STRESS ANALYSIS OF A POROUS DEFORMABLE SEABED UNDER WAVE LOADING

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by

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A thesis submitted to the School of Graduate Studies in partial fulfillment of the requirements for the degree of Master of Engineering

Faculty of Engineering and Applied Science
Memorial University of Newfoundland

September 1984

St. John's Newfoundland

To my parents

Jonathan and Beryl Raman-Nair

Abstract

Wave-induced pore pressure and effective stresses in sea beds of finite and infinite depth are computed using the Biot theory of deformation of porous media (1962). The Biot theory is simplified to a quasi-static case and the sea bed response is determined using the Papkovich-Neuber technique of the theory of elasticity. The general theory (Biot, 1962) is compared with the quasi-static theory.

The results are used to determine the depth of liquefaction and the depth of "sliding" failure, the latter being investigated via the Mohr-Coulomb failure criterion.

Acknowledgements

I should like to thank my supervisor Dr. Gary Sabin for his patient guidance at every stage of the work. I am also grateful for the helpful suggestions of the Associate Dean, Dr. T.R. Chari. For assistance with the plotting of the graphs, I am indebted to Mr. Mervin Marshall and Mr. Ramakanth Reddy, and for the drawings in sections 2, 3 and 6 I thank Mr. Daryl Thompson. In addition, I gratefully acknowledge the financial support of the School of Graduate Studies and the Faculty of Engineering and Applied Science.

Finally, I express my thanks to God for his guidance in my work and in my life.

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List of Symbols

Latin symbols

a_o	wave amplitude
c	consolidation coefficient (equation 5.18)
$\bar{c}$	cohesion intercept (clay)
C	an elastic modulus of soil (equation 2.38)
e_{ij}	strain tensor
f	porosity of soil
f_i	force vector on solid part of an elemental area of soil
F_i	force vector on fluid part of an elemental area of soil
F_{ij}	sum of all inter-granular forces on an area δA of soil normal to the j axis.
g	acceleration due to gravity
$G(t)$	relaxation modulus
$\bar{G}^*$	complex modulus
h	water depth
H	an elastic modulus of soil (equation 2.38)
i	$= \sqrt{-1}$
I_1, I_2, I_3	strain invariants
k	permeability of soil in m^2
k_o	permeability of soil in ms^{-1}
K	kinetic energy of a volume Ω of soil
K_b	bulk modulus of soil skeleton
K_f	bulk modulus of water
K_r	bulk modulus of soil grains

L	wavelength; Lagrangian of a volume Ω of soil
M	an elastic modulus of soil (equation 2.38)
n_f	dynamic viscosity of water (kg m^{-1} s^{-1})
n_i	unit normal vector to a surface area Σ of soil
p	pore pressure
P	water pressure above sea bed
P_o	amplitude of wave-induced pressure at the mudline
t	time
T	wave period; kinetic energy per unit volume at a point in a volume Ω of soil
T_i	surface traction on a surface Σ of soil
$\underline{u} = (u_1,u_2,u_3)$	displacement vector of the solid matrix at a point in the soil continuum
$\underline{U} = (U_1,U_2,U_3)$	displacement vector of fluid at a point in the soil continuum
$U_i(z)$	the depth dependent part of the function u_i, $i = 1,2$
V	Speed of water particles at the free surface; potential energy of a volume Ω of soil
$\underline{V} = (V_1,V_2)$	velocity of water particles above the sea bed
$\underline{w} = (w_1,w_2,w_3) = f(\underline{U} - \underline{u})$	
W	strain energy per unit volume at a point in the soil continuum
$W_i(z)$	the depth dependent part of the function w_i, $i = 1,2$
x	the horizontal coordinate direction (wave direction)
$(x_1,x_2,x_3) = (x,y,z)$, Cartesian coordinates	
X_i	dissipative body force due to fluid-solid relative motion
z	the vertical coordinate direction (depth)
z_o	depth of sea bed

Greek symbols

α	ratio of the elastic moduli C to M
α_o	soil parameter (equation 5.18)
γ	unit weight of water; "coefficient of fluid content"
γ_b	buoyant unit weight of water
δ	variational operator; damping parameter
δA	small area of soil
δ_{ij}	Kronecker delta
Δ	"unjacketed" compressibility
ε	dilatation of solid matrix (div $\underline{u}$)
ζ	"increment of fluid content" (-div $\underline{w}$)
$\eta(x,t)$	water surface elevation above still water level
λ	wave number
λ'	parameter dependent on λ, ω, c (equation 5.26)
λ_c	an elastic modulus of soil (equation 2.39)
λ^*	Lamé constant
$\overline{\lambda}^*$	complex Lamé constant
μ	shear modulus of soil
$\overline{\mu}$	complex shear modulus of soil
ν	Poisson's ratio
ρ	density of saturated soil
ρ_a	"added mass" parameter (kg/m^3)
ρ_f	density of water
ρ_s	density of soil grains
σ_{ij}	stress tensor on solid part of an area δA of soil, averaged over entire area δA

Σ	surface area bounding a volume Ω of soil
τ_{ij}	total stress tensor (wave-induced)
τ'_{ij}	effective stress tensor (wave-induced)
$\overline{\tau}'_{ij}$	net effective stress tensor
$\tau_{ij}^{'(o)}$	initial static effective stress tensor
ϕ	stress function; stress angle
ϕ	water velocity potential above sea bed
ψ	stress function
ω	circular frequency of wave
Ω	an arbitrary volume of soil

Symbols on the graphs

Real (P/P_o)	: Normalized wave-induced pore pressure
Arg(P)	: Pore pressure phase shift
SX	: Wave-induced horizontal effective stress
SZ	: Wave-induced vertical effective stress
SXZ	: Wave-induced shear stress
DELTA	: Damping parameter δ

INTRODUCTION

The cyclic loading of the sea floor by ocean waves during a storm can have a significant effect on the pore pressures and effective stresses within the bed. The dynamic pore pressures (i.e. pore pressures in excess of hydrostatic) may attain values which reduce the effective stresses to zero. In such a case the soil is said to be liquefied and in fact it behaves as a dense liquid. Another mode of failure may be termed "sliding" failure, in which the effective stresses overcome the soil internal friction. Sliding failure may be investigated via the Mohr-Coulomb failure criterion.

The literature contains several examples of soil failure due to the above mechanisms. The most striking example is the failure of two shell platforms in the Gulf of Mexico during Hurricane Camille of August 1969. (Bea, 1971). One platform was displaced at the mudline about three or four feet from its installed position. The other platform had fallen on its side and was displaced about one hundred feet opposite to the wave direction. Both wells were plugged and abandoned. Several intensive investigations established without a doubt that soil movements were the major contributing factor in these failures.

Another example was the failure of a ten foot diameter steel pipeline in Lake Ontario. Failure has occurred during storms when sections of the pipe rose and translated laterally. The specific gravity of the pipe is less than the specific gravity of liquefied sand (about 1.75 to 1.80). Liquefaction of the sand by storm waves caused flotation of the pipe which was then translated by current forces.

The stability of the seabed is thus an important consideration in the design of offshore structures. This thesis is addressed to the determination of the instantaneous response (i.e. pore pressures and effective stresses) of a porous deformable seabed under wave loading. The bed is assumed to be homogeneous, isotropic and of uniform porosity. Biot's linear theory of poroelastic media (1962) is used to model the soil. The development of residual pore pressures due to soil densification over several loading cycles is not considered here. From the instantaneous values of effective stress, the depth of liquefaction is determined and the Mohr-Coulomb failure criterion is used to determine the depth to which "sliding" failure occurs. The problem is solved for beds of finite depth as well as for infinite depth.

The theory is applicable to both sands and clays and it is shown how the general theory (Biot, 1962) may be simplified for the modelling of sand beds. The simplified theory is a quasi-static one and it approximates the Biot theory of three dimensional consolidation (Biot, 1941). A solution technique to the equations of the simplified theory is illustrated, using the Papkovich-Neuber solution of the equilibrium equations of the theory of elasticity.

Following the procedure of Yamamoto (1983) the effects of soil internal damping is introduced into the Biot theory. An alternative viscoelastic model is also discussed. The seabed responses predicted by the general dynamic theory and the quasi-static theory are compared.

1. SURVEY OF PREVIOUS WORK

The response of the seabed to wave loading has been investigated by a number of authors. As pointed out by Finn et al. (1983) there are two aspects to the problem. Both transient and residual pore water pressures are generated in the seabed due to wave loads. The transient pore pressures and stresses are the instantaneous response of the pore water and seabed to the loading. Residual pore pressures represent the pore pressure build-up over several loading cycles, a phenomenon which is dependent on the drainage characteristics of the seafloor and the duration and intensity of the loading. Both aspects of the problem have been examined by researchers.

1.1 Transient response of the seabed

Many models have been proposed for the prediction of transient pore pressures and stresses. Putnam (1949) assumed the soil skeleton to be incompressible and hydraulically isotropic (i.e. having equal horizontal and vertical permeabilities). He assumed that the flow through the porous soil is governed by Darcy's Law and that sea-water is incompressible. Sleath (1970) added to this the assumption that the soil skeleton is hydraulically anisotropic. For the hydraulically isotropic case the governing equation for the excess (wave-induced) pore pressure p is the same for both authors:

$$\nabla^2 p = 0 \qquad (1.1)$$

Moshagen and Tørum (1975) assumed that the soil skeleton is rigid and non-deformable but that the pore water is compressible. The governing equation in the case of hydraulic isotropy is

$$\frac{k_o}{\gamma} \nabla^2 p = \frac{f}{K'} \frac{\partial p}{\partial t} \tag{1.2}$$

where k_o is the permeability of the soil, γ is the unit weight of water, f is the porosity of the soil, and K' is the apparent bulk modulus of elasticity of water.

The above models do not take into account the deformation of the seabed and the coupling of pore water and soil skeleton. Consequently they lead to incomplete results in general. In 1941 M.A. Biot presented a general theory of three dimensional consolidation which incorporates these aspects of the seafloor behaviour. The seabed is modelled as a porous elastic solid skeleton containing compressible pore water. It is assumed that the equilibrium equations of elasticity and Hooke's law are valid for the soil skeleton and that the flow of pore water is governed by Darcy's Law. The equation for excess pore pressure p is

$$\frac{k_o}{\gamma} \nabla^2 p = \frac{f}{K'} \frac{\partial p}{\partial t} + \frac{\partial \varepsilon}{\partial t} \tag{1.3}$$

where ε is the volumetric strain of the soil. This equation has also been derived by Verruijt (1969). Yamamoto (1978) was the first to apply the Biot theory to the present problem and he derived analytical solutions for both an infinitely deep seabed and beds of finite thickness. Yamamoto obtained excellent correlation between experimental results for sand beds and the Biot theory. Madsen (1978) performed a similar analysis, also based on Biot's equations, but he considered only beds of infinite thickness. He also included the effect of hydraulic anisotropy while assuming soil isotropy. However, such an assumption is not physically consistent, as pointed out by Biot, and the poroelastic theory of anisotropic media (Biot, 1955) must be used.

An approximate solution to the problem was developed by Mei (1982) and is also based on Biot's theory (1941). In this model the seabed is divided into a poroelastic boundary layer near the mudline and an elastic sublayer. Within the boundary layer there is significant relative movement between the pore water and soil particles, whereas below the boundary layer the pore water and solid skeleton move approximately in unison. Effective stresses and pore pressures were determined for sand beds of infinite and finite depths. The results approximate those of Yamamoto (1978).

Siddharthan and Finn (1979) have also used the Biot theory (1941) to determine transient pore pressures and effective stresses. Their analysis includes the case of layered soils with hydraulic anisotropy and variable soil depth. The Mohr-Coulomb failure criterion is used to determine seabed stability, as was done by Yamamoto (1978).

As mentioned above, Yamamoto obtained excellent agreement between experiment and the Biot theory of 1941 for sand beds. For clay beds, however, the theory does not adequately model the bed response because it does not include the effects of soil inertia, such effects being significant for the relatively soft clay beds. In 1962 Biot published a more general theory in which soil inertia is taken into account and a more complete constitutive relation replaces Hooke's law. We show in this thesis, however, that the constitutive relation is approximated very closely by Hooke's law. Yamamoto (1983) has used Biot's theory (1962) to determine the response of beds of infinite depth. As suggested by Stoll and Bryan (1970), he has incorporated the Coulomb damping effects of the soil grain-to-grain friction by using appropriate complex values for the elastic moduli of the soil. The Biot theory of 1962 provides a comprehensive model for the determination of the instantaneous response of the seabed, and it is used in this thesis.

1.2 Residual pore pressures.

It has been observed that when saturated soil is subjected to cyclic loading the pore pressure increases over several cycles. This is due to the fact that permanent shear strains and permanent volume contraction (densification) occurs after each cycle of loading. A model for the determination of these pore pressure increases (residuals) has been developed by Seed and Rahman (1978). Their model is based on Terzaghi's theory of one dimensional consolidation and the governing equation for excess pore pressure is

$$\frac{\partial}{\partial z}\left(\frac{k_o}{\gamma}\frac{\partial p}{\partial z}\right) = \frac{\partial e}{\partial t} \qquad (1.4)$$

where e is the volumetric strain of the soil due to the above-mentioned densification phenomenon, and z is the vertical coordinate direction. This is a simplified form of equation (1.3) of the Biot theory. It should be noted, however, that the volumetric strain ε in (1.3) is due to elastic deformation at constant density whereas in (1.4) the volumetric strain e is due to densification. Zienkiewicz et al. (1982) have suggested a method of incorporating the effects of densification into equation (1.3). In the Seed-Rahman model an equation is developed to relate the strain rate $\frac{\partial e}{\partial t}$ to the rate of pore pressure dissipation. Siddharthan and Finn (1979) have generalized the Seed-Rahman method to include the effects of changes in the shear and bulk moduli of the soil due to increasing pore pressures.

2. THEORY

We will model the soil as a solid, porous, deformable skeleton, the pores being filled with fluid. We assume that the submarine soil is completely saturated, i.e. the air content is zero or negligible. A linear theory of deformation of such a medium was published by M.A. Biot in 1962 and will be outlined here. We assume that the soil is statistically isotropic and is of uniform porosity. A simplified theory, applicable only to sand beds will also be presented.

Notation Conventions

We will denote the Cartesian coordinates x,y and z by x_1, x_2 and x_3 respectively. Subscripts will assume the values 1, 2, 3. A repeated Latin suffix indicates summation with respect to that suffix over the range 1, 2, 3 unless otherwise specified, e.g. $A_{ii} \equiv A_{11} + A_{22} + A_{33}$.

A comma followed by a Latin suffix indicates partial differentiation with respect to the appropriate coordinate direction, e.g. $A_{,i} \equiv \frac{\partial A}{\partial x_i}$.

2.1 Development of Constitutive Equations

We let $\underline{u} = (u_1, u_2, u_3)$ be the displacement of the solid matrix at any point; $\underline{U} = (U_1, U_2, U_3)$ be the displacement of the pore water at any point; and f be the porosity of the soil, i.e. the fraction of voids in a given volume of soil. The vector $\underline{U}$ is defined in such a way that the volume of fluid displaced through unit areas normal to the x_1, x_2 and x_3 axes are fU_1, fU_2 and fU_3 respectively.

We denote by τ_{ij} the total stress components of the bulk material. τ_{ij} is an i-direction stress acting on a plane normal to the j-axis. Further we let p be the pressure of water in the pores. This is called the pore pressure.

The total stress components τ_{ij} may be written as the sum of stresses acting on the solid and liquid portions of soil. We consider a plane area of soil δA normal to the j-axis, shown schematically in Figure (2.1). We denote by F_{ij} the sum of all inter-granular forces acting on the area δA in the i direction. The pore pressure p exerts a force $p\delta A$ in opposition to the normal inter-granular forces. Since τ_{ij} is the net total stress we have the following balance of forces on the area δA:

$$\tau_{ij}\,\delta A = F_{ij} - p\delta A\,\delta_{ij}$$

where δ_{ij} is the Kronecker delta, i.e.

$$\delta_{ij} = \begin{cases} 0 \;, & i \neq j \\ 1 \;, & i = j \end{cases}$$

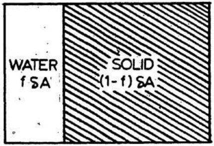

FIG. 2.1 PLANE AREA SA NORMAL TO J-AXIS

We define the effective stress components τ'_{ij} by

$$\tau'_{ij} = \frac{F_{ij}}{\delta A}$$

Hence the above equation becomes

$$\tau_{ij} = \tau'_{ij} - p\,\delta_{ij} \tag{2.1}$$

A different representation may be obtained by defining σ_{ij} as the net i-direction stress on the solid area averaged over the entire area δA i.e. $\sigma_{ij}\,\delta A$ is the net i-direction force on the solid area. With this definition we find

$$\sigma_{ij}\,\delta A = F_{ij} - \delta_{ij}\,(1 - f)\,\delta A\,p$$

which simplifies to

$$\sigma_{ij} = \tau'_{ij} - \delta_{ij}\,(1 - f)p \tag{2.2}$$

Eliminating τ'_{ij} between (2.1) and (2.2) gives

$$\tau_{ij} = \sigma_{ij} - \delta_{ij}\,fp \tag{2.3}$$

To develop the constitutive equations we assume an equilibrium condition ...

$$\tau_{ij,j} = 0 \ , \tag{2.4}$$

with the fluid in steady motion.

We denote by W the strain energy per unit volume of soil. For a volume Ω of soil bounded by a surface Σ the variation of the strain energy is equal to the virtual work of the body and surface forces i.e.

$$\iiint_{\Omega} \delta W d\Omega = \iint_{\Sigma} (f_i \, \delta u_i + F_i \, \delta U_i) d\Sigma + \iiint_{\Omega} X_i \, \delta w_i \, d\Omega \tag{2.5}$$

where $f_i = \sigma_{ij} n_j$ are the components of stresses acting on the solid part of an element of surface $d\Sigma$; $F_i = -fp \, \delta_{ij} n_j$ are the components of stresses acting on the fluid part of an element of surface $d\Sigma$; n_j are the components of the outward unit normal to the surface Σ; and X_i are the components of a dissipative body force per unit volume due to relative motion between the fluid and solid skeleton. The vector w_i is a measure of the relative displacement between the fluid and solid particles and is defined by

$$w_i = f(U_i - u_i) \tag{2.6}$$

The body force X_i is defined in such a way that the product of X_i and w_i gives the work done per unit volume by fluid-solid friction. We obtain the relationship between X_i and p as follows.

Consider an infinitesimal volume of soil of length dx_i in the i direction and unit area of cross-section normal to the i-direction. The force responsible for fluid motion in the i-direction is $\frac{-dp}{dx_i} \cdot dx_i \cdot (f \cdot 1)$

(no sum on i). This force acts in the positive i-direction if the pressure gradient $\frac{dp}{dx_i}$ is negative, and is balanced by the frictional forces $-X_i$. (X_i is defined as acting in the positive i-direction). The work done by the pressure force against friction is $\frac{-dp}{dx_i} \cdot dx_i \cdot f(U_i - u_i) = -p_{,i}\, w_i\, dx_i$ (no sum on i).

By the definition of X_i, this is equal to $-X_i \cdot (1 \cdot dx_i) w_i$ (no sum on i) so that

$$p_{,i} = X_i \, . \tag{2.7}$$

Using (2.3) we may write

$$\begin{aligned} f_i &= (\tau_{ij} + \delta_{ij}\, fp) n_j \\ F_i &= -fpn_i \end{aligned} \tag{2.8}$$

Equations (2.5) and (2.8) combine to give

$$\begin{aligned} \int\!\!\int\!\!\int_\Omega \delta W d\Omega &= \int\!\!\int_\Sigma [(\tau_{ij} + \delta_{ij}\, fp) n_j \cdot \delta u_i + (-fpn_j)\delta U_j] d\Sigma \\ &\quad + \int\!\!\int\!\!\int_\Omega X_i\, \delta w_i \, d\Omega \\ &= \int\!\!\int_\Sigma (\tau_{ij} n_j\, \delta u_i - pn_j\, \delta w_j) d\Sigma + \int\!\!\int\!\!\int_\Omega X_i\, \delta w_i d\Omega \end{aligned} \tag{2.9}$$

By the Divergence Theorem we obtain

$$\int\int\int_{\Omega} \delta W d\Omega = \int\int\int_{\Omega} [(\tau_{ij}\,\delta u_i)_{,j} - (p\delta w_j)_{,j} + X_j \delta w_j] d\Omega$$

$$\int\int\int_{\Omega} \delta W d\Omega = \int\int\int_{\Omega} [\tau_{ij}\,\delta u_{i,j} - p\,\delta w_{j,j}] d\Omega \qquad (2.10)$$

where we have used equations (2.4) and (2.7), and the fact that the variational symbol δ commutes with the differential operator. Since the volume Ω is arbitrary we deduce from (2.10) that

$$\delta W = \tau_{ij}\,\delta u_{i,j} + p\,\delta\zeta \qquad (2.11)$$

$$\text{where } \zeta = -w_{j,j} \qquad (2.12)$$

We should like to comment that the form of the strain energy function (equation 2.11) is valid whether or not an equilibrium condition exists. The variable ζ is called the "increment of fluid content".

We note that

$$\tau_{ij}\,\delta u_{i,j} = \tau_{ij}[\tfrac{1}{2}(\delta u_{i,j} + \delta u_{j,i}) + \tfrac{1}{2}(\delta u_{i,j} - \delta u_{j,i})]$$

$$= \tau_{ij}\,\delta e_{ij} + \tau_{ij}\,\delta\alpha_{ij}$$

$$\text{where } e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) , \qquad (2.13)$$

$$\alpha_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) .$$

Since τ_{ij} is a symmetric tensor and α_{ij} is anti-symmetric we have

$$\tau_{ij}\alpha_{ij} = 0$$

Hence we have

$$\tau_{ij}\,\delta u_{i,j} = \tau_{ij}\,\delta e_{ij} .$$

We substitute this result into (2.11) to get

$$\delta W = \tau_{ij}\,\delta e_{ij} + p\delta\zeta \tag{2.14}$$

Thus $W = W(e_{ij}, \zeta)$ (2.15)

Since W has a physical meaning that is independent of the choice of coordinate axes, it is invariant with respect to all transformations of the axes. Hence from (2.15) we can deduce that W is a function of the three strain invariants I_1, I_2, I_3 and the parameter ζ, i.e.

$$W = W(I_1, I_2, I_3, \zeta) \tag{2.16}$$

where $I_1 = e_{11} + e_{22} + e_{33} \equiv \varepsilon$

$$I_2 = e_{11}e_{22} + e_{22}e_{33} + e_{33}e_{11} - (e_{12}^2 + e_{23}^2 + e_{31}^2)$$

$$I_3 = \begin{vmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{vmatrix} \tag{2.17}$$

In order that we have linear stress-strain relations, W must be a quadratic function of the strains e_{ij} and the parameter ζ. Hence W cannot depend explicitly on I_3, which is of order 3. Therefore we have

$$W = W(\varepsilon, I_2, \zeta) \tag{2.18}$$

We let

$$W = \frac{1}{2} H\varepsilon^2 - 2\mu I_2 - C\varepsilon\zeta + \frac{1}{2} M\zeta^2 \tag{2.19}$$

which is the general form of a quadratic function in ε, I_2 and ζ. The constants μ, H, C and M will be interpreted later. From (2.19) we have

$$\delta W = (H\varepsilon - C\zeta)\delta\varepsilon - 2\mu\delta I_2 + (-C\varepsilon + M\zeta)\delta\zeta \tag{2.20}$$

From (2.17) we substitute for ε and I_2 into (2.20) and compare with equation (2.14) to obtain the stress-strain relations as

$$\tau_{11} = H\varepsilon - 2\mu(e_{22} + e_{33}) - C\zeta \quad ,$$

$$\tau_{22} = H\varepsilon - 2\mu(e_{33} + e_{11}) - C\zeta \quad ,$$

$$\tau_{33} = H\varepsilon - 2\mu(e_{11} + e_{22}) - C\zeta \quad ,$$

$$\tau_{12} = 2\mu e_{12}, \; \tau_{23} = 2\mu e_{23}, \; \tau_{31} = 2\mu e_{31} \quad ,$$

$$p = -C\varepsilon + M\zeta \quad .$$

In condensed notation we have the constitutive equations

$$\begin{aligned} \tau_{ij} &= [(H - 2\mu)\varepsilon - C\zeta]\delta_{ij} + 2\mu e_{ij} \\ p &= -C\varepsilon + M\zeta \end{aligned} \qquad (2.21)$$

We replace the constants H and C by the constants λ_c and α defined by

$$\begin{aligned} \lambda_c &= H - 2\mu \\ \alpha &= \frac{C}{M} \end{aligned} \qquad (2.22)$$

We can thus rewrite (2.21) in the form

$$\begin{aligned} \tau_{ij} &= 2\mu e_{ij} + (\lambda_c \varepsilon - \alpha M\zeta)\delta_{ij} \quad , \\ p &= -\alpha M\varepsilon + M\zeta \quad . \end{aligned} \qquad (2.23)$$

2.2 Interpretation of the Elastic Moduli

It is necessary at this stage to gain an appreciation for the meaning of the elastic moduli μ, H, C and M (and thus λ_c, α), and relate them to the more familiar elastic moduli. The parameter μ is easily recognized by noting from (2.21) that

$$\tau_{ij} = 2\mu e_{ij} \quad \text{for } i \neq j .$$

Further from equation (2.1) we see that

$$\tau_{ij} = \tau'_{ij} \quad \text{for } i \neq j$$

so that $\tau'_{ij} = 2\mu e_{ij}$ for $i \neq j$.

Hence μ is the familiar shear modulus of the soil skeleton.

To examine the parameters H, C and M we consider the following theoretical tests on a soil sample (Biot, 1957; Stoll, 1974).

(a) The "jacketed" test.

The saturated soil is placed in an impervious flexible bag and loaded by an external pressure p' as shown in Figure (2.2). Pore fluid is free to flow out of the bag via a tube so that the induced fluid pressure p remains at zero during slow loading.

(b) The "unjacketed" test.

An uncased sample of soil is completely immersed in fluid which is subsequently pressurized with constant pressure p' from an external source. See Figure (2.3).

In both tests, $\tau_{11} = \tau_{22} = \tau_{33} = -p'$, and $\tau_{ij} = 0$ for $i \neq j$.

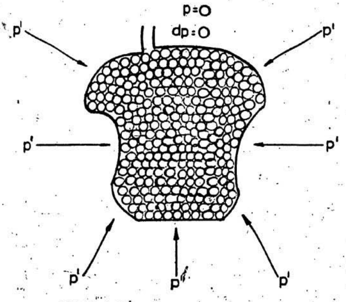

FIG. 2.2 THE "JACKETED" TEST

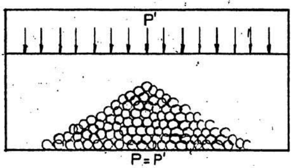

FIG. 2.3 THE "UNJACKETED" TEST

From (2.21) we have

$$\tau_{kk} = [(H - 2\mu)\varepsilon - C\zeta]3 + 2\mu\varepsilon$$

$$= \varepsilon(3H - 4\mu) - 3C\zeta$$

Since $\tau_{kk} = -3p'$ we have

$$-p' = (H - \tfrac{4}{3}\mu)\varepsilon - C\zeta \qquad (2.24)$$

In the "jacketed" test, we let K_b be the bulk modulus of the free draining soil skeleton. This is given by $K_b = \frac{-p'}{\varepsilon}$ and from (2.24) we deduce that

$$K_b = H - \tfrac{4}{3}\mu - \frac{C\zeta}{\varepsilon}.$$

Since p = 0 in this test, the last of equations (2.21) gives

$$\zeta = \frac{C\varepsilon}{M}$$

Hence we have

$$K_b = H - \tfrac{4}{3}\mu - \frac{C^2}{M} \qquad (2.25)$$

In the "unjacketed" test we have $p' = p$ so that the last of equations (2.21) becomes

$$p' = -C\varepsilon + M\zeta \qquad (2.26)$$

Combining (2.24) and (2.26) we have

$$M\zeta - C\varepsilon = C\zeta - (H - \tfrac{4}{3}\mu)\varepsilon$$

which gives

$$\zeta = \frac{\varepsilon(C - H + \frac{4}{3}\mu)}{M - C} \tag{2.27}$$

Substituting (2.27) in (2.26) we get

$$p' = \frac{-\varepsilon M(H - \frac{4\mu}{3}) + \varepsilon C^2}{M - C} \tag{2.28}$$

We define Δ, the "unjacketed compressibility" by

$$\Delta = \frac{-\varepsilon}{p'} \tag{2.29}$$

Using (2.28) we find

$$\Delta = \frac{1 - \frac{C}{M}}{H - \frac{4}{3}\mu - \frac{C^2}{M}} \tag{2.30}$$

We define γ, the "coefficient of fluid content" as

$$\gamma = \frac{\zeta}{p'} \tag{2.31}$$

From (2.27) and (2.28) we obtain

$$\gamma = \frac{H - \frac{4\mu}{3} - C}{(H - \frac{4\mu}{3} - \frac{C^2}{M})M} \tag{2.32}$$

We note that by the definition of ζ (equation 2.12) we have

$$\gamma = \frac{f(u_{i,i} - U_{i,i})}{p'} \tag{2.33}$$

We let K_f be the bulk modulus of the water and K_r be the bulk modulus of the soil grains.

$$\text{i.e.}\quad \frac{1}{K_f} = -\frac{U_{i,i}}{p'}, \qquad \frac{1}{K_r} = -\frac{u_{i,i}}{p'} \tag{2.34}$$

We note that K_r, the bulk modulus of the soil grains, is determined from the "unjacketed" test and is distinct from K_b, the bulk modulus of the soil skeleton in free drainage ("jacketed" test).

Using (2.34) we re-write (2.33) as

$$\gamma = f(\frac{1}{K_f} - \frac{1}{K_r}) \tag{2.35}$$

From (2.29) and the second of equations (2.34) we have

$$\Delta = \frac{1}{K_r} \tag{2.36}$$

We solve the three equations (2.25), (2.30) and (2.32) for H, C and M and then use (2.35) and (2.36) to eliminate γ and Δ.

From (2.25),

$$H = K_b + \frac{4}{3}\mu + \frac{C^2}{M} \qquad (2.25a)$$

Substituting in (2.30) and (2.32) we have

$$\Delta = \frac{1 - \frac{C}{M}}{K_b} \qquad (2.30a)$$

$$\gamma = \frac{K_b + \frac{C^2}{M} - C}{K_b M} \qquad (2.32a)$$

From (2.30a) we have

$$\frac{1}{M} = \frac{1 - \Delta K_b}{C} \qquad (2.30b)$$

Substituting in (2.32a) we get

$$\gamma = \frac{(1 - \Delta C)(1 - \Delta K_b)}{C}$$

which gives

$$C = \frac{1 - \Delta K_b}{\gamma + \Delta - \Delta^2 K_b}$$

We now substitute for γ and Δ from (2.35) and (2.36) respectively to get

$$C = \frac{K_r(K_r - K_b)}{D - K_b} \quad ,$$

where $D = K_r[1 + f(\frac{K_r}{K_f} - 1)]$.

From (2.30b), (2.36) and the above expression for C we obtain

$$M = \frac{K_r^{\,2}}{D - K_b}$$

From (2.25a) we have

$$H = K_b + \frac{4}{3}\,\mu + \frac{(K_r - K_b)^2}{D - K_b}$$

Summarising we have

$$H = \frac{(K_r - K_b)^2}{D - K_b} + K_b + \frac{4\mu}{3} \quad ,$$

$$C = \frac{K_r(K_r - K_b)}{D - K_b} \qquad (2.38)$$

$$M = \frac{K_r^{\,2}}{D - K_b}$$

where

$$D = K_r[1 + f(\frac{K_r}{K_f} - 1)]$$

From (2.22) we obtain λ_c and α as

$$\lambda_c = \frac{(K_r - K_b)^2}{D - K_b} + K_b - \frac{2\mu}{3} ,$$

$$\alpha = 1 - \frac{K_b}{K_r} \qquad (2.39)$$

2.3 The Constitutive Equations Related to Hooke's Law

From the second of equations (2.23) we have

$$\zeta = \frac{1}{M}(p + \alpha M\varepsilon)$$

Substituting this into the first of equations (2.23) gives

$$\tau_{ij} = 2\mu e_{ij} + \delta_{ij}[(\lambda_c - \alpha^2 M)\varepsilon - \alpha p] \qquad (2.40)$$

From (2.38) and (2.39) we find that

$$\lambda_c - \alpha^2 M = K_b - \frac{2\mu}{3} \equiv \lambda^* \qquad (2.41)$$

λ^* is the standard Lamé constant of the theory of elasticity. Equation (2.40) may now be re-written as

$$\tau_{ij} = 2\mu e_{ij} + \delta_{ij}\lambda^*\varepsilon - \delta_{ij}\alpha p \qquad (2.42)$$

Using (2.1) we write this equation in terms of effective stress:

$$\tau'_{ij} = 2\mu e_{ij} + \delta_{ij}\lambda^*\varepsilon + \delta_{ij}(1 - \alpha)p \qquad (2.43)$$

The constitutive relations are thus given by equation (2.43) which differs from Hooke's law by the term $\delta_{ij}(1 - \alpha)p$. For most sands and clays the ratio $\frac{K_b}{K_r}$ is very small (a typical value is 10^{-4}), so that α is very close to unity (see equation 2.39). Thus the constitutive equation (2.43) is approximated very closely by Hooke's law, viz

$$\tau'_{ij} = 2\mu e_{ij} + \delta_{ij}\,\lambda^{*}\varepsilon \qquad (2.44)$$

2.4 Models for Energy Dissipation in the Solid

The model of a purely elastic matrix saturated with viscous fluid must be modified in order to include the effects of energy dissipation in the solid. Such energy dissipation may be due to the relaxation properties of the solid under load or to intergranular solid friction. Two models will be discussed: (a) a linear viscoelastic model; (b) a non-linear model. We will show that these models lead to very simple modifications in the constitutive relations already derived. Energy dissipation due to fluid-solid friction is considered in Section 2.5.2 (equation 2.74).

(a) Linear Viscoelastic Model

Without loss of generality we consider a one-dimensional linear viscoelastic stress-strain relation

$$\sigma(t) = \int_{-\infty}^{t} G(t - \tau)\frac{d\varepsilon(\tau)}{d\tau}\, d\tau \tag{2.45}$$

where $\sigma(t)$, $\varepsilon(t)$ are stress and strain respectively at time t. The function G is called the relaxation modulus. We will discuss three methods of replacing (2.45) by an equivalent relation involving a complex modulus $\overline{G}$.

(1) Under sinusoidal loading the strain history may be represented by the complex function

$$\varepsilon(t) = \varepsilon_o e^{i\omega t} \tag{2.46}$$

where ω is the circular frequency of the loading, the real part of the function denotes the physical strain, and ε_o is constant.

Substituting (2.46) into (2.45) we have

$$\sigma(t) = i\omega\varepsilon_o \int_{-\infty}^{t} G(t-\tau)e^{i\omega\tau}\, d\tau$$

$$= -i\omega\varepsilon_o \int_{\infty}^{o} G(\eta)e^{i\omega(t-\eta)}\, d\eta$$

i.e.

$$\sigma(t) = i\omega\left[\int_{o}^{\infty} G(\eta)e^{-i\omega\eta}\, d\eta\right]\varepsilon_o e^{i\omega t}$$

Thus we may write

$$\sigma(t) = \overline{G}(\omega)\ \varepsilon(t) \qquad (2.47)$$

where $$\overline{G}(\omega) = i\omega \int_{o}^{\infty} G(\eta)e^{-i\omega\eta}\, d\eta\ . \qquad (2.48)$$

Breaking up the right hand side of equation (2.48) into real and imaginary parts we find

$$\overline{G}(\omega) = G'(\omega) + i\, G''(\omega) \qquad (2.49)$$

$$\text{where} \quad G'(\omega) = \omega \int_0^\infty G(\eta)\sin(\omega\eta)\,d\eta$$

$$G''(\omega) = \omega \int_0^\infty G(\eta)\cos(\omega\eta)\,d\eta \tag{2.50}$$

$G'(\omega)$ is called the storage modulus and $G''(\omega)$ is called the loss modulus.

(2) An alternate approach is to take the Laplace transform of (2.45) to find

$$\sigma^*(s) = sG^*(s)\,\varepsilon^*(s) \tag{2.51}$$

where for any function $f(t)$, the Laplace transform $f^*(s)$ is defined by

$$f^*(s) = \int_0^\infty f(t)\,e^{-st}\,dt. \tag{2.52}$$

In (2.51) we have assumed that $\varepsilon(0) = 0$. By replacing s by $i\omega$ in (2.52) we deduce that

$$G^*(i\omega) = \int_0^\infty G(t)\,e^{-i\omega t}\,dt$$

Using equation (2.48) we obtain

$$\overline{G}(\omega) = i\omega\, G^*(i\omega)$$

Hence putting $s = i\omega$ in (2.51) we have the result

$$\sigma^*(i\omega) = \overline{G}(\omega)\ \varepsilon^*(i\omega) \tag{2.53}$$

(3) The last method is to take the Fourier transform of (2.45) and find

$$\bar{\sigma}(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{t} G(t - \tau) \frac{d\varepsilon(\tau)}{d\tau}\, d\tau \right] e^{-i\omega t}\, dt \tag{2.54}$$

where

$$\bar{\sigma}(\omega) = \int_{-\infty}^{\infty} \sigma(t)\ e^{-i\omega t}\, dt$$ is the Fourier transform of $\sigma(t)$.

Interchanging the order of integration in (2.54) and using (2.48) we have

$$\bar{\sigma}(\omega) = \int_{-\infty}^{\infty} \int_{\tau}^{\infty} G(t - \tau) \frac{d\varepsilon(\tau)}{d\tau}\, e^{-i\omega t}\, dt\, d\tau$$

$$= \int_{-\infty}^{\infty} \frac{d\varepsilon(\tau)}{d\tau} \left[\int_{0}^{\infty} G(\eta)\ e^{-i\omega(\eta + \tau)} d\eta \right] d\tau$$

$$= \int_{-\infty}^{\infty} \frac{d\varepsilon(\tau)}{d\tau}\, e^{-i\omega\tau} \cdot \frac{\overline{G}(\omega)}{i\omega}\, d\tau$$

$$= \frac{\overline{G}(\omega)}{i\omega} \int_{-\infty}^{\infty} \frac{d\varepsilon(\tau)}{d\tau} e^{-i\omega\tau} d\tau$$

$$= \frac{\overline{G}(\omega)}{i\omega} \left\{ [\varepsilon(\tau) e^{-i\omega\tau}]_{\tau=-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} \varepsilon(\tau) e^{-i\omega\tau} d\tau \right\}$$

We assume that $\varepsilon(-\infty) = 0$ so that the above equation reduces to

$$\tilde{\sigma}(\omega) = \overline{G}(\omega) \tilde{\varepsilon}(\omega) \tag{2.55}$$

where $\tilde{\varepsilon}(\omega) = \int_{-\infty}^{\infty} \varepsilon(\tau) e^{-i\omega\tau} d\tau$ is the Fourier transform of $\varepsilon(t)$.

We note that the same complex modulus $\overline{G}(\omega)$ relates stress and strain in equations (2.47), (2.53) and (2.55). In each case, however, the stress and strain are interpreted differently. In equation (2.47) they are the time dependent stress and strain while in equations (2.53) and (2.55) they are the Laplace and Fourier transforms respectively. Equation (2.47) is applicable only to sinusoidal loading, but equations (2.53) and (2.55) are quite general.

(b) A non-linear model

It has been pointed out by Stoll and Bryan (1970) that the elastic moduli and damping of soils are nearly independent of loading frequency. To model such behaviour we consider a one-dimensional stress-strain relation of the form

$$\sigma(t) = c \left|\frac{\varepsilon(t)}{\dot{\varepsilon}(t)}\right| \dot{\varepsilon}(t) + k\varepsilon(t) \tag{2.56}$$

where c is a damping constant and k is an elastic modulus; $\dot{\varepsilon}(t)$ denotes the time derivative of $\varepsilon(t)$. Under sinusoidal loading we represent strain by the complex form (2.46) and equation (2.56) becomes

$$\sigma(t) = (k + ic)\ \varepsilon(t) \tag{2.57}$$

Equation (2.57) suggests that in the case of sinusoidal loading, the effects of damping may be included by replacing the real elastic modulus k by the complex modulus $\overline{k}$ defined by

$$\overline{k} = k(1 + i\delta) \quad \text{where} \quad \delta = \frac{c}{k}$$

Hence in equation (2.43) we replace μ and λ^* by $\overline{\mu}$ and $\overline{\lambda}^*$ respectively. We note that λ^* is related to μ via the Poisson's ratio ν:

$$\lambda^* = \left(\frac{2\nu}{1 - 2\nu}\right)\mu \tag{2.58}$$

Hence we may write

$$\overline{\lambda}^* = \left(\frac{2\nu}{1 - 2\nu}\right)\overline{\mu} \tag{2.59}$$

and $\overline{\mu} = \mu(1 + i\delta)$ (2.60) where δ is a damping parameter. Values of δ that have been quoted for marine sediments vary from 0.02 for small strains to 0.20 for large strains (Yamamoto, 1983).

Stoll and Bryan (1970) have also suggested the use of constant complex moduli to model frequency-independent damping, and this method was used by Yamamoto (1983). It is uncertain whether such a model describes an internally damped system except possibly for purely sinusoidal loading. Graham (1973) in his excellent review on material damping suggests that for a low frequency range, such as that encountered in offshore wave loading, a viscoelastic model is adequate.

2.5 The Equations of Motion

Before deriving the equations of motion we will establish Hamilton's principle for an elastic continuum. Although saturated soil is a two phase material the general principle derived here is still applicable.

(2.5.1) Hamilton's Principle for an Elastic Continuum

The equations of motion for an elastic continuum are (Sokolnikoff, equation 25.1)

$$\tau_{ij,j} + X_i = \rho \ddot{u}_i = \frac{\partial^2 u_i}{\partial t^2} \tag{2.61}$$

where X_i is the body force per unit volume; ρ is the density of the material; u_i is the displacement vector at any point; τ_{ij} is the stress tensor at any point; and t is time.

Consider an arbitrary volume Ω bounded by a surface Σ. For arbitrary virtual displacements δu_i we have

$$\int\int\int_\Omega (\rho \ddot{u}_i - \tau_{ij,j} - X_i)\delta u_i \, d\Omega = 0 \,. \tag{2.62}$$

We observe that

$$\frac{d}{dt}(\dot{u}_i \, \delta u_i) = \ddot{u}_i \, \delta u_i + \dot{u}_i \, \delta \dot{u}_i$$

We multiply this equation by ρ and integrate over the volume Ω to find

$$\int\int\int_{\Omega} \rho \frac{d}{dt} (\dot{u}_i \ \delta u_i) d\Omega = \int\int\int_{\Omega} \rho \ddot{u}_i \ \delta u_i \ d\Omega + \int\int\int_{\Omega} \rho \dot{u}_i \ \delta \dot{u}_i \ d\Omega \ ,$$

$$= \int\int\int_{\Omega} \rho \ddot{u}_i \ \delta u_i \ d\Omega + \frac{1}{2} \int\int\int_{\Omega} \rho \delta (\dot{u}_i \ \dot{u}_i) d\Omega \ ,$$

$$= \int\int\int_{\Omega} \rho \ddot{u}_i \ \delta u_i \ d\Omega + \delta K \ , \qquad (2.63)$$

$$\text{where} \ K = \frac{1}{2} \int\int\int_{\Omega} \rho \dot{u}_i \ \dot{u}_i \ d\Omega \ . \qquad (2.64)$$

K is called the <u>kinetic energy</u> of the volume Ω. From (2.63) we have

$$\int\int\int_{\Omega} \rho \ddot{u}_i \ \delta u_i \ d\Omega = \int\int\int_{\Omega} \rho \frac{d}{dt} (\dot{u}_i \ \delta u_i) d\Omega - \delta K$$

Substituting this result into (2.62) we get

$$\int\int\int_{\Omega} \rho \frac{d}{dt} (\dot{u}_i \ \delta u_i) d\Omega = \delta K + \int\int\int_{\Omega} (\tau_{ij,j} + X_i) \delta u_i \ d\Omega \qquad (2.65)$$

Using the fact that

$$\tau_{ij,j}\,\delta u_i = (\tau_{ij}\,\delta u_i)_{,j} - \tau_{ij}\,\delta u_{i,j}$$

we integrate both sides of this equation to find

$$\begin{aligned}\iiint_{\Omega} \tau_{ij,j}\,\delta u_i\,d\Omega &= \iiint_{\Omega} [(\tau_{ij}\,\delta u_i)_{,j} - \tau_{ij}\,\delta u_{i,j}]d\Omega\,,\\ &= \iint_{\Sigma} \tau_{ij}\,\delta u_i\,\nu_j\,d\Sigma - \iiint_{\Omega} \tau_{ij}\,\delta u_{i,j}\,d\Omega\end{aligned} \tag{2.66}$$

where we have used the divergence theorem, and ν_j represents the jth component of the unit normal to the surface Σ.

We define the strain energy per unit volume, W, by

$$\begin{aligned}\delta W &= \tau_{ij}\,\delta u_{i,j}\\ \text{i.e.}\ \delta W &= \tau_{ij}\,\delta e_{ij}\end{aligned} \tag{2.67}$$

as shown in section (2.1).

If the force per unit area acting on the surface Σ is denoted by T_i, we have (Sokolnikoff, equation 24.4).

$$T_i = \tau_{ij}\,\nu_j \tag{2.68}$$

Substituting in (2.66) we find

$$\int\int\int_{\Omega} \tau_{ij,j}\,\delta u_i\,d\Omega = \int\int_{\Sigma} T_i\,\delta u_i\,d\Sigma - \int\int\int_{\Omega} \delta W\,d\Omega\ .$$

Using this in (2.65) we have

$$\int\int\int_{\Omega} \rho\,\frac{d}{dt}\,(\dot{u}_i\,\delta u_i)d\Omega = \delta K + \int\int_{\Sigma} T_i\,\delta u_i\,d\Sigma - \int\int\int_{\Omega} \delta W\,d\Omega + \int\int\int_{\Omega} X_i\,\delta u_i\,d\Omega \tag{2.69}$$

We define the potential energy V by

$$V = \int\int\int_{\Omega} W d\Omega - \int\int_{\Sigma} T_i\,u_i\,d\Sigma - \int\int\int_{\Omega} X_i\,u_i\,d\Omega \tag{2.70}$$

Using (2.70), equation (2.69) becomes

$$\begin{aligned}\int\int\int_{\Omega} \rho\,\frac{d}{dt}\,(\dot{u}_i\,\delta u_i)d\Omega &= \delta K - \delta V \\ &= \delta(K - V) \\ &= \delta L\end{aligned} \tag{2.71}$$

where $L = K - V$ (2.72)

L is called the Lagrangian.

We integrate equation (2.71) with respect to time t, between the limits $t = t_1$ and $t = t_2$ where it is assumed that

$$\delta u_i(t_1) = \delta u_i(t_2) = 0 :$$

$$\int_{t_1}^{t_2} \delta L \, dt = 0 \tag{2.73}$$

Equation (2.73) is called Hamilton's principle.

This derivation is analogous to that used by Meirovitch (1970) for a system of particles.

(2.5.2) Derivation of Equations of Motion

We consider a volume Ω of soil bounded by a surface Σ. In keeping with equation (2.70) we define the potential energy V of the volume Ω by

$$V = \iiint_{\Omega} W d\Omega - \iint_{\Sigma} f_i u_i d\Sigma - \iint_{\Sigma} F_i U_i d\Sigma - \iiint_{\Omega} X_i w_i d\Omega \tag{2.74}$$

where f_i are the components of stresses acting on the solid part of an element of surface dΣ; F_i are the components of stresses acting on the fluid part of an element of surface dΣ; X_i are the components of a dissipative body force per unit volume due to the relative motion between the fluid and solid skeleton; and W is the strain energy per unit volume. We do not consider gravity forces since we are concerned only with wave-induced effective stresses and pore pressures.

The force X_i is defined in such a way that the product of X_i and w_i gives the energy dissipated due to fluid-solid friction.

From equations (2.11) and (2.12) we have

$$W = \tau_{ij}\, u_{i,j} + p\zeta$$

i.e.

$$W = \tau_{ij}\, u_{i,j} - pw_{i,i}$$

or

$$W = (\tau_{ij}\, u_i)_{,j} - \tau_{ij,j}\, u_i - (pw_i)_{,i} + p_{,i}\, w_i \tag{2.75}$$

Integrating (2.75) over the volume Ω we have

$$\int\int\int_\Omega W d\Omega = \int\int\int_\Omega [(\tau_{ij}\, u_i)_{,j} - (pw_i)_{,i}] d\Omega$$

$$- \int\int\int_\Omega (\tau_{ij,j}\, u_i - p_{,i}\, w_i) d\Omega$$

Applying the divergence theorem to the first integral on the right hand side gives

$$\int\int\int_\Omega W d\Omega = \int\int_\Sigma \tau_{ij}\, u_i\, n_j\, d\Sigma - \int\int_\Sigma pw_i\, n_i\, d\Sigma$$

$$- \int\int\int_\Omega (\tau_{ij,j}\, u_i - p_{,i}\, w_i) d\Omega \tag{2.76}$$

where n_i is the ith component of the unit outward normal to the surface Σ.

Substituting (2.76) into (2.74) we obtain

$$V = \iint_\Sigma (\tau_{ij} u_i n_j - p w_i n_i - f_i u_i - F_i U_i) d\Sigma$$

$$- \iiint_\Omega \tau_{ij,j} u_i \, d\Omega - \iiint_\Omega (X_i - p_{,i}) w_i \, d\Omega \tag{2.77}$$

Substituting for τ_{ij}, f_i and F_i from (2.3) and (2.6) we can write the surface integral on the right hand side of (2.77), I, as

$$I = \iint_\Sigma [(\sigma_{ij} - fp\,\delta_{ij}) u_i n_j - p w_i n_i - \sigma_{ij} u_i n_j + fp\, U_i n_i] d\Sigma$$

$$I = \iint_\Sigma [pf(U_i - u_i) n_i - p w_i n_i] d\Sigma = 0$$

Hence (2.77) reduces to

$$V = - \iiint_\Omega \tau_{ij,j} u_i \, d\Omega - \iiint_\Omega (X_i - p_{,i}) w_i \, d\Omega \, . \tag{2.78}$$

We define T as the kinetic energy per unit volume at any point in the medium. It is clear that T is a function of $\dot{u}_i$ and $\dot{U}_i$. If we use u_i w_i as generalized coordinates we can write

$$T = T(\dot{u}_i, \dot{w}_i) \tag{2.79}$$

The total kinetic energy K of the volume Ω is given by

$$K = \int\int\int_{\Omega} T d\Omega \tag{2.80}$$

The Lagrangian L is defined by

$$L = K - V \tag{2.81}$$

and from Hamilton's principle we have

$$\int_{t_1}^{t_2} \delta L \; dt = 0 \tag{2.82}$$

where the virtual quantities δu_i and δw_i are zero at times $t = t_1$ and $t = t_2$.

From equation (2.82) we write

$$\int_{t_1}^{t_2} \delta \; [\int\int\int_{\Omega} T d\Omega + \int\int\int_{\Omega} \tau_{ij,j} \; u_i \; d\Omega$$

$$+ \int\int\int_{\Omega} (X_i - p_{,i}) w_i \; d\Omega] dt = 0 \tag{2.83}$$

Considering the first term in this integral we find

$$\int_{t_1}^{t_2}[\delta\int\int\int_\Omega T d\Omega]dt = \int_{t_1}^{t_2}[\int\int\int_\Omega \delta T\, d\Omega]dt$$

$$= \int_{t_1}^{t_2}[\int\int\int_\Omega(\frac{\partial T}{\partial \dot{u}_k}\delta\dot{u}_k + \frac{\partial T}{\partial \dot{w}_k}\delta\dot{w}_k)d\Omega]dt$$

$$= \int\int\int_\Omega[\int_{t_1}^{t_2}\frac{\partial T}{\partial \dot{u}_k}\frac{d}{dt}(\delta u_k)dt + \int_{t_1}^{t_2}\frac{\partial T}{\partial \dot{w}_k}\frac{d}{dt}(\delta w_k)dt]d\Omega$$

$$= \int\int\int_\Omega[\frac{\partial T}{\partial \dot{u}_k}\delta u_k + \frac{\partial T}{\partial \dot{w}_k}\delta w_k]_{t=t_1}^{t=t_2}d\Omega$$

$$+ \int\int\int_\Omega[\int_{t_1}^{t_2}\{-\frac{d}{dt}(\frac{\partial T}{\partial \dot{u}_k})\delta u_k - \frac{d}{dt}(\frac{\partial T}{\partial \dot{w}_k})\delta w_k\}\, dt]d\Omega$$

$$= \int\int\int_\Omega[\int_{t_1}^{t_2}\{-\frac{d}{dt}(\frac{\partial T}{\partial \dot{u}_k})\delta u_k - \frac{d}{dt}(\frac{\partial T}{\partial \dot{w}_k})\delta w_k\}\, dt]d\Omega$$

since $\delta u_k(t_1) = \delta u_k(t_2) = 0$,

and $\delta w_k(t_1) = \delta w_k(t_2) = 0$.

Substituting the last equation above into (2.83) we get

$$\int\int\int_{\Omega}\left[\int_{t_1}^{t_2}\{\tau_{ij,j} - \frac{d}{dt}(\frac{\partial T}{\partial \dot{u}_i})\}\delta u_i + \{X_i - p_{,i} - \frac{d}{dt}(\frac{\partial T}{\partial \dot{w}_i})\}\,\delta w_i\;dt\right]d\Omega = 0 \qquad (2.84)$$

Since the δu_i and δw_i are arbitrary we obtain from (2.84) the Lagrangian equations of motion,

$$\tau_{ij,j} = \frac{d}{dt}(\frac{\partial T}{\partial \dot{u}_i}) \qquad (2.85)$$

$$-p_{,i} = \frac{d}{dt}(\frac{\partial T}{\partial \dot{w}_i}) - X_i \qquad (2.86)$$

We assume that the dissipative body force X_i is directly proportional to the variable $\dot{w}_i$. Further, we expect X_i to depend on the fluid viscosity n_f (units: kg m^{-1} s^{-1}) and the coefficient of permeability of the soil k (units: m^2). Dimensional analysis then leads to

$$X_i = \frac{n_f}{k}\dot{w}_i \qquad (2.87)$$

which renders (2.86) consistent with Darcy's law. Equation (2.86) is in fact an extended form of Darcy's law with the additional term $\frac{d}{dt}(\frac{\partial T}{\partial \dot{w}_i})$. The permeability k_o usually described in texts on soil mechanics has units of ms^{-1} and is related to k by the equation

$$k_o = \frac{\rho_f\, g\, k}{n_f}$$

where g is the acceleration due to gravity.

The kinetic energy per unit volume, T, is given by

$$2T = (1 - f)\rho_s \dot{u}_k\dot{u}_k + f\rho_f \dot{U}_k\dot{U}_k + \rho_a(\dot{U}_k - \dot{u}_k)(\dot{U}_k - \dot{u}_k) \tag{2.88}$$

where ρ_s is the density of the soil grains and ρ_f is the density of the pore water. The quantity ρ_a has units of density and is an "added mass" parameter due to the coupling of soil grains and fluid in relative motion. The value of ρ_a depends on the orientation of the pores in the soil. For uniform circular pores with axes parallel to the pressure gradient, ρ_a would be zero, whereas for an arbitrary orientation of uniform pores the theoretical value of ρ_a is $2f\rho_f$ (Stoll and Bryan, 1970).

From (2.8) we have $U_k = \frac{w_k}{f} + u_k$

Substituting for U_k in (2.88) we get

$$2T = \rho\dot{u}_k\dot{u}_k + 2\rho_f \dot{u}_k\dot{w}_k + m\,\dot{w}_k\dot{w}_k \tag{2.89}$$

where $\rho = (1 - f)\rho_s + f\rho_f$

$$m = \frac{f\rho_f + \rho_a}{f^2} \tag{2.90}$$

From (2.89) we have

$$\frac{\partial T}{\partial \dot{u}_i} = \rho\dot{u}_i + \rho_f\dot{w}_i$$

$$\frac{\partial T}{\partial \dot{w}_i} = \rho_f \dot{u}_i + m \dot{w}_i$$

Substituting these equations into (2.85) and (2.86) we have the equations of motion

$$\tau_{ij,j} = \rho \ddot{u}_i + \rho_f \ddot{w}_i \ , \tag{2.91}$$

$$-p_{,i} = \rho_f \ddot{u}_i + m \ddot{w}_i + \frac{n_f}{k} \dot{w}_i \tag{2.92}$$

where (2.87) has been used.

From the constitutive equations (2.23) we find

$$\tau_{ij,j} = 2(\mu e_{ij})_{,j} + (\lambda_c \varepsilon - \alpha M \zeta)_{,j} \delta_{ij}$$

i.e.

$$\tau_{ij,j} = 2(\mu e_{ij})_{,j} + (\lambda_c \varepsilon - \alpha M \zeta)_{,i}$$

and

$$p_{,i} = (-\alpha M \varepsilon + M \zeta)_{,i}$$

We substitute these two equations into (2.91) and (2.92) to get

$$2(\mu e_{ij})_{,j} + (\lambda_c \varepsilon - \alpha M \zeta)_{,i} = \rho \ddot{u}_i + \rho_f \ddot{w}_i \tag{2.93}$$

$$(\alpha M\varepsilon - M\zeta)_{,i} = \rho_f \ddot{u}_i + m\, \ddot{w}_i + \frac{n_f}{k}\, \dot{w}_i \tag{2.94}$$

Putting $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ and $C = \alpha M$ in (2.93) and (2.94) we have the equations of motion in the form

$$\mu u_{i,jj} + (\lambda_c + \mu) u_{j,ji} + C\, w_{j,ji} = \rho \ddot{u}_i + \rho_f \ddot{w}_i \tag{2.95}$$

$$C\, u_{j,ji} + M\, w_{j,ji} = \rho_f \ddot{u}_i + m\, \ddot{w}_i + \frac{n_f}{k}\, \dot{w}_i \tag{2.96}$$

3. FORMULATION OF THE PROBLEM

We consider a flat seabed under the loading of sinusoidal waves. The problem may be considered two dimensional with x and z axes chosen as shown in Figure (3.1). Suffixes i and j will take the values 1 and 2, with 1 referring to the x direction and 2 referring to the z direction. The soil bed is of finite depth z_o.

In the stress analysis, conditions of plane strain will be assumed. For a sinusoidal wave propagating in the x-direction we may write the water surface elevation (above still water level) $\eta(x,t)$ as

$$\eta(x,t) = \mathrm{Re}\ [a_o e^{i(\lambda x + \omega t)}] \tag{3.1}$$

where a_o is the wave amplitude, λ is the wave number, and ω is the circular frequency of the wave. The positive direction for $\eta(x,t)$ is vertically upwards. It is assumed that the flow above the seabed ($-h < z < 0$) is inviscid, incompressible and irrotational. The condition of irrotationality allows the water velocity components V_1 and V_2 to be expressed in terms of a velocity potential $\Phi(x,z,t)$ as follows:

$$V_1 = \Phi_{,1} \quad ; \quad V_2 = \Phi_{,2} \tag{3.2}$$

Φ is defined for the range $-h \leq z \leq 0$. The equation of continuity requires that

$$\nabla^2 \Phi = 0 \tag{3.3}$$

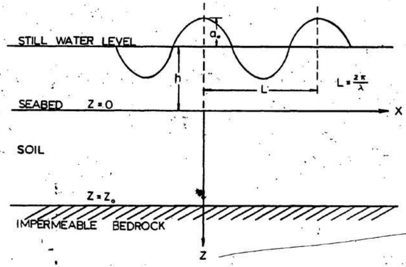

FIG. 3.1 WAVE LOADING OF SEABED

For an inviscid, irrotational flow, Bernoulli's equation takes the form

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} \Phi_{,i} \Phi_{,i} + \frac{P}{\rho_f} + g \bar{z} = 0 \tag{3.4}$$

where $\bar{z} = -(z + h)$, P is the fluid pressure and g is the acceleration due to gravity.

At the free surface, $z = -(h + \eta)$ so that $\bar{z} = \eta$. Also $P = 0$ at the free surface. Thus using (3.4) at the free surface we get

$$\left.\frac{\partial \Phi}{\partial t}\right|_{z = -(h + \eta)} + \frac{1}{2} V^2 + g\eta = 0 \tag{3.5}$$

where V is the speed of the water particles at the free surface.

For small displacements of the free surface we may neglect the term in V^2, and (3.5) reduces to

$$\left.\frac{\partial \Phi}{\partial t}\right|_{z = -(h + \eta)} + g\eta = 0 \tag{3.6}$$

Hence we find

$$\eta = -\frac{1}{g}\left.\frac{\partial \Phi}{\partial t}\right|_{z = -(h + \eta)} \approx -\frac{1}{g}\left.\frac{\partial \Phi}{\partial t}\right|_{z = -h} \tag{3.7}$$

since η is small.

These approximations are reasonable if $\frac{\omega^2 a_o}{2g} << 1$. This criterion is justified in the Appendix. The kinematic free surface condition is

$$\underline{V}(x,z)\Big|_{z = -(h+\eta)} = \underline{V}_n \tag{3.8}$$

where $\underline{V} = (V_1, V_2)$ is the water particle velocity and $\underline{V}_n$ is the normal velocity of the free surface. From (3.2) we have that $\underline{V} = \nabla\Phi$, and since at the free surface $\underline{V}$ must be directed along the normal to the surface, we may write

$$\underline{V}\Big|_{z = -(h+\eta)} = \nabla\Phi\Big|_{z = -(h+\eta)} \equiv \frac{\partial\Phi}{\partial n}\Big|_{z = -(h+\eta)} \hat{\underline{n}} \tag{3.9}$$

where n denotes the direction normal to the free surface and $\hat{\underline{n}}$ is the unit normal vector at the free surface.

We assume that η is sufficiently small compared to L, so that we may replace the normal coordinate n by z and the vector $\hat{\underline{n}}$ by $\hat{\underline{k}}$, the unit vector in the z direction; i.e. (3.9) may be written

$$\underline{V}\Big|_{z = -(h+\eta)} = \frac{\partial\Phi}{\partial z}\Big|_{z = -(h+\eta)} \hat{\underline{k}} \simeq \frac{\partial\Phi}{\partial z}\Big|_{z = -h} \hat{\underline{k}} \tag{3.10}$$

Similarly we may write

$$\underline{V}_n \simeq \frac{\partial\eta}{\partial t}\,(-\hat{\underline{k}}) \tag{3.11}$$

Using (3.10) and (3.11) in (3.8) we find

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=-h} = -\frac{\partial\eta}{\partial t} \tag{3.12}$$

It will now be convenient to work with complex forms for η and ϕ, it being understood that the real parts of the complex functions apply to the physical problem. Thus in lieu of (3.1) we write

$$\eta(x,t) = a_o e^{i(\lambda x + \omega t)} \tag{3.13}$$

We assume that ϕ takes the form

$$\phi(x,z,t) = \phi'(z)e^{i(\lambda x + \omega t)} \qquad (-h \leq z \leq 0) \tag{3.14}$$

From (3.3) we have

$$\frac{d^2\phi'}{dz^2} - \lambda^2\phi' = 0$$

which has the general solution

$$\phi'(z) = A\cosh\lambda z + B\sinh\lambda z \tag{3.15}$$

where A and B are arbitrary constants.

From (3.7) we have

$$a_o = -\frac{i\omega}{g}\phi'(z)\Big|_{z=-h} = -\frac{i\omega}{g}(A\cosh\lambda h - B\sinh\lambda h) \tag{3.16}$$

From (3.12) we have

$$-\lambda A\sinh\lambda h + \lambda B\cosh\lambda h = -i\omega a_o \tag{3.17}$$

We solve (3.16) and (3.17) for A and B to get

$$A = \frac{ia_o g}{\omega}\left(\cosh\lambda h - \frac{\omega^2}{\lambda g}\sinh\lambda h\right)$$

$$B = \frac{ia_o g}{\omega}\left(\sinh\lambda h - \frac{\omega^2}{\lambda g}\cosh\lambda h\right) \tag{3.18}$$

Substituting (3.18) into (3.15) we get

$$\phi'(z) = \frac{ia_o g}{\omega}\left[\cosh\lambda(z+h) - \frac{\omega^2}{\lambda g}\sinh\lambda(z+h)\right] \tag{3.19}$$

This is consistent with the real forms of velocity potential derived in standard texts on wave mechanics.

Boundary Value Problem

The boundary conditions at the mudline are that the vertical effective stress and horizontal effective shear stress are zero i.e.

$$\tau'_{zz} \equiv \tau_{zz} + p = 0 \text{ at } z = 0 \tag{3.20}$$

$$\tau'_{xz} \equiv \tau_{xz} = 0 \text{ at } z = 0 \tag{3.21}$$

Further, the wave-induced pore pressure in the bed at $z = 0$ must be the same as the wave-induced pressure at the mudline.

Putting $z = 0$ in Bernoulli's equation (3.4) and neglecting the velocity term we have

$$P = -\rho_f \left.\frac{\partial \Phi}{\partial t}\right|_{z=0} + \rho_f g h$$

The term $\rho_f g h$ represents hydrostatic pressure, so that the wave-induced dynamic pressure at the mudline is given by $-\rho_f \left.\frac{\partial \Phi}{\partial t}\right|_{z=0}$.

Hence

$$p \text{ (in bed)} = -\rho_f \frac{\partial \Phi}{\partial t} \text{ at } z = 0 \tag{3.22}$$

For a bed of finite depth we require that vertical soil and water displacements at the impermeable bedrock underlying the bed, be zero i.e.

$$u_2 = 0 \quad \text{at } z = z_o \tag{3.23}$$

$$w_2 = 0 \quad \text{at } z = z_o \tag{3.24}$$

We will also assume that horizontal soil displacements are negligible at

$z = z_o$ i.e.

$$u_1 = 0 \quad \text{at} \quad z = z_o\,. \tag{3.25}$$

For a bed of infinite depth, soil displacements and wave-induced pore pressure vanish as $z \to \infty$ i.e.

$$u_1, u_2 \to 0 \quad \text{as} \quad z \to \infty \tag{3.26}$$

$$p \to 0 \quad \text{as} \quad z \to \infty \tag{3.27}$$

A dispersion relation may be obtained by invoking continuity of flow into the seabed at the mudline. Considering vertical flow through unit area of soil at the mudline, we must have

$$\frac{\partial \phi}{\partial z} = f \frac{\partial U_2}{\partial t} + (1 - f) \frac{\partial u_2}{\partial t} \quad \text{at} \quad z = 0$$

i.e.

$$\frac{\partial \phi}{\partial z} = \frac{\partial u_2}{\partial t} + \frac{\partial w_2}{\partial t} \quad \text{at} \quad z = 0 \tag{3.28}$$

An approximate dispersion relation is obtained by assuming that the right hand side of (3.28) is negligible i.e. $\frac{\partial \phi}{\partial z} = 0$ and hence $\frac{d\phi'}{dz} = 0$ at $z = 0$. We then find from equation (3.19) that

$$\tanh \lambda h = \frac{\omega^2}{\lambda g} \tag{3.29}$$

which is the familiar dispersion relation of linear wave theory.

4. SOLUTION OF EQUATIONS OF MOTION FOR WAVE LOADING

For convenience we record again the equations to be solved, viz. equations (2.95) and (2.96):

$$\mu u_{i,jj} + (\lambda_c + \mu)u_{j,ji} + Cw_{j,ji} = \rho\ddot{u}_i + \rho_f\ddot{w}_i \tag{4.1}$$

$$Cu_{j,ji} + Mw_{j,ji} = \rho_f\ddot{u}_i + m\ddot{w}_i + \frac{n_f}{k}\dot{w}_i \tag{4.2}$$

As noted in section 3, subscripts i and j take the values 1 and 2 with 1 referring to the x direction and 2 referring to the z direction. The above equations have been solved by Yamamoto (1983) for wave loading and for comparison purposes we present a different method of solution.

In view of the waveform (3.13) we assume solutions of the form

$$\begin{aligned} u_j &= U_j(z)\, e^{i(\lambda x + \omega t)} \\ w_j &= W_j(z)\, e^{i(\lambda x + \omega t)} \end{aligned} \tag{4.3}$$

We remark that the use of the notation U_1, U_2 here is not to be confused with its use in section 2.

Substituting (4.3) into (4.1) and (4.2) gives the four equations

$$(-\lambda^2 U_1 + \frac{d^2U_1}{dz^2}) + i\lambda(\lambda_c + \mu)F + i\lambda CG = -\omega^2(\rho U_1 + \rho_f W_1) \tag{4.4}$$

$$\mu(-\lambda^2 U_2 + \frac{d^2U_2}{dz^2}) + (\lambda_c + \mu)\frac{dF}{dz} + C\,\frac{dG}{dz} = -\omega^2(\rho U_2 + \rho_f W_2) \tag{4.5}$$

$$i\lambda CF + i\lambda MG = -\omega^2(\rho_f U_1 + mW_1) + \frac{i\omega n_f}{k} W_1 \tag{4.6}$$

$$C\frac{dF}{dz} + M\frac{dG}{dz} = -\omega^2(\rho_f U_2 + mW_2) + \frac{i\omega n_f}{k} W_2 \tag{4.7}$$

where

$$F(z) = i\lambda U_1 + \frac{dU_2}{dz}$$

$$G(z) = i\lambda W_1 + \frac{dW_2}{dz} \tag{4.8}$$

We assume that the functions $U_j(z)$ and $W_j(z)$ take the form

$$U_j(z) = P_j\, e^{sz}$$

$$W_j(z) = Q_j\, e^{sz} \tag{4.9}$$

Substituting this into equations (4.4) to (4.7) gives an equation of the form

$$[A]\begin{pmatrix}P_1\\P_2\\Q_1\\Q_2\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \tag{4.10}$$

where the matrix [A] is given by

$$
\begin{bmatrix}
\mu s^2 - \lambda^2(\lambda_c + 2\mu) + \omega^2\rho & i\lambda s(\lambda_c + \mu) & -\lambda^2 C + \omega^2\rho_f & i\lambda Cs \\
i\lambda s(\lambda_c + \mu) & s^2(\lambda_c + 2\mu) - \mu\lambda^2 + \omega^2\rho & i\lambda Cs & s^2C + \omega^2\rho_f \\
-\lambda^2 C + \omega^2\rho_f & i\lambda Cs & -\lambda^2 M + \omega^2 m - \frac{i\omega n_f}{k} & i\lambda Ms \\
i\lambda Cs & s^2C + \omega^2\rho_f & i\lambda Ms & s^2 M + \omega^2 m - \frac{i\omega n_f}{k}
\end{bmatrix}
$$

Matrix [A] (4.11)

For non-trivial solutions of (4.10) we require that the determinant of [A] be zero i.e.

$$|[A]| = 0 \tag{4.12}$$

Equation (4.12) is a cubic in s^2 and it takes the form

$$s^6 + \alpha_1 s^4 + \alpha_2 s^2 + \alpha_3 = 0 \tag{4.13}$$

where α_1, α_2, α_3 are complex constants. The six roots of (4.12) may be written $\pm s_1$, $\pm s_2$, $\pm s_3$ and are, in general, complex. We will take s_1, s_2, s_3 to be the roots with positive real parts.

Equation (4.9) may now be replaced by

$$\begin{aligned}
U_1(z) &= \sum_{k=1}^{3} a_k e^{-s_k z} + \sum_{k=1}^{3} a_k' e^{s_k z} \\
U_2(z) &= \sum_{k=1}^{3} b_k e^{-s_k z} + \sum_{k=1}^{3} b_k' e^{s_k z} \\
W_1(z) &= \sum_{k=1}^{3} c_k e^{-s_k z} + \sum_{k=1}^{3} c_k' e^{s_k z} \\
W_2(z) &= \sum_{k=1}^{3} d_k e^{-s_k z} + \sum_{k=1}^{3} d_k' e^{s_k z}
\end{aligned} \tag{4.14}$$

The coefficients a_k, a_k^1, . . . d_k, d_k' are determined from the governing equations (4.4) to (4.7), and the boundary conditions. We substitute (4.14) into equtions (4.4) to (4.7) to obtain four equations of the form

$$\sum_{k=1}^{3} (A_1^{(k)} a_k + A_2^{(k)} b_k + A_3^{(k)} c_k + A_4^{(k)} d_k) e^{-s_k z}$$

$$+ \sum_{k=1}^{3} (A_1'^{(k)} a_k' + A_2'^{(k)} b_k' + A_3'^{(k)} c_k' + A_4'^{(k)} d_k') e^{s_k z} = 0$$

$$\sum_{k=1}^{3} (B_1^{(k)} a_k + B_2^{(k)} b_k + B_3^{(k)} c_k + B_4^{(k)} d_k) e^{-s_k z}$$

$$+ \sum_{k=1}^{3} B_1'^{(k)} a_k' + B_2'^{(k)} b_k' + B_3'^{(k)} c_k' + B_4'^{(k)} d_k') e^{s_k z} = 0$$

$$\sum_{k=1}^{3} (C_1^{(k)} a_k + C_2^{(k)} b_k + C_3^{(k)} c_k + C_4^{(k)} d_k) e^{-s_k z}$$

$$+ \sum_{k=1}^{3} (C_1'^{(k)} a_k' + C_2'^{(k)} b_k' + C_3'^{(k)} c_k' + C_4'^{(k)} d_k') e^{s_k z} = 0$$

$$\sum_{k=1}^{3} (D_1^{(k)} a_k + D_2^{(k)} b_k + D_3^{(k)} c_k + D_4^{(k)} d_k) e^{-s_k z}$$

$$+ \sum_{k=1}^{3} D_1'^{(k)} a_k' + D_2'^{(k)} b_k' + D_3'^{(k)} c_k' + D_4'^{(k)} d_k') e^{s_k z} = 0 \qquad (4.15)$$

where

$$A_1^{(k)} = \mu s_k^2 - \lambda^2(\lambda_c + 2\mu) + \omega^2\rho$$

$$A_2^{(k)} = -i\lambda s_k(\lambda_c + \mu)$$

$$A_3^{(k)} = -\lambda^2 C + \omega^2 \rho_f$$

$$A_4^{(k)} = -i\lambda C s_k$$

$$B_1^{(k)} = -i\lambda s_k(\lambda_c + \mu)$$

$$B_2^{(k)} = s_k^2(\lambda_c + 2\mu) - \mu\lambda^2 + \omega^2\rho$$

$$B_3^{(k)} = -i\lambda C s_k$$

$$B_4^{(k)} = s_k^2 C + \omega^2\rho_f$$

$$C_1^{(k)} = -\lambda^2 C + \omega^2\rho_f$$

$$C_2^{(k)} = -i\lambda C s_k$$

$$C_3^{(k)} = -\lambda^2 M + \omega^2 m - i\omega \frac{n_f}{k}$$

$$C_4^{(k)} = -i\lambda M s_k$$

$$D_1^{(k)} = -i\lambda C s_k$$

$$D_2^{(k)} = s_k^2 C + \omega^2\rho_f$$

$$D_3^{(k)} = -i\lambda M s_k$$

$$D_4^{(k)} = s_k^2 M + \omega^2 m - \frac{i\omega n_f}{k}$$

The coefficients $A_1'^{(k)}$, $A_2'^{(k)}$, . . . $D_4'^{(k)}$ are obtained from $A_1^{(k)}$, $A_2^{(k)}$, . . . $D_4^{(k)}$ respectively by replacing s_k by $-s_k$.

Since equations (4.15) must hold for all z, the coefficients of $e^{-s_k z}$ and $e^{s_k z}$ must be identically zero for each k, i.e.

$$\begin{bmatrix} A_1^{(k)} & A_2^{(k)} & A_3^{(k)} & A_4^{(k)} \\ B_1^{(k)} & B_2^{(k)} & B_3^{(k)} & B_4^{(k)} \\ C_1^{(k)} & C_2^{(k)} & C_3^{(k)} & C_4^{(k)} \\ D_1^{(k)} & D_2^{(k)} & D_3^{(k)} & D_4^{(k)} \end{bmatrix} \begin{Bmatrix} a_k \\ b_k \\ c_k \\ d_k \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4.16}$$

for k = 1, 2, 3

$$\begin{bmatrix} A_1'^{(k)} & A_2'^{(k)} & A_3'^{(k)} & A_4'^{(k)} \\ B_1'^{(k)} & B_2'^{(k)} & B_3'^{(k)} & B_4'^{(k)} \\ C_1'^{(k)} & C_2'^{(k)} & C_3'^{(k)} & C_4'^{(k)} \\ D_1'^{(k)} & D_2'^{(k)} & D_3'^{(k)} & D_4'^{(k)} \end{bmatrix} \begin{Bmatrix} a_k' \\ b_k' \\ c_k' \\ d_k' \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4.17}$$

From (4.16) we may express b_k, c_k and d_k in terms of a_k' as

$$b_k = \varepsilon_k^{(1)} a_k', \quad c_k = \varepsilon_k^{(2)} a_k, \quad d_k = \varepsilon_k^{(3)} a_k \tag{4.18}$$

for k = 1, 2, 3.

Similarly, from (4.17) we determine b_k', c_k' and d_k' in terms of a_k':

$$b_k' = \varepsilon_k^{(1)'} a_k' \quad , \quad c_k' = \varepsilon_k^{(2)'} a_k' \quad , \quad d_k' = \varepsilon_k^{(3)'} a_k' \qquad (4.19)$$

for k = 1, 2, 3.

Case (a): Bed of Finite Depth.

By using the constitutive relations (2.23) and equations (4.18) and (4.19), the boundary conditions (3.20) to (3.25) may be written in the form

$$[C]\underline{a} = \underline{r} \qquad (4.20)$$

where

$$\underline{a} = (a_1, a_2, a_3, a_1', a_2', a_3')^T$$

$$\underline{r} = (0, 0, P_0, 0, 0, 0)^T$$

with

$$P_0 = \rho_f a_0 g(\cosh \lambda h - \frac{\omega^2}{\lambda g} \sinh \lambda h) \qquad (4.21)$$

The elements c_{ij} of the matrix [C] are given by

$$c_{1,k} = i\lambda(\lambda_c - C) + (\lambda_c + 2\mu - C)(-s_k\varepsilon_k^{(1)}) + i\lambda(C - M)\varepsilon_k^{(2)} + (C - M)(-s_k\varepsilon_k^{(3)}) \qquad k = 1, 2, 3$$

$$c_{1,k+3} = i\lambda(\lambda_c - C) + (\lambda_c + 2\mu - C)(s_k\varepsilon_k^{(1)'}) + i\lambda(C - M)\varepsilon_k^{(2)'} + (C - M)(s_k\varepsilon_k^{(3)'}) \qquad k = 1, 2, 3$$

$$c_{2,k} = -s_k + i\lambda\varepsilon_k^{(1)} \qquad k = 1, 2, 3$$

$$c_{2,k+3} = s_k + i\lambda\varepsilon_k^{(1)'} \qquad k = 1, 2, 3$$

$$c_{3,k} = -i\lambda C + Cs_k\varepsilon_k^{(1)} - i\lambda M\varepsilon_k^{(2)} + Ms_k\varepsilon_k^{(3)} \qquad k = 1, 2, 3 \tag{4.22}$$

$$c_{3,k+3} = -i\lambda C - Cs_k\varepsilon_k^{(1)'} - i\lambda M\varepsilon_k^{(2)'} - Ms_k\varepsilon_k^{(3)'} \qquad k = 1, 2, 3$$

$$c_{4,k} = \varepsilon_k^{(1)} e^{-s_k z_o} \qquad k = 1, 2, 3$$

$$c_{4,k+3} = \varepsilon_k^{(1)'} e^{s_k z_o} \qquad k = 1, 2, 3$$

$$c_{5,k} = \varepsilon_k^{(3)} e^{-s_k z_o} \qquad k = 1, 2, 3$$

$$c_{5,k+3} = \varepsilon_k^{(3)'} e^{s_k z_o} \qquad k = 1, 2, 3$$

$$c_{6,k} = e^{-s_k z_o} \qquad k = 1, 2, 3$$

$$c_{6,k+3} = e^{s_k z_o} \qquad k = 1, 2, 3$$

Equation (4.20) is solved to obtain the coefficients a_k, a_k' (k = 1, 2, 3). The coefficients b_k, b_k', c_k, c_k', d_k, d_k' (k = 1, 2, 3) are then found from (4.18) and (4.19) and $U_j(z)$, $W_j(z)$ (j = 1, 2) determined from (4.14). Pore pressures and effective stresses are now easily computed from the constitutive relations (2.23).

Case (b): Bed of Infinite Depth.

In this case we immediately deduce that $a_k' = b_k' = c_k' = d_k' = 0$ (k = 1, 2, 3) in order to satisfy boundary conditions (3.26) and (3.27). Boundary conditions (3.20) to (3.22) result in the equation

$$[C]\underline{a} = \underline{r} \qquad (4.23)$$

where

$$\underline{a} = (a_1, a_2, a_3)^T$$

$$\underline{r} = (0, 0, P_0)^T$$

where P_0 is given by (4.21). The matrix [C] is of order 3 and its elements are $c_{1,k}$, $c_{2,k}$, $c_{3,k}$ (k = 1, 2, 3) as defined in (4.22).

5. QUASI-STATIC THEORY FOR SAND BEDS

5.1 Theory

The general theory described in section 2 may be simplified for the analysis of sand beds. Because of the stiffness of sand beds (as compared to clays, for example) we may neglect the acceleration terms in the equations of motion, viz the terms in $\ddot{u}_i$ and $\ddot{w}_i$. The equations of motion (2.91) and (2.92) become

$$\tau_{ij,j} = 0 \tag{5.1}$$

$$-p_{,i} = \frac{n_f}{k} \dot{w}_i \tag{5.2}$$

From the constitutive equation in the form (2.42) we have

$$\tau_{ij,j} = \mu\, u_{i,jj} + (\lambda^* + \mu) u_{j,ji} - \alpha p_{,i}$$

Using this in equation (5.1) gives

$$\mu\, u_{i,jj} + (\lambda^* + \mu) u_{j,ji} = \alpha p_{,i} \tag{5.3}$$

We will rewrite equation (5.2) in terms of the soil displacement vector u_i (rather than w_i). Differentiating (5.2) with respect to x_i gives

$$p_{,ii} = -\frac{n_f}{k} \dot{w}_{i,i} = \frac{n_f}{k} \dot{\zeta}$$

$$= \frac{n_f}{k} \frac{\partial}{\partial t} \left(\frac{p + \alpha M \varepsilon}{M}\right)$$

where we have used the last of equations (2.23).

Hence we obtain

$$\frac{k}{n_f} \nabla^2 p = \frac{1}{M} \frac{\partial p}{\partial t} + \alpha \frac{\partial \varepsilon}{\partial t} \tag{5.4}$$

We have already noted in section (2.3) that for most soils $K_r >> K_b$ so that α is very close to unity. Also, from (2.38) we have

$$\frac{1}{M} = \frac{K_r[1 + f\,(\frac{K_r}{K_f} - 1)] - K_b}{K_r^2}$$

$$= \frac{1 - f}{K_r} - \frac{K_b}{K_r} + \frac{f}{K_r}$$

If $K_r \to \infty$, $\frac{1}{M} \to \frac{f}{K_f}$

Thus for soils for which $K_r \to \infty$, equation (5.4) becomes

$$\frac{k}{n_f} \nabla^2 p = \frac{f}{K_f} \frac{\partial p}{\partial t} + \frac{\partial \varepsilon}{\partial t} \tag{5.5}$$

Equation (5.5) is the so-called storage equation derived in a different way by Verruijt (1969) and Biot (1941). The Biot theory of 1941 is in fact represented by equation (5.3) with $\alpha = 1$, and equation (5.5).

We will solve the system of equations (5.3) and (5.4) subject to appropriate boundary conditions. For a bed of finite depth z_o we have

$$\tau'_{zz} = \tau'_{xz} = 0 \quad \text{at} \quad z = 0$$

$$p \text{ (in bed)} = -\rho_f \frac{\partial \phi}{\partial t} \quad \text{at} \quad z = 0 \tag{5.6}$$

$$u_1 = u_2 = 0 \quad \text{at} \quad z = z_o$$

$$\frac{\partial p}{\partial z} = 0 \quad \text{at} \quad z = z_o \tag{5.7}$$

For an infinite bed we replace the conditions (5.7) by the condition that u_1, u_2 and p tend to zero as z goes to infinity i.e.

$$u_1, u_2, p \to 0 \quad \text{as } z \to \infty \tag{5.8}$$

5.2 General Solution of the System of Equations (5.3) and (5.4)

As in section 4, subscripts i and j will take the values 1 and 2. Equation (5.3) is satisfied by the function

$$u_i = (\phi + x_j \psi_j)_{,i} - 4(1 - \nu)\psi_i \qquad (5.9)$$

where

$$\nu = \frac{\lambda^*}{2(\lambda^* + \mu)} \qquad (5.10)$$

provided that

$$\frac{2\mu(1 - \nu)}{1 - 2\nu} [\nabla^2\phi + x_j\nabla^2 \psi_j]_{,i} - 4\mu(1 - \nu)\nabla^2 \psi_i = \alpha p_{,i} \qquad (5.11)$$

Equation (5.9) is the well known Papkovich-Neuber solution of the equilibrium equations of the theory of elasticity. ϕ and ψ_i are functions of x,z and t. Equation (5.11) is in fact the equilibrium equation (5.3) written in terms of ϕ and ψ_i. The parameter ν is Poisson's ratio.

In the theory of linear elasticity any one of the functions ϕ, ψ_1, ψ_2 may be taken to be zero without loss of completeness provided that the coordinate system is chosen in an appropriate way and 4ν is not a positive integer (Eubanks and Sternberg, 1956). We will make the same assumption in the present case.

We choose $\psi_1 = 0$ and $\psi_2 = \psi$. Then equations (5.11) become

$$2\mu\beta \frac{\partial}{\partial x} (\nabla^2\phi + z\, \nabla^2\psi) = \alpha \frac{\partial p}{\partial x} \qquad (5.12)$$

$$2\mu\beta \frac{\partial}{\partial z} (\nabla^2\phi + z\nabla^2\psi) - 4\mu(1 - \nu)\nabla^2\psi = \alpha \frac{\partial p}{\partial z} \tag{5.13}$$

where

$$\beta = \frac{1 - \nu}{1 - 2\nu} \tag{5.14}$$

Using (5.9) we write equation (5.4) in terms of ϕ and ψ:

$$\frac{k}{n_f} \nabla^2 p - \frac{1}{M} \frac{\partial p}{\partial t} = \alpha \frac{\partial}{\partial t} [\nabla^2\phi + z\nabla^2\psi - 2(1 - 2\nu) \frac{\partial\psi}{\partial z}] \tag{5.15}$$

A solution to (5.12) and (5.13) is given by

$$\nabla^2\psi = 0$$
$$\nabla^2\phi = \frac{\alpha p}{2\mu\beta} \tag{5.16}$$

Substituting (5.16) into (5.15) gives

$$c \nabla^2 p = \frac{\partial p}{\partial t} + \alpha_o \frac{\partial}{\partial t} (\frac{\partial\psi}{\partial z}) \tag{5.17}$$

where

$$c = \frac{k}{n_f} [\frac{1}{M} + \frac{\alpha^2}{2\mu\beta}]^{-1}$$
$$\alpha_o = -2(1 - 2\nu)\alpha[\frac{1}{M} + \frac{\alpha^2}{2\mu\beta}]^{-1} \tag{5.18}$$

We set

$$p(x,z,t) = q(x,z,t) + r(x,z,t) \tag{5.19}$$

where q satisfies the homogeneous equation

$$c\ \nabla^2 q = \frac{\partial q}{\partial t} \tag{5.20}$$

We substitute (5.19) into (5.17) and use (5.20) to obtain the following equation for r:

$$c\ \nabla^2 r = \frac{\partial r}{\partial t} + \alpha_o \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial z}\right) \tag{5.21}$$

A solution of (5.21) is given by

$$r = -\alpha_o \frac{\partial \psi}{\partial z} \tag{5.22}$$

since by the first of equations (5.16),

$$\nabla^2 r = \nabla^2\left(-\alpha_o \frac{\partial \psi}{\partial z}\right) = -\alpha_o \frac{\partial}{\partial z}(\nabla^2 \psi) = 0\ .$$

Thus a solution to (5.17) is given by (5.19) subject to conditions (5.20) and (5.22). We note at this point that p is the sum of an harmonic function r and a function q which satisfies the "heat equation" form. The wave-induced pore pressure p is thus a combination of the responses suggested in previous studies (see equations (1.1) and (1.2)).

The above approach provides a general solution to the system of equations (5.3) and (5.4). The problem reduces to solving the equations

$$\nabla^2\psi = 0$$

$$\nabla^2\phi = \frac{\alpha}{2\mu\beta}(q + r) \tag{5.23}$$

where q and r satisfy

$$c\nabla^2 q = \frac{\partial q}{\partial t}$$

$$r = -\alpha_o \frac{\partial\psi}{\partial z} \tag{5.24}$$

5.3 Solution for Wave Loading

In view of the complex wave form (3.13) we assume that the wave-induced pore pressure is given by

$$p(x,z,t) = P(z)e^{i(\lambda x + \omega t)} \tag{5.25}$$

Since $p = q + r$ we let

$$q(x,z,t) = Q(z)e^{i(\lambda x + \omega t)}$$

$$r(x,z,t) = R(z)e^{i(\lambda x + \omega t)}$$

To satisfy (5.24) we must have

$$R(z) = A_1 e^{-\lambda z} + A_2 e^{\lambda z}$$

$$Q(z) = A_3 e^{-\lambda' z} + A_4 e^{\lambda' z}$$

where

$$(\lambda')^2 = \lambda^2 + \frac{i\omega}{c} \tag{5.26}$$

and the A_i's are constants. The form of $R(z)$ was deduced from the fact that r is harmonic.

Combining the above results with (5.19) we have

$$p(x,z,t) = (A_1 e^{-\lambda z} + A_2 e^{\lambda z} + A_3 e^{-\lambda' z} + A_4 e^{\lambda' z})e^{i(\lambda x + \omega t)} \tag{5.27}$$

The functions ϕ and ψ must be chosen in accordance with (5.16) and (5.22). To choose the form of ϕ, we note from the last of equations (5.16) and equation (5.19) that we may decompose ϕ into

$$\phi = \phi_1 + \phi_2$$

where

$$\nabla^2\phi_1 = \frac{\alpha q}{2\mu\beta} \quad \text{and} \quad \nabla^2\phi_2 = \frac{\alpha r}{2\mu\beta} \tag{5.28}$$

Applying the operator ∇^2 to the first of equations (5.28) and using (5.20) gives

$$\nabla^4\phi_1 = \frac{\alpha\nabla^2 q}{2\mu\beta} = \frac{\alpha}{2\mu\beta c}\frac{\partial q}{\partial t}$$

$$= \frac{\alpha}{2\mu\beta c}\frac{\partial}{\partial t}\left(\frac{2\mu\beta}{\alpha}\nabla^2\phi_1\right)$$

i.e.

$$\nabla^4\phi_1 = \frac{1}{c}\frac{\partial}{\partial t}(\nabla^2\phi_1) \tag{5.29}$$

If we assume that

$$\phi_1 = \Phi_1(z)e^{i(\lambda x + \omega t)} ,$$

equation (5.29) gives

$$\phi_1(z) = b_1 e^{-\lambda z} + b_2 e^{\lambda z} + b_3 e^{-\lambda' z} + b_4 e^{\lambda' z} \tag{5.30}$$

From the second of equations (5.28) we have similarly

$$\nabla^4 \phi_2 = \frac{\alpha \nabla^2 r}{2\mu\beta} = 0$$

Writing $\phi_2 = \phi_2(z) e^{i(\lambda x + \omega t)}$ we find

$$\phi_2(z) = b_5 z\, e^{-\lambda z} + b_6 z\, e^{\lambda z} \tag{5.31}$$

From (5.30) and (5.31) we deduce the form of ϕ as

$$\phi(x,z,t) = [(a_1 + a_2 z)e^{-\lambda z} + (a_3 + a_4 z)e^{\lambda z} + a_5 e^{-\lambda' z} + a_6 e^{\lambda' z}] e^{i(\lambda x + \omega t)} \tag{5.32}$$

where a_k; $k = 1,2 \ldots 6$ are constants. Substituting (5.32) and (5.27) into the second of equations (5.16) and equating coefficients of $e^{-\lambda z}$, $e^{\lambda z}$, $e^{-\lambda' z}$, $e^{\lambda' z}$ we get

$$A_1 = \frac{-4\mu\beta\lambda a_2}{\alpha}$$

$$A_2 = \frac{4\mu\beta\lambda a_4}{\alpha} \tag{5.33}$$

$$A_3 = \frac{2\mu\beta[(\lambda')^2 - \lambda^2] a_5}{\alpha}$$

$$A_4 = \frac{2\mu\beta[(\lambda')^2 - \lambda^2] a_6}{\alpha}$$

We choose $\psi = (Be^{-\lambda z} + De^{\lambda z})e^{i(\lambda x + \omega t)}$

which satisfies the first of equations (5.16) and is in accordance with (5.22). From (5.22) we find

$$B = \frac{A_1}{\alpha_o \lambda} = \frac{-4\mu\beta a_2}{\alpha\alpha_o}$$

$$D = \frac{-A_2}{\alpha_o \lambda} = \frac{-4\mu\beta a_4}{\alpha\alpha_o} \qquad (5.34)$$

Using (5.33) and (5.34), p and ψ may be expressed in terms of the six constants $a_1, a_2, \ldots, a_6$. Using the constitutive relations we write the effective stress components in terms of ϕ and ψ:

$$\tau'_{ij} = 2\mu[\phi_{,ij} + z\psi_{,ij} - (1 - 2\nu)(\psi_{i,j} + \psi_{j,i})]$$

$$+ \delta_{ij}\left(\frac{2\mu\nu}{1 - 2\nu}\right)[\nabla^2\phi + z\nabla^2\psi - 2(1 - 2\nu)\frac{\partial\psi}{\partial z}] \qquad (5.35)$$

$$+ \delta_{ij}(1 - \alpha)\frac{2\mu\beta}{\alpha}\nabla^2\phi$$

Hence the effective stress components can be expressed in terms of the constants $a_1, a_2, \ldots a_6$. The six boundary conditions (5.6) and (5.7) lead to an equation of the form

$$[A]\underline{x} = \underline{b} \qquad (5.36)$$

where

$$\underline{x} = (a_1, a_2, a_3, a_4, a_5, a_6)^r$$

$$\underline{b} = (0, 0, P_o, 0, 0, 0)^r \qquad (5.37)$$

P_o is the amplitude of the pressure wave at the mudline i.e.

$$-\rho_f \left.\frac{\partial\phi}{\partial t}\right|_{z=0} = P_o e^{i(\lambda x + \omega t)}$$

The expression for P_o is given by equation (4.21).

The elements a_{ij} of the matrix [A] are given on the following page.

$$\begin{bmatrix} \lambda^2 & a_{12} & \lambda^2 & a_{14} & \lambda^2 + \frac{\beta[(\lambda')^2 - \lambda^2]}{\alpha} & \lambda^2 + \frac{\beta[(\lambda')^2 - \lambda^2]}{\alpha} \\ -\lambda & 1 + \frac{4(1-\nu)\mu}{\alpha\alpha_o} & \lambda & 1 + \frac{4(1-\nu)\mu}{\alpha\alpha_o} & -\lambda' & \lambda' \\ 0 & -2\lambda & 0 & 2\lambda & (\lambda')^2 - \lambda^2 & (\lambda')^2 - \lambda^2 \\ e^{-\lambda z_o} & z_o e^{-\lambda z_o}\left(1 - \frac{4\mu\beta}{\alpha\alpha_o}\right) & e^{\lambda z_o} & z_o e^{\lambda z_o}\left(1 - \frac{4\mu\beta}{\alpha\alpha_o}\right) & e^{-\lambda' z_o} & e^{\lambda' z_o} \\ -\lambda e^{-\lambda z_o} & a_{52} & \lambda e^{\lambda z_o} & a_{54} & -\lambda' e^{-\lambda' z_o} & \lambda' e^{\lambda' z_o} \\ 0 & 2\lambda^2 e^{-\lambda z_o} & 0 & 2\lambda^2 e^{\lambda z_o} & -\lambda'[(\lambda')^2 - \lambda^2]e^{-\lambda' z_o} & \lambda'[(\lambda')^2 - \lambda^2]e^{\lambda' z_o} \end{bmatrix}$$

Matrix [A] (5.38)

$$a_{12} = -\frac{2\lambda\beta}{\alpha}\left[1 + \frac{4(1-\nu)\mu}{\alpha_o}\right] \qquad a_{14} = -a_{12}$$

$$a_{52} = \left[-\lambda z_o + 1 + (\lambda z_o + 3 - 4\nu)\frac{4\mu\beta}{\alpha\alpha_o}\right]e^{-\lambda z_o}$$

$$a_{54} = \left[\lambda z_o + 1 + (-\lambda z_o + 3 - 4\nu)\frac{4\mu\beta}{\alpha\alpha_o}\right]e^{\lambda z_o}$$

Equation (5.36) is solved for the coefficients $a_1, a_2, \ldots a_6$ and hence the functions ϕ and ψ are determined. The wave-induced pore pressure p is found from the second of equations (5.16) and the wave-induced effective stress components τ'_{ij} are found from (5.35).

For a bed of infinite depth we must have $a_3 = a_4 = a_6 = 0$. The matrix equation (5.36) reduces to

$$\begin{bmatrix} a_{11} & a_{12} & a_{15} \\ a_{21} & a_{22} & a_{25} \\ a_{31} & a_{32} & a_{35} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ P_o \end{bmatrix} \qquad (5.39)$$

and this equation is solved for the coefficients a_1, a_2 and a_5.

6. FAILURE ANALYSIS

The sign convention most commonly used in soil mechanics is that a compressive stress is considered positive. Adopting this convention we write

$$\bar{\tau}'_{ij} = \tau'^{(o)}_{ij} - \tau'_{ij} \tag{6.1}$$

where τ'_{ij} are the wave-induced effective stress components as computed from (2.43) or (5.35), $\tau'^{(o)}_{ij}$ are the initial static effective stress components, and $\bar{\tau}'_{ij}$ are the net effective stress components.

The static effective stresses are given by

$$\begin{aligned} \tau'^{(o)}_{11} &= \left(\frac{\nu}{1-\nu}\right)\gamma_b\, z \\ \tau'^{(o)}_{22} &= \gamma_b\, z \\ \tau'^{(o)}_{12} &= 0 \end{aligned} \tag{6.2}$$

where γ_b is the buoyant unit weight of soil and is given by

$$\begin{aligned} \gamma_b &= \gamma_{sat} - \rho_f g \\ &= (\rho_s - \rho_f)(1 - f)g \end{aligned} \tag{6.3}$$

where g is the acceleration due to gravity and γ_{sat} is the saturated unit weight of the soil.

Types of Failure

(a) Liquefaction of Sands.

When at least one of the net effective stresses $\overline{\tau}'_{11}$, $\overline{\tau}'_{22}$ is zero or negative (tensile) the sand is said to be liquefied and in fact behaves like a dense liquid.

(b) "Sliding" Failure (Mohr-Coulomb) failure)

When the net effective stress components $\overline{\tau}'_{ij}$ are such that the internal frictional resistance of the soil is exceeded on some plane, "sliding" failure is said to have occurred. This is investigated using the Mohr-Coulomb failure criterion.

Figure (6.1) shows the Mohr circle corresponding to the state of stress $\overline{\tau}'_{ij}$ at some depth z. Denoting the centre of the circle by (a,0) and the radius by r we have

$$a = \frac{1}{2} (\overline{\tau}'_{11} + \overline{\tau}'_{22})$$

$$r = \frac{1}{2} [(\overline{\tau}'_{11} - \overline{\tau}'_{22})^2 + 4(\overline{\tau}'_{12})^2]^{\frac{1}{2}} \qquad (6.4)$$

Figure (6.1) also shows the failure envelope which intercepts the τ axis at $(0,\overline{c})$ and is inclined at an angle ϕ_f to the σ axis. The angle ϕ_f is known as the angle of internal friction of the soil. $\overline{c}$ is called the cohesion intercept.

The tangent from $(0,\overline{c})$ to the circle makes an angle ϕ with the σ axis. The angle ϕ will be called the stress angle corresponding to the state of stress $\overline{\tau}'_{ij}$.

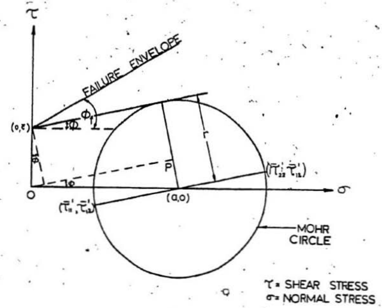

FIG. 6.1 THE STRESS ANGLE φ

By constructing the line OP as shown parallel to the tangent it is easily shown that

$$r = a \sin \phi + \bar{c} \cos \phi \qquad (6.5)$$

It is clear from the diagram that "sliding" failure occurs when

$$\phi \geq \phi_f' \qquad (6.6)$$

The angle ϕ is determined by solving equation (6.5). For sands, $\bar{c} = 0$ and hence

$$\sin \phi = \frac{r}{a} = \frac{[(\bar{\tau}_{11}' - \bar{\tau}_{22}')^2 + 4(\bar{\tau}_{12}')^2]^{\frac{1}{2}}}{\bar{\tau}_{11}' + \bar{\tau}_{22}'} \qquad (6.7)$$

The angle ϕ provides a convenient measure of the "distance" between the existing stress state in the soil and the failure state, the failure criterion being (6.6).

7. RESULTS

The wave-induced pore pressure, effective stresses and stress angle under a wave crest were computed for a fine sand, a coarse sand and a clay using both the dynamic theory (Biot, 1962) and the quasi-static theory described in section 5. Figures (7.1) to (7.15) show the results for a bed of finite depth; figures (7.16) to (7.30) show the results for a bed of infinite depth. The following data were used (Yamamoto, 1978 and 1983):

		Sand	Clay
Shear Modulus	$\mu(N/m^2)$	1.0×10^7	1.0×10^6
Damping parameter	δ	0.05	0.05
Poisson's ratio	ν	0.333	0.45
Porosity	f	0.3	0.4
Bulk modulus of grain	$K_r(N/m^2)$	3.60×10^{10}	3.60×10^{10}
Bulk modulus of water	$K_f(N/m^2)$	2.30×10^9	2.30×10^9
Density of soil grains	$\rho_s(kg/m^3)$	2.7×10^3	2.65×10^3
Density of water	$\rho_f(kg/m^3)$	1.0×10^3	1.0×10^3
Permeability	k(m/s)	1.0×10^{-4} (fine sand) 1.0×10^{-2} (coarse sand)	9.81×10^{-7}
"Added mass" parameter	$\rho_a(kg/m^3)$	$0.25 \cdot f\ \rho_f$	$0.25\ f\ \rho_f$
Soil depth	$z_o(m)$	25.0	25.0
Cohesion intercept (clay)	$\bar{c}(N/m^2)$		12.0×10^3
		Wave Parameters	
Wave Period	T(s)	15.0	15.0

Water depth	h(m)	70.0	50.0
Wavelength	L(m)	311.812	256.456
Wave amplitude	a_o(m)	12.0	12.0

The values of the wavelength L were calculated by solving equation (3.29).

The following notation has been used on the graphs:

Real(P)/P_o = Normalized wave-induced pore pressure

Arg(P) = Pore pressure phase shift

SX = Real (τ'_{11}) = Wave-induced horizontal effective stress

SZ = Real (τ'_{22}) = Wave-induced vertical effective stress

SXZ = Real (τ'_{12}) = Wave-induced shear stress

DELTA = δ = Damping parameter

The results were obtained by solving the matrix equations (4.20) (dynamic theory) and (5.36) (quasi-static theory) for a finite bed; and equations (4.23) (dynamic theory) and (5.39) (quasi-static theory) for an infinite bed. Required input for the computer program of the dynamic theory includes the roots of equation (4.13) and these were obtained using an IMSL subroutine on the university's VAX 11/780 computer system. The matrix equations were also solved with the aid of an IMSL subroutine. In the case of clay beds, overflow problems were encountered due to the large moduli of the parameters s_3 and λ' and the computer program had to be modified to deal with these problems.

As suggested by equations (2.60) and (2.59) the effects of damping in the soil may be incorporated into the Biot theory by replacing the real elastic moduli μ and λ^* by complex moduli $\bar{\mu}$ and $\bar{\lambda}^*$. The value of the damping parameter δ varies from 0.02 for very small strains to 0.20 for large strains, and a conservative value of $\delta = 0.05$ was used for both the sand and clay beds (Yamamoto, 1983).

The results indicate the following:

(1) For sand beds under wave loading there is virtually no difference between the quasi-static and dynamic theories except for the shear stress response. However, the magnitude of the shear stress is insignificant compared to the other stress components required for the Mohr-Coulomb failure criterion, and has negligible effect on the depth to which the soil fails. This can be seen from figures (7.5), (7.10), (7.20) and (7.25) in which the stress angle variation is the same for both theories.

(2) For clay beds the quasi-static and dynamic theories do not agree. This suggests that for the soft clay beds the dynamic terms $\ddot{u}_i$ and $\ddot{w}_i$ in the equations of motion are significant, so that a quasi-static theory does not adequately predict the bed response. However, there is at present insufficient experimental data to test this claim.

The graphs of stress angle vs. depth may be used to determine the soil failure depths by liquefaction or "sliding" failure. For sands the value of the stress angle ϕ was set at 90° whenever $\bar{\tau}'_{11} \leq 0$ (tensile), so that the depth of liquefaction is the greatest depth for which $\phi = 90°$. The net vertical effective stress $\bar{\tau}'_{22}$ was always positive (compressive). The depth to which "sliding" failure occurs is determined from the Mohr-Coulomb failure criterion (6.6).

The sensitivity of the failure depths to the following parameters was examined for a coarse sand using the quasi-static theory: shear modulus G, Poisson's ratio ν, porosity f, permeability $k_o(ms^{-1})$, wavelength L, wave amplitude a_o. The friction angle ϕ_f was assumed to be 30°. The results are shown in Figures (7.31) to (7.35). No graph was drawn for the shear modulus G because the results showed no change in the failure depth as G was changed from 10^7 N/m^2 to 10^8 N/m^2. We observe that liquefaction depth is far less sensitive to changes in the above parameters than "sliding" failure depth. The graph for Poisson's ratio (Figure 7.31) deserves special comment, this being a parameter which may be difficult to determine accurately. The liquefaction depth is relatively insensitive to the value of ν but the "sliding" failure depth is strongly dependent on ν. For instance, a Poisson's ratio of $\nu = 0.30$ gives a "sliding" failure depth of 10 m., whereas $\nu = 0.33$ gives a "sliding" failure depth of 6.5 m.

We note that the failure depths are quite sensitive to the wave amplitude a_o, as is to be expected, while changes in porosity, permeability and wavelength have less significant effects on the failure depths.

Finally, in order to examine the effect of the small damping parameter, wave-induced horizontal effective stresses in a finite coarse sand bed were computed for the cases $\delta = 0.05$ and $\delta = 0.00$ (undamped) and the results are shown in Figure (7.36). The graphs show no appreciable difference in the stresses for the two cases.

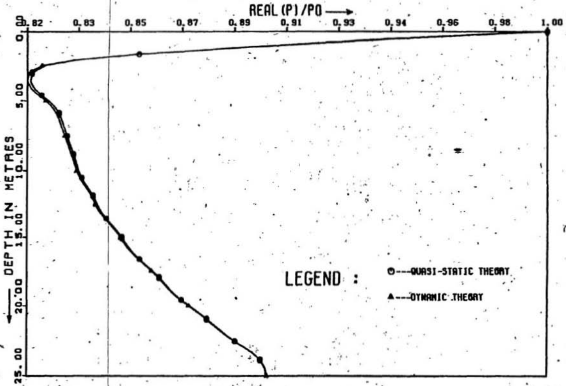

FIG. 7.1a NORMALIZED WAVE-INDUCED PORE PRESSURE
[BED OF FINITE DEPTH (FINE SAND)]

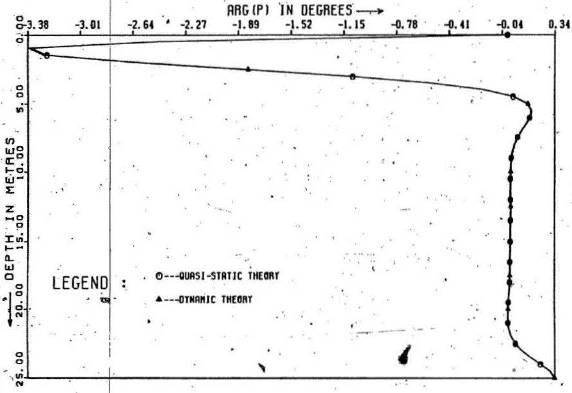

FIG. 7.1b PORE PRESSURE PHASE SHIFT
[BED OF FINITE DEPTH (FINE SAND)]

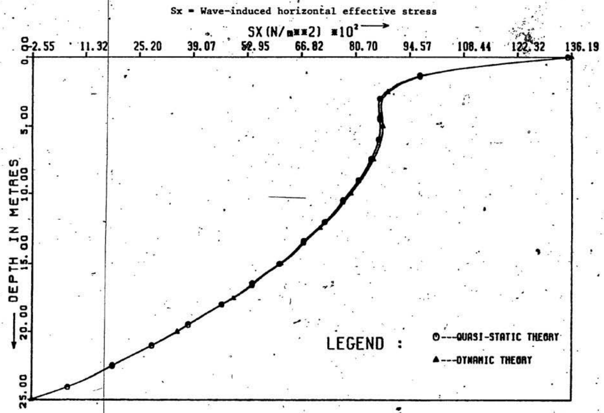

FIG. 7.2 WAVE-INDUCED HORIZONTAL EFFECTIVE STRESS
[BED OF FINITE DEPTH (FINE SAND)]

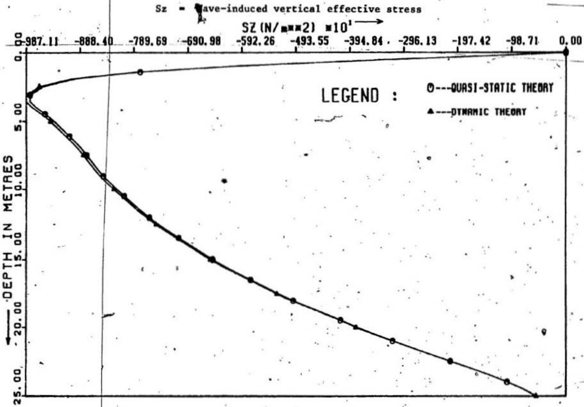

FIG. 7.3 WAVE-INDUCED VERTICAL EFFECTIVE STRESS
[BED OF FINITE DEPTH (FINE SAND)]

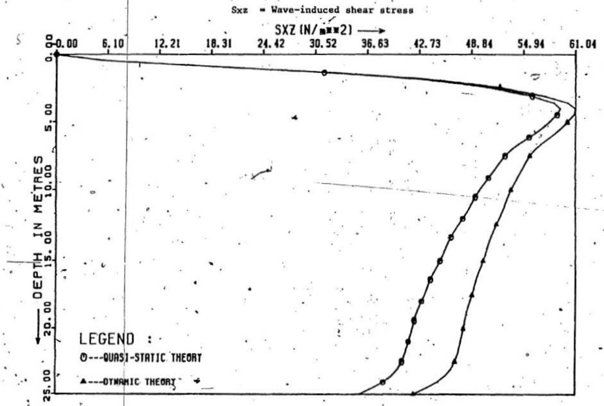

FIG. 7.4 WAVE-INDUCED SHEAR STRESS
[BED OF FINITE DEPTH (FINE SAND)]

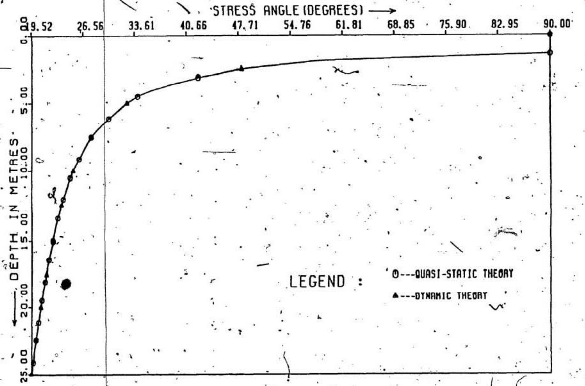

FIG. 7.5 STRESS ANGLE
[BED OF FINITE DEPTH (FINE SAND)]

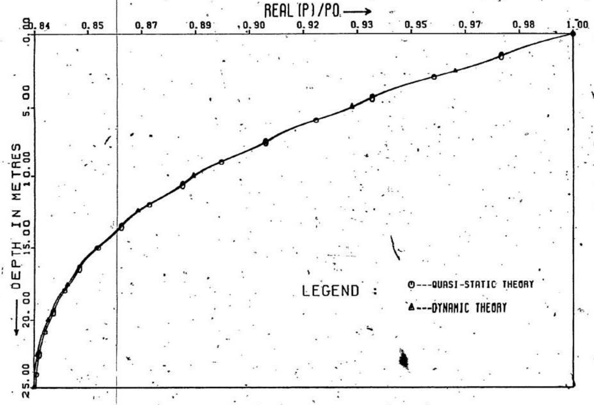

FIG. 7.6a NORMALIZED WAVE-INDUCED PORE PRESSURE

[BED OF FINITE DEPTH (COARSE SAND)]

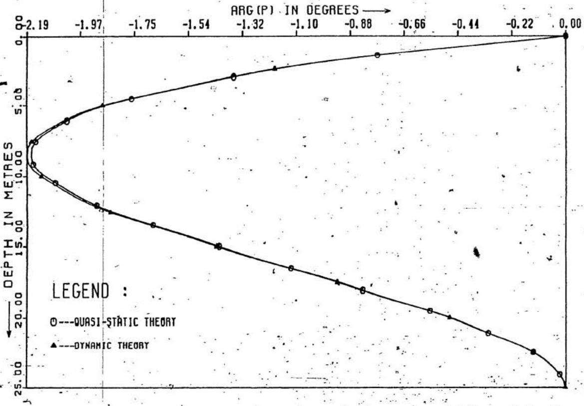

FIG. 7.6b PORE PRESSURE PHASE SHIFT

[BED OF FINITE DEPTH (COARSE SAND)]

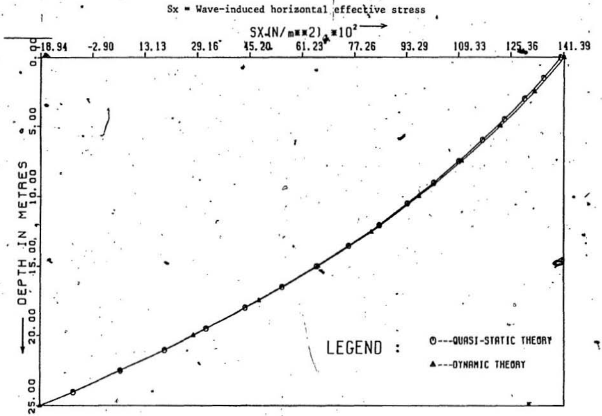

FIG. 7.7 WAVE-INDUCED HORIZONTAL EFFECTIVE STRESS
[BED OF FINITE DEPTH (COARSE SAND)]

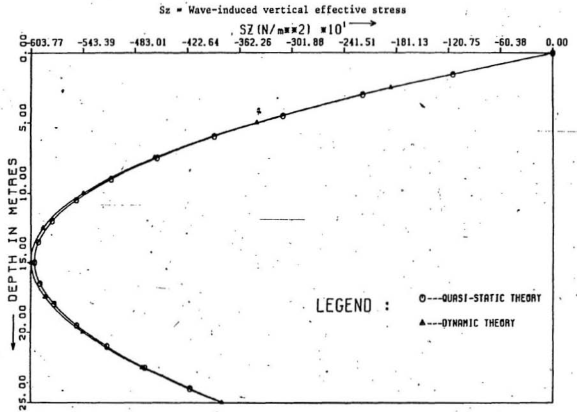

FIG. 7.8 WAVE-INDUCED VERTICAL EFFECTIVE STRESS
[BED OF FINITE DEPTH (COARSE SAND)]

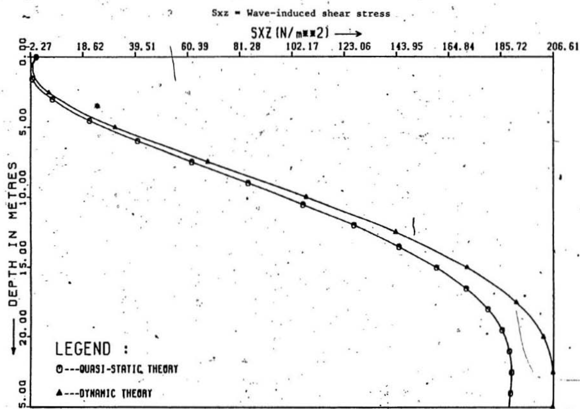

FIG. 7.9 WAVE-INDUCED SHEAR STRESS
[BED OF FINITE DEPTH (COARSE SAND)]

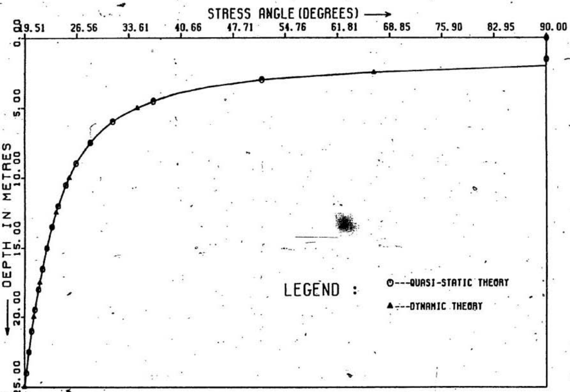

FIG. 7.10 STRESS ANGLE
[BED OF FINITE DEPTH (COARSE SAND)]

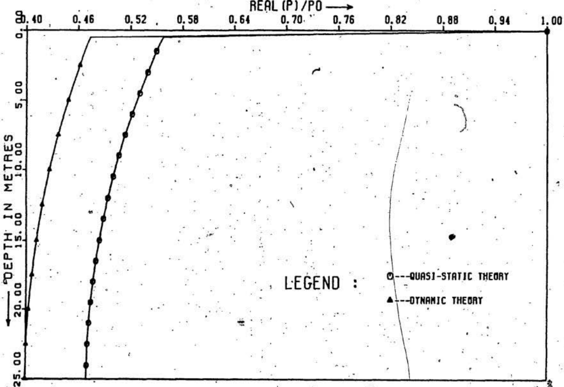

FIG. 7.11a NORMALIZED WAVE-INDUCED PORE PRESSURE
[BED OF FINITE DEPTH (CLAY)]

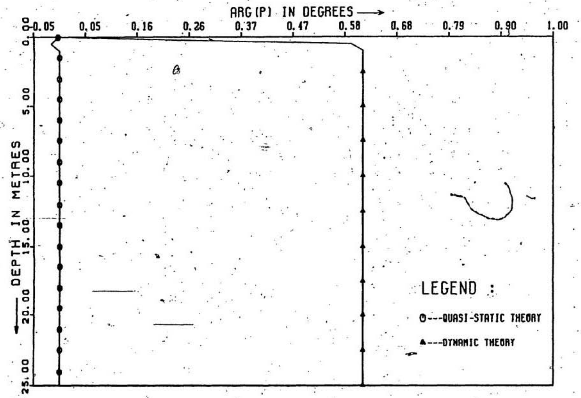

FIG. 7.11b PØRE PRESSURE PHASE SHIFT
[BED OF FINITE DEPTH (CLAY)]

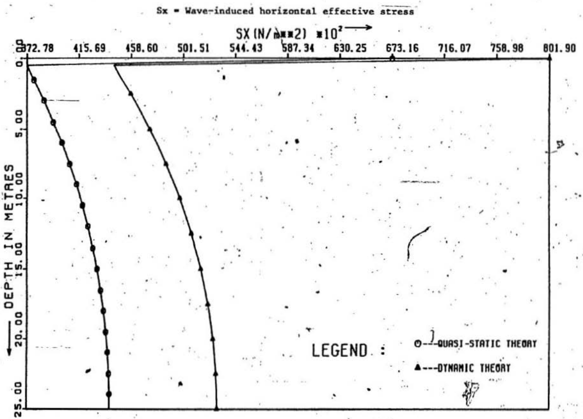

FIG. 7.12 WAVE-INDUCED HORIZONTAL EFFECTIVE STRESS
[BED OF FINITE DEPTH (CLAY)]

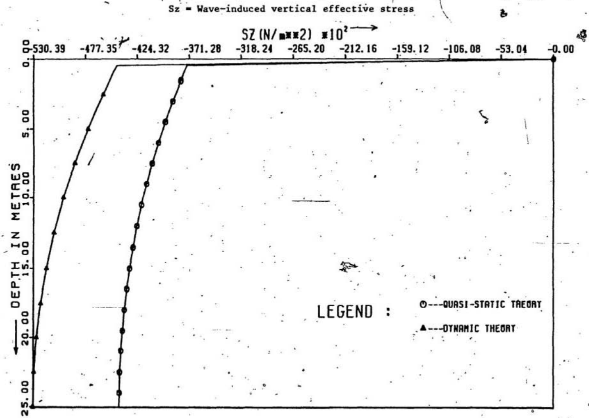

FIG. 7.13 WAVE-INDUCED VERTICAL EFFECTIVE STRESS
[BED OF FINITE DEPTH (CLAY)]

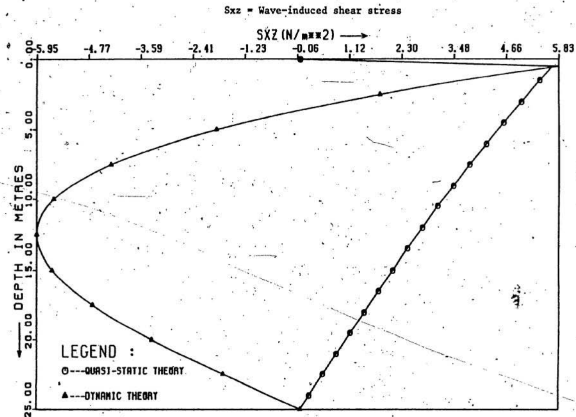

FIG. 7.14 WAVE-INDUCED SHEAR STRESS
[BED OF FINITE DEPTH (CLAY)]

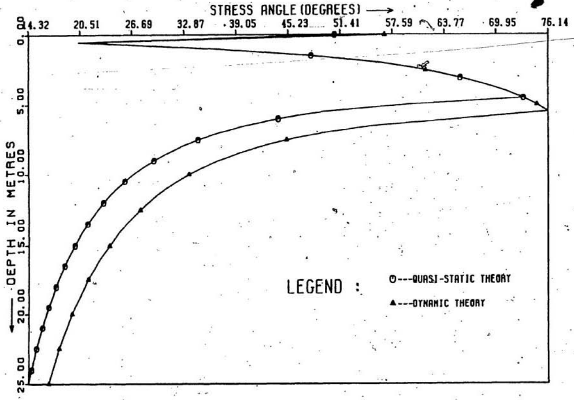

FIG. 7.15 STRESS ANGLE
[BED OF FINITE DEPTH (CLAY)]

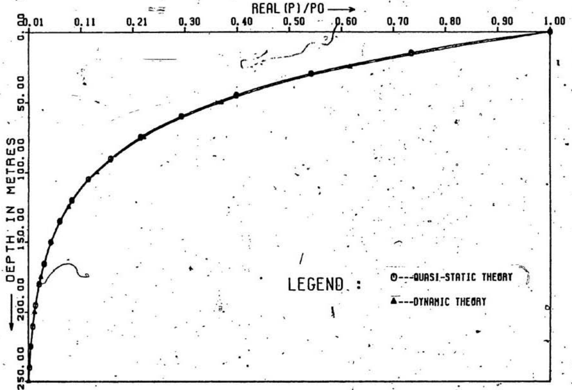

FIG. 7.16a-NORMALIZED WAVE-INDUCED PORE PRESSURE
[BED OF INFINITE DEPTH (FINE SAND)]

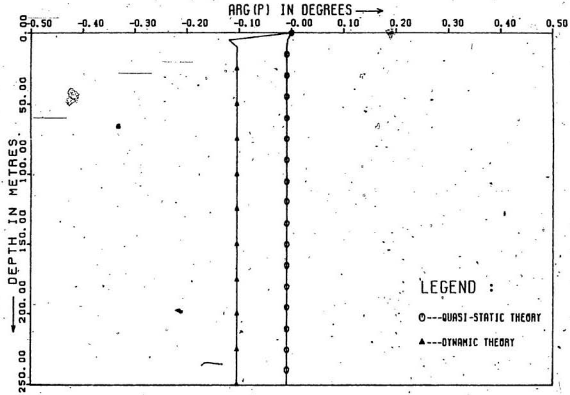

FIG. 7.16b PORE PRESSURE PHASE SHIFT
[BED OF INFINITE DEPTH (FINE SAND)]

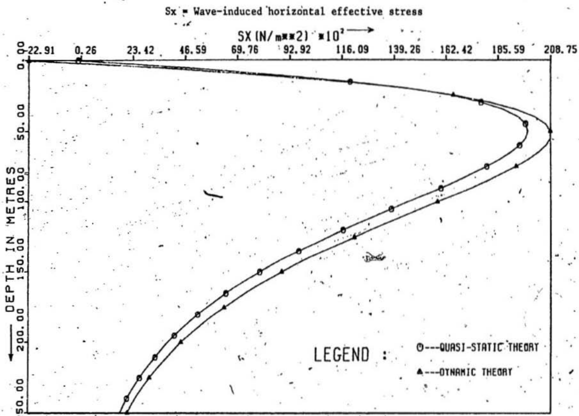

FIG. 7.17 WAVE-INDUCED HORIZONTAL EFFECTIVE STRESS
[BED OF INFINITE DEPTH (FINE SAND)]

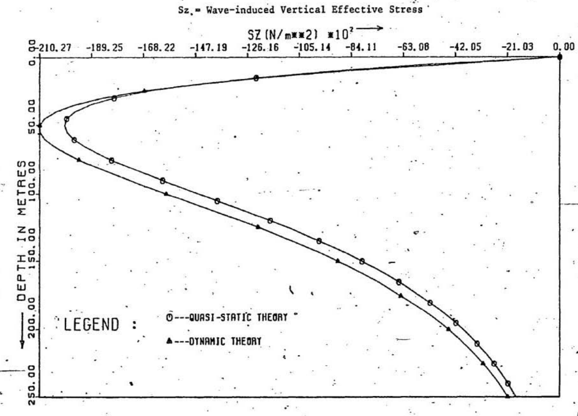

FIG. 7.18 WAVE-INDUCED VERTICAL EFFECTIVE STRESS
[BED OF INFINITE DEPTH (FINE SAND)]

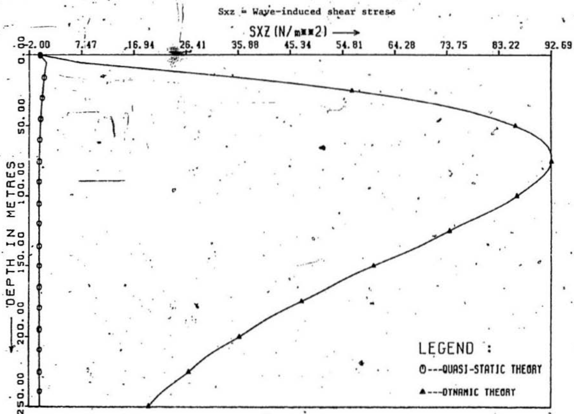

FIG. 7.19 WAVE-INDUCED SHEAR STRESS
[BED OF INFINITE DEPTH (FINE SAND)]

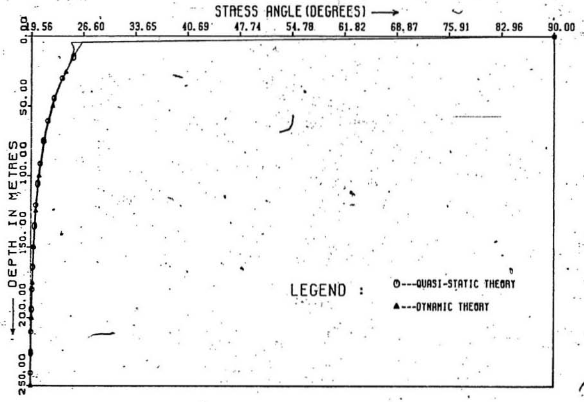

FIG. 7.20 STRESS ANGLE
[BED OF INFINITE DEPTH (FINE SAND)]

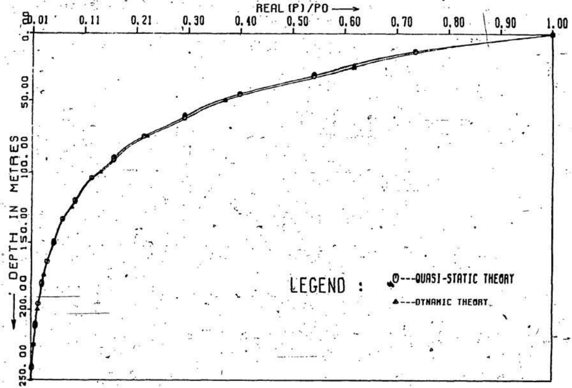

FIG. 7.21a NORMALIZED WAVE-INDUCED PORE PRESSURE
[BED OF INFINITE DEPTH (COARSE SAND)]

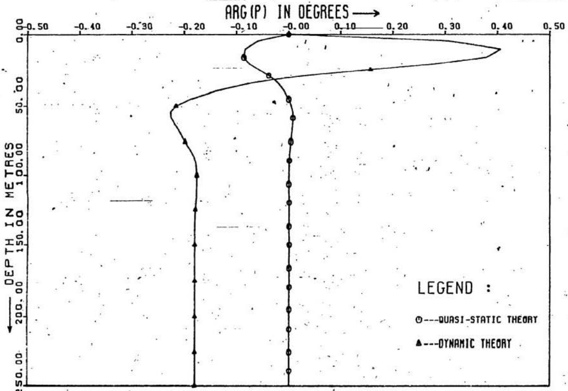

FIG. 7.21b PORE PRESSURE PHASE SHIFT
[BED OF INFINITE DEPTH (COARSE SAND)]

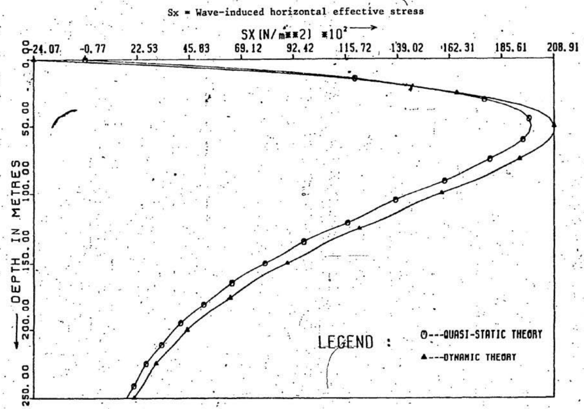

FIG. 7.22 WAVE-INDUCED HORIZONTAL EFFECTIVE STRESS

[BED OF INFINITE DEPTH (COARSE SAND)]

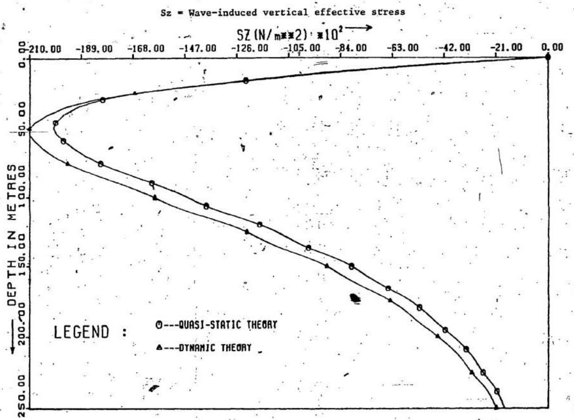

FIG. 7.23 WAVE-INDUCED VERTICAL EFFECTIVE STRESS
[BED OF INFINITE DEPTH (COARSE SAND)]

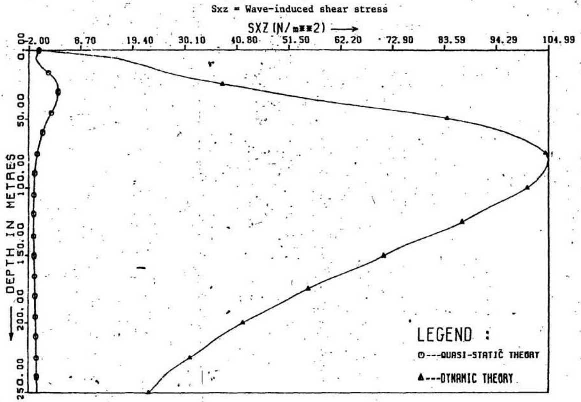

FIG. 7.24 WAVE-INDUCED SHEAR STRESS
[BED OF INFINITE DEPTH (COARSE SAND)]

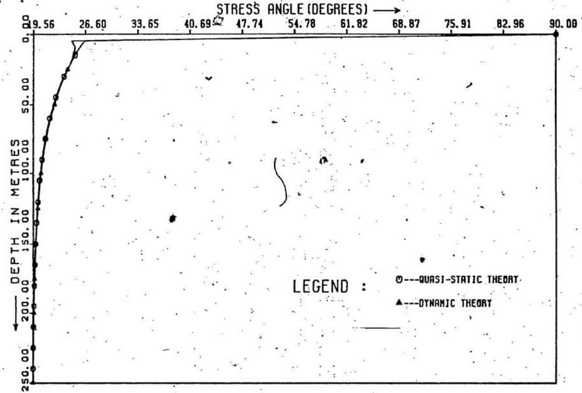

FIG. 7.25 STRESS ANGLE
[BED OF INFINITE DEPTH (COARSE SAND)]

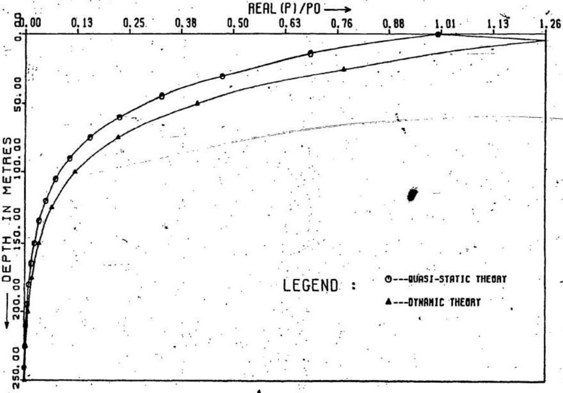

FIG. 7.26a NØRMALIZED WAVE-INDUCED PØRE PRESSURE
[BED OF INFINITE DEPTH (CLAY)]

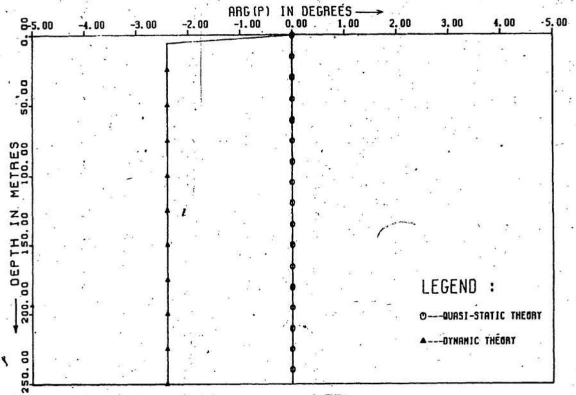

FIG. 7.26b PØRE PRESSURE PHASE SHIFT
[BED OF INFINITE DEPTH (CLAY)]

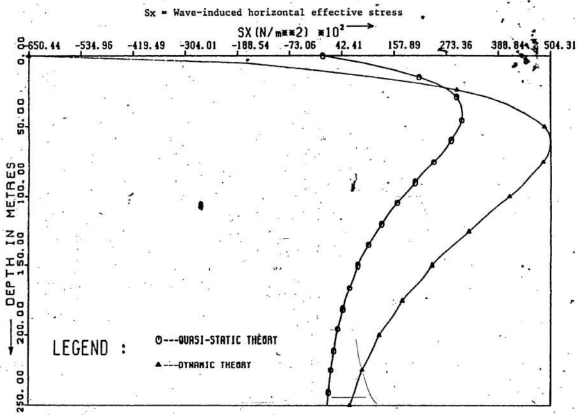

FIG. 7.27 WAVE-INDUCED HORIZONTAL EFFECTIVE STRESS
[BED OF INFINITE DEPTH (CLAY)]

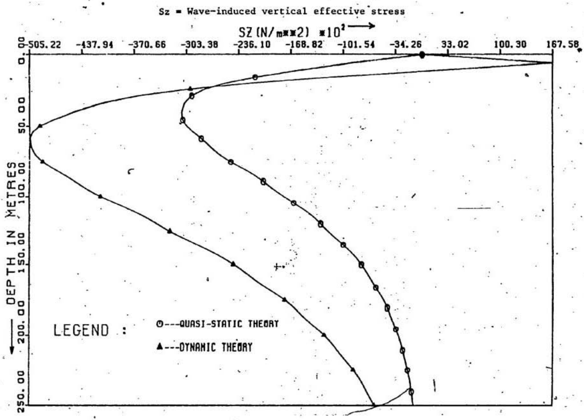

FIG. 7.28 WAVE-INDUCED VERTICAL EFFECTIVE STRESS

[BED OF INFINITE DEPTH (CLAY)]

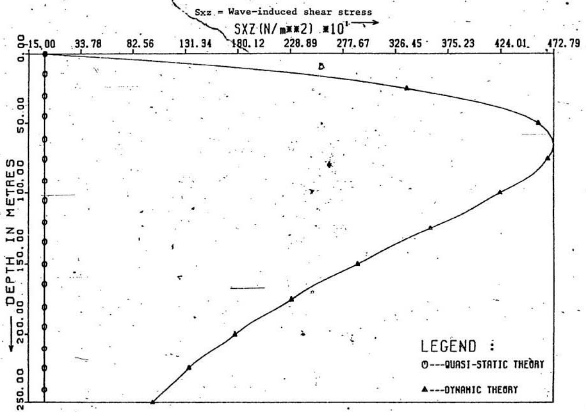

FIG. 7.29 WAVE-INDUCED SHEAR STRESS
[BED OF INFINITE DEPTH (CLAY)]

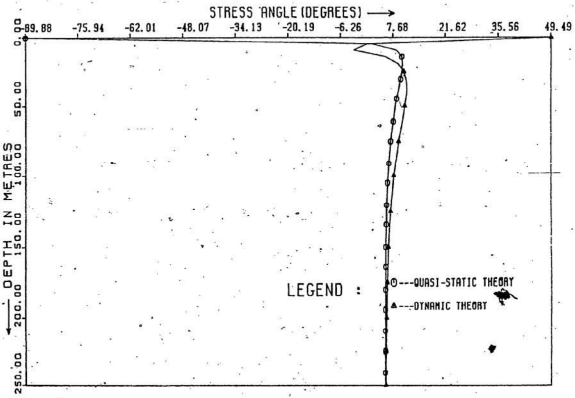

FIG. 7.30 STRESS ANGLE
[BED OF INFINITE DEPTH (CLAY)]

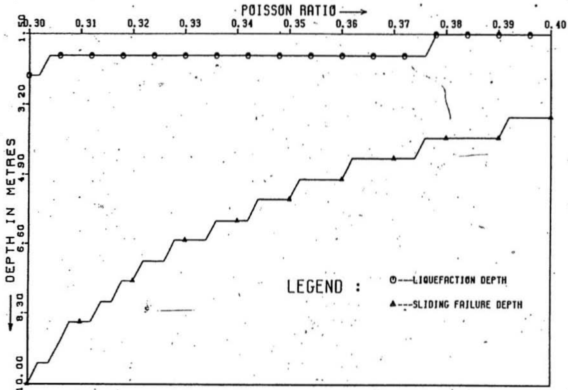

FIG. 7.31 FAILURE DEPTHS VS. POISSON RATIO
[BED OF FINITE DEPTH (COARSE SAND)]

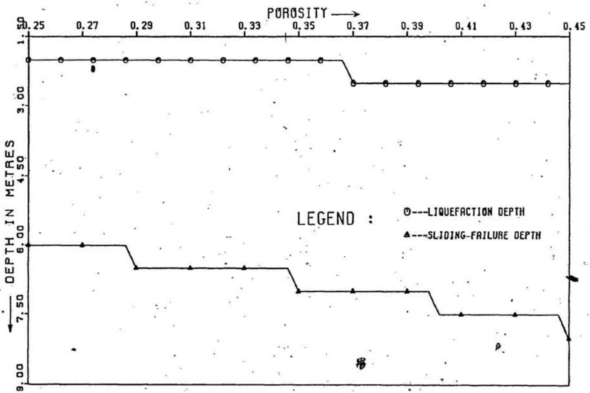

FIG. 7.32 FAILURE DEPTHS VS. POROSITY
[BED OF FINITE DEPTH (COARSE SAND)]

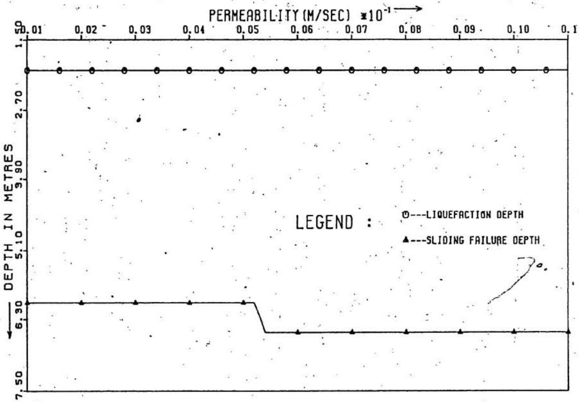

FIG. 7.33 FAILURE DEPTHS VS. PERMEABILITY

[BED OF FINITE DEPTH (COARSE SAND)]

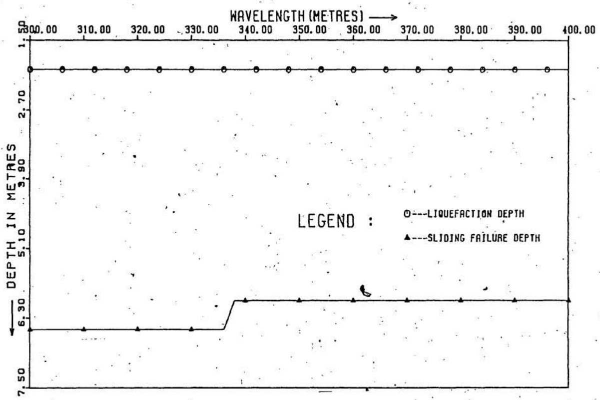

FIG. 7.34 FAILURE DEPTHS VS. WAVELENGTH
[BED OF FINITE DEPTH (COARSE SAND)]

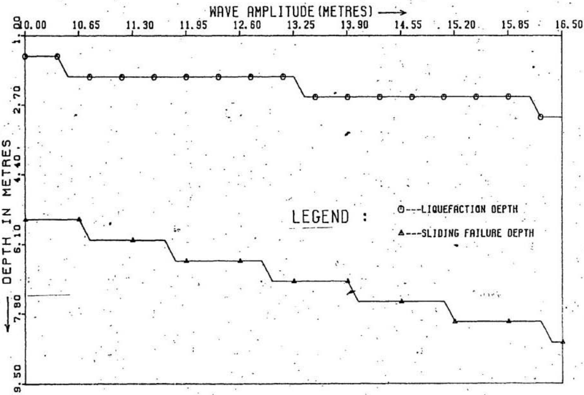

FIG. 7.35 FAILURE DEPTHS VS. WAVE AMPLITUDE
[BED OF FINITE DEPTH (COARSE SAND)]

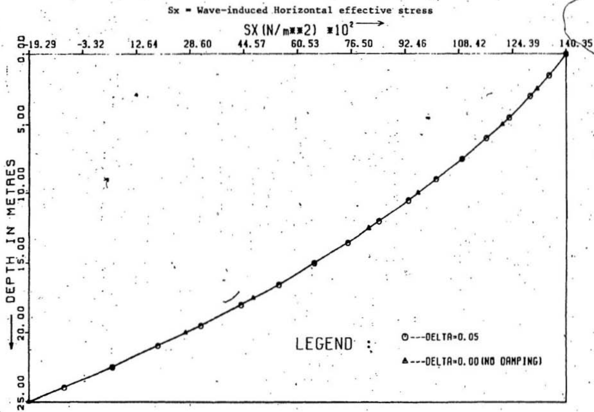

FIG. 7.36 WAVE INDUCED HORIZONTAL EFFECTIVE STRESS
[BED OF FINITE DEPTH (COARSE SAND)]

8. SUMMARY AND CONCLUSIONS

The scope of the thesis may be summarised as follows:

(1) A theory of deformation of porous media proposed by Biot in 1962 has been re-examined and elucidated. This theory is appropriate as a continuum model for offshore soils.

(2) The relationship between Biot's general constitutive relation and Hooke's law has been derived.

(3) Methods of incorporating the effects of soil damping into the constitutive relations have been examined.

(4) A quasi-static theory has been outlined and shown to be closely approximated by the Biot theory of 1941. The responses of seabeds of finite and infinite depth predicted by the quasi-static and dynamic theories have been compared. The results show that the quasi-static theory provides an excellent model for sand beds but may not be adequate for clay beds. Some of these results have appeared in the literature in various forms (Yamamoto, 1978 and 1983; Madsen, 1978), but the equations and graphs for the finite bed using the dynamic theory, are new. Further, we have given graphs for all the stresses, some of which have not appeared in the literature previously.

(5) The sensitivity of failure depths to the following parameters has been examined: shear modulus, Poisson's ratio, porosity, permeability, wavelength, wave amplitude. Of these, the Poisson's ratio and the wave amplitude have the greatest effect on failure depths, in particular the depth of "sliding" failure. In general, the depth of liquefaction is not as sensitive to the above parameters as is the "sliding" failure depth. No graph is shown for shear modulus because changes in this parameter (for the range considered) had no effect on the failure depth.

(6) A general solution technique for the equations of the quasi-static theory has been presented. The solution technique was suggested by the Papkovich-Neuber method of the theory of elasticity. This method is considerably less tedious than the solutions presented by Yamamoto (1978) and Madsen (1978). Further, the method is applicable to more complex problems such as wave loading of the seabed with a body resting on the bed. Methods of approaching such problems have not previously appeared in the literature.

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Appendix

Criterion for the validity of the assumptions of linear wave theory (Section 3, pg. 51).

From equation (3.1) we have

$$\eta(x,t) = a_o \; \text{Cos}(\lambda x + \omega t) \tag{A.1}$$

For small displacements of the free surface we may assume that the speed V of the water particles at the free surface is given approximately by

$$V = \frac{\partial \eta}{\partial t} \tag{A.2}$$

From (A.1) and (A.2) we have

$$V = -\omega a_o \; \text{Sin}(\lambda x + \omega t) \tag{A.3}$$

Using (A.1) and (A.3) we write the expression $\frac{1}{2} V^2 + g\eta$ occurring in equation (3.5) as

$$\frac{1}{2} V^2 + g\eta = \frac{1}{2} \omega^2 a_o^2 \; \text{Sin}^2(\lambda x + \omega t) + g a_o \; \text{Cos} \; (\lambda x + \omega t)$$

The term in V^2 may be neglected if

$$\frac{1}{2} \omega^2 a_o^2 << g a_o$$

i.e. if $\frac{\omega^2 a_o}{2g} << 1$ (A.4)

In order to obtain a condition involving wavelength L, we use the dispersion relation (3.29).

From (3.29), $\omega^2 = \lambda g \tanh \lambda h$ so that condition (A.4) becomes

$$\frac{1}{2} \lambda a_o \tanh \lambda h << 1$$

or, since $\lambda = \frac{2\pi}{L}$,

$$\frac{\pi a_o}{L} \tanh \left(\frac{2\pi h}{L}\right) << 1 \tag{A.5}$$