THE EFFECT OF DIFFERENT WEIGHTING FUNCTIONS WHEN USING THE METHOD OF MOMENTS IN LINEAR ANTENNA ANALYSIS: A DIFFERENT APPROACH

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THE EFFECT OF DIFFERENT WEIGHTING FUNCTIONS
WHEN USING THE METHOD OF MOMENTS IN LINEAR ANTENNA ANALYSIS:
A DIFFERENT APPROACH

by

© NAGWA HAMED MOHAMED-NASR, B.Sc. (Eng.)

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ABSTRACT

It is the purpose of the present investigation to obtain and study mathematical expressions for the currents of a center-fed linear antenna subjected to a known incident electrical field. In the analysis process the direct solution of an integral equation for the current expression is replaced by the solution of an integral equation for a certain intermediate function. This method which is discussed by Walsh[27-30] eliminates the need for applying boundary conditions to an integral operator. The proposed method has been examined using the moment method in conjunction with three different weighting functions for solving the integral equation and a comparative study of the results is made. A set of Walsh functions is used for the basis functions. The use of Walsh functions for weighting functions allows efficient calculations of the matrix elements, and yields accurate results for certain antenna lengths. However, for other lengths the use of cosine functions for weighting functions is found to be more efficient. The third set of weighting functions used is piece-wise sinusoidal functions, which yields the most complicated calculations. Several examples are given to illustrate the application of the proposed method, when the moment method is used in conjunction with three different weighting functions. The method has a rate of convergence which is in general much faster than the methods used by Srivastava[24] and Thiele[25]. The results obtained by Srivastava are based on the same method of analysis, with the same basis functions used in this thesis, but in conjunction with point-matching, while Thiele's results are based on solving Pocklington's equation using point-matching in conjunction with different basis functions. It is of interest to note that, the proposed method has the same flexibility as Pocklington's

equation, but is better behaved numerically. Furthermore, the use of the moment method in conjunction with different weighting functions rather than using the point-matching technique, yields a smaller matrix to be inverted. In general, the results are in good agreement with the results obtained by other researchers.

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LIST OF SYMBOLS

$\underline{A}$	magnetic vector potential
a	radius of the antenna
B_{in}	input susceptance
$\underline{E}$	electrical field strength
E_z	axial component of the electrical field
E_ρ	radial component of the electrical field
E_ϕ	angular component of the electrical field
E_z^s	axial component of the scattered electrical field
E_z^i	axial component of the incident electrical field
G_{in}	input conductance
G	Green's function
$\underline{H}$	magnetic field strength
h	half length of the antenna
I_z	axial component of the current
$\underline{J}$	current density
J_z	axial component of the current density
J_ρ	radial component of the current density
J_ϕ	angular component of the current density
K_z	exact kernel for axial field
K_{za}	approximate kernel for axial field
K_{zc}	exact composite kernel for axial field
K_{zac}	approximate composite kernel for axial field
K_ρ	exact kernel for radial field
$K_{\rho a}$	approximate kernel for radial field
$K_{\rho c}$	exact composite kernel for radial field

$K_{\rho ac}$	approximate composite kernel for radial field
k	wave number
L	linear integral operator
R_{in}	input resistance
S	linear differential operator
V	scalar potential
X_{in}	input reactance
Z_c	characteristic impedance
Z_{in}	input impedance
η_o	intrinsic impedance of free space
μ	permeability of the medium
ε	permittivity of the medium
ω	radian frequency
∇	del operator
δ	Dirac-delta function
λ	wave length
ρ	charge density
γ	constant for transmission line
α	attenuation constant
β	phase constant

Other symbols are defined as they occur.

CHAPTER I

INTRODUCTION

1.1 General

In the study of antenna theory, a knowledge of the current distribution is of fundamental importance. Such data may be obtained either by measurement or by solving the antenna integral equations[7]. Two of the most popular current integral equations are Pocklington's and Hallén's equations. Many workers have advanced various solutions to Hallén's integral equation. Particularly significant is the work of Mie[17] which extended Hallén's equation to antennas of arbitrary geometry. Many other works related to Hallén's integral equation has been given by King[12]. Pocklington's form of the integral equation for a thin wire antenna is essentially that used by Richmond[19]. Pocklington's equation is more general than Hallén's in the sense that it can be applied for arbitrary incident fields, while the latter is confined to the use of a delta-gap generator.

Integral equations are difficult to solve even for the simplest case of a dipole antenna. However, as a result of the development of modern high speed computers, the range of application of the integral equation method has been greatly enlarged.

1.2 Scope of the Present Investigation

In this thesis another method for obtaining mathematical expressions for the antenna currents, subjected to a known incident electrical field is presented. The proposed method which is discussed by Walsh[27-30] is based on expressing the radiated electrical field by a product of two linear operators acting on the current expression. One of these is a convolution operator and the other is a differential operator. These operators are treated in the sense of generalized functions[3,4], and are shown to commute. The differential operator which is invertible in a closed form is replaced by an intermediate function. In order to solve the problem of interest, the solution of an integral equation for the intermediate function is first obtained, then by simple integration the current distribution of the antenna is found. The proposed method has the flexibility of Pocklington's formulation, in the sense that arbitrary incident fields may be handled, and the kernel of the integral equation is basically of the same form as that of Hallén's equation.

The unifying concept in the numerical treatment of integral equations is the method of moments. This general approach to radiation problems is essentially a reduction of the associated integral equation to a system of linear algebraic equations. The unknowns of the equations are usually coefficients in some approximate expansion of the current. In the application of the method of moments to thin-wire antenna problems, many different choices of expansion and testing functions have been successfully used. The primary differences being in numerical efficiency and rate of convergence.

In the present investigation a set of Walsh functions is used as expansion or basis functions. Three different sets of functions; Walsh, cosine and piece-wise sinusoidal functions are used as testing or weighting functions. A comparative study between the obtained results is presented. A parametric study is also made to determine the effect of antenna length, antenna radius and the feeding line dimensions on antenna characteristics. Walsh functions are used as weighting function in this parametric analysis.

CHAPTER II

GENERAL FORMULATION

In practice most generators produce voltage and currents, and hence electric and magentic fields, which vary sinusoidally with time. Even where this is not the case any periodic variation can always be analyzed in terms of sinusoidal variations with fundamental and harmonic frequencies, so it is customary in most problems to assume sinusoidal time variations. Consider the situation where a current source $\underline{J}$ is restricted to a bounded region of three-dimensional space R^3, and immersed in a medium which is assumed to be homogeneous, isotropic and perfect dielectric with permeability μ and permittivity ε. In the sinusoidal steady state, Maxwell's equations may be expressed in phasor form as[8]

$$\nabla \times \underline{H} = j\omega\varepsilon \underline{E} + \underline{J} \qquad [2.1]$$

$$\nabla \times \underline{E} = j\omega\mu \underline{H} \qquad [2.2]$$

$$\nabla \cdot \underline{E} = \rho/\varepsilon \qquad [2.3]$$

$$\nabla \cdot \underline{H} = 0 \qquad [2.4]$$

where

$\underline{H}$ = magnetic field strength vector

$\underline{E}$ = electrical field strength vector

$\underline{J}$ = current density vector of the source

ρ = charge density

The above equations contain the equation of continuity,

$$\nabla \cdot \underline{J} = -j\omega\rho \qquad [2.5]$$

If the vector magnetic potential is designated by the vector $\underline{A}$, then $\underline{H}$ can be obtained as a space derivative of $\underline{A}$. The curl is the space-derivative operation which can be used for such a relation. Let

$$\nabla \times \underline{A} = \mu \underline{H} \qquad [2.6]$$

Substituting equation [2.6] into [2.2]

$$\nabla \times \underline{E} = -j\omega \nabla \times \underline{A}$$

or $\nabla \times (\underline{E} + j\omega \underline{A}) = 0$ [2.7]

Equation [2.7] is satisfied if ($\underline{E} + j\omega \underline{A}$) is represented as the gradient of a scalar. Setting ($\underline{E} + j\omega \underline{A}$) equal to ($-\nabla V$) defines the scalar potential V. Thus the electric field strength may be expressed as

$$\underline{E} = -\nabla V - j\omega \underline{A} \qquad [2.8]$$

From equations [2.1] and [2.6]

$$\nabla \times \nabla \times \underline{A} = j\omega\mu\varepsilon \underline{E} + \mu \underline{J} \qquad [2.9]$$

If we use the vector identity

$$\nabla \times \nabla \times \underline{A} = \nabla\nabla \cdot \underline{A} - \nabla^2 \underline{A} \qquad [2.10]$$

then equations [2.8], [2.9] and [2.10], may be combined to give

$$\nabla\nabla \cdot \underline{A} - \nabla^2 \underline{A} = -j\omega\mu\varepsilon \nabla V + \omega^2 \mu\varepsilon \underline{A} + \mu \underline{J}$$

or $\nabla^2 \underline{A} + k^2 \underline{A} = \nabla (\nabla . \underline{A} + j\omega\mu\varepsilon V) - \mu \underline{J}$ [2.11]

where k is the wave number and equals $\omega \sqrt{\mu\varepsilon}$

Using the Lorentz gauge condition;

$$\nabla . \underline{A} = - j\omega\mu\varepsilon V$$

equation [2.11] becomes

$$\nabla^2 \underline{A} + k^2 \underline{A} = - \mu \underline{J} \qquad [2.12]$$

This equation is known as the phasor Helmholtz equation. Of all the equations having this form, the simplest is the one having a unit point source represented by a three-dimensional Dirac-delta function δ at the origin. We write the function which satisfies this equation as K, thus

$$\nabla^2 K - \gamma^2 K = - \delta \qquad [2.13]$$

where

$$\gamma = \alpha + j\beta$$

The solution of equation [2.13], is

$$K = \frac{e^{-\gamma r}}{4\pi r} \qquad [2.14]$$

where

$$r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$$

If the medium is assumed to be lossless, then $\gamma = j\beta$ and [2.14] becomes

$$K = \frac{e^{-j\beta r}}{4\pi r} \qquad [2.15]$$

where

$$\beta = k$$

Applying the previous derivation, equation [2.12] will have the solution

$$A(r) = \mu \, \underline{J}(r) * K(r) \qquad [2.16]$$

where * defines convolution of the expression for $\underline{J}$ and K in the spatial variables x, y and z.

For equation [2.16] to be meaningful, it is sufficient that $\underline{J}$ exist in a bounded region.

Let the convolution be denoted by

$$\underline{J}(r) * K(r) = T[\underline{J}(r)] \qquad [2.17]$$

In this sense, equation [2.16] can be rewritten as

$$\underline{A}(r) = \mu \, T[\underline{J}(r)] \qquad [2.18]$$

From equations [2.9], [2.10] and [2.12], $\underline{E}(r)$ can be expressed as

$$\underline{E}(r) = \frac{1}{j\omega\mu\varepsilon} [\nabla \nabla \cdot \underline{A}(r) + k^2 \underline{A}(r)]$$

or $$\underline{E}(r) = \frac{1}{j\omega\mu\varepsilon} S[\underline{A}(r)] \qquad [2.19]$$

where S is the differential operator $(\cancel{\nabla\nabla} \cdot + k^2)$. Applying equation [2.18] to [2.19]

$$\underline{E}\,(r) = S_0\,\{\,T\,[\underline{J}\,(r)]\,\} \qquad [2.20a]$$

or $\underline{E}\,(r) = L_{op}\,[\underline{J}\,(r)\,]$ [2.20b]

where L_{op} is a linear operator defined as $S_0 T$, $S_0 = S/j\omega\varepsilon$

It is of interest to note that the linear operators T, S_0 commute, in the sense that

$$S_0\,\{\,T\,[\,\underline{J}\,(r\,)]\} = T\,\{\,S_0\,[\underline{J}\,(r\,)]\}\;, \qquad [2.21]$$

when the expressions are interpreted in the sense of generalized functions[3,4].

Consider a perfect conductor with known incident electric field $\underline{E}^i$, due to a source located anywhere on or outside the conductor. The total tangential electric field is zero everywhere on the surface of the conductor,

$$E^s_{tan}\,(r) + E^i_{tan}\,(r) = 0$$

or

$$E^s_{tan}\,(r) = -\,E^i_{tan}\,(r) \qquad [2.22]$$

where E^s_{tan} is the scattered electric field radiated by the current density $\underline{J}\,(r')$.

Equation [2.20b] becomes

$$-E^{i}_{tan}(r) = L_{op}\,[\underline{J}\,(r)\,] \qquad [2.23]$$

L_{op} must be inverted to solve the problem of interest. $E^{i}_{tan}(r)$ is the known excitation function, and $\underline{J}$ (r) is the unknown response function to be determined.

In linear antennas, the differential operator S is invertible in a closed form. To obtain T^{-1}, any standard method for solving integral equations can be used. So, for the known incident field the current distributions can be determined as will be considered in the next chapter.

CHAPTER III

LINEAR CYLINDRICAL DIPOLE ANTENNAS

3.1 The Linear Current Element

An isolated current element may appear to be a very unreal concept, but it is evident that any physical circuit or antenna carrying current may be considered to consist of a large number of such elements joined end to end. Therefore, if the characteristics of this "building block" is known, the characteristics of any actual antenna may be calculated.

Now, consider a linear cylindrical tube of radius a and length 2h, which is assumed to fulfil certain assumptions. These parallel closely the assumptions used by other workers in this field:

[a] The tube is a perfect conductor

[b] The tube thickness is negligible

[c] Current is taken to flow along the tube only in a longitudinal direction and is confined to the surface.

[d] The tube radius is assumed to be much smaller than both the tube length and the wave length, so the current may be assumed to have uniform distribution over the tube.

The axis of the tube is assumed to lie along the Z-axis of a system of cylindrical coordinates $[z,\rho,\phi]$ with the origin of the tube at the center. Now, considering the current element $\underline{J}$; when the tube is carrying a current I_z, we have in cylindrical coordinates,

$$\underline{J}_\rho = \underline{J}_\phi = 0$$

$$\underline{J}_z = \frac{I_z(z)}{2\pi a} \delta(\rho - a) \qquad \text{for } -h \leq z \leq h$$

$$\underline{J}_z = 0 \qquad \text{for } |z| > h \qquad [3.1]$$

where

δ is the Dirac-delta function

ρ is the cylindrical radial coordinate

It is shown in Appendix A that the radiated electrical field from such a source can be expressed as

$$E_z\,(z,\rho) = S_z\,[I\,(z)\,] * K_z\,(z,\rho) \qquad [3.2]$$

$$E_\rho\,(z,\rho) = -S_z\,[I\,(z)\,] * K_\rho\,(z,\rho) \qquad [3.3]$$

$$E_\phi = 0 \qquad [3.4]$$

where E_z, E_ρ, E_ϕ are axial, radial and angular components of the electrical field respectively, and S_z is a differential operator defined by

$$S_z = \left(\frac{\partial}{\partial z^2} + k^2\right) \qquad [3.5]$$

$$K_z\,(z,\rho) = \frac{1}{2\pi} \int_0^{2\pi} K_0\,(z,\rho,\phi)\, d\phi \qquad [3.6]$$

$$K_\rho\,(z,\rho) = \frac{1}{2\pi} \int_0^{2\pi} \frac{z\,(\rho - a\cos\phi)}{(\rho^2 + a^2 - 2\rho a\cos\phi)}\; K_0\,(z,\rho,\phi)\, d\phi \qquad [3.7]$$

where

$$K_0\,(z,\rho,\phi) = \frac{\exp\,[-jkr(z,\rho,\phi)\,]}{j4\pi\omega\varepsilon r(z,\rho,\phi)} \qquad [3.8a]$$

and $$r\,(z,\rho,\phi) = (z^2 + \rho^2 + a^2 - 2\rho a\cos\phi)^{1/2} \qquad [3.8b]$$

The convolution in [3.2] and [3.3] is restricted to z only.

3.1.1 The Differential Operator for the Linear Current Element

The differential operator expressed by

$$S_z\ [I_z] = \frac{d^2 I_z}{dz^2} + k^2\ I_z \qquad [3.9]$$

has to be defined for $-\infty < z < \infty$. However I_z exists only in the interval -h to h, so $I_z \equiv 0$ for $|z| > h$. Using an auxiliary function $E_0\ (z)$ to express the restriction of $S_z\ [I_z]$ to the open interval (-h,h), gives

$$S_z\ [I_z] = \frac{d^2 I_z}{dz^2} + k^2\ I_z = E_0\ (z) \qquad -h < z < h \qquad [3.10]$$

where

$$E_0\ (z) = 0 \qquad |z| > h \qquad [3.11]$$

Equations [3.10] and [3.11] give

$$S_z\ [I_z] = E_0\ (z) + I_z\ (-h)\ \delta'\ (z+h) - I_z\ (h)\ \delta'\ (z-h) + I_z'\ (-h)\ \delta\ (z+h) - I_z'\ (h)\ \delta\ (z-h) \qquad [3.12]$$

in which $I_z\ (-h)$ and $I_z\ (h)$ are the end currents at $z = -h$ and $z = h$ respectively, and the primes denote differentiation with respect to z. $I_z'\ (-h)$ and $I_z'\ (h)$ define the limiting values at -h and h within the interval (-h,h), as follows

$$I_z'\ (-h) = \lim_{\varepsilon \to 0}\ I_z'\ (-h + \varepsilon) \qquad , \varepsilon \geq 0$$

$$I_z'(h) = \lim_{\varepsilon \to o} I_z'(h - \varepsilon) \quad , \varepsilon \geq o \qquad [3.13]$$

Consider the differential equation

$$\frac{d^2 I_z}{dz^2} + k^2 I_z = E_o(z) \quad , -h \leq z \leq h \qquad [3.14]$$

The solution of this equation depends on some boundary conditions for the current. Using the end currents $I_z(-h)$ and $I_z(h)$ for these boundary conditions, the solution is

$$I_z(z) = I_z(-h)\,\frac{\sin[k(h-z)]}{\sin(2kh)} + I_z(h)\,\frac{\sin[k(h+z)]}{\sin(2kh)} + \int_{-h}^{h} G(z/z')\,E_o(z)\,dz' \qquad [3.15]$$

where $G(z/z')$ is the Green's function given by

$$G(z/z') = -\begin{cases} \dfrac{\sin[k(h+z)]\,\sin[k(h-z')]}{k\sin(2kh)} & , z < z' \\[2ex] \dfrac{\sin[k(h-z)]\,\sin[k(h+z')]}{k\sin(2kh)} & , z > z' \end{cases} \qquad [3.16]$$

with the condition $\sin(2kh) \neq o$, i.e. $h \neq n\lambda/4$, where λ is the wave-length and n is a positive integer. For a practical case this singularity can be avoided either by treating the element as two different elements axially joined together in a straight line, or, by setting $h = (n\lambda/4) - \Delta$, where Δ is a very small quantity compared to $n\lambda/4$. The latter case will be illustrated by examples in the section on calculation of antenna characteristics.

Equation [3.2] and [3.3] can be rewritten as

$$E_z(z,\rho) = \int_{-h}^{h} E_0(z')\, K_{zc}(z/z'/\rho)\, dz' + I_z(-h)\, K'_z(z+h,\rho) - I_z(h)\, K'_z(z-h,\rho) + b\, K_z(z+h,\rho) - c\, K_z(z-h,\rho) \quad [3.17]$$

$$E_\rho(z,\rho) = -\{ \int_{-h}^{h} E_0(z')\, K_{\rho c}(z/z'/\rho)\, dz' + I_z(-h)\, K'_\rho(z+h,\rho) - I_z(h)\, K'_\rho(z-h,\rho) + b\, K_\rho(z+h,\rho) - c\, K_\rho(z-h,\rho)\} \quad [3.18]$$

where the primes denote differentiation with respect to z, and

$$K_{zc}(z/z'/\rho) = K_z(z-z',\rho) + K_z(z+h,\rho)\, G'(-h/z') - K_z(z-h,\rho)\, G'(h/z') \quad [3.19]$$

$$K_{\rho c}(z/z'/\rho) = K_\rho(z-z',\rho) + K_\rho(z+h,\rho)\, G'(-h/z') - K_\rho(z-h,\rho)\, G'(h/z') \quad [3.20]$$

$$b = -I_z(-h)\, \frac{k \cos(2kh)}{\sin(2kh)} + I_z(h)\, \frac{k}{\sin(2kh)} \quad [3.21]$$

$$c = -I_z(-h)\, \frac{k}{\sin(2kh)} + I_z(h)\, \frac{k \cos(2kh)}{\sin(2kh)} \quad [3.22]$$

$$G'(-h/z') = -\frac{\sin[k(h-z')]}{\sin(2kh)}, \quad z' > -h \quad [3.23]$$

$$G'(h/z') = \frac{\sin[k(h+z')]}{\sin(2kh)}, \quad z' < h \quad [3.24]$$

Equations [3.15] and [3.17] are used as a basis for the analysis of thin, tubular linear antennas. If the scattered electrical field E_z^s is specified, a solution of equation [3.17] yields E_0 (z) and the end currents. The antenna current I_z may then be determined from [3.15].

3.2 Calculation of Antenna Characteristics

3.2.1 The Dipole

Consider a center-driven cylindrical dipole antenna that extends from -h to h along the Z-axis of a rectangular coordinate system. The radius of the antenna is a, and the antenna is centered at the origin as illustrated in figure 3.1. The assumptions for the dipole antenna are the same as those for the linear current element in Section 3.1. The antenna is driven by a magnetic frill current[25,26]. This case may represent an actual physical situation, an example of which is an antenna located over a ground plane and fed through a coaxial-cable. The ratio of the outer to the inner radius of the frill is taken 2.23, which gives a characteristic impedance of an airfilled coaxial-cable of 50 ohms. This impedance is standard for a cable normally used to feed such antennas.

3.2.2 The Approximate Kernel

Traditionally, thin wire antenna problems have been dealt with using Pocklington's and Hallén's integral equations. A principal difficulty in the solution of the exact integral equations is the complicated kernel given by equations [3.8]. To solve equation [3.2] a significant simplification may be achieved if an approximate kernel[9,27] is introduced as

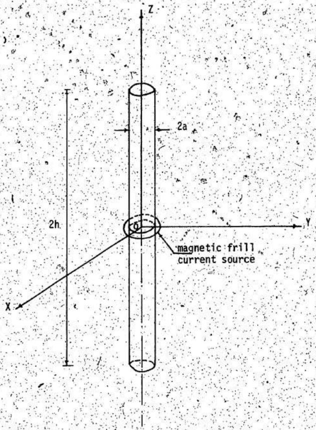


Figure 3.1 Geometry of a symmetrical center-fed linear cylindrical dipole.

$$K_{za}(z,\rho) = \frac{\exp[-jk(z^2+\rho^2+a^2)^{\frac{1}{2}}]}{j4\pi\omega\varepsilon(z^2+\rho^2+a^2)^{\frac{1}{2}}} \qquad [3.25]$$

Since for the center-fed dipole antenna the observation point of the scattered field lies on the axis, so, ρ equals 0. The surface current is then represented by an equivalent filamentary line source parallel to the Z-axis and located at a radial distance a from the observation point. Hence, equation [3.25]reduces to

$$K_{za}(z,0) = \frac{\exp[-jk(z^2+a^2)^{\frac{1}{2}}]}{j4\pi\omega\varepsilon(z^2+a^2)^{\frac{1}{2}}} \qquad [3.26]$$

3.2.3 Integral Equations for Dipole Antennas

3.2.3.1 Pocklington's Integral Equation

Pocklington's form of the integral equation for thin wire antennas may be obtained from equation [3.2] by applying the operator S_z defined by [3.5] to K_{za} defined by [3.26] instead of I_z. This yields

$$E_z^s(z,0) = I_z(z) * S_z[K_{za}(z,0)] \qquad [3.27]$$

in which the convolution is restricted to z only. The incident field E_z^i may be expressed as

$$I_z * S_z[K_{za}(z,0)] = -E_z^i(z,0) \qquad [3.28]$$

or in integral form

$$\int_{-h}^{h} I_z(z')\left[\frac{\partial^2}{\partial z^2}K_{za}(z-z',0) + k^2K_{za}(z-z',0)\right]dz' = -E_z^i(z,0) \qquad [3.29]$$

Equation [3.29] has been used extensively by many workers e.g., Richmond[19] and Harrington[6]. This equation is more flexible than Hallén's equation because it can be used for arbitrary incident fields.

3.2.3.2 Hallén's Integral Equation

For a center-fed dipole antenna, driven by a delta-gap voltage generator V the incident field can be expressed as

$$E_z^i\ (z,\rho) = V\ \delta(z) \qquad [3.30]$$

Hence, the scattered electric field can be obtained from equation [3.2] using the approximate kernel defined by [3.26]

$$E_z^s\ (z,0) = S_z\ [\ I_z\ (z) \ * \ K_{za}\ (z,0)\ ] \qquad [3.31]$$

or $$S_z\ [\ I_z\ (z) \ * \ K_{za}\ (z,0)] = -\ V\ \delta(z) \qquad [3.32]$$

The solution of equation [3.32] is

$$\int_{-h}^{h} I_z\ (z')\ K_{za}\ (z-z',0)\ dz' = \frac{j}{2\ \eta_o}\ (\ C \cos\ kz\ +\ V \sin k|z|) \qquad [3.33]$$

where η_o is the intrinsic impedance of free space. This is Hallén's integral equation for the perfectly conducting dipole antenna. The constant C must be evaluated from the condition that the current vanishes at the endpoints of the antenna.

Many researchers have advanced various solutions of Hallén's integral equation. Particularly significant is the work of Mie[17], which extended Hallén's equation to antennas of arbitrary geometry.

An excellent summary of various works related to Hallén's integral equation has been given by King[9-13].

Since Pocklington's integral equation involves the derivative of the kernel K_z, or its approximation K_{za}, the resulting kernel is of a higher order than the kernel of Hallén's integral equation. Therefore, numerical problems may be expected to be more prevalent with Pocklington's than with Hallén's equation.

In the proposed method the integral equation has the advantage of Hallén's equation in the sense that it contains no derivatives of the composite kernel K_{zc}, or its approximation K_{zac} (the approximate composite kernel can be obtained from equation [3.19] by replacing K_z by K_{za}). The composite kernel K_{zc} or K_{zac} vanishes at $z' = -h$ and h for all z with the possible exception of $z = -h$ and h, whereas Hallén's kernel does not. It is evident that the composite kernel of the proposed method is even better behaved numerically than Hallén's kernel.

3.2.4 Solution of Antenna Current

Using the approximate kernel of equation [3.26] and the homogeneous boundary conditions $I(h) = I(-h) = 0$, equations [3.15] and [3.17] can be reduced to

$$-E_z^i\,(z,0) = \int_{-h}^{h} E_o\,(z')\,K_{zac}\,(z/z'/0)\,dz' \qquad [3.34]$$

and

$$I_z\,(z) = \int_{-h}^{h} G\,(z/z')\,E_o\,(z')\,dz' \qquad [3.35]$$

where K_{zac} $(z/z'/0)$ is given by [3.19] with K_z (z,ρ) replaced by

K_{za} (z,0) and E_z^i is the known axial incident field defined by

$$E_z^i(z,0) = \frac{1}{2\ln(b/a)}\left\{\frac{\exp[-jk(z^2+a^2)^{\frac{1}{2}}]}{(z^2+a^2)^{\frac{1}{2}}} - \frac{\exp[-jk(z^2+b^2)^{\frac{1}{2}}]}{(z^2+b^2)^{\frac{1}{2}}}\right\} \qquad [3.36]$$

where a,b are the inner and outer radii of the magnetic frill current respectively.

Equation [3.34] may be solved by any standard technique such as the method of moments which will be discussed in the next section. When E_o (z') is determined, the approximate current distribution may be found from equations [3.35].

3.2.5 Method of Moments

The unifying concept in the numerical treatment of radiation problems is the method of moments[5,20,25]. This approach is essentially a reduction of the associated integral equation to a system of linear algebraic equations in, say, N unknowns where the N unknowns are usually coefficients in some appropriate expansion of the unknown function, such as E_o in equation [3.14].

Rewriting equation [3.34] in the form

$$L\,[E_o] = -\,E_z^i \qquad [3.37]$$

where L is a linear integral operator (see Appendix B.1), E_z^i is a known excitation function and E_o is the unknown response function. The integral form for L is

$$L[E_0] = \int_{-h}^{h} E_0(z')\, K_{zac}(z/z'/0)\, dz' \qquad [3.38]$$

We may expand the response function E_0 in a series of basis functions $E_{o_1}, E_{o_2}, E_{o_3}, \ldots$ spanning the domain of L. That is, let

$$E_0 = \sum_{j=1}^{N} c_j E_{oj} \qquad [3.39]$$

where the c_j's are unknown expansion coefficients. For an exact solution, N in general is an infinite number but for practical approximations we may take N finite. Substituting equation [3.39] into [3.37] gives

$$L\left[\sum_{j=1}^{N} c_j E_{oj}\right] = -E_z^i \qquad [3.40]$$

Then, because of the linearity of L,

$$\sum_{j=1}^{N} c_j L[E_{oj}] = -E_z^i \qquad [3.41]$$

In solving such problems one may need an inner product $< E_0, E_z >$, which is a scalar defined by

$$< E_0, E_z > = \int_{-h}^{h} E_0(z) E_z(z)\, dz \qquad [3.42]$$

Inner products are discussed in more details in Appendix (B.2). We now define a set of weighting functions $W_1, W_2, \ldots$ in the domain of L and then form the inner product

$$\sum_{j=1}^{N} c_j < W_i, L[E_{oj}] > = - < W_i, E_z^i > \qquad [3.43]$$

where $i = 1,2, \ldots m$. The concept of weighting functions is discussed in more details in Appendix (B.3).

3.2.5.1 Bases

Basis functions are divided into two classes. The first class is the basis functions defined and non-zero over the entire domain of L. This class of basis functions is called entire-domain bases. The second class is sub-domain basis function. These are defined in the domain of L but are zero over part of the domain. In the case of wire radiators this constitutes dividing the antenna into sections. It appears generally true that the closer the basis functions resemble the actual current distribution on the radiator, the better the rate of convergence.

i) Entire-domain Bases

As illustrative examples of basis functions, consider the following entire-domain functions that could be used for dipole antennas:

$$\text{Fourier:}\quad E_0(z) = c_1 \cos(\pi z/2h) + c_2 \cos(3\pi z/2h) + c_3 \cos(5\pi z/2h) + \cdots \qquad [3.44]$$

$$\text{Chebyshev:}\quad E_0(z) = c_1 T_0(z/h) + c_2 T_2(z/h) + c_3 T_4(z/h) + \ldots \qquad [3.45]$$

ii) Sub-domain Bases

As examples of sub-domain bases consider the following:

$$\text{Pulse function } E_o(z) = \begin{cases} E_{oj} & \text{for } z \text{ in } \Delta z_j , \\ 0 & \text{otherwise} \end{cases} \qquad [3.46]$$

$$\text{Picewise-linear } E_o(z) = \begin{cases} E_{oj}(z_{j+1} - z) + E_{oj+1}(z - z_j) & \text{for } z \text{ in } \Delta z_j , \\ 0 & \text{otherwise} \end{cases} \qquad [3.47]$$

3.2,5.2 Galerkin's Method

From equation [3.43], if the weighting functions are taken the same as basis functions i.e., $W_i = E_{oi}$, then the formulation of the problem is known as Galerkin's method. Thus, equation [3.43] becomes

$$\sum_{j=1}^{N} c_j < E_{oi} , L[E_{oj}] > = < E_{oi} , E_z^i > \qquad [3.48]$$

and the set of all such equations written in matrix form is

$$\begin{bmatrix} <E_{o_1},L[E_{o_1}]> & <E_{o_1},L[E_{o_2}]> & \cdots & <E_{o_1},L[E_{oN}]> \\ <E_{o_2},L[E_{o_1}]> & <E_{o_2},L[E_{o_2}]> & \cdots & <E_{o_2},L[E_{oN}]> \\ \vdots & \vdots & & \vdots \\ <E_{oN},L[E_{o_1}]> & <E_{oN},L[E_{o_2}]> & & <E_{oN},L[E_{oN}]> \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = - \begin{bmatrix} <E_{o_1},E_z^i> \\ <E_{o_2},E_z^i> \\ \vdots \\ <E_{oN},E_z^i> \end{bmatrix} \qquad [3.49]$$

In practice, the evaluation of the matrix of equation [3.49] may be very laborious even in a high-speed computer since at least two integrations may have to be done. However, one of these integrations may be avoided as will be discussed in the following section.

3.2.5.3 Point-matching Method

Using Dirac-delta functions as weighting functions will minimize the difficulty of calculating the matrix of equation [3.49]. The matrix equation will be

$$\begin{bmatrix} <\delta(z-z_1),L[E_{O_1}]> & \cdots & <\delta(z-z_1),L[E_{ON}]> \\ <\delta(z-z_2),L[E_{O_1}]> & \cdots & <\delta(z-z_2),L[E_{ON}]> \\ \vdots & & \vdots \\ <\delta(z-z_N),L[E_{O_1}]> & & <\delta(z-z_N),L[E_{ON}]> \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = - \begin{bmatrix} <\delta(z-z_1),E_z^i> \\ <\delta(z-z_2),E_z^i> \\ \vdots \\ <\delta(z-z_N),E_z^i> \end{bmatrix} \quad [3.50]$$

The use of Dirac-delta functions limits the integrations to these defined by L. In this method the electric field is matched only at discrete points. The accuracy of the solution depends on the number of points, and their location. The calculation of near-field data such as impedance is sensitive to the number and location of the match points.

3.2.6 Effect of Different Weighting Functions

In the application of the method of moments to thin-wire antenna problems, many different choices of basis and weighting functions have been successfully used.[15] The guidelines on choosing these functions are:

1) the functions within each set should be linearly independent, 2) a linear combination of the modeling functions should be able to approximate the corresponding exact functions, and 3) the resulting matrix equation should be well conditioned.[14]

3.2.7 Comparative Study

A comparative study for three different weighting functions will be presented here. The first weighting functions used are Walsh functions, the second are cosine functions and finally picewise-sinusoidal functions. For a center-fed dipole antenna, because of symmetry these functions must be even. These three sets of weighting functions will be discussed in the next three sections. Even Walsh functions are used as basis functions.

Equation [3.34] can be approximately solved by using Walsh functions Wal, to approximate E_o. Using equation [3.39], Equation [3.43] becomes

$$\sum_{j=1}^{N} c_j < W_i , L [Wal_j] > = - < W_i , E_z^i > \qquad [3.51]$$

where

Wal_j = Walsh functions shown in figure (3.2)

N = number of basis functions

The matrix form of equation [3.51] is

$$\begin{bmatrix} \langle W_1, L[Wal_1]\rangle & \langle W_1, L[Wal_2]\rangle & \cdots & \langle W_1, L[Wal_N]\rangle \\ \langle W_2, L[Wal_1]\rangle & \langle W_2, L[Wal_2]\rangle & \cdots & \langle W_2, L[Wal_N]\rangle \\ \vdots & \vdots & & \vdots \\ \langle W_N, L[Wal_1]\rangle & \langle W_N, L[Wal_2]\rangle & \cdots & \langle W_N, L[Wal_N]\rangle \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = - \begin{bmatrix} \langle W_1, E_z^i \rangle \\ \langle W_2, E_z^i \rangle \\ \vdots \\ \langle W_N, E_z^i \rangle \end{bmatrix} \qquad [3.52]$$

The general entry of the matrix AM (i,j), and the right-hand side vector EV(i) of equation [3.52] is given by

$$\sum_{j=1}^{N} c_j \int_{-h}^{h} \int_{-h}^{h} W_i(z)\ Wal_j\ (z')\ K_{zac}\ (z/z'/0)\ dz'\,dz$$

$$= - \int_{-h}^{h} W_i(z)\ E_z^i\ (z,0)\ dz \qquad [3.53]$$

Since Walsh functions are constant over sub-intervals, the integral domain can be divided into sub-intervals, and equation [3.53] can be rewritten as

$$\sum_{j=1}^{N} c_j \sum_{jj=1}^{NWD} Wal_{j,jj} \int_{-h}^{h} \int_{L_{jj}}^{U_{jj}} W_i(z)\ K_{zac}\ (z/z'/0)\ dz'dz$$

$$= - \int_{-h}^{h} W_i(z)\ E_z^i\ (z,0)\ dz \qquad [3.54]$$

where

NWD = number of sub-intervals

L_{jj}, U_{jj} = lower and upper limits of the sub-intervals respectively

$Wal_{j,jj}$ = value of the j^{th} Walsh function in the interval jj

3.2.7.1 Walsh Functions

A set of Walsh functions are used as weighting functions. Some of these functions are shown in figure 3.2. The method of derivation of the Walsh functions is given in Appendix C. A computer program was developed to generate the Walsh functions, and was introduced as a subroutine to the program. A listing of this subroutine is given in Appendix E.

The general entry of the matrix AM (i,j) of equation [3.52] is given by

$$AM\,(i,j) = \sum_{ii=1}^{NWD} \sum_{jj=1}^{NWD} Wal_{i,ii}\, Wal_{j,jj} \int_{L_{ii}}^{U_{ii}} \int_{L_{jj}}^{U_{jj}} K_{zac}(z/z'/0)\,dz'\,dz \qquad [3.55]$$

The general entry of the right-hand side vector EV(i) of equation [3.52] is given by

$$EV(i) = -\sum_{ii=1}^{NWD} Wal_{i,ii} \int_{L_{ii}}^{U_{ii}} E_z^i\,(z)\,dz \qquad [3.56]$$

A computer program was developed to determine the current distribution and the antenna input impedance using the Walsh functions as weighting functions.

To reduce the computer time of the central processing unit (CPU), a set of subroutines were developed to replace the general library subroutines supplied by International Business Machine (IBM) for single

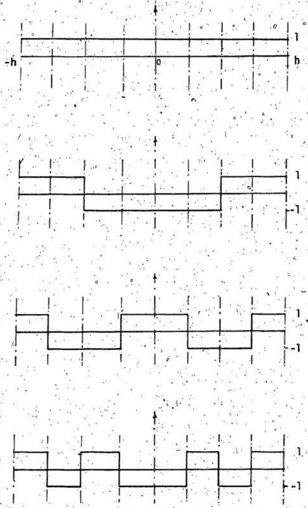


Figure 3.2 A set of even Walsh functions.

and double integration calculations. In these subroutines, the sine and the cosine functions were expanded as power series. A brief description of the method is given in Appendix D.

3.2.7.2 Cosine Functions

A set of cosine functions is used as weighting functions, some of these functions are shown in figure 3.3. The i^{th} function is defined by

$$F_i(z) = \cos\left[(2i-1)\frac{\pi z}{2h}\right] \qquad [3.57]$$

The general entry of the matrix of equation [3.52] AM(i,j) is expressed as

$$AM(i,j) = \sum_{jj=1}^{NWD} Wa1_{j,jj} \int_{-h}^{h} \int_{L_{jj}}^{U_{jj}} \cos\left[(2i-1)\frac{\pi z}{2h}\right] K_{zac}(z/z'/0)\,dz'\,dz \qquad [3.58]$$

Also, the general entry of the right-hand side vector EV(i) of equation [3.52] is given by

$$EV(i) = -\int_{-h}^{h} \cos\left[(2i-1)\frac{\pi z}{2h}\right] E_z^i(z)\,dz \qquad [3.59]$$

A computer program was developed to obtain the current distribution and the input impedance. In order to evaluate equation [3.59] and the real part of equation [3.58], both the sine and cosine functions were approximated using power series expansion. Computer subroutines were developed to replace the IBM library subroutines. For the imaginary part of [3.58] the approximation of cosine and sine functions

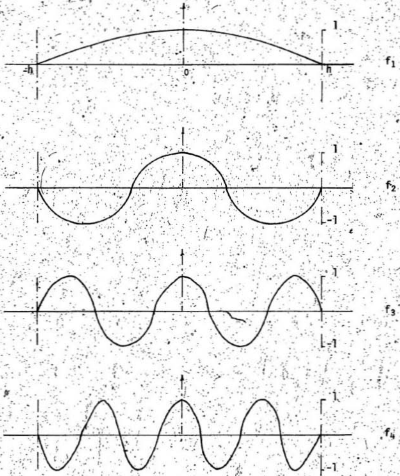


Figure 3.3 A set of cosine functions used as weighting functions.

is not appropriate, in the sense that more terms of the series are required, and the calculations will be complicated. IBM library subroutines were used for evaluating the imaginary parts.

3.2.7.3 Piece-wise-sinusoidal Functions

The piece-wise-sinusoidal functions shown in fugure 3.4 were also used as weighting functions. The general entry of the matrix of equation [3.52] AM(i,j) is given by

$$AM(i,j) = \sum_{jj=1}^{NWD} Wal_{j,jj} \sum_{ii=1}^{(2i-1)} \int_{L_{i,ii}}^{U_{i,ii}} \int_{L_{jj}}^{U_{jj}} F_{i,ii}(z)\, K_{zac}(z/z'/0)\,dz'\,dz \qquad [3.60]$$

where $L_{i,ii}$ and $U_{i,ii}$ are the lower and upper limits of subdivisions for the weighting functions. As an example, taking four basis functions and hence, four weighting functions to form the N x N matrix (N = 4), and dividing the range of the weighting function to NWD divisions (NWD = 8), this gives

$$Wal_{j,jj} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix} \qquad [3.61]$$

$$L_{i,ii} = \begin{bmatrix} -h & & & & & & \\ -h & -h/2 & h/2 & & & & \\ -h & -3h/4 & -h/4 & h/4 & 3h/4 & & \\ -h & -3h/4 & -h/2 & -h/4 & h/4 & h/2 & 3h/4 \end{bmatrix} \qquad [3.62a]$$

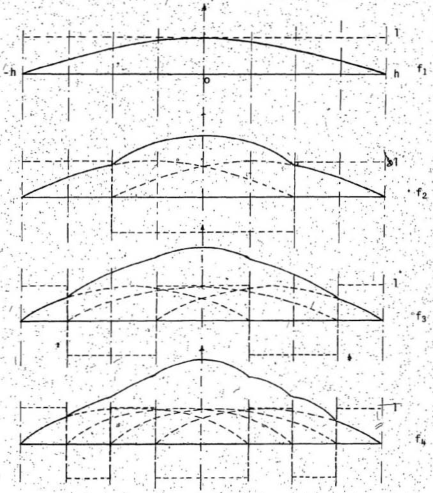


Figure 3.4 Piece-wise-sinusoidal functions used as weighting functions.

$$U_{i,ii} = \begin{bmatrix} h & & & & & & \\ -h/2 & h/2 & h & & & & \\ -3h/4 & -h/4 & h/4 & 3h/4 & h & & \\ -3h/4 & -h/2 & -h/4 & h/4 & h/2 & 3h/4 & h \end{bmatrix} \qquad [3.62b]$$

$$L_{jj} = \begin{bmatrix} -h & -3h/4 & -h/2 & -h/4 & 0 & h/4 & h/2 & 3h/4 \end{bmatrix}^T \qquad [3.63a]$$

$$U_{jj} = \begin{bmatrix} -3h/4 & -h/2 & -h/4 & 0 & h/4 & h/2 & 3h/4 & h \end{bmatrix}^T \qquad [3.63b]$$

$$F_{i,ii}(z) = \begin{bmatrix} f_{11} & & & & & & \\ f_{21} & f_{22} & f_{23} & & & & \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} & & \\ f_{41} & f_{42} & f_{43} & f_{44} & f_{45} & f_{46} & f_{47} \end{bmatrix} \qquad [3.64a]$$

where

$$f_{11} = \cos\left[\pi\left(\frac{z}{2h}\right)\right]$$

$$f_{21} = \cos\left[\pi\left(\frac{z+h/4}{3h/2}\right)\right]$$

$$f_{22} = \cos\left[\pi\left(\frac{z+h/4}{3h/2}\right)\right] + \cos\left[\pi\left(\frac{z-h/4}{3h/2}\right)\right]$$

$$f_{23} = \cos\left[\pi\left(\frac{z-h/4}{3h/2}\right)\right]$$

$$f_{31} = \cos\left[\pi\left(\frac{z+3h/8}{5h/4}\right)\right]$$

$$f_{32} = \cos\left[\pi\left(\frac{z+3h/8}{5h/4}\right)\right] + \cos\left[\pi\left(\frac{z}{3h/2}\right)\right]$$

$$f_{33} = \cos\left[\pi\left(\frac{z+3h/8}{5h/4}\right)\right] + \cos\left[\pi\left(\frac{z}{3h/2}\right)\right] + \cos\left[\pi\left(\frac{z-3h/8}{5h/4}\right)\right]$$

$$
\left.
\begin{aligned}
f_{34} &= \cos\left[\pi\left(\frac{z}{3h/2}\right)\right] + \cos\left[\pi\left(\frac{z-3h/8}{5h/4}\right)\right] \\
f_{35} &= \cos\left[\pi\left(\frac{z-3h/8}{5h/4}\right)\right] \\
f_{41} &= \cos\left[\pi\left(\frac{z+3h/8}{5h/4}\right)\right] \\
f_{42} &= \cos\left[\pi\left(\frac{z+3h/8}{5h/4}\right)\right] + \cos\left[\pi\left(\frac{z+h/8}{5h/4}\right)\right] \\
f_{43} &= \cos\left[\pi\left(\frac{z+3h/8}{5h/4}\right)\right] + \cos\left[\pi\left(\frac{z+h/8}{5h/4}\right)\right] + \cos\left[\pi\left(\frac{z-h/8}{5h/4}\right)\right] \\
f_{44} &= \cos\left[\pi\left(\frac{z+3h/8}{5h/4}\right)\right] + \cos\left[\pi\left(\frac{z+h/8}{5h/4}\right)\right] + \cos\left[\pi\left(\frac{z-h/8}{5h/4}\right)\right] \\
&\qquad + \cos\left[\pi\left(\frac{z-3h/8}{5h/4}\right)\right] \\
f_{45} &= \cos\left[\pi\left(\frac{z+h/8}{5h/4}\right)\right] + \cos\left[\pi\left(\frac{z-h/8}{5h/4}\right)\right] + \cos\left[\pi\left(\frac{z-3h/8}{5h/4}\right)\right] \\
f_{46} &= \cos\left[\pi\left(\frac{z-h/8}{5h/4}\right)\right] + \cos\left[\pi\left(\frac{z-3h/8}{5h/4}\right)\right] \\
f_{47} &= \cos\left[\pi\left(\frac{z-3h/8}{5h/4}\right)\right]
\end{aligned}
\right\rbrace
\qquad [3.64b]
$$

These functions were chosen to satisfy the condition that the points of discontinuous slope must be coincident with those of the basis functions. They also must vanish at the endpoints of the antenna, i.e., at -h and h, and have their maximum at the origin, i.e., the center of the antenna.

The general entry of the right-hand side vector of equation [3.52] is given by

$$EV(i) = - \sum_{ii=1}^{(2i-1)} \int_{L_{i,ii}}^{U_{i,ii}} F_{i,ii}(z)\, E_z^i(z)\, dz \qquad [3.65]$$

A computer program was developed to determine the current distribution and the antenna input impedance using the previous set of piece-wise-sinusoidal functions.

3.2.8 Numerical Results and Comparison

In this section the current distribution and the input impedance for the dipole of figure 3.1 in free space are studied with the proposed method. A set of even Walsh functions are used as basis functions for E_0 (z') in equation [3.34]. For weighting functions three different sets of functions are used. A comparison between the results obtained and the results of other researchers is presented.

3.2.8.1 Antenna Input Impedance

The input impedances of two dipoles similar to that of figure 3.1 are studied. These dipoles differ in their lengths and radii and the results are as follows.

a) Dipole of Length 0.45λ

For this dipole the rate of convergence for the input impedance is studied. A comparison is made between the results of the proposed method and those of Thiele[25]. In obtaining the results of the proposed method, the moment method is used. Thiele obtained his results by solving Pocklington's equation using point-matching in conjunction with entire-domain cosine basis functions. The results are also compared with those of Srivastava[24] using the proposed method and the same basis functions, but applying point-matching technique.

Figure 3.5 shows the rate of convergence of the input impedance for the dipole calculated by the proposed method using the moment method for different numbers of basis functions up to 24 functions. The weighting functions used in obtaining the results of figure 3.5 are even Walsh functions. It is seen that the rate of convergence for the resistance is the same as those of Thiele and Srivastava, but it is slower for the reactance.

Figure 3.6 shows the rate of convergence of the input impedance for the dipole calculated by the proposed method using the method of moments for different numbers of basis functions up to 8 functions. The weighting functions used in obtaining the results of figure 3.6 are cosine functions given by equation [3.57]. Again, it is seen that the rate of convergence for the resistance is faster than that for the reactance. It is of interest to note that the rate of convergence in figure 3.6 is faster than that of figure 3.5.

Figure 3.7 shows the rate of convergence of the input impedance for the dipole calculated by the proposed method using the method of moments for different numbers of basis functions up to 4 functions only. The weighting functions used in obtaining the results of figure 3.7 are picewise-sinusoidal functions given by equations [3.64]. It is seen that the rate of convergence for the resistance is much faster than that of the reactance. Furthermore, the convergence of figure 3.7 is much faster than those of figure 3.5 and 3.6.

b) <u>Dipole of Length 0.47λ</u>

Another example for studying the rate of convergence of the dipole input impedance as the number of basis functions is increased, is presented in figure 3.8. A comparison is made between the results of the proposed method using the moment method and the results of Thiele[25]

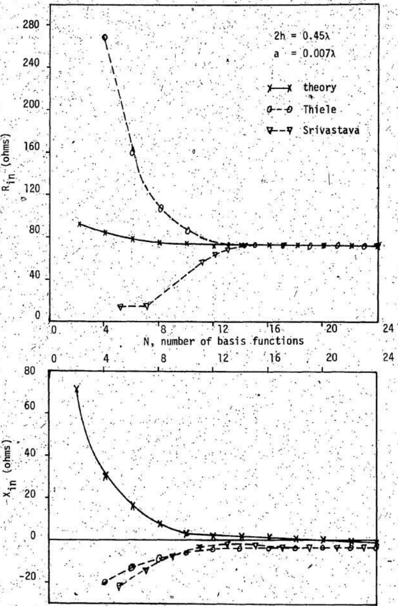


Figure 3.5 Curves showing convergence of input impedance of a center-fed dipole antenna using even Walsh Functions as weighting functions.

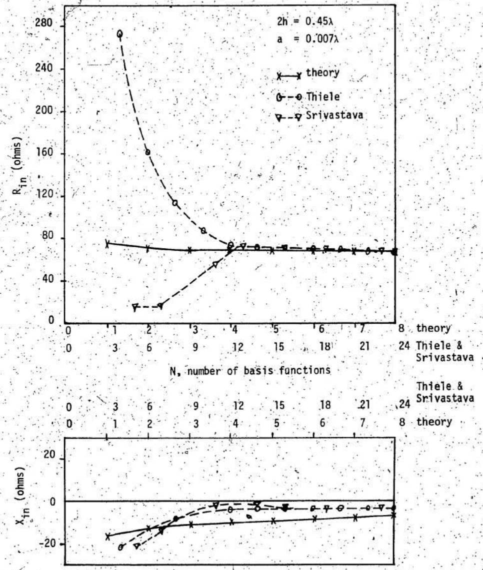


Figure 3.6 Curves showing convergence of input impedance of a center-fed dipole antenna using cosine functions as weighting functions.

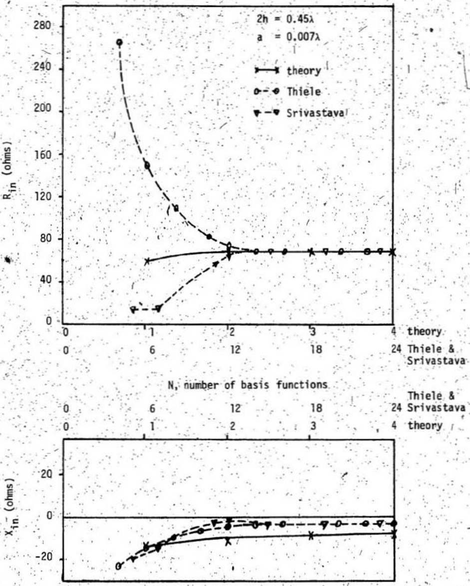


Figure 3.7 Curves showing convergence of input impedance of a center-fed dipole antenna using picewise-sinusoidal functions as weighting functions.

and Srivastava[24]. Thiele obtained his results by solving Pocklington's equation using point-matching technique in conjunction with sub-domain pulse functions. Srivastava obtained his results using the proposed method with the same basis function of the previous example, in conjunction with point-matching.

Figure 3.8 shows the rate of convergence of the input impedance for the dipole calculated by the proposed method using the method of moments for different numbers of basis functions up to 32 functions. The weighting functions used in obtaining the results of figure 3.8 are even Walsh functions. It is seen that the rate of convergence of the proposed method for the resistance is faster than those of Srivastava and Thiele. In the case of the reactance the convergence rate is still much faster than those of Thiele but, slower than those of Srivastava.

Figure 3.9 shows the rate of convergence of the dipole input impedance calculated by the proposed method using the method of moments for different number of basis functions up to 8 functions. Cosine functions are used as weighting functions to obtain the results of figure 3.9. It is seen that the rate of convergence for the proposed method is faster than those of Thiele and Srivastava, when cosine functions are used as weighting functions.

Figure 3.10 shows the rate of convergence of the input impedance of the dipole calculated by the proposed method using the method of moments for different numbers of basis functions up to 4 functions only. The weighting functions used in obtaining the results of figure 3.10 are picewise-sinusoidal functions. It is seen that the rate of convergence for the proposed method is much faster than those of Thiele and

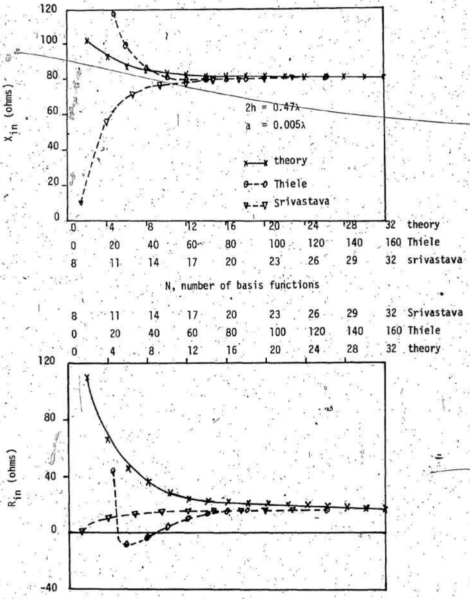


Figure 3.8 Curves showing convergence of input impedance of a center-fed dipole antenna using even Walsh functions as weighting functions.

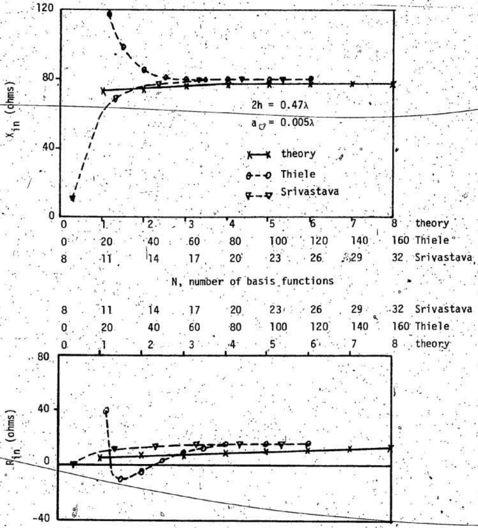
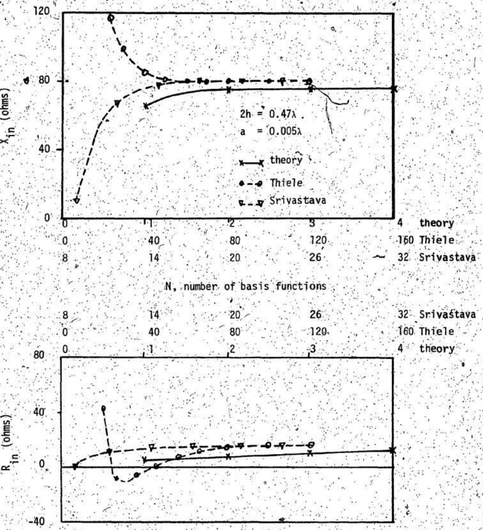


Figure 3.9 Curves showing convergence of input impedance of a center-fed dipole antenna using cosine functions as weighting functions.



3.10 Curves showing convergence of input impedance of a center-fed dipole antenna using picewise-sinusoidal functions as weighting functions.

Srivastava.

3.2.8.2 Antenna Currents

Current distributions for several antennas of different lengths and radii are presented. All the antennas are similar to that of figure 3.1. The results for each of these dipoles will be discussed separately. In this study, three sets of weighting functions are used. A set of 16 even Walsh functions, a second set of 4 cosine functions, and finally a set of 4 piece-wise-sinusoidal functions are considered. The number of functions used in each set was chosen based on the analysis which was presented in the previous section such that the corresponding results are generally in agreement with those obtained by other researchers. On the other hand, using more weighting functions means more computer time which will make it practically expensive to use the developed computer programs.

a) Half Wave Dipole

The distribution of currents along a half wave dipole is studied. A comparison is made between the results of the proposed method and the measured results of Mack as taken from Silvester[22].

Figure 3.11 shows the current distributions along the dipole calculated by the proposed method using 16 basis functions. The weighting functions used to obtain the results of figure 3.11 are 16 even Walsh functions. The results show good agreement with those of Mack.

Figure 3.12 shows the current distributions along the dipole calculated by the proposed method using 4 basis functions. The weighting functions used in obtaining the results of figure 3.12 are 4 cosine functions. The results are in good agreement with those of Mack.

Figure 3.13 shows the current distributions along the dipole calculated by the proposed method using 4 basis functions. The weighting functions used in obtaining the results of figure 3.13 are 4 piece-wise-sinusoidal functions. The results are in good agreement with those of Mack.

b) Three-quarter Wave Dipole

Current distributions along a three-quarter wave dipole are studied. A comparison between the results of the proposed method and the measured results of Mack as taken from Popović[18], is made.

Figure 3.14 shows the current distributions along the dipole calculated by the proposed method using 16 basis functions. The weighting functions used in obtaining the results of figure 3.14 are 16 even Walsh functions. The results are in good agreement with those of Mack.

Figure 3.15 shows the current distributions along the dipole calculated by the proposed method using 4 basis functions. The weighting functions used in obtaining the results of figure 3.15 are 4 cosine functions. These results show close agreement with those of Mack for the real part of the current and differ for the imaginary part of it.

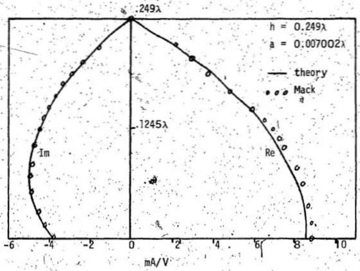


Figure 3.11 Current distributions along half wave center-fed dipole antenna, using even Walsh functions as weighting functions.

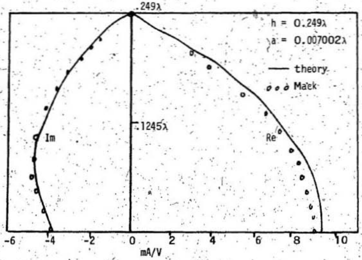


Figure 3.12 Current distributions along half wave center-fed dipole antenna, using cosine functions as weighting functions.

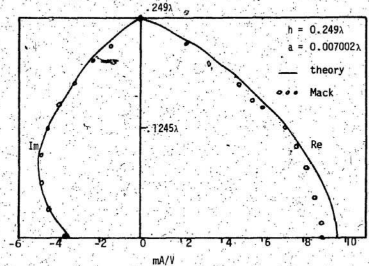


Figure 3.13 Current distributions along half wave center-fed dipole antenna, using picewise-sinusoidal functions as weighting functions.

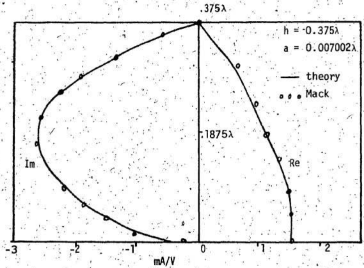


Figure 3.14 Current distributions along three-quarter wave center-fed dipole antenna, using even Walsh functions as weighting functions.

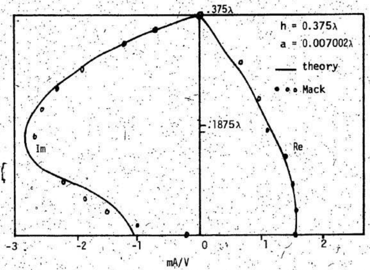


Figure 3.15 Current distributions along three-quarter wave center-fed dipole antenna, using cosine functions as weighting functions.

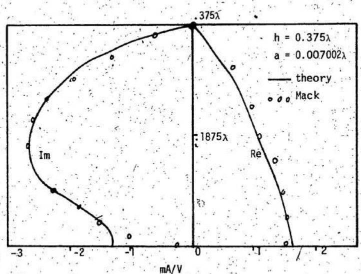


Figure 3.16 Current distributions along three-quarter wave center-fed dipole antenna, using picewise-sinusoidal functions as weighting functions.

Figure 3.16 shows the current distributions along the dipole calculated by the proposed method using 4 basis functions. The weighting functions used in obtaining the results of figure 3.16 are 4 piece-wise-sinusoidal functions. The results of the proposed method are in close agreement with those of Mack for the real part of the current, but they differ for the imaginary part. This deviation in the results may be due to the numerical accuracy of the integration subroutines.

c) One-and-a-quarter Wave Dipole

Current distributions along a one-and-a-quarter wave dipole are studied. A comparison is made between the results of the proposed methods and the measured results of Mack as taken from Popović[18].

Figure 3.17 shows the distributions of currents along the dipole calculated by the proposed method using 16 basis functions. The weighting functions used in obtaining the results of figure 3.17 are 16 even Walsh functions. The results are in good agreement with those of Mack.

Figure 3.18 shows the distribution of currents along the dipole calculated by the proposed method using 4 basis functions. The weighting functions used in obtaining the results of figure 3.18 are 4 cosine functions. The results are in close agreement with those of Mack.

Figure 3.19 shows the current distributions of the dipole calculated by the proposed method using 4 basis functions. The weighting functions used in obtaining the results of figure 3.18 are 4 piece-wise sinusoidal functions. The results show close agreement with those of

Mack.

d) Two-and-a-half Wave Dipole

Current distributions along a two-and-a-half wave dipole are studied. A comparison is made between the results obtained by the proposed method and those obtained by Silvester[21].

Figure 3.20 shows the current distributions of the dipole calculated by the proposed method using 16 basis functions. The weighting functions used to obtain the results of figure 3.20 are 16 even Walsh functions. It is seen that the currents obtained by the proposed method are smaller than those of Silvester.

Figure 3.21 shows the current distributions along the dipole calculated by the proposed method using 4 basis functions. The weighting functions used in obtaining the results of figure 3.21 are 4 cosine functions. The results are in close agreement with those of Silvester.

Figure 3.22 shows the current distributions along the dipole calculated by the proposed method using 4 basis functions. In obtaining the results of figure 3.22, the weighting functions are 4 piece-wise sinusoidal functions. It is seen that the results are far from those of Silvester. This difference may be due to the numerical accuracy of the integration subroutines when larger values of dipole lengths are considered. Also, the number of weighting functions used in this case may have some effect on the accuracy of the results.

3.2.8.3 Computer Core Storage Requirements and Running Time

Five computer programs PW1, PW2, PC1, PC2 and PP1 have been

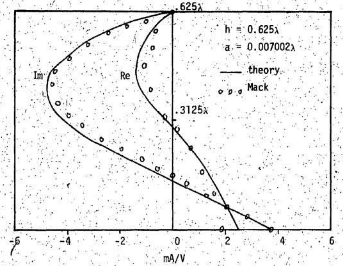


Figure 3.17 Current distributions along one-and-a-quarter wave center-fed ipole antenna, using even Walsh functions as weighting functions.

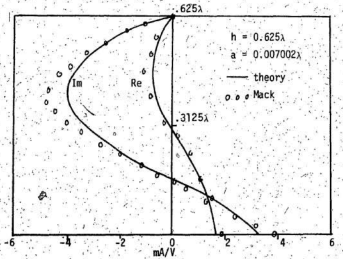


Figure 3.18 Current distributions along one-and-a-quarter wave center-fed dipole antenna, using cosine functions as weighting functions.

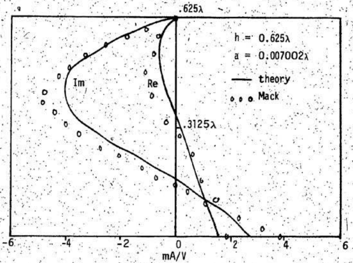


Figure 3.19 Current distributions along one-and-a-quarter wave center-fed dipole antenna, using picewise-sinusoidal functions as weighting functions.

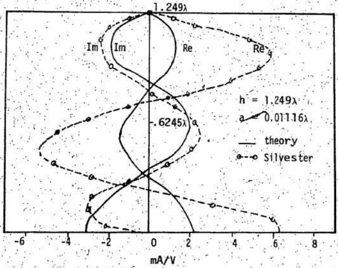


Figure 3.20 Current distributions along two-and-a-half wave center-fed dipole antenna, using even Walsh functions as weighting functions.

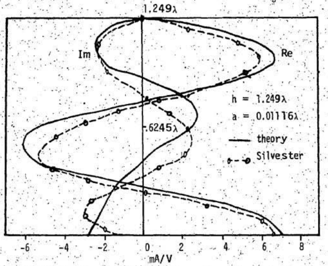


Figure 3.21 Current distributions along two-and-a-half wave center-fed dipole antenna, using cosine functions as weighting functions.

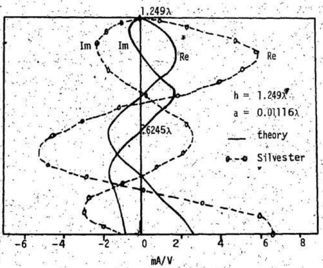


Figure 3.22 Current distributions along two-and-a-half wave center-fed dipole antenna, using picewise-sinusoidal functions as weighting functions.

developed to study a center-fed dipole antenna. These programs have been written for the PORTRAN H compiler of an International Business Machine (IBM 370/158). The input data consists of a list of parameters. These parameters are the antenna length, the antenna radius, the frequency, the permittivity of the medium, the inner and the outer radii of the magnetic frill current. These computer programs are useful for calculating the current distribution, input impedance and the input admittance for the dipole. In order to solve equation [3.54] the exponential terms have been divided into real and imaginary parts. The basis functions are even Walsh functions for each of the five programs.

a) The Computer Program PW1

This program calculates dipole characteristics with even Walsh functions as weighting functions. The real part of equation [3.54] contains the integration of sin x/x which is not in closed form. An approximation for the terms sin x/x and cos x/x is made using the power series, taking five terms of the series. The integral operations have been carried out[31] for both single and double integrals. More details on these integrations are discussed in Appendix D. Also, a listing of the computer program PW1 is presented in Appendix E. This program requires 104K of storage for compilation and 108K for execution of one problem. The central processing unit (CPU) time for compilation is 11.4 sec. and 3.6 sec. for execution. This program is useful for antenna lengths up to 0.75λ.

b) The Computer Program PW2

This program calculates dipole characteristics with even Walsh functions as weighting functions. IBM library subroutines for integration have been used. This program requires 108K of the core storage for compilation and 172K for execution of one problem. The CPU time for compilation is 7.8 sec. and for execution is 124.8 sec. This program was used for antenna of lengths above 0.75λ and up to 2.498λ.

Since the execution CPU time of the computer program PW2 is more than 30 times that of the program PW1, it is evident that the approximate subroutines for integration are much faster than those of the IBM library. But these approximate subroutines are accurate only for antenna lengths up to 0.75λ.

c) The Computer Program PC1

This program is developed to calculate antenna characteristics, using cosine functions as weighting functions. The sin x/x terms are approximated using a power series and the integrations are accordingly determined. Since the imaginary part of equation [3.54] contains two cosine functions, the approximation is of higher order than that of the real part. It was found that the results obtained using these high order approximation are poor. Hence, the power series subroutines for integration are used to determine the real part of equation [3.54] and IBM library subroutines for integration are used for the imaginary part of the equation. A partial listing of the program PC1 is presented in Appendix E. This program requires 104K of the core storage for compilation and 180K for execution of one problem. The CPU time required for compilation is 15.4 sec. and for execution is 37.6 sec. This program can be used for antenna lengths up to 0.75λ.

d) The Computer Program PC2

This program is used to calculate antenna characteristics using cosine functions as weighting functions. The integrations of equation [3.54] are carried out using IBM library subroutines. The core storage required to solve one problem by this program is 108K for compilation and 172K for execution. The required CPU time is 7.9 sec. for compilation and 68.4 sec. for execution. This program was used for antenna of lengths up to 2.498λ.

Since the time required for execution of one problem using the program PC2 is about twice that of PC1, the saving in time between the approximate subroutines and IBM library subroutines for integration is not that significant.

e) <u>The Computer Program PP1</u>

This program is used to calculate antenna characteristics using picewise-sinusoidal functions for the weighting functions. IBM library subroutines are used for the integrations of equation [3.54]. The required core storage to solve one problem is 156K for compilation and 216K for execution. The CPU time is 37.9 sec. for compilation and for execution it is 304.6 sec. This program was used for antenna lengths up to 2.498λ. A partial listing of the program PP1 is presented in Appendix E. The reason for the relatively enormous CPU time required for this program is that it contains a large number of integrations compared to the other four programs because of the nature of the piece-wise sinusoidal functions.

CHAPTER IV

PARAMETRIC ANALYSIS

4.1 General

In the process of analyzing center-fed linear dipole antennas, a number of variables are considered in determining antenna characteristics. These include the properties of the antennas as well as those of the feeding source. The antenna under analysis is assumed to be perfect conductor immersed in a perfect insulator, which is a free space. A perfect conductor is one which has infinite conductivity. Most actual conductors have a finite conductivity, however, the actual conductivity may be very large and for many practical applications it is useful to assume it to be infinite.

Parametric studies are made to determine the effect of 1) antenna half length, 2) antenna radius and 3) the ratio of outer to inner diameter of the feeding source, i.e., the magnetic frill current. The effect of each of these variables on the antenna characteristics is studied. For these studies even Walsh functions have been used for both basis and weighting functions. For half lengths up to 0.4λ, the computer program PW1 is used and for lengths above 0.4λ, the computer program PW2 is used.

4.2 Practical Range of Input Data

In order to analyze a linear dipole antenna, the practical range of the parameters under study has to be determined. Since the analyzed antenna is thin, its radius, a, must be much smaller than the wave length; it is taken in this analysis to satisfy the inequality

$$k\,a \leq 0.06 \quad \text{or} \quad a \leq 0.01\,\lambda \qquad [4.1]$$

where k is the wave number defined by $\omega\sqrt{\mu\varepsilon}$.

Thin dipole antennas are commonly driven from coaxial lines with characteristic impedances between 25 and 100 ohms, that is, b/a ranges between 1.5 to 5.3, so that

$$k\,b \leq 0.32 \quad \text{or} \quad b \leq 0.53\,\lambda \qquad [4.2]$$

where a and b represent the inner and outer radii of the coaxial cable respectively.

The antennas are considered to be relatively short with half length $h < \lambda$ or in other words with $k\,h < 2\pi$. In this analysis h is taken up to $1.01\,\lambda$.

4.3 Parametric Studies

The effect of the above parameters has been studied within their practical range given in Section 4.2. Different characteristics of the antenna have been studied such as input admittance, input impedance and characteristic impedance of the antenna. Also, the effect of the feeding coaxial-cable characteristic impedance on the antenna input impedance has been investigated.

4.3.1 Effect of Antenna Length

Typical graphs showing the effect of the antenna length are shown in Figure 4.1 and 4.2. In Figure 4.1, the input admittance for different lengths of dipole antennas are shown. Sixteen even Walsh functions are used for both basis and weighting functions. For half

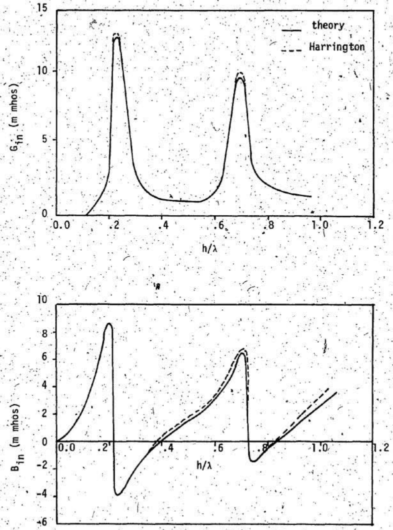


Figure 4.1 Input admittance of center-fed dipole antennas of radius a = h/72.4, for different lengths 2h of dipoles.

lengths up to 0.4 λ, the approximate integration subroutines in the computer program PW1 are used, and for half lengths above 0.4 λ and up to 1.01 λ, IBM subroutines for integration in conjunction with the computer program PW2 are used. The results are in good agreement with the results obtained by Harrington[6], except for some parts of the input susceptance. The difference in the suceptance is expected because the incident field in Harrington's calculations is different from that of the magnetic frill field. In the calculations carried out by Harrington the current is approximated by piece-wise linear functions which gives faster convergence than a step approximation. Harrington used 32 segments for his calculations.

Figure 4.2 shows the effect of antenna lengths on its characteristic impedance[8] given by

$$Z_c = 120 \left[\ln \left(\frac{h}{a}\right) - 1 - \frac{1}{2} \ln \left(\frac{2h}{\lambda}\right)\right] \qquad [4.3]$$

The range of the parameter h, is taken the same as in the previous case. It is shown that the characteristic impedance of the antenna increases with an increase in its length.

4.3.2. Effect of Antenna Radius

Typcial graphs showing the effect of antenna radius are shown in figures 4.3, 4.4 and 4.5. In figure 4.3 the effect of the antenna radius on its characteristic impedance is presented. Figure 4.3 indicates that the characteristic impedance of the antenna Z_c, decreases with an increase in antenna radius a. The characteristic impedance Z_c is given by equation [4.3].

Figure 4.4 shows the effect of the antenna radius for a half wave

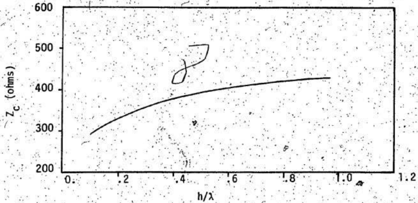


Figure 4.2 Characteristic impedance of center-fed dipole antennas of radius a = 0.007022λ, for different lenghts 2h of dipoles.

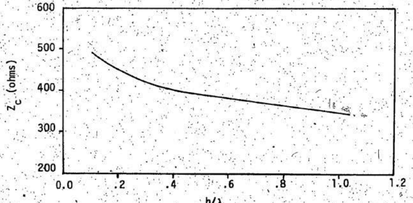


Figure 4.3 Characteristic impedance of center-fed dipole antemnas of radius a = h/72.4, for different lengths 2h of dipoles.

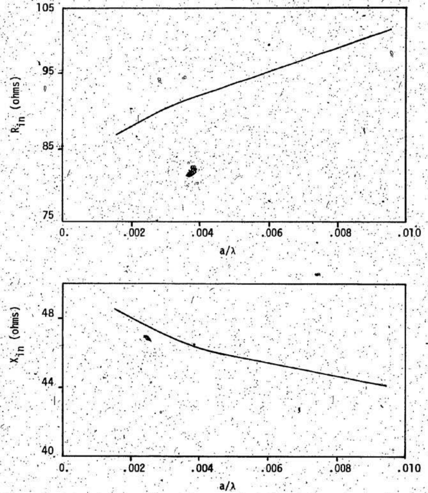


Figure 4.4 Input impedance of half wave center-fed dipole antennas for different radii a of dipoles.

dipole antenna on its input impedance. The figure indicates that the input resistance of the antenna R_{in}, increases with an increase of the antenna radius a , while the input reactance X_{in}, decreases with an increase of the antenna radius.

Figure 4.5 shows the effect of the antenna radius on the distributions of current along a half wave dipole antenna. It is seen that the slope of the distribution is the same but the magnitude is different. For antenna radius a equals to 0.009525 λ, the magnitude of the current is larger than that when a equals 0.001588 λ. This shows that the magnitude of current along the half wave dipole antenna increases with an increase of the antenna radius.

For the previous three examples the antenna radius is within the practical range for electrically thin antennas satisfying the condition

$$a \leq .01 \lambda \qquad [4.4]$$

4.3.3 Effect of Feeding Line Dimensions

The effect of the ratio of the outer to inner radius of the feeding coaxial-cable has also been studied. Figure 4.6 shows the effect of the ratio between the outer radius of the coaxial-cable b, and its inner radius a , on the cable characteristic impedance. The figure indicates that an increase of the ratio b/a causes an increase in the cable characteristic impedance.

Figure 4.7 shows the effect of changing the feeding line characteristic impedance on the half wave dipole antenna input impedance. It is seen that an increase in the characteristic impedance of the feeding line causes a decrease in the input resistance of the antenna and an increase in the input reactance of the antenna.

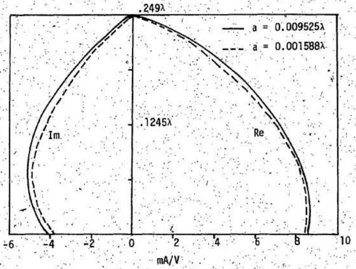


Figure 4.5 Current distributions along half wave center-fed dipole antennas, for different radii a.

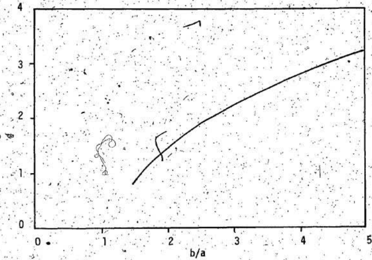


Figure 4.6 Characteristic impedance of the feeding coaxial-cables, for different ratios of the outer to the inner radius b/a.

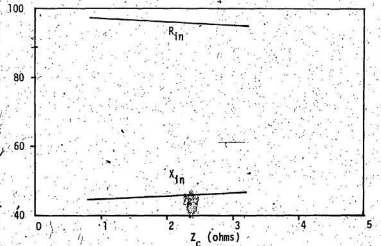


Figure 4.7 Input impedance of half wave dipole antenna center-fed by coaxial-cables of different characteristic impedances.

CHAPTER V

CONCLUSIONS

This thesis has presented an investigation of a method of analysis of linear antenna systems. The proposed method of analysis, which is discussed by Walsh[27-30], eliminates the need of imposing boundary conditions on the solution of an integral equation by the use of an intermediate differential equation. The proposed method has been studied using the method of moments in conjunction with three different sets of weighting functions.

General formulations were given in the analysis of antennas in chapter 2. In this thesis, the utilization of these formulations is confined to linear antenna systems. However, it should be noted that they can also be applied to other antenna systems, particularly those for which the differential operator is invertible in a closed form.

Two basic integral equations used in wire antenna investigations have been discussed in Section 3.2.3. Although Hallén's integral equation is favored by some researchers for wire antenna problems, Pocklington's integral equation is more general and therefore more flexible. This is because Hallén's equation exclusively uses a fictitious delta-gap to represent the feed-point. Whereas Pocklington's equation permits one to use a wider range of sources, such as the magnetic frill current model of a true physical feed-point configuration. The proposed method has the simplicity of Hallén's formulation and the flexibility of Pocklington's equation.

The general method of moments has been discussed in section 3.2.5. This method is the unifying concept in the numerical solution of certain electromagnetic field problems. In essence, the method of moments provides a means of accurately approximating an integral equation with a system of simultaneous linear algebraic equations. There are two special cases resulting from a particular choice of weighting functions. The first special case is Galerkin's method which uses weighting functions that are the same as the basis functions. Galerkin's method, however, generally requires two integrations in order to obtain the matrix element. One can reduce the number of integrations required in the calculation of the matrix by choosing Dirac-delta functions as weighting functions. This second special case of the method of moments is usually referred to as point-matching. That is, a boundary condition is satisfied at a discrete number of points rather than continuously over some surfaces. Point-matching is a more straightforward approach than Galerkin's method and is sufficiently adequate for many problems. It tends, however, to be a slower converging method.

A very important decision to be made in formulating a problem is in choosing the basis functions of the current. Thus, considerable discussion has been devoted in Section 3.2.5.1 to bases. Two basic philosophies as to the choice of basis functions have been presented. That is, entire-domain bases and sub-domain bases with the latter being used in the present investigation.

Another important decision to be made in formulating a problem is in choosing the weighting functions. The principles in choosing the weighting functions have been discussed in Section 3.2.6. These functions must be linearly independent and should be able to approximate the corresponding exact functions and the resulting matrix should be well conditioned.

In this thesis three sets of weighting functions have been considered. Sets of even Walsh functions, cosine functions, and piece-wise-sinusoidal functions have been used.

The input impedance of a center-driven dipole antenna has been obtained for two different lengths applying the proposed method with the three sets of weighting functions. A comparison has been made between the results of the proposed method and those of Thiele[25] and Srivastava[24]. The results obtained by Thiele are based on the solution of Pocklington's equation applying the point-matching technique. In the first example he used cosine functions as entire-domain basis functions, while he uses pulse functions as sub-domain basis functions in the second example. Srivastava obtained his results using the same method of analysis and the same basis functions used in this investigation, but in conjunction with the point-matching. The results of the proposed method (shown in figures 3.5 to 3.10) show good agreement with those of Thiele and Srivastava. Of the three sets of weighting functions used, the picewise-sinusoidal functions showed the fastest rate of convergence. However, this advantage is offset by the complicated form of the resulting integral.

The current distributions of a center-fed dipole antenna have been obtained for four different antenna lengths. In the first three examples shown in figures 3.11 to 3.19, a comparison has been made between the results of the proposed method using the three sets of weighting functions and those measured by Mack taken from Popović[18] and Silvester[22]. In the fourth example a comparison has been made with the results obtained by Silvester[21].

The effect of antenna length on its input admittance has been presented in Section 4.3.1. A comparison has been made between the results of the proposed method using even Walsh functions for the weighting functions and those of Harrington[6]. Also, the effect of antenna length on its characteristic impedance has been discussed in Section 4.3.1.

The effect of antenna radius on its characteristic and input impedance has been discussed in Section 4.3.2. It is shown that the characteristic impedance decreases with an increase in antenna radius. For a half wave dipole antenna the input resistance of the antenna increases and the input reactance decreases with an increase in the antenna radius.

From the discussion of numerical results in Section 3.2.8, it can be concluded that the use of Walsh functions for weighting functions means less consumed computer time than using cosine or piecewise-sinusoidal functions. Hence, for the parametric studies presented in Chapter 4, Walsh functions were used as weighting functions.

An improvement in results may be achieved either by increasing the number of used basis and weighting functions or by increasing the accuracy of integration routines. This improvement has not been done in this thesis because of the relatively large amount of the required computer time. Moreover, the use of piecewise-sinusoidal functions as weighting functions gives faster convergence than using Walsh or cosine functions. However, the use of piecewise-sinusoidal functions is costly in computer time, unless more efficient numerical techniques of integration can be found. Finding the best numerical integral technique is a mathematical problem that remains to be investigated.

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APPENDIX A

THE ELECTRICAL FIELDS

A.1 The Axial Electrical Field[24]

The axial electrical field E_z (x) can be expressed as follows using equation [2.20a]

$$E_z(\underline{x}) = S_o \{ T[\underline{J}(\underline{x})]\}$$
$$= \{ S [\underline{J}(\underline{x})]\}_z * K_o(\underline{x}) \qquad [A.1]$$

where

$$K_o(\underline{x}) = \frac{\exp[-jkr(\underline{x})]}{j4\pi\omega\varepsilon\, r(\underline{x})} \qquad [A.2]$$

$$\{S[\underline{J}(\underline{x})]\}_z = \frac{\partial^2 J_x}{\partial z\, \partial x} + \frac{\partial^2 J_y}{\partial z\, \partial y} + \frac{\partial^2 J_z}{\partial z^2} + k^2 J_z \qquad [A.3]$$

For cylindrical coordinates (z,ρ,ϕ) equation [A.3] becomes

$$\{S[\underline{J}(z,\rho,\phi)]\}_z = \frac{1}{\rho}\frac{\partial^2(\rho J_\rho)}{\partial z\, \partial \rho} + \frac{\partial^2 J_\phi}{\rho \partial z\, \partial \phi} + \frac{\partial^2 J_z}{\partial z^2} + k^2 J_z \qquad [A.4]$$

For the linear current element of equation [3.1], equation [A.4] is reduced to

$$\{S[\underline{J}(z,\rho)]\}_z = [\frac{d^2}{dz^2} + k^2]\, J_z$$
$$= [\frac{d^2}{dz^2} + k^2]\, \frac{I_z(z)}{2\pi a}\, \delta(\rho-a)$$
$$= S_z[I_z(z)]\, \delta\frac{(\rho-a)}{2\pi a}, \quad z \leq h \qquad [A.5]$$

where $S_z = (\frac{d^2}{dz^2} + k^2)$ [A.6]

Applying equation [A.5], equation [A.1] can be expressed as

$$E_z\ (z,\rho,\phi) = S_z\ [I_z\ (z)]\ \frac{\delta(\rho-a)}{2\pi a} * K_o\ (x) \qquad [A.7]$$

For equation [A.2]

$$r(x) = (x^2 + y^2 + z^2)^{\frac{1}{2}} \qquad [A.8]$$

hence

$$r(x-x')^2 = \{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{1}{2}}$$

$$= \{(z-z')^2 + \rho^2 + \rho'^2 - 2\rho\rho'\cos(\phi-\phi')\}^{\frac{1}{2}} \qquad [A.9]$$

Integrating equation [A.7] with respect to ρ' and ϕ'

$$E_z\ (z,\rho,\phi) = S_z\ [I_z] * K_z\ (z,\rho,\phi) \qquad [A.10]$$

in which

$$K_z\ (z,\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} K_o\ (z,\rho,\phi-\phi')\ d\phi' \qquad [A.11]$$

where K_o is given by [A.2] with

$$r(z,\rho,\phi) = \{z^2 + \rho^2 + a^2 - 2\rho a\cos\phi\}^{\frac{1}{2}} \qquad [A.12]$$

It is evident that equation [A.11] is independent of ϕ, hence equation [A.7] may be rewritten as

$$E_z\ (z,\rho) = S_z[I_z\ (z)] * K_z\ (z,\rho) \qquad [A.13]$$

where

$$K_z\ (z,\rho) = \frac{1}{2\pi}\int_0^{2\pi} K_o\ (z,\rho,\phi)\ d\phi \qquad [A.14]$$

A.2 The Radial Electrical Field [24]

The radial electrical field E_ρ can be obtained from equation [2.20a]

$$E_\rho\ (x) = \{S[\ J\ (x)\ *\ K_o\ (x)\ ]\}_\rho \qquad [A.15]$$

where $K_o\ (x)$ is given by equation [A.2] and

$$\{S[J\ (x)\ *\ K_o\ (x)]\}_\rho = \frac{\partial}{\partial\rho}\left[\frac{\partial[\rho J_\rho * K_o\ (x)]}{\rho\partial\rho}\right] + \frac{\partial}{\partial\rho}\left[\frac{\partial\ [J_\phi * K_o\ (x)]}{\rho\partial\phi}\right] + \frac{\partial^2}{\partial\rho\ \partial z}[\ J_z\ *\ K_o\ (x)] + k^2\ J_\rho \qquad [A.16]$$

Applying equation [A.16] for the linear current element of equation [3.1], equation [A.15] becomes

$$E_\rho(x) = \frac{\partial^2}{\partial\rho\ \partial z}\ [J_z\ *\ K_o\ (x)] = \frac{\partial^2}{\partial\rho\ \partial z}\ [I_z\ (z)\ \frac{\delta\ (\rho-a)}{2\pi a}\ *\ K_o\ (z)\ ] \qquad [A.17]$$

Applying the convolution with respect to x and y, as carried out in equation [A.3], equation [A.17] reduces to

$$E_\rho(z,\rho) = \frac{\partial^2}{\partial\rho\ \partial z}\ [\ I_z\ (z)\ *\ K_z\ (z,\rho)] \qquad [A.18]$$

where $K_z\ (z,\rho)$ is given by equation [A.14], with the restriction of the convolution to z only. Since the convolution and differentiations commute, equation [A.18] can be rewritten as

$$E_\rho(z,\rho) = I_z(z) * \frac{\partial^2}{\partial\rho\,\partial z} K_z(z,\rho) \qquad [A.19]$$

and $$\frac{\partial^2}{\partial\rho\,\partial z} [K_z(z,\rho)] = -S_z [K_\rho(z,\rho)] \qquad [A.20]$$

where S_z is defined by equation [A.6] and,

$$K_\rho(z,\rho) = \frac{1}{2\pi} \int_0^{2\pi} \frac{z(\rho - a\cos\phi)}{\rho^2 + a^2 - 2\rho a\cos\phi} K_0(z,\rho,\phi)\, d\phi \qquad [A.21]$$

From equations [A.19], [A.20] and [A.21]

$$E(z,\rho) = I_z(z) * \{-S_z [K_\rho(z,\rho)]\}$$

$$= -S_z [I_z(z)] * K_\rho(z,\rho) \qquad [A.22]$$

A.3 The Angular Electrical Field[24]

The angular electrical field E_ϕ can be obtained from equation [2.20a]

$$E_\phi(x) = \{S[J(x)]\}_\phi * K_0'(x) \qquad [A.23]$$

where

$$\{S[J(x)]\}_\phi = \frac{1}{\rho}\frac{\partial}{\partial\phi}\left[\frac{\partial(\rho J_\rho)}{\rho\partial\rho}\right] + \frac{1}{\rho^2}\frac{\partial^2 J_\phi}{\partial\phi} + \frac{1}{\rho}\frac{\partial^2 J_z}{\partial\phi\,\partial z} + k^2 J_\phi \qquad [A.24]$$

For the linear current element of equation [3.1], equation [A.24] becomes

$$\{S[J(z,\rho)]\}_\phi = \frac{1}{\rho}\frac{\partial^2 J_z}{\partial\phi\,\partial z}$$

$$= \frac{1}{\rho} \frac{\partial^2}{\partial\phi\, \partial z} \left[\frac{I_z(z) \quad \delta(\rho - a)}{2\pi a} \right]$$

$$= 0 \qquad [A.25]$$

Applying equation [A.25] to equation [A.23], the latter will be

$$E_\phi(x) = 0$$

APPENDIX B

LINEAR OPERATORS AND WEIGHTING FUNCTIONS

B.1 The Linear Operator [6]

Consider an equation of inhomogeneous type

$$L\ (f)\ =\ g \qquad [B.1]$$

Where L is a linear operator, g is the excitation or source (known function), and f is the field or response (unknown to be determined). The problem of analysis involves determining f when L and g are given.

Given a deterministic problem of the form [B.1], it is desired to identify the operator L, its domain (the functions f on which it operates), and its range (the functions g resulting from the operation). In addition to determining the domain and range of the operator, it is often necessary to formulate an inner product $<f,g>$, which will be discussed in the next section.

Properties of the solution of [B.1] depend on properties of the operator L. The adjoint operator L^a and its domain are defined by

$$<Lf,g>\ =\ <f,L^a g> \qquad [B.2]$$

for all f in the domain of L. An operator is self adjoint if $L^a = L$ and the domain of L^a is that of L. An operator is real if Lf is real whenever f is real. An operator is positive definite if

$$<f^*\ ,\ Lf>\ >\ 0 \qquad [B.3]$$

for all $f \neq 0$ in its domain, where * denotes complex conjugate. It is positive semidefinite if > is replaced by $\geq$ in [B.3], negative definite if > is replaced by < in [B.3].

If the solution of L (f) = g exists and is unique for all g, then the inverse operator L^{-1} exists such that

$$f = L^{-1} (g) \qquad [B.4]$$

If g is known, then [B.4] represents the solution to the original problem. However, [B.4] is itself an inhomogeneous equation for g if f is known, and its solution is $L(f) = g$. Hence, L and L^{-1} form a pair of operators, each of which is the inverse of the other.

B.2 Inner Product [16]

One usually needs an inner product $<f,g>$, which is a scalar defined to satisfy

$$<f,g> = <g,f>^* \qquad [B.5]$$

$$<\alpha f + \beta g,h> = \alpha <f,h> + \beta <g,h> \qquad [B.6]$$

$$\begin{aligned} <f^*,f> &> 0, \quad \text{if } f \neq 0 \\ &= 0, \quad \text{if } f = 0 \end{aligned} \qquad [B.7]$$

The norm of a function is denoted $||f||$ and defined by

$$||f|| = [< f,f^* >]^{\frac{1}{2}} \qquad [B.8]$$

It corresponds to Euclidean vector concept of length. The metric d of two functions is

$$d(f,g) = ||f-g|| \qquad [B.9]$$

and corresponds to the Euclidean vector concept of distance between two points.

B.3 Weighting Functions [6]

Let f in equation [B.1] be expanded in a series of functions $f_1, f_2, f_3, \ldots$ in the domain of L, as

$$f = \sum_n \alpha_n f_n \tag{B.10}$$

where the α_n are constants. The f_n are called basis functions. Substituting [B.10] into [B.1], and using the linearity of L, one has

$$\sum_n \alpha_n L(f_n) = g \tag{B.11}$$

Now define a set of weighting functions, or testing functions, $W_1, W_2, W_3, \ldots$ in the range of L, and take the inner product of [B.11] with each W_m. The result is

$$\sum_n \alpha_n < W_m, Lf_n > = < W_m, g > \tag{B.12}$$

$m = 1,2,3,\ldots$. This set of equations can be written in matrix form as

$$[l_{mn}]\,[\alpha_n] = [g_m] \tag{B.13}$$

where

$$[l_{mn}] = \begin{bmatrix} <W_1, Lf_1> & <W_1, Lf_2> \ldots & <W_1, Lf_n> \\ <W_2, Lf_1> & <W_2, Lf_2> \ldots & <W_2, Lf_n> \\ \vdots & \vdots & \vdots \\ <W_m, Lf_1> & <W_m, Lf_2> \ldots & <W_m, Lf_n> \end{bmatrix} \tag{B.14}$$

$$[\alpha_n] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}, \quad [g]_m = \begin{bmatrix} < W_1, g> \\ < W_2, g> \\ \vdots \\ < W_m, g> \end{bmatrix} \tag{B.15}$$

If the matrix $[l]$ is nonsingular its inverse $[l^{-1}]$ exists. The α_n are then given by

$$[\alpha_n] = [l_{mn}^{-1}] \; [g_m] \qquad [B.16]$$

and the solution for f is given by [B.10]. For concise expression of this result, define the matrix of functions

$$[\tilde{f}] = [f_1 \; f_2 \; f_3 \ldots\ldots] \qquad [B.17]$$

and $$f = [\tilde{f}_n] \; [\alpha_n] = [\tilde{f}_n] \; [l_{mn}^{-1}] \; [g_m] \qquad [B.18]$$

This solution may be exact or approximate, depending upon the choice of the f_n and W_m. If n and m are finite, the matrix is of finite order, and it can be inverted by known computational algorithms. The W_m should be linearly independent and chosen so that the products $\langle W_m, g\rangle$ depend on relatively independent properties of g. Some additional factors which affect the choice of f_n and W_m are: a) the accuracy of the desired solution, b) the ease of evaluation of the matrix elements, c) the size of the matrix that can be inverted, and d) the realization of a well-conditioned matrix $[l]$.

APPENDIX C

GENERATION OF WALSH FUNCTIONS

Consider as an example eight Walsh functions. Sampling of these functions in figure [C.1] at 8 equidistant points results in the (8x8) matrix shown in figure [C.2]. In general an (NxN) matrix would be obtained. Such matrices are denoted by $H_w(n)$, $n = \log_2 N$, since they can be obtained by reordering the rows of a class of matrices called Hadmard matrices[1].

Let u_i and v_i denote the i^{th} bit in the binary representations of the integers u and v respectively, that is

$$(u)_{decimal} = (u_{n-1}\ u_{n-2} \cdots u_1 u_o)_{binary} \qquad [C.1]$$

and $$(v)_{decimal} = (v_{n-1}\ v_{n-2} \cdots v_1 v_o)_{binary} \qquad [C.2]$$

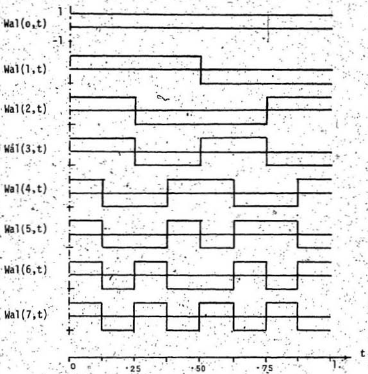
Then the elements $h_{uv}(w)$ of $H_w(n)$ can be generated as follows:

$$h_{uv}(w) = (-1)^p \qquad [C.3a]$$

where

$$p = \sum_{i=0}^{n-1} r_i(u)\, v_i \quad , \quad u,v = 0,1, \ldots N-1 \qquad [C.3b]$$

$$\left.\begin{aligned} r_o(u) &= u_{n-1}\,, \\ r_1(u) &= u_{n-1} + u_{n-2}\,, \\ r_2(u) &= u_{n-2} + u_{n-3}\,, \\ &\vdots \\ r_{n-1}(u) &= u_1 + u_0 \end{aligned}\right\} \qquad [C.4]$$



$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1
\end{bmatrix}
$$

Figure C.1 Walsh-ordered continuous Walsh functions, N = 8
Figure C.2 Walsh-ordered discrete Walsh functions , N = 8

As an example, the first eight Walsh functions can be obtained using the information in table [C.1].

Table [C.1] The binary representation of the decimal numbers.

$i_{decimal}$	i_{binary}
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

A computer program was developed to generate Walsh functions. This program is presented in Appendix E as a subroutine.

APPENDIX D

APPROXIMATE EXPRESSIONS

FOR

SOME INTEGRALS

D.1 Approximation of Some Functions

The composite kernel K_{zac} given by equation [3.19] in conjunction with [3.26] can be rewritten in the sense of real and imaginary parts. The real part is given by

$$\mathrm{Re}\,K_{zac} = -\frac{k}{4\pi\omega\varepsilon}\left\{\frac{\sin\left[k[(z-z')^2+a^2]^{\frac{1}{2}}\right]}{k\,[(z-z')^2+a^2]^{\frac{1}{2}}}\right.$$

$$-\frac{\sin\left[k[(z+h)^2+a^2]^{\frac{1}{2}}\right]}{k\,[(z+h)^2+a^2]^{\frac{1}{2}}}\;\frac{\sin\left[k\,(h-z')\right]}{\sin\,(2kh)}$$

$$\left.-\frac{\sin\left[k[(z-h)^2+a^2]^{\frac{1}{2}}\right]}{k\,[(z-h)^2+a^2]^{\frac{1}{2}}}\;\frac{\sin\left[k\,(h+z')\right]}{\sin\,(2kh)}\right\} \qquad [D.1a]$$

$$\mathrm{Im}\,K_{zac} = -\frac{k}{4\pi\omega\varepsilon}\left\{\frac{\cos\left[k[(z-z')^2+a^2]^{\frac{1}{2}}\right]}{k\,[(z-z')^2+a^2]^{\frac{1}{2}}}\right.$$

$$-\frac{\cos\left[k[(z+h)^2+a^2]^{\frac{1}{2}}\right]}{k\,[(z+h)^2+a^2]^{\frac{1}{2}}}\;\frac{\sin\left[k\,(h-z')\right]}{\sin\,(2kh)}$$

$$\left.-\frac{\cos\left[k[(z-h)^2+a^2]^{\frac{1}{2}}\right]}{k\,[(z-h)^2+a^2]^{\frac{1}{2}}}\;\frac{\sin\left[k(h+z')\right]}{\sin\,(2kh)}\right\} \qquad [D.1b]$$

To calculate the matrix entries of equation [3.52], the integration will not be available in a closed form as seen from equations [D.1]. To find an approximate expression for these integrals the functions sin x/x and cos x/x can be approximated by considering the first five terms of power series[2], where x takes different expressions according to equations [D.1]. The approximations are given by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \qquad [D.2a]$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} \qquad [D.2b]$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \qquad [D.3a]$$

$$\frac{\cos x}{x} = \frac{1}{x} - \frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \frac{x^7}{8!} \qquad [D.3b]$$

D.2 <u>Some Useable Integrals</u>

$$I_1 = \int_A^B \frac{\sin\left[k[(z+a_1)^2 + b_1^2]^{\frac{1}{2}}\right]}{k\left[(z+a_1)^2 + b_1^2\right]^{\frac{1}{2}}}\, dz \qquad [D.4]$$

substituting $x = z + a_1$

then $dx = dz$

at $z = A$, $x = A + a_1$, and at $z = B$, $x = B + a_1$

$$I_1 = \int_{A+a_1}^{B+a_1} \frac{\sin\left[k[x^2 + b_1^2]^{\frac{1}{2}}\right]}{k\left[x^2 + b_1^2\right]^{\frac{1}{2}}}$$

$$= \int_{A+a_1}^{B+a_1} [c_1 + c_2 u^2 + c_3 u^4 + c_4 u^6 + c_5 u^8]\, dx \qquad [D.5]$$

where

$$\left.\begin{aligned} u &= [x^2 + b_1^2]^{\frac{1}{2}} \\ c_1 &= 1 \;,\; c_2 = -k^2/3! \;,\; c_3 = k^4/5! \\ c_4 &= -k^6/7! \;,\; c_5 = k^8/9! \end{aligned}\right\} \qquad [D.6]$$

$$I_1 = \int_{A+a_1}^{B+a_1} [c_{11} + c_{22} x^2 + c_{33} x^4 + c_{44} x^6 + c_{55} x^8]\, dx \qquad [D.7]$$

$$\text{or } I_1 = \left[c_{11} x + c_{22} \frac{x^3}{3} + c_{33} \frac{x^5}{5} + c_{44} \frac{x^7}{7} + c_{55} \frac{x^9}{9} \right]_{A+a_1}^{B+a_1} \qquad [D.8]$$

where

$$\left.\begin{array}{l} c_{11} = c_1 + c_2\, b_1^2 + c_3\, b_1^4 + c_4\, b_1^6 + c_5\, b_1^8 \\ c_{22} = c_2 + c_3\, (2b_1^2) + c_4\, (3b_1^4) + c_5\, (4b_1^6) \\ c_{33} = c_3 + c_4\, (3b_1^2) + c_5\, (6b_1^4) \\ c_{44} = c_4 + c_5\, (4b_1^2)\, , \quad c_{55} = c_5 \end{array}\right\} \qquad [D.9]$$

$$I_2 = \int_A^B \int_C^D \frac{\sin\, [k[(x-y)^2 + b_1^2]^{\frac{1}{2}}\,]}{k[(x-y)^2 + b_1^2]^{\frac{1}{2}}}\, dy\, dx \qquad [D.10]$$

Applying equation [D.8]

$$\begin{aligned} I_2 = & \int_C^D \{ c_{11}\, (B-y) + \frac{c_{22}}{3}\, (B-y)^3 + \frac{c_{33}}{5}(B-y)^5 + \frac{c_{44}}{7}(B-y)^7 \\ & \qquad + \frac{c_{55}}{9}(B-y)^9 \}\, dy \\ & - \int_C^D \{ c_{11}\, (A-y) + \frac{c_{22}}{3}\, (A-y)^3 + \frac{c_{33}}{5}\, (A-y)^5 + \frac{c_{44}}{7}(A-y)^7 \\ & \qquad + \frac{c_{55}}{9}\, (A-y)^9 \}\, dy \end{aligned} \qquad [D.11]$$

substituting z_1 = B-y and z_2 = A-y, equation [D.11] reduces to

$$I_2 = -\int_{B-C}^{B-D} \{ c_{11} z_1 + \frac{c_{22}}{3} z_1^3 + \frac{c_{33}}{5} z_1^5 + \frac{c_{44}}{7} z_1^7 + \frac{c_{55}}{9} z_1^9 \} \, dz_1$$

$$+ \int_{A-C}^{A-D} \{ c_{11} z_2 + \frac{c_{22}}{3} z_2^3 + \frac{c_{33}}{5} z_2^5 + \frac{c_{44}}{7} z_2^7 + \frac{c_{55}}{9} z_2^9 \} \, dz_2$$

$$= \left[c_{11} \frac{z_1^2}{2} + \frac{c_{22}}{3} \frac{z_1^4}{4} + \frac{c_{33}}{5} \frac{z_1^6}{6} + \frac{c_{44}}{7} \frac{z_1^8}{8} + \frac{c_{55}}{9} \frac{z_1^{10}}{10} \right]_{B-D}^{B-C}$$

$$+ \left[c_{11} \frac{z_2^2}{2} + \frac{c_{22}}{3} \frac{z_2^4}{4} + \frac{c_{33}}{5} \frac{z_2^6}{6} + \frac{c_{44}}{7} \frac{z_2^8}{8} + \frac{c_{55}}{9} \frac{z_2^{10}}{10} \right]_{A-C}^{A-D}$$

[D.12]

$$I_3 = \int_A^B \frac{\cos [k[(z+a_1)^2 + b_1^2]^{1/2}]}{k[(z+a_1)^2 + b_1^2]^{1/2}} \, dz \qquad \text{[D.13]}$$

putting $x = z+a_1$, equation [D.13] becomes

$$I_3 = \int_{A+a_1}^{B+a_1} \frac{\cos (ku)}{ku} \, dx \qquad \text{[D.14]}$$

where

$$u = [x^2 + b_1^2]^{1/2} \qquad \text{[D.15]}$$

Applying equation [D.3b], equation [D.14] reduces to

$$I_3 = \int_{A+a_1}^{B+a_1} \{ c_1 \frac{1}{u} + c_2 u + c_3 u^3 + c_4 u^5 + c_5 u^7 \} dx$$

$$= c_1 I_{3\cdot1} + c_2 I_{3\cdot2} + c_3 I_{3\cdot3} + c_4 I_{3\cdot4} + c_5 I_{3\cdot5} \qquad [D.16]$$

where

$$\left.\begin{array}{ll} c_1 = \frac{1}{k} , & c_2 = -\frac{k}{2!} \\ c_3 = \frac{k^3}{4!} , & c_4 = -\frac{k^5}{6!} \\ c_5 = \frac{k^7}{8!} & \end{array}\right\} \qquad [D.17]$$

$$I_{3\cdot1} = \int_{A+a_1}^{B+a_1} \frac{1}{u}\, dx = [\ln (x+u)]_{A+a_1}^{B+a_1}$$

$$I_{3\cdot2} = \int_{A+a_1}^{B+a_1} u dx = \frac{1}{2} [xu + b_1^2 \ln (x+u)]_{A+a_1}^{B+a_1}$$

$$I_{3\cdot3} = \int_{A+a_1}^{B+a_1} u^3 dx = \frac{3}{8} [\frac{2}{3} xu^3 + b_1^2 xu + b_1^4 \ln (x+u)]_{A+a_1}^{B+a_1}$$

$$I_{3\cdot4} = \int_{A+a_1}^{B+a_1} u^5 dx = \frac{5}{16} [\frac{8}{15} xu^5 + \frac{2}{3} b_1^2 xu^3 + b_1^4 xu + b_1^6 \ln (x+u)]_{A+a_1}^{B+a_1}$$

$$I_{3\cdot5} = \int_{A+a_1}^{B+a_1} u^7 dx = \int_{A+a_1}^{B+a_1} u^4 u^3 dx = \int_{A+a_1}^{B+a_1} (x^2 + b_1^2)^2 u^3 dx$$

$$= \int_{A+a_1}^{B+a_1} (b_1^4 + 2b_1^2x^2 + x^4)\, u^3\, dx$$

$$= c_{11}\; I_{3\cdot5\cdot1} + c_{22}\; I_{3\cdot5\cdot2} + c_{33}\; I_{3\cdot5\cdot3} \qquad [D.18]$$

$$c_{11} = b_1^4 \quad , \quad c_{22} = 2b_1^2 \quad , \quad c_{33} = 1$$

$$I_{3\cdot5\cdot1} = \int_{A+a_1}^{B+a_1} u^3 dx = I_{3\cdot3}$$

$$I_{3\cdot5\cdot2} = \int_{A+a_1}^{B+a_1} x^2u^3dx = \frac{1}{16}\Big[\frac{8}{3}xu^5 - \frac{2}{3}b_1^2xu^3 - b_1^4xu - b_1^6\ln(x+u)\Big]_{A+a_1}^{B+a_1}$$

$$I_{3\cdot5\cdot3} = \int_{A+a_1}^{B+a_1} x^4u^3\,dx = \frac{3}{128}\Big[\frac{16}{3}x^3u^5 - \frac{8}{3}b_1^2xu^5 + \frac{2}{3}b_1^4xu^3 + b_1^6xu + b_1^8\ln(x+u)\Big]_{A+a_1}^{B+a_1}$$

$$[D.19]$$

$$I_3 = \Big[c_{111}\ln(x+u) + c_{222}\,xu + c_{333}\,xu^3 + c_{444}\,xu^5 + c_{555}\,x^3u^5\Big]_{A+a_1}^{B+a_1} \qquad [D.20]$$

$$c_{111} = c_1 + c_2\left(\frac{1}{2}b_1^2\right) + c_3\left(\frac{3}{8}b_1^4\right) + c_4\left(\frac{5}{16}b_1^6\right) + c_5$$

$$\left\{c_{11}\left(\frac{3}{8}b_1^4\right) + c_{22}\left(\frac{-1}{16}b_1^6\right) + \left(\frac{3}{128}b_1^8\right)\right\}$$

$$\left.\begin{aligned}
c_{222} &= c_2(\tfrac{1}{2}) + c_3(\tfrac{3}{8}b_1^2) + c_4(\tfrac{5}{16}b_1^4) + c_5 \\
&\quad \{c_{11}(\tfrac{3}{8}b_1^2) + c_{22}(\tfrac{-1}{16}b_1^4) + (\tfrac{3}{128}b_1^6)\} \\
c_{333} &= c_3(\tfrac{1}{4}) + c_4(\tfrac{5}{16}\cdot\tfrac{2}{3}b_1^2) + c_5 \\
&\quad \{c_{11}(\tfrac{1}{4}) + c_{22}(\tfrac{1}{16}\cdot\tfrac{-2}{3}b_1^2) + (\tfrac{1}{64}b_1^4)\} \\
c_{444} &= c_4(\tfrac{1}{6}) + c_5\{c_{22}(\tfrac{1}{6}) + (\tfrac{-1}{16}b_1^2)\} \\
c_{555} &= c_5(\tfrac{1}{8})
\end{aligned}\right\} \qquad [D.21]$$

$$I_4 = \int_A^B \int_C^D \frac{\cos[k[(z-z_1)^2 + b_1^2]^{\frac{1}{2}}]}{k[(z-z_1)^2 + b_1^2]^{\frac{1}{2}}}\, dz\, dz_1 \qquad [D.22]$$

Applying equation [D.20] and using the following substitutions

$$x_1 = B-z_1 \quad , \quad x_2 = A-z_1$$

$$u_1 = [x_1^2 + b_1^2]^{\frac{1}{2}} \quad , \quad u_2 = [x_2^2 + b_1^2]^{\frac{1}{2}}$$

equation [D.22] reduces to

$$I_4 = \int_{B-D}^{B-C} II\, dx_1 + \int_{A-C}^{A-D} II\, dx_2 \qquad [D.23]$$

where

$$II = c_{111} \ln (x+u) + c_{222}\, xu + c_{333}\, xu^3 + c_{444}\, xu^5 + c_{555}\, x^3u^5 \qquad [D.24]$$

or

$$II = c_{111}\, II_1 + c_{222}\, II_2 + c_{333}\, II_3 + c_{444}\, II_4 + c_{555}\, II_5 \qquad [D.25]$$

where

$$\left.\begin{aligned}
II_1 &= \int \ln (x+u)\, dx = \{ x \ln (x+u) - u \} \\
II_2 &= \int xu\, dx = \{ \tfrac{1}{3} u^3 \} \\
II_3 &= \int xu^3 dx = \{ \tfrac{1}{5} u^5 \} \\
II_4 &= \int xu^5 dx = \{ \tfrac{1}{7} u^7 \} \\
II_5 &= \int x^3u^5 dx = \int \frac{x^3}{u} u^6\, dx \\
&= \int \frac{x^3}{u} (x^2 + b_1^2)^3\, dx \\
&= \int \frac{x^9}{u} + 3b_1^2 \frac{x^7}{u} + 3b_1^4 \frac{x^5}{u} + b_1^6 \frac{x^3}{u} \} \, dx
\end{aligned}\right\} \qquad [D.26a]$$

where

$$\int \frac{x^{2m}}{u}\, dx = \frac{(2m)!}{2^{2m}(m!)^2} \left[u \sum_{r=1}^{m} \frac{r!(r-1)!}{(2r)!} (-b_1^2)^{m-r} (2x)^{2r-1} + (-b_1^2)^m \ln (x+u) \right]$$

and

$$\int \frac{x^{2m+1}}{u}\, dx \;=\; u \sum_{r=0}^{m} \frac{(2r)!\,(m!)^2}{(2m+1)!\,(r!)^2} \left(-4b_1^2\right)^{m-r} x^{2r} \qquad [D.26b]$$

Note that x in equations [D.24, D.25, D.26] represents x_1 for the first part of equation [D.23] and represents x_2 for the second part of it.

$$I_5 = \int_A^B \frac{\sin\left[k[(z+a_1)^2 + b_1^2]^{\frac{1}{2}}\right]}{k[(z+a_1)^2 + b_1^2]^{\frac{1}{2}}} \cos gz \; dz \qquad [D.27]$$

$$I_5 = \int_A^B \{ c_1 + c_2 u_1^2 + c_3 u_1^4 + c_4 u_1^6 + c_5 u_1^8 \} \cos gz \; dz \qquad [D.28]$$

where c_1, c_2, c_3, c_4 and c_5 are given by equation [D.6] and

$$u_1 = [(z+a_1)^2 + b_1^2]^{\frac{1}{2}}$$

Equation [D.28] can be reduced to

$$I_5 = \int_A^B \{ c_{11} + c_{22}\, z + c_{33}\, z^2 + c_{44}\, z^3 + c_{55}\, z^4 + c_{66}\, z^5 + c_{77}\, z^6 + c_{88}\, z^7 + c_{99}\, z^8 \} \cos gz \; dz \qquad [D.29]$$

where

$$\begin{aligned}
c_{11} = {} & [c_1 + c_2 b_1^2 + c_3 b_1^4 + c_4 b_1^6 + c_5 b_1^8] + [c_2 + c_3(2b_1^2) \\
& + c_4(2b_1^4) + c_5(3b_1^6)]\; a_1^2 + [c_3 + c_4(3b_1^2) + c_5(5b_1^4)]\; a_1^4 \\
& + [c_4 + c_5(4b_1^2)]\; a_1^6 + c_5 a_1^8 \\[1em]
c_{22} = {} & [c_2(2) + c_3(4\,b_1^2) + c_4(6b_1^4) + c_5(8b_1^6)]\; a_1 \\
& + [c_3(4) + c_4(12b_1^2) + c_5(22b_1^4)]\; a_1^3 + [c_4(6) + c_5(24b_1^2)]\, a_1^5 \\
& + [c_5(8)]\; a_1^7
\end{aligned}$$

$$c_{33} = [c_2 + c_3(2b_1^2) + c_4(3b_1^4) + c_5(4b_1^6)] + [c_3(6) + c_4(18b_1^2) + c_5(35b_1^4)]\, a_1^2 + [c_4(15) + c_5(60b_1^2)]\, a_1^4 + [c_5(28)]\, a_1^6$$

$$c_{44} = [c_3(4) + c_4(12b_1^2) + c_5(24b_1^4)]\, a_1 + [c_4(20) + c_5(80b_1^2)]\, a_1^3 + [c_5(56)]\, a_1^5$$

$$c_{55} = [c_3 + c_4(3b_1^2) + c_5(6b_1^4)] + [c_4(15) + c_5(60b_1^2)]a_1^2 + [c_5(70)]\, a_1^4$$

$$c_{66} = [c_4(6) + c_5(24b_1^2)]\, a_1 + [c_5(56)]\, a_1^3$$

$$c_{77} = [c_4 + c_5(4b_1^2)] + [c_5(28)]\, a_1^2$$

$$c_{88} = [c_5(8)]\, a_1 \quad , \quad c_{99} = [c_5] \qquad [D.30]$$

It is seen that equation [D.29] contains integral operations of the form $z^m \cos gz\, dz$ which are tabulated[31] as

$$\int z^m (\cos gz)\, dz = \sin gz \sum_{r=0}^{[\frac{m}{2}]} (-1)^r \frac{m!}{(m-2r)!} \cdot \frac{z^{m-2r}}{g^{2r+1}} + \cos gz \sum_{r=0}^{[\frac{m-1}{2}]} (-1)^r \frac{m!}{(m-2r-1)!} \cdot \frac{z^{m-2r-1}}{g^{2r+2}} \qquad [D.31]$$

Hence, equation [D.27] can be evaluated

$$I_6 = \int_A^B \int_C^D \frac{\sin [k[(z-z_1)^2 + b_1^2]^{\frac{1}{2}}]}{k[(z-z_1)^2 + b_1^2]^{\frac{1}{2}}} \cos gz\, dz_1\, dz \qquad [D.32]$$

Substitution in equation [D.29] for $a_1 = -z_1$ gives

$$I_6 = \sum_{i=1}^{9} \int_A^B f_i(z)\, dz \int_C^D f_i(z_1)\, dz_1 \qquad [D.33]$$

where

$$f_i(z) = z^{i-1} \cos gz$$

and $f_i(z_1)$ can be obtained using equations [D.30]. Hence, I_6 can be evaluated.

$$I_7 = \int_A^B \frac{\cos\,[k[(z+a_1)^2 + b_1^2]^{\frac{1}{2}}]}{k[(z+a_1)^2 + b_1^2]} (\cos gz)\, dz \qquad [D.34]$$

$$I_8 = \int_A^B \int_C^D \frac{\cos\,[k[(z-z_1)^2 + b_1^2]^{\frac{1}{2}}]}{k[(z-z_1)^2 + b_1^2]} (\cos gz)\, dz_1\, dz \qquad [D.35]$$

To calculate equation [D.34],

substitute $x = z+a_1$, $z = x-a_1$

$$I_7 = \int_{A+a_1}^{B+a_1} \frac{\cos\,(ku)}{ku} \cos g\,(x-a_1)\, dx \qquad [D.36]$$

where

$$\frac{\cos\,(ku)}{ku} = c_1 \frac{1}{u} + c_2 u + c_3 u^3 + c_4 u^5 + c_5 u^7 \qquad [D.37]$$

where c_i's are obtained according to equation [D.3b] as follows

$$c_1 = \frac{1}{k} \quad , \quad c_2 = \frac{-k}{2!} \quad ,$$

$$c_3 = \frac{k^3}{4!} \quad , \quad c_4 = \frac{-k^5}{6!} \quad ,$$

$$c_5 = \frac{k^7}{8!} \quad \text{and}$$

$$u = [x^2 + b_1^2]^{\frac{1}{2}}$$

$\cos g\,(x-a_1)$ can also be approximated according to equation [D.3a] as

$$\cos g\,(x-a_1) = 1 + e_1(x-a_1)^2 + e_2(x-a_1)^4 + e_3(x-a_1)^6 + e_4(x-a_1)^8 \qquad \text{[D.38]}$$

where

$$e_1 = -\frac{g^2}{2!} \quad , \quad e_2 = \frac{g^4}{4!}$$

$$e_3 = -\frac{g^6}{6!} \quad , \text{ and } \quad e_4 = \frac{g^8}{8!}$$

Substituting $\alpha = -a_1$, for equation [D.38], then we have

$$\cos g\,(x+\alpha) = c_{11} + c_{22}\,x + c_{33}\,x^2 + c_{44}\,x^3 + c_{55}\,x^4 + c_{66}\,x^5 + c_{77}\,x^6 + c_{88}\,x^7 + c_{99}\,x^8 \qquad \text{[D.39a]}$$

where

$$c_{11} = 1 + c_1\alpha^2 + c_2\alpha^4 + c_3\alpha^6 + c_4\alpha^8$$

$$c_{22} = c_1(2\alpha) + c_2(4\alpha^3) + c_3(6\alpha^5) + c_4(8\alpha^7)$$

$$\left.\begin{aligned}
c_{33} &= c_1 + c_2(6\alpha^2) + c_3(15\alpha^4) + c_4(28\alpha^6) \\
c_{44} &= c_2(4\alpha) + c_3(20\alpha^3) + c_4(56\alpha^5) \\
c_{55} &= c_2 + c_3(15\alpha^2) + c_4(70\alpha^4) \\
c_{66} &= c_3(6\alpha) + c_4(56\alpha^3) \\
c_{77} &= c_3 + c_4(28\alpha^2) \\
c_{88} &= c_4(8\alpha) \\
c_{99} &= c_4
\end{aligned}\right| \qquad [D.39b]$$

Equation [D.36] will be reduced to

$$I_7 = \int_{A+a_1}^{B+a_1} (c_1\frac{1}{u} + c_2 u + c_3 u^3 + c_4 u^5 + c_5 u^7)$$
$$(c_{11} + c_{22} x + c_{33} x^2 + c_{44} x^3 + c_{55} x^4 + c_{66} x^5 + c_{77} x^6$$
$$+ c_{88} x^7 + c_{99} x^8) \qquad [D.40]$$

Studying equation [D.40], it is seen that it contains some typical forms of integration i.e. $\int \frac{x^{2m}}{u} dx$ and $\int \frac{x^{2m+1}}{u} dx$ which are given by equation [D.26b]. Another form required to evaluate this equation is

$$\int x^m u^n dx = \frac{1}{m+1} [f_{(\frac{n-1}{2})} (m,n)] \qquad [D.41a]$$

where $f_0 (m,n) = x^{m+n} u - \int \frac{x^{m+n+1}}{u} dx$

and $f_i (m,n) = x^{(m+n-2i)} u^{(2i+1)} + \frac{[-(2i+1)]}{[m+n-2(i-1)]} f_{(i-1)} (m,n)$

[D.41b]

Equation [D.35] can be evaluated as follows

substitute $x = z-z_1$

Hence, equation [D.35] reduces to

$$I_8 = -\int_A^B \int_{z-C}^{z-D} \frac{\cos(ku)}{ku} \cos gz \, dx \, dz$$

$$\text{or} \quad I_8 = \int_A^B \cos gz \left\{ \int_{z-D}^{z-C} \frac{\cos(ku)}{ku} \, dx \right\} dz \qquad [D.42]$$

Integration of (cos ku/ku) was previously obtained in equation [D.20], Hence, equation [D.42] becomes

$$I_8 = \int_A^B \cos gz \, [f(z-c) - f(z-D)] \, dz \qquad [D.43]$$

where

$$f(x) = c_{111} \ln(x+u) + c_{222} xu + c_{333} xu^3 + c_{444} xu^5 + c_{555} x^3u^5 \qquad [D.44]$$

where c_{111}'s are given by equations [D.21]

substituting $y_1 = z-c$, $y_2 = z-D$

$$I_8 = \int_{A-C}^{B-C} \cos g(y_1+c) \, f(y_1) \, dy_1 - \int_{A-D}^{B-D} \cos g(y_2+D) \, f(y_2) \, dy_2 \qquad [D.45]$$

Using equation [D.39a] in conjunction with [D.44], equation [D.45] can be evaluated. Another typical integral form used in the calculations is

$$\int x^m \ln (x+u)\, dx = \frac{x^{m+1}}{m+1} \ln (x+u) - \frac{1}{m+1} \int \frac{x^{m+1}}{u}\, dx \quad [D.46]$$

where the last term can be evaluated from equation [D.26b].

APPENDIX E

COMPUTER PROGRAMS

E.I PROGRAM PW1

```
C
C
C
C
C     ****************************************************************
C     ****************************************************************
C     **                                                            **
C     **THIS PROGRAM TO CALCULATE THE CURRENTS USING THE MOMENT METHOD**
C     **                                                            **
C     **                  WEIGHTING FUNCTIONS ARE                   **
C     **                                                            **
C     **                      WALSH FUNCTIONS                       **
C     **                                                            **
C     ****************************************************************
C     ****************************************************************
C
C
C      F      IS THE FREQUENCY
C      VEL    IS THE SPACE VELOCITY
C      WL     IS THE WAVE LENGTH
C      L      IS THE ANTENNA HALF LENGTH
C      R1     IS THE INNER RADIUS OF FRILL
C      R2     IS THE OUTER RADIUS OF FRILL
C      EPS    IS THE ABSOLUTE DIELECTRIC CONSTANT (PERMITTIVITY)
C      N      IS THE NUMBER OF USED WALSH FUNCTIONS
C      NWD    IS THE NUMBER OF DIVISIONS FOR WALSH FUNCTICNS
C
C
C
C
C
      IMPLICIT REAL * 8 (A-H , O-Z)
C
      REAL * 8 L,K,LL
C
      COMPLEX * 16 DCMPLX
      COMPLEX * 16 EZ(32),E(32,1)
      COMPLEX * 16 DIKCF(32,32),A(32,32)
      COMPLEX * 16 E0(32),CURR(30)
      COMPLEX * 16 ZIN
C
      DIMENSION LL(32),UL(32),WF(32,32),W(32,32)
      DIMENSION XR(32),XI(32)
      DIMENSION XR1(32),XI1(32)
      DIMENSION WA(32)
      DIMENSION Z(30),G1(32),G2(32)
C
      COMMON  K,L,R1,R2,ZZ
C
C
C
      F   = 7.D6
      VEL = 3.D8
      WL  = VEL / F
```

```
      L    = 0.375 * WL
      R1   = 0.007022 * WL
      R2   = 2.23 * R1
      PI   = 4. * DATAN(1.D0)
      OMEGA = 2*PI*F
      EPS  = 1.D-9 / (36.*PI)
      K    = OMEGA / VEL
      CC   = 4.*PI*OMEGA*EPS
C
      WRITE (6,10) F,VEL,WL,L,R1,R2,EPS
C
C
      N   = 16
      NWD = 32
C
C
C
C
C     -----------------------------------------------------------------
C     CALCULATE THE LIMITS OF THE INTEGRALS
C     -----------------------------------------------------------------
C
      STEP = 2*L/NWD
      LL(1) =-L
      DO 100 K1=1,NWD
      UL(K1) = LL(K1) + STEP
      IF (K1 .EQ. NWD/2) UL(K1) = 0.0
      IF (K1 .EQ. NWD) UL(K1) = L
      M = K1 + 1
      IF(M .GT. NWD) GO TO 100
      LL(M) = UL(K1)
  100 CONTINUE
C
      WRITE (6,20)
      WRITE (6,21) (LL(I),UL(I), I=1,NWD)
C
C     -----------------------------------------------------------------
C     GENERATE THE WALSH FUNCTIONS
C     -----------------------------------------------------------------
C
      CALL WALSHF(NWD,WF)
      DO 200 I=1,N
      DO 200 J=1,NWD
  200 W(I,J) = WF(2*I-1 , J)
C
      WRITE (6,30)
      DO 210 I=1,N
  210 WRITE (6,31) (W(I,J), J=1,NWD)
C
C     -----------------------------------------------------------------
C     CALCULATE THE INCIDENT FIELDS
C     -----------------------------------------------------------------
      CALL CONST1
```

```
      CALL CONST3
      DO 300 II=1,NWD
      XR1(II) = EZR (LL(II),UL(II))
      XI1(II) = EZI (LL(II),UL(II))
  300 CONTINUE
      DO 305 I=1,N
      XR(I) = 0.0D0
      XI(I) = 0.0D0
      DO 305 II=1,NWD
      XR(I) = XR(I) + W(I,II) * XR1(II)
      XI(I) = XI(I) + W(I,II) * XI1(II)
  305 CONTINUE
C
      DEN = 2. * DLOG(R2/R1)
      DO 310 K1=1,N
      EZ(K1) = DCMPLX(XR(K1) , XI(K1)) / DEN
      E(K1,1) = -K * EZ(K1)
  310 CONTINUE
C
      WRITE (6,40)
      WRITE (6,41) (I,E(I,1) , I=1,N)
C
C     ------------------------------------------------------------------
C     CALCULATE THE ENTRIES OF THE MATRIX
C     ------------------------------------------------------------------
C
      DO 430 II=1,NWD
      DO 430 JJ=1,NWD
      XREAL = VKCR (LL(II),UL(II),LL(JJ),UL(JJ))
      XIMAG = VKCI (LL(II),UL(II),LL(JJ),UL(JJ))
  430 DIKCF(II,JJ) = DCMPLX(XREAL,XIMAG) * (-K/CC)
C
      DO 440 I=1,N
      DO 440 J=1,N
      A(I,J) = (0.0D0,0.0D0)
      DO 440 JJ=1,NWD
      DO 440 II=1,NWD
  440 A(I,J) = A(I,J) + DIKCF(II,JJ) * W(J,JJ) * W(I,II)
C
      WRITE (6,50)
      DO 450 I=1,N
  450 WRITE (6,51) (A(I,J), J=1,N)
C
C     ------------------------------------------------------------------
C     CALCULATE THE COEFFICIENTS OF THE MATRIX
C     ------------------------------------------------------------------
C
      CALL LEQTIC(A,N,32,E,1,32,0,WA,IER)
C
      WRITE (6,60)
      WRITE (6,61) (I,E(I,1), I=1,N)
C
C
```

```
C     ------------------------------------------------------------------
C     CALCULATE E0
C     ------------------------------------------------------------------
C
      DO 500 K1=1,NWD
      E0(K1) = (0.0D0,0.0D0)
      DO 500 J=1,N
      E0(K1) = E0(K1) + E(J,1)*W(J,K1)
  500 CONTINUE
C
C     ------------------------------------------------------------------
C     CALCULATE THE CURRENTS
C     ------------------------------------------------------------------
C
      NPC = 21
      Z(1) = 0.0
      DL = L / (NPC-1)
      NOHA = NPC
      DO 600 I=2,NPC
      Z(I) = Z(I-1) + DL
      IF(I .EQ. NOHA) Z(I)= L
  600 CONTINUE
      WRITE (6,70)
      DO 610 I=1,NPC
  610 WRITE (6,71) I,Z(I)
C
      DO 660 I=1,NPC
      ZZ = Z(I)
      DO 620 K1=1,NWD
      G1(K1) = GRL( UL(K1) ) - GRL( LL(K1) )
      G2(K1) = GRG( UL(K1) ) - GRG( LL(K1) )
  620 CONTINUE
      DO 630 K1=1,NWD
      IF(ZZ .GT. UL(K1))GO TO 630
      M = K1
      GO TO 640
  630 CONTINUE
  640 IF(DABS( LL(M)-ZZ ) .GT. 0.0001 ) GO TO 641
      G11 = G1(M)
      G22 = 0.0
      GO TO 650
  641 IF(DABS( UL(M)-ZZ ) .GT. 0.0001 ) GO TO 642
      G11 = 0.0
      G22 = G2(M)
      GO TO 650
  642 G11 = GRL( UL(M) ) - GRL( ZZ )
      G22 = GRG( ZZ ) - GRG( LL(M) )
C
  650 CURR(I) = (0.0D0,0.0D0)
      M1 = M-1
      IF(M .EQ. 1)GO TO 652
      DO 651 K1=1,M1
  651 CURR(I) = CURR(I) + E0(K1)*G2(K1)
```

```
652 M2 = M+1
    IF(M .EQ. NWD)GO TO 654
    DO 653 K1=M2,NWD
653 CURR(I) = CURR(I) + E0(K1)*G1(K1)
654 CURR(I) = CURR(I) + E0(M)*(G11+G22)
C
660 CONTINUE
C
    WRITE (6,72)
    WRITE (6,73) (I,CURR(I), I=1,NPC)
C
C     ------------------------------------------------------------------
C     CALCULATE THE INPUT IMPEDANCE
C     ------------------------------------------------------------------
C
    ZIN = 1. / CURR(1)
C
    WRITE (6,80) ZIN
C
C
C     ------------------------------------------------------------------
C     FORMAT STATEMENTS
C     ------------------------------------------------------------------
C
 10 FORMAT ('1',5(/),10X,'INPUT DATA'/10X,10(1H*)////
   1                    15X,'FREQUENCY                ,F   = ',D15.7//
   2                    15X,'SPACE VELOCITY           ,VEL = ',D15.7//
   3                    15X,'WAVE LENGTH              ,WL  = ',D15.7//
   4                    15X,'ANTENNA HALF LENGTH      ,L   = ',D15.7//
   5                    15X,'INNER RADIUS             ,R1  = ',D15.7//
   6                    15X,'OUTER RADIUS             ,R2  = ',D15.7//
   7                    15X,'PERMITTIVITY             ,EPS = ',D15.7)
 20 FORMAT ('1',5(/),10X,'LIMITS OF THE INTEGRALS'/10X,23(1H*)/////)
 21 FORMAT (15X,D15.7,10X,D15.7/)
 30 FORMAT ('1',5(/),10X,'WALSH FUNCTIONS'/10X,15(1H*)////)
 31 FORMAT ((15X,8(D9.2,4X)/))
 40 FORMAT ('1',5(/),10X,'INCIDENT FIELDS'/10X,15(1H*)////)
 41 FORMAT ((15X,'EZ(',I2,') = ',2(D15.7,5X)/))
 50 FORMAT ('1',5(/),10X,'THE MATRIX ENTRIES'/10X,18(1H*)////)
 51 FORMAT ((' ',4(2D15.7,3X)/))
 60 FORMAT ('1',5(/),10X,'THE COEFFICIENTS OF THE MATRIX'/
   1                    10X,30(1H*)////)
 61 FORMAT ((15X,'A(',I2,') = ',2(D15.7,5X)/))
 70 FORMAT ('1',5(/),10X,'THE POINTS'/10X,10(1H*)////)
 71 FORMAT (15X,'Z(',I2,') = ',D15.7/)
 72 FORMAT ('1',5(/),10X,'THE CURRENTS'/10X,12(1H*)////)
 73 FORMAT ((15X,'CURR(',I2,') = ',2(D15.7,5X)/))
 80 FORMAT ('1',5(/),10X,'INPUT IMPEDANCE'/10X,15(1H*)////
   1                    15X,'ZIN = ',2(D15.7,5X))
C
C
    STOP
    END
```

```
C
C
C
      SUBROUTINE WALSHF(N,W)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      INTEGER U,V,R,P
      DIMENSION W(32,32)
      DIMENSION U(5),V(5),R(32)
      TN = N
      EXPN = DLOG(TN) / DLOG(2.0 D0)
      NS = EXPN + 0.1
      DO 1000 IU=1,N
      IUM = IU - 1
      CALL BIN(IUM,U)
      DO 1000 JV=1,N
      JVM = JV - 1
      CALL BIN(JVM,V)
      P = U(NS)*V(1)
      DO 100 I=2,NS
      R(I) = U(NS - (I-1)) + U(NS - (I-2))
  100 P = P + R(I)*V(I)
      W(IU,JV) = (-1)**P
 1000 CONTINUE
      RETURN
      END
C
C
C
      SUBROUTINE BIN(DEC,BINV)
C
      INTEGER DEC,BINV
      DIMENSION BINV(5)
      DO 100 I=1,5
  100 BINV(I) = 0
      INT = DEC
      IF(INT .EQ. 0) GO TO 300
      DO 200 I=1,5
      IF( ((INT/2)*2) .NE. INT ) BINV(I)=1
      INT = INT/2
      IF(INT .EQ. 0) GO TO 300
  200 CONTINUE
  300 RETURN
      END
```

```
C
C
C
      FUNCTION GRL(X)
C
C     ******************************************************************
C     *                                                                *
C     *          THIS FUNCTION TO CALCULATE G WITH  Z < ZP             *
C     *                                                                *
C     ******************************************************************
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,ZZ
      B1 = 2*K*L
      S1 =-K*DSIN(B1)
      B2 = K*(L+ZZ)
      S2 = DSIN(B2)
      B3 = K*(L-X)
      S3 = DCOS(B3)/K
      GRL = S2*S3/S1
      RETURN
      END
C
C
C
      FUNCTION GRG(X)
C
C     ******************************************************************
C     *                                                                *
C     *          THIS FUNCTION TO CALCULATE G WITH  Z > ZP             *
C     *                                                                *
C     ******************************************************************
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,ZZ
      B1 = 2*K*L
      S1 =-K*DSIN(B1)
      B2 = K*(L-ZZ)
      S2 = DSIN(B2)
      B3 = K*(L+X)
      S3 =-DCOS(B3)/K
      GRG = S2*S3/S1
      RETURN
      END
```

```
C
C
      FUNCTION EZR (A,B)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,ZZ
      T1 = SINCOS (A,B,0.0D0,R1)
      T2 = SINCOS (A,B,0.0D0,R2)
      EZR =  (T1 - T2)
      RETURN
      END
C
C
C
      FUNCTION EZI (A,B)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      REAL * 8 K,L
      T1 = SINSIN (A,B,0.0D0,R1)
      T2 = SINSIN (A,B,0.0D0,R2)
      EZI = -(T1 - T2)
      RETURN
      END
C
C
C
      FUNCTION VKCR (A,B,C,D)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,ZZ
      TA = DINSIN (A,B,C,D,R1)
      TB = SINSIN (A,B, L,R1)
      TC = SINSIN (A,B,-L,R1)
      TD = ( DCOS(K*(L-D)) - DCOS(K*(L-C)) ) /   K
      TE = ( DCOS(K*(L+D)) - DCOS(K*(L+C)) ) / (-K)
      TF = DSIN (2.*K*L)
      VKCR = TA - (TB*TD + TC*TE) / TF
      RETURN
      END
C
C
C
      FUNCTION VKCI (A,B,C,D)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,ZZ
      TA = DINCOS (A,B,C,D,R1)
      TB = SINCOS (A,B, L,R1)
      TC = SINCOS (A,B,-L,R1)
      TD = ( DCOS(K*(L-D)) - DCOS(K*(L-C)) ) /   K
      TE = ( DCOS(K*(L+D)) - DCOS(K*(L+C)) ) / (-K)
```

```
      TF = DSIN (2.*K*L)
      VKCI = TA - (TB*TD + TC*TE) / TF
      RETURN
      END
C
C
C
      FUNCTION IFAC(N)
C
      IFAC = 1.
      IF (N .LE. 1) GO TO 200
      DO 100 I=2,N
  100 IFAC = IFAC * I
  200 RETURN
      END
C
C
C
      SUBROUTINE CONST1
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      REAL * 8 K
      COMMON K
      COMMON /BLOC01/ C1,C2,C3,C4,C5
      FC3 = IFAC(3)
      FC5 = IFAC(5)
      FC7 = IFAC(7)
      FC9 = IFAC(9)
      C1 =  1.
      C2 = -(K**2) / FC3
      C3 =  (K**4) / FC5
      C4 = -(K**6) / FC7
      C5 =  (K**8) / FC9
      RETURN
      END
C
C
C
      SUBROUTINE CONST2 (B,C111,C222,C333,C444,C555)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      COMMON /BLOC01/ C1,C2,C3,C4,C5
      B2 = B**2
      B4 = B**4
      B6 = B**6
      B8 = B**8
      C111 = C1 + C2*B2 + C3*B4     + C4*B6     + C5*B8
      C222 =      C2    + C3*B2*2.  + C4*B4*3.  + C5*B6*4.
      C333 =              C3        + C4*B2*3.  + C5*B4*6.
      C444 =                          C4        + C5*B2*4.
      C555 =                                      C5
      RETURN
      END
```

```
C
C
C
      SUBROUTINE CONST3
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      REAL * 8 K
      COMMON K
      COMMON /BLOCO2/ C1,C2,C3,C4,C5
      FC2 = IFAC(2)
      FC4 = IFAC(4)
      FC6 = IFAC(6)
      FC8 = IFAC(8)
      C1 =         1. / K
      C2 =        -K  / FC2
      C3 =   (K**3)   / FC4
      C4 = -(K**5)    / FC6
      C5 =   (K**7)   / FC8
      RETURN
      END
C
C
C
      SUBROUTINE CONST4 (B,CP111,CP222,CP333,CP444,CP555)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      COMMON /BLOCO2/ C1,C2,C3,C4,C5
      C11 = B**4
      C22 = 2.*B*B
      CP111 = C1 + C2*(1./2.)*(B*B) + C3*(3./8.)*(B**4) +
     1        C4*(5./16.)*(B**6) + C5*( C11*(3./8.)*(B**4) +
     2        C22*(-1./16.)*(B**6) + (3./128.)*(B**8) )
      CP222 = C2*(1./2.) + C3*(3./8.)*(B*B) +
     1        C4*(5./16.)*(B**4) + C5 * ( C11*(3./8.)*(B**2) +
     2        C22*(-1./16.)*(B**4) + (3./128.)*(B**6) )
      CP333 = C3*(1./4.) + C4*(5./24.)*(B**2) + C5 * ( C11*(1./4.) +
     1        C22*(-1./24.)*(B*B) + (1./64.)*(B**4) )
      CP444 = C4*(1./6.) + C5 * ( C22*(1./6.) + (-1./16.)*(B**2) )
      CP555 = C5*(1./8.)
      RETURN
      END
C
C
C
C
C
      FUNCTION SINSIN (A,B,A1,B1)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      REAL * 8 LL
      SS(Z) = C111*Z + (C222/3.)*(Z**3) + (C333/5.)*(Z**5)
     1               + (C444/7.)*(Z**7) + (C555/9.)*(Z**9)
      CALL CONST2 (B1,C111,C222,C333,C444,C555)
```

```
      LL = A + A1
      UL = B + A1
      SINSIN = SS(UL) - SS(LL)
      RETURN
      END
C
C
C
      FUNCTION DINSIN (A,B,C,D,B1)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      REAL * 8 LL1,LL2
      DS(Z) = (C111/2.)*(Z*Z) + (C222/12.)*(Z**4) + (C333/30.)*(Z**6)
     1                        + (C444/56.)*(Z**8) + (C555/90.)*(Z**10)
      CALL CONST2 (B1,C111,C222,C333,C444,C555)
      LL1 = B - D
      UL1 = B - C
      LL2 = A - C
      UL2 = A - D
      DINSIN = ( DS(UL1) - DS(LL1) ) + ( DS(UL2) - DS(LL2) )
      RETURN
      END
C
C
C
      FUNCTION SINCOS (A,B,A1,B1)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      REAL * 8 LL
      U (Z) = DSQRT (Z*Z + B1*B1)
      F1(Z) = DLOG (Z + U(Z))
      F2(Z) = Z * U(Z)
      F3(Z) = Z * (U(Z)**3)
      F4(Z) = Z * (U(Z)**5)
      F5(Z) = (Z**3) * (U(Z)**5)
      SC(Z) = CP111*F1(Z) + CP222*F2(Z) + CP333*F3(Z)
     1                    + CP444*F4(Z) + CP555*F5(Z)
      CALL CONST4 (B1,CP111,CP222,CP333,CP444,CP555)
      LL = A + A1
      UL = B + A1
      SINCOS = SC(UL) - SC(LL)
      RETURN
      END
C
C
C
      FUNCTION DINCOS (A,B,C,D,B1)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      REAL * 8 LL1,LL2
      U  (Z) = DSQRT (Z*Z + B1*B1)
      F11(Z) = Z * DLOG(Z + U(Z)) - U(Z)
      F22(Z) = (U(Z)**3) / 3.
```

```
      F33(Z) = (U(Z)**5) / 5.
      F44(Z) = (U(Z)**7) / 7.
      DC (Z) = CP111*F11(Z) + CP222*F22(Z) + CP333*F33(Z)
     1                      + CP444*F44(Z) + CP555*F55(Z,B1)
      CALL CONST4 (B1,CP111,CP222,CP333,CP444,CP555)
      LL1 = B - D
      UL1 = B - C
      LL2 = A - C
      UL2 = A - D
      DINCOS = ( DC(UL1) - DC(LL1) ) + ( DC(UL2) - DC(LL2) )
      RETURN
      END
C
C
C
      FUNCTION F55 (Z,B1)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      F551 = FODD(Z,B1,9)
      F552 = FODD(Z,B1,7)
      F553 = FODD(Z,B1,5)
      F554 = FODD(Z,B1,3)
      F55 = F551 + 3.*(B1**2)*F552 + 3.*(B1**4)*F553 + (B1**6)*F554
      RETURN
      END
C
C
C
      FUNCTION FODD (Z,B1,MM)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      INTEGER R
      U(Z) = DSQRT (Z*Z + B1*B1)
      M = (MM-1) / 2
      FCM = IFAC(M)
      FCMM = IFAC(MM)
      M1 = M + 1
      TT = 0.0
      DO 100 I=1,M1
      R = I - 1
      FC2R = IFAC(2*R)
      FCR = IFAC(R)
      T1 = (FC2R*(FCM**2)) / (FCMM*(FCR**2))
      T2 = 1.
      IF (R .NE. M) T2 = (-4.*B1*B1)**(M-R)
      T3 = 1.
      IF (R .NE. 0) T3 = Z**(2*R)
  100 TT = TT + (T1 * T2 * T3)
      FODD = U(Z) * TT
      RETURN
      END
```

E.II PROGRAM PC1

```
C
C
C
C
C     ******************************************************************
C     ******************************************************************
C     **                                                              **
C     **THIS PROGRAM TO CALCULATE THE CURRENTS USING THE MOMENT METHOD**
C     **                                                              **
C     **                 WEIGHTING FUNCTIONS ARE                      **
C     **                                                              **
C     **                   COSINE FUNCTIONS                           **
C     **                                                              **
C     ******************************************************************
C     ******************************************************************
C
C
C      F     IS THE FREQUENCY
C      VEL   IS THE SPACE VELOCITY
C      WL    IS THE WAVE LENGTH
C      L     IS THE ANTENNA HALF LENGTH
C      R1    IS THE INNER RADIUS OF FRILL
C      R2    IS THE OUTER RADIUS OF FRILL
C      EPS   IS THE ABSOLUTE DIELECTRIC CONSTANT (PERMITTIVITY)
C      N     IS THE NUMBER OF USED WALSH FUNCTIONS
C      NWD   IS THE NUMBER OF DIVISIONS FOR WALSH FUNCTIONS
C
C
C
      IMPLICIT REAL * 8 (A-H , O-Z)
C
      REAL * 8 L,K,LL
C
      COMPLEX * 16 DCMPLX
      COMPLEX * 16 EZ(32),E(32,1)
      COMPLEX * 16 DIKCF(32,32),A(32,32)
      COMPLEX * 16 E0(32),CURR(30)
      COMPLEX * 16 ZIN
C
      DIMENSION LL(32),UL(32),WF(32,32),W(32,32)
      DIMENSION XR(32),XI(32)
      DIMENSION X1(40),Y1(40),F2(40,40),WK(2031)
      DIMENSION WA(32)
      DIMENSION Z(30),G1(32),G2(32)
C
      EXTERNAL EZRF
C
      COMMON  K,L,R1,R2,ZZ,G
C
C
      F   = 7.D6
      VEL = 3.D8
      WL  = VEL / F
      L   = 0.375 * WL
```

```
      R1   = 0.007022 * WL
      R2   = 2.23 * R1
      PI   = 4. * DATAN(1.D0)
      OMEGA = 2*PI*F
      EPS = 1.D-9 / (36.*PI)
      K    = OMEGA / VEL
      CC   = 4.*PI*OMEGA*EPS
C
      WRITE (6,10) F,VEL,WL,L,R1,R2,EPS
C
C
      N    = 4
      NWD = 8
C
C
C
C
C    ------------------------------------------------------------------
C     CALCULATE THE LIMITS OF THE INTEGRALS
C    ------------------------------------------------------------------
C
      STEP = 2*L/NWD
      LL(1) =-L
      DO 100 K1=1,NWD
      UL(K1) = LL(K1) + STEP
      IF (K1 .EQ. NWD/2) UL(K1) = 0.0
      IF (K1 .EQ. NWD) UL(K1) = L
      M = K1 + 1
      IF(M .GT. NWD) GO TO 100
      LL(M) = UL(K1)
  100 CONTINUE
C
      WRITE (6,20)
      WRITE (6,21) (LL(I),UL(I), I=1,NWD)
C
C    ------------------------------------------------------------------
C     GENERATE THE WALSH FUNCTIONS
C    ------------------------------------------------------------------
C
      CALL WALSHF(NWD,WF)
      DO 200 I=1,N
      DO 200 J=1,NWD
  200 W(I,J) = WF(2*I-1 , J)
C
      WRITE (6,30)
      DO 210 I=1,N
  210 WRITE (6,31) (W(I,J), J=1,NWD)
C
C    ------------------------------------------------------------------
C     CALCULATE THE INCIDENT FIELDS
C    ------------------------------------------------------------------
      CALL CONST1
      DO 300 K1=1,N
```

```
      G = ((2.*K1 - 1) / 2.) * (PI/L)
      XR(K1) = DCADRE(EZRF,-L,L,0.01,ERROR,IER)
      XI(K1) = EZI(-L,L)
  300 CONTINUE
C
      DEN = 2. * DLOG(R2/R1)
      DO 310 K1=1,N
      EZ(K1) = DCMPLX(XR(K1) , XI(K1)) / DEN
      E(K1,1) = -K * EZ(K1)
  310 CONTINUE
C
      WRITE (6,40)
      WRITE (6,41) (I,E(I,1) , I=1,N)
C
C     ----------------------------------------------------------------
C     CALCULATE THE ENTRIES OF THE MATRIX
C
C     ----------------------------------------------------------------
      NN  = 40
      NX  = 40
      NY  = 40
C
      DELX1 = 2.*L / (NX-2)
      X1(1) = -L - DELX1/2.
      DO 400 III=2,NX
  400 X1(III) = X1(III-1) + DELX1
      DO 430 I =1,N
      G = ((2.*I - 1) / 2.) * (PI/L)
      DO 430 JJ=1,NWD
      XREAL = VKCR (-L,L,LL(JJ),UL(JJ))
      DELY1 = (UL(JJ) - LL(JJ)) / (NY-2)
      Y1(1) = LL(JJ) - DELY1/2.
      DO 410 JJJ=2,NY
  410 Y1(JJJ) = Y1(JJJ-1) + DELY1
      DO 420 III=1,NX
      DO 420 JJJ=1,NY
      F2(III,JJJ) = FUI(X1(III),Y1(JJJ))
  420 CONTINUE
      CALL DBCQDU(F2,NN,X1,NX,Y1,NY,-L   , L  ,LL(JJ),UL(JJ),QI   ,WK,IER)
      XIMAG = QI
  430 DIKCF(I,JJ) = DCMPLX(XREAL,XIMAG) * (-K/CC)
C
      DO 440 I=1,N
      DO 440 J=1,N
      A(I,J) = (0.0D0,0.0D0)
      DO 440 JJ=1,NWD
  440 A(I,J) = A(I,J) + DIKCF(I,JJ) * W(J,JJ)
C
      WRITE (6,50)
      DO 450 I=1,N
  450 WRITE (6,51) (A(I,J), J=1,N)
```

```
C
C
C
      FUNCTION EZRF(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,ZZ,G
      T1 = K * DSQRT(X*X + R1*R1)
      T2 = K * DSQRT(X*X + R2*R2)
      EZR =  DCOS(T1)/T1 - DCOS(T2)/T2
      F = DCOS(G * X)
      EZRF = EZR * F
      RETURN
      END
C
C
C
      FUNCTION FUI(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,ZZ,G
      R12 = R1 * R1
      T1 = K * DSQRT((X-Y)**2 + R12)
      T2 = K * DSQRT((X+L)**2 + R12)
      T3 = K * DSQRT((X-L)**2 + R12)
      T4 = K * (L-Y)
      T5 = K * (L+Y)
      T6 = 2.*K*L
      TA = DCOS(T1) / T1
      TB = DCOS(T2) / T2
      TC = DCOS(T3) / T3
      TD = DSIN(T4)
      TE = DSIN(T5)
      TF = DSIN(T6)
      FI = TA - (TB*TD + TC*TE) / TF
      F = DCOS(G * X)
      FUI = FI * F
      RETURN
      END
```

```
C
C
C
      FUNCTION EZI (A,B)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,ZZ,G
      T1 = SINSC   (A,B,0.0D0,R1)
      T2 = SINSC   (A,B,0.0D0,R2)
      EZI = -(T1 - T2)
      RETURN
      END
C
C
C
      FUNCTION VKCR (A,B,C,D)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,ZZ,G
      TA = DINSC   (A,B,C,D,R1)
      TB = SINSC   (A,B, L,R1)
      TC = SINSC   (A,B,-L,R1)
      TD = ( DCOS(K*(L-D)) - DCOS(K*(L-C)) ) /    K
      TE = ( DCOS(K*(L+D)) - DCOS(K*(L+C)) ) / (-K)
      TF = DSIN (2.*K*L)
      VKCR = TA - (TB*TD + TC*TE) / TF
      RETURN
      END
C
C
C
      FUNCTION IFAC(N)
C
      IFAC = 1.
      IF (N .LE. 1) GO TO 200
      DO 100 I=2,N
  100 IFAC = IFAC * I
  200 RETURN
      END
C
C
C
      SUBROUTINE CONST1
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      REAL * 8 K
      COMMON K
      COMMON /BLOC01/ C1,C2,C3,C4,C5
      FC3 = IFAC(3)
      FC5 = IFAC(5)
      FC7 = IFAC(7)
      FC9 = IFAC(9)
```

```
      C1 =   1.
      C2 = -(K**2) / FC3
      C3 =  (K**4) / FC5
      C4 = -(K**6) / FC7
      C5 =  (K**8) / FC9
      RETURN
      END
C
C
C
      SUBROUTINE CONST2 (B,CV)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      DIMENSION CV(25)
      COMMON /BLOC01/ C1,C2,C3,C4,C5
      B2 = B**2
      B4 = B**4
      B6 = B**6
      B8 = B**8
      CV( 1) = C1 + C2*B2 + C3*B4    + C4*B6     + C5*B8
      CV( 2) =      C2    + C3*B2*2. + C4*B4*2.  + C5*B6*3.
      CV( 3) =              C3       + C4*B2*3.  + C5*B4*5.
      CV( 4) =                         C4        + C5*B2*4.
      CV( 5) =                                     C5
C
      CV( 6) = C2*2. + C3*B2*4. + C4*B4* 6. + C5*B6* 8.
      CV( 7) =         C3*4.    + C4*B2*12. + C5*B4*22.
      CV( 8) =                    C4*6.     + C5*B2*24.
      CV( 9) =                                C5*8.
C
      CV(10) = C2 + C3*B2*2. + C4*B4* 3. + C5*B6* 4.
      CV(11) =      C3*6.    + C4*B2*18. + C5*B4*35.
      CV(12) =                 C4*15.    + C5*B2*60.
      CV(13) =                             C5*28.
C
      CV(14) = C3*4. + C4*B2*12. + C5*B4*24.
      CV(15) =         C4*20.    + C5*B2*80.
      CV(16) =                     C5*56.
C
      CV(17) = C3 + C4*B2* 3. + C5*B4* 6.
      CV(18) =      C4*15.    + C5*B2*60.
      CV(19) =                  C5*70.
C
      CV(20) = C4*6. + C5*B2*24.
      CV(21) =         C5*56.
      CV(22) = C4    + C5*B2* 4.
      CV(23) =         C5*28.
      CV(24) =         C5* 8.
      CV(25) =         C5
      RETURN
      END
```

```
C
C
C
      FUNCTION SINSC (A,B,A1,B1)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      DIMENSION CV(25),T1(9),T2(9)
      CALL CONST2 (B1,CV)
      A12 = A1**2
      A13 = A1**3
      A14 = A1**4
      A15 = A1**5
      A16 = A1**6
      A17 = A1**7
      A18 = A1**8
      T1(1) = CV(1) + CV(2)*A12 + CV(3)*A14 + CV(4)*A16 + CV(5)*A18
      T1(2) = CV(6)*A1 + CV(7)*A13 + CV(8)*A15 + CV(9)*A17
      T1(3) = CV(10) + CV(11)*A12 + CV(12)*A14 + CV(13)*A16
      T1(4) = CV(14)*A1 + CV(15)*A13 + CV(16)*A15
      T1(5) = CV(17) + CV(18)*A12 + CV(19)*A14
      T1(6) = CV(20)*A1 + CV(21)*A13
      T1(7) = CV(22) + CV(23)*A12
      T1(8) = CV(24)*A1
      T1(9) = CV(25)
      DO 100 I=1,9
      M = I-1
  100 T2(I) = XMCOS(B,M) - XMCOS(A,M)
      TT = 0.
      DO 200 I=1,9
  200 TT = TT + T1(I)*T2(I)
      SINSC = TT
      RETURN
      END
C
C
C
      FUNCTION DINSC (A,B,C,D,B1)
C
      IMPLICIT REAL * 8 (A-H , O-Z)
      DIMENSION CV(25),T1(9),T2(9)
      F1(Y) = CV01  * Y + CV02 /3. * Y**3 + CV03 /5. * Y**5
     1               + CV04 /7. * Y**7 + CV05 /9. * Y**9
      F2(Y) = -( CV06 /2. * Y**2 + CV07 /4. * Y**4 + CV08 /6. * Y**6
     1                            + CV09 /8. * Y**8 )
      F3(Y) = CV10  * Y + CV11 /3. * Y**3 + CV12 /5. * Y**5
     1               + CV13 /7. * Y**7
      F4(Y) = -(CV14 /2. * Y**2 + CV15 /4. * Y**4 + CV16 /6. * Y**6)
      F5(Y) = CV17  * Y + CV18 /3. * Y**3 + CV19 /5. * Y**5
      F6(Y) = -( CV20 /2. * Y**2 + CV21 /4. * Y**4 )
      F7(Y) = CV22  * Y + CV23 /3. * Y**3
      F8(Y) = -( CV24 /2. * Y**2 )
      F9(Y) = CV25  * Y
      CALL CONST2 (B1,CV)
      CV01 = CV( 1)
```

```
      CV02 = CV( 2)
      CV03 = CV( 3)
      CV04 = CV( 4)
      CV05 = CV( 5)
      CV06 = CV( 6)
      CV07 = CV( 7)
      CV08 = CV( 8)
      CV09 = CV( 9)
      CV10 = CV(10)
      CV11 = CV(11)
      CV12 = CV(12)
      CV13 = CV(13)
      CV14 = CV(14)
      CV15 = CV(15)
      CV16 = CV(16)
      CV17 = CV(17)
      CV18 = CV(18)
      CV19 = CV(19)
      CV20 = CV(20)
      CV21 = CV(21)
      CV22 = CV(22)
      CV23 = CV(23)
      CV24 = CV(24)
      CV25 = CV(25)
      T1 (1) = F1(D) - F1(C)
      T1 (2) = F2(D) - F2(C)
      T1 (3) = F3(D) - F3(C)
      T1 (4) = F4(D) - F4(C)
      T1 (5) = F5(D) - F5(C)
      T1 (6) = F6(D) - F6(C)
      T1 (7) = F7(D) - F7(C)
      T1 (8) = F8(D) - F8(C)
      T1 (9) = F9(D) - F9(C)
      DO 100 I=1,9
      M = I-1
  100 T2(I) = XMCOS(B,M) - XMCOS(A,M)
      TT = 0.
      DO 200 I=1,9
  200 TT = TT + T1(I)*T2(I)
      DINSC = TT
      RETURN
      END
C
C
C
      FUNCTION XMCOS (X,M)
      IMPLICIT REAL *8 (A-H , O-Z)
      REAL * 8 K,L
      INTEGER R
      COMMON  K,L,R1,R2,ZZ,G
      IF (M .NE. 0) GO TO 100
      XMCOS = DSIN(G*X) / G
      GO TO 400
```

```
100 FCM = IFAC(M)
    M1 = M / 2
    M11 = M1 + 1
    TT = 0.
    DO 200 I=1,M11
    R = I - 1
    FCM2R = IFAC(M - 2*R)
    T1 = 1.
    IF (R .NE. 0) T1=(-1)**R
    T2 = FCM / FCM2R
    T3 = 1.
    IF (M .NE. 2*R) T3=X**(M - 2*R)
    T4 = G**(2*R + 1)
200 TT = TT + T1 * T2 * T3 / T4
    TT1 = TT * DSIN(G*X)
    M2 = (M-1) / 2
    M22 = M2 + 1
    TT = 0.
    DO 300 I=1,M22
    R = I - 1
    FCM2R1 = IFAC(M - 2*R - 1)
    T1 = 1.
    IF (R .NE. 0) T1 = (-1)**R
    T2 = FCM / FCM2R1
    T3 = 1.
    IF (M .NE. (2*R + 1)) T3 = X**(M - 2*R - 1)
    T4 = G**(2*R + 2)
300 TT = TT + T1 * T2 * T3 / T4
    TT2 = TT * DCOS(G*X)
    XMCOS = TT1 + TT2
400 RETURN
    END
```

E.III PROGRAM PP1

```
C
C
C
C
C     ******************************************************************
C     ******************************************************************
C     **                                                              **
C     **THIS PROGRAM TO CALCULATE THE CURRENTS USING THE MOMENT METHOD**
C     **                                                              **
C     **                 WEIGHTING FUNCTIONS ARE                      **
C     **                                                              **
C     **              PIECEWISE SINUSOIDAL FUNCTIONS                  **
C     **                                                              **
C     ******************************************************************
C     ******************************************************************
C
C
C      F     IS THE FREQUENCY
C      VEL   IS THE SPACE VELOCITY
C      WL    IS THE WAVE LENGTH
C      L     IS THE ANTENNA HALF LENGTH
C      R1    IS THE INNER RADIUS OF FRILL
C      R2    IS THE OUTER RADIUS OF FRILL
C      EPS   IS THE ABSOLUTE DIELECTRIC CONSTANT (PERMITTIVITY)
C      N     IS THE NUMBER OF USED WALSH FUNCTIONS
C      NWD   IS THE NUMBER OF DIVISIONS FOR WALSH FUNCTIONS
C
C
C
      IMPLICIT REAL * 8 (A-H , O-Z)
C
      REAL * 8 L,K,LL
      REAL * 8 LP,LM,L2P,L2M,L4P,L4M,L34P,L34M
C
      COMPLEX * 16 DCMPLX
      COMPLEX * 16 EZ(32),E(32,1)
      COMPLEX * 16 DIKCF(32,32),A(32,32)
      COMPLEX * 16 E0(32),CURR(30)
      COMPLEX * 16 ZIN
      COMPLEX * 16 A1(32,32),E1(32,1)
C
      DIMENSION LL(32),UL(32),WF(32,32),W(32,32)
      DIMENSION XR(32),XI(32)
      DIMENSION X1(40),X2(40),X3(40),X4(40),X5(40),X6(40),X7(40),Y1(40),
     1          F1(40,40),F2(40,40),WK(2037),WA(32)
      DIMENSION Z(30),G1(32),G2(32)
C
      EXTERNAL EZR1,EZI1
      EXTERNAL EZR21,EZR22,EZR23,EZI21,EZI22,EZI23
      EXTERNAL EZR31,EZR32,EZR33,EZR34,EZR35,
     1         EZI31,EZI32,EZI33,EZI34,EZI35
      EXTERNAL EZR41,EZR42,EZR43,EZR44,EZR45,EZR46,EZR47,
     1         EZI41,EZI42,EZI43,EZI44,EZI45,EZI46,EZI47
```

```
C
      COMMON  K,L,R1,R2,PI,ZZ
C
C
C
C
      F   = 7.96
      VEL = 3.08
      WL  = VEL / F
      L   = 0.375 * WL
      R1  = 0.007022 * WL
      R2  = 2.23 * R1
      PI  = 4. * DATAN(1.D0)
      OMEGA = 2*PI*F
      EPS = 1.D-9 / (36.*PI)
      K   = OMEGA / VEL
      CC  = 4.*PI*OMEGA*EPS
C
      WRITE (6,10) F,VEL,WL,L,R1,R2,EPS
C
C
      N   = 4
      NWD = 8
C
C
C
C
C     ------------------------------------------------------------
C     CALCULATE THE LIMITS OF THE INTEGRALS
C     ------------------------------------------------------------
C
      STEP = 2*L/NWD
      LL(1) =-L
      DO 100 K1=1,NWD
      UL(K1) = LL(K1) + STEP
      IF (K1 .EQ. NWD/2) UL(K1) = 0.0
      IF (K1 .EQ. NWD) UL(K1) = L
      M = K1 + 1
      IF(M .GT. NWD) GO TO 100
      LL(M) = UL(K1)
  100 CONTINUE
C
      LP   =  L
      LM   = -L
      L2P  =  L/2.
      L2M  = -L/2
      L4P  =  L/4.
      L4M  = -L/4.
      L34P =  3.*L/4.
      L34M = -3.*L/4.
C
      WRITE (6,20)
      WRITE (6,21) (LL(I),UL(I), I=1,NWD)
```

```
      WRITE (6,22)
      WRITE (6,21) LM,LP,L34M,L34P,L2M,L2P,L4M,L4P
C
C     ------------------------------------------------------------------
C     GENERATE THE WALSH FUNCTIONS
C     ------------------------------------------------------------------
C
      CALL WALSHF(NWD,WF)
      DO 200 I=1,N
      DO 200 J=1,NWD
  200 W(I,J) = WF(2*I-1 , J)
C
      WRITE (6,30)
      DO 210 I=1,N
  210 WRITE (6,31) (W(I,J), J=1,NWD)
C
C     ------------------------------------------------------------------
C     CALCULATE THE INCIDENT FIELDS
C     ------------------------------------------------------------------
C
      XR(1) = DCADRE (EZR1,LM  ,LP  ,0.0,0.01,ERROR,IER)
      XI(1) = DCADRE (EZI1,LM  ,LP  ,0.0,0.01,ERROR,IER)
C
      ER21 = DCADRE (EZR21,LM  ,L2M ,0.0,0.01,ERROR,IER)
      ER22 = DCADRE (EZR22,L2M ,L2P ,0.0,0.01,ERROR,IER)
      ER23 = DCADRE (EZR23,L2P ,LP  ,0.0,0.01,ERROR,IER)
      EI21 = DCADRE (EZI21,LM  ,L2M ,0.0,0.01,ERROR,IER)
      EI22 = DCADRE (EZI22,L2M ,L2P ,0.0,0.01,ERROR,IER)
      EI23 = DCADRE (EZI23,L2P ,LP  ,0.0,0.01,ERROR,IER)
      XR(2) = ER21 + ER22 + ER23
      XI(2) = EI21 + EI22 + EI23
C
      ER31 = DCADRE (EZR31,LM  ,L34M,0.0,0.01,ERROR,IER)
      ER32 = DCADRE (EZR32,L34M,L4M ,0.0,0.01,ERROR,IER)
      ER33 = DCADRE (EZR33,L4M ,L4P ,0.0,0.01,ERROR,IER)
      ER34 = DCADRE (EZR34,L4P ,L34P,0.0,0.01,ERROR,IER)
      ER35 = DCADRE (EZR35,L34P,LP  ,0.0,0.01,ERROR,IER)
      EI31 = DCADRE (EZI31,LM  ,L34M,0.0,0.01,ERROR,IER)
      EI32 = DCADRE (EZI32,L34M,L4M ,0.0,0.01,ERROR,IER)
      EI33 = DCADRE (EZI33,L4M ,L4P ,0.0,0.01,ERROR,IER)
      EI34 = DCADRE (EZI34,L4P ,L34P,0.0,0.01,ERROR,IER)
      EI35 = DCADRE (EZI35,L34P,LP  ,0.0,0.01,ERROR,IER)
      XR(3) = ER31 + ER32 + ER33 + ER34 + ER35
      XI(3) = EI31 + EI32 + EI33 + EI34 + EI35
C
      ER41 = DCADRE (EZR41,LM  ,L34M,0.0,0.01,ERROR,IER)
      ER42 = DCADRE (EZR42,L34M,L2M ,0.0,0.01,ERROR,IER)
      ER43 = DCADRE (EZR43,L2M ,L4M ,0.0,0.01,ERROR,IER)
      ER44 = DCADRE (EZR44,L4M ,L4P ,0.0,0.01,ERROR,IER)
      ER45 = DCADRE (EZR45,L4P ,L2P ,0.0,0.01,ERROR,IER)
      ER46 = DCADRE (EZR46,L2P ,L34P,0.0,0.01,ERROR,IER)
      ER47 = DCADRE (EZR47,L34P,LP  ,0.0,0.01,ERROR,IER)
      EI41 = DCADRE (EZI41,LM  ,L34M,0.0,0.01,ERROR,IER)
```

```
      EI42 = DCADRE (EZI42,L34M,L2M  ,0.0,0.01,ERROR,IER)
      EI43 = DCADRE (EZI43,L2M ,L4M  ,0.0,0.01,ERROR,IER)
      EI44 = DCADRE (EZI44,L4M ,L4P  ,0.0,0.01,ERROR,IER)
      EI45 = DCADRE (EZI45,L4P ,L2P  ,0.0,0.01,ERROR,IER)
      EI46 = DCADRE (EZI46,L2P ,L34P,0.0,0.01,ERROR,IER)
      EI47 = DCADRE (EZI47,L34P,LP   ,0.0,0.01,ERROR,IER)
      XR(4) = ER41 + ER42 + ER43 + ER44 + ER45 + ER46 + ER47
      XI(4) = EI41 + EI42 + EI43 + EI44 + EI45 + EI46 + EI47
C
      DEN = 2. * DLOG(R2/R1)
      DO 300 K1=1,N
      EZ(K1) = DCMPLX(XR(K1) , XI(K1)) / DEN
      E(K1,1) = -K * EZ(K1)
  300 CONTINUE
C
      WRITE (6,40)
      WRITE (6,41) (I,E(I,1) , I=1,N)
C
C     ------------------------------------------------------------
C     CALCULATE THE ENTRIES OF THE MATRIX
C
C     ------------------------------------------------------------
      NN = 40
      NX = 20
      NY = 20
C
      DO 450 I =1,N
      DO 450 JJ=1,NWD
C
      DELY1 = (UL(JJ) - LL(JJ)) / (NY-2)
      Y1(1) = LL(JJ) - DELY1/2.
      DO 400 JJJ=2,NY
  400 Y1(JJJ) = Y1(JJJ-1) + DELY1
C
      GO TO (410,420,430,440) , I
C
  410 CONTINUE
      IF (JJ .GT. 1) GO TO 412
      DELX1 = (LP - LM) / (NX-2)
      X1(1) = LM - DELX1/2.
      DO 411 III=2,NX
  411 X1(III) = X1(III-1) + DELX1
C
  412 DO 413 III=1,NX
      DO 413 JJJ=1,NY
      F1(III,JJJ) = FUR1(X1(III),Y1(JJJ))
      F2(III,JJJ) = FUI1(X1(III),Y1(JJJ))
  413 CONTINUE
      CALL DBCQDU(F1,NN,X1,NX,Y1,NY,LM  ,LP  ,LL(JJ),UL(JJ),QR  ,WK,IER)
      CALL DBCQDU(F2,NN,X1,NX,Y1,NY,LM  ,LP  ,LL(JJ),UL(JJ),QI  ,WK,IER)
C
      XREAL = QR
      XIMAG = QI
```

```
      GO TO 450
C
  420 CONTINUE
      IF (JJ .GT. 1) GO TO 422
      DELX1 = (L2M -LM  ) / (NX-2)
      DELX2 = (L2P -L2M ) / (NX-2)
      DELX3 = (LP  -L2P ) / (NX-2)
      X1(1) = LM   - DELX1/2.
      X2(1) = L2M  - DELX2/2.
      X3(1) = L2P  - DELX3/2.
      DO 421 III=2,NX
      X1(III) = X1(III-1) + DELX1
      X2(III) = X2(III-1) + DELX2
      X3(III) = X3(III-1) + DELX3
  421 CONTINUE
C
  422 DO 423 III=1,NX
      DO 423 JJJ=1,NY
      F1(III,JJJ) = FUR21 (X1(III),Y1(JJJ))
      F2(III,JJJ) = FUI21 (X1(III),Y1(JJJ))
  423 CONTINUE
      CALL DBCQDU(F1,NN,X1,NX,Y1,NY,LM  ,L2M ,LL(JJ),UL(JJ),QR21,WK,IER)
      CALL DBCQDU(F2,NN,X1,NX,Y1,NY,LM  ,L2M ,LL(JJ),UL(JJ),QI21,WK,IER)
      DO 424 III=1,NX
      DO 424 JJJ=1,NY
      F1(III,JJJ) = FUR22 (X2(III),Y1(JJJ))
      F2(III,JJJ) = FUI22 (X2(III),Y1(JJJ))
  424 CONTINUE
      CALL DBCQDU(F1,NN,X2,NX,Y1,NY,L2M ,L2P ,LL(JJ),UL(JJ),QR22,WK,IER)
      CALL DBCQDU(F2,NN,X2,NX,Y1,NY,L2M ,L2P ,LL(JJ),UL(JJ),QI22,WK,IER)
      DO 425 III=1,NX
      DO 425 JJJ=1,NY
      F1(III,JJJ) = FUR23 (X3(III),Y1(JJJ))
      F2(III,JJJ) = FUI23 (X3(III),Y1(JJJ))
  425 CONTINUE
      CALL DBCQDU(F1,NN,X3,NX,Y1,NY,L2P ,LP  ,LL(JJ),UL(JJ),QR23,WK,IER)
      CALL DBCQDU(F2,NN,X3,NX,Y1,NY,L2P ,LP  ,LL(JJ),UL(JJ),QI23,WK,IER)
C
      XREAL = QR21 + QR22 + QR23
      XIMAG = QI21 + QI22 + QI23
      GO TO 450
C
  430 CONTINUE
      IF (JJ .GT. 1) GO TO 432
      DELX1 = (L34M-LM  ) / (NX-2)
      DELX2 = (L4M -L34M) / (NX-2)
      DELX3 = (L4P -L4M ) / (NX-2)
      DELX4 = (L34P-L4P ) / (NX-2)
      DELX5 = (LP  -L34P) / (NX-2)
      X1(1) = LM   - DELX1/2.
      X2(1) = L34M - DELX2/2.
      X3(1) = L4M  - DELX3/2.
      X4(1) = L4P  - DELX4/2.
```

```
      X5(1) = L34P - DELX5/2.
      DO 431 III=2,NX
      X1(III) = X1(III-1) + DELX1
      X2(III) = X2(III-1) + DELX2
      X3(III) = X3(III-1) + DELX3
      X4(III) = X4(III-1) + DELX4
      X5(III) = X5(III-1) + DELX5
  431 CONTINUE
C
  432 DO 433 III=1,NX
      DO 433 JJJ=1,NY
      F1(III,JJJ) = FUR31 (X1(III),Y1(JJJ))
      F2(III,JJJ) = FUI31 (X1(III),Y1(JJJ))
  433 CONTINUE
      CALL DBCQDU(F1,NN,X1,NX,Y1,NY,LM   ,L34M,LL(JJ),UL(JJ),QR31,WK,IER)
      CALL DBCQDU(F2,NN,X1,NX,Y1,NY,LM   ,L34M,LL(JJ),UL(JJ),QI31,WK,IER)
      DO 434 III=1,NX
      DO 434 JJJ=1,NY
      F1(III,JJJ) = FUR32 (X2(III),Y1(JJJ))
      F2(III,JJJ) = FUI32 (X2(III),Y1(JJJ))
  434 CONTINUE
      CALL DBCQDU(F1,NN,X2,NX,Y1,NY,L34M,L4M ,LL(JJ),UL(JJ),QR32,WK,IER)
      CALL DBCQDU(F2,NN,X2,NX,Y1,NY,L34M,L4M ,LL(JJ),UL(JJ),QI32,WK,IER)
      DO 435 III=1,NX
      DO 435 JJJ=1,NY
      F1(III,JJJ) = FUR33 (X3(III),Y1(JJJ))
      F2(III,JJJ) = FUI33 (X3(III),Y1(JJJ))
  435 CONTINUE
      CALL DBCQDU(F1,NN,X3,NX,Y1,NY,L4M ,L4P ,LL(JJ),UL(JJ),QR33,WK,IER)
      CALL DBCQDU(F2,NN,X3,NX,Y1,NY,L4M ,L4P ,LL(JJ),UL(JJ),QI33,WK,IER)
      DO 436 III=1,NX
      DO 436 JJJ=1,NY
      F1(III,JJJ) = FUR34 (X4(III),Y1(JJJ))
      F2(III,JJJ) = FUI34 (X4(III),Y1(JJJ))
  436 CONTINUE
      CALL DBCQDU(F1,NN,X4,NX,Y1,NY,L4P ,L34P,LL(JJ),UL(JJ),QR34,WK,IER)
      CALL DBCQDU(F2,NN,X4,NX,Y1,NY,L4P ,L34P,LL(JJ),UL(JJ),QI34,WK,IER)
      DO 437 III=1,NX
      DO 437 JJJ=1,NY
      F1(III,JJJ) = FUR35 (X5(III),Y1(JJJ))
      F2(III,JJJ) = FUI35 (X5(III),Y1(JJJ))
  437 CONTINUE
      CALL DBCQDU(F1,NN,X5,NX,Y1,NY,L34P,LP   ,LL(JJ),UL(JJ),QR35,WK,IER)
      CALL DBCQDU(F2,NN,X5,NX,Y1,NY,L34P,LP   ,LL(JJ),UL(JJ),QI35,WK,IER)
C
      XREAL = QR31 + QR32 + QR33 + QR34 + QR45
      XIMAG = QI31 + QI32 + QI33 + QI34 + QI35
      GO TO 450
C
  440 CONTINUE
      IF (JJ .GT. 1) GO TO 442
      DELX1 = (L34M-LM  ) / (NX-2)
      DELX2 = (L2M -L34M) / (NX-2)
```

2

```
      DELX3 = (L4M -L2M ) / (NX-2)
      DELX4 = (L4P -L4M ) / (NX-2)
      DELX5 = (L2P -L4P ) / (NX-2)
      DELX6 = (L34P-L2P ) / (NX-2)
      DELX7 = (LP  -L34P) / (NX-2)
      X1(1) = LM   - DELX1/2.
      X2(1) = L34M - DELX2/2.
      X3(1) = L2M  - DELX3/2.
      X4(1) = L4M  - DELX4/2.
      X5(1) = L4P  - DELX5/2.
      X6(1) = L2P  - DELX6/2.
      X7(1) = L34P - DELX7/2.
      DO 441 III=2,NX
      X1(III) = X1(III-1) + DELX1
      X2(III) = X2(III-1) + DELX2
      X3(III) = X3(III-1) + DELX3
      X4(III) = X4(III-1) + DELX4
      X5(III) = X5(III-1) + DELX5
      X6(III) = X6(III-1) + DELX6
      X7(III) = X7(III-1) + DELX7
  441 CONTINUE
C
  442 DO 443 III=1,NX
      DO 443 JJJ=1,NY
      F1(III,JJJ) = FUR41 (X1(III),Y1(JJJ))
      F2(III,JJJ) = FUI41 (X1(III),Y1(JJJ))
  443 CONTINUE
      CALL DBCQDU(F1,NN,X1,NX,Y1,NY,LM   ,L34M,LL(JJ),UL(JJ),QR41,WK,IER)
      CALL DBCQDU(F2,NN,X1,NX,Y1,NY,LM   ,L34M,LL(JJ),UL(JJ),QI41,WK,IER)
      DO 444 III=1,NX
      DO 444 JJJ=1,NY
      F1(III,JJJ) = FUR42 (X2(III),Y1(JJJ))
      F2(III,JJJ) = FUI42 (X2(III),Y1(JJJ))
  444 CONTINUE
      CALL DBCQDU(F1,NN,X2,NX,Y1,NY,L34M,L2M ,LL(JJ),UL(JJ),QR42,WK,IER)
      CALL DBCQDU(F2,NN,X2,NX,Y1,NY,L34M,L2M ,LL(JJ),UL(JJ),QI42,WK,IER)
      DO 445 III=1,NX
      DO 445 JJJ=1,NY
      F1(III,JJJ) = FUR43 (X3(III),Y1(JJJ))
      F2(III,JJJ) = FUI43 (X3(III),Y1(JJJ))
  445 CONTINUE
      CALL DBCQDU(F1,NN,X3,NX,Y1,NY,L2M ,L4M ,LL(JJ),UL(JJ),QR43,WK,IER)
      CALL DBCQDU(F2,NN,X3,NX,Y1,NY,L2M ,L4M ,LL(JJ),UL(JJ),QI43,WK,IER)
      DO 446 III=1,NX
      DO 446 JJJ=1,NY
      F1(III,JJJ) = FUR44 (X4(III),Y1(JJJ))
      F2(III,JJJ) = FUI44 (X4(III),Y1(JJJ))
  446 CONTINUE
      CALL DBCQDU(F1,NN,X4,NX,Y1,NY,L4M ,L4P ,LL(JJ),UL(JJ),QR44,WK,IER)
      CALL DBCQDU(F2,NN,X4,NX,Y1,NY,L4M ,L4P ,LL(JJ),UL(JJ),QI44,WK,IER)
      DO 447 III=1,NX
      DO 447 JJJ=1,NY
      F1(III,JJJ) = FUR45 (X5(III),Y1(JJJ))
```

```
      F2(III,JJJ) = FUI45 (X5(III),Y1(JJJ))
  447 CONTINUE
      CALL DBCQDU(F1,NN,X5,NX,Y1,NY,L4P ,L2P ,LL(JJ),UL(JJ),QR45,WK,IER)
      CALL DBCQDU(F2,NN,X5,NX,Y1,NY,L4P ,L2P ,LL(JJ),UL(JJ),QI45,WK,IER)
      DO 448 III=1,NX
      DO 448 JJJ=1,NY
      F1(III,JJJ) = FUR46 (X6(III),Y1(JJJ))
      F2(III,JJJ) = FUI46 (X6(III),Y1(JJJ))
  448 CONTINUE
      CALL DBCQDU(F1,NN,X6,NX,Y1,NY,L2P ,L34P,LL(JJ),UL(JJ),QR46,WK,IER)
      CALL DBCQDU(F2,NN,X6,NX,Y1,NY,L2P ,L34P,LL(JJ),UL(JJ),QI46,WK,IER)
      DO 449 III=1,NX
      DO 449 JJJ=1,NY
      F1(III,JJJ) = FUR47 (X7(III),Y1(JJJ))
      F2(III,JJJ) = FUI47 (X7(III),Y1(JJJ))
  449 CONTINUE
      CALL DBCQDU(F1,NN,X7,NX,Y1,NY,L34P,LP  ,LL(JJ),UL(JJ),QR47,WK,IER)
      CALL DBCQDU(F2,NN,X7,NX,Y1,NY,L34P,LP  ,LL(JJ),UL(JJ),QI47,WK,IER)
C
      XREAL = QR41 + QR42 + QR43 + QR44 + QR45 + QR46 + QR47
      XIMAG = QI41 + QI42 + QI43 + QI44 + QI45 + QI46 + QI47
C
  450 DIKCF(I,JJ) = DCMPLX(XREAL,XIMAG) * (-K/CC)
C
      DO 460 I=1,N
      DO 460 J=1,N
      A(I,J) = (0.0D0,0.0D0)
      DO 460 JJ=1,NWD
  460 A(I,J) = A(I,J) + DIKCF(I,JJ) * W(J,JJ)
C
      WRITE (6,50)
      DO 470 I=1,N
  470 WRITE (6,51) (A(I,J), J=1,N)
```

```
C
C
C
      FUNCTION EZR(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      T1 = K * DSQRT(X*X + R1*R1)
      T2 = K * DSQRT(X*X + R2*R2)
      EZR =  DCOS(T1)/T1 - DCOS(T2)/T2
      RETURN
      END
C
C
C
      FUNCTION EZR1(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * X/(2.*L))
      EZR1  = EZR(X) * F
      RETURN
      END
C
C
C
      FUNCTION EZR21(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X+L/4.)/(L*3./2.))
      EZR21 = EZR(X) * F
      RETURN
      END
C
C
C
      FUNCTION EZR22(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X+L/4.)/(L*3./2.))
     1   + DCOS (PI * (X-L/4.)/(L*3./2.))
      EZR22 = EZR(X) * F
      RETURN
      END
```

```
C
C
      FUNCTION EZR23(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =    DCOS (PI * (X-L/4.)/(L*3./2.))
      EZR23 = EZR(X) * F
      RETURN
      END
C
C
C
      FUNCTION EZR31(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =    DCOS (PI * (X+L*3./8.)/(L*5./4.))
      EZR31 = EZR(X) * F
      RETURN
      END
C
C
C
      FUNCTION EZR32(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =    DCOS (PI * (X+L*3./8.)/(L*5./4.))
     1     + DCOS (PI * (X)/(L*3./2.))
      EZR32 = EZR(X) * F
      RETURN
      END
C
C
C
      FUNCTION EZR33(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =    DCOS (PI * (X+L*3./8.)/(L*5./4.))
     1     + DCOS (PI * (X)/(L*3./2.))
     2     + DCOS (PI * (X-L*3./8.)/(L*5./4.))
      EZR33 = EZR(X) * F
      RETURN
      END
```

```
C
C
C
      FUNCTION EZR34(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X)/(L*3./2.))
     1     + DCOS (PI * (X-L*3./8.)/(L*5./4.))
      EZR34 = EZR(X) * F
      RETURN
      END
C
C
C
      FUNCTION EZR35(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X-L*3./8.)/(L*5./4.))
      EZR35 = EZR(X) * F
      RETURN
      END
C
C
C
      FUNCTION EZR41(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X+L*3./8.)/(L*5./4.))
      EZR41 = EZR(X) * F
      RETURN
      END
C
C
C
      FUNCTION EZR42(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X+L*3./8.)/(L*5./4.))
     1     + DCOS (PI * (X+L*1./8.)/(L*5./4.))
      EZR42 = EZR(X) * F
      RETURN
      END
```

```
C
C
C
      FUNCTION EZR43(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X+L*3./8.)/(L*5./4.))
     1    + DCOS (PI * (X+L*1./8.)/(L*5./4.))
     2    + DCOS (PI * (X-L*1./8.)/(L*5./4.))
      EZR43 = EZR(X) * F
      RETURN
      END
C
C
C
      FUNCTION EZR44(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X+L*3./8.)/(L*5./4.))
     1    + DCOS (PI * (X+L*1./8.)/(L*5./4.))
     2    + DCOS (PI * (X-L*1./8.)/(L*5./4.))
     3    + DCOS (PI * (X-L*3./8.)/(L*5./4.))
      EZR44 = EZR(X) * F
      RETURN
      END
C
C
C
      FUNCTION EZR45(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X+L*1./8.)/(L*5./4.))
     1    + DCOS (PI * (X-L*1./8.)/(L*5./4.))
     2    + DCOS (PI * (X-L*3./8.)/(L*5./4.))
      EZR45 = EZR(X) * F
      RETURN
      END
```

```
C
C
C
      FUNCTION EZR46(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =    DCOS (PI * (X-L*1./8.)/(L*5./4.))
     1    + DCOS (PI * (X-L*3./8.)/(L*5./4.))
      EZR46 = EZR(X) * F
      RETURN
      END
C
C
C
      FUNCTION EZR47(X)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =    DCOS (PI * (X-L*3./8.)/(L*5./4.))
      EZR47 = EZR(X) * F
      RETURN
      END
```

```
C
C
C
      FUNCTION FUR(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      R12 = R1 * R1
      T1 = K * DSQRT((X-Y)**2 + R12)
      T2 = K * DSQRT((X+L)**2 + R12)
      T3 = K * DSQRT((X-L)**2 + R12)
      T4 = K * (L-Y)
      T5 = K * (L+Y)
      T6 = 2.*K*L
      TA = DSIN(T1) / T1
      TB = DSIN(T2) / T2
      TC = DSIN(T3) / T3
      TD = DSIN(T4)
      TE = DSIN(T5)
      TF = DSIN(T6)
      FUR = TA - (TB*TD + TC*TE) / TF
      RETURN
      END
C
C
C
      FUNCTION FUR1(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * X/(2.*L))
      FUR1  = FUR(X,Y) * F
      RETURN
      END
C
C
C
      FUNCTION FUR21(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X+L/4.)/(L*3./2.))
      FUR21 = FUR(X,Y) * F
      RETURN
      END
```

```
C
C
C
      FUNCTION FUR22(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X+L/4.)/(L*3./2.))
     1    + DCOS (PI * (X-L/4.)/(L*3./2.))
      FUR22 = FUR(X,Y) * F
      RETURN
      END
C
C
C
      FUNCTION FUR23(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X-L/4.)/(L*3./2.))
      FUR23 = FUR(X,Y) * F
      RETURN
      END
C
C
C
      FUNCTION FUR31(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X+L*3./8.)/(L*5./4.))
      FUR31 = FUR(X,Y) * F
      RETURN
      END
C
C
C
      FUNCTION FUR32(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X+L*3./8.)/(L*5./4.))
     1    + DCOS (PI * (X)/(L*3./2.))
      FUR32 = FUR(X,Y) * F
      RETURN
      END
```

```
C
C
C
      FUNCTION FUR33(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =    DCOS (PI  * (X+L*3./8.)/(L*5./4.))
     1    + DCOS (PI  * (X)/(L*3./2.))
     2    + DCOS (PI  * (X-L*3./8.)/(L*5./4.))
      FUR33 = FUR(X,Y) * F
      RETURN
      END
C
C
C
      FUNCTION FUR34(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =    DCOS (PI  * (X)/(L*3./2.))
     1    + DCOS (PI  * (X-L*3./8.)/(L*5./4.))
      FUR34 = FUR(X,Y) * F
      RETURN
      END
C
C
C
      FUNCTION FUR35(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =    DCOS (PI  * (X-L*3./8.)/(L*5./4.))
      FUR35 = FUR(X,Y) * F
      RETURN
      END
```

```
C
C
C
      FUNCTION FUR41(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X+L*3./8.)/(L*5./4.))
      FUR41 = FUR(X,Y) * F
      RETURN
      END
C
C
C
      FUNCTION FUR42(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X+L*3./8.)/(L*5./4.))
     1    + DCOS (PI * (X+L*1./8.)/(L*5./4.))
      FUR42 = FUR(X,Y) * F
      RETURN
      END
C
C
C
      FUNCTION FUR43(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X+L*3./8.)/(L*5./4.))
     1    + DCOS (PI * (X+L*1./8.)/(L*5./4.))
     2    + DCOS (PI * (X-L*1./8.)/(L*5./4.))
      FUR43 = FUR(X,Y) * F
      RETURN
      END
C
C
C
      FUNCTION FUR44(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =   DCOS (PI * (X+L*3./8.)/(L*5./4.))
     1    + DCOS (PI * (X+L*1./8.)/(L*5./4.))
     2    + DCOS (PI * (X-L*1./8.)/(L*5./4.))
     3    + DCOS (PI * (X-L*3./8.)/(L*5./4.))
      FUR44 = FUR(X,Y) * F
      RETURN
      END
```

```
C
C
C
      FUNCTION FUR45(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =    DCOS (PI * (X+L*1./8.)/(L*5./4.))
     1    + DCOS (PI * (X-L*1./8.)/(L*5./4.))
     2    + DCOS (PI * (X-L*3./8.)/(L*5./4.))
      FUR45 = FUR(X,Y) * F
      RETURN
      END
C
C
C
      FUNCTION FUR46(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =    DCOS (PI * (X-L*1./8.)/(L*5./4.))
     1    + DCOS (PI * (X-L*3./8.)/(L*5./4.))
      FUR46 = FUR(X,Y) * F
      RETURN
      END
C
C
C
      FUNCTION FUR47(X,Y)
C
      IMPLICIT REAL*8(A-H , O-Z)
      REAL * 8 K,L
      COMMON  K,L,R1,R2,PI,ZZ
      F =    DCOS (PI * (X-L*3./8.)/(L*5./4.))
      FUR47 = FUR(X,Y) * F
      RETURN
      END
```