkr_1}}{4\pi r_1}\right]\hat{z} \qquad (2.115)$$

As we are concerned with the far field of the dipole, the first term in equation (2.115) may be neglected for h small (ie: the dipole is close to the surface) as this term gives rise to the near field of the dipole (ie: it will contain terms with decreasing powers of r_1). The far field of the dipole is therefore given as

$$\vec{E}_s = \frac{1}{j\omega\epsilon_0}\ k^2 IdL\ \frac{e^{-jkr_1}}{4\pi r_1}\hat{z} \qquad (2.116)$$

or denoting the z component of $\vec{E}_s$ as E_{zs}

$$E_{zs} = C\ \frac{e^{-jkr_1}}{4\pi r_1} \qquad (2.117)$$

where

$$C = -j\omega\mu_0 IdL \quad : \text{dipole moment} \qquad (2.118)$$

The far field derived here agrees with that given by Jordan and Balmain, (1968).

The spatial Fourier transform of E_{zs} will be

$$\underline{E_{zs}} = C\ \frac{e^{-|z-h|u_0}}{2u_0}$$

Therefore for $z = z^-$,

$$\underline{E_{zs}}^{z^-} = C\ \frac{e^{-|z^- - h|u_0}}{2u_0}.$$

and since $z^- < 0 < h$ then,

$$z^- - h < 0 \quad \text{and} \quad -| z^- - h | = (z^- - h)$$

$$\underline{E_{zs}}^{z^-} = C\,\frac{e^{(z^- - h)u_0}}{2u_0} \qquad (2.119)$$

2.7.2 Inversion of the Integral Equation

Summarizing, we have obtained an equation for the vertical component of the surface field, E_z^+, in terms of a linear operator **T**, (2.109) which was given as

$$E_z^+ = [\, I - T \,]^{-1} f(x,y) \qquad (2.120)$$

where

$$T = g(x,y) * h_2(\cdot) \qquad (2.121)$$

$$g(x,y) = F^{-1}\left\{\frac{jk(\Delta_1 - \Delta_2)}{u_0 + jk\Delta_1}\right\} \qquad (2.122)$$

$$f(x,y) = F^{-1}\left[\underline{E_{zs}}^{z^-}\,\frac{e^{-z^- u_0}\; 2u_0}{u_0 + jk\Delta_1}\right] \qquad (2.123)$$

The spatial Fourier transform of the source field, $\underline{E_{zs}}^{z^-}$, is given by (2.119) and therefore (2.123) may be written as

$$f(x,y) = F^{-1}\left[C\,\frac{e^{-hu_0}}{u_0 + jk\Delta_1}\right] \qquad (2.124)$$

Now for g(x, y) we have

$$g(x,y) = F^{-1}\left\{\frac{jk(\Delta_1-\Delta_2)}{u_0 + jk\Delta_1}\right\}$$

$$= jk(\Delta_1-\Delta_2)\ F^{-1}\left\{\frac{1}{u_0 + jk\Delta_1}\right\}$$

$$= jk(\Delta_1-\Delta_2)\ F^{-1}\left\{\frac{1}{u_0} - \frac{jk\Delta_1}{u_0(u_0 + jk\Delta_1)}\right\}$$

$$= jk(\Delta_1-\Delta_2)\left\{\frac{e^{-jk\rho}}{2\pi\rho} - \right.$$

$$\left. - \frac{1}{(2\pi)^2}\int_{k_x=-\infty}^{\infty}\int_{k_y=-\infty}^{\infty}\frac{jk\Delta_1}{u_0(u_0 + jk\Delta_1)}\, e^{-jk_x x\ -jk_y y}\, dk_x dk_y\right\}$$

or

$$g(x,y) = \frac{jk(\Delta_1-\Delta_2)}{2\pi}\left[\frac{e^{-jk\rho}}{\rho} - P\right] \qquad (2.125)$$

where

$$\rho = \sqrt{x^2 + y^2}$$

and

$$P = \frac{1}{2\pi}\int_{k_x=-\infty}^{\infty}\int_{k_y=-\infty}^{\infty}\frac{jk\Delta_1}{u_0(u_0 + jk\Delta_1)}\, e^{-jk_x x\ -jk_y y}\, dk_x dk_y \qquad (2.126)$$

We may write the integral P in cylindrical coordinates where

$$x = \rho\cos\theta \qquad k_x = \lambda\cos\phi$$

$$y = \rho\sin\theta \qquad k_y = \lambda\sin\phi$$

and

$$u_0 = \sqrt{\lambda^2 - k^2} \quad \text{where} \quad \lambda^2 = k_x^2 + k_y^2$$

$$P = \frac{1}{2\pi}\int_{\lambda=0}^{\infty}\int_{\phi=0}^{2\pi} \frac{jk\Delta_1}{u_0(u_0 + jk\Delta_1)} e^{\lambda\rho\cos(\phi-\theta)} \lambda d\phi d\lambda$$

$$= \frac{1}{2\pi}\int_0^{\infty} \frac{jk\Delta_1}{u_0(u_0 + jk\Delta_1)}\int_0^{2\pi} e^{\lambda\rho\cos(\phi-\theta)} d\phi d\lambda$$

$$= \int_0^{\infty} \frac{jk\Delta_1\lambda}{u_0(u_0 + jk\Delta_1)} J_0(\lambda\rho) d\lambda \qquad (2.127)$$

where J_0 is a Bessel function of the first kind of zero order.

This integral P may be evaluated asymptotically using the saddle point method as shown by Wait (1964). The result is subject to the conditions $|\Delta_1| \ll 1$ and $k\rho \gg 1$; and is given as

$$P = -j\sqrt{\pi p}\, e^{-p}\, erfc(j\sqrt{p})\, \frac{e^{-jk\rho}}{\rho} \qquad (2.128)$$

where $p = \frac{-jk\Delta_1^2}{2}\rho$: numerical distance

and

$$erfc = \frac{2}{\sqrt{\pi}}\int_{j\sqrt{p}}^{\infty} e^{-y^2} dy \quad : \text{complementary error function}$$

Inserting this expression for P into (2.125) we obtain for $g(x,y)$

$$g(x,y) = \frac{jk(\Delta_1 - \Delta_2)}{2\pi}\frac{e^{-jk\rho}}{\rho} F_1 \qquad (2.129)$$

where $F_1 = 1 - j\sqrt{\pi p}\, e^{-p}\, erfc(j\sqrt{p})$ is interpreted as a Norton attenuation function, the subscript indicating that this attenuation function applies for propagation over medium 1.

Similarly we obtain for (2.124), as $h \to 0$,

$$f(x,y) = C\, F^{-1}\left\{\frac{1}{u_0 + jk\Delta_1}\right\}$$

$$= \frac{C}{2\pi}\frac{e^{-jk\rho}}{\rho} F_1 \qquad (2.130)$$

Rewriting equation (2.109) using (2.110) the surface field is given as,

$$E_z^+ = \sum_{i=0}^{\infty} T^i f(x,y)$$

$$= f(x,y) + T f(x,y) + T^2 f(x,y) + \ldots \qquad (2.131)$$

where $T = \frac{jk(\Delta_1 - \Delta_2)}{2\pi}\frac{e^{-jk\rho}}{\rho} F_1(\rho) * h_2(\cdot)$

Denote each term of (2.131) as M_i, M_i being the i^{th} order term in the solution for E_z^+.

Accordingly,

$$M_0 = f(x,y) = \frac{C}{2\pi}\frac{e^{-jk\rho}}{\rho} F_1(\rho) \qquad (2.132)$$

$$M_1 = T f(x,y) = \frac{C}{2\pi}\frac{jk(\Delta_1-\Delta_2)}{2\pi}\frac{e^{-jk\rho}}{\rho} F_1(\rho) * h_2 \frac{e^{-jk\rho}}{\rho} F_1(\rho) \qquad (2.133)$$

$$M_2 = T^2 f(x,y) = \frac{C}{2\pi}\frac{jk(\Delta_1-\Delta_2)}{2\pi}\frac{e^{-jk\rho}}{\rho} F_1(\rho) *$$

$$h_2\left[\frac{jk(\Delta_1-\Delta_2)}{2\pi}\frac{e^{-jk\rho}}{\rho} F_1(\rho) * h_2\frac{e^{-jk\rho}}{\rho} F_1(\rho)\right] \qquad (2.134)$$

Similarly all higher orders may be expressed.

The zeroth order term of the solution, M_0, represents the undisturbed field propagating with the Norton attenuation function of medium 1 (ie: F_1). The higher order terms represent the modification of the field due to the presence of the second medium. It is these higher order terms (ie: i= 1 to ∞) which will contribute to the backscattered field.

Equation (2.133) may be written as,

$$M_1 = \frac{C}{2\pi}\frac{jk(\Delta_1-\Delta_2)}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-jk(\rho-\rho')}}{(\rho-\rho')}F_1(\rho-\rho')\ h_2(x')\frac{e^{-jk\rho'}}{(\rho')}F_1(\rho')\,dx'dy' \tag{2.135}$$

where,

$$(\rho-\rho') = ((x-x')^2+(y-y')^2)^{1/2} \quad \text{and} \quad (\rho') = ((x')^2+(y')^2)^{1/2}$$

A graphical interpretation of the first order term of the solution, M_1, is shown in Figure 3 where this term is interpreted as a spatial convolution over the region $x>d$.

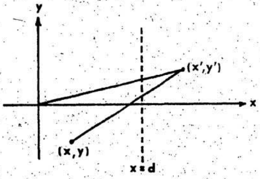


Figure 3 Graphical Interpretation of the First Order Solution

The convolution may be obtained for any (x,y) however the point (x',y') is restricted to the region $x>d$. The first order term, M_1, may thereby be interpreted as the sum of single reflections of the source field in medium 2 (ie: $x>d$).

The second order term, M_2, is given graphically in Figure 4 where M_2, is interpreted as a spatial convolution over the region $x>d$ to a point (x',y') and another spatial convolution over this region from (x',y') to any (x,y).

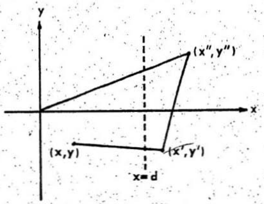


Figure 4 Graphical Interpretation of the Second Order Solution

The third order term, M_3, is the result of two convolutions in the region $x>d$. That is, each point (x',y') receives energy that has been twice reflected in medium 2. Similarly the l^{th} order represents $l-1$ such reflections in medium 2. Although this ray type interpretation of the field may not be strictly applicable to this type of propagation it certainly provides a correct and convenient interpretation of the convolution integrals in the solution.

Consider the second order term, M_2.

$$M_2 = \frac{C}{2\pi}\left[\frac{jk(\Delta_1-\Delta_2)}{2\pi}\right]^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk(\rho-\rho')}}{(\rho-\rho')} F_1(\rho-\rho')\, h_2(x')$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk(\rho'-\rho'')}}{(\rho'-\rho'')} F_1(\rho'-\rho'') h_2(x'') \frac{e^{-jk(\rho'')}}{\rho''} F_1(\rho'')\, dx''dy''dx'dy' \quad (2.136)$$

If we let $r_a = \rho''$ and $r_b = \rho'-\rho''$ in the integral over x'', y'' in (2.136) we may write the inner integral as,

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk(r_a+r_b)}}{r_a r_b} F_1(r_a)\, h_2(x'')\, F_1(r_b)\, dx''dy'' \quad (2.137)$$

The integrand in (2.137) contains an exponential function which varies much more rapidly than its other factors. This suggests the application of a saddle point approximation at the point of stationary phase. To this end we introduce elliptical coordinates. In the notation of King(1968),

$$\frac{r_a + r_b}{\rho} = \cosh\mu$$

$$\frac{r_a - r_b}{\rho} = \cos\theta$$

and,

$$dx'dy' = r_a\, r_b\, d\mu\, d\theta$$

Equation (2.137) in elliptical coordinates will be,

$$I = \int_{\mu=-\infty}^{\infty}\int_{\theta=0}^{\pi} \frac{e^{-jk\rho'\cosh\mu}}{r_a r_b} F_1(r_a) h_2\left(\frac{\rho'}{2}\cosh\mu\cos\theta\right) F_1(r_b) r_a r_b\, d\mu\, d\theta$$

and taking the saddle point approximation at the point of stationary phase (μ=0) we have,

$$\cosh\mu \approx 1 + \frac{\mu^2}{2}$$

The integral I becomes,

$$I = e^{-jk\rho'} \int_0^{\pi} F_1(\rho_a) h_2(\frac{\rho'}{2}\cos\theta) F_1(\rho_b) \int_{-\infty}^{\infty} e^{-jk\rho'\frac{\mu^2}{2}} d\mu d\theta \qquad (2.138)$$

where,

$$\rho_a = \frac{\rho'}{2}(1+\cos\theta)$$

$$\rho_b = \frac{\rho'}{2}(1-\cos\theta)$$

Noting that,

$$\int_{-\infty}^{\infty} e^{-jk\rho'\frac{\mu^2}{2}} d\mu = e^{-j\frac{\pi}{4}} \sqrt{\frac{2\pi}{k\rho'}} \qquad (2.139)$$

(2. 137) becomes,

$$I \approx \frac{e^{-jk\rho'}}{\sqrt{\rho'}} \sqrt{\frac{2\pi}{jk}} \int_0^{\pi} F_1(\rho_a) h_2(\frac{\rho'}{2}+\rho_a) F_1(\rho_b) d\theta$$

Making a change of variable to ρ_a.

$$d\theta = \frac{-d\rho_a}{\sqrt{\rho_a(\rho'-\rho_a)}}$$

$$I \approx \frac{e^{-jk\rho'}}{\sqrt{\rho'}} \sqrt{\frac{2\pi}{jk}} \int_0^{\rho'} \frac{F_1(\rho_a) h_2(\frac{\rho'}{2}+\rho_a) F_1(\rho_b)}{\sqrt{\rho_a(\rho'-\rho_a)}} d\rho_a \qquad (2.140)$$

Equation (2. 137) may also be written as,

$$I \approx \frac{e^{-jk\rho'}}{\sqrt{\rho'}} \sqrt{\frac{2\pi}{jk}} \int_0^{\rho'} \frac{F_1(\rho_a)\ F_1(\rho'-\rho_a)}{\sqrt{\rho_a(\rho'-\rho_a)}} d\rho_a \qquad (2.141)$$

where the two-dimensional convolution has been reduced to a one-dimensional Volterra type integral.

We may now return the result of (2.141) to equation (2.135).

$$M_2 \sim \frac{C}{2\pi}\left[\frac{jk(\Delta_1-\Delta_2)}{2\pi}\right]^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk(\rho-\rho')}}{\rho-\rho'} F_1(\rho-\rho')\ h_2(x')\cdot$$

$$\frac{e^{-jk\rho'}}{\sqrt{\rho'}}\sqrt{\frac{2\pi}{jk}}\int_d^{\rho'} \frac{F_1(\rho_a)\ F_1(\rho'-\rho_a)}{\sqrt{\rho_a(\rho'-\rho_a)}} d\rho_a dx'dy' \qquad (2.142)$$

The crux of the matter is that the inner integral over ρ_a is zero for $\rho_a < d$ and consequently the Heaviside function outside the integral is redundant and may be omitted.

Now consider third order.

$$M_3 = \frac{C}{2\pi}\left[\frac{jk(\Delta_1-\Delta_2)}{2\pi}\right]^3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk(\rho-\rho')}}{\rho-\rho'} F_1(\rho-\rho')\ h_2(x')\cdot$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk(\rho'-\rho'')}}{\rho'-\rho''} F_1(\rho'-\rho'')h_2(x'')\cdot$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk(\rho''-\rho''')}}{\rho''-\rho'''} F_1(\rho''-\rho''')h_2(x''')\frac{e^{-jk\rho'''}}{\rho'''}F_1(\rho''')\cdot$$

$$dx'''dy'''dx''dy''dx'dy' \qquad (2.143)$$

As it was for second order the inner integral may be reduced to a line integral by a transformation to elliptical coordinates and stationary phase integration.

$$M_3 \sim \frac{C}{2\pi}\left[\frac{jk(\Delta_1-\Delta_2)}{2\pi}\right]^3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk(\rho-\rho')}}{\rho-\rho'} F_1(\rho-\rho')\ h_2(x')\cdot$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk(\rho'-\rho'')}}{\rho'-\rho''} F_1(\rho'-\rho'')h_2(x'')\cdot$$

$$\frac{e^{-jk\rho}}{\sqrt{\rho}}\sqrt{\frac{2\pi}{jk}}\int_{d}^{\rho''}\frac{F_1(\rho_a)\,F_1(\rho''-\rho_a)}{\sqrt{\rho_a(\rho''-\rho_a)}}\,d\rho_a\,dx''dy''dx'dy' \qquad (2.144)$$

As for second order the Heaviside function $h_2(x''')$ is redundant and may be removed.

Now let,

$$\frac{G(\rho'')}{\sqrt{\rho''}}=\int_{d}^{\rho''}\frac{F_1(\rho_a)\,F_1(\rho''-\rho_a)}{\sqrt{\rho_a(\rho''-\rho_a)}}\,d\rho_a \qquad (2.145)$$

where $G(\rho'')$ is some function which satisfies this relationship.

Substituting (2.145) into (2.144),

$$M_3\sim\frac{C}{2\pi}\left[\frac{jk(\Delta_1-\Delta_2)}{2\pi}\right]^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-jk(\rho-\rho')}}{\rho-\rho'}F_1(\rho-\rho')\,h_2(x')\cdot$$

$$\sqrt{\frac{2\pi}{jk}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-jk(\rho'-\rho'')}}{\rho'-\rho''}F_1(\rho'-\rho'')\cdot$$

$$\frac{e^{-jk\rho}}{\rho}G(\rho'')\,dx''dy''dx'dy'$$

Again changing the inner integral to elliptical coordinates and taking the stationary phase approximation we obtain

$$M_3\sim\frac{C}{2\pi}\left[\frac{jk(\Delta_1-\Delta_2)}{2\pi}\right]^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-jk(\rho-\rho')}}{\rho-\rho}F_1(\rho-\rho')h_2(x')\cdot$$

$$\frac{e^{-jk\rho}}{\sqrt{\rho}}\left[\sqrt{\frac{2\pi}{jk}}\right]^2\int_{0}^{\rho'}\frac{F_1(\rho_2)\,G(\rho'-\rho_2)}{\sqrt{\rho_2(\rho'-\rho_2)}}\,d\rho_2\,dx'dy' \qquad (2.146)$$

Now the expression for G must be put back into (2.146). Consider the integral,

$$\int_{0}^{\rho'}\frac{F_1(\rho_2)\,G(\rho'-\rho_2)}{\sqrt{\rho_2(\rho'-\rho_2)}}\,d\rho_2$$

This integral may also be written as.

$$\int_0^{\rho'} \frac{F_1(\rho'-\rho_2)G(\rho_2)}{\sqrt{(\rho'-\rho_2)\rho_2}} d\rho_2$$

Making use of this relationship and the expression for $\frac{G(\rho'')}{\sqrt{\rho''}}$ (2.146) may be written as.

$$M_3 \approx \frac{C}{2\pi}\left\{\frac{jk(\Delta_1-\Delta_2)}{2\pi}\right\}^3 \left\{\sqrt{\frac{2\pi}{jk}}\right\}^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk(\rho-\rho')}}{\rho-\rho'} F_1(\rho-\rho')h_2(x')\cdot$$

$$\frac{e^{-jk\rho}}{\sqrt{\rho}} \int_0^{\rho'} \frac{F_1(\rho'-\rho_2)}{\sqrt{\rho'-\rho_2}}\cdot$$

$$\int_d^{\rho_2} \frac{F_1(\rho_a)F_1(\rho_2-\rho_a)}{\sqrt{\rho_a(\rho_2-\rho_a)}} d\rho_a d\rho_2 dx'dy' \qquad (2.148)$$

Similarly fourth order term will be.

$$M_4 \approx \frac{C}{2\pi}\left\{\frac{jk(\Delta_1-\Delta_2)}{2\pi}\right\}^4 \left\{\sqrt{\frac{2\pi}{jk}}\right\}^3 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk(\rho-\rho')}}{(\rho-\rho')} F_1(\rho-\rho')h_2(x')\cdot$$

$$\frac{e^{-jk\rho}}{\sqrt{\rho}} \int_0^{\rho'} \frac{F_1(\rho'-\rho_3)}{\sqrt{\rho'-\rho_3}} \int_0^{\rho_3} \frac{F_1(\rho_3-\rho_2)}{\sqrt{\rho_3-\rho_2}}\cdot$$

$$\int_d^{\rho_2} \frac{F_1(\rho_a)F_1(\rho_2-\rho_a)}{\sqrt{\rho_a(\rho_2-\rho_a)}} d\rho_a d\rho_2 d\rho_3 dx'dy' \qquad (2.149)$$

and the n^{th} order term will be given by.

$$M_n \approx \frac{C}{2\pi}\left\{\frac{jk(\Delta_1-\Delta_2)}{2\pi}\right\}^n \left\{\sqrt{\frac{2\pi}{jk}}\right\}^{n-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk(\rho-\rho')}}{(\rho-\rho')} F_1(\rho-\rho')h_2(x')\cdot$$

$$\frac{e^{-jk\rho}}{\sqrt{\rho}} \int_0^{\rho'} \frac{F_1(\rho'-\rho_n)}{\sqrt{\rho'-\rho_n}} \int_0^{\rho_{n-1}} \frac{F_1(\rho_{n-1}-\rho_{n-2})}{\sqrt{\rho_{n-1}-\rho_{n-2}}} \cdots$$

$$\int_d^{\rho_2} \frac{F_1(\rho_a)F_1(\rho_2-\rho_a)}{\sqrt{\rho_a(\rho_2-\rho_a)}} d\rho_a d\rho_2 d\rho_3 \cdots \rho_n dx'dy' , \qquad (2.150)$$

2.7.3 The Propagated Field

It is worthwhile at this point to derive an expression for the field propagated past the vertical discontinuity at x=d for comparison with the work of other investigators. This may be achieved if the stationary phase integration procedure is carried out for the outermost integral in all orders of the solution.

The vertical component of the surface field may be written as

$$E_z^+ = \sum_{i=1}^{\infty} M_i \qquad (2.151)$$

where

$$M_0 = \frac{C}{2\pi} \frac{e^{-jk\rho}}{\rho} F_1(\rho) \qquad (2.152)$$

$$M_1 = \frac{C}{2\pi} \frac{jk(\Delta_1-\Delta_2)}{2\pi} \sqrt{\frac{2\pi}{jk}} \frac{e^{-jk\rho}}{\sqrt{\rho}} \int_0^{\rho} \frac{F_1(\rho-\rho_a)h_2(\rho_a)F_1(\rho_a)}{\sqrt{\rho_a(\rho-\rho_a)}} d\rho_a \qquad (2.153)$$

$$M_2 = \frac{C}{2\pi} \left[\frac{jk(\Delta_1-\Delta_2)}{2\pi}\right]^2 \left[\sqrt{\frac{2\pi}{jk}}\right]^2 \frac{e^{-jk\rho}}{\sqrt{\rho}} \int_0^{\rho} \frac{F_1(\rho-\rho_2)}{\sqrt{(\rho-\rho_2)}} \cdot$$

$$\int_0^{\rho_2} \frac{F_1(\rho_2-\rho_a)h_2(\rho_a)F_1(\rho_a)}{\sqrt{(\rho_2-\rho_a)\rho_a}} d\rho_a d\rho_2 \qquad (2.154)$$

$$M_3 = \frac{C}{2\pi} \left[\frac{jk(\Delta_1-\Delta_2)}{2\pi}\right]^3 \left[\sqrt{\frac{2\pi}{jk}}\right]^3 \frac{e^{-jk\rho}}{\sqrt{\rho}} \int_0^{\rho} \frac{F_1(\rho-\rho_3)}{\sqrt{\rho-\rho_3}} \cdot$$

$$\int_0^{\rho_3} \frac{F_1(\rho_3-\rho_2)}{\sqrt{\rho_3-\rho_2}} \cdot$$

$$\int_0^{\rho_2} \frac{F_1(\rho_2-\rho_a)h_2(\rho_a)F_1(\rho_a)}{\sqrt{(\rho_2-\rho_a)\rho_a}} d\rho_a d\rho_2 d\rho_3 \qquad (2.155)$$

If we note that the outer integrals are complete convolutions in the Laplace sense, we may utilize the Laplace transform to reduce these convolutions to multiplications.

Let,

$$B_1 = \frac{jk(\Delta_1-\Delta_2)}{2\pi} \sqrt{\frac{2\pi}{jk}}$$

$$= \frac{j(\sqrt{c_1} - \sqrt{c_2})}{\sqrt{\pi}} \qquad (2.156)$$

where $c_1 = \frac{-jk\Delta_1^2}{2}$ and $c_2 = \frac{-jk\Delta_2^2}{2}$.

Taking the Laplace transform, L, of M_1 to M_3 as given by (2.153) to (2.155) we have

$$\mathrm{L}\left[M_1\right] = \frac{C}{2\pi} \mathrm{L}\left[\frac{e^{-jk\rho}}{\sqrt{\rho}}\right] \cdot B_1 \mathrm{L}\left[\frac{u(\rho)F_1(\rho)}{\sqrt{\rho}}\right] \mathrm{L}\left[\frac{h_2(\rho)F_1(\rho)}{\sqrt{\rho}}\right] \qquad (2.157)$$

$$\mathrm{L}\left[M_2\right] = \frac{C}{2\pi} \mathrm{L}\left[\frac{e^{-jk\rho}}{\sqrt{\rho}}\right] \cdot B_1^2 \left[\mathrm{L}\left[\frac{u(\rho)F_1(\rho)}{\sqrt{\rho}}\right]\right]^2 \mathrm{L}\left[\frac{h_2(\rho)F_1(\rho)}{\sqrt{\rho}}\right] \qquad (2.158)$$

$$\mathrm{L}\left[M_3\right] = \frac{C}{2\pi} \mathrm{L}\left[\frac{e^{-jk\rho}}{\sqrt{\rho}}\right] \cdot B_1^3 \left[\mathrm{L}\left[\frac{u(\rho)F_1(\rho)}{\sqrt{\rho}}\right]\right]^3 \mathrm{L}\left[\frac{h_2(\rho)F_1(\rho)}{\sqrt{\rho}}\right] \qquad (2.159)$$

where $u(\rho)$ is Heaviside's function (ie: the unit step) and $*$ represents a convolution.

Similarly the n^{th} order term may be transformed to give

$$\mathrm{L}\left[M_n\right] = \frac{C}{2\pi}\,\mathrm{L}\left[\frac{e^{-jk\rho}}{\sqrt{\rho}}\right] * B_1^n\left[\mathrm{L}\left[\frac{u(\rho)F_1(\rho)}{\sqrt{\rho}}\right]\right]^n \mathrm{L}\left[\frac{h_2(\rho)F_1(\rho)}{\sqrt{\rho}}\right] \quad (2.160)$$

Now,

$$\mathrm{L}\left[\frac{u(\rho)F_1(\rho)}{\sqrt{\rho}}\right] = \frac{\sqrt{\pi}}{\sqrt{s}+j\sqrt{c_1}} \qquad \text{(King , 1968)}$$

therefore using (2.156) for B_1

$$B_1^n\left[\mathrm{L}\left[\frac{u(\rho)F_1(\rho)}{\sqrt{\rho}}\right]\right]^n = \left[\frac{j(\sqrt{c_1}-\sqrt{c_2})}{\sqrt{s}+j\sqrt{c_1}}\right]^n \quad (2.161)$$

These Laplace transformed versions of the solution are now in a form which allows the series to be summed. Forming the sum of $\mathrm{L}\left[M_1\right]$ to $\mathrm{L}\left[M_n\right]$ we have,

$$\sum_{i=1}^{n}\mathrm{L}\left[M_i\right] = \frac{C}{2\pi}\,\mathrm{L}\left[\frac{e^{-jk\rho}}{\sqrt{\rho}}\right] * \mathrm{L}\left[h_2(\rho)\frac{F_1(\rho)}{\sqrt{\rho}}\right]\frac{j(\sqrt{c_1}-\sqrt{c_2})}{\sqrt{s}+j\sqrt{c_1}}$$

$$\left[1+\frac{j(\sqrt{c_1}-\sqrt{c_2})}{\sqrt{s}+j\sqrt{c_1}}+\left[\frac{j(\sqrt{c_1}-\sqrt{c_2})}{\sqrt{s}+j\sqrt{c_1}}\right]^2+\cdots\right.$$

$$\cdots + \left[\frac{j(\sqrt{c_1} - \sqrt{c_2})}{\sqrt{s}+j\sqrt{c_1}}\right]^n \quad (2.162)$$

As the expanded Neumann series for the original integral equation (2.103) is now in a summable form any concern about the convergence of the solution may be eliminated by taking the sum of the series as the solution to (2.103). Denoting the sum of the series by S, we have,

$$S = 1 + X + X^2 + X^3 + \cdots + X^n + \cdots$$

where,

$$X = \frac{j(\sqrt{c_1} - \sqrt{c_2})}{\sqrt{s}+j\sqrt{c_1}}$$

As $n \to \infty$,

$$S = \frac{1}{1-X}$$

$$= \frac{1}{1 - \dfrac{j(\sqrt{c_1} - \sqrt{c_2})}{\sqrt{s}+j\sqrt{c_1}}}$$

$$= \frac{\sqrt{s}+j\sqrt{c_1}}{\sqrt{s}+j\sqrt{c_2}} \quad (2.163)$$

Replacing the infinite series in (2.162) by S we have,

$$\sum_{l=1}^{\infty} \mathrm{L}\left[M_l\right] = \frac{C}{2\pi}\, \mathrm{L}\left[\frac{e^{-jk\rho}}{\sqrt{\rho}}\right].$$

$$\left[L \left[h_2(\rho) \frac{F_1(\rho)}{\sqrt{\rho}} \right] \frac{j(\sqrt{c_1} - \sqrt{c_2})}{\sqrt{s} + j\sqrt{c_1}} \frac{\sqrt{s} + j\sqrt{c_1}}{\sqrt{s} + j\sqrt{c_2}} \right] \tag{2.164}$$

$$= \frac{C}{2\pi} L \left[\frac{e^{-jk\rho}}{\sqrt{\rho}} \right] * \left[L \left[h_2(\rho) \frac{F_1(\rho)}{\sqrt{\rho}} \right] \frac{j(\sqrt{c_1} - \sqrt{c_2})}{\sqrt{s} + j\sqrt{c_2}} \right] \tag{2.165}$$

Taking the inverse Laplace transform of equation (2.165) we may obtain the vertical Electric field propagated beyond the vertical discontinuity. From (2.151),

$$E_z^+ = \sum_{l=1}^{\infty} M_l$$

$$= \frac{C}{2\pi} \frac{e^{-jk\rho}}{\rho} F_1(\rho) + \frac{C}{2\pi} \frac{j(\sqrt{c_1} - \sqrt{c_2})}{\sqrt{\pi}} \frac{e^{-jk\rho}}{\sqrt{\rho}} \int_d^{\rho} \frac{F_1(\rho_a) F_2(\rho - \rho_a)}{\sqrt{\rho_a(\rho - \rho_a)}} d\rho_a \tag{2.166}$$

where,

$$\frac{F_2(\rho)}{\sqrt{\rho}} = L^{-1} \left[\frac{\sqrt{\pi}}{\sqrt{s} + j\sqrt{c_2}} \right]$$

$$= \frac{1}{\sqrt{\rho}} - j\sqrt{\pi c_2 \rho}\, e^{-c_2 \rho} \operatorname{erfc}(j\sqrt{c_2 \rho})$$

Normalizing this equation to the field that would be present for the case of a perfectly conducting medium, $2E_0$, where,

$2E_0 = \dfrac{C}{2\pi} \dfrac{e^{-jk\rho}}{\rho}$ *we obtain*,

$$\frac{E_z^+}{2E_0} = F_1(\rho) + \frac{j(\sqrt{c_1} - \sqrt{c_2})}{\sqrt{\pi}} \sqrt{\rho} \int_d^{\rho} \frac{F_1(\rho_a) F_2(\rho-\rho_a)}{\sqrt{\rho_a(\rho-\rho_a)}} d\rho_a \qquad (2.167)$$

This is the result derived by both Bremmer (1954) and Wait (1964) for the propagated field. In the next section we turn our attention to the derivation of the backscattered field.

2.7.4 The Backscattered Field

We shall now consider again the equations (2.142), (2.146) and (2.149) which represent the second, third and forth orders of the solution. In these equations we have performed stationary phase integration on all but the last integral.

We have for (2.142),

$$M_2 = \frac{C}{2\pi} \frac{jk(\Delta_1-\Delta_2)}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk(\rho-\rho')}}{(\rho-\rho')} F_1(\rho-\rho') h_2(x') \frac{e^{-jk\rho'}}{\sqrt{\rho'}} N_2 dx'dy' \qquad (2.168)$$

where,

$$N_2 = \frac{jk(\Delta_1-\Delta_2)}{2\pi} \sqrt{\frac{2\pi}{jk}} \int_d^{\rho'} \frac{F_1(\rho_a) F_2(\rho'-\rho_a)}{\sqrt{\rho_a(\rho'-\rho_a)}} d\rho_a$$

and similarly, (2.146) may be written as,

$$M_3 = \frac{C}{2\pi} \frac{jk(\Delta_1-\Delta_2)}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk(\rho-\rho')}}{(\rho-\rho')} F_1(\rho-\rho') h_2(x') \frac{e^{-jk\rho'}}{\sqrt{\rho'}} N_3 dx'dy' \qquad (2.169)$$

where,

$$N_3 = \left[\frac{jk(\Delta_1-\Delta_2)}{2\pi}\right]^2 \left[\sqrt{\frac{2\pi}{jk}}\right]^2 \int_0^{\rho'} \frac{F_1(\rho'-\rho_2)}{\sqrt{(\rho'-\rho_2)}} \int_d^{\rho_2} \frac{F_1(\rho_a)F_2(\rho_2-\rho_a)}{\sqrt{(\rho_a(\rho_2-\rho_a))}} \, d\rho_a d\rho_2 \, .$$

Proceeding in the same manner, all terms of the solution may be written as M_2 and M_3. Forming the sum of M_2 to M_n we have,

$$\sum_{i=2}^{n} M_i = \frac{C}{2\pi} \frac{jk(\Delta_1-\Delta_2)}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk(\rho-\rho')}}{(\rho-\rho')} F_1(\rho-\rho')h_2(x')$$

$$\frac{e^{-jk\rho'}}{\sqrt{\rho'}} \left[\sum_{i=2}^{n} N_i\right] dx'dy' \qquad (2.170)$$

The sum of N_i for i from 2 to ∞ is exactly the same series considered in the previous section for the propagated field. Utilizing the Laplace transform, this infinite series may be summed as before (see equation 2.163) to yield,

$$\sum_{i=2}^{\infty} L\left[N_i\right] = L\left[h_2(\rho)\frac{F_1(\rho)}{\sqrt{\rho}}\right] \frac{j(\sqrt{c_1} - \sqrt{c_2})}{\sqrt{s}+j\sqrt{c_2}} \qquad (2.171)$$

and taking the inverse Laplace transform,

$$\sum_{i=2}^{\infty} N_i = \frac{j(\sqrt{c_1} - \sqrt{c_2})}{\sqrt{\pi}} \int_d^{\rho'} \frac{F_2(\rho'-\rho_a)\, F_1(\rho_a)}{\sqrt{(\rho'-\rho_a)\rho_a}} d\rho_a \qquad (2.172)$$

The backscattered field will be given as the sum of M_1 to M_∞, excluding M_0 as this term represents the undisturbed forward propagating field. We have for the backscattered field, E_{bs},

$$E_{bs} = \sum_{i=1}^{\infty} M_i$$

$$= M_1 + M_s \tag{2.173}$$

where M_1 is the first order and M_s is the sum of all higher order terms. Using (2.170) and (2.172), M_s is given as,

$$M_s = \sum_{l=2}^{\infty} M_l$$

$$= \frac{C}{2\pi} \frac{jk(\Delta_1-\Delta_2)}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk(\rho-\rho')}}{(\rho-\rho')} F_1(\rho-\rho') h_2(x')$$

$$\frac{j(\sqrt{c_1} - \sqrt{c_2})}{\sqrt{\pi}} \frac{e^{-jk\rho'}}{\sqrt{\rho'}} \int_d^{\rho'} \frac{F_1(\rho_a)\ F_2(\rho'-\rho_a)}{\sqrt{(\rho_a(\rho'-\rho_a))}} d\rho_a dx'dy' \tag{2.174}$$

Accordingly, using (2.135) and (2.174),

$$E_{bs} = \frac{C}{2\pi} \frac{jk(\Delta_1-\Delta_2)}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-jk(\rho-\rho')}}{(\rho-\rho')} F_1(\rho-\rho') h_2(x') \frac{e^{-jk\rho'}}{\rho'}$$

$$\left[F_1(\rho') + \frac{j(\sqrt{c_1} - \sqrt{c_2})}{\sqrt{\pi}} \sqrt{\rho'} \int_d^{\rho'} \frac{F_1(\rho_a)\ F_2(\rho'-\rho_a)}{(\rho_a(\rho'-\rho_a))} d\rho_a \right] dx'dy' \tag{2.175}$$

For the practical application of this equation we will assume that the observation point is at the origin (ie: $(x,y)=(0,0)$) and for convenience change to polar coordinates.

$$\rho = \sqrt{x^2 + y^2} = 0$$

$$x' = \rho'\cos\theta$$

$$y' = \rho'\sin\theta$$

$$E_{bs} = \frac{C}{2\pi}\,\frac{jk(\Delta_1-\Delta_2)}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{d\sec\theta}^{\infty}\frac{e^{-jk2\rho'}}{\rho'^2}\,F_1(\rho')G(\rho')\,\rho' d\rho' d\theta \qquad (2.176)$$

where

$$G(\rho') = F_1(\rho') + \frac{j(\sqrt{c_1}-\sqrt{c_2})}{\sqrt{\pi}}\sqrt{\rho'}\int_{d\sec\theta}^{\rho'}\frac{F_1(\rho_a)\;F_2(\rho'-\rho_a)}{\sqrt{(\rho_a(\rho'-\rho_a))}}\,d\rho_a \qquad (2.177)$$

We will seek an asymptotic solution of equation (2.177). Since the integrand contains no stationary points, the maximum contribution to the integral will come from the endpoints. This suggests that an expansion for the integral about $\rho_a = d\sec\theta$ will provide a good approximation the integral. In the vicinity of $\rho_a = d\sec\theta$ the function $\frac{F_1(\rho_a)}{\sqrt{\rho_a}}$ is slowly varying and may be removed from the integral.

Let $a = d\sec\theta$

$$G(\rho') \sim F_1(\rho')\left[\,1 + \frac{j(\sqrt{c_1}-\sqrt{c_2})}{\sqrt{\pi}}\int_{a}^{\rho'}\frac{F_2(\rho'-\rho_a)}{\sqrt{(\rho'-\rho_a)}}\,d\rho_a\right]$$

Making a change of variable $x = \rho'-\rho_a$

$$G(\rho') \sim F_1(\rho')\left[\,1 + \frac{j(\sqrt{c_1}-\sqrt{c_2})}{\sqrt{\pi}}\int_{0}^{\rho'-a}\frac{F_2(x)}{\sqrt{x}}\,dx\right] \qquad (2.178)$$

The integral of (2.178) may be evaluated using the definition of $F_1(x)$ and a series expansion for the complementary error function.

$$\frac{F_1(x)}{\sqrt{x}} = \frac{1}{\sqrt{x}} - j\sqrt{\pi c_2}\;e^{-c_2 x}\;\mathrm{erfc}(j\sqrt{c_2 x}) \qquad (2.179)$$

From Abromowitz and Stegun (1972),

$$e^{-c_2 x}\ erfc(j\sqrt{c_2 x}) = \sum_{n=0}^{\infty} \frac{(-j\sqrt{c_2 x})^n}{\Gamma(\frac{n}{2}+1)} \tag{2.180}$$

Equation (2.178) becomes,

$$G(\rho') \sim F_1(\rho')\left[1 + \frac{j(\sqrt{c_1}-\sqrt{c_2})}{\sqrt{\pi}}\left\{\int_0^{\rho'-a}\left[\frac{1}{\sqrt{x}} - j\sqrt{\pi c_2}\,\frac{(-j\sqrt{c_2 x})^n}{\Gamma(\frac{n}{2}+1)}\right]dx\right\}\right] \tag{2.181}$$

Equation (2.181) may be integrated term by term to yield,

$$G(\rho') \sim F_1(\rho')\left[1 + \frac{j(\sqrt{c_1}-\sqrt{c_2})}{\sqrt{\pi}}\left\{-2\sqrt{\rho'-a} - j\sqrt{\pi c_2}\left[(\rho'-a) + \frac{(-j\sqrt{c_2})(\rho'-a)^{3/2}}{\Gamma(3/2)\,3/2} + \frac{(-j\sqrt{c_2})^2(\rho'-a)^2}{\Gamma(2)2} + \frac{(-\sqrt{c_2})^3(\rho'-a)^{5/2}}{\Gamma(5/2)\,5/2} + \frac{c_2^{\,2}(\rho'-a)^3}{\Gamma(3)(3)} + \ldots\right]\right\}\right] \tag{2.182}$$

Using (2.182) in (2.176) we obtain,

$$E_{bs} \sim \frac{C}{2\pi}\,\frac{jk(\Delta_1-\Delta_2)}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_a^{\infty}\frac{e^{-jk2\rho'}}{\rho'}F_1^{\,2}(\rho')\cdot$$

$$\left[1 + \frac{j(\sqrt{c_1}-\sqrt{c_2})}{\sqrt{\pi}}\left\{-2\sqrt{\rho'-a} - \right.\right.$$

$$- j\sqrt{\pi c_2}\left[(\rho-a) + \frac{(-j\sqrt{c_2})(\rho'-a)^{3/2}}{\Gamma(3/2)\,3/2} + \frac{(-j\sqrt{c_2})^2(\rho'-a)^2}{\Gamma(2)(2)} + \right.$$

$$\left. + \frac{(-j\sqrt{c_2})^3(\rho'-a)^{5/2}}{\Gamma(5/2)\,5/2} + \frac{c_2^{\,2}(\rho'-a)^3}{\Gamma(3)(3)} + \ldots \right]\Bigg]\Bigg] d\rho' d\theta \qquad (2.183)$$

We may also expand $\frac{F_1^{\,2}(\rho')}{\rho'}$ in a Taylor series about $\rho' = a$,

$$\frac{F_1^{\,2}(\rho')}{\rho'} = \sum_{n=0}^{\infty} \left[\frac{F_1^{\,2}(a)}{a}\right]^{(n)} \frac{(\rho'-a)^n}{n!}$$

where (n) denotes the n^{th} derivative with respect to a.

Utilizing this expansion in (2.183) and making a change of variables, $\rho' = ax + a$ to shift the endpoint of the inner integral, we obtain,

$$E_{bs} \approx \frac{C}{2\pi} \frac{jk(\Delta_1-\Delta_2)}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\, e^{-jk2a} \int_0^{\infty} e^{-jk2ax} \Bigg[$$

$$\frac{F_1^{\,2}(a)}{a}\left[1 + \frac{j(\sqrt{c_1}-\sqrt{c_2})}{\sqrt{\pi}}\left[-2\sqrt{ax} - j\sqrt{\pi c_2}\left[(ax) - \frac{j\sqrt{c_2}(ax)^{3/2}}{\Gamma(5/2)} - \ldots\right]\right]\right] +$$

$$+ \left[\frac{F_1^{\,2}(a)}{a}\right]^{(1)} \left[ax + \frac{j(\sqrt{c_1}-\sqrt{c_2})}{\sqrt{\pi}}\left[-2(ax)^{3/2} - \right.\right.$$

$$\left.\left. - j\sqrt{\pi c_2}\left[(ax)^2 - \frac{j\sqrt{c_2}(ax)^{5/2}}{\Gamma(7/2)} - \ldots\right]\right]\right] +$$

$$+\frac{1}{2!}\left[\frac{F_1^2(a)}{a}\right]^{(2)}\left[(ax)^2+\frac{j(\sqrt{c_1}-\sqrt{c_2})}{\sqrt{\pi}}\left[-2(ax)^{5/2}-\right.\right.$$

$$\left.\left.\left.-j\sqrt{\pi c_2}\left[(ax)^3-\frac{j\sqrt{c_2}(ax)^{7/2}}{\Gamma(9/2)}-\dots\right]\right]\right]+$$

$$+\dots\dots\dots\dots\Big]\,dx\,d\theta \qquad (2.184)$$

Integrating (2.184) term by term, one obtains

$$E_{bs}\approx\frac{C}{2\pi}\frac{jk(\Delta_1-\Delta_2)}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}a\,e^{-jk2a}\Bigg[$$

$$\frac{F_1^2(a)}{a}\left[\frac{1}{jk2a}+\frac{j(\sqrt{c_1}-\sqrt{c_2})}{\sqrt{\pi}}\left[\frac{-2\sqrt{a}\,\Gamma(3/2)}{(jk2a)^{3/2}}-\right.\right.$$

$$\left.\left.-j\sqrt{\pi c_2}\left[\frac{a\,\Gamma(2)}{(jk2a)^2}-\frac{-j\sqrt{c_2}\,\Gamma(5/2)a^{3/2}}{\Gamma(5/2)(jk2a)^{5/2}}-\dots\right]\right]\right]$$

$$+\left[\frac{F_1^2(a)}{a}\right]^{(1)}\left[\frac{1}{(jk2a)^2}+\frac{j(\sqrt{c_1}-\sqrt{c_2})}{\sqrt{\pi}}\left[\frac{-2a^{3/2}\,\Gamma(5/2)}{(jk2a)^{5/2}}\right.\right.$$

$$\left.\left.-j\sqrt{\pi c_2}\left[\frac{a^2\,\Gamma(3)}{(jk2a)^3}-\frac{-j\sqrt{c_2}\,\Gamma(7/2)a^{5/2}}{\Gamma(5/2)(jk2a)^{7/2}}-\dots\right]\right]\right]$$

$$+\ldots\ldots\ldots\ldots \Bigg] d\theta \tag{2.185}$$

Now,

$$\left[\frac{F_1^{\,2}(a)}{a}\right]^{(1)} = -\frac{F_1^{\,2}(a)}{a^2} + \frac{1}{a}\left[2F_1(a)\ \frac{d}{da}\left[F_1(a)\right]\right]$$

$$= -\frac{F_1^{\,2}(a)}{a^2} + \frac{2F_1^{\,2}(a)}{a}\left[c_1 - \frac{1}{2a} + \frac{1}{2aF_1(a)}\right] \tag{2.186}$$

For large $c_1 a$ the first term of the asymptotic expansion of $F_1(a)$ is given as,

$$F_1(a) \sim \frac{-1}{2c_1 a} \tag{2.187}$$

and we have,

$$\left[\frac{F_1^{\,2}(a)}{a}\right]^{(1)} \sim -\frac{F_1^{\,2}(a)}{a^2} + \frac{2F_1^{\,2}(a)}{a}\left[\frac{-1}{2a}\right]$$

$$\sim \frac{-2F_1^{\,2}(a)}{a^2}$$

Therefore using (2.187),

$$\frac{F_1^{\,2}(a)}{a} = O\left[\frac{1}{a^3}\right]$$

and,

$$\left[\frac{F_1^{\,2}(a)}{a}\right]^{(1)} = O\left[\frac{1}{a^4}\right]$$

Hence (2.185) contains an asymptotic series in a and it will therefore be sufficient to approximate the complete series by.

$$\frac{F_1^{\,2}(a)}{a}\,\frac{1}{jk2a}\left[1+\frac{(\Delta_1-\Delta_2)}{2}+\frac{\Delta_1(\Delta_1-\Delta_2)}{4}+\frac{\Delta_1^{\,2}(\Delta_1-\Delta_2)}{8}+\ldots\right] \quad (2.188)$$

This result may also be obtained by using Bremmer's expansion for the field in media 2 near the discontinuity (1954, p453).

Let,

$$B_2 = 1+\frac{(\Delta_1-\Delta_2)}{2}+\frac{\Delta_1(\Delta_1-\Delta_2)}{4}+\frac{\Delta_1^{\,2}(\Delta_1-\Delta_2)}{8}+\cdots \quad (2.189)$$

Accordingly, the backscattered field is given as

$$E_{bs} \sim \frac{C}{2\pi}\,\frac{jk(\Delta_1-\Delta_2)}{2\pi}\,B_2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-jk2a}\,\frac{F_1^{\,2}(a)}{jk2a}\,d\theta$$

or as $a = d\sec\theta$

$$E_{bs} \sim \frac{C}{2\pi}\,\frac{jk(\Delta_1-\Delta_2)}{2\pi}\,B_2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-jk2d\sec\theta}\,\frac{F_1^{\,2}(d\sec\theta)}{jk2d\sec\theta}\,d\theta \quad (2.190)$$

Since d is large, the exponential in the integrand will be much more rapidly varying than the other factors, thus allowing the use of stationary phase integration on this final integral.

We have,

$f(\theta) = \sec\theta$

$f'(\theta) = \tan\theta\,\sec\theta$

There is a stationary point at $f'(\theta) = 0$ or $\theta = 0$ and,

$$f''(\theta) = \cos\theta + \tan^2\theta \sec\theta$$

$$f''(0) = 1$$

Evaluating (2.190) we have,

$$E_{bs} \sim \frac{C}{2\pi} \frac{jk(\Delta_1-\Delta_2)}{2\pi} B_2 \frac{e^{-jk2d}}{-jk2\sqrt{-k2d}} \frac{F_1^{\,2}(d)}{d} (2\pi/1)\, e^{j\frac{\pi}{4}} \tag{2.191}$$

where B_2 is given by equation (2.189).

Alternatively, since $\frac{\Delta_1}{2} < 1$, the series of B_2 has a known sum and may be written as,

$$B_2 = 1 + \frac{(\Delta_1-\Delta_2)}{2} \left[1 + \frac{\Delta_2}{2} + \left[\frac{\Delta_2}{2}\right]^2 + \left[\frac{\Delta_2}{2}\right]^3 + \cdots \right]$$

$$= 1 + \frac{(\Delta_1-\Delta_2)}{2-\Delta_2} \tag{2.192}$$

Equation (2.191) represents the vertical component of the surface field backscattered to the source point from a vertical discontinuity located at a distance d from the source.

CHAPTER 3

THE RADAR EQUATION

The expression derived for the backscattered field may be utilized to form a radar type equation. The power available at the receiving antenna is given by the Poynting power $\vec{S}$.

$$\vec{S} = \frac{1}{2} \vec{E} \times H$$

$$= \frac{1}{2} \frac{| E_{bs} |^2}{\eta} \qquad (3.1)$$

where η is the intrinsic impedence of free space.

Inserting (2.191) into (3.1) we have,

$$S = \frac{1}{2\eta} \frac{|C|^2 F_1^4 \lambda}{(2\pi)^4 \; 8d^3} \left| \Delta_1 - \Delta_2 + \frac{(\Delta_1 - \Delta_2)^2}{2 - \Delta_2} \right|^2 \qquad (3.2)$$

where λ is the radar wavelength.

The power received, P_r, from the antenna is given as.

$$P_r = S \cdot A_e \qquad (3.3)$$

where A_e is the effective area of the antenna and is given by,

$$A_e = \frac{G \lambda^2}{4\pi} \; ; \quad G = \text{antenna gain}$$

The transmitted power, P_t, and the gain of the antenna, G, are related to the dipole constant, C, by the relationship.

$$|C|^2 = 8\pi\eta\ G\ P_t \qquad (3.4)$$

Utilizing (3.2) and (3.4) in (3.3) the received power is given as,

$$P_r = \frac{8\pi\eta\ G\ P_t\ G\ \lambda^3 F_1^{\ 4}}{2\eta(2\pi)^4\ 4\pi 8d^3}\ |\Delta_1 - \Delta_2 + \frac{(\Delta_1-\Delta_2)^2}{2-\Delta_2}\ |^2$$

$$P_r = \frac{P_t\ G^2\ \lambda^2\ F_1^{\ 4}}{(4\pi)^3\ d^4}\sigma \qquad (3.5)$$

where,

$$\sigma = \frac{d\lambda}{2\pi}\ |\Delta_1 - \Delta_2 + \frac{(\Delta_1-\Delta_2)^2}{2-\Delta_2}\ |^2 \qquad (3.6)$$

Equations (3.5) and (3.6) may now be utilized to evaluate the ice hazard detection capability of HF radar.

CHAPTER 4

ICE HAZARD DETECTION

We have developed a useful equation, (3.6), for the calculation of the radar cross-section of a vertical discontinuity separating two semi-infinite media. This equation may now be applied to the problem of interest, namely, that of ice hazard detection.

When navigating or exploring in Arctic waters it is of extreme importance to know where multi-year ice floes and pressure ridges are located. Utilizing equation (3.6) the problem is reduced to a standard radar analysis of the situation. The ice edge will be detected if its reflected signal is greater than the thermal noise in the receiving system. In the following analysis it is assumed that the probability of detection of the ice hazard is dependent on the signal-to-noise ratio (S/N) present at the output of the receiver, (Skolnik, 1970).

Several naturally occuring ice hazard situations are considered for analysis, including

(i) Sea to First-Year Ice (FYI) transition

(ii) FYI to Sea

(iii) Sea to Multi-Year Ice (MYI)

(iv) FYI to MYI

(v) MYI to FYI

(vi) MYI to Sea

4.1 Electrical Properties of Sea and Sea Ice

The electrical properties for typical sea water, first-year ice (FYI) and multi-year ice (MYI) are given in Table 1 where ϵ_r and σ for sea ice are taken from Parashar (1977). The conductivity and permittivity of the sea ice is dependent on both the temperature and brine volume. The brine volume in turn depends on the salinity and temperature of the ice, both of which vary with the ice thickness and vertical location. First-year sea ice as the name indicates is the ice from one winter's growth and multi-year ice is ice which has weathered more than one year. First-year ice seldom exceeds a 2 meter thickness except where ridging may cause far greater vertical dimensions. First-year ice normally exhibits both a larger permittivity and conductivity than multi-year ice as indicated in Table 1. These differences may be attributed to the lower salinity of multi-year ice which is due to the loss of brine volume over several years of thawing and freezing.

For the range of values of electrical properties given in Table 1 the following relationship is valid,

$$\Delta \approx \frac{1}{n}$$

$$\approx \frac{1}{\sqrt{\epsilon_r - j\,\frac{\sigma}{\omega\epsilon_0}}} \qquad (4.1)$$

where,

Δ is defined on page 38

n = the refractive index

ϵ_r = the relative permittivity

σ = the conductivity, mho/m and,

$\omega = 2\pi f$; f = frequency, Hz

ϵ_0 = permittivity of free space

$$= \frac{1}{36\pi} \times 10^{-9}$$

Table 1

The Electrical Properties of Sea and Sea Ice at 30 Megahertz (Ice at -5 degrees centigrade)

	ϵ_r	$\frac{\sigma}{\omega\epsilon_0}$	Δ
Sea	80	2400	$2\times10^{-2}e^{-j.77}$
FYI Salinity =10%	6.141	4.514	$0.36\ e^{-j.32}$
FYI Salinity =15%	6.856	6.527	$0.33\ e^{-j.38}$
MYI Salinity =1%	4.853	0.967	$0.45\ e^{-j.10}$

Utilizing these values for Δ the radar cross-sections for the previously mentioned cases are calculated and tabulated in Table 2 as a function of range, d.

Table 2

Radar Cross-sections as a Function of Range, d
(from equation 3.6)
λ =10 m (f = 30 MHz)

		Media 2		
	σ	Sea	FYI	MYI
	Sea	-------	0.113 d	0.173 d
Media 1	FYI	0.229 d	-------	0.028 d
	MYI	0.318 d	0.038 d	-------

A typical HF radar will have the following specifications,

P_t = 8 kW ; peak power

τ = 8 μsec ; pulse length

B_n = 125 kHz ; noise bandwidth

λ = 10 meters ; radar wavelength

$G_t = G_r$ = 8 dB ; antenna gain

The radar equation may be written as,

$$\frac{S}{N} = \frac{P_t \, G_t \, G_r \, \lambda^2 \, \sigma}{(4\pi)^3 \, N_0 \, B_n} \frac{F_1^4}{R^4} \qquad (4.2)$$

where,

N_0 = noise power spectral density, watts/Hz

$\frac{S}{N}$ = signal-to-noise ratio at the output of the receiver

At 30 MHz the total average noise power spectral density refered to receiver input, N_0, is,

$$N_0 = kT + F_{am} \quad (4.3)$$

where kT is the internal thermal noise in the front end of the radar receiver (k = Boltzmann's constant = 1.38×10^{-23} and T is the system temperature ≈ 300 degrees Kelvin). F_{am} is the median atmospheric noise factor (≈ 20 db at 30 MHz , Barrick 1976).

Therefore,

$$N_0 = 10\log(kT) + 20 \quad \text{dBwatts/Hz}$$

$$= -184 \text{ dBwatts/Hz}$$

or,

$$N_0 = 3.98 \times 10^{-19} \text{ watts/Hz} \quad (4.4)$$

and the radar equation may be given as,

$$\frac{S}{N} = 3.226 \times 10^{17} \, \sigma \, \frac{F_1^4}{R^4} \quad (4.5)$$

Table 3 gives values of F_1^4 for ranges from 1 km to 80 km for each type of media (ie: Sea, FYI, MYI).

Table 3

Tabulated Values of the Propagation Factor, F_1^4, for Sea and Sea Ice

Range(km)	Sea	FYI(15%)	MYI(1%)
1	1.0	4.6E(-8)	3.8E(-9)
2	0.92	2.9E(-9)	2.4E(-10)
5	0.62	7.3E(-11)	6.1E(-12)
10	0.25	4.6E(-12)	< 1E(-12)
20	4.0E(-3)	< 1E(-12)	
30	3.3E(-4)		
40	4.6E(-5)		
50	9.6E(-6)		
60	2.5E(-6)		
70	7.6E(-7)		
80	2.4E(-7)		

Note: 1E(-10) = 1×10^{-10}

The values for F^4 over sea water are for a sea state 2 (significant wave height ≈ 1 meter) and were calculated with a spherical earth model, Barrick(1976). The values for first-year ice (FYI) and multi-year ice (MYI) were computed by the author with a flat earth model.

If we let $\sigma = \sigma_e d$ where σ_e is the coefficient in Table 2 and as $d = R$, equation (4.5) may be written as,

$$\frac{S}{N} = 3.226 \times 10^{17} \frac{\sigma_e F^4}{R^3} \qquad (4.6)$$

It is interesting to note that the signal-to-noise ratio exhibits an R^{-3} dependency whereas for a standard microwave radar the signal-to-noise ratio exhibits an R^{-4} dependency in the near range and an R^{-8} in the interference region. This dependency is due in part to the assumption of infinite edge length. For large ice sheets close to the radar this assumption is valid which demonstrates an advantage of the surface wave mode of detection over standard radar detection.

Writing equation (4.6) in decibels,

$$\frac{S}{N} = 175.1 + 10\log(\sigma_e) + 10\log(F^4) - 30\log(R) \qquad (4.7)$$

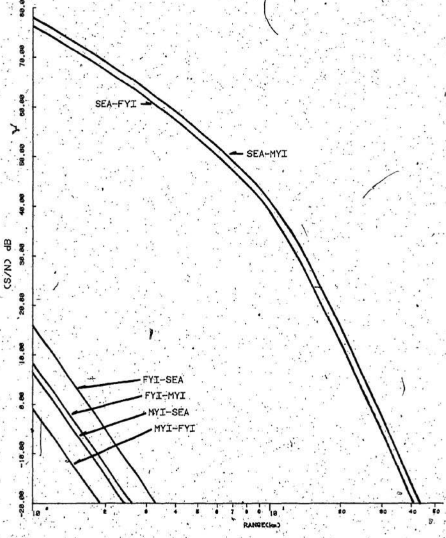
Assuming a signal-to-noise ratio of 21 dB is required at the output of the receiver to give a probability of detection of 0.9 for the vertical discontinuity, equation (4.7) may be plotted as a function of range as in Figure 5 and the range corresponding to 21 dB found. Figure 5 shows that the detection of a multi-year ice sheet beyond first year ice is very limited for single pulse detection. This is due to the surface wave attenuation in the first-year ice. In certain situations first-year ice may exhibit a higher conductivity than that accounted for in Table 1 thereby decreasing the attenuation; however it is expected that for typical first-year ice these tabulated values are representative.

There is very little difference in the detection ranges for boundaries between sea and first-year ice and sea and multi-year ice, both being about 18 km. These detection ranges appear to be not too promising, however most HF radars offer coherent processing features which in effect increase the signal-to-noise ratio by an amount proportional to the number of pulses coherently integrated. An improvement of $10\log(n)$ dB in signal-to-noise may be achieved for n pulses correlated. This translates into an increase in the detection range of a multi-year ice sheet in open sea to 30 km and in first-year ice to 1.4 km for n = 128 (CODAR correlates 128 pulses).

The normal microwave radar horizon for an antenna height of 25 meters is 20 km indicating that the HF radar will offer a definite advantage in range capability. The other advantage of course is that even the detection of the ice within the radar horizon is unreliable with the microwave radar as microwave is prone to propagation and clutter problems.

Figure 5

Backscattered Signal to Noise Ratio as a Function of Range for Various Interfaces



The analysis indicates that the detection of a multi-year ice sheet beyond first-year ice will be limited; however, it should still be possible. As indicated in the previous discussion with the proper application of available signal processing techniques these ranges may be further increased.

The influence of sea clutter on the detection of the ice edge in open water may be analyzed by investigating the doppler spectrum of the ocean. As the ice edge is expected to have a small velocity it will appear in the vicinity of the zero doppler part of the spectrum. From the work of Srivastava (1983) the radar cross-section of the sea surface at the zero doppler frequency, $\sigma_c(0)$, for a radar operating at 30 MHz and having a horizontal beamwidth of 6 degrees is given by,

$$\sigma_c(0) = 3.84\times10^{-3}\ d \qquad (4.8)$$

for a windspeed of 15 knots, a wind direction of 45 degrees and assuming a Pierson-Moskowitz Spectrum.

We may write the signal-to-clutter ratio, (S/C),

$$\frac{S}{C} = \frac{\sigma}{\sigma_c(0)} \qquad (4.9)$$

where σ is taken from Table 2.

Therefore, for a multi-year ice edge in open sea with a wind of 15 knots the signal-to-clutter ratio will be,

$$\frac{S}{C} = \frac{0.173}{3.84\times10^{-3}} \qquad (4.10)$$

$$= 45$$

or,

$$\frac{S}{C} = 16.5\ \text{dB}$$

From Blake (1980) this signal-to-clutter ratio will give a probability of detection of 0.75 for a false alarm probability of 10^{-6} (single pulse). Also, it should be noted that there is no range dependence as is normally the case for microwave radar. Thus it would appear that for moderate seas the detection of the ice edge will be limited mainly by the noise level in the receiving system.

CHAPTER 5

CONCLUSION

A theoretical analysis of the electromagnetic scattering from a vertical discontinuity has been carried out with reference to the problem of ice hazard detection. The method is based on a method of Space/Field decomposition which allows a three dimensional space to be decomposed into regions according to their electrical properties. Maxwell's equations were used to derive a partial differential equation for the electric field which was decomposed into three field equations and a boundary equation. An appropriate Green's function was taken as the fundamental solution for each of these field equations and the spatial Fourier transform utilized to simplify these equations to a single integral equation. Assuming an elementary vertical electric dipole as the source, this integral equation was written in an operator form and the solution for the vertical component of the surface field derived in terms of a Neumann series expansion of the inverse operator.

Utilizing stationary phase integration and the Laplace transform the series solution was summed to give expressions for the propagated and backscattered fields. The propagated field agrees with the results of both Bremmer and Wait. The expression for the backscattered field has been used to derive the radar cross-section of the vertical discontinuity.

The model chosen for analysis represents the boundary as an abrupt discontinuty. Wait (1963) has discussed the use of this type of model and proposes a model which consists of a gradual transition from one media to another. This would seem appropriate when treating both the problems of propagation across a coastline and the one at hand, however, Wait shows that far beyond the boundary there is no difference in the two models. This is expected to be true also for the backscattered field when the boundary is far away.

It has been demonstrated that with a moderate amount of processing (correlation of 128 pulses) a pulsed radar operating at 30 MHz will detect a multi-year ice edge out to a distance of 30 km in open sea with a probability of 0.9. The effect of sea clutter has been shown to be minimal for moderate seas. The detection of a multi-year ice sheet beyond first-year ice is limited due to greater attenuation of the surface wave in the first-year ice.

As both the radar cross-section and the surface wave attenuation are dependent on the radar wavelength (i.e., an increase in wavelength increases the radar cross-section by a proportional amount and decreases the surface wave attenuation) a longer wavelength may provide for greater detection ranges. However, for the practical application of this formulation the infinite edge assumption may dictate the use of the shorter wavelength to obtain the derived radar cross-section.

In conclusion, it appears that a HF radar will offer an increase in ice hazard detection capability over a microwave radar due to both its over-the-horizon capability and the fact that it will not be affected by the propagation and clutter problems which plague radars operating in the microwave region. Furthermore, the detection mode of the HF radar is not dependent on above water height and may therefore provide detection of ice features which would not be detected with a microwave radar.

Future work in this area might include the derivation of the backscattered field from multiple discontinuities. By generalizing this formulation to cases where the discontinuities are no longer straight edges with infinite length this derivation may then be applied to the problem of the detection of smaller hazards with irregular shapes such as icebergs. In addition, it would be useful to obtain the backscattered field for the case when the observation point is no longer at the source point and thereby allow the derivation of the bi–static radar cross-section for the various types of ice hazards.

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