ELECTROMAGNETIC SCATTERING FROM A HALF-SPACE VERTICAL DISCONTINUITY: OPERATOR DECOMPOSITION APPROACH

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ELECTROMAGNETIC SCATTERING FROM A HALF-SPACE VERTICAL DISCONTINUITY: OPERATOR DECOMPOSITION APPROACH

by

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ABSTRACT

In this thesis the so-called "mixed-path" propagation problem is tackled by modelling the physical geometry and the various electromagnetic field quantities with the aid of Heaviside step functions in the manner of Walsh (29). Operating on the regional field quantities with the usual vector differential operators produces the classical boundary conditions of electromagnetism that are normally supplied in the classical analyses. The Helmholtz-type equations obtained for the regional field quantities are convolved with the proper Green functions and then Fourier transformed. An algebraic equation in the transform domain for the surface field to the right of the media interface is thus obtained. In order that this equation may be solved, various approximations are made in order that Wiener-Hopf method may be succesfully applied. The final answer, which is in the form of an asymptotic expansion of a Fourier inversion integral, is of the wrong order in field strength fall-off when compared with other authors' results and empirical evidence. Discussion is given as to the suitability and limitations of the Wiener-Hopf method for application to the problem.

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LIST OF SYMBOLS

μ	Permeability of the medium
ϵ	Permittivity of the medium
σ	Conductivity of the medium
ϵ_r	Relative permittivity of the medium
k	Wave number
ω	Radian frequency
E	Electric field vector
D	Displacement field vector
H	Magnetic field vector
B	Magnetic flux density vector
$\mathbf{J}_s$	Source current density vector
$\mathbf{J}_c$	Conduction current density vector
∇	Del differential operator
δ	Dirac delta function
h_1	Unit step function in z direction

h_2	Unit step function in x direction
a_z	Unit vector in z-direction, for example
n_{01}	Complex refractive index, for example, of medium 1.
K_{02}	Green Function in medium 2, for example
$*$	Convolution operator
x_0	Distance of source from interface
ξ	x Fourier transform variable
η	y Fourier transform variable
ζ	z Fourier transform variable
I	Dipole current.
dl	Infinitesimal dipole length

Other symbols are defined as and when they occur and must be interpreted within the context of the situation.

CHAPTER I: INTRODUCTION

General

We shall be concerned with the so-called "mixed-path" problem of electromagnetic propagation, which may be concisely described as follows. We assume the earth to be flat, and consisting of two [electrically] different linear, isotropic, and homogeneous media, as shown in Diagram 1.1, separated by a vertical interface. Given a transmitting source situated in the [assumed] vacuum above medium 1 we wish to determine the field strength on the earth's surface above medium 2.

This is easily seen to be a generalisation of Sommerfeld's classical homogeneous flat earth propagation problem. The mixed-path problem enjoyed a considerable amount of attention in the ten or so years after the second World War [see Chapter 2]. Owing to the complexity of the problem, most investigators were forced to consider the case of transmitting and receiving points on the earth's surface.

The solution of the mixed-path problem is important from an electrical engineering viewpoint. Both the amplitude and the phase characteristics of the surface field have been found [see Millington [15]] to change when crossing the boundary. Indeed, the so-called "recovery-effect" describes the sudden increase in field strength, as compared with that of the corresponding homogeneous earth result, when making the transition from earth to seawater, for example, [again, see Millington [15]].

Scope of this Thesis

In view of the close agreement between the existing analytical models for mixed-path propagation, as discussed in chapter 2, with the experimental observations, we should hardly be in a position to expect any startling new results by our approach. Rather, it is the aim of this thesis to show that an "acceptable" theoretical solution can be obtained through an entirely different analysis.

The electrical properties and the electromagnetic field vectors themselves are written, for each of the three distinct media, with the aid of unit step (Heaviside) functions. Application of the various scalar and vector differential operators to these quantities provides us with three Helmholtz-type differential equations for the three regional fields, together with a boundary equation. Convolving the three field equations with the appropriate "Green function", we arrive at three convolution equations for the regional fields. As with other, classical, studies we neglect the fields below the earth's surface. Thus, the convolution equations involve only functions of the surface field. To eliminate the convolutions, we apply a two dimensional x-y Fourier transform. The boundary equation allows us to substitute for some of the surface field quantities in terms of others. By considering the transformed equations on various fixed horizontal planes, either above or below the earth's surface, we arrive at a transform domain equation involving the surface field to the right of the vertical interface.

To obtain the spatial domain solution for the surface field we resort to the Wiener-Hopf technique. This requires that we "decompose" certain functions of the x transform variable, either "multiplicatively" or "additively", in terms of functions which are analytic in either the upper or lower complex half-planes. Applying the usual Wiener-Hopf decomposition formulas, we arrive at equations for the first, "multiplicative", decompostion functions that contain intractable integrals. To surmount this difficulty we resort to the practice of approximating the function to be decomposed by another, with similar analytic properties, but whose decomposition may be effected much more easily. This allows us to proceed to a transform domain algebraic equation for the surface field.

Standard asymptotic methods are employed to calculate the two-dimensional inverse transform of our transform domain surface field. The resulting asymptotic expansion indicates a weaker fall-off of field strength than is predicted by other theories and measured experimentally. Indeed, the fall-off is more indicative of a line source than the chosen dipole source. Various conclusions are drawn about the suitability of the Wiener-Hopf technique to this problem, particularly in terms of the approximation made to the function to be decomposed.

CHAPTER 2: PREVIOUS WORK

Outline

Feinberg [7] seems to have first considered a general method for treating the problem of mixed-path attenuation over a flat earth. Employing a semi-empirical approach Millington [15] obtained the so-called "geometric mean" formula for calculating mixed path field strengths.

In 1953 Clemmow [4] attacked the flat-earth mixed-path problem with the aid of his plane-wave spectrum representation technique. This method is applicable to a wide range of electromagnetic radiation and scattering problems, and with it Clemmow obtained results that supported Millington's work. Dual integral equations are established for the problem, and Clemmow employs a technique akin to the Wiener-Hopf method, as used in this thesis, for their solution. For this reason we shall discuss Clemmow's work in a certain amount of detail.

In 1954 Bremmer [2] was successful in applying the rationale of "operational calculus" [23] to the flat earth mixed-path problem. His approach was different from Clemmow's in that his starting point was a statement of Green's integral theorem for the situation. Bremmer shows, though, how Clemmow's formula can be derived from his own work. We present an outline of Bremmer's work here.

Later, Wait [24] showed how Monteath's [16] compensation theorem could be applied to the mixed-path flat earth integral formula, indicating also how the results could be modified to treat the more general case of spherical earth mixed-path propagation. Wait went on to derive, with Householder [25], impressive numerical calculations for the spherical earth model. Furutsu ([9] & [10]) and Wait' ([26] & [27]) have paid further analytical attention to the problem of propagation over a spherical earth in mixed path cases.

Bremmer's Work

Bremmer's work initially parallels that of Hufford [13] in that his basic integral equation is derived by an application of Green's integral theorem. The fundamental quantity is the vertical component of the Hertz vector, Π. This function is known [see, e.g. Stratton [22]] to satisfy the homogeneous wave equation everywhere in a volume V consisting of the semi-infinite half-space above the [flat] earth with the exception of a small sphere surrounding the transmitter T, and a small hemisphere surrounding the receiver, or observation point, P on the earth's surface. The surface S of the volume V is shown in dotted lines in diagram 2.1, along with the basic geometry of the problem.

Bremmer assumes the transmitting source to be an elementary electric dipole, so that the Hertz potential at the source T takes the form $\Pi_{pr} = e^{-ik_0 TP}/TP$ [the "pr" subscript refers to the primary stimulation, and TP is the length of the line segment from the transmitter T to the observation point P]. Green's integral theorem for the volume V yields

$$\iint_S dS \left[\Pi(Q) \cdot \frac{\partial}{\partial n} \frac{e^{ik_0 QP}}{QP} - \frac{\partial \Pi(Q)}{\partial n} \cdot \frac{e^{ik_0 QP}}{QP} \right] = 0 \qquad (2.1)$$

wherein Q is a point on the surface S of V. Letting the surface S tend to infinity, and assuming that Π and its normal derivative vanish along with it, equation (2.1) allows us to write, formally, for the function Π at the point P:

$$\Pi(P) = \frac{2e^{ik_0 TP}}{TP} + \frac{1}{2\pi}\iint dS \left[\Pi(Q)\frac{\partial}{\partial z}\frac{e^{ik_0 QP}}{QP} - \frac{e^{ik_0 QP}}{QP}\frac{\partial\Pi}{\partial z}\right] \qquad (2.2)$$

wherein the range of integration corresponds to the surface of the earth, with the exception of the immediate vicinity of P, and Q is the integration point on the surface.

Since Q and [fixed] P are both on the earth's surface [z=0], the partial derivative with respect to z causes the first term in the integral above to vanish. To deal with the second term in the integral in (2.2), Bremmer makes the well established assumption that the ratio of ∂Π/∂z to Π at the earth's surface depends only on the soil conditions; i.e. the ratio is independent of the particular wave function Π. If we consider the primary [transmitter] field to be a superposition of plane waves, the previous assumption corresponds to the physical situation wherein all the composite plane waves of the primary source have the same direction of propagation upon reflection into the earth. This is also known as a "Leontovich boundary condition" [see [8]]. That common direction of propagation is the same as that which would arise if the primary source waves glanced the earth at grazing incidence, so that we have

$$\frac{\partial\Pi}{\partial z} = -ik\left[1 - \frac{1}{n^2}\right]^{1/2}\Pi \quad (z<0) \qquad (2.3)$$

wherein k is the wave number within the earth and n^2 is the refractive index of the earth as given by

$$n^2 = \frac{\epsilon}{\epsilon_0} + \frac{i\sigma}{\omega\epsilon_0} = \frac{k^2}{k_0^2} \qquad (2.4).$$

Additionally, the continuity of $\frac{\partial\Pi}{\partial z}$ and $k^2\Pi$ is implied by the classical boundary conditions of electromagnetism as applied to the Hertz vector of an elementary vertical dipole [see e.g. [22] or [21]]. Hence, equation (2.3) may be rewritten

$$\frac{\partial\Pi}{\partial z} \approx \left[\frac{-i}{k}\right]\left[1 - \frac{1}{n^2}\right]k_0^2\Pi \qquad (z = 0_+) \qquad (2.5).$$

Bremmer introduces the quantity

$$\Delta = \left\{\frac{ik}{n^2}\left[1 - \frac{1}{n^2}\right]\right\}^{1/2} \qquad (2.6).$$

By considering the classical Sommerfeld formula for propagation over a homogeneous flat earth, *viz.*

$$\frac{\Pi}{2\Pi_{pr}} \approx y(\rho) = 1 + \sqrt{D}\,e^{-D}\left[i\sqrt{\pi} - 2\int_0^{\sqrt{D}} ds\, e^{s^2}\right] \qquad (2.7).$$

wherein D is known as the "numerical distance" between the point source and the observation point (separated by an actual distance, say, x) and is defined by

$$D = \left[\frac{ik_0}{n^2}\left\{1 - \frac{1}{n^2}\right\}\right]x \qquad (2.8).$$

we see that our quantity Δ^2 is equivalent to the ratio of the numerical to the actual distances in classical flat earth theory. Finally then, the boundary condition (2.5) may be rewritten

$$\frac{\partial \Pi}{\partial z} \approx -\Delta\sqrt{2ik_0}\,\Pi \quad , (z=0_+) \tag{2.9}$$

By assuming that Δ is a function of the position on the earth's surface, determined only by the soil conditions, equation (2.2) becomes approximately

$$\Pi(P) = \frac{2e^{ik_0TP}}{TP} + \frac{\sqrt{ik_0/2}}{\pi}\iint dS \left\{\frac{e^{ik_0QP}}{QP}\Delta(Q)\Pi(Q)\right\} \tag{2.10}$$

wherein the integration region now consists of the entire surface of the [flat] earth including the vicinity of P.

Bremmer now assumes that both the transmitting and the receiving points, T and P, lie on the earth's surface. The function $\Pi(Q)$ is decomposed into the product of the field existing in the case of infinitely conducting soil, i.e. $2\Pi_{pr}$, and an attenuation factor W(Q) so that we may write

$$\Pi(Q) = 2\frac{e^{ik_0QT}}{QT}\cdot W(Q) \tag{2.11}$$

Making this substitution in the integral in equation (2.10) allows us to convert to elliptical coordinates with the following changes of variable

$$\frac{PQ + QT}{PT} = \cosh(u), \quad \frac{PQ - QT}{PT} = \cos(v)$$

whereupon the integral in equation (2.10) becomes

$$2\int_0^{\pi} dv \int_{-\infty}^{\infty} du\, e^{ik_0PT\cosh(u)}\,\Delta(u,v)\,W(u,v) \tag{2.12}$$

It is seen that the integrand in equation (2.12) has a saddle point at $u=0$. The elliptical coordinates used in the integral in equation (2.12) are shown in diagram 2.2.

Assuming that the non-exponential factors in (2.12) are slowly varying at the saddle point, they may be approximated by their value at the saddle point. The change of variable, say,

$$\cosh(u) = 1 + \tau^2$$

thus reduces equation (2.12), eventually, to the single integral

$$2\left[\frac{2\pi i}{k_0 PT}\right]^{1/2} e^{ik_0 PT} \int_0^\pi dv\, \Delta(0,v)\, W(0,v) \qquad (2.13),$$

which runs along the line segment joining T and P in the u-v plane [see diagram 2.2]. Bremmer then introduces the distance ξ of the integration point from T on the line segment PT, and defines the distance PT as x [again, see diagram 2.2], so that

$$\cos(v) = 1 - \frac{2\xi}{x}\ ; \quad \rightarrow \quad dv = \frac{d\xi}{\sqrt{\xi(x-\xi)}} .$$

The integral in (2.13) then becomes

$$2\left[\frac{2\pi i}{k_0 x}\right]^{1/2} e^{ik_0 x} \int_0^x d\xi \frac{\Delta(\xi)\, W(\xi)}{\sqrt{\xi(x-\xi)}} \qquad (2.14).$$

Substituting result (2.14) for the integral in equation (2.10) and dividing both sides by $2e^{ik_0 TP}/TP$ [$= 2e^{ik_0 x}/x$] we arrive at

$$W(x) = 1 + i\sqrt{\frac{x}{\pi}} \int_0^x d\xi \frac{\Delta(\xi)\, W(\xi)}{\sqrt{\xi(x-\xi)}} \qquad (2.15)$$

as an integral equation for the attenuation factor W, expressing its dependence on the distance x from the transmiter T. By defining the intermediate quantity

$$\psi(x) = \frac{W(x)}{\sqrt{x}} = \frac{1}{\sqrt{x}} \frac{\Pi(x)}{\Pi_{pr}(x)} \quad (2.16).$$

equation (2.15) may be written

$$\psi(x) = \frac{1}{\sqrt{x}} + \frac{i}{\sqrt{\pi}} \int_0^x d\xi \, \Delta(\xi) \frac{\psi(\xi)}{\sqrt{(x-\xi)}} \quad (2.17).$$

Equation (2.17) is a Volterra-type integral equation. From the integration limits we see that the implication is that propagation up to the point P [i.e. at x] is entirely independent of the Δ function [i.e. the soil conditions] beyond P.

The case is now considered where the earth consists of two [electrically] different but homogeneous regions separated by an interface at a distance d_1 from the transmitter, as shown in diagram 2.3. In view of the preceeding paragraph, the function ψ in the first medium is independent of the second medium, and therefore must equal the classical Sommerfeld homogeneous flat earth solution as given in equation (2.7). That is to say, bearing in mind equations (2.6), (2.8), & (2.16),

$$\psi(x) = \frac{y(\Delta^2 x)}{\sqrt{x}} \quad \text{for } x < d_1 \quad (2.18).$$

To obtain the corresponding function in the second medium, Bremmer employs the standard "operational-calculus" [see Van der Pol & Bremmer, 1950]. With this technique one defines the "operational transform" of a function f(x) as

$$F(p) = p \int_{-\infty}^{\infty} dx\, e^{-px} f(x) \qquad (2.19),$$

or, symbolically

$$F(p) \leftrightarrow f(x) \qquad (2.20).$$

Incidentally, from equation (2.19) we see that the operational transform of a suitably well-behaved function f(x) amounts to taking the two-sided Laplace transform of the derivative of f(x). Bremmer neglects detailed consideration of the strip of convergence in the p-plane of the operational transform defined above.

Analogously to the Laplace and Fourier transforms, a "convolution theorem" exists for the operational transform. Indeed, it is readily verified that

$$\frac{1}{p} F_1(p) F_2(p) \leftrightarrow \int_{-\infty}^{\infty} d\xi\, f_1(\xi) f_2(x-\xi) = f_1(x) * f_2(x) \qquad (2.21),$$

wherein the symbol "*" denotes [one-dimensional] convolution.

Equation (2.17) may now be broken down into its two "constituent" equations: i.e. one for each of the two different media. These two equations may be concisely "joined" by

$$\psi_1(x) + \psi_2(x) = \frac{h(x)}{\sqrt{x}} + \frac{i}{\sqrt{\pi}} \int_0^x d\xi\, [\Delta_1 \psi_1(\xi) + \Delta_2 \psi_2(\xi)] \frac{h(x-\xi)}{\sqrt{x-\xi}} \qquad (2.22).$$

wherein

$$\Psi_1(x) = \begin{cases} \psi(x) & 0 < x < d_1 \\ 0 & otherwise \end{cases} \qquad (2.22a).$$

$$\Psi_2(x) = \begin{cases} \psi(x) & d_1 < x \\ 0 & otherwise \end{cases} \qquad (2.22b).$$

and h(x) is the [Heaviside] unit step function as defined by

$$h(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}.$$

Taking the operational transform of equation (2.22) results in

$$\Psi_1 + \Psi_2 = \sqrt{\pi p} + \frac{1}{\sqrt{\pi}}\frac{1}{p} \times [\Delta_1 \Psi_1 + \Delta_2 \Psi_2] \times \sqrt{\pi p} \qquad (2.23),$$

wherein use has been made of the operational transform convolution theorem as given in equation (2.21); $\Psi_1(p)$ and $\Psi_2(p)$ denote the operational transforms, respectively, of ψ_1 and ψ_2. Equation (2.23) may be solved for Ψ_2 to give

$$\Psi_2 = \sqrt{\pi p} - \frac{(\sqrt{p} - I\Delta_1)\Psi_1}{\sqrt{p} - I\Delta_2} \qquad (2.24),$$

which, in order to facilitate the subsequent analysis, may be written as

$$\Psi_2 = \frac{\sqrt{\pi p}}{\sqrt{p} - I\Delta_2} - \Delta_1 + \frac{I(\Delta_1 - \Delta_2)}{\sqrt{\pi}}\frac{1}{p} \times \Psi_1 \times \frac{\sqrt{\pi p}}{\sqrt{p} - I\Delta_2} \qquad (2.25).$$

Now, consider the function

$$\frac{y(\Delta^2 x)}{\sqrt{x}} h(x) .$$

Applying an operational transform to this function and using equation (2.7), we find, after an integration by parts, that

$$\frac{y(\Delta^2 x)}{\sqrt{x}} h(x) \leftrightarrow \frac{\sqrt{\pi p}}{\sqrt{p} - i\Delta} \quad (Re(p) > 0) \tag{2.26}$$

Additionally, in view of equation (2.22a) and our previous discussions about the solutions in the first medium, we may write

$$\Psi_1(p) \leftrightarrow \psi_1(x) = \frac{y(\Delta_1^2 x)}{\sqrt{x}} h(x)\, h(d_1 - x) \tag{2.27}$$

The inverse operational transform of the third term on the right-hand side of equation (2.25) is readily obtained, in view of equations (2.21) and (2.26).

From the preceeding paragraph then we see that the inverse operational transform of equation (2.25) is

$$\psi_2(x) = \frac{y(\Delta_2^2 x)}{\sqrt{x}} h(x) - \frac{y(\Delta_1^2 x)}{\sqrt{x}} h(x) h(d_1 - x) \tag{2.28}$$

$$+ \frac{i(\Delta_1 - \Delta_2)}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \frac{y(\Delta_1^2 \xi)}{\sqrt{\xi}} h(\xi) h(d_1 - \xi) \frac{y[\Delta_2^2 (x-\xi)]}{\sqrt{x-\xi}} h(x-\xi)$$

It is seen from equation (2.22b) that this function $\psi_2(x)$ should vanish for $x < d_1$. The heaviside functions ensure this if $x < 0$. However, the implication of this condition in the region $0 < x < d_1$ is that

$$\frac{y(\Delta_2^2 x) - y(\Delta_1^2 x)}{\sqrt{x}} = \frac{i(\Delta_2 - \Delta_1)}{\sqrt{\pi}} \int_0^x d\xi \frac{y(\Delta_1^2 \xi)}{\sqrt{\xi}} \frac{y[\Delta_2^2 (x-\xi)]}{\sqrt{x-\xi}} \quad (2.29).$$

Bremmer observes that equation (2.29) also follows by taking the inverse operational transform of the algebraic identity

$$\frac{\sqrt{\pi p}}{\sqrt{p} - i\Delta_2} - \frac{\sqrt{\pi p}}{\sqrt{p} - i\Delta_1} = \frac{i(\Delta_2 - \Delta_1)}{\sqrt{\pi}} \frac{1}{p} \times \frac{\sqrt{\pi p}}{\sqrt{p} - i\Delta_1} \times \frac{\sqrt{\pi p}}{\sqrt{p} - i\Delta_2} .$$

If $x > d_1$, then equation (2.28) reduces to

$$\Psi_2(x) = \frac{1}{\sqrt{x}} \frac{\Pi(x)}{2\Pi_{pr}(x)} = \frac{y(\Delta_2^2 x)}{\sqrt{x}}$$

$$+ \frac{i(\Delta_1 - \Delta_2)}{\sqrt{\pi}} \int_0^{d_1} d\xi \frac{y(\Delta_1^2 \xi)}{\sqrt{\xi}} \frac{y[\Delta_2^2 (x-\xi)]}{\sqrt{x-\xi}} \quad (2.30).$$

Since x is equivalent to the distance $\rho = d_1 + d_2$ between the source and observation points [see diagram 2.3], equation (2.30) becomes

$$\frac{\Pi}{2\Pi_{pr}} = y(\Delta_2^2 \rho) + \frac{i(\Delta_1 - \Delta_2)}{\sqrt{\pi}} \sqrt{\rho} \int_0^{d_1} d\xi \frac{y(\Delta_1^2 \xi)}{\sqrt{\xi}} \frac{y[\Delta_2^2 (\rho-\xi)]}{\sqrt{\rho-\xi}} \quad (2.31).$$

Suppose now that the two media were juxtaposed. That is, the medium nearest to the transmitter extends over a distance d_2 with soil conditions dictated by Δ_2, and the medium near the observation point (parameter Δ_1) over a distance d_1. In this event instead of equation (2.31) we should arrive at

$$\frac{\Pi}{2\Pi_{pr}} = y(\Delta_1^2 \rho) + \frac{i(\Delta_2 - \Delta_1)}{\sqrt{\pi}} \sqrt{\rho} \int_0^{d_1} d\xi \frac{y(\Delta_2^2 \xi)}{\sqrt{\xi}} \frac{y[\Delta_1^2 (\rho-\xi)]}{\sqrt{\rho-\xi}} .$$

which becomes, after changing the variable of integration,

$$\frac{\Pi}{2\Pi_{pr}} = y(\Delta_1^2 \rho) + \frac{i(\Delta_2 - \Delta_1)}{\sqrt{\pi}} \sqrt{\rho} \int_{d_1}^{d} d\xi \frac{y(\Delta_1^2 \xi)}{\sqrt{\xi}} \frac{y[\Delta_2^2(\rho-\xi)]}{\sqrt{\rho-\xi}} \qquad (2.32).$$

The difference between equations (2.31) and (2.32) is zero, in view of the result displayed in equation (2.29). However, this is none other than the confirmation of the reciprocity principle, which states that the field strength should remain unchanged after the exchange of transmitter with receiver.

From equation (2.16) we see that the left-hand side of equations (2.31) and (2.32) is equivalent to the attenuation function W described previuously. Bremmer attaches a physical significance to, say, equation (2.31) in the following manner. With regard to diagram 2.3 it is seen that the field in the second medium is composed of a contribution which represents propagation from the source to the observation point over a homogeneous earth with the electrical properties of medium 2, together with a contribution of "weighted" secondary waves, with excitation point in the first medium.

In order to develop asymptotic expansions for the field in the second medium, Bremmer initially returns to the operational transform domain and, obtains an expression for Ψ_2 by substituting for Ψ_1, as obtained through equation (2.27), in equation (2.24). Omitting the analytical details, the result is

$$\Psi_2(p) = \frac{2\sqrt{p} \int_{\sqrt{d_1 p}}^{\infty} du\, e^{-u^2} - \frac{[1-y(\Delta_1^2 d_1)]}{\sqrt{d_1}} e^{-d_1 p}}{\left[1 + \frac{i\Delta_1}{\sqrt{p}}\right]\left[1 - \frac{i\Delta_2}{\sqrt{p}}\right]} \qquad (2.33).$$

Bremmer then goes on to develop asymptotic series for the fields both far beyond and near to the boundary at $x = d_1$. The leading terms of the former series are

$$\frac{\Pi}{2\Pi_{pR}} \sim \frac{-1}{2\sqrt{\Delta_1^2 \rho}\,\sqrt{\Delta_2^2 \rho}} \left[1 + \frac{i(\Delta_1 - \Delta_2)}{\Delta_1 \Delta_2} \frac{1 - y(\Delta_1^2 d_1)}{\sqrt{\pi d_1}} \right] \qquad (2.34a)$$

and of the latter are (2.34b).

$$\frac{\Pi}{2\Pi_{pr}} \sim y(\Delta_1^2 \rho) + i(\Delta_2 \Delta_1) \frac{\sqrt{\rho}}{\sqrt{d_1}} \left[\frac{2}{\sqrt{\pi}} y(\Delta_1^2 d_1) \sqrt{\rho - d_1} + i\Delta_2 y(\Delta_1^2 d_1)(\rho - d_1) \right]$$

in which the "recovery effect" is readily seen from the second term.

Clemmow's Work

Clemmow has considered the "mixed-path" propagation problem through the method of expanding the relevant electromagnetic field component in terms of a "spectrum of plane waves". This method is closely associated with classical Fourier analytical techniques.

As an initial simplification, Clemmow first considers the problem of propagation over a "two-component" surface. The physical geometry is depicted in diagram 2.4. The half-space $y > 0$ is a vacuum and the half-space $y < 0$ consists of a homogeneous medium with electrical properties as shown. At the interface $y = 0$ there lies a perfectly conducting plane $x > 0$. The z-axis is assumed to be normal to the page. As Clemmow points out, the assumption that the second surface has infinite conductivity is met, to a certain [relative] degree by sea water. Moreover, in most practical cases electromagnetic radiation penetrates negligibly into the earth, so that the "large scale" results of the model chosen should not differ significantly from a more accurate model.

Clemmow's methodology can be outlined as follows. It is easy to demostrate that an arbitrary electromagnetic plane wave whose direction of propagation lies, say, in the xy-plane, may be considered as the superposition of two "basic" plane waves travelling in the same direction [with the geometry of diagram 2.4]: one whose **H** vector is parallel to the z-axis [denoted as an "H-polarised field"] and one whose **E** vector is parallel to the z-axis [denoted as the "E-polarised field"]: the **E** and **H** vectors of these two basic plane waves, respectively, will therefore be orthogonal to the z-axis. Further, the characteristics of, say, the E-polarised field, for a given arbitrary plane wave, are readily deduced from the corresponding H-polarised field via the Maxwell equations. Therefore, the behaviour of any arbitrary plane wave may be completely determined from an understanding of its H-polarised component.

Clemmow demonstrates how the electromagnetic fields propagating from a wide variety of different sources may be written as the general superposition of a [possibly infinite] number of plane waves travelling in different directions. Thus, knowing the exact "combination" of plane waves that need to be superposed to simulate the effects of a given source, the propagation from that source, in any given geometry, may be studied by considering what happens to a general "component" plane wave and then combining the results in the correct manner to form the behaviour of the

and

$$\cos(\alpha) = \eta \cos(\gamma) \quad (2.41),$$

wherein

$$n^2 = \left[\frac{k}{k_0}\right]^2 = \frac{\epsilon}{\epsilon_0} - \frac{i\sigma}{\omega\epsilon_0} \quad (2.42)$$

is the square of the refractive index of the medium [earth]. The incident radiation is assumed to have an oscillatory component of the form $e^{+i\omega t}$.

With the perfectly conducting sheet present, in addition to the fields described above there will be a scattered electromagnetic field geenerated by the surface currents which have been excited in the sheet. As was implied above, the scattered field may be expressed as a general superposition, or "spectrum", of plane waves as given by

$$H_z^{sc} = \int_C d\beta\, P(\cos\beta)\, e^{-ik_0 r\cos(\theta-\beta)} \qquad y > 0 \quad (2.43),$$

and correspondingly

$$E_x^{sc} = -Z_0 \int_C d\beta \sin\beta\, P(\cos\beta)\, e^{-ik_0 r\cos(\theta-\beta)} \qquad y > 0 \quad (2.44),$$

wherein $Z_0 = \sqrt{\mu_0/\epsilon_0}$ and the equations (2.43) and (2.44) are linked via the Maxwell equation, say, $\nabla\times H = i\omega\epsilon_0 E$ [since $k_0^2 = \omega^2\epsilon_0\mu_0$]. The integration contour C in equations (2.43) and (2.44) is shown in diagram 2.6.

Let us consider what the choice of the integration contour C implies in terms of the superposition of plane waves in the integrand. The portion given source field.

From diagram 2.4 it is clear that we shall first consider the effect of an H-polarised field

$$H_z^{inc} = e^{ik_0 r\cos(\theta-\alpha)} \qquad 0<\alpha<\pi \qquad (2.36)$$

incident from $y>0$. The solution for a [infinite] line source will then be derived by a suitably weighted integration over the variable α. More will be said about this later. In equation (2.36) k_0 again denotes the free-space wave number.

If the perfectly conducting sheet were absent, the incident field as given in equation (2.36) would give rise to both a reflected and a transmitted wave [see diagram 2.5]. The reflected wave is given by

$$H_z^{ref} = \rho_H e^{ik_0 r\cos(\theta+\alpha)} \qquad (2.37)$$

in $y>0$, and the transmitted wave by

$$H_z^{trans} = \tau_H e^{ikr\cos(\theta-\gamma)} \qquad (2.38)$$

in $y<0$, wherein k is the wave number in the medium [earth]. In equations (2.37) and (2.38) respectively, the reflection coefficient ρ_H and the transmission coefficient τ_H are determined by employing the usual boundary conditions of classical electromagnetism: i.e. continuity of the phase and of the tangential components at the interface. The results are

$$\rho_H = \frac{\sin(\alpha) - \sin(\gamma)/n}{\sin(\alpha) + \sin(\gamma)/n} \qquad (2.39)$$

$$\tau_H = \frac{2\sin(\alpha)}{\sin(\alpha) + n\sin(\gamma)} \qquad (2.40)$$

$$H_z = \begin{cases} H_z^{inc} + H_z^{ref} + H_z^{sc} & \text{for } y>0 \\ H_z^{trans} + H_z^{sc} & \text{for } y<0 \end{cases} \qquad (2.48).$$

The remaining boundary conditions that Clemmow invokes are:

(i) H_z is continuous at $y=0$, $x<0$ since there is no other surface current apart from that on the perfectly conducting sheet.

(ii) E_x must vanish at $y=0$, $x>0$; this merely expresses the boundary condition on the tangential component of **E** at a perfect conductor.

However, it is also known that at $y=0$, the following hold; $E_x^{inc} + E_x^{ref} = E_x^{trans}$, $H_z^{inc} + H_z^{ref} = H_z^{trans}$. Therefore, in light of equation (2.48), the above boundary conditions (i) and (ii) become

$$H_z^{sc} \textit{ is continuous at } y=0,\ x<0 \qquad (2.49)$$

$$E_x^{sc} = -E_x^{trans} \textit{ at } y=0,\ x>0 \qquad (2.50).$$

From equations (2.43) and (2.46) the boundary condition expressed in (2.49) becomes, upon making the change of variable,

$$\lambda = \cos\beta \qquad (2.51),$$

$$\int_{-\infty}^{\infty} d\lambda \left[\frac{\sqrt{1-\lambda^2} + \frac{\sqrt{1-\lambda^2/n^2}}{n}}{\sqrt{1-\lambda^2}\sqrt{1-\lambda^2/n^2}} \right] P(\lambda)\, e^{-ik_0 x\lambda} = 0 \quad \text{for } x<0 \qquad (2.52).$$

Similarly, from the representation (2.44) or (2.45), together with equation on the real β-axis lying between 0 and π corresponds to normal, or what Clemmow calls "homogeneous", plane waves travelling in the direction β. The parts of the integration contour lying off the Real axis correspond to what Clemmow calls "inhomogeneous plane waves": i.e. plane waves that decay exponentially away from the plane $y=0$. These are also known as "evanescent" or "surface waves".

The term $P(\cos\beta)$, known as the "spectrum function", specifies [in terms of amplitude and phase] the "weight" to be attached to each plane wave in the spectrum. As stated previously, Clemmow has shown, by finding the appropriate spectrum function P, that the radiated fields from most common electromagnetic sources can be described with the use of integrals of the form of (2.43) and (2.44).

The equations (2.43) and (2.44) have their counterparts in the region $y<0$. Clemmow invokes the boundary condition that E_x^{sc} must be continuous across $y=0$, and accounts for this continuity by writing

$$E_x^{sc} = -Z_0 \int_C d\beta \sin\beta \; P(\cos\beta)\, e^{-ik_0 x\cos\beta}\, e^{iky\sin\beta'} \qquad y<0 \qquad (2.45),$$

and [since $\nabla \times \mathbf{H} = i\omega\epsilon\mathbf{E}$]

$$H_z^{sc} = -n \int_C d\beta \frac{\sin\beta}{\sin\beta'} P(\cos\beta)\, e^{-ik_0 x\cos\beta}\, e^{iky\sin\beta'} \qquad y<0 \qquad (2.46),$$

wherein

$$\cos\beta = n\cos\beta' \qquad (2.47).$$

The entire field E, H can obviously be specified by:

wherein $L(\lambda)$ is a function which is analytic and non-zero in the lower complex λ half-plane. Clemmow argues effectively that the path of integration in equation (2.53) should be indented above the real pole at $\lambda = \lambda_0$.

Eliminating $P(\lambda)$ between equations (2.55) and (2.56) gives

$$\sqrt{1-\lambda^2}\,\frac{\sqrt{1-\lambda^2/n^2}}{\sqrt{1-\lambda^2}+\dfrac{\sqrt{1-\lambda^2/n^2}}{n}} = \frac{-1\;2}{2\pi i n}\,\frac{L(\lambda)}{U(\lambda)L(-\lambda_0)}\,\frac{K(\lambda_0)}{\lambda+\lambda_0} \qquad (2.57)$$

wherein $K(\lambda_0)$ is the constant as given by the expression inside the large brackets ( ) in equation (2.56). Note that, for example, $1/L(\lambda)$ is also analytic and non-zero in the lower complex λ half-plane by definition.

In equation (2.57) the factorisation of the term $\sqrt{1-\lambda^2}$ into either the product or quotient of a U and a L function is trivial. However, the second term on the left-hand side presents difficulties. Formally, we may write, for example,

$$\frac{2}{n}\,\frac{\sqrt{1-\lambda^2/n^2}}{\sqrt{1-\lambda^2}+\dfrac{\sqrt{1-\lambda^2/n^2}}{n}} = \frac{1}{U_1(\lambda)L_1(\lambda)} \qquad (2.58)$$

as one form of the factorisation.

Clemmow notes that since $U_1(-\lambda)L_1(-\lambda) = U_1(\lambda)L_1(\lambda)$, from equation (2.58), and since $U_1(-\lambda)$ must be analytic and non-zero in the lower complex λ half-plane, then $U_1(-\lambda)$ must be a constant multiple of $L_1(\lambda)$; similarly, $L_1(-\lambda)$ must be a constant multiple of $U_1(\lambda)$. It is

(2.38), we get the equivalent form of the boundary condition (2.50)

$$\int_{-\infty}^{\infty} d\lambda\, P(\lambda)\, e^{-ik_0 x\lambda} = \frac{2}{n} \frac{\sqrt{1-\lambda_0^2}\sqrt{1-\lambda_0^2/n^2}}{\sqrt{1-\lambda_0^2}+\frac{\sqrt{1-\lambda_0^2/n^2}}{n}} e^{ik_0 x\lambda_0}$$

for $x > 0$ (2.53).

wherein we've written

$$\lambda_0 = \cos(\alpha) \tag{2.54}$$

Equations (2.52) and (2.53) are dual integral equations which must be solved for $P(\lambda)$. Clemmow adopts a method of solution that is similar to the Wiener-Hopf technique which we shall employ later on.

By closing the contour in the **upper** half-plane in (2.52) we see that the equation would be satisfied if

$$\left[\frac{\sqrt{1-\lambda^2}+\frac{\sqrt{1-\lambda^2/n^2}}{n}}{\sqrt{1-\lambda^2}\sqrt{1-\lambda^2/n^2}}\right] P(\lambda) = U(\lambda) \tag{2.55}$$

wherein $U(\lambda)$ is a function analytic and non-zero in the upper complex λ half-plane.

By closing the integration contour in the **lower** half-plane in (2.53) we see that a solution of the equation is offered by

$$P(\lambda) = \frac{-1}{2\pi i}\frac{2}{n}\left[\frac{\sqrt{1-\lambda_0^2}\sqrt{1-\lambda_0^2/n^2}}{\sqrt{1-\lambda_0^2}+\frac{\sqrt{1-\lambda_0^2/n^2}}{n}}\right]\frac{L(\lambda)}{L(-\lambda_0)(\lambda+\lambda_0)} \tag{2.56}$$

$$H_z^g = \begin{cases} e^{ik_0 r\cos(\theta-\alpha)} + e^{ik_0 r\cos(\theta+\alpha)} & \text{for } 0 \leq \theta \leq \pi - \alpha \\ e^{ik_0 r\cos(\theta-\alpha)} + e^{ik_0 r\cos(\theta+\alpha)} & \text{for } \pi - \alpha \leq \theta \leq \pi \end{cases} \qquad (2.63),$$

and

$$H_z^g = \frac{i\cos(\alpha/2)}{\pi L_1(\cos\alpha)} \int_{S(\theta)} d\beta \frac{\cos(\beta/2)}{L_1(\cos\beta)[\cos\beta + \cos\alpha]} e^{-ik_0 r\cos(\theta-\beta)} \qquad (2.64).$$

The integration contour in equation (2.64) has been deformed to the steepest descent path $S(\theta)$, as shown in diagram 2.7, passing through the saddle point at $\beta = \pi - \theta$.

Clemmow now considers the case of a line source situated at (r_0, θ_0). The spectrum function corresponding to a line source is found to be

$$\frac{e^{-i\pi/4}}{\sqrt{2\pi}} e^{-ik_0 r_0 \cos(\theta_0 - \alpha)} \qquad (2.65)$$

Therefore, the solution for a line source is found by multiplying the result in equation (2.62) by the spectrum function in (2.65) and then integrating over a contour C in the complex α-plane exactly as shown in diagram 2.6 for the β variable case. As can be seen from the form of (2.65), the α integration will involve a saddle point in the α-plane at the point $\alpha = \pi - \theta_0$. The integration contour C may be deformed to the steepest descent path $S(\theta_0)$ which passes through the saddle point. The steepest descent path $S(\theta_0)$ in the α-plane turns out to be the same as the path $S(\theta)$ in the β-plane [except for the precise location of the saddle point], as indicated in diagram 2.7. Again, without going into details, the result for a convenient to choose the constant of proportionality as unity. Then, after some manipulation we may solve for $P(\lambda)$ to find

$$P(\lambda) = \frac{-1}{2\pi i} \frac{\sqrt{1+\lambda_0}\sqrt{1+\lambda_0}}{L_1(\lambda_0)L_1(\lambda)} \frac{1}{\lambda+\lambda_0} \qquad (2.59),$$

or, alternately,

$$P(\cos\beta) = \frac{i}{\pi} \frac{1}{L_1(\cos\alpha)L_1(\cos\beta)} \frac{\cos(\alpha/2)\cos(\beta/2)}{\cos\alpha + \cos\beta} \qquad (2.60).$$

Substituting from equation (2.60) into equation (2.43) we arrive at an expression for the scattered field as generated by the currents in the perfectly conducting half-plane. Restricting our attention now exclusively to the half-space $y>0$ we thus find the total field to be given by [see equation (2.48)]

$$H_z = e^{ik_0 r\cos(\theta-\alpha)} + \rho_H e^{ik_0 r\cos(\theta+\alpha)} \qquad (2.61).$$
$$+ \frac{i\cos(\alpha/2)}{\pi L_1(\cos\alpha)} \int_C d\beta \frac{\cos(\beta/2)}{L_1(\cos\beta)[\cos\beta + \cos\alpha]} e^{-ik_0 r\cos(\theta-\beta)}$$

As mentioned previously, the solution for a line source may now be obtained by multiplying equation (2.61) by a [known] spectrum function and then integrating over a suitable path in the complex α-plane. To this end Clemmow shows that it is desirable to express H_z as given in equation (2.61) as the superposition of a "geometrical optics" and a "diffraction" field. Without elaborating on the details the result is

$$H_z = H_z^g + H_z^d \qquad (2.62),$$

wherein

ally obtain the function $L_1(\ )$. Rather, he is able to exploit certain properties of the function that appear when both the source and observation points lie on the earth's surface. These properties arise because the variables α and β have a certain physical significance.

Clemmow derives an asymptotic expansion, for the integrals occurring in equation (2.68), which is valid for the case $k_0 r \gg 1$, $k_0 r_0 \gg 1$, $\theta_0 \approx \pi$, and $\theta \approx 0$. We shall again not elaborate on the details, which are manifest. We shall quote the result for the special case when $\theta = 0$, $\theta_0 = \pi$, $R = S = \rho$, and $\sqrt{k_0 \rho / 2} / |n| \gg 1$. As with Bremmer, it is convenient to work in terms of the factor by which the undisturbed source field must be multiplied in order to obtain the actual field. We find that factor in this case to be approximately

$$\frac{8k}{\rho k_0 \Delta^2} + \frac{2\sqrt{2k}}{\sqrt{\pi k_0}} \frac{\sqrt{r}}{\sqrt{r_0 \rho}} \qquad (2.69)$$

wherein Δ is the quantity as defined in equation (2.6). The "recovery effect" is apparent in the second term of this last result.

Walsh's Work

Walsh [30] has independently applied an analysis identical to that employed, in this thesis to arrive at an integral equation like that in (3.185). For the solution of the integral equation Walsh uses a Neumann series operator expansion in the transform domain, which he then inverts and sums to obtain an approximation to the physical domain solution.

line source turns out to be

$$H_z = H_z^g + H_z^d \qquad (2.66)$$

wherein

$$H_z^g = \begin{cases} \sqrt{\pi/2}\, e^{-i\pi/4} [H_0^{(2)}(k_0 R) + H_0^{(2)}(k_0 S)] & \text{for } 0 \leq \theta \leq \pi - \theta_0 \\ \sqrt{\pi/2}\, e^{-i\pi/4} H_0^{(2)}(k_0 R) + \frac{e^{-i\pi/4}}{\sqrt{2\pi}} \int_{S(\pi/2)} d\alpha\, \rho_H e^{ik_0 S\cos(\psi+\alpha)} & \\ & \text{for } \pi - \theta_0 \leq \theta \leq \pi \end{cases} \qquad (2.67).$$

and

$$H_z^d = \frac{e^{i\pi/4}}{\pi\sqrt{2\pi}} \int_{S(\theta_0)} d\alpha \int_{S(\theta)} d\beta \frac{\cos(\alpha/2)\cos(\beta/2)}{L_\nu(\cos\alpha) L_1(\cos\beta)[\cos\alpha + \cos\beta]} \times e^{-ik_0[r_0\cos(\theta_0 - \alpha) + r\cos(\theta - \beta)]} \qquad (2.68).$$

In equations (2.67) the terms R, S, and ψ are geometrical quantities dependent on the location of the source, as shown in diagram 2.8. The quantities $H_0^{(2)}(\ )$ denote the well known zeroth order Hankel function of the second kind.

Much of Clemmow's analysis is spent on rewriting the integral in equation (2.68) in a tractable form. Difficulties arise due to the possible existence of poles in the integrand lying close to the steepest descent path, as well as the multivalued behaviour of the $L_1(\ )$ functions, as can be seen from equation (2.58), for example.

It should be stated explicitly at this point that Clemmow does not actu-

Now, depending on whether $|c_2(\rho - x_0)| \ll 1$ or $|c_2(\rho - x_0)| \gg 1$, we can employ either the convergent series or asymptotic expansion representation, respectively, for the complementary error function appearing in equation (2.74). This corresponds to the cases, respectively, when the observation point is very close to or very far beyond the interface. In both cases, if we assume that the interface is sufficiently far away from the source to ensure that $|c_1(x_0)| \gg 1$, we may make use of the asymptotic expansion representations of the complementary error functions inherent in the F_1 functions [see equation (2.71)].

For example, for the surface field far from the interface we obtain

$$E_z^+ \approx \frac{I\,dlk^2}{2\pi i\omega\epsilon_0}\frac{e^{-ik\rho}}{\rho}\left[\frac{-1}{2c_1\rho}\left(1 + \frac{e^{-i\pi/4}\sqrt{k}}{\sqrt{2c_2}}(\Delta_1 - \Delta_2)\right) - \sqrt{\frac{ik}{2\pi}}\frac{(\Delta_1 - \Delta_2)}{c_2}\frac{1}{2c_1x_0^{3/2}}\frac{\sqrt{\rho}}{\sqrt{\rho - x_0}}\left(1 + \frac{1}{2c_2(\rho - x_0)}\right)\right] \qquad (2.75)$$

Walsh arrives at the result

$$E_z^+ = \frac{I\,dlk^2}{2\pi i\omega\epsilon_0}\frac{e^{-ik\rho}}{\rho}\left[F_1(\rho) + \sqrt{\frac{ik}{2\pi}}(\Delta_1-\Delta_2)\sqrt{\rho}\right.$$

$$\left.\times\int_{x_0}^{\rho} d\rho' \frac{F_2(\rho-\rho')F_1(\rho')}{\sqrt{\rho-\rho'}\quad\sqrt{\rho'}}\right] \qquad (2.70),$$

for the z-component of the surface electric field to the right of the interface, wherein, for example,

$$F_1(\rho) = 1 - i\sqrt{\pi c_1\rho}\,e^{-c_1\rho}\,erfc(i\sqrt{c_1\rho}) \qquad (2.71),$$

$$c_1 = -ik\frac{\Delta_1^2}{2} \qquad (2.72),$$

$$\Delta_1^2 = \frac{(n_1^2-1)}{n_1^4} \qquad (2.73),$$

and x_0 is the distance of the interface from the dipole, which is assumed to be of length dl and carry current I. Note that this result is equivalent to Bremmer's [c.f. equation (2.32), say].

Integrating by parts and retaining only the most significant terms allows us to approximate equation (2.70) by

$$E_z^+ = \frac{I\,dlk^2}{2\pi i\omega\epsilon_0}\frac{e^{-ik\rho}}{\rho}\left[F_1(\rho)\right.$$

$$\left.+\frac{i\sqrt{\pi}}{\sqrt{c_2}}\left[\frac{F_1(x_0)}{\sqrt{x_0}}e^{-c_2(\rho-x_0)}\,erfc[i\sqrt{c_2(\rho-x_0)}] - \frac{F_1(\rho)}{\sqrt{\rho}}\right]\right] \qquad (2.74).$$

CHAPTER 3: FORMULATION OF THE PROBLEM

Physical Geometry of the Problem

We shall consider the problem of propagation over a flat earth consisting of two different but homogeneous, linear, and isotropic media as shown in diagram 3.1, separated by a vertical interface at $x=x_0$. The region above the earth is assumed to be a vacuum. The three electromagnetic scalar quantities μ, σ, ϵ are shown in the diagram for each medium, as well as the coordinate system used.

In the manner of Walsh [29] the physical geometry may be concisely described with the aid of two "step" functions as defined by

$$h_1(z) = \begin{cases} 0 & \text{if } z < 0 \\ 1 & \text{if } z > 0 \end{cases} \tag{3.1}$$

and

$$h_2(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} \tag{3.2}$$

Using these step functions we may "decompose" the electrical properties of the entire space into the "sum" of their components to the left of the interface and below the surface, to the right of the interface and below the surface, and above the earth's surface as follows:

$$\sigma = (1 - h_1)(1 - h_2)\sigma_1 + (1 - h_1)h_2\sigma_2 \tag{3.3}$$

$$\epsilon = \epsilon_0 h_1 + (1 - h_1)(1 - h_2)\epsilon_1 + (1 - h_1)h_2\epsilon_2 \tag{3.4}$$

wherein we've dropped the explicit argument labelling of h_1 and h_2 as given in equations (3.1) and (3.2), respectively.

We shall be solving the point-form Maxwell equations

$$\nabla\times\mathbf{E} = -i\omega\mathbf{B} \tag{3.5}$$

$$\nabla\times\mathbf{H} = i\omega\mathbf{D} + \mathbf{J} \tag{3.6}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{3.7}$$

and

$$\nabla\cdot\mathbf{D} = \rho \tag{3.8}$$

A comparison of equations (3.5) and (3.6) with their general forms

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{3.9}$$

and

$$\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} + \mathbf{J} \tag{3.10}$$

respectively, shows that we have assumed an $e^{i\omega t}$ time dependence for the electromagnetic field vectors. Alternatively, we may interpret equations (3.5) and (3.6) as being the Fourier-transformed versions of equations (3.9) and (3.10), respectively.

Using the assumed isotropy of the media, we have the following constitutive relations linking the electromagnetic field vectors

$$\begin{aligned}
\mathbf{B} &= \mu_0\mathbf{H} && \text{for } all\ z \\
\mathbf{D} &= \epsilon_0\mathbf{E} && \text{for } z>0 \\
\mathbf{D} &= \epsilon_1\mathbf{E} && \text{for } z<0,\ x<x_0 \\
\mathbf{D} &= \epsilon_2\mathbf{E} && \text{for } z<0,\ x>x_0 \\
\mathbf{J}_c &= 0 && \text{for } z>0 \\
\mathbf{J}_{c_1} &= \sigma_1\mathbf{E} && \text{for } z<0,\ x<x_0 \\
\mathbf{J}_{c_2} &= \sigma_2\mathbf{E} && \text{for } z<0,\ x>x_0
\end{aligned} \tag{3.11}$$

The equations in (3.11) may be summarised in terms of the step functions in equations (3.1) and (3.2) as follows:

$$D = \{\epsilon_0 h_1 + \epsilon_1(1 - h_2)(1 - h_1) + \epsilon_2(1 - h_1)h_2\}E \qquad (3.12)$$

$$J_c = \{[\sigma_1(1 - h_2) + \sigma_2 h_2](1 - h_1)\}E \qquad (3.13)$$

$$B = \mu_0 H \quad \text{for all } z \qquad (3.14).$$

Basic Electric Field Equation

Taking the curl of equation (3.5) yields

$$\nabla\times\nabla\times E = -i\omega\nabla\times B$$

$$= -i\omega\mu_0\nabla\times H \qquad (3.15).$$

Hence, substituting from equation (3.6) for $\nabla\times H$ gives

$$\nabla\times\nabla\times E = -i\omega\mu_0(J + i\omega D) \qquad (3.16).$$

The current vector J may be divided into the sum of two components

$$J = J_s + J_c$$

wherein:

(a) J_s is a "source current". We restrict J_s such that its support must lie in the upper half-space $z > 0$.

(b) J_c represents the "conduction current".

Using this decomposition we may rewrite equation (3.16)

$$\nabla\times\nabla\times E = -i\omega\mu_0(J_s + J_c + i\omega D) \qquad (3.17)$$

We shall make use of the vector identity

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

which holds in Cartesian coordinate systems. This last equation, along with equations (3.12) and (3.13), allows us to rewrite equation (3.17) as

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -i\omega\mu_0 \left[\mathbf{J}_s + \left[[\sigma_1 (1-h_2) + \sigma_2 h_2](1-h_1) \right] \mathbf{E} + i\omega \left[\epsilon_0 h_1 + \epsilon_1 (1-h_1)(1-h_2) + \epsilon_2 (1-h_1) h_2 \right] \mathbf{E} \right]$$

Therefore,

$$\nabla(\nabla \cdot \mathbf{E}) + i\omega\mu_0 \mathbf{J}_s = \nabla^2 \mathbf{E} - i\omega\mu_0 \left[(1-h_1)[\sigma_1 (1-h_2) + \sigma_2 h_2] \right] \mathbf{E} + \mu_0 \omega^2 \left[\epsilon_0 h_1 + \epsilon_1 (1-h_1)(1-h_2) + \epsilon_2 (1-h_1) h_2 \right] \mathbf{E}$$

and so

$$\nabla(\nabla \cdot \mathbf{E}) + i\omega\mu_0 \mathbf{J}_s = \nabla^2 \mathbf{E} + \omega^2 \mu_0 \epsilon_0 \left[\frac{1}{i\omega\epsilon_0} \right] \left[(1-h_1)[\sigma_1 (1-h_2) + \sigma_2 h_2] \right] \mathbf{E} + \omega^2 \mu_0 \epsilon_0 \left[h_1 + \epsilon_{r1} (1-h_1)(1-h_2) + \epsilon_{r2} (1-h_1) h_2 \right] \mathbf{E},$$

wherein

$$\epsilon_{r1} = \frac{\epsilon_1}{\epsilon_0} \quad \text{and} \quad \epsilon_{r2} = \frac{\epsilon_2}{\epsilon_0} \qquad (3.18).$$

Thusly, we may write

$$\nabla(\nabla \cdot \mathbf{E}) + i\omega\mu_0 \mathbf{J}_s = \nabla^2 \mathbf{E} + \omega^2 \mu_0 \epsilon_0 \left\{ \left[(1-h_1)(\sigma_1 (1-h_2) + \sigma_2 h_2) \right] \left[\frac{1}{i\omega\epsilon_0} \right] + \left[h_1 + \epsilon_{r1} (1-h_1)(1-h_2) + \epsilon_{r2} (1-h_1) h_2 \right] \right\} \mathbf{E}$$

$$= \nabla^2 \mathbf{E} + \omega^2 \mu_0 \epsilon_0 \left\{ (1-h_1) \left[\left[\frac{1}{i\omega\epsilon_0}\right] (\sigma_1 (1-h_2) + \sigma_2 h_2) + (\epsilon_{r1}(1-h_2) + \epsilon_{r2} h_2) \right] + h_1 \right\} \mathbf{E} \qquad (3.19).$$

From equations (3.12) and (3.13) we may equate

$$\mathbf{J}_c + i\omega \mathbf{D} = \left\{ (1-h_1)[\sigma_1(1-h_2) + \sigma_2 h_2] + i\omega[\epsilon_0 h_1 + \epsilon_1 (1-h_1)(1-h_2) + \epsilon_2 (1-h_1) h_2] \right\} \mathbf{E}.$$

Thus,

$$\mathbf{J}_c + i\omega \mathbf{D} = i\omega \left\{ (1-h_1) \left[\left[\frac{\sigma_1}{i\omega}\right](1-h_2) + \left[\frac{\sigma_2}{i\omega}\right] h_2 \right] + \left[\epsilon_0 h_1 + \epsilon_1 (1-h_1)(1-h_2) + \epsilon_2 (1-h_1) h_2 \right] \right\} \mathbf{E}.$$

So, if we define

$$\mathbf{D}_c = \mathbf{D} + \frac{\mathbf{J}_c}{i\omega} \qquad (3.20),$$

we may write

$$\mathbf{D}_c = \left\{ (1-h_1) \left[\left[\frac{\sigma_1}{i\omega}\right](1-h_2) + \left[\frac{\sigma_2}{i\omega}\right] h_2 + \epsilon_1 (1-h_2) + \epsilon_2 h_2 \right] + \epsilon_0 h_1 \right\} \mathbf{E}.$$

In this last equation we can bring together terms with the same x- and z-axis supports to obtain

$$\mathbf{D}_c' = \left\{ (1-h_1) \left[(1-h_2)(\epsilon_1 + \frac{\sigma_1}{i\omega}) + h_2 (\epsilon_2 + \frac{\sigma_2}{i\omega}) \right] + \epsilon_0 h_1 \right\} \mathbf{E}.$$

By defining the complex permittivities

$$\epsilon_{1c} = \epsilon_1 + \frac{\sigma_1}{i\omega}$$
$$\epsilon_{2c} = \epsilon_2 + \frac{\sigma_2}{i\omega} \qquad (3.21).$$

we have then

$$D_c = \left\{(1-h_1)\left[(1-h_2)\epsilon_{1c} + h_2\epsilon_{2c}\right] + \epsilon_0 h_1\right\}E \qquad (3.22).$$

Equation (3.22) may be "inverted" to give E in terms of D_c. The result is

$$E = \left\{(1-h_1)\left[\frac{(1-h_2)}{\epsilon_{1c}} + \frac{h_2}{\epsilon_{2c}}\right] + \frac{h_1}{\epsilon_0}\right\}D_c \qquad (3.23).$$

The Maxwell equation (3.6) may be rearranged as follows:

$$\nabla\times H = i\omega D + J$$
$$= i\omega D + J_c + J_s$$
$$= i\omega\left[D + \frac{J_c}{i\omega}\right] + J_s,$$

and so, from equation (3.20),

$$\nabla\times H = J_s + i\omega D_c.$$

Taking the divergence of both sides of this equation yields

$$\nabla\cdot(\nabla\times H) = \nabla\cdot J_s + i\omega\nabla\cdot D_c,$$

but, since

$$\nabla\cdot(\nabla\times H) = 0,$$

we're left with

$$\nabla \cdot \mathbf{D}_c = \frac{-1}{i\omega} \nabla \cdot \mathbf{J}_s \qquad (3.24).$$

Recall that the support of $\mathbf{J}_s$ lies in the upper half-space $z > 0$. Thus, from equation (3.24) we see that

$$\nabla \cdot \mathbf{D}_c = 0$$

in a neighbourhood of the surface $z = 0$.

We are able to rewrite equation (3.23), which gave us $\mathbf{E}$ in terms of $\mathbf{D}_c$, in two ways. On the one hand we have

$$\mathbf{E} = \left\{ h_1 \left[\frac{1}{\epsilon_0} - \left[\frac{(1-h_2)}{\epsilon_{1c}} + \frac{h_2}{\epsilon_{2c}} \right] \right] + \left[\frac{(1-h_2)}{\epsilon_{1c}} + \frac{h_2}{\epsilon_{2c}} \right] \right\} \mathbf{D}_c \qquad (3.25a).$$

but, alternatively,

$$\mathbf{E} = \left\{ (1-h_1) \left[\frac{(1-h_2)}{\epsilon_{1c}} + \frac{h_2}{\epsilon_{2c}} \right] - \frac{(1-h_1)}{\epsilon_0} + \frac{1}{\epsilon_0} \right\} \mathbf{D}_c.$$

which gives

$$\mathbf{E} = \left\{ (1-h_1) \left[\frac{(1-h_2)}{\epsilon_{1c}} + \frac{h_2}{\epsilon_{2c}} - \frac{1}{\epsilon_0} \right] + \frac{1}{\epsilon_0} \right\} \mathbf{D}_c \qquad (3.25b).$$

It is seen immediately that both equation (3.25a) and (3.25b) contain the term

$$\frac{(1-h_2)}{\epsilon_{1c}} + \frac{h_2}{\epsilon_{2c}}$$

Each of equations (3.25a) and (3.25b) may be interpreted in two different, but consistent" ways, depending on the formulation of this last expression. Specifically, we may write either

$$\frac{(1-h_2)}{\epsilon_{1c}} + \frac{h_2}{\epsilon_{2c}} = (1-h_2) \left[\frac{1}{\epsilon_{1c}} - \frac{1}{\epsilon_{2c}} \right] + \frac{1}{\epsilon_{2c}},$$

or

$$\frac{(1-h_2)}{\epsilon_{1c}} + \frac{h_2}{\epsilon_{2c}} = h_2\left[\frac{1}{\epsilon_{2c}} - \frac{1}{\epsilon_{1c}}\right] + \frac{1}{\epsilon_{1c}}.$$

Using these last two expressions, equations (3.25a) may be rewritten as either

$$E = \left\{h_1\left[\frac{1}{\epsilon_0} - (1-h_2)\left[\frac{1}{\epsilon_{1c}} - \frac{1}{\epsilon_{2c}}\right] - \frac{1}{\epsilon_{2c}}\right] + (1-h_2)\left[\frac{1}{\epsilon_{1c}} - \frac{1}{\epsilon_{2c}}\right] + \frac{1}{\epsilon_{2c}}\right\}D_c \qquad (3.26),$$

or as

$$E = \left\{h_1\left[\frac{1}{\epsilon_0} - h_2\left[\frac{1}{\epsilon_{2c}} - \frac{1}{\epsilon_{1c}}\right] - \frac{1}{\epsilon_{1c}}\right] + h_2\left[\frac{1}{\epsilon_{2c}} - \frac{1}{\epsilon_{1c}}\right] + \frac{1}{\epsilon_{1c}}\right\}D_c \qquad (3.27).$$

Similarly, we may rewrite equation (3.25b) as either

$$E = \left\{(1-h_1)\left[(1-h_2)\left[\frac{1}{\epsilon_{1c}} - \frac{1}{\epsilon_{2c}}\right] + \frac{1}{\epsilon_{2c}} - \frac{1}{\epsilon_0}\right] + \frac{1}{\epsilon_0}\right\}D_c \qquad (3.28),$$

or as

$$E = \left\{(1-h_1)\left[h_2\left[\frac{1}{\epsilon_{2c}} - \frac{1}{\epsilon_{1c}}\right] + \frac{1}{\epsilon_{1c}} - \frac{1}{\epsilon_0}\right] + \frac{1}{\epsilon_0}\right\}D_c \qquad (3.29).$$

By making the definition

$$\frac{1}{\epsilon_{1c}} - \frac{1}{\epsilon_{2c}} = \epsilon_{12c} \qquad (3.30)$$

we may rewrite equations (3.26) through (3.29) as

$$E = \left\{h_1\left[\frac{1}{\epsilon_0} - (1-h_2)\epsilon_{12c} - \frac{1}{\epsilon_{2c}}\right] + (1-h_2)\epsilon_{12c} + \frac{1}{\epsilon_{2c}}\right\}D_c \quad (3.31),$$

$$E = \left\{h_1\left[\frac{1}{\epsilon_0} + h_2\epsilon_{12c} - \frac{1}{\epsilon_{1c}}\right] - h_2\epsilon_{12c} + \frac{1}{\epsilon_{1c}}\right\}D_c \quad (3.32),$$

$$E = \left\{(1-h_1)\left[(1-h_2)\epsilon_{12c} + \frac{1}{\epsilon_{2c}} - \frac{1}{\epsilon_0}\right] + \frac{1}{\epsilon_0}\right\}D_c \quad (3.33),$$

and

$$E = \left\{(1-h_1)\left[-h_2\epsilon_{12c} + \frac{1}{\epsilon_{1c}} - \frac{1}{\epsilon_0}\right] + \frac{1}{\epsilon_0}\right\}D_c \quad (3.34).$$

By further making the definitions

$$\frac{1}{\epsilon_0} - \frac{1}{\epsilon_{1c}} = \epsilon_{01c} \quad (3.35),$$

and

$$\frac{1}{\epsilon_0} - \frac{1}{\epsilon_{2c}} = \epsilon_{02c} \quad (3.36),$$

equations (3.31) through (3.34) become

$$E = \left\{h_1\left[\epsilon_{02c} - (1-h_2)\epsilon_{12c}\right] + (1-h_2)\epsilon_{12c} + \frac{1}{\epsilon_{2c}}\right\}D_c \quad (3.37),$$

$$E = \left\{h_1\left[\epsilon_{01c} + h_2\epsilon_{12c}\right] - h_2\epsilon_{12c} + \frac{1}{\epsilon_{1c}}\right\}D_c \quad (3.38),$$

$$E = \left\{(1-h_1)\left[(1-h_2)\epsilon_{12c} - \epsilon_{02c}\right] + \frac{1}{\epsilon_0}\right\}D_c \quad (3.39),$$

and

$$E = \left\{(1-h_1)\left[-h_2\epsilon_{12c} - \epsilon_{01c}\right] + \frac{1}{\epsilon_0}\right\}D_c \quad (3.40).$$

From equations (3.37) through (3.40) we shall proceed to obtain expressions for the divergence of E. Taking the divergence of equation (3.37) yields

$$\nabla\cdot E = \epsilon_{02C}\nabla\cdot[h_1 D_C] - \epsilon_{12C}\nabla\cdot[h_1(1-h_2)D_C]$$

$$+ \epsilon_{12C}\nabla\cdot[(1-h_2)D_C] + \frac{1}{\epsilon_{2C}}\nabla\cdot D_C \qquad (3.41).$$

Care must be taken when we carry out the differentiation in these expressions. Let us consider the physical scenario again [see diagram 3.2]. The vector normal to the surface $z = 0$ in the direction of increasing z is a_z. The vector normal to the surface $x = 0$, $z < 0$, is given by a_x. Bearing diagram 3.2 in mind, we may expand equation (3.41) as follows:

$$\nabla\cdot E = \epsilon_{02C}\left\{h_1\nabla\cdot D_C + a_z\cdot D_C^{+}\delta(z)\right\}$$

$$- \epsilon_{12C}\left\{(1-h_2)a_z\cdot D_C^{+}\delta(z) - h_1 a_x\cdot D_C^{L}\delta(x) + h_1(1-h_2)\nabla\cdot D_C\right\}$$

$$+ \epsilon_{12C}\left\{-a_x\cdot D_C^{L}\delta(x) + (1-h_2)\nabla\cdot D_C\right\} + \frac{1}{\epsilon_{2C}}\nabla\cdot D_C \qquad (3.42),$$

wherein we've used

$$D_C^{+} = \lim_{z\to 0^+} D_C, \quad \text{and} \quad D_C^{L} = \lim_{x\to 0^-} D_C \qquad (3.43).$$

Recall equation (3.24), which gave us

$$\nabla\cdot D_C = \frac{-1}{i\omega}\nabla\cdot J_s ,$$

and also remember that the support of the source-current J_s lies in the half-space $z > 0$. Therefore

$$h_1 \nabla \cdot D_c = \frac{-1}{I\omega} h_1 \nabla \cdot J_s = \frac{-1}{I\omega} \nabla \cdot J_s \qquad (3.44).$$

Hence, equation (3.42) becomes

$$\nabla \cdot E = \epsilon_{02c} \left[\frac{-1}{I\omega} \nabla \cdot J_s + a_z \cdot D_c^{+} \delta(z) \right]$$

$$- \epsilon_{12c} \left[(1-h_2)\delta(z) a_z \cdot D_c^{+} - h_1 a_x \cdot D_c^{L} \delta(x) - \frac{(1-h_2)}{I\omega} \nabla \cdot J_s \right]$$

$$+ \epsilon_{12c} \left[-a_x \cdot D_c^{L} \delta(x) - \frac{(1-h_2)}{I\omega} \nabla \cdot J_s \right] - \frac{1}{I\omega\epsilon_{2c}} \nabla \cdot J_s$$

If we combine terms from the second and third brackets ( ) in this last expression, we arrive at

$$\nabla \cdot E = \epsilon_{02c} \left[\frac{-1}{I\omega} \nabla \cdot J_s + \delta(z) a_z \cdot D_c^{+} \right]$$

$$+ \epsilon_{12c} \left[-(1-h_2)\delta(z) a_z \cdot D_c^{+} - (1-h_1)\delta(x) a_x \cdot D_c^{L} \right] - \frac{1}{I\omega\epsilon_{2c}} \nabla \cdot J_s \, .$$

Recall from equation (3.36) that

$$\epsilon_{02c} = \frac{1}{\epsilon_0} - \frac{1}{\epsilon_{2c}}$$

so that our previous equation becomes

$$\nabla \cdot E = \frac{-1}{I\omega\epsilon_0} \nabla \cdot J_s + \epsilon_{02c} (a_z \cdot D_c^{+} \delta(z))$$

$$+ \epsilon_{12c} (-(1-h_1)\delta(x) a_x \cdot D_c^{L} - (1-h_2)\delta(z) a_z \cdot D_c^{+}) \qquad (3.45).$$

With the aid of diagram 3.3 we can define the various components of D_c in terms of their surface supports. In other words

$$
\begin{aligned}
\mathbf{D}_c^{+} &= (1-h_2)\mathbf{D}_c^{+} + h_2\mathbf{D}_c^{+} \\
\mathbf{D}_c^{-} &= (1-h_2)\mathbf{D}_c^{-} + h_2\mathbf{D}_c^{-}
\end{aligned}
\qquad (3.46)
$$

so that equation (3.45) may be rewritten

$$
\nabla\cdot\mathbf{E} = \frac{-1}{i\omega\epsilon_0}\nabla\cdot\mathbf{J}_s + \epsilon_{02c}\delta(z)\mathbf{a}_z\cdot[(1-h_2)\mathbf{D}_c^{+} + h_2\mathbf{D}_c^{+}]
$$

$$
+ \epsilon_{12c}\left\{-(1-h_1)\delta(x)\mathbf{a}_x\cdot\mathbf{D}_c^{L} - (1-h_2)\delta(z)\mathbf{a}_z\cdot\mathbf{D}_c^{+}\right\} .
$$

Finally, combining terms with like surface supports we obtain

$$
\nabla\cdot\mathbf{E} = \frac{-1}{i\omega\epsilon_0}\nabla\cdot\mathbf{J}_s + \epsilon_{02c}\delta(z)h_2\mathbf{a}_z\cdot\mathbf{D}_c^{+}
$$

$$
+ \epsilon_{01c}\delta(z)(1-h_2)\mathbf{a}_z\cdot\mathbf{D}_c^{+} - \epsilon_{12c}\delta(x)(1-h_1)\mathbf{a}_x\cdot\mathbf{D}_c^{L} \qquad (3.47).
$$

Equation (3.47) was derived from equation (3.37). By repeating the analysis for equations (3.38), (3.39), and (3.40) we obtain, respectively,

$$
\nabla\cdot\mathbf{E} = \frac{-1}{i\omega\epsilon_0}\nabla\cdot\mathbf{J}_s + \epsilon_{01c}\delta(z)(1-h_2)\mathbf{a}_z\cdot\mathbf{D}_c^{+}
$$

$$
+ \epsilon_{02c}\delta(z)h_2\mathbf{a}_z\cdot\mathbf{D}_c^{+} - \epsilon_{12c}\delta(x)(1-h_1)\mathbf{a}_x\cdot\mathbf{D}_c^{R} \qquad (3.48),
$$

$$
\nabla\cdot\mathbf{E} = \frac{-1}{i\omega\epsilon_0}\nabla\cdot\mathbf{J}_s + \epsilon_{02c}\delta(z)h_2\mathbf{a}_z\cdot\mathbf{D}_c^{-}
$$

$$
+ \epsilon_{01c}\delta(z)(1-h_2)\mathbf{a}_z\cdot\mathbf{D}_c^{-} - \epsilon_{12c}\delta(x)(1-h_1)\mathbf{a}_x\cdot\mathbf{D}_c^{L} \qquad (3.49),
$$

and

$$
\nabla\cdot\mathbf{E} = \frac{-1}{i\omega\epsilon_0}\nabla\cdot\mathbf{J}_s + \epsilon_{02c}\delta(z)h_2\mathbf{a}_z\cdot\mathbf{D}_c^{-}
$$

$$
+ \epsilon_{01c}\delta(z)(1-h_2)\mathbf{a}_z\cdot\mathbf{D}_c^{-} - \epsilon_{12c}\delta(x)(1-h_1)\mathbf{a}_x\cdot\mathbf{D}_c^{R} \qquad (3.50).
$$

Equations (3.25a) and (3.25b) give us two expressions for the electric field vector $\mathbf{E}$ in terms of $\mathbf{D}_c$. Taking the divergence of both sides of equation (3.25a) gives us

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\nabla\cdot[h_1\mathbf{D}_c] - \frac{1}{\epsilon_{1c}}\nabla\cdot[h_1(1-h_2)\mathbf{D}_c] - \frac{1}{\epsilon_{2c}}\nabla\cdot[h_1h_2\mathbf{D}_c]$$
$$+ \frac{1}{\epsilon_{1c}}\nabla\cdot[(1-h_2)\mathbf{D}_c] + \frac{1}{\epsilon_{2c}}\nabla\cdot[h_2\mathbf{D}_c]$$

Expanding the divergence of the brackets [ ] and using equations (3.24) and (3. [illegible]) gives us,

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\left\{\delta(z)\mathbf{a}_z\cdot\mathbf{D}_c^+ - \frac{1}{i\omega}\nabla\cdot\mathbf{J}_s\right\}$$
$$- \frac{1}{\epsilon_{1c}}\left\{(1-h_2)\delta(z)\mathbf{a}_z\cdot\mathbf{D}_c^+ - h_1\delta(x)\mathbf{a}_x\cdot\mathbf{D}_c^L - \frac{(1-h_2)}{i\omega}\nabla\cdot\mathbf{J}_s\right\}$$
$$- \frac{1}{\epsilon_{2c}}\left\{h_2\delta(z)\mathbf{a}_z\cdot\mathbf{D}_c^+ + h_1\delta(x)\mathbf{a}_x\cdot\mathbf{D}_c^R - \frac{h_2}{i\omega}\nabla\cdot\mathbf{J}_s\right\}$$
$$+ \frac{1}{\epsilon_{1c}}\left\{-\delta(x)\mathbf{a}_x\cdot\mathbf{D}_c^L - \frac{(1-h_2)}{i\omega}\nabla\cdot\mathbf{J}_s\right\}$$
$$+ \frac{1}{\epsilon_{2c}}\left\{\delta(x)\mathbf{a}_x\cdot\mathbf{D}_c^R - \frac{h_2}{i\omega}\nabla\cdot\mathbf{J}_s\right\}.$$

Therefore

$$\nabla\cdot\mathbf{E} = \frac{-1}{i\omega\epsilon_0}\nabla\cdot\mathbf{J}_s + \frac{1}{\epsilon_0}\delta(z)\mathbf{a}_z\cdot[(1-h_2)\mathbf{D}_c^+ + h_2\mathbf{D}_c^+]$$
$$- \frac{1}{\epsilon_{1c}}\left\{(1-h_2)\delta(z)\mathbf{a}_z\cdot\mathbf{D}_c^+ - h_1\delta(x)\mathbf{a}_x\cdot\mathbf{D}_c^L\right\}$$
$$- \frac{1}{\epsilon_{2c}}\left\{h_2\delta(z)\mathbf{a}_z\cdot\mathbf{D}_c^+ + h_1\delta(x)\mathbf{a}_x\cdot\mathbf{D}_c^R\right\}$$

$$+ \frac{1}{\epsilon_{1c}}\left\{-\delta(x)\mathbf{a}_x\cdot\mathbf{D}_c^L\right\} + \frac{1}{\epsilon_{2c}}\left\{\delta(x)\mathbf{a}_x\cdot\mathbf{D}_c^R\right\} .$$

Combining terms with the same coefficient of ϵ gives

$$\nabla\cdot\mathbf{E} = \frac{-1}{i\omega\epsilon_0}\nabla\cdot\mathbf{J}_s + \frac{1}{\epsilon_0}\delta(z)\mathbf{a}_z\cdot[(1-h_2)\mathbf{D}_c^+ + h_2\mathbf{D}_c^+]$$

$$- \frac{1}{\epsilon_{1c}}\left\{(1-h_2)\delta(z)\mathbf{a}_z\cdot\mathbf{D}_c^+ + (1-h_1)\delta(x)\mathbf{a}_x\cdot\mathbf{D}_c^L\right\}$$

$$- \frac{1}{\epsilon_{2c}}\left\{h_2\delta(z)\mathbf{a}_z\cdot\mathbf{D}_c^+ + (1-h_1)\delta(x)\mathbf{a}_x\cdot\mathbf{D}_c^R\right\} .$$

which may finally be written as

$$\nabla\cdot\mathbf{E} = \frac{-1}{i\omega\epsilon_0}\nabla\cdot\mathbf{J}_s + \epsilon_{01c}\delta(z)(1-h_2)\mathbf{a}_z\cdot\mathbf{D}_c^+$$

$$+ \epsilon_{02c}\delta(z)h_2\mathbf{a}_z\cdot\mathbf{D}_c^+ - \frac{1}{\epsilon_{1c}}(1-h_1)\delta(x)\mathbf{a}_x\cdot\mathbf{D}_c^L$$

$$+ \frac{1}{\epsilon_{2c}}(1-h_1)\delta(x)\mathbf{a}_x\cdot\mathbf{D}_c^R \qquad (3.51).$$

Taking the divergence of equation (3.25b) gives

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_{1c}}\nabla\cdot[(1-h_1)(1-h_2)\mathbf{D}_c] + \frac{1}{\epsilon_{2c}}\nabla\cdot[(1-h_1)h_2\mathbf{D}_c]$$

$$- \frac{1}{\epsilon_0}\nabla\cdot[(1-h_1)\mathbf{D}_c] + \frac{1}{\epsilon_0}\nabla\cdot[\mathbf{D}_c] .$$

Expanding the divergence of the brackets [ ] and using equations (3.24), and (3.44), we obtain

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_{1c}}\left[-(1-h_2)\delta(z)\mathbf{a}_z\cdot\mathbf{D}_c^- - (1-h_1)\delta(x)\mathbf{a}_x\cdot\mathbf{D}_c^L - 0\right]$$

$$+ \frac{1}{\epsilon_{2c}}\left[-h_2\delta(z)\mathbf{a}_z\cdot\mathbf{D}_c^- + (1-h_1)\delta(x)\mathbf{a}_x\cdot\mathbf{D}_c^R - 0\right]$$

$$-\frac{1}{\epsilon_0}\left\{-\delta(z)\,\mathbf{a}_z\cdot\mathbf{D}_c^- - 0\right\} - \frac{1}{i\omega\epsilon_0}\nabla\cdot\mathbf{J}_s \, .$$

Note that in deriving this last equation we have used the fact that

$$(1-h_1)\,\nabla\cdot\mathbf{J}_s = 0\, ,$$

since the support of $\mathbf{J}_s$ lies entirely in the half-space $z > 0$.

Our last expression for $\nabla\cdot\mathbf{E}$ may be further reduced, as before, to

$$\nabla\cdot\mathbf{E} = -\frac{1}{i\omega\epsilon_0}\nabla\cdot\mathbf{J}_s - \frac{1}{\epsilon_{1c}}\left\{(1-h_2)\,\delta(z)\,\mathbf{a}_z\cdot\mathbf{D}_c^- + (1-h_1)\,\delta(x)\,\mathbf{a}_x\cdot\mathbf{D}_c^L\right\}$$

$$-\frac{1}{\epsilon_{2c}}\left\{(1-h_2)\,\delta(z)\,\mathbf{a}_z\cdot\mathbf{D}_c^- - (1-h_1)\,\delta(x)\,\mathbf{a}_z\cdot\mathbf{D}_c^R\right\}$$

$$+\frac{1}{\epsilon_0}\delta(z)\,\mathbf{a}_z\cdot[h_2\mathbf{D}_c^- + (1-h_2)\,\mathbf{D}_c^-]\, .$$

which finally yields

$$\nabla\cdot\mathbf{E} = -\frac{1}{i\omega\epsilon_0}\nabla\cdot\mathbf{J}_s + \epsilon_{01c}\,\delta(z)\,(1-h_2)\,\mathbf{a}_z\cdot\mathbf{D}_c^- + \epsilon_{02c}\,\delta(z)\,h_2\,\mathbf{a}_z\cdot\mathbf{D}_c^-$$

$$-\frac{1}{\epsilon_{1c}}\delta(x)\,(1-h_1)\,\mathbf{a}_x\cdot\mathbf{D}_c^L + \frac{1}{\epsilon_{2c}}\delta(x)\,(1-h_1)\,\mathbf{a}_x\cdot\mathbf{D}_c^R \qquad (3.52).$$

If we compare equations (3.51) and (3.52) and postulate the uniqueness of $\nabla\cdot\mathbf{E}$, we see that

$$(1-h_2)\,\mathbf{a}_z\cdot\mathbf{D}_c^+ = (1-h_2)\,\mathbf{a}_z\cdot\mathbf{D}_c^-$$

$$\text{and} \quad h_2\,\mathbf{a}_z\cdot\mathbf{D}_c^+ = h_2\,\mathbf{a}_z\cdot\mathbf{D}_c^- \qquad (3.53).$$

Equation (3.53) is none other than the classical boundary condition of electromagnetism, which usually has to be supplied, *viz:* "The normal component of $\mathbf{D}_c$ is continuous across the interface $z = 0$, provided the support of J_s lies entirely above the interface [See Ideman [14]]. Note that this boundary condition falls out of the analysis naturally as a result of modelling the physical system in terms of the step functions given in equations (3.1) and (3.2).

Similarly, from a comparison of, say, equations (3.47) and (3.48), the uniqueness of $\nabla \cdot \mathbf{E}$ tells us that

$$(1-h_1)\mathbf{a}_x \cdot \mathbf{D}_c^L = (1-h_1)\mathbf{a}_x \cdot \mathbf{D}_c^R \qquad (3.54)$$

that is, the normal component of $\mathbf{D}_c$ is continuous across the interface $x=0,\ z<0$.

Let us reconsider equation (3.22):

$$\mathbf{D}_c = \left\{ (1-h_1)[(1-h_2)\epsilon_{1c} + h_2\epsilon_{2c}] + \epsilon_0 h_1 \right\} \mathbf{E} .$$

By decomposing the left-hand side in terms of its support components in the various regions, we may see that

$$\begin{aligned} (1-h_2)\mathbf{D}_c^- &= \epsilon_{1c}(1-h_2)\mathbf{E}^- \\ h_2\mathbf{D}_c^- &= \epsilon_{1c}h_2\mathbf{E}^- \\ \mathbf{D}_c^+ &= \epsilon_0\mathbf{E}^+ \\ (1-h_1)\mathbf{D}_c^L &= (1-h_1)\epsilon_{1c}\mathbf{E}^L \\ (1-h_1)\mathbf{D}_c^R &= (1-h_1)\epsilon_{2c}\mathbf{E}^R \end{aligned} \qquad (3.55)$$

Using these last results, we may rewrite equations (3.47) through (3.52) as, respectively,

$$\nabla \cdot \mathbf{E} = -\frac{1}{i\omega\epsilon_0}\nabla \cdot \mathbf{J}_s + \epsilon_0\epsilon_{02c}\delta(z)h_2\mathbf{a}_z \cdot \mathbf{E}^+ \qquad (3.56)$$

$$= 1 - \frac{1}{\left[\epsilon_0/\epsilon_{2c}\right]^{-1}}$$

$$= \frac{\left[\epsilon_0/\epsilon_{2c}\right]^{-1} - 1}{\left[\epsilon_0/\epsilon_{2c}\right]^{-1}}$$

and so

$$\epsilon_0 \epsilon_{02c} = \frac{n_{02}^2 - 1}{n_{02}^2} \qquad (3.62)$$

wherein we've used the definition

$$n_{02}^2 = \frac{\epsilon_{2c}}{\epsilon_0} = \frac{\epsilon_2}{\epsilon_0} + \frac{\sigma_2}{i\omega\epsilon_0} = \epsilon_{r_{02}} + \frac{\sigma_2}{i\omega\epsilon_0} \qquad (3.63)$$

Similarly,

$$\epsilon_0 \epsilon_{01c} = \frac{n_{01}^2 - 1}{n_{01}^2},$$

wherein (3.64)

$$n_{01}^2 = \epsilon_{r_{01}} + \frac{\sigma_1}{i\omega\epsilon_0} = \frac{\epsilon_{1c}}{\epsilon_0}.$$

Now, we may write

$$\epsilon_{1c}\epsilon_{12c} = \epsilon_{1c}\left[\frac{1}{\epsilon_{1c}} - \frac{1}{\epsilon_{2c}}\right]$$

$$= 1 - \frac{1}{(\epsilon_{2c}/\epsilon_{1c})}$$

$$= \frac{(\epsilon_{2c}/\epsilon_{1c}) - 1}{(\epsilon_{2c}/\epsilon_{1c})}$$

$$+ \epsilon_0 \epsilon_{01c} \delta(z)(1-h_2) \mathbf{a}_z \cdot \mathbf{E}^+ - \epsilon_{1c} \epsilon_{12c} \delta(x)(1-h_1) \mathbf{a}_x \cdot \mathbf{E}^L$$

$$\nabla \cdot \mathbf{E} = -\frac{1}{i\omega\epsilon_0} \nabla \cdot \mathbf{J}_s + \epsilon_0 \epsilon_{02c} \delta(z) \mathbf{a}_z \cdot \mathbf{E}^+ \tag{3.57}$$

$$+ \epsilon_0 \epsilon_{01c} \delta(z)(1-h_2) \mathbf{a}_z \cdot \mathbf{E}^+ - \epsilon_{2c} \epsilon_{12c} \delta(x)(1-h_1) \mathbf{a}_x \cdot \mathbf{E}^R$$

$$\nabla \cdot \mathbf{E} = -\frac{1}{i\omega\epsilon_0} \nabla \cdot \mathbf{J}_s + \epsilon_2 \epsilon_{02c} \delta(z) h_2 \mathbf{a}_z \cdot \mathbf{E}^- \tag{3.58}$$

$$+ \epsilon_1 \epsilon_{01c} \delta(z)(1-h_2) \mathbf{a}_z \cdot \mathbf{E}^- - \epsilon_{1c} \epsilon_{12c} \delta(x)(1-h_1) \mathbf{a}_x \cdot \mathbf{E}^L$$

$$\nabla \cdot \mathbf{E} = -\frac{1}{i\omega\epsilon_0} \nabla \cdot \mathbf{J}_s + \epsilon_2 \epsilon_{02c} \delta(z) h_2 \mathbf{a}_z \cdot \mathbf{E}^- \tag{3.59}$$

$$+ \epsilon_1 \epsilon_{01c} \delta(z)(1-h_2) \mathbf{a}_z \cdot \mathbf{E}^- - \epsilon_{2c} \epsilon_{12c} \delta(x)(1-h_1) \mathbf{a}_x \cdot \mathbf{E}^R$$

$$\nabla \cdot \mathbf{E} = -\frac{1}{i\omega\epsilon_0} \nabla \cdot \mathbf{J}_s + \epsilon_0 \epsilon_{02c} \delta(z) h_2 \mathbf{a}_z \cdot \mathbf{E}^+ + \epsilon_0 \epsilon_{01c} \delta(z)(1-h_2) \mathbf{a}_z \cdot \mathbf{E}^+$$

$$- (1-h_1)\delta(x) \mathbf{a}_x \cdot \mathbf{E}^L + (1-h_1)\delta(x) \mathbf{a}_x \cdot \mathbf{E}^R \tag{3.60}$$

and

$$\nabla \cdot \mathbf{E} = -\frac{1}{i\omega\epsilon_0} \nabla \cdot \mathbf{J}_s + \epsilon_{1c} \epsilon_{01c} \delta(z)(1-h_2) \mathbf{a}_z \cdot \mathbf{E}^- + \epsilon_{2c} \epsilon_{02c} \delta(z) h_2 \mathbf{a}_z \cdot \mathbf{E}^-$$

$$-\delta(x)(1-h_1) \mathbf{a}_x \cdot \mathbf{E}^L + \delta(x)(1-h_1) \mathbf{a}_x \cdot \mathbf{E}^R \tag{3.61}$$

Let us consider the various constants appearing in equations (3.56) through (3.61). To begin with,

$$\epsilon_0 \epsilon_{02c} = \epsilon_0 \left[\frac{1}{\epsilon_0} - \frac{1}{\epsilon_{2c}} \right] = 1 - \frac{\epsilon_0}{\epsilon_{2c}}$$

Therefore, since

$$\frac{\epsilon_{2c}}{\epsilon_{1c}} = \frac{(\epsilon_{2c}/\epsilon_0)}{(\epsilon_{1c}/\epsilon_0)} = \frac{n_{02}^2}{n_{01}^2},$$

we readily see from above that

$$\epsilon_{1c}\epsilon_{12c} = \frac{n_{02}^2 - n_{01}^2}{n_{02}^2} \quad (3.65).$$

Similarly,

$$\epsilon_{2c}\epsilon_{12c} = \frac{n_{02}^2 - n_{01}^2}{n_{01}^2} \quad (3.66).$$

Finally,

$$\epsilon_{2c}\epsilon_{02c} = \epsilon_{2c}\left[\frac{1}{\epsilon_0} - \frac{1}{\epsilon_{2c}}\right] = n_{02}^2 - 1$$

and (3.67).

$$\epsilon_{1c}\epsilon_{01c} = n_{01}^2 - 1$$

Using the constants defined in equations (3.62) through (3.67) we can take the gradient of equations (3.56) through (3.61) to obtain

$$\begin{aligned}\nabla(\nabla\cdot\mathbf{E}) = -\frac{1}{i\omega\epsilon_0}\nabla(\nabla\cdot\mathbf{J}_s) &+ \left[\frac{(n_{02}^2 - 1)}{n_{02}^2}\right]\nabla\left[\delta(z)h_2\mathbf{a}_z\cdot\mathbf{E}^+\right] \\ &+ \left[\frac{(n_{01}^2 - 1)}{n_{01}^2}\right]\nabla\left[\delta(z)(1-h_2)\mathbf{a}_z\cdot\mathbf{E}^+\right] \\ &- \left[\frac{n_{02}^2 - n_{01}^2}{n_{02}^2}\right]\nabla\left[\delta(x)(1-h_1)\mathbf{a}_x\cdot\mathbf{E}^L\right]\end{aligned} \quad (3.68).$$

$$\nabla(\nabla\cdot\mathbf{E}) = -\frac{1}{i\omega\epsilon_0}\nabla(\nabla\cdot\mathbf{J}_s) + \left[\frac{(n_{02}^2-1)}{n_{02}^2}\right]\nabla\left\{\delta(z)h_2\mathbf{a}_z\cdot\mathbf{E}^+\right\}$$
$$+ \left[\frac{(n_{01}^2-1)}{n_{01}^2}\right]\nabla\left\{\delta(z)(1-h_2)\mathbf{a}_z\cdot\mathbf{E}^+\right\} \qquad (3.69).$$
$$- \left[\frac{n_{02}^2-n_{01}^2}{n_{01}^2}\right]\nabla\left\{\delta(x)(1-h_1)\mathbf{a}_x\cdot\mathbf{E}^R\right\}$$

$$\nabla(\nabla\cdot\mathbf{E}) = -\frac{1}{i\omega\epsilon_0}\nabla(\nabla\cdot\mathbf{J}_s) + (n_{02}^2-1)\nabla\left\{\delta(z)h_2\mathbf{a}_z\cdot\mathbf{E}^-\right\}$$
$$+ (n_{01}^2-1)\nabla\left\{\delta(z)(1-h_2)\mathbf{a}_z\cdot\mathbf{E}^-\right\} \qquad (3.70).$$
$$- \left[\frac{n_{02}^2-n_{01}^2}{n_{02}^2}\right]\nabla\left\{\delta(x)(1-h_1)\mathbf{a}_x\cdot\mathbf{E}^L\right\}$$

$$\nabla(\nabla\cdot\mathbf{E}) = -\frac{1}{i\omega\epsilon_0}\nabla(\nabla\cdot\mathbf{J}_s) + (n_{02}^2-1)\nabla\left\{\delta(z)h_2\mathbf{a}_z\cdot\mathbf{E}^-\right\}$$
$$+ (n_{01}^2-1)\nabla\left\{\delta(z)(1-h_2)\mathbf{a}_z\cdot\mathbf{E}^-\right\} \qquad (3.71).$$
$$- \left[\frac{n_{02}^2-n_{01}^2}{n_{01}^2}\right]\nabla\left\{\delta(x)(1-h_1)\mathbf{a}_x\cdot\mathbf{E}^R\right\}$$

$$\nabla(\nabla\cdot\mathbf{E}) = -\frac{1}{i\omega\epsilon_0}\nabla(\nabla\cdot\mathbf{J}_s) + \left[\frac{(n_{02}^2-1)}{n_{02}^2}\right]\nabla\left\{\delta(z)h_2\mathbf{a}_z\cdot\mathbf{E}^+\right\}$$
$$+ \left[\frac{(n_{01}^2-1)}{n_{01}^2}\right]\nabla\left\{\delta(z)(1-h_2)\mathbf{a}_z\cdot\mathbf{E}^+\right\}$$

$$-\nabla\left\{\delta(x)(1-h_1)\mathbf{a}_x\cdot\mathbf{E}^L\right\} + \nabla\left\{\delta(x)(1-h_1)\mathbf{a}_x\cdot\mathbf{E}^R\right\} \qquad (3.72).$$

and

$$\nabla(\nabla\cdot\mathbf{E}) = -\frac{1}{i\omega\epsilon_0}\nabla(\nabla\cdot\mathbf{J}_s) + (n_{01}^2-1)\nabla\left\{\delta(z)(1-h_2)\mathbf{a}_z\cdot\mathbf{E}^-\right\}$$

$$+ (n_{02}^2-1)\nabla\left\{\delta(z)h_2\mathbf{a}_z\cdot\mathbf{E}^-\right\}$$

$$-\nabla\left\{\delta(x)(1-h_1)\mathbf{a}_x\cdot\mathbf{E}^L\right\} + \nabla\left\{\delta(x)(1-h_1)\mathbf{a}_x\cdot\mathbf{E}^R\right\} \qquad (3.73).$$

Equations (3.53) and (3.54), which gave us the continuity of the $\mathbf{D}_c$ field normal components across the various boundaries, may be written in terms of the **E** field from equation (3.22) as follows:

$$\epsilon_0(1-h_2)\mathbf{a}_z\cdot\mathbf{E}^+ = \epsilon_{1c}(1-h_2)\mathbf{a}_z\cdot\mathbf{E}^-,$$

$$\epsilon_0 h_2\mathbf{a}_z\cdot\mathbf{E}^+ = \epsilon_{2c}h_2\mathbf{a}_z\cdot\mathbf{E}^- .$$

and

$$\epsilon_{1c}(1-h_1)\mathbf{a}_x\cdot\mathbf{E}^L = \epsilon_{2c}(1-h_1)\mathbf{a}_x\cdot\mathbf{E}^R .$$

Hence, from equations (3.63), (3.64), and (3.65), we may write

$$(1-h_2)\mathbf{a}_z\cdot\mathbf{E}^+ = (1-h_2)E_z^+ = n_{01}^2(1-h_2)E_z^- ,$$

$$h_2\mathbf{a}_z\cdot\mathbf{E}^+ = h_2E_z^+ = n_{02}^2h_2E_z^- \qquad (3.74).$$

and

$$(1-h_1)\mathbf{a}_x\cdot\mathbf{E}^L = (1-h_1)E_x^L = \frac{n_{02}^2}{n_{01}^2}(1-h_1)E_x^R$$

Note in equation (3.74) that a subscript on the letter E denotes that component of the electric field vector **E** along that axis.

Recall equation (3.19), which was our original equation containing $\nabla \cdot \mathbf{E}$:

$$\nabla(\nabla\cdot\mathbf{E}) + i\omega\mu_0\mathbf{J}_s = \nabla^2\mathbf{E} + \omega^2\mu_0\epsilon_0\left\{(1-h_1)\left[\frac{1}{i\omega\epsilon_0}\left[\sigma_1(1-h_2) + \sigma_2h_2\right] + \left[\epsilon_{r1}(1-h_2) + \epsilon_{r2}h_2\right]\right] + h_1\right\}\mathbf{E}.$$

This may be rewritten

$$\nabla(\nabla\cdot\mathbf{E}) = \frac{1}{i\omega\epsilon_0}(\omega^2\epsilon_0\mu_0)\mathbf{J}_s + \nabla^2\mathbf{E} + \omega^2\mu_0\epsilon_0\left\{(1-h_1)\left[\frac{1}{i\omega\epsilon_0}(\sigma_1(1-h_2) + \sigma_2h_2) + (\epsilon_{r1}(1-h_2) + \epsilon_{r2}h_2)\right] + h_1\right\}\mathbf{E}.$$

and hence as

$$\nabla(\nabla\cdot\mathbf{E}) = \frac{k^2}{i\omega\epsilon_0}\mathbf{J}_s + \nabla^2\mathbf{E} + k^2\left\{(1-h_1)\left[\frac{1}{i\omega\epsilon_0}(\sigma_1(1-h_2) + \sigma_2h_2) + (\epsilon_{r1}(1-h_2) + \epsilon_{r2}h_2)\right] + h_1\right\}\mathbf{E} \qquad (3.75).$$

wherein we've used the quantity k^2 as defined by

$$k^2 = \omega^2 \mu_0 \epsilon_0 \qquad (3.76).$$

We shall now rewrite the quantity in brackets { } in equation (3.75) in a form that allows us to relate this equation with equations (3.68) through (3.72).

$$\left\{(1-h_2)\left[\frac{1}{i\omega\epsilon_0}(\sigma_1(1-h_2) + \sigma_2 h_2) + (\epsilon_{r1}(1-h_2) + \epsilon_{r2}h_2)\right] + h_1\right\}$$

$$= \left\{(1-h_1)\left[(1-h_2)\frac{[\epsilon_1 + \sigma_1/i\omega]}{\epsilon_0} + h_2\frac{[\epsilon_2 + \sigma_2/i\omega]}{\epsilon_0}\right] + h_1\right\}.$$

Hence, from equations (3.21), (3.63), and (3.64), we have

$$\left\{(1-h_1)\left[(1-h_2)\frac{[\epsilon_1 + \sigma_1/i\omega]}{\epsilon_0} + h_2\frac{[\epsilon_2 + \sigma_2/i\omega]}{\epsilon_0}\right] + h_1\right\}$$

$$= \left\{(1-h_1)\left[(1-h_2)\frac{\epsilon_{1c}}{\epsilon_0} + h_2\frac{\epsilon_{2c}}{\epsilon_0}\right] + h_1\right\}$$

$$= \left\{(1-h_1)\left[n_{01}^2(1-h_2) + n_{02}^2 h_2\right] + h_1\right\}.$$

Thus, equation (3.75) becomes

$$\nabla(\nabla\cdot\mathbf{E}) = \frac{k^2}{i\omega\epsilon_0}\mathbf{J}_s + \nabla^2\mathbf{E}$$

$$+ k^2\left\{(1-h_1)[n_{01}^2(1-h_2) + n_{02}^2 h_2] + h_1\right\}\mathbf{E} \qquad (3.77).$$

This expression for $\nabla(\nabla\cdot\mathbf{E})$ may be substituted into, say, equation (3.68) to obtain

$$\nabla^2\mathbf{E} + k^2\left\{(1-h_1)[n_{01}^2(1-h_2) + n_{02}^2 h_2] + h_1\right\}\mathbf{E}$$

$$= -\frac{1}{i\omega\epsilon_0}[\nabla(\nabla\cdot\ ) + k^2]J_s + \left[\frac{(n_{02}^2 - 1)}{n_{02}^2}\right]\nabla(\delta(z)h_2E_z^+)$$

$$+ \left[\frac{(n_{01}^2 - 1)}{n_{01}^2}\right]\nabla(\delta(z)(1-h_2)E_z^+) - \left[\frac{n_{02}^2 - n_{01}^2}{n_{02}^2}\right]\nabla(\delta(z)(1-h_1)E_x^L) .$$

That is to say

$$\nabla^2E + \gamma_0^2E = -T_{sE}(J_s) + \left[\frac{(n_{02}^2 - 1)}{n_{02}^2}\right]\nabla(\delta(z)h_2E_z^+)$$

$$+ \left[\frac{(n_{01}^2 - 1)}{n_{01}^2}\right]\nabla(\delta(z)(1-h_2)E_z^+)$$

$$- \left[\frac{n_{02}^2 - n_{01}^2}{n_{02}^2}\right]\nabla(\delta(x)(1-h_1)E_x^L) \qquad (3.78).$$

wherein we've used the quantities defined by

$$\gamma_0^2 = k^2\left\{(1-h_1)[n_{01}^2(1-h_2) + n_{02}^2h_2] + h_1\right\} \qquad (3.79).$$

and

$$T_{sE} = \frac{1}{i\omega\epsilon_0}[\nabla(\nabla\cdot\ ) + k^2] \qquad (3.80).$$

The operator defined in equation (3.80) will be known as the "Electrical Source Operator".

Equation (3.78) was arrived at by substituting for $\nabla(\nabla\cdot E)$ from equation (3.77) arbitrarily into equation (3.68). It is seen, because of the relationships given in equation (3.74), that had we substituted from equation (3.77) into any of equations (3.69) through (3.73), we should arrive at an equation identical to (3.78).

Electric Field Decomposition

We now decompose the electric field vector as

$$\mathbf{E} = h_1 \mathbf{E} + (1-h_1)[(1-h_2) + h_2]\mathbf{E} \qquad (3.81);$$

we shall proceed to derive an expression for $\nabla^2 \mathbf{E} + \gamma_0^2 \mathbf{E}$ to use in equation (3.78). From equation (3.81) we may immediately write

$$\nabla^2 \mathbf{E} = \nabla^2(h_1 \mathbf{E}) + \nabla^2((1-h_1)(1-h_2)\mathbf{E}) + \nabla^2((1-h_1)h_2 \mathbf{E}) \qquad (3.82),$$

and shall deal with each of the terms appearing on the right-hand side separately. We understand $\nabla^2 \mathbf{E}$ to mean that vector quantity obtained from the action of the scalar operator ∇^2 on the individual components of $\mathbf{E}$.

Thus

$$\nabla^2(h_1 \mathbf{E}) = [\nabla^2(h_1 E_x)]\mathbf{a}_x + [\nabla^2(h_1 E_y)]\mathbf{a}_y + [\nabla^2(h_1 E_z)]\mathbf{a}_z \qquad (3.83).$$

Considering the first term on the right-hand side of equation (3.83), we see that the coefficient of $\mathbf{a}_x$ is

$$\nabla^2(h_1 E_x) = \nabla \cdot [\nabla(h_1 E_x)] .$$

Now,

$$\nabla(h_1 E_x) = h_1 \nabla E_x + (\delta(z) E_x^+)\mathbf{a}_z \qquad (3.84)$$

wherein, analogously to equation (3.43), we have

$$E_x^+ = \lim_{z \to 0^+} E_x = E_x^+(x,y) \qquad (3.85).$$

Similarly,

$$\nabla(h_1 E_y) = h_1 \nabla E_y + (\delta(z) E_y^+) \mathbf{a}_z \qquad (3.86),$$

and

$$\nabla(h_1 E_z) = h_1 \nabla E_z + (\delta(z) E_z^+) \mathbf{a}_z \qquad (3.87).$$

Therefore

$$\nabla \cdot (\nabla(h_1 E_x)) = \nabla \cdot (h_1 \nabla E_x) + \nabla \cdot ([\delta(z) E_x^+] \mathbf{a}_z) \qquad (3.88),$$

and

$$\nabla \cdot (h_1 \nabla E_x) = h_1 \nabla^2 E_x + \delta(z) (\mathbf{a}_z \cdot (\nabla E_x)^+) \qquad (3.89),$$

wherein we've used the quantity defined by

$$(\nabla E_x)^+ = \lim_{z \to 0^+} (\nabla E_x) = (\nabla E_x)^+(x,y) \qquad (3.90).$$

Substituting from (3.89) into (3.88) gives

$$\nabla \cdot (\nabla(h_1 E_x)) = \nabla^2 (h_1 E_x)$$

$$= h_1 \nabla^2 E_x + \delta(z) (\mathbf{a}_z \cdot (\nabla E_x)^+) + \nabla \cdot ([\delta(z) E_x^+] \mathbf{a}_z) \qquad (3.91).$$

Similarly,

$$\nabla \cdot (\nabla(h_1 E_y)) = \nabla^2 (h_1 E_y)$$

$$= h_1 \nabla^2 E_y + \delta(z) (\mathbf{a}_z \cdot (\nabla E_y)^+) + \nabla \cdot ([\delta(z) E_y^+] \mathbf{a}_z) \qquad (3.92),$$

and

$$\nabla \cdot (\nabla(h_1 E_z)) = \nabla^2 (h_1 E_z)$$

$$= h_1 \nabla^2 E_z + \delta(z) (\mathbf{a}_z \cdot (\nabla E_z)^+) + \nabla \cdot ([\delta(z) E_z^+] \mathbf{a}_z) \qquad (3.93).$$

The combination of equations (3.91), (3.92), and (3.93) results in the single vector equation

$$\nabla^2(h_1 \mathbf{E}) = h_1 \nabla^2 \mathbf{E} + \delta(z) \Big\{ [\mathbf{a}_z \cdot (\nabla E_x)^+] \mathbf{a}_x + [\mathbf{a}_z \cdot (\nabla E_y)^+] \mathbf{a}_y + [\mathbf{a}_z \cdot (\nabla E_z)^+] \mathbf{a}_z \Big\} + \Big[\nabla \cdot \Big\{ [\delta(z) E_x^+] \mathbf{a}_z \Big\} \Big] \mathbf{a}_x + \Big[\nabla \cdot \Big\{ [\delta(z) E_y^+] \mathbf{a}_z \Big\} \Big] \mathbf{a}_y + \Big[\nabla \cdot \Big\{ [\delta(z) E_z^+] \mathbf{a}_z \Big\} \Big] \mathbf{a}_z ,$$

which may be written

$$\nabla^2(h_1 \mathbf{E}) = h_1 \nabla^2 \mathbf{E} + \delta(z) \left[\frac{\partial \mathbf{E}}{\partial z}\right]^+ + \Big[\nabla \cdot \Big\{ [\delta(z) E_x^+] \mathbf{a}_z \Big\} \Big] \mathbf{a}_x + \Big[\nabla \cdot \Big\{ [\delta(z) E_y^+] \mathbf{a}_z \Big\} \Big] \mathbf{a}_y + \Big[\nabla \cdot \Big\{ [\delta(z) E_z^+] \mathbf{a}_z \Big\} \Big] \mathbf{a}_z \quad (3.94),$$

wherein $\left[\frac{\partial \mathbf{E}}{\partial z}\right]^+$ denotes the normal derivative of E immediately above the surface $z = 0$. Specifically,

$$\left[\frac{\partial \mathbf{E}}{\partial z}\right]^+ = \Big[\mathbf{a}_z \cdot (\nabla E_x)^+ \Big] \mathbf{a}_x + \Big[\mathbf{a}_z \cdot (\nabla E_y)^+ \Big] \mathbf{a}_y + \Big[\mathbf{a}_z \cdot (\nabla ez)^+ \Big] \mathbf{a}_z \quad (3.95).$$

We shall now consider the second term on the right-hand side of equation (3.82).

$$\nabla^2[(1-h_1)(1-h_2)\mathbf{E}] = \Big\{ \nabla^2[(1-h_1)(1-h_2)E_x] \Big\} \mathbf{a}_x + \Big\{ \nabla^2[(1-h_1)(1-h_2)E_y] \Big\} \mathbf{a}_y + \Big\{ \nabla^2[(1-h_1)(1-h_2)E_z] \Big\} \mathbf{a}_z \quad (3.96).$$

The coefficient of $\mathbf{a}_x$ in equation (6.96) is

$$\nabla^2[(1-h_1)(1-h_2)E_x] = \nabla\cdot\left\{\nabla[(1-h_1)(1-h_2)E_x]\right\} .$$

and

$$\nabla[(1-h_1)(1-h_2)E_x] = [-\delta(z)(1-h_2)E_x^-]\mathbf{a}_z$$

$$+ [-\delta(x)(1-h_1)E_x^L]\mathbf{a}_x + (1-h_1)(1-h_2)\nabla E_x .$$

so that

$$\nabla\cdot\left\{\nabla[(1-h_1)(1-h_2)E_x]\right\} = \nabla\cdot\left\{(1-h_1)(1-h_2)\nabla E_x\right\}$$

$$+ \nabla\cdot\left\{[-\delta(z)(1-h_2)E_x^-]\mathbf{a}_z\right\}$$

$$+ \nabla\cdot\left\{[-\delta(x)(1-h_1)E_x^L]\mathbf{a}_x\right\} \qquad (3.97).$$

Now

$$\nabla\cdot\left\{(1-h_1)(1-h_2)\nabla E_x\right\} = (1-h_1)(1-h_2)\nabla^2 E_x$$

$$- \delta(z)(1-h_2)\left[\mathbf{a}_z\cdot(\nabla E_x 0^-\right]$$

$$- \delta(x)(1-h_1)\left[\mathbf{a}_x\cdot(\nabla E_x)^L\right] \qquad (3.98).$$

wherein

$$(\nabla E_x)^L = \lim_{x\to 0^-}(\nabla E_x) = (\nabla E_x)^L(y,z) \qquad (3.99).$$

Substituting from equation (3.98) into (3.97) thus yields

$$\nabla^2\left[(1-h_1)(1-h_2)E_x\right] = (1-h_1)(1-h_2)\nabla^2 E_x - \delta(z)(1-h_2)[\mathbf{a}_z\cdot(\nabla E_x)^-]$$

$$- \delta(x)(1-h_1)[\mathbf{a}_x\cdot(\nabla E_x)^L]$$

$$+ \left\{[-\delta(z)(1-h_2)E_x^-]\mathbf{a}_z\right\}$$

$$+ \nabla\cdot\left\{[-\delta(x)(1-h_1)E_x^L]\mathbf{a}_x\right\} \qquad (3.100).$$

Similarly, we have

$$\nabla^2\left[(1-h_1)(1-h_2)E_y\right] = (1-h_1)(1-h_2)\nabla^2 E_y - \delta(z)(1-h_2)[\mathbf{a}_z\cdot(\nabla E_y)^-]$$

$$- \delta(x)(1-h_1)[\mathbf{a}_x\cdot(\nabla E_y)^L]$$

$$+ \nabla\cdot\left\{[-\delta(z)(1-h_2)E_y^-]\mathbf{a}_z\right\}$$

$$+ \nabla\cdot\left\{[-\delta(x)(1-h_1)E_y^L]\mathbf{a}_x\right\} \qquad (3.101)$$

and

$$\nabla^2\left[(1-h_1)(1-h_2)E_z\right] = (1-h_1)(1-h_2)\nabla^2 E_z - \delta(z)(1-h_2)[\mathbf{a}_z\cdot(\nabla E_z)^-]$$

$$- \delta(x)(1-h_1)[\mathbf{a}_x\cdot(\nabla E_z)^L]$$

$$+ \nabla\cdot\left\{[-\delta(z)(1-h_2)E_z^-]\mathbf{a}_z\right\}$$

$$+ \nabla \cdot \left\{ [-\delta(x)(1-h_1)E_z^L]\mathbf{a}_x \right\} \qquad (3.102).$$

Equations (3.100), (3.101), and (3.102) may be combined into a single vector equation:

$$\nabla^2 \left\{ (1-h_1)(1-h_2)\mathbf{E} \right\}$$

$$= (1-h_1)(1-h_2)\nabla^2\mathbf{E}$$

$$- \delta(z)(1-h_2) \left\{ [\mathbf{a}_z \cdot (\nabla E_x)^-]\mathbf{a}_x + [\mathbf{a}_z \cdot (\nabla E_y)^-]\mathbf{a}_y + [\mathbf{a}_z \cdot (\nabla E_z)^-]\mathbf{a}_z \right\}$$

$$- \delta(x)(1-h_1) \left\{ [\mathbf{a}_x \cdot (\nabla E_x)^L]\mathbf{a}_x + [\mathbf{a}_x \cdot (\nabla E_y)^L]\mathbf{a}_y + [\mathbf{a}_x \cdot (\nabla E_z)^L]\mathbf{a}_z \right\}$$

$$- \left[\nabla \cdot \left\{ [\delta(z)(1-h_2)E_x^-]\mathbf{a}_z \right\} + \nabla \cdot \left\{ [\delta(x)(1-h_1)E_x^L]\mathbf{a}_x \right\} \right] \mathbf{a}_x$$

$$- \left[\nabla \cdot \left\{ [\delta(z)(1-h_2)E_y^-]\mathbf{a}_z \right\} + \nabla \cdot \left\{ [\delta(x)(1-h_1)E_y^L]\mathbf{a}_x \right\} \right] \mathbf{a}_y$$

$$- \left[\nabla \cdot \left\{ [\delta(z)(1-h_2)E_z^-]\mathbf{a}_z \right\} + \nabla \cdot \left\{ [\delta(x)(1-h_1)E_z^L]\mathbf{a}_x \right\} \right] \mathbf{a}_z .$$

Hence, as in equation (3.94), we may write

$$\nabla^2((1-h_1)(1-h_2)\mathbf{E}) =$$

$$(1-h_1)(1-h_2)\nabla^2\mathbf{E} - \delta(z)(1-h_2)\left[\frac{\partial \mathbf{E}}{\partial z}\right]^- - \delta(x)(1-h_1)\left[\frac{\partial \mathbf{E}}{\partial x}\right]^L$$

$$- \left[\nabla \cdot \left\{ [\delta(z)(1-h_2)E_x^-]\mathbf{a}_z + [\delta(x)(1-h_1)E_x^L]\mathbf{a}_x \right\} \right] \mathbf{a}_x$$

$$- \left[\nabla\cdot\left\{[\delta(z)(1-h_2)E_y^-]a_z + [\delta(x)(1-h_1)E_y^L]a_y\right\}\right]a_y$$

$$- \left[\nabla\cdot\left\{[\delta(x)(1-h_2)E_z^-]a_z + [\delta(x)(1-h_1)E_z^L]a_z\right\}\right]a_z \qquad (3.103).$$

We see from equation (3.82) that there is the remaining term $\nabla^2((1-h_1)h_2E)$ to evaluate. Consideration of the steps leading from equation (3.96) to (3.103) allows us [*mutatis mutandis*] to write down by analogy

$$\nabla^2((1-h_1)h_2E) = (1-h_1)h_2\nabla^2E - \delta(z)h_2\left[\frac{\partial E}{\partial z}\right]^- + \delta(x)(1-h_1)\left[\frac{\partial E}{\partial x}\right]^R$$

$$- \left[\nabla\cdot\left\{[\delta(z)h_2E_x^-]a_z - [\delta(x)(1-h_1)E_x^R]a_x\right\}\right]a_x$$

$$- \left[\nabla\cdot\left\{[\delta(z)h_2E_y^-]a_z - [\delta(x)(1-h_1)E_y^R]a_x\right\}\right]a_y$$

$$- \left[\nabla\cdot\left\{[\delta(z)h_2E_z^-]a_z - [\delta(x)(1-h_1)E_z^R]a_x\right\}\right]a_z \qquad (3.104).$$

Consider equation (3.78), which is the basic electric field equation:

$$\nabla^2E + \gamma_0^2E = \left[\nabla^2(h_1E) + \nabla^2((1-h_1)(1-h_2)E) + \nabla^2((1-h_1)h_2E)\right] + \gamma_0^2E$$

$$= -T_{sE}(J_s) + \left[\frac{(n_{02}^2-1)}{n_{02}^2}\right]\nabla\left\{\delta(z)h_2E_z^+\right\}$$

$$+ \left[\frac{(n_{01}^2-1)}{n_{01}^2}\right]\nabla\left\{\delta(z)(1-h_2)E_z^+\right\}$$

$$- \left[\frac{n_{02}^2-n_{01}^2}{n_{02}^2}\right]\nabla\left\{\delta(x)(1-h_1)E_x^L\right\} \qquad (3.105).$$

wherein, from equation (3.79),

$$\gamma_0^2 = k^2 \left\{ (1-h_1)[n_{01}^2(1-h_2) + n_{02}^2 h_2] + h_1 \right\} .$$

From equations (3.94), (3.103), and (3.104), we see that equation (3.105) is satisfied if the electric field is specified by the following equations

$$h_1 \left\{ \nabla^2 \mathbf{E} + k^2 \mathbf{E} \right\} = -T_{s\mathbf{E}}(\mathbf{J}_s) \qquad (3.106),$$

$$(1-h_1)(1-h_2) \left\{ \nabla^2 \mathbf{E} + \gamma_{01}^2 \mathbf{E} \right\} = 0 \qquad (3.107),$$

and

$$(1-h_1) h_2 \left\{ \nabla^2 \mathbf{E} + \gamma_{02}^2 \mathbf{E} \right\} = 0 \qquad (3.108),$$

wherein we've used the quantities defined from γ_0^2 above as

$$\gamma_{01}^2 = k^2 n_{01}^2 = k^2 \left[\epsilon_{r_{01}} - \frac{i\sigma_1}{\omega\epsilon_0} \right]$$

and $\qquad (3.109).$

$$\gamma_{02}^2 = k^2 n_{02}^2 = k^2 \left[\epsilon_{r_{02}} - \frac{i\sigma_2}{\omega\epsilon_0} \right]$$

We are also left with the rather formidable boundary equation

$$\delta(z) \left[\frac{\partial \mathbf{E}}{\partial z}\right]^{+} - (1-h_2)\delta(z) \left[\frac{\partial \mathbf{E}}{\partial z}\right]^{-} - h_2 \delta(z)' \left[\frac{\partial \mathbf{E}}{\partial z}\right]^{-}$$

$$- \delta(x)(1-h_1) \left[\frac{\partial \mathbf{E}}{\partial z}\right]^{L} + \delta(x)(1-h_1) \left[\frac{\partial \mathbf{E}}{\partial x}\right]^{R}$$

$$+ \left[\nabla\cdot\left\{[\delta(z)E_x^+]\mathbf{a}_z\right\}\right]\mathbf{a}_x + \left[\nabla\cdot\left\{[\delta(z)E_y^+]\mathbf{a}_z\right\}\right]\mathbf{a}_y + \left[\nabla\cdot\left\{[\delta(z)E_z^+]\mathbf{a}_z\right\}\right]\mathbf{a}_z$$

$$- \left[\nabla\cdot\left\{\left[[\delta(z)(1-h_2)E_x^-] + [\delta(z)h_2E_x^-]\right]\mathbf{a}_z + \left[[\delta(x)(1-h_1)E_x^L] - [\delta(x)(1-h_1)E_x^R]\right]\mathbf{a}_x\right\}\right]\mathbf{a}_x$$

$$- \left[\nabla\cdot\left\{\left[[\delta(z)(1-h_2)E_y^-] + [\delta(z)h_2E_y^-]\right]\mathbf{a}_z + \left[[\delta(x)(1-h_1)E_y^L] - [\delta(x)(1-h_1)E_y^R]\right]\mathbf{a}_x\right\}\right]\mathbf{a}_y$$

$$- \left[\nabla\cdot\left\{\left[[\delta(x)(1-h_2)E_z^-] + [\delta(z)h_2E_z^-]\right]\mathbf{a}_z + \left[[\delta(x)(1-h_1)E_z^L] - [\delta(x)(1-h_1)E_z^R]\right]\mathbf{a}_x\right\}\right]\mathbf{a}_z$$

$$= \left[\frac{(n_{02}^2-1)}{n_{02}^2}\right]\nabla(\delta(z)h_2E_z^+) + \left[\frac{(n_{01}^2-1)}{n_{01}^2}\right]\nabla(\delta(z)(1-h_2)E_z^+)$$

$$- \left[\frac{n_{02}^2-n_{01}^2}{n_{02}^2}\right]\nabla(\delta(x)(1-h_1)E_x^L) .$$

which may be simplified somewhat as

$$\delta(z)\left\{\left[\frac{\partial\mathbf{E}}{\partial z}\right]^+ - \left[\frac{\partial\mathbf{E}}{\partial z}\right]^-\right\} + (1-h_1)\delta(x)\left\{\left[\frac{\partial\mathbf{E}}{\partial x}\right]^R - \left[\frac{\partial\mathbf{E}}{\partial x}\right]^L\right\}$$

$$+ \left[\nabla\cdot\left\{[\delta(z)(E_x^+ - E_x^-)]\mathbf{a}_z + [\delta(x)(1-h_1)(E_x^R - E_x^L)]\mathbf{a}_x\right\}\right]\mathbf{a}_x$$

$$+ \left[\nabla\cdot\left\{[\delta(z)(E_y^+ - E_y^-)]\mathbf{a}_z + [\delta(x)(1-h_1)(E_y^R - E_y^L)]\mathbf{a}_x\right\}\right]\mathbf{a}_y$$

$$+ \left[\nabla\cdot\left\{[\delta(z)(E_z^+ - E_z^-)]\mathbf{a}_z + [\delta(x)(1-h_1)(E_z^R - E_z^L)]\mathbf{a}_x\right\}\right]\mathbf{a}_z$$

$$= \left[\frac{(n_{02}^2 - 1)}{n_{02}^2}\right]\nabla(\delta(z)h_2E_z^+) + \left[\frac{(n_{02}^2 - 1)}{n_{01}^2}\right]\nabla(\delta(z)(1-h_2)E_z^+)$$

$$- \left[\frac{n_{02}^2 - n_{01}^2}{n_{02}^2}\right]\nabla(\delta(x)(1-h_1)E_x^L) \qquad (3.110).$$

Reduction to an Integral Equation

We shall need the radiation-condition "tempered" solutions given by

$$K_{00}(x,y,z) = \frac{e^{-ikr}}{4\pi r},$$

$$K_{01}(x,y,z) = \frac{e^{-i\gamma_{01}r}}{4\pi r}, \qquad (3.111),$$

and

$$K_{02}(x,y,z) = \frac{e^{-i\gamma_{02}r}}{4\pi r}$$

wherein γ_{01}^2 and γ_{02}^2 are as given in (3.109),

$$r = (x^2 + y^2 + z^2)^{1/2},$$

and

$$k^2 = \omega^2\mu_0\epsilon_0.$$

The functions K_{00}, K_{01}, and K_{02} satisfy the following equations:

$$\nabla^2 K_{00} + k^2 K_{00} = -\delta(x)\delta(y)\delta(z) \quad (3.112),$$

$$\nabla^2 K_{01} + \gamma_{01}^2 K_{01} = -\delta(x)\delta(y)\delta(z) \quad (3.113),$$

and

$$\nabla^2 K_{02} + \gamma_{02}^2 K_{02} = -\delta(x)\delta(y)\delta(z) \quad (3.114)$$

Additionally, we make use of the following relations, provided that the convolutions exist:

$$\nabla^2 (h_1 E) * K_{00} = (h_1 E) * \nabla^2 K_{00} \quad (3.115),$$

$$\nabla^2 ((1-h_1)(1-h_2)E) * K_{01} = ((1-h_1)(1-h_2)E) * \nabla^2 K_{01} \quad (3.116),$$

and

$$\nabla^2 ((1-h_1)h_2 E) * K_{02} = ((1-h_1)h_2 E) * \nabla^2 K_{02} \quad (3.117),$$

wherein "$*$" denotes the spatial three-dimensional convolution operator, for the time being.

Consider equation (3.115). Using equation (3.94) the left-hand hand side may be written

$$\nabla^2 (h_1 E) * K_{00} = (h_1 \nabla^2 E) * K_{00} + (\delta(z) \left[\frac{\partial E}{\partial z}\right]^+) * K_{00}$$
$$+ \left\{ \left[\nabla \cdot ([\delta(z) E_x^+] \mathbf{a}_z)\right] \mathbf{a}_x + \left[\nabla \cdot ([\delta(z) E_y^+] \mathbf{a}_z)\right] \mathbf{a}_y \right.$$
$$\left. + \left[\nabla \cdot ([\delta(z) E_z^+] \mathbf{a}_z)\right] \mathbf{a}_z \right\} * K_{00} \quad (3.118).$$

The right-hand side of equation (3.115) may be rewritten, with the aid of equation (3.112), as

$$(h_1 E) * \nabla^2 K_{00} = (h_1 E) * \left\{ -\delta(x)\delta(y)\delta(z) - k^2 K_{00} \right\}$$

$$= -h_1 E - k^2 (h_1 E) * K_{00} \qquad (3.119),$$

and hence, from equations (3.118) and (3.119), equation (3.115) becomes

$$h_1 E + \left\{ h_1 [\nabla^2 E + k^2 E] \right\} * K_{00}$$

$$= -\left\{ \delta(z) \left[\frac{\partial E}{\partial z}\right]^+ \right\} * K_{00} - \left\{ \left[\nabla \cdot ([\delta(z) E_x^+] \mathbf{a}_z)\right] \mathbf{a}_x \right.$$

$$\left. + \left[\nabla \cdot ([\delta(z) E_y^+] \mathbf{a}_z)\right] \mathbf{a}_y + \left[\nabla \cdot ([\delta(z) E_z^+] \mathbf{a}_z)\right] \mathbf{a}_z \right\} * K_{00} \qquad (3.120).$$

This last equation may be simplified somewhat. Consider the expression

$$\left\{ \left[\nabla \cdot ([\delta(z) E_x^+] \mathbf{a}_z)\right] \mathbf{a}_x + \left[\nabla \cdot ([\delta(z) E_y^+] \mathbf{a}_z)\right] \mathbf{a}_y \right.$$

$$\left. + \left[\nabla \cdot ([\delta(z) E_z^+] \mathbf{a}_z)\right] \mathbf{a}_z \right\} * K_{00} \qquad (3.121).$$

We may expand each of the terms in the large bracket { } as

$$\nabla \cdot ([\delta(z) E_x^+] \mathbf{a}_z) = \frac{\partial}{\partial z}[\delta(z) E_x^+] \qquad (3.122),$$

$$\nabla \cdot ([\delta(z) E_y^+] \mathbf{a}_z) = \frac{\partial}{\partial z}[\delta(z) E_y^+] \qquad (3.123),$$

and

$$\nabla \cdot ([\delta(z) E_z^+] \mathbf{a}_z) = \frac{\partial}{\partial z}[\delta(z) E_z^+] \qquad (3.124).$$

Hence, the expression in equation (3.121) becomes

$$\left\{\frac{\partial}{\partial z}[\delta(z)E_x^+]\mathbf{a}_x + \frac{\partial}{\partial z}[\delta(z)E_y^+]\mathbf{a}_y + \frac{\partial}{\partial z}[\delta(z)E_z^+]\mathbf{a}_z\right\} * K_{00}$$

which may be written concisely as

$$\left\{\frac{\partial}{\partial z}[\delta(z)\mathbf{E}^+]\right\} * K_{00} .$$

Equation (3.120) now becomes

$$h_1\mathbf{E} + \left\{h_1^{\circ}[\nabla^2\mathbf{E} + k^2\mathbf{E}]\right\} * K_{00}$$

$$= -\left\{\delta(z)\left[\frac{\partial \mathbf{E}}{\partial z}\right]^+ + \frac{\partial}{\partial z}[\delta(z)\mathbf{E}^+]\right\} * K_{00} \qquad (3.125).$$

Substituting into equation (3.125) from equation (3.106) for the expression $h_1[\nabla^2\mathbf{E} + k^2\mathbf{E}]$, we see that equation (3.106) will be satisfied if

$$h_1\mathbf{E} = (T_{sE}(\mathbf{J}_s)) * K_{00} - \left\{\delta(z)\left[\frac{\partial \mathbf{E}}{\partial z}\right]^+ + \frac{\partial}{\partial z}[\delta(z)\mathbf{E}^+]\right\} * K_{00} \qquad (3.126).$$

Proceeding with similar analyses applied to equations (3.116) and (3.117) we find that equations (3.107) and (3.108) will be satisfied if

$$(1-h_1)(1-h_2)\mathbf{E}$$

$$= \left\{(\delta(z)(1-h_2^{\circ})\left[\frac{\partial \mathbf{E}}{\partial z}\right]^-) + (\delta(x)(1-h_1^{\circ})\left[\frac{\partial \mathbf{E}}{\partial x}\right]^L)\right.$$

$$\left. + \frac{\partial}{\partial z}[\delta(z)(1-h_2)\mathbf{E}^-] + \frac{\partial}{\partial x}[\delta(x)(1-h_1)\mathbf{E}^L]\right\} * K_{01} \qquad (3.127),$$

and

$$(1-h_1)h_2\mathbf{E}$$

$$= \left\{ (\delta(z) h_2 \left[\frac{\partial E}{\partial z}\right]^-) - (\delta(x)(1-h_1)\left[\frac{\partial E}{\partial x}\right]^R) \right.$$

$$\left. + \frac{\partial}{\partial z}[\delta(z) h_2 E^-] - \frac{\partial}{\partial x}[\delta(x)(1-h_1) E^R] \right\} * K_{02} \qquad (3.128),$$

respectively.

Furthermore, in view of the analyses in equations (3.122) through (3.124), the boundary equation (3.110) may be expressed more concisely as

$$\delta(z)\left\{\left[\frac{\partial E}{\partial z}\right]^+ - \left[\frac{\partial E}{\partial z}\right]^-\right\} + (1-h_1)\delta(x)\left\{\left[\frac{\partial E}{\partial x}\right]^R - \left[\frac{\partial E}{\partial x}\right]^L\right\}$$

$$+ \frac{\partial}{\partial z}[\delta(z)(E^+ - E^-)] + \frac{\partial}{\partial x}[\delta(x)(1-h_1)(E^R - E^L)]$$

$$= (n_{01}^2 - 1)\nabla(\delta(z)(1-h_2)E_z^-) + (n_{02}^2 - 1)\nabla(\delta(z)h_2 E_z^-)$$

$$- \left[\frac{n_{02}^2 - n_{01}^2}{n_{01}^2}\right]\nabla(\delta(x)(1-h_1)E_x^R) \qquad (3.129),$$

or

$$= \left[\frac{(n_{01}^2 - 1)}{n_{01}^2}\right]\nabla(\delta(z)(1-h_2)E_z^+) \qquad (3.130).$$

$$+ \left[\frac{(n_{02}^2 - 1)}{n_{02}^2}\right]\nabla(\delta(z)h_2 E_z^+) - \left[\frac{n_{02}^2 - n_{01}^2}{n_{02}^2}\right]\nabla(\delta(x)(1-h_1)E_x^L)$$

In going from equation (3.130) to (3.129), use has been made of the relationships between E^+ and E^-, E^L and E^R, as given in equations (3.74).

Consider equation (3.126) again:

$$h_1 E = (T_{sE}(J_s)) * K_{00} - (\delta(z)\left(\frac{\partial E}{\partial z}\right)^+ + \frac{\partial}{\partial z}(\delta(z) E^+)) * K_{00} .$$

Using a well known property of the convolution operator, we can "shift" the differentiation in the last term to obtain

$$h_1 E = E_s - (\delta(z)\left[\frac{\partial E}{\partial z}\right]^+) * K_{00} - (\delta(z) E^+) * \frac{\partial}{\partial z} K_{00} \qquad (3.131).$$

wherein we've used the term E_s as defined by

$$E_s = (T_{sE}(J_s)) * K_{00} \qquad (3.132).$$

and known as the "source field".

Similarly, equations (3.127) and (3.128) give us

$$(1-h_1)(1-h_1)E = (\delta(z)(1-h_2)\left[\frac{\partial E}{\partial z}\right]^-) * K_{01} + (\delta(x)(1-h_1)\left[\frac{\partial E}{\partial x}\right]^L) * K_{01}$$

$$+ (\delta(z)(1-h_2)E^-) * \frac{\partial}{\partial z} K_{01}$$

$$+ (\delta(x)(1-h_1)E^L) * \frac{\partial}{\partial x} K_{01} \qquad (3.133).$$

and

$$(1-h_1)h_2 E = (\delta(z)h_2\left[\frac{\partial E}{\partial z}\right]^-) * K_{02} - (\delta(x)(1-h_1)\left[\frac{\partial E}{\partial x}\right]^R) * K_{02}$$

$$+ (\delta(z)h_2 E^-) * \frac{\partial}{\partial z} K_{02}$$

$$- (\delta(x)(1-h_1)E^R) * \frac{\partial}{\partial x} K_{02} \qquad (3.134).$$

We shall now introduce a considerable simplification, by making the (reasonable) assumption that the surface fields below the surface (i.e.

those surface fields existing on the vertical interface) are negligible. This assumption, which was made by both Clemmow and Bremmer, is physically acceptable on the grounds that at the frequencies of interest the penetration depth into the earth is quite small for "normal" media such as damp earth or seawater. As a result, our equations (3.126) throught (3.128) become

$$h_1 E = E_s - \left\{ (\delta(z) \left[\frac{\partial E}{\partial z}\right]^+) + \frac{\partial}{\partial z}[\delta(z) E^+] \right\} * K_{00} \qquad (3.135),$$

$$(1-h_1)(1-h_2)E = \left\{ (\delta(z)(1-h_2) \left[\frac{\partial E}{\partial z}\right]^-) + \frac{\partial}{\partial z}[\delta(z)(1-h_2)E^-] \right\} * K_{01} \qquad (3.136),$$

$$(1-h_1)h_2 E = \left\{ (\delta(z) h_2 \left[\frac{\partial E}{\partial z}\right]^-) + \frac{\partial}{\partial z}[\delta(z) h_2 E^-] \right\} * K_{02} \qquad (3.137),$$

and the boundary equations (3.129) and (3.130) simplify to

$$\delta(z) \left\{ \left[\frac{\partial E}{\partial z}\right]^+ - \left[\frac{\partial E}{\partial z}\right]^- \right\} + \frac{\partial}{\partial z}[\delta(z)(E^+ - E^-)]$$

$$= (n_{01}^2 - 1)\nabla(\delta(z)(1-h_2)E_z^-) + (n_{02}^2 - 1)\nabla(\delta(z) h_2 E_z^-) \qquad (3.138),$$

or

$$= \left[\frac{(n_{01}^2 - 1)}{n_{01}^2}\right] \nabla(\delta(z)(1-h_2)E_z^+) + \left[\frac{(n_{02}^2 - 1)}{n_{02}^2}\right] \nabla(\delta(z) h_2 E_z^+) \qquad (3.139).$$

Equation (3.139) may be rewritten

$$(\delta(z) \left[\frac{\partial E}{\partial z}\right]^-) + \frac{\partial}{\partial z}[\delta(z) E^-]$$

$$= (\delta(z)\left[\frac{\partial \mathbf{E}}{\partial z}\right]^{+}) + \frac{\partial}{\partial z}[\delta(z)\mathbf{E}^{+}] - \left[\frac{(n_{01}^{2}-1)}{n_{01}^{2}}\right]\nabla(\delta(z)(1-h_{2})E_{z}^{+})$$

$$- \left[\frac{(n_{02}^{2}-1)}{n_{02}^{2}}\right]\nabla(\delta(z)h_{2}E_{z}^{+}) \qquad (3.140).$$

Furthermore, this equation may be broken down into its "support equations": one with support $(1-h_2)$ and the other with support h_2. The results are

$$(\delta(z)(1-h_{2})\left[\frac{\partial \mathbf{E}}{\partial z}\right]^{-}) + \frac{\partial}{\partial z}[\delta(z)(1-h_{2})\mathbf{E}^{-}]$$

$$= (\delta(z)(1-h_{2})\left[\frac{\partial \mathbf{E}}{\partial z}\right]^{+}) + \frac{\partial}{\partial z}[\delta(z)(1-h_{2})\mathbf{E}^{+}]$$

$$- \left[\frac{(n_{01}^{2}-1)}{n_{01}^{2}}\right]\nabla(\delta(z)(1-h_{2})E_{z}^{+}) \qquad (3.141),$$

and

$$(\delta(z)h_{2}\left[\frac{\partial \mathbf{E}}{\partial z}\right]^{-}) + \frac{\partial}{\partial z}[\delta(z)h_{2}\mathbf{E}^{-}]$$

$$= (\delta(z)h_{2}\left[\frac{\partial \mathbf{E}}{\partial z}\right]^{+}) + \frac{\partial}{\partial z}[\delta(z)h_{2}\mathbf{E}^{+}] - \left[\frac{(n_{02}^{2}-1)}{n_{02}^{2}}\right]\nabla(\delta(z)h_{2}E_{z}^{+}) \qquad (3.142).$$

We see that the left-hand sides of equations (3.141) and (3.142) appear, respectively, in equations (3.136) and (3.137). Hence, substituting for the latter from the former gives

$$(1-h_{1})(1-h_{1})\mathbf{E} = \left\{(\delta(z)(1-h_{2})\left[\frac{\partial \mathbf{E}}{\partial z}\right]^{+}) + \frac{\partial}{\partial z}[\delta(z)(1-h_{2})\mathbf{E}^{+}]\right.$$

$$- \left[\frac{(n_{01}^2 - 1)}{n_{01}^2}\right] \nabla(\delta(z)(1-h_2)E_z^+)\Bigg\} * K_{01} \qquad (3.143).$$

and

$$(1-h_1)h_2 E = \Bigg\{(\delta(z)h_2\left[\frac{\partial E}{\partial z}\right]^+) + \frac{\partial}{\partial z}[\delta(z)h_2 E^+]$$

$$- \left[\frac{(n_{02}^2 - 1)}{n_{02}^2}\right] \nabla(\delta(z)h_2 E_z^+)\Bigg\} * K_{02} \qquad (3.144).$$

We shall now restrict our attention to the z-component of the electric field. In the case of a vertical dipole source, it is this component which is least attenuated in terrestial propagation, and hence is of most importance. Considering the z-component then, equations (3.135), (3.143), and (3.144) become

$$h_1 E_z = E_{s_z} - \Bigg\{(\delta(z)\left(\frac{\partial E_z}{\partial z}\right)^+ + \frac{\partial}{\partial z}[\delta(z)E_z^+]\Bigg\} * K_{00} \qquad (3.145),$$

$$(1-h_1)(1-h_2)E_z = \Bigg\{(\delta(z)(1-h_2)\left[\frac{\partial E_z}{\partial z}\right]^+) + \frac{\partial}{\partial z}[\delta(z)(1-h_2)E_z^+]$$

$$- \left[\frac{(n_{01}^2 - 1)}{n_{01}^2}\right] \frac{\partial}{\partial z}(\delta(z)(1-h_2)E_z^+)\Bigg\} * K_{01} \qquad (3.146),$$

and

$$(1-h_1)h_2 E_z = \Bigg\{(\delta(z)h_2\left[\frac{\partial E_z}{\partial z}\right]^+) + \frac{\partial}{\partial z}[\delta(z)h_2 E_z^+]$$

$$- \left[\frac{(n_{02}^2 - 1)}{n_{02}^2}\right]\frac{\partial}{\partial z}(\delta(z)\,(1-h_2)\,E_z^+)\Bigg] * K_{02} \qquad (3.147).$$

respectively.

Tidying up equations (3.146) and (3.147) gives

$$(1-h_1)(1-h_2)E_z = \Bigg[(\delta(z)\,(1-h_2)\left[\frac{\partial E_z}{\partial z}\right]^+)$$

$$+ \frac{1}{n_{01}^2}\frac{\partial}{\partial z}[\delta(z)(1-h_2)E_z^+]\Bigg] * K_{01} \qquad (3.148).$$

and

$$(1-h_1)h_2E_z = \Bigg[(\delta(z)h_2\left[\frac{\partial E_z}{\partial z}\right]^+) + \frac{1}{n_{02}^2}\frac{\partial}{\partial z}[\delta(z)h_2E_z^+]\Bigg] * K_{02} \qquad (3.149).$$

respectively.

We shall now "shift" the z-differentiation in equations (3.148) and (3.149) to write them as

$$(1-h_1)(1-h_2)E_z = (\delta(z)\,(1-h_2)\left[\frac{\partial E_z}{\partial z}\right]^+) * K_{01}$$

$$+ \frac{1}{n_{01}^2}(\delta(z)\,(1-h_2)\,E_z^+) * \frac{\partial}{\partial z}K_{01} \qquad (3.150),$$

and

$$(1-h_1)h_2E_z = (\delta(z)h_2\left[\frac{\partial E_z}{\partial z}\right]^+)*K_{02} + \frac{1}{n_{02}^2}(\delta(z)h_2E_z^+)*\frac{\partial}{\partial z}K_{02} \qquad (3.151)$$

respectively.

Performing the convolution with respect to z results in

$$(1-h_1)(1-h_2)E_z = ((1-h_2)\left[\frac{\partial E_z}{\partial z}\right]^+)*K_{01}$$

$$+ \frac{1}{n_{01}^2}((1-h_2)E_z^+)*\frac{\partial}{\partial z}K_{01} \qquad (3.152)$$

and

$$(1-h_1)h_2E_z = (h_2\left[\frac{\partial E_z}{\partial z}\right]^+)*K_{02} + \frac{1}{n_{02}^2}(h_2E_z^+)*\frac{\partial}{\partial z}K_{02} \qquad (3.153)$$

Of course, in equations (3.152) and (3.153) we have not written the explicit functional dependence of the terms like K_{02} and K_{01} which should now of course be functions of z since they have been convolved with $\delta(z)$. For example

$$K_{02} = K_{02}(x,y,z) \quad \text{and} \quad \frac{\partial}{\partial z}K_{02} = \frac{\partial}{\partial z}K_{02}(x,y,z) \qquad (3.154)$$

Moreover, the "*" symbol now denotes two dimensional (x,y) convolution.

The ("forward") two-dimensional Fourier transform of a function f(x,y) may be defined as

$$\overline{f(\xi,\eta)} = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx\, e^{-i\xi x - i\eta y} f(x,y) \qquad (3.155)$$

with its corresponding inverse transform

$$f(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\xi\, e^{i\xi x + i\eta y}\, \overline{\overline{f(\xi,\eta)}} \tag{3.156}$$

Taking the two-dimensional Fourier transform of equations (3.152) and (3.153) gives

$$(1-h_1)\overline{\overline{[(1-h_2)E_z]}} = \overline{\overline{[(1-h_2)\left[\frac{\partial E_z}{\partial z}\right]^+]}}\,\overline{\overline{K_{01}}} + \frac{1}{n_{01}^2}\overline{\overline{[(1-h_2)E_z^+]}}\frac{\partial}{\partial z}\overline{\overline{K_{01}}} \tag{3.157}$$

and

$$(1-h_1)\overline{\overline{[h_2E_z]}} = \overline{\overline{[h_2\left[\frac{\partial E_z}{\partial z}\right]^+]}}\,\overline{\overline{K_{02}}} + \frac{1}{n_{02}^2}\overline{\overline{[h_2E_z^+]}}\frac{\partial}{\partial z}\overline{\overline{K_{02}}} \tag{3.158}$$

respectively, wherein the explicit functional dependence of the transformed quantities on the transform variables (ξ,η) has not been written.

Now [see Appendix A]

$$\overline{\overline{K_{01}}}(\xi,\eta,z) = \frac{e^{-|z|u_1}}{2u_1} \tag{3.159}$$

and

$$\overline{\overline{K_{02}}}(\xi,\eta,z) = \frac{e^{-|z|u_2}}{2u_2} \tag{3.160}$$

wherein we've used the quantities u_1 and u_2 as defined by

$$u_1 = \sqrt{\xi^2 + \eta^2 - k_0^2 n_{01}^2} \tag{3.161}$$

and

$$u_2 = \sqrt{\xi^2 + \eta^2 - k_{02}^2 n_{02}^2} \qquad (3.162)$$

In equations (3.161) and (3.162) we choose that branch of the multivalued square-root function that has a positive real component. This is equivalent to specifying

$$\sqrt{\xi^2 + \eta^2 - k_{01}^2 n_{01}^2}\,\Big|_{\xi,\eta = 0} = ik_{01}n_{01} \qquad (3.163);$$

and similarly for equation (3.162).

Using equations (3.159) and (3.160), the equations (3.157) and (3.158) become

$$(1-h_1)\overline{\overline{[(1+h_2)E_z]}} = [(1-h_2)\overline{\overline{\left[\frac{\partial E_z}{\partial z}\right]^+}}]\frac{e^{-|z|u_1}}{2u_1}$$

$$+\frac{1}{n_{01}^2}\overline{\overline{[(1-h_2)E_z^+]}}\frac{\partial}{\partial z}\frac{e^{-|z|u_1}}{2u_1} \qquad (3.164)$$

and

$$(1-h_1)\overline{\overline{[h_2E_z]}} = [h_2\overline{\overline{\left[\frac{\partial E_z}{\partial z}\right]^+}}]\frac{e^{-|z|u_2}}{2u_2} + \frac{1}{n_{02}^2}\overline{\overline{[h_2E_z^+]}}\frac{\partial}{\partial z}\frac{e^{-|z|u_2}}{2u_2} \qquad (3.165),$$

respectively.

We shall now consider equations (3.164) and (3.165) on a fixed horizontal plane $z = z_0 > 0$. This being the case, our equations become

$$\frac{\partial}{\partial z}\frac{e^{-|z|u_1}}{2u_1} = \frac{\partial}{\partial z}\frac{e^{-zu_1}}{2u_1} = -u_1\frac{e^{-zu_1}}{2u_1} = \frac{-e^{-zu_1}}{2} \qquad (3.166),$$

and, similarly

$$\frac{\partial}{\partial z}\frac{e^{-|z|u_2}}{2u_2} = \frac{-e^{-zu_2}}{2} \qquad (3.167).$$

Therefore, in the case of a fixed horizontal plane $z = z_0 > 0$, equations (3.164), and (3.165) become

$$0 = \overline{\overline{[(1-h_2)\left[\frac{\partial E_z}{\partial z}\right]^+]}}\left[\frac{1}{2u_1}\right] - \frac{1}{2n_{01}^2}\overline{\overline{[(1-h_2)E_z^+]}}$$

and

$$0 = \overline{\overline{[h_2\left[\frac{\partial E_z}{\partial z}\right]^+]}}\left[\frac{1}{2u_2}\right] - \frac{1}{2n_{01}^2}\overline{\overline{[h_2E_z^+]}}$$

respectively. These last two equations may be rewritten as

$$\overline{\overline{[(1-h_2)\left[\frac{\partial E_z}{\partial z}\right]^+]}} = \frac{u_1}{n_{01}^2}\overline{\overline{[(1-h_2)E_z^+]}} \qquad (3.168),$$

and

$$\overline{\overline{[h_2\left[\frac{\partial E_z}{\partial z}\right]^+]}} = \frac{u_2}{n_{02}^2}\overline{\overline{[h_2E_z^+]}} \qquad (3.169).$$

Note that both equations (3.168) and (3.169), which were derived from equations (3.164) and (3.165) by considering a fixed plane $z = z_0 > 0$, are independent of z.

We shall rewrite our remaining field equation (3.145), for the field above the surface, as

$$h_1 E_z = E_{a_z} - (\theta(z) \left[\frac{\partial E_z}{\partial z}\right]^+) * K_{00} - (\theta(z) E_z^+) * \frac{\partial}{\partial z} K_{00}$$

Performing the convolution with respect to z allows us to write this last equation as

$$h_1 E_z = E_{a_z} - (\left[\frac{\partial E_z}{\partial z}\right]^+) * K_{00}(x,y,z) - (E_z^+) * \frac{\partial}{\partial z} K_{00}(x,y,z) \; ,$$

which may be x-y Fourier transformed to yield

$$h_1 \overline{\overline{E_z}} = \overline{\overline{E_{a_z}}} - \overline{\overline{(\left[\frac{\partial E_z}{\partial z}\right]^+)}}\, \overline{\overline{K_{00}}} - \overline{\overline{(E_z^+)}} \left[\frac{\partial}{\partial z} \overline{\overline{K_{00}}}\right] \qquad (3.170).$$

Once again, in equation (3.170) and its predecessor, the symbol "*" is now understood to mean two-dimensional x-y convolution.

Now [see Appendix A]

$$\overline{\overline{K_{00}}}(\xi,\eta,z) = \frac{e^{-|z|u_0}}{2u_0} \qquad (3.171),$$

wherein we've used the quantity u_0 as defined by

$$u_0 = \sqrt{\xi^2 + \eta^2 - k^2} \qquad (3.172).$$

In equation (3.172), the same branch of the multivalued square root function has been chosen as in equations (3.161) and (3,162).

Now, fixing a plane $z = z_1 < 0$, we find that

$$\frac{\partial}{\partial z}\overline{\overline{K_{00}}} = \frac{\partial}{\partial z}\frac{e^{+zu_0}}{2u_0} = \frac{e^{zu_0}}{2},$$

so that, for a fixed plane $z = z_1 < 0$, equation (3.170) becomes

$$0 = \overline{\overline{E_{s_z}}}\, e^{-z_1 u_0} - \overline{\overline{\left(\left[\frac{\partial E_z}{\partial z}\right]^+\right)\left[\frac{1}{2u_0}\right]}} - \frac{1}{2}\overline{\overline{E_z^+}} \qquad (3.173).$$

Of course we must realize in future that E_{s_z} must be evaluated at the fixed plane $z = z_1 < 0$. Since we may evidently write

$$\overline{\overline{\left[\frac{\partial E_z}{\partial z}\right]^+}} = \overline{\overline{[(1-h_2)\left[\frac{\partial E_z}{\partial z}\right]^+]}} + \overline{\overline{[h_2\left[\frac{\partial E_z}{\partial z}\right]^+]}},$$

substituting from both equations (3.168) and (3.169) into equation (3.173) results in

$$2\overline{\overline{E_{s_z}}}\, e^{-z_1 u_0} = \frac{1}{n_{01}^2}\left[\frac{u_1}{n_{01}^2}\overline{\overline{[(1-h_2)E_z^+]}} + \frac{u_2}{n_{02}^2}\overline{\overline{[h_2 E_z^+]}}\right]$$

$$+ \overline{\overline{[(1-h_2)E_z^+]}} + \overline{\overline{[h_2 E_z^+]}} \qquad (3.174).$$

This last equation may be rewritten

$$2u_0\overline{\overline{E_{s_z}}}\, e^{-z_1 u_0} = \left[u_0 + \frac{u_1}{n_{01}^2}\right]\overline{\overline{[(1-h_2)E_z^+]}}$$

$$+ \left[u_0 + \frac{u_2}{n_{02}^2} \right] \overline{\overline{(h_2 E_z^+)}} \qquad (3.175).$$

Since we may decompose

$$\overline{\overline{(1-h_2)E_z^+}} = \overline{\overline{E_z^+}} - \overline{\overline{(h_2 E_z^+)}} .$$

equation (3.175) may be rewritten as

$$2u_0 \overline{\overline{E_{s_z}}} e^{-z_1 u_0} = \left[u_0 + \frac{u_1}{n_{01}^2} \right] \overline{\overline{(E_z^+)}} + \left[\frac{u_2}{n_{02}^2} - \frac{u_1}{n_{01}^2} \right] \overline{\overline{(h_2 E_z^+)}} \qquad (3.176)$$

Equation (3.176) may be rearranged to show

$$\overline{\overline{E_z^+}} + \frac{\dfrac{u_2}{n_{02}^2} - \dfrac{u_1}{n_{01}^2}}{u_0 + \dfrac{u_1}{n_{01}^2}} \overline{\overline{(h_2 E_z^+)}} = \frac{2u_0}{u_0 + \dfrac{u_1}{n_{01}^2}} \overline{\overline{E_{s_z}}} e^{-z_1 u_0} \qquad (3.177).$$

If we now define

$$\overline{\overline{D(x,y)}} = \frac{2u_0 \overline{\overline{E_{s_z}}} e^{-z_1 u_0}}{u_0 + \dfrac{u_1}{n_{01}^2}} \qquad (3.178).$$

and

$$\overline{\overline{W(x,y)}} = \frac{\dfrac{u_2}{n_{02}^2} - \dfrac{u_1}{n_{01}^2}}{u_0 + \dfrac{u_1}{n_{01}^2}} \qquad (3.179).$$

equation (3.177) becomes, formally,

$$\overline{\overline{E_z^+}} + \overline{\overline{W}}\,\overline{\overline{[h_2 E_z^+]}} = \overline{\overline{D}} \qquad (3.180).$$

Applying an inverse two-dimensional Fourier transform to equation (3.180) shows that

$$E_z^+ + ([h_2 E_z^+] * W(x,y)) = D(x,y) \quad ,$$

which may further be decomposed into the two surface-support equations

$$(1-h_2)E_z^+ + (1-h_2)([h_2 E_z^+] * W(x,y)) = (1-h_2)D(x,y) \quad ,$$

and

$$h_2 E_z^+ + h_2([h_2 E_z^+] * W(x,y)) = h_2 D(x,y) \qquad (3.181).$$

We shall now concentrate on solving for $h_2 E_z^+$. Equation (3.181) may be rewritten as

$$h_2 E_z^+ + ([h_2 E_z^+] * W(x,y)) = h_2 D(x,y) + (1-h_2)([h_2 E_z^+] * W(x,y)) \quad ,$$

which may be Fourier transformed to give

$$(1 + \overline{\overline{W(x,y)}})\overline{\overline{[h_2 E_z^+]}} = \overline{\overline{[h_2 D(x,y)]}} + \overline{\overline{[(1-h_2)([h_2 E_z^+] * W(x,y))]}} \qquad (3.182).$$

By defining

$$\overline{\overline{V(x,y)}} = 1 + \overline{\overline{W(x,y)}} \qquad (3.183)$$

we see from equation (3.179) that

$$\overline{\overline{V(x,y)}} = \frac{u_0 + \dfrac{u_2}{n_{02}^2}}{u_0 + \dfrac{u_1}{n_{01}^2}} \qquad (3.184).$$

so that equation (3.182) becomes

$$\overline{\overline{V(x,y)}}\,\overline{\overline{[h_2 E_z^+]}} = \overline{\overline{[h_2 D(x,y)]}} + \overline{\overline{[(1-h_2)([h_2 E_z^+]*W(x,y))]}} \quad (3.185),$$

wherein V(x,y), D(x,y), and W(x,y) are as defined in equations (3.184), (3.178), and (3.179), respectively. Note that D(x,y) is not to be confused with the electromagnetic vector quantity D appearing previously.

The next chapter is concerned with a solution of equation (3.185) for the surface field $h_2 E_z^+$.

CHAPTER 4: SOLUTION OF THE INTEGRAL EQUATION

Rationale

Equation (3.185) must now be solved for the quantity $\overline{\overline{h_2 E_z^+}}$: an inverse Fourier transform will then provide us with $h_2 E_z^+$: i.e. the surface electric field component to the right of the vertical interface.

Many techniques exist, in theory, for solving equations of the form of equation (3.185). It is difficult, *ab initio*, to select any one method which will hopefully lead straight to a solution. In choosing a particular approach it is worthwhile considering what has transpired so far.

The preceeding analysis has been, compared with other electromagnetic methods, straightforward and rather elegant. We have avoided the use of intermediate Hertz potentials which are usually employed in the classical analyses. Moreover, the boundary conditions have fallen neatly into our lap as a result of modelling the physical geometry of the problem with the aid of step functions.

Thus, from an aesthetic point of view, equation (3.185) of the preceeding chapter should be solved in a manner befitting the hitherto steps. Indeed, if we could paraphrase Dirac's [6] now famous remark, "It is more important to have beauty in one's equations than to have them fit physical reality" we should perhaps choose as our guiding principal: "It is more important to solve an equation elegantly then expeditiously".

In view of these statements, the Wiener-Hopf method has been chosen to solve the desired equation. Because of the relative complexity of the functions involved, the Wiener–Hopf method is fraught with pitfalls and deadends. In the end we shall be forced to make some [seemingly reasonable] approximations before headway can be made with the problem.

Outline of the Wiener–Hopf Procedure

In order to effect a Wiener–Hopf solution of equation (3.185), viz

$$\overline{V(x,y)}\,\overline{\overline{[h_2 E_z^+]}} = \overline{[h_2 D(x,y)]} + \overline{\overline{[(1-h_2)([h_2 E_z^+]*W(x,y))]}} \qquad (4.1),$$

we must study the analytic properties, in the complex ξ-plane, of the terms in this equation.

Consider the term $\overline{\overline{[h_2 E_z^+]}}$, and suppose that we could inverse Fourier transform this quantity from the ξ-plane back into the x-domain. The resulting function

$$\overline{[h_2 E_z^+]}(x,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{i\xi x}\, \overline{\overline{[h_2 E_z^+]}} \qquad (4.2),$$

(wherein we've explicitly written the functional dependence of the left hand side) is clearly zero for all $x < 0$. In other words

$$\int_{-\infty}^{\infty} d\xi\, e^{i\xi x}\, \overline{\overline{[h_2 E_z^+]}} = 0 \qquad x < 0 \qquad (4.3).$$

However, in order to actually evaluate the integral occurring in equations (4.3) or (4.2) in the case $x < 0$, we can close the contour in the lower complex ξ-plane. The Cauchy Residue Theorem then tells us that

$$\overline{\overline{[h_2 E_z^+]}} \textit{ is analytic in the lower complex } \xi\textit{-plane} \qquad (4.4).$$

We shall denote this property of the function $\overline{\overline{[h_2 E_z^+]}}$ by attaching a subscript "–" to denote analyticity in the lower complex ξ-plane. Thus

$$\overline{\overline{[h_2 E_z^+]}} = \overline{\overline{[h_2 E_z^+]}}_- \tag{4.5}$$

Similar analyses reveal that

$$\overline{\overline{[h_2 D(x,y)]}} = \overline{\overline{[h_2 D(x,y)]}}_- \tag{4.6}$$

and

$$\overline{\overline{[(1-h_2)([h_2 E_z^+]*W(x,y))]}} = \overline{\overline{[(1-h_2)([h_2 E_z^+]*W(x,y))]}}_+ \tag{4.7}$$

wherein the subscript "+" obviously denotes analyticity in the upper half complex ξ-plane. Equation (4.1) may now be written

$$\overline{\overline{V(x,y)}}\;\overline{\overline{[h_2 E_z^+]}}_- = \overline{\overline{[h_2 D(x,y)]}}_- + \overline{\overline{[(1-h_2)([h_2 E_z^+]*W(x,y))]}}_+ \tag{4.8}$$

Now suppose that we could "decompose" $\overline{\overline{V(x,y)}}$ into the quotient of two functions as

$$\overline{\overline{V(x,y)}} = \frac{v_-(\xi)}{v_+(\xi)} \tag{4.9}$$

wherein, again, the subscripts denote analyticity, but, additionally, freedom of zeros as well, in the respective complex half ξ-planes [see Hochstadt [12]]. In this event equation (4.8) becomes

$$v_-(\xi)\,\overline{\overline{[h_2 E_z^+]}}_- = v_+(\xi)\,\overline{\overline{[h_2 D(x,y)]}}_-$$

$$+ v_+(\xi)\overline{\overline{[(1-h_2)([h_2E_z^+]*W(x,y))]}}_+ \qquad (4.10).$$

The product of two "+" subscripted or two "-" subscripted functions obviously yields, respectively, a "+" or a "-" subscripted function. The product of a "+" with a "-" subscripted function is not, in general, analytic in either complex half ξ-plane. However, if we could make the further additive decomposition

$$v_+(\xi)\overline{[h_2D(x,y)]}_- = f_+(\xi) + f_-(\xi) \qquad (4.11),$$

then equation (4.10) becomes

$$v_-(\xi)\overline{\overline{[h_2E_z^+]}}_- - f_-(\xi) = f_+(\xi) + v_+(\xi)\overline{\overline{[(1-h_2)([h_2E_z^+]*W(x,y))]}}_+ \quad (4.12).$$

The left-hand side of equation (4.12) represents a function which is analytic in the lower complex ξ half-plane, and the right-hand side a function analytic in the upper complex ξ half-plane. By definition of the Fourier transform, equation (4.12) must hold on the real ξ-axis. Thus, both the right- and left-hand sides are representations of an entire function of ξ: the left-hand side is a faithful representation of this entire function in the lower complex ξ half-plane, and the right-hand side is a faithful representation of this entire function in the upper complex ξ half-plane. Both right- and left-hand sides faithfully represent this entire function on the real ξ-axis. This is equivalent to saying that the left-hand side of equation (4.12) is the analytic continuation of the right-hand side into the lower complex ξ half-plane, and vice-versa.

That equation (4.12) holds only on the real ξ-axis tells us that the entire function in question must be identically zero [again, see Hochstadt [12]], so that we have, for example,

$$\overline{\overline{[h_2 E_z^+]}} = \frac{f_-(\xi)}{v_-(\xi)}$$

which may finally be inverse Fourier transformed to give

$$h_2 E_z^+ = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\xi \, e^{i\xi x + i\eta y} \frac{f_-(\xi)}{v_-(\xi)} \qquad (4.13).$$

Thus we have, in theory, formally solved for $h_2 E_z^+$.

Our first hurdle comes when we try to effect the decomposition in equation (4.9). Recall from equation (3.184) that the function $\overline{\overline{V(x,y)}}$ is given by

$$\overline{\overline{V(x,y)}} = \frac{u_0 + \dfrac{u_2}{n_{02}^2}}{u_0 + \dfrac{u_1}{n_{01}^2}} \qquad (4.14),$$

wherein u_0, u_1, and u_2 are determined from equations (3.172), (3.161), and (3.162), respectively, as being

$$\left.\begin{aligned} u_0 &= \sqrt{\xi^2 + \eta^2 - k^2} \\ u_1 &= \sqrt{\xi^2 + \eta^2 - k^2 n_{01}^2} \\ u_2 &= \sqrt{\xi^2 + \eta^2 - k^2 n_{02}^2} \end{aligned}\right\} \qquad (4.15).$$

In order to make the decomposition of $\overline{\overline{V(x,y)}}$ somewhat more tractable we shall adopt the standard "surface impedance" boundary condition approximation, which is well accepted in the literature [see Wait [28], for a comprehensive dicussion]. This allows us to approximate, for example

$$\frac{\sqrt{\xi^2 + \eta^2 - k^2 n_{01}^2}}{n_{01}^2} \approx \frac{\sqrt{k^2(1 - n_{01}^2)}}{n_{01}^2} \tag{4.16}$$

and we shall now define

$$\mu = \frac{\sqrt{k^2(1 - n_{01}^2)}}{n_{01}^2} \tag{4.17}$$

Similarly, we approximate

$$\frac{\sqrt{\xi^2 + \eta^2 - k^2 n_{02}^2}}{n_{02}^2} \approx \frac{\sqrt{k^2(1 - n_{02}^2)}}{n_{02}^2} \tag{4.18}$$

and make the definition

$$\nu = \frac{\sqrt{k^2(1 - n_{02}^2)}}{n_{02}^2} \tag{4.19}$$

These approximations allow us to rewrite equation (4.14) as

$$\overline{\overline{V(x,y)}} \approx \frac{\sqrt{\xi^2 + \eta^2 - k^2} + \nu}{\sqrt{\xi^2 + \eta^2 - k^2} + \mu} \tag{4.20}$$

Now, because our Wiener-Hopf decomposition procedure will be carried out in the complex ξ-plane we shall, for the time being, consider η as being held constant. Thus, by defining

$$\eta^2 - k^2 = \alpha^2 \tag{4.21}$$

we may rewrite equation (4.20) as

$$\overline{\overline{V(x,y)}} = \frac{\sqrt{\xi^2 + \alpha^2} + \nu}{\sqrt{\xi^2 + \alpha^2} + \mu} \tag{4.22}$$

approximately. It is this last function $\overline{\overline{V(x,y)}}$ which we must decompose in the manner of equation (4.9) above. It is somewhat cumbersome to apply the Wiener-Hopf decomposition formulas to the function $\overline{\overline{V(x,y)}}$ as it stands. It is simpler to rewrite $\overline{\overline{V(x,y)}}$ as the product of two other functions: say

$$\overline{\overline{V(x,y)}} = R(\xi)\,S(\xi) \tag{4.23}$$

wherein

$$R(\xi) = \frac{\sqrt{\xi^2 + \alpha^2} + \nu}{\sqrt{\xi^2 + \alpha^2}} \tag{4.24}$$

and

$$S(\xi) = \frac{\sqrt{\xi^2 + \alpha^2}}{\sqrt{\xi^2 + \alpha^2} + \mu} \tag{4.25}$$

The exact form of $R(\xi)$ and $S(\xi)$ appears arbitrary, but in fact must satisfy certain conditions as $\xi \to \infty$ in order that we may apply the "standard" Wiener–Hopf decomposition formulas [see e.g., Noble [18], or Carrier, Krook, & Pearson [3]].

If we could make the decompositions

$$R(\xi) = \frac{r_-(\xi)}{r_+(\xi)}$$

and

$$S(\xi) = \frac{s_-(\xi)}{s_+(\xi)},$$

then, from equations (4.9) and (4.23), we could identify

$$v_-(\xi) = r_-(\xi)\,s_-(\xi) \qquad (4.26),$$

and

$$v_+(\xi) = r_+(\xi)\,s_+(\xi). \qquad (4.27)$$

Furthermore, we note that there is a certain "symmetry" between $R(\xi)$ and $S(\xi)$ as given in equations (4.24) and (4.25). Specifically

$$S(\xi) = \left\{\frac{1}{R(\xi)}\right\}_{\nu \to \mu} \qquad (4.28).$$

Consider, for example, $r_+(\xi)$. Since this function is analytic and non-vanishing in the upper complex ξ half-plane, so is the function $1/r_+(\xi)$. Applying a similar argument to $r_-(\xi)$ we may write

$$\frac{1}{R(\xi)} = \frac{\left[\frac{1}{r_-(\xi)}\right]_-}{\left[\frac{1}{r_+(\xi)}\right]_+}$$

so that

$$S(\xi) = \frac{s_-(\xi)}{s_+(\xi)} = \frac{\left[\frac{1}{r_-(\xi)}\right]_-}{\left[\frac{1}{r_+(\xi)}\right]_+}\Bigg|_{\nu\to\mu} \qquad (4.29).$$

The "bottom line" of this discussion is that we need only work to find $r_-(\xi)$ and $r_+(\xi)$, for $S(\xi)$ is readily decomposed according to equation (4.29).

Application of the standard Wiener–Hopf decomposition formulas to the functions $R(\xi)$ and $S(\xi)$ requires a detailed knowledge of the behaviour of the "constants" appearing in eqsations (4.24) and (4.25). A description of the properties of the function $R(\xi)$ has been given in Appendix B.

Applying the results of Appendix B to the standard Wiener–Hopf decomposition formulas [see e.g. Noble [18], or Carrier, Krook, and Pearson [3]] tells us that the functions in the decomposition

$$R(\xi) = \frac{r_-(\xi)}{r_+(\xi)} \qquad (4.30)$$

are given by

$$\log[r_-(\xi)] = \frac{-1}{2\pi i}\int_{\Gamma_1} d\zeta\, \frac{\log[R(\zeta)]}{\zeta - \xi} \qquad (4.31),$$

and

$$\log[r_+(\xi)] = \frac{-1}{2\pi i}\int_{\Gamma_2} d\zeta\, \frac{\log[R(\zeta)]}{\zeta - \xi} \qquad (4.32).$$

The integration contours in equations (4.31) and (4.32) are shown in diagram (4.1), where we have also included information about the analytic structure of $R(\xi)$.

Purely for [the author's] convenience we shall rewrite equation (4.31) as follows

$$\log[r_-(\lambda)] = \frac{-1}{2\pi i}\int_{\Gamma_1} d\xi \frac{\log[R(\xi)]}{\xi - \lambda} \qquad (4.33),$$

wherein λ is a point below the contour Γ_1 as shown in diagram (4.1). Note that ξ in the integral above is merely an integration variable, and hence must **NOT** be confused with the x Fourier transform variable. The distinction will be clear in proceeding work from the context. Substituting from equation (4.24) for an explicit representation of $R(\xi)$, we find that

$$\log[r_-(\lambda)] = \frac{-1}{2\pi i}\int_{\Gamma_1} \frac{d\xi}{\xi - \lambda}\log\left[\frac{\sqrt{\xi^2+\alpha^2} + \nu}{\sqrt{\xi^2+\alpha^2}}\right] \qquad (4.34)$$

Similarly, equation (4.32) becomes

$$\log[r_+(\lambda)] = \frac{-1}{2\pi i}\int_{\Gamma_2} \frac{d\xi}{(\xi - \lambda)}\log\left[\frac{\sqrt{\xi^2+\alpha^2} + \nu}{\sqrt{\xi^2+\alpha^2}}\right] \qquad (4.35),$$

wherein λ is now taken to be any point above the contour Γ_2 as shown in diagram (4.1).

Carrier, Krook, & Pearson have considered the decomposition of a function akin to, but somewhat simpler than our function $R(\xi)$ [see Carrier, Krook, & Pearson [3], pp. 382-384]. The integrals occurring in equations (4.34) and (4.35) are, to the best of this author's knowledge, intractable. Some progress can be made if we first differentiate with respect to λ, in each equation, which is usually done in the Wiener-Hopf method for relatively complicated functions [again, see e.g., Carrier, Krook, & Pearson, [3], or Noble [18]]. We obtain

$$\frac{d}{d\lambda}(\log[r_-(\lambda)]) = \frac{-1}{2\pi i}\int_{\Gamma_1}\frac{d\xi}{(\xi-\lambda)^2}\log\left[\frac{\sqrt{\xi^2+\alpha^2}+\nu}{\sqrt{\xi^2+\alpha^2}}\right] \qquad (4.36),$$

and

$$\frac{d}{d\lambda}(\log[r_+(\lambda)]) = \frac{-1}{2\pi i}\int_{\Gamma_2}\frac{d\xi}{(\xi-\lambda)^2}\log\left[\frac{\sqrt{\xi^2+\alpha^2}+\nu}{\sqrt{\xi^2+\alpha^2}}\right] \qquad (4.37).$$

The right-hand side of equation (4.36) may be integrated by parts as

$$\frac{d}{d\lambda}(\log[r_-(\lambda)]) = \frac{1}{2\pi i}\left[\frac{1}{(\xi-\lambda)}\log\left[\frac{\sqrt{\xi^2+\alpha^2}+\nu}{\sqrt{\xi^2+\alpha^2}}\right]\right]_{\Gamma_1;\ endpoints}$$

$$-\frac{1}{2\pi i}\int_{\Gamma_1}\frac{d\xi}{(\xi-\lambda)}\frac{d}{d\xi}\left[\log\left[\frac{\sqrt{\xi^2+\alpha^2}+\nu}{\sqrt{\xi^2+\alpha^2}}\right]\right] \qquad (4.38).$$

The contour Γ_1 lies within the strip of analyticity of the function $R(\xi)$. Moreover, our choice of branch of the multivalued square-root function ensures that $\log[R(\xi)] \to 0$ as $\xi \to \pm\infty$ within the strip of analyticity. Hence, the first term coming from the right-hand side of equation (4.38) vanishes, and we are left with

$$\frac{d}{d\lambda}(\log[r_-(\lambda)]) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{d\xi}{(\xi-\lambda)}\frac{d}{d\xi}\log\left[\sqrt{\xi^2+\alpha^2}\right]$$

$$- \frac{1}{2\pi i}\int_{\Gamma_1} \frac{d\xi}{(\xi - \lambda)} \frac{d}{d\xi}\log\left\{\sqrt{\xi^2+\alpha^2} + \nu\right\}.$$

which may be simplified to give finally

$$\frac{d}{d\lambda}\{\log[r_-(\lambda)]\} = \frac{1}{2\pi i}\int_{\Gamma_1} d\xi \frac{\xi}{(\xi - \lambda)} \frac{1}{\xi^2 + \alpha^2}$$

$$- \frac{1}{2\pi i}\int_{\Gamma_1} d\xi \frac{\xi}{(\xi - \lambda)} \frac{1}{[\sqrt{\xi^2+\alpha^2} + \nu]\sqrt{\xi^2+\alpha^2}}.$$

By defining [c.f. Appendix B, equation (B.8)]

$$\tau^2 = \alpha^2 - \nu^2 \qquad (4.39),$$

this last equation becomes

$$\frac{d}{d\lambda}\{\log[r_-(\lambda)]\} = \frac{1}{2\pi i}\int_{\Gamma_1} d\xi \frac{\xi}{(\xi - \lambda)} \frac{1}{\xi^2 + \alpha^2}$$

$$- \frac{1}{2\pi i}\int_{\Gamma_1} d\xi \frac{\xi}{(\xi - \lambda)} \frac{(\sqrt{\xi^2+\alpha^2} - \nu)}{(\xi^2 + \tau^2)} \frac{1}{\sqrt{\xi^2+\alpha^2}};$$

which may be simplified somewhat to yield

$$\frac{d}{d\lambda}\{\log[r_-(\lambda)]\} = \frac{1}{2\pi i}\int_{\Gamma_1} d\xi \frac{\xi}{(\xi - \lambda)} \frac{1}{\xi^2 + \alpha^2} - \frac{1}{2\pi i}\int_{\Gamma_1} \frac{d\xi}{(\xi - \lambda)} \frac{\xi}{(\xi^2 + \tau^2)}$$

$$+ \frac{\nu}{2\pi i}\int_{\Gamma_1} \frac{d\xi}{(\xi - \lambda)} \frac{\xi}{(\xi^2 + \tau^2)\sqrt{\xi^2+\alpha^2}} \qquad (4.40).$$

The same analysis [*mutatis mutandis*] of equation (4.35) tells us that

$$\frac{d}{d\lambda}\{\log[r_+(\lambda)]\} = \frac{1}{2\pi i}\int_{\Gamma_2} d\xi \frac{\xi}{(\xi - \lambda)} \frac{1}{(\xi^2 + \alpha^2)} - \frac{1}{2\pi i}\int_{\Gamma_2} \frac{d\xi}{(\xi - \lambda)} \frac{\xi}{(\xi^2 + \tau^2)}$$

$$+\frac{\nu}{2\pi i}\int_{\Gamma_2}\frac{d\xi}{(\xi-\lambda)}\frac{\xi}{(\xi^2+\tau^2)\sqrt{\xi^2+\alpha^2}} \tag{4.41}$$

It must be remembered that equations (4.40) and (4.41) differ in two important respects: the integration contour is different in each equation and the half-plane in which λ lies is different in each equation.

The integrals in both equations (4.40) and (4.41) may be evaluated to yield [see Appendix C]

$$\frac{d}{d\lambda}\{\log[r_-(\lambda)]\} = \frac{-1}{2(\lambda-i\alpha)}+\frac{1}{\lambda-i\tau}-\frac{\nu\lambda(\pi/2-\phi)}{\pi(\lambda^2+\tau^2)\sqrt{\lambda^2+\alpha^2}}$$

$$-\frac{(\lambda\pi/2+\psi i\tau)}{\pi(\lambda^2+\tau^2)} \tag{4.42}$$

and

$$\frac{d}{d\lambda}\{\log[r_+(\lambda)]\} = \frac{1}{2(\lambda+i\alpha)}-\frac{1}{\lambda+i\tau}+\frac{\nu\lambda(\pi/2+\phi)}{\pi(\lambda^2+\tau^2)\sqrt{\lambda^2+\alpha^2}}$$

$$+\frac{(\lambda\pi/2-\psi i\tau)}{\pi(\lambda^2+\tau^2)} \tag{4.43}$$

wherein ψ and ϕ are defined through equations (C.21) and (C.32), respectively, as being

$$\sin\psi = \tau/\alpha \tag{4.44}$$

and

$$\sin\phi = i\lambda/\alpha \tag{4.45}$$

It is readily verified that the expressions in equations (4.42) and (4.43) are analytic in their respective half-planes. For example, the apparent multivaluedness of the right-hand side of equation (4.42) about the point $\lambda = i\alpha$ vanishes on noting that

$$\phi \to -\pi - \phi \tag{4.46}$$

as we turn about the point $\lambda = i\alpha$. Moreover, from equations (4.42) and (4.43) we find that

$$\frac{d}{d\lambda}(\log[r_-(\lambda)]) - \frac{d}{d\lambda}(\log[r_+(\lambda)])$$
$$= \frac{-\lambda}{\lambda^2 + \alpha^2} + \frac{\lambda}{\lambda^2 + \tau^2} - \frac{\nu\lambda}{(\lambda^2 + \tau^2)\sqrt{\lambda^2 + \alpha^2}} \tag{4.47}$$

Since, from equation (4.24),

$$\log[R(\lambda)] = \log[\sqrt{\lambda^2 + \alpha^2} + \nu] - (1/2)\log(\lambda^2 + \alpha^2),$$

we have that

$$\frac{d}{d\lambda}(\log[R(\lambda)]) = \frac{\lambda}{[\sqrt{\lambda^2 + \alpha^2} + \nu]\sqrt{\lambda^2 + \alpha^2}} - \frac{\lambda}{(\lambda^2 + \alpha^2)}.$$

This last equation may be simplified to give

$$\frac{d}{d\lambda}(\log[R(\lambda)]) = \frac{\lambda}{\lambda^2 + \tau^2} - \frac{\nu\lambda}{(\lambda^2 + \tau^2)\sqrt{\lambda^2 + \alpha^2}} - \frac{\lambda}{\lambda^2 + \alpha^2},$$

so that indeed the equation

$$\frac{d}{d\lambda}(\log[R(\lambda)]) = \frac{d}{d\lambda}(\log[r_-(\lambda)]) - \frac{d}{d\lambda}(\log[r_+(\lambda)]),$$

coming from equation (4.30), is satisfied by equations (4.42) and (4.43).

Equations (4.42) and (4.43) may be formally integrated to give [see

Appendix D]

$$\log[r_-(\lambda)] = -(1/2)\log(\lambda - i\alpha) + \log(\lambda - i\tau)$$
$$+ \frac{\psi}{2\pi}\log\left[\frac{\lambda + i\tau}{\lambda - i\tau}\right] - (1/2)\log[\sqrt{\lambda^2 + \alpha^2} - \nu]$$
$$+ \frac{\sin^{-1}(i\lambda/\alpha)}{2\pi}\log\left[\frac{\sqrt{\lambda^2 + \alpha^2} - \nu}{\sqrt{\lambda^2 + \alpha^2} + \nu}\right]$$
$$+ \frac{i}{2\pi}\int^{\lambda} \frac{d\xi}{\sqrt{\xi^2 + \alpha^2}}\log\left[\frac{\sqrt{\xi^2 + \alpha^2} + \nu}{\sqrt{\xi^2 + \alpha^2} - \nu}\right] \qquad (4.48),$$

and

$$\log[r_+(\lambda)] = (1/2)\log(\lambda + i\alpha) - \log(\lambda + i\tau)$$
$$+ \frac{\psi}{2\pi}\log\left[\frac{\lambda + i\tau}{\lambda - i\tau}\right] + (1/2)\log[\sqrt{\lambda^2 + \alpha^2} - \nu]$$
$$+ \frac{\sin^{-1}(i\lambda/\alpha)}{2\pi}\log\left[\frac{\sqrt{\lambda^2 + \alpha^2} - \nu}{\sqrt{\lambda^2 + \alpha^2} + \nu}\right]$$
$$+ \frac{i}{2\pi}\int_{i}^{\lambda} \frac{d\xi}{\sqrt{\xi^2 + \alpha^2}}\log\left[\frac{\sqrt{\xi^2 + \alpha^2} + \nu}{\sqrt{\xi^2 + \alpha^2} - \nu}\right] \qquad (4.49),$$

respectively.

It is seen that differentiating the expressions in equations (4.48) and (4.49) leads us back to equations (4.42) and (4.43), respectively, as they should. Moreover, from the equations above we find that

$$\log[r_-(\lambda)] - \log[r_+(\lambda)] = -\log\sqrt{\lambda^2+\alpha^2} + \log(\lambda^2+\tau^2)$$

$$-\log[\sqrt{\lambda^2+\alpha^2} - \nu] .$$

which is equivalent to

$$\log[r_-(\lambda)] - \log[r_+(\lambda)] = \log[\sqrt{\lambda^2+\alpha^2} + \nu] - \log\sqrt{\lambda^2+\alpha^2} .$$

Comparing this last expression with equation (4.24) we see that the equation

$$\log[R(\lambda)] = \log[r_-(\lambda)] - \log[r_+(\lambda)]$$

is indeed satisfied by the quantities given in equations (4.48) and (4.49).

Much effort has been spent in trying to evaluate the integral [i.e. inverse derivative] occurring in both equations (4.48) and (4.49), to no avail. It might seem that, since the "same" term will appear in both $r_-(\lambda)$ and $r_+(\lambda)$ in equation (4.30), we might cancel this term from both expressions. However, this is not the case. An inspection of either equation (4.48) or (4.49) shows that the last integral is required to make the relevant function analytic in its particular half-plane. Without this trait the terms $r_-(\lambda)$ and $r_+(\lambda)$ lose their significance in equation (4.30) as a valid Wiener-Hopf decomposition.

Approximation of the Function to be Decomposed

To circumvent this apparent impasse we shall resort to the method of approximating the function to be decomposed in the Wiener-Hopf manner. This practice exists to some extent in the literature and has been invoked for the solution of certain homogeneous problems when the decomposition has proved to be impossible [e.g. see Noble[18]: pp. 160–164, or Carrier, Krook, & Pearson[3]: pp. 393 ff.]. We shall be approximating the function at hand by another which replicates its primary features, such as singularities in the x-domain, area, and first few moments. As such, we shall make the following approximation

$$R(\xi) = \frac{\sqrt{\xi^2+\alpha^2} + \nu}{\sqrt{\xi^2+\alpha^2}} \approx R_1(\xi) = \frac{(\xi^2 + \tau^2)(\xi^2 + b^2)}{\sqrt{\xi^2+\alpha^2}\sqrt{\xi^2 + a^2}(\xi^2 + c^2)} \quad (4.50)$$

wherein a, b, and c will be chosen so that both the inverse Fourier transform of both $R_1(\xi)$ and $R(\xi)$ share the same singularity, area, and second moment [no first moments exists]. This leaves one parameter "free" [see Carrier, Krook, & Pearson [3]: pp. 397] to ensure that no zeros or poles lie near the real ξ-axis. Enforcing these restrictions leads us [see Appendix E] to the approximation

$$R(\xi) = \frac{\sqrt{\xi^2+\alpha^2} + \nu}{\sqrt{\xi^2+\alpha^2}} \approx R_1(\xi) = \frac{(\xi^2 + \tau^2)(\xi^2 + b^2)}{(\xi^2 + \alpha^2)(\xi^2 + c^2)} \quad (4.51)$$

wherein [see Appendix E, equation (E.12) & (E.11)]

$$b^2 = 2\alpha^2 \quad (4.52)$$

and

$$c^2 = 2\alpha(\alpha - \nu) \quad (4.53)$$

We shall also be approximating $S(\xi)$, which is given in equation (4.25), by $S_1(\xi)$. We have specifically [see Appendix E]

$$S(\xi) = \frac{\sqrt{\xi^2+\alpha^2}}{\sqrt{\xi^2+\alpha^2} \pm \mu} \approx S_1 = \frac{(\xi^2 + \alpha^2)(\xi^2 + d^2)}{(\xi^2 + \sigma^2)(\xi^2 + b^2)} \tag{4.54}$$

wherein

$$d^2 = 2\alpha(\alpha - \mu) \tag{4.55}$$

and b^2 is as given in equation (4.52).

From equation (4.23) we have thus the approximation, using equations (4.51) and (4.54),

$$\overline{\overline{V(x,y)}} = R(\xi)S(\xi) \approx \overline{\overline{V_1(x,y)}} = \frac{(\xi^2 + \tau^2)(\xi^2 + d^2)}{(\xi^2 + \sigma^2)(\xi^2 + c^2)} \tag{4.56}$$

Our analysis in Appendix E allows us to decompose $\overline{\overline{V_1(x,y)}}$ as

$$\overline{\overline{V_1(x,y)}} = \frac{v_{1_-}(\xi)}{v_{1_+}(\xi)} = \frac{\left[\frac{(\xi - i\tau)(\xi - id)}{(\xi - i\sigma)(\xi - ic)}\right]_-}{\left[\frac{(\xi + i\sigma)(\xi + ic)}{(\xi + i\tau)(\xi + id)}\right]_-} \tag{4.57}$$

Obtaining an Approximate Solution

The "approximated" version of equation (4.10) would read as

$$v_{1_-}(\xi)\overline{\overline{[h_2E_z^+]}} = v_{1_+}(\xi)\overline{\overline{[h_2D(x,y)]}}_- \tag{4.58}$$

$$+ v_{1_+}(\xi)\overline{\overline{[(1-h_2)([h_2E_z^+]*W(x,y))]}}_+ \tag{4.58}$$

with $v_{1_-}(\xi)$ and $v_{1_+}(\xi)$ as given in equation (4.57). Our next task is the additive decomposition of $v_{1_+}(\xi)\overline{\overline{[h_2D(x,y)]}}_-$ as given in equation

(4.11). As such, an investigation of $\overline{\overline{[h_2 D(x,y)]_-}}$ is in order. From the convolution theorem for Fourier transforms [see Papoulis [19]] we may write

$$\overline{\overline{[h_2 D(x,y)]_-}} = \frac{1}{(2\pi)^2}\,\overline{\overline{h_2}} * \overline{\overline{D(x,y)}} \qquad (4.59).$$

It follows from the results pertaining to the one dimensional Fourier transforms of a constant and of a step function [again, see Papoulis [19]] that

$$\overline{\overline{h_2}} = 2\pi\delta(\eta)\,[\pi\delta(\xi) + \frac{1}{i\xi}] \qquad (4.60).$$

From equation (3.178) we have

$$\overline{\overline{D(x,y)}} = \frac{2u_0 \overline{\overline{E_{s_z}}}\, e^{-z_1 u_0}}{u_0 + \dfrac{u_1}{n_{01}^2}} \qquad (4.61)$$

wherein, from equation (3.132), we have defined

$$E_s = (T_{sE}(J_s)) * K_{00}.$$

In the "far-field" case, we have the result [see Appendix F]

$$E_{s_z} = \frac{Idlk^2}{4\pi i\omega\epsilon_{00}}\,\frac{e^{-ikr'}}{r'} \qquad (4.62).$$

wherein r' is the distance measured from the source, which is assumed to be an elementary electric dipole located at a height h above the ground at a distance x_0 to the left of the vertical interface, to the observation point of E_{s_z}. From equation (3.173) we recall that E_{s_z} must be evaluated on an arbitrary but fixed plane $z = z_1 < 0$. That is, from Appendix F, equation (F.7)

$$r' = \left[(x + x_0)^2 + y^2 + (z_1 - h)^2\right]^{1/2} \qquad (4.63).$$

Fourier transforming equation (4.61) tells us that, approximately [again, see Appendix F],

$$\overline{\overline{E_{s_z}}} = \frac{I\,dl\,k^2}{I\omega\epsilon_0} e^{I\xi x_0} \frac{e^{(z_1 - h)u_0}}{2u_0} \qquad (4.64)$$

Therefore, as an approximation to equation (4.61) we have, in the far-field case,

$$\overline{\overline{D(x,y)}} = \frac{I\,dl\,k^2}{I\omega\epsilon_0} \frac{e^{I\xi x_0} e^{-hu_0}}{u_0 + \frac{u_1}{n_{01}^2}}.$$

By making the "surface-impedance" boundary condition approximation as given by equations (4.15), (4.16), & (4.17), and using the fact that

$$k^2 = \omega^2 \mu_0 \epsilon_0 ,$$

our previous expression becomes

$$\overline{\overline{D(x,y)}} = -I\omega\mu_0 I\,dl \frac{e^{I\xi x_0} e^{-hu_0}}{\sqrt{\xi^2 + \alpha^2} + \mu} \qquad (4.65).$$

With the aid of equation (4.60), equation (4.59) becomes

$$\overline{\overline{h_2 D(x,y)}} = \frac{1}{(2\pi)^2}\left\{\left[2\pi\delta(\eta)(\pi\delta(\xi) + \frac{1}{I\xi})\right] * \overline{\overline{D(x,y)}}\right\}.$$

Evaluating the delta function convolutions gives

$$\overline{\overline{[h_2 D(x,y)]}} = (1/2)\overline{\overline{D(x,y)}} + \frac{1}{2\pi i}\frac{1}{\xi} * \overline{\overline{D(x,y)}} \qquad (4.66)$$

Unfortunately the convolution in equation (4.66) cannot be evaluated exactly. In fact, we are forced to consider an aysmptotic expansion for the convolution integral which is valid for $x_0 \gg 1$ and in the limit as $h \to 0$: i.e. the source is a large distance from the interface and its height above ground tends to zero. Additionally, we assume that $|\mu|^2 \ll 1$ [see Appendix B], which certainly holds true for, say, seawater or soil of average conductivity. We find [see Appendix G] that, approximately,

$$\lim_{h \to 0} \overline{\overline{[h_2 D(x,y)]}}_- = \frac{\mu_0 \omega I\, dl}{2\pi} \left[\frac{2\pi\mu e^{-\alpha x_0}}{\sigma(\xi - i\alpha)} - \frac{\sqrt{2\pi}\, e^{-x_0 \alpha}}{\sqrt{x_0 \alpha}\,(\xi - i\alpha)} \right] \qquad (4.67)$$

Henceforth we shall tacitly assume that $\overline{\overline{[h_2 D(x,y)]}}_-$ has been approximated in the limit as the source height h tends to zero, and drop any explicit limit reference. As mentioned after equation (4.58), we must effect the additive decomposition as given in equation (4.11). In our case equation (4.11) reduces to

$$v_{1_+}(\xi)\, \overline{\overline{[h_2 D(x,y)]}}_-$$

$$= \frac{\mu_0 \omega I\, dl}{2\pi} \frac{(\xi + i\alpha)(\xi + ic)}{(\xi + i\tau)(\xi + id)} \left[\frac{2\pi\mu e^{-\alpha x_0}}{\sigma(\xi - i\alpha)} - \frac{\sqrt{2\pi}\, e^{-x_0 \alpha}}{\sqrt{x_0 \alpha}\,(\xi - i\alpha)} \right]$$

$$= f_+(\xi) + f_-(\xi). \qquad (4.68)$$

as can be seen from equations (4.57) and (4.67). The additive split in this last equation is relatively trivial because of the polynomial behaviour of the left hand side. We find that

$$v_{1_+}(\xi)\,\overline{[h_2 D(x,y)]}_- \tag{4.69}$$

$$= \Bigg\{ \frac{\mu\mu_0 \omega l\, dl\, e^{-\sigma x_0}}{\sigma(\xi - i\sigma)} \left[\frac{(\xi + i\sigma)(\xi + ic)}{(\xi + i\tau)(\xi + id)} - \frac{2\sigma(\sigma+c)}{(\sigma+\tau)(\sigma+d)}\right]$$

$$- \frac{\mu_0 \omega l\, dle^{-x_0\alpha}}{\sqrt{2\pi x_0}\,\alpha(\xi - i\alpha)} \left[\frac{(\xi + i\sigma)(\xi + ic)}{(\xi + i\tau)(\xi + id)} - \frac{(\alpha+\sigma)(\alpha+c)}{(\alpha+\tau)(\alpha+d)}\right] \Bigg\}_+$$

$$+ \Bigg\{ \frac{2\mu\mu_0 \omega l\, dle^{-\sigma x_0}}{(\xi - i\sigma)} \left[\frac{(\sigma+c)}{(\sigma+\tau)(\sigma+d)}\right] - \frac{\mu_0 \omega l\, dle^{-x_0\alpha}}{\sqrt{2\pi x_0}\,\alpha(\xi - i\alpha)} \left[\frac{(\alpha+\sigma)(\alpha+c)}{(\alpha+\tau)(\alpha+d)}\right] \Bigg\}_-$$

Comparing equations (4.68) and (4.69) we readily identify

$$f_-(\xi) = \frac{2\mu\mu_0 \omega l\, dle^{-\sigma x_0}}{(\xi - i\sigma)} \frac{(\sigma+c)}{(\sigma+\tau)(\sigma+d)}$$

$$- \frac{\mu_0 \omega l\, dle^{-x_0\alpha}}{\sqrt{2\pi x_0}\,\alpha(\xi - i\alpha)} \frac{(\alpha+\sigma)(\alpha+c)}{(\alpha+\tau)(\alpha+d)} \tag{4.70}$$

Therefore, from equation (4.57) together with that immediately preceeding equation (4.13), we find that

$$\overline{[h_2 E_z^+]}_- = 2\mu\mu_0 \omega l\, dl \frac{e^{-\sigma x_0}(\xi - ic)(\sigma+c)}{(\xi - i\tau)(\xi - id)(\sigma+\tau)(\sigma+d)}$$

$$-\frac{\mu_0 \omega i\, dl}{\sqrt{2\pi x_0}} \frac{e^{-x_0 \alpha}(\xi - i\sigma)(\xi - ic)(\alpha+\sigma)(\alpha+c)}{(\xi - i\alpha)(\xi - i\tau)(\xi - id)(\alpha+\tau)(\alpha+d)\sqrt{\alpha}} \tag{4.71}$$

wherein, from equations (4.21), (B.10), (B.8), (4.53), and (4.55) we have

$$\begin{aligned} \alpha^2 &= \eta^2 - k^2 \\ \sigma^2 &= \alpha^2 - \mu^2 \\ \tau^2 &= \alpha^2 - \nu^2 \\ c^2 &= 2\alpha(\alpha - \nu) \\ d^2 &= 2\alpha(\alpha - \mu) \end{aligned} \tag{4.72}$$

Equation (4.71) must now be subjected to a two dimensional inverse Fourier transform to give $h_2 E_z^+$. This has been attempted in Appendix H, under several assumptions. It has been assumed, consistent with the working so far, that the quantities $|\mu|$ and $|\nu|$ are both small compared with unity. Moreover, as was assumed in Appendix G, the distance of the source from the interface, x_0, was considered to be large compared with unity. With these assumptions we arrive at the most significant terms in the asymptotic expansion of the inverse two dimensional Fourier transform as being:

$$h_2 E_z^+ = \frac{k^2 i\, dl}{2\pi i \omega \epsilon_0} \frac{e^{-ik\rho}}{\rho}$$

$$\times \left\{ \frac{(1+2\nu i/k)}{1+2\mu i/k} \left[c_2 (\sqrt{\rho} - \frac{1}{2c_1^2 k \sqrt{\rho}}] - c_2 \frac{\sqrt{\rho}}{\sqrt{x_0}} [1 - \frac{1}{2c_1^2 k \rho}] \right] \right.$$

$$- \frac{(\mu - \nu)}{c_1^4 k^2 \sqrt{\rho}} [c_2 - \frac{c_3}{\sqrt{x_0}}] \Bigg] \qquad (4.73).$$

In equation (4.73) we have used the quantities defined by [c.f. equations (H.39), (H.40), & (H.41), respectively]

$$c_1 = l[1 + 2\mu l/k] \qquad (4.75),$$

$$c_2 = \frac{2\sqrt{2}e^{i\pi/4}\mu\sqrt{\pi}}{\sqrt{k}[1 + 2\mu l/k]} \qquad (4.76),$$

and

$$c_3 = \frac{2(1 + \mu^2/\nu^2)}{l(1 + 2\mu l/k)} \qquad (4.77).$$

Additionally, the variable ρ and the constant x_0 are the distances, respectively, of the observation point and the interface from the source, as shown in diagram 4.2.

The term outside the brackets ( ) in equation (4.73) represents the free space field of the elementary electric dipole of length dl, carrying current I [see Appendix F]. It is immediately seen that the asymptotic expansion in equation (4.73) is of the wrong "order" in the leading terms in ρ, as compared with any of the analyses presented in Chapter 2. Indeed, since the leading terms is of order $1/\sqrt{\rho}$, this would seemingly indicate the far field of a line source [cf. the leading term in the asymptotic expansion of the Hankel function $H_0^2(k\rho)$]. Some disagreement in the "constants" between ours and the other analyses is perhaps to be expected in view of the approximations introduced in our Wiener-Hopf procedure. The presence of the term $\sqrt{\rho}/\sqrt{x_0}$ guarantees that we have some semblance of a "recovery effect", although not in the same explicit form as, for example, in Bremmer's work.

The conclusions and suggestions for possible modification are discussed in the next chapter.

CHAPTER 5: SUMMARY AND CONCLUSIONS

Summary

The analysis carried out in chapter 3, which was later verified independently by Walsh [30] is representative of a new approach to electromagnetic boundary-value problems. By describing the physical geometry, the electrical properties of the media, and the electromagnetic fields themselves in terms of the step functions introduced in equation (3.1), we have bypassed the use of the indirect Hertz potential normally used in the classical analyses. Considering the Maxwell equations in a generalized-functional sense, we operated on the various "decomposed" quantities. The result was that the boundary conditions normally supplied in the classical approaches were presented to us as a by-product of the analysis.

The differential equations obtained for the electric fields in the three regions were convolved with the appropriate "Green functions" (see equations (3.111)) to obtain three convolution equations for the various regional electric fields (equations (3.126) through (3.128)). In this working convenient use was made of the property of the convolution operator that permits differentiation to be shifted from either of the convolved quantities to the other. The three equations so obtained expressed the regional electric fields as convolutions of the appropriate Green functions with functions of the surface fields present on the boundaries of the region, together with any sources that may exist. This is equivalent to invoking Green's theorem in the classical analyses.

A considerable simplification was made by neglecting the surface fields on the vertical sides of the interface. For media such as earth and sea-water, with sufficient conductivities, this assumption is physically reasonable on the grounds that the penetration depth is very small at the frequencies of interest.

The boundary equation was broken down into its various "support" equations, and from these we substituted for terms in the regional electric field equations so that the functions of the surface fields in the resulting equations had their support only on the upper side of the horizontal interface. Additionally, we focussed our attention on the z-component of the electric field since this is the predominant propagation component in the case of a vertical elementary dipole source.

A two dimensional x-y Fourier transform of the three regional electric field equations was taken. From a consideration of the form of the resulting equations on various fixed horizontal planes, either above or below the interface, we were able to eliminate the "total" electric field terms entirely and obtain an equation (see equation (3.175)) involving only the Fourier transforms of both the source term and the z-components of the surface electric fields on either side of the vertical interface. This equation was manipulated to give, finally, an equation (see equation (3.185)) in the transform domain for the z-component of the surface electric field to the right of the vertical interface.

We were thus faced with the solution of an integral equation for the relevant surface electric field component. Having arrived at the same point independently. Walsh [30] elected to solve the equation using first the technique of operator expansion by Neumann series, followed by an inverse Fourier transform. Walsh was able to sum the terms in his series, and obtained a result (see equation (2.70)) which is equivalent to that of Bremmer (see equations (2.31) or (2.32)). As such, the operator expansion approach would seem to have been better suited to the problem than the one chosen here for the solution of the integral equation.

The discontinuity of the media in the x-direction, along with the ensuing use of the multiplier functions h_2 and $(1-h_2)$ (see equation (3.2)), suggested the Wiener-Hopf method of solution in the ξ domain (wherein ξ is the x Fourier transform variable). In order to use this method we had two Wiener-Hopf decompositions to effect. The first was the decomposition of the function $\overline{\overline{V(x,y)}}$ into the quotient of the two functions $v_-(\xi)$ and $v_+(\xi)$ (see equation (4.9)), wherein the subscripts "-" and "+" denoted analyticity and freedom of zeros, respectively, in the lower and upper complex ξ half-planes. The next step was the decomposition of the function $v_+(\xi)\overline{\overline{[h_2 D(x,y)]}}_-$ into the sum of two functions $f_+(\xi) + f_-(\xi)$. This having been accomplished, we should have arrived at an algebraic equation for the Fourier transform of the z-component of the surface electric field to the right of the vertical interface. This equation could then have been subjected to an inverse Fourier transform to obtain the relevant spatial domain surface field component (see equation (4.13)).

In order to simplify the function $\overline{\overline{V(x,y)}}$ somewhat we made the "surface impedance" boundary-condition assumptions that are well established in the classical literature (see Wait [28] for a more complete discussion). This allowed us to replace the terms u_2/n_{02}^2 and u_1/n_{01}^2 in equation (4.14) (and as defined in equations (4.15)) by their values when $\xi^2 + \eta^2 = k^2$; i.e. by the quantities ν and μ respectively, as defined through equations (4.19) and (4.18). $\overline{\overline{V(x,y)}}$ was thus approximated by the expression (4.20).

The Wiener-Hopf decompositions, as was stated, were to be carried out in the ξ-plane so that during the decomposition procedure the y Fourier transform variable, η, was to be considered as arbitrary but fixed. The form of $\overline{\overline{V(x,y)}}$ to be decomposed was thus as given in equation (4.22). This form was then broken down into the product of the two functions $R(\xi)$ and $S(\xi)$, as given in equations (4.24) and (4.25), and the obvious "inverse symmetry" between these functions was noted. Thus, we concentrated on the decomposition of the function $R(\xi)$ into the quitient of two functions $r_-(\xi)$ and $r_+(\xi)$ as given in equation (4.30), since the decomposition of $S(\xi)$ could then be written down by inspection.

Taking the logarithm of equation (4.30), we were able to utilize the standard Wiener-Hopf Cauchy integral formulas, for the additive decomposition of a function, to obtain integral relations for the functions $log[r_-(\xi)]$ and $log[r_+(\xi)]$ (as given in equations (4.31) and (4.32), respectively). In order to evaluate these integrals we resorted to the usual trick of differentiating under the integral sign, to obtain integral expressions for $d\,log[r_+(\xi)]/d\xi$ and $d\,log[r_-(\xi)]/d\xi$. The resulting integrals turned out to be tractable, although a certain amount of caution had to be exercised in their evaluation (see Appendix [C]). This left us with the expressions as given in equations (4.42) and (4.43) for the quantities $d\,\log[r_-(\xi)]/d\xi$ and $d\,\log[r_+(\xi)]/d\xi$. At this stage we were able to verify that the difference of these two expressions did indeed give us $d\,\log[R(\xi)]/d\xi$, thus providing a useful check.

Unfortunately, the full integration of equations (4.42) and (4.43) proved to be impossible, and we were left with equations (4.48) and (4.49), giving us, respectively, expressions for $\log[r_-(\xi)]$ and $\log[r_+(\xi)]$ that involved the "same" intractable integral. However, at this stage we were again able to verify that the difference of the two expressions for $\log[r_-(\xi)]$ and $\log[r_+(\xi)]$ agreed with $\log[R(\xi)]$, thus again ensuring that we hadn't erred so far.

It was at this apparent impasse that we resorted to the method of approximating the function $R(\xi)$ to be decomposed by another function $R_1(\xi)$ whose inverse Fourier transform possessed the same large scale analytical features as that of $R(\xi)$ (i.e. the same singularity at the origin in the x-domain, same area under the x-axis, and same second moment), but whose Wiener-Hopf decomposition could be effected much more readily. This approximation technique has been used successfully in certain homogeneous Wiener-Hopf problems (see Carrier, Krook, and Pearson [3], or Noble [18]). The result was an approximation of the function $\overline{\overline{V(x,y)}}$ by the function $\overline{\overline{V_1(x,y)}}$, whose decomposition into the quotient of the two functions $v_{1_-}(\xi)$ and $v_{1_+}(\xi)$ was accomplished by inspection, and given by equation (4.57).

The Wiener-Hopf equation for the Fourier transform of the z-component of the surface electric field to the right of the vertical interface thus became as given in equation (4.58). As mentioned above, the next stage of the solution involved the additive decomposition of the term $v_{1_+}(\xi)\overline{[h_2D(x,y)]}_-$ into the sum of two functions $f_+(\xi)$ and $f_-(\xi)$. It was first required to determine $\overline{[h_2D(x,y)]}_-$ exactly. Exploiting various properties of the Fourier transform, we were able to obtain an expression for this quantity that included a convolution integral containing the function $\overline{D(x,y)}$ (see equation (4.59)). This function, defined in equation (4.61), contains the Fourier transform of the z-component of the source field.

In order to approximately evaluate the convolution integral in the expression for $\overline{[h_2D(x,y)]}_-$ several assumptions were made. It was assumed that the source, an elementary electric dipole, was located on the earth's surface in medium 1 at a large distance x_0 from the vertical interface, and also that the quantity $|\mu|$ is very small (corresponding to the assumption that the magnitude of the refractive index in medium 1 is large). These assumptions allowed us to take the first term of the asymptotic expansion of an otherwise intractable part of the convolution integral in question (see Appendix [G]). This left us with the approximation for the convolution integral as given in equation (4.67).

Because of the polynomial characteristics of the components of $v_{1_+}(\xi)\overline{\overline{[h_2 D(x,y)]}}_-$, its additive decomposition into the sum of the functions $f_+(\xi)$ and $f_-(\xi)$ was straightforward (see equation (4.69)). We were thus left with an approximate expression for $\overline{\overline{[h_2 E_z^+]}}_-$ (see equation (4.71)). This expression consisted of quotients of products of monomials in the transform variable ξ, with various functions of the transform variable η appearing as "constants".

The final stage of the solution involved applying an inverse x-y Fourier transform to the expression for $\overline{\overline{[h_2 E_z^+]}}_-$. The integration with respect to ξ was trivial, in view of the simplicity of the integrand when considered as a function of ξ. However, the resulting integration with respect to the variable η was exceedingly complicated, since most of the "constants" appearing in the integrand were in fact multivalued functions of η.

In order to obtain an approximation for the resulting integrals, we were forced to make some approximations in the integrands. Assuming that the quantities $|\mu|$ and $|\nu|$ were both large, we neglected $O(\mu^2)$ and $O(\nu^2)$ terms in the integrands. Additionally, we observed that, in the approximation of $\overline{\overline{V(x,y)}}$ by $\overline{\overline{V_1(x,y)}}$, certain spurious poles and zeros (the location of which was a function of the transform variable η) had crept in, so we neglected their contribution in the light of some order-of-magnitude calculations based on the assumption that x_0 was large.

The remaining two integrals were amenable to solution by the classical steepest-descent method. Changing to polar coordinates in both the x-y spatial and $\xi-\eta$ transform domains we deformed the common integration contour in both integrals so that it lay on the steepest descent path. We observed that, provided the spatial polar coordinate θ was not close to $\pm\pi/2$, no poles of the integrand should be crossed by such a contour deformation, nor should they lie close to the steepest descent path. Fixing θ to be zero (see diagram (4.2)), we applied the usual asymptotic expansion methods to the remaining integrals. We thus arrived at equation (4.73), giving us the most significant terms in an expansion for $h_2E_z^+$.

Conclusions

It may be concluded that the method of modelling electromagnetic propagation and scattering phenomena in the manner of this work constitutes a definite simplification as compared to the classical techniques. We have worked directly with the field quantities themselves and thus sidestepped the use of intermediate Hertz potentials. Operating on the field quantities, when they had been decomposed into the sums of their various support region fields, we obtained the standard boundary conditions of electromagnetism that are normally supplied in the classical analyses.

The "mixed-path" propagation problem tackled here broke down naturally into two stages:

(a) derivation of an equation in the transform domain for the z-component of the surface electric field to the right of the interface (i.e. equation (3.185)), and

(b) solution of the [integral] equation to obtain a spatial domain expression for the relevant field quantity (i.e. equation (4.73)).

Having obtained equation (3.185) we chose the Wiener-Hopf method of solution. This method suggested itself because of its [formal] elegance. In addition to the discontinuity of the media in the x direction and the inherent use of the step function multipliers h_2 and $(1-h_2)$, which corresponded to analyticity in either the lower or upper complex half-planes of the x transform variable, ξ. The ensuing analysis was somewhat involved, and the failure of the solution obtained (i.e. equation (4.73)) to agree with the classical results indicates that this problem is too complicated for conventional Wiener-Hopf analysis. That Walsh managed independently to obtain a "successful" solution of equation (3.185) using an entirely different approach convinces us that our analysis up to that point was correct.

We conclude that the chief source of the apparent "impotency" of the Wiener-Hopf method lay in our inability, using the conventional theory, to completely decompose the function $\overline{\overline{V(x,y)}}$, and the consequent approximation of this "accurate" function by the function $\overline{\overline{V_1(x,y)}}$. However, it seems somewhat doubtful, in view of the relative complexity of equations (4.48) and (4.49), that we could have managed the second [additive] decomposition of the function $v_{1_+}(\xi)\overline{\overline{[h_2D(x,y)]}}$, even if we'd derived the first intended decomposition of $\overline{\overline{V(x,y)}}$. Therefore, perhaps we could only have staved off eventual "large scale" approximation by one stage in the solution, although this may have been enough to guarantee a closer resemblance with classical results.

In Appendix G we obtained an approximation for the function $\overline{[h_2 D(x,y)]}_-$ consisting of two terms. The first term was "exact", coming from the contribution from a pole of the integrand of the convolution integral. The second was obtained as the most significant term from the asymptotic expansion of a branch-cut integral. We are thus reasonably certain of our approximation to $\overline{[h_2 D(x,y)]}_-$ and are convinced of the inadequacy of the function $\overline{V_1(x,y)}$ as an approximation to $\overline{V(x,y)}$ for the purposes of Wiener-Hopf decomposition.

It is to be remembered that, in our approximation of the function $\overline{V(x,y)}$, and indeed, throughout the course of our attempted Wiener-Hopf decompositions, the y transform variable η was considered as fixed. As such, no heed was paid to the analytic nature of either $\overline{V(x,y)}$ or of $\overline{[h_2 D(x,y)]}_-$ when considered as a function of η. As a result, the final solution was of the wrong order in the distance variable ρ [being, in fact, somewhat more characteristic of cylindrical wave propagation from an infinite line source]. It may be concluded that this is attributable to the integrand in the last inverse transform [see, for example, equation (H.4)] being of the wrong order in the transform variable η. It is suggested that, in future, in any attempt at a "one-dimensional" Wiener-Hopf decomposition of a function of two transform variables, particular attention be paid to the analytic behaviour of the "redundant" variable in the process.

As a further suggestion, the "surface impedance" boundary condition approximations that allowed us to write the function $\overline{\overline{V(x,y)}}$ as given in, for example, equation (4.20), needn't have been made. We might just as well have proceeded with the approximation of $\overline{\overline{V(x,y)}}$ as given more accurately in equation (4.14), for example, and left any implications of the nature of the refractive indices of the media for the final solution.

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APPENDIX A: Derivation of Equations (3.159), (3.160), & (3.171)

We shall focus our attention on the derivation of equation (3.171), since the other two equations can be derived by analogy.

It is possible to obtain equation (3.171), giving us an expression for $\overline{\overline{K_{00}}}$, by directly applying a two dimensional x–y Fourier transform to the quantity

$$K_{00}(x,y,z) = \frac{e^{-ikr}}{4\pi r}.$$

The procedure is clear: the x and y integration variables as well as the two transform variables are converted to pairs of plane polar coordinates. Of the two integrals, the one with respect to the angular variable yields a Bessel function of order zero. The remaining [radial] integral, with integration limits 0 and ∞, can be found from a table of integrals [for example, Gradshteyn & Ryzhik [11]].

However, we shall indicate how the same result can be derived via a somewhat more subtle method. From equation (3.112), we know that the function $K_{00}(x,y,z)$ satisfies the equation

$$\nabla^2 K_{00} + k^2 K_{00} = -\delta(x)\delta(y)\delta(z) \qquad \text{(A.1)},$$

wherein

$$K_{00}(x,y,z) = \frac{e^{-ikr}}{4\pi r} \qquad \text{(A.2)}.$$

Taking a three dimensional Fourier transform of equation (A.1) results in

$$\overline{\overline{K_{00}}}(\xi,\eta,\zeta) = \frac{1}{\zeta^2 + \eta^2 + \xi^2 - k^2} \qquad (A.3),$$

wherein $\xi, \eta,$ & ζ correspond, respectively, to the x, y, & z Fourier transform variables, as defined partially in equation (3.155). If we denote, say

$$u_0^2 = \xi^2 + \eta^2 - k^2 \qquad (A.4),$$

then applying an inverse Fourier transform from the ζ to the z domain results in

$$\overline{K_{00}}(\xi,\eta,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\zeta \frac{e^{i\zeta z}}{\zeta^2 + u_0^2} \qquad (A.5).$$

If we assume that $u_0^2 > 0$ then the integral in equation (A.5) is readily evaluated. If $z > 0$ we close the integration contour in the upper half-plane and capture the pole at $\zeta = iu_0$; if $z < 0$ we close the contour in the lower half-plane and capture the pole at $\zeta = -iu_0$. Hence, we find that

$$\overline{K_{00}}(\xi,\eta,z) = \frac{e^{-|z|u_0}}{2u_0}, \quad u_0^2 > 0 \qquad (D.6).$$

If $u_0^2 < 0$ the situation is far more complicated, and many of the questions connected with the solutions of the partial differential equation (A.1) are manifested.

To begin with, in the absence of any boundary conditions, it is clear that both $e^{-ikr}/(4\pi r)$ and $e^{ikr}/(4\pi r)$ are solutions of equation (A.1): the general solution will thus consist of a linear solution of these two functions. Therefore, our expression (A.5) for the Fourier transform of K_{00} wil contain "information" not only on $e^{-ikr}/(4\pi r)$, but also on $e^{+ikr}/(4\pi r)$ as well, since we have no *a priori* grounds for excluding the latter function.

Since $u_0^2 < 0$ it is clear that the integrand in (A.5) will have poles on the real axis. In order to evaluate the integral using the residue theorem in this case, the usual method is to consider the result when the poles lie off the real axis, and then take the limit as the poles tend to the real axis: i.e. the limit as the imaginary parts of the poles tend to zero. This method allows the real axis integration contour to be indented appropriately about the corresponding pole.

The two dimensional Fourier transform is [formally] defined for the transform variables ξ, η real. The limiting process described in the preceeding paragraph can be accomplished by allowing the wave number k to have either a small positive or negative imaginary part, and letting this part tend to zero. However, if we, say, choose k to have a small negative imaginary part [as we do in the remainder of the text to facilitate the Wiener-Hopf decomposition process], this is equivalent to applying a so-call "radiation-condition" boundary condition to the solutions of (A.1) in the sense that e^{ikr}/r will grow exponentially as $r \to \infty$ whereas e^{-ikr}/r will decay.

Additionally, our choice of branch of the multivalued square root function, for a given chosen time dependence, is intimately connected with the "choice" of solution of (A.1): either $e^{-ikr}/(4\pi r)$ or $e^{ikr}/(4\pi r)$. This will effect the value of the function u_0, which is in general multivalued in view of equation (A.4), and hence effect in turn the nature of the poles of the integrand in equation (A.5).

Without going into the manifest analytical details we quote the result again

$$\overline{\overline{K_{00}}}(\xi,\eta,z) = \frac{e^{-|z|u_0}}{2u_0} \qquad \text{(A.8)}$$

APPENDIX B: Properties of the Function $A(\xi)$

From equation (4.24) we find that

$$A(\xi) = \frac{\sqrt{\xi^2+\alpha^2} + \nu}{\sqrt{\xi^2+\alpha^2}} \tag{B.1}$$

wherein, from equation (4.21),

$$\alpha^2 = \eta^2 - k^2 \tag{B.2}$$

and, from equation (4.19),

$$\nu = \frac{\sqrt{k^2(1 - n_{02}^2)}}{n_{02}^2} \tag{B.3}$$

From equation (3.63) we have that

$$n_{02}^2 = \epsilon_{r_{02}} + \frac{\sigma_2}{i\omega\epsilon_0} \tag{B.4}$$

To obtain "order of magnitude" estimates on the size of these various constants, we shall assume that medium 2 consists of seawater, whose conductivity σ_2 is approximately 4 [mhos], and that the source frequency is 18 MegaHertz. For seawater we find that $\epsilon_{r_{02}}$ is about 80. Therefore, from (B.4), we find

$$n_{02}^2 \approx 80 - \frac{4i}{(2\pi)\times(1.8\times10^7)\times\left[\frac{10^{-9}}{36\pi}\right]} = 80 - 4000i \tag{B.5}$$

for seawater. That is to say, the magnitude of the imaginary part of n_{02}^2 is much greater than that of the real part. Thus, in the complex plane,

n_{02}^2 lies in the fourth quadrant close to the negative imaginary axis. Our convention in choosing square roots tells us then that n_{02} lies in the fourth quadrant at approximately $\pi/4$ radians [actually, slightly less than this] to the real axis. This situation may be depicted, although not to scale, by diagram B.1.

Now, in order to facilitate most of the subsequent analysis, we shall assume that k has a small negative imaginary part. In the final result we shall consider the limit as this imaginary part of k tends to zero. Considering equations (3.111), we see that this assumption corresponds to the assumption that the Green functions K_{00}, K_{01}, and K_{02} "die out" [if ever so slightly] as $r \to \infty$.

Since, by definition,

$$k^2 = \left[\frac{2\pi}{\lambda}\right]^2$$

we see that, for 18 MHz waves,

$$k \approx 0.4$$

Since η is the y Fourier transform variable, we know that it varies [formally] along the real axis from $-\infty$ to ∞. Hence, η^2 lies along the real axis between the origin and $+\infty$.

In view of the preceeding assumptions and discussion, equation (B.2) tells us that α^2 lies in the upper complex half-plane, off the real axis. Thus, because of our choice of the branch of the multivalued square root function, α lies in the first quadrant, off the real axis. These results are depicted in diagram B.2, although not to scale.

It is worthwhile to consider the effect of our assumption that k has a [small] negative imaginary part. Had we not made this assumption, then α^2 would lie entirely on the real axis between $-k^2$ and $+\infty$; and so would vary along part of the positive imaginary and all of the positive real axes. Our assumption has thus "pushed α off the fence" so that it lies definitely within the first quadrant for all values of η. We shall see that this will facilitate our Wiener-Hopf decomposition.

Since, for seawater, n_{02}^2 has a fairly large magnitude, we may write equation (B.3) as

$$\nu = \frac{\left[-k^2 n_{02}^2 (1 - 1/n_{02}^2)\right]^{1/2}}{n_{02}^2}$$

so that

$$\nu \approx \frac{\sqrt{-k^2 n_{02}^2}}{n_{02}^2}$$

From diagram B.1 and our choice of branch of the square root function, we find that

$$\nu \approx \frac{ikn_{02}}{n_{02}^2} = \frac{ik}{n_{02}} \qquad \text{(B.6)}$$

Information from diagrams B.1 and B.2 may be combined to give diagram B.3 [again, not to scale though], revealing that ν has an approximate phase of $3\pi/4$. Moreover, since n_{02} is relatively large [at least O(10)] and k is about 0.4 at 18 MHz, we find that ν is disturbingly close to the origin. This information is most useful in the asymptotic analysis of results to come.

We are now in a position to discuss the analytic properties of $R(\xi)$. From (B.1) it is evident that $R(\xi)$ has branch-point singularities at the points $\xi = \pm i\alpha$. The zeros of $R(\xi)$, however, are more elusive. We recall that we have chosen that branch of the multivalued square root function such that $\sqrt{\xi^2+\alpha^2} \to +\infty$ as $\xi \to \pm\infty$ on the real axis. This is equivalent to saying that we are on that sheet of the Riemann surface of the function $\sqrt{\xi^2+\alpha^2}$ for which $|arg[\sqrt{\xi^2+\alpha^2}]| < \pi/2$, i.e. we're choosing the square root with the **positive** real part.

Now, let's consider

$$R(\xi) = \frac{\sqrt{\xi^2+\alpha^2} + \nu}{\sqrt{\xi^2+\alpha^2}}.$$

We shall select branch cuts as shown in diagram (B.4). From diagram (B.3) we see that $-\nu$ will lie in the fourth quadrant, with a phase angle of between $-\pi/4$ [perfect conductor] and $-\pi/2$ [perfect insulator]. From the preceeding paragraph we realise that we may write

$$\sqrt{\xi^2+\alpha^2} = \exp\Big[(1/2)\log|\xi + i\alpha| + (1/2)\log|\xi - i\alpha| + (i/2)\arg|\xi + i\alpha| + (i/2)\arg|x - i\alpha|\Big].$$

The lengths and phase angles of the "vectors" [complex numbers] $\xi + i\alpha$ and $\xi - i\alpha$ are shown in diagram (B.4), along with the approximate positions of ν and $-\nu$, although not to scale. From the diagram we realise that, provided $\alpha \neq 0$, then $R(\xi)$ will have two zeros on "our" sheet of the Riemann surface, at the points

$$\xi = \pm i\tau \tag{B.7}$$

as shown approximately in the diagram, wherein

$$\tau^2 = \alpha^2 - \nu^2 \tag{B.8}$$

Since $\alpha^2 = \eta^2 - k^2$, and k was assumed initially to have a small negative imaginary part, α will always be non-zero, because the transform variable η is (formally) defined to lie on the real axis. In any case, subsequent asymptotic analysis of the Fourier inversion integral will reveal that the significant contribution comes from the immediate vicinity of $\eta = 0$, at least for the physical geometry to which we shall restrict ourselves. Thus, our analytic claims will be justified *a posteriori*.

From equation (4.17) we see that if the magnitude of n_{01}^2 is sufficiently large, we may make the approximation

$$\mu \approx \frac{ik}{n_{01}} \tag{B.9}$$

analogous to equation (B.6). For example, if medium 1 is chosen to be earth, the magnitude of the refractive index will be large enough so that (B.9) will be a reasonable approximation.

APPENDIX C: Derivation of Equations (4.42) & (4.43)

Equation (4.40) tells us that

$$\frac{d}{d\lambda}(\log[r_-(\lambda)]) = \frac{1}{2\pi i}\int_{\Gamma_1} d\xi \frac{\xi}{(\xi-\lambda)}\frac{1}{(\xi^2+\alpha^2)} - \frac{1}{2\pi i}\int_{\Gamma_1}\frac{d\xi}{(\xi-\lambda)}\frac{\xi}{(\xi^2+\tau^2)}$$

$$+ \frac{\nu}{2\pi i}\int_{\Gamma_1}\frac{d\xi}{(\xi-\lambda)}\frac{\xi}{(\xi^2+\tau^2)\sqrt{\xi^2+\alpha^2}} \qquad (C.1),$$

wherein Γ_1 is the contour as shown in diagram 4.1, and λ is a point lying below the contour. Equation (C.1) may be written

$$\frac{d}{d\lambda}(\log[r_-(\lambda)]) = \frac{1}{2\pi i}I_1 - \frac{1}{2\pi i}I_2 + \frac{\nu}{2\pi i}I_3 \qquad (C.2),$$

wherein

$$I_1 = \int_{\Gamma_1} d\xi \frac{\xi}{(\xi-\lambda)}\frac{1}{(\xi^2+\alpha^2)} \qquad (C.3),$$

$$I_2 = \int_{\Gamma_1} d\xi \frac{\xi}{(\xi-\lambda)}\frac{1}{(\xi^2+\tau^2)} \qquad (C.4),$$

and

$$I_3 = \int_{\Gamma_1}\frac{d\xi}{(\xi-\lambda)}\frac{\xi}{(\xi^2+\tau^2)\sqrt{\xi^2+\alpha^2}} \qquad (C.5)$$

I_3 as given in equation (C.5) may be further decomposed as

$$I_3 = (1/2)[J_1 + J_2] \qquad (C.6),$$

wherein

$$J_1 = \int_{\Gamma_1}\frac{d\xi}{(\xi-\lambda)}\frac{1}{(\xi+i\tau)\sqrt{\xi^2+\alpha^2}} \qquad (C.7).$$

and

$$J_2 = \int_{\Gamma_1} \frac{d\xi}{(\xi - \lambda)} \frac{1}{(\xi - i\tau)\sqrt{\xi^2+\alpha^2}} \qquad (C.8).$$

Furthermore, we have the decompositions

$$J_1 = \frac{1}{\lambda + i\tau}[K_1 - L_1] \qquad (C.9),$$

$$J_2 = \frac{1}{\lambda - i\tau}[K_2 - L_2] \qquad (C.10),$$

wherein

$$K_1 = \int_{\Gamma_1} \frac{d\xi}{(\xi - \lambda)} \frac{1}{\sqrt{\xi^2+\alpha^2}} \qquad (C.11),$$

$$L_1 = \int_{\Gamma_1} \frac{d\xi}{(\xi + i\tau)} \frac{1}{\sqrt{\xi^2+\alpha^2}} \qquad (C.12),$$

$$K_2 = \int_{\Gamma_1} \frac{d\xi}{(\xi - \lambda)} \frac{1}{\sqrt{\xi^2+\alpha^2}} = K_1 \qquad (C.13),$$

and

$$L_2 = \int_{\Gamma_1} \frac{d\xi}{(\xi - i\tau)} \frac{1}{\sqrt{\xi^2+\alpha^2}} \qquad (C.14).$$

The integration contour Γ_1 is shown in diagram C.1. We may evaluate each of the integrals given in (C.3), (C.4), and (C.11) through (C.14) by closing the integration contour in the upper half-plane. In this event, the pole at $\xi = i\tau$ may be captured.

The contribution to each of the preceding integrals from the semicircular portion of the integration contour tends to zero as the radius of the semicircle tends to infinty. L_2 is the only integral among those given in equations (C.11) through (C.14) which captures the pole at $\xi = i\tau$. Considering equation (C.14) and diagram C.1 we see that

$$L_2 = \frac{2\pi i}{\sqrt{\alpha^2 - \tau^2}} - \left[-\int_{i\infty}^{i\alpha} \frac{d\xi}{(\xi - i\tau)} \frac{1}{\sqrt{\xi^2 + \alpha^2}} + \int_{i\alpha}^{i\infty} \frac{d\xi}{(\xi - i\tau)} \frac{1}{\sqrt{\xi^2 + \alpha^2}} \right] \quad (C.15).$$

In writing equation (C.15), we of course recognise that the phase of $\sqrt{\xi^2 + \alpha^2}$ changes as we swing about the branch point $\xi = i\alpha$. From equation (4.39) we may simplify the expression for L_2 as given in equation (C.15) somewhat to show that

$$L_2 = \frac{2\pi i}{\nu} - 2\int_{i\alpha}^{i\infty} \frac{d\xi}{(\xi - i\tau)} \frac{1}{\sqrt{\xi^2 + \alpha^2}} \quad (C.16);$$

that is to say

$$L_2 = \frac{2\pi i}{\nu} - 2M_2 \quad (C.17),$$

wherein

$$M_2 = \int_{i\alpha}^{i\infty} \frac{d\xi}{(\xi - i\tau)} \frac{1}{\sqrt{\xi^2 + \alpha^2}} \quad (C.18).$$

To evaluate the integral M_2 we make the change of variable

$$\xi = i\alpha \sin\theta \quad (C.19).$$

whereupon equation (C. 18) becomes

$$M_2 = i\alpha \int_{\pi/2}^{\pi/2+i\infty} d\theta \frac{\cos\theta}{i\alpha\sin\theta - i\tau} \frac{1}{\sqrt{\alpha^2 - \alpha^2\sin^2\theta}} .$$

Therefore we have that

$$M_2 = \frac{1}{\alpha} \int_{\pi/2}^{\pi/2+i\infty} \frac{d\theta}{\sin\theta - \sin\psi} \qquad (C.20).$$

wherein

$$\sin\psi = \tau/\alpha \qquad (C.21).$$

The integration contour in equation (C. 20) is shown in diagram C. 2. Now, M_2 may be rewritten

$$M_2 = \frac{1}{2\alpha} \int_{\pi/2}^{\pi/2+i\infty} \frac{d\theta}{\cos\left[\frac{\theta+\psi}{2}\right] \sin\left[\frac{\theta-\psi}{2}\right]} \qquad (C.22).$$

We see that

$$\frac{d}{d\theta}\log\left[\frac{\cos\left[\frac{\theta+\psi}{2}\right]}{\sin\left[\frac{\theta-\psi}{2}\right]}\right]^{1} = \frac{-1/2\cos\psi}{\cos\left[\frac{\theta+\psi}{2}\right]\sin\left[\frac{\theta-\psi}{2}\right]} \qquad (C.23)$$

so that

$$M_2 = -\frac{1}{\alpha\cos\psi}\left[\log\left[\frac{\cos\left[\frac{\theta+\psi}{2}\right]}{\sin\left[\frac{\theta-\psi}{2}\right]}\right]\right]_{\pi/2}^{\pi/2+i\infty} \qquad (C.24).$$

At the upper limit,

$$\log\left[\frac{\cos\left[\frac{\theta+\psi}{2}\right]}{\sin\left[\frac{\theta-\psi}{2}\right]}\right]^{1}\Bigg|_{\theta=\pi/2+i\infty} = \log(-ie^{-i\psi}) .$$

and at the lower limit

$$\log\left\{\frac{\cos\left[\frac{\theta+\psi}{2}\right]}{\sin\left[\frac{\theta-\psi}{2}\right]}\right\}\Bigg|_{\theta=\pi/2} = \log(1) \;.$$

so that

$$\log\left\{\frac{\cos\left[\frac{\theta+\psi}{2}\right]}{\sin\left[\frac{\theta-\psi}{2}\right]}\right\}\Bigg|_{\pi/2}^{\pi/2+i\infty} = \log(-ie^{-i\psi}) \qquad \text{(C. 25)}$$

To determine the correct branch of the log function in equation (C. 25) we consider the special case when $\psi=0$. In this event

$$\log\left\{\frac{\cos\left[\frac{\theta+\psi}{2}\right]}{\sin\left[\frac{\theta-\psi}{2}\right]}\right\}\Bigg|_{\pi/2}^{\pi/2+i\infty} \to \Big[\log(\cot\theta/2)\Big]_{\pi/2}^{\pi/2+i\infty} \;.$$

We find that

$$\Big[\log(\cot\theta/2)\Big]_{\pi/2}^{\pi/2+i\infty} = \log\left[\frac{1-i}{1+i}\right] + \log(1) \;,$$

that is:

$$\Big[\log(\cot\theta/2)\Big]_{\pi/2}^{\pi/2+i\infty} = \log[e^{-\pi/2}] = \frac{i\pi}{2} \qquad \text{(C. 26)}.$$

Thus, in equation (C. 25) we know that

$$\lim_{\psi\to 0}\left[\log(-ie^{-i\psi}\right] = \frac{i\pi}{2} \;,$$

which suffices to fix the correct branch of the logarithm in equation (C. 25). We have then that

$$\log\left\{\frac{\cos\left[\frac{\theta+\psi}{2}\right]}{\sin\left[\frac{\theta-\psi}{2}\right]}\right\}\Bigg]_{\pi/2}^{\pi/2+i\infty} = -\frac{i\pi}{2}$$

$$= -i(\pi/2 + \psi) \qquad (C.27),$$

so that equation (C.24) becomes

$$M_2 = \frac{i(\pi/2 + \psi)}{\alpha\cos\psi} \qquad (C.28),$$

and hence, from equation (C.17),

$$L_2 = \frac{2\pi i}{\nu} - \frac{2i(\pi/2 + \psi)}{\alpha\cos\psi} \qquad (C.29),$$

wherein ψ is as given in equation (C.21).

The evaluation of L_1, as given in equation (C.21) parallels the above analysis. Closing the contour in the upper half-plane, we do not capture any pole. Hence, analogously to equation (C.18) we find

$$L_1 = -2\int_{i\alpha}^{i\infty}\frac{d\xi}{(\xi + i\tau)}\frac{1}{\sqrt{\xi^2+\alpha^2}} .$$

Making the same changes of variable as in equations (C.19) and (C.21) we obtain

$$L_1 = -\frac{2}{\alpha}\int_{\pi/2}^{\pi/2+i\infty}\frac{d\theta}{\sin\theta + \sin\psi} \qquad (C.30),$$

which may hence be written

$$L_1 = -\frac{1}{\alpha}\int_{\pi/2}^{\pi/2+i\infty}\frac{d\theta}{\sin\left[\frac{\theta+\psi}{2}\right]\cos\left[\frac{\theta-\psi}{2}\right]} .$$

On noting that

$$\frac{d}{d\theta}\log\left\{\frac{\sin\left[\frac{\theta+\psi}{2}\right]}{\cos\left[\frac{\theta-\psi}{2}\right]}\right\} = \frac{\cos\psi}{2\sin\left[\frac{\theta+\psi}{2}\right]\cos\left[\frac{\theta-\psi}{2}\right]} .$$

our expression for L_1 becomes

$$L_1 = \frac{2}{\alpha\cos\psi}\left[\log\left\{\frac{\sin\left[\frac{\theta+\psi}{2}\right]}{\cos\left[\frac{\theta-\psi}{2}\right]}\right\}\right]_{\pi/2}^{\pi/2+i\infty}$$

Evaluating the expression on the right-hand side of this last equation gives us, analogously to equation (C.25),

$$L_1 = -\frac{2}{\alpha\cos\psi}\log[ie^{-i\psi}] .$$

As before [c.f. steps leading from equation (C.25) to (C.27)] we can determine the correct branch of the log function occurring in this last expression. Our result is

$$L_1 = \frac{-2i[\pi/2 - \psi]}{\alpha\cos\psi} \qquad \text{(C.31)}.$$

wherein ψ is as given in equation (C.21).

In order that we may determine I_3 as given in equation (C.6), we have remaining the integral for K_1 as given in equation (C.11). Again, closing the contour in the upper half-plane captures no pole and we find that

$$K_1 = -2\int_{\alpha}^{i\infty}\frac{d\xi}{(\xi-\lambda)}\frac{1}{\sqrt{\xi^2+\alpha^2}} .$$

Making the substitution as given in (C. 19) leads us to

$$K_1 = -\frac{2}{\alpha}\int_{\pi/2}^{\pi/2+i\infty}\frac{d\theta}{\sin\theta - \lambda/i\alpha}$$

so that by defining

$$\sin\phi = i\lambda/\alpha \tag{C. 32}$$

we should obtain

$$K_1 = -\frac{2}{\alpha}\int_{\pi/2}^{\pi/2+i\infty}\frac{d\theta}{\sin\theta + \sin\phi}.$$

This integral expression for K_1 may be evaluated like that in equation (C. 30), with ϕ replacing ψ. Hence, from (C. 31) and this last result we obtain

$$K_1 = \frac{-2i[\pi/2 - \phi]}{\alpha\cos\phi} \tag{C. 33}$$

wherein ϕ is as given in equation (C. 32).

The results displayed in equations (C. 29), (C. 31), and (C. 33) may be employed in equations (C. 9) and (C. 10) to give

$$J_1 = \frac{1}{\lambda + i\tau}\left\{\frac{-2i[\pi/2-\phi]}{\alpha\cos\phi} + \frac{2i[\pi/2-\psi]}{\alpha\cos\phi}\right\}.$$

and

$$J_2 = \frac{1}{\lambda - i\tau}\left\{\frac{-2i[\pi/2-\phi]}{\alpha\cos\phi} + \frac{2\pi i}{\nu} + \frac{2i[\pi/2+\psi]}{\alpha\cos\psi}\right\}$$

respectively. Thus, our expression for I_3 from equation (C. 6) becomes

$$I_3 = \frac{\pi i}{\nu(\lambda - i\tau)} - \frac{2i\lambda(\pi/2-\phi)}{(\lambda^2+\tau^2)\alpha\cos\phi}$$

$$+ \frac{i\pi\lambda}{(\lambda^2+\tau^2)\alpha\cos\psi} - \frac{2\psi\tau}{\alpha\cos\psi(\lambda^2+\tau^2)} \tag{C. 34}$$

We may solve for $\cos\psi$ and $\cos\phi$ from equations (C. 21) and (C. 32), respectively, to rewrite equation (C. 34) as

$$I_3 = \frac{\pi i}{\nu(\lambda - i\tau)} - \frac{2i\lambda(\pi/2-\phi)}{(\lambda^2+\tau^2)\sqrt{\lambda^2+\alpha^2}}$$

$$+ \frac{2i(\lambda\pi/2+\psi i\tau)}{(\lambda^2+\tau^2)\sqrt{\alpha^2-\tau^2}} ,$$

which, with the aid of equation (4. 39), becomes

$$I_3 = \frac{\pi i}{\nu(\lambda - i\tau)} - \frac{2i\lambda(\pi/2-\phi)}{(\lambda^2+\tau^2)\sqrt{\lambda^2+\alpha^2}}$$

$$- \frac{2i(\lambda\pi/2+\psi i\tau)}{\nu(\lambda^2+\tau^2)} \tag{C. 35}$$

We have left the integrals for the terms I_1 and I_2 as given, respectively, in equations (C. 3) and (C. 4). Since λ is a point below the contour Γ_1 we shall evaluate both integrals by closing the contour in the upper half-plane as shown in diagram (C. 3).

From equations (C. 3) and (C. 4) we may thus write

$$I_1 = \frac{\pi i}{(i\alpha - \lambda)} \tag{C. 36}$$

and

$$I_2 = \frac{\pi i}{(i\tau - \lambda)} \tag{C. 37}$$

Gathering the results arrayed in equations (C. 35), (C. 36), and (C. 37) we may solve for the left-hand side of equation (C. 2) to yield

$$\frac{d}{d\lambda}\left\{\log\left[r_{-}(\lambda)\right]\right\}$$

$$= \frac{1}{2(i\alpha-\lambda)} + \frac{1}{2(\lambda-i\tau)}$$

$$+ \frac{\nu}{2\pi i}\left[\frac{\pi i}{\nu(\lambda-i\tau)} - \frac{2i\lambda(\pi/2-\phi)}{(\lambda^2+\tau^2)\sqrt{\lambda^2+\alpha^2}} - \frac{2i(\lambda\pi/2+\psi i\tau)}{\nu(\lambda^2+\tau^2)}\right].$$

This last equation may be simplified somewhat to produce

$$\frac{d}{d\lambda}\left\{\log\left[r_{-}(\lambda)\right]\right\} = \frac{1}{2(i\alpha-\lambda)} + \frac{1}{(\lambda-i\tau)} - \frac{\nu\lambda(\pi/2-\phi)}{\pi(\lambda^2+\tau^2)\sqrt{\lambda^2+\alpha^2}} - \frac{(\lambda\pi/2+\psi i\tau)}{\pi(\lambda^2+\tau^2)} \quad (C.38),$$

hence equation (4.42).

Now we shall consider (4.41), which told us that

$$\frac{d}{d\lambda}\left\{\log\left[r_{+}(\lambda)\right]\right\} = \frac{1}{2\pi i}\int_{\Gamma_2} d\xi \frac{\xi}{(\xi-\lambda)}\frac{1}{(\xi^2+\alpha^2)} - \frac{1}{2\pi i}\int_{\Gamma_2}\frac{d\xi}{(\xi-\lambda)}\frac{\xi}{(\xi^2+\tau^2)} + \frac{\nu}{2\pi i}\int_{\Gamma_2}\frac{d\xi}{(\xi-\lambda)}\frac{\xi}{(\xi^2+\tau^2)\sqrt{\xi^2+\alpha^2}} \quad (C.39),$$

wherein Γ_2 is the integration contour as shown in diagram (4.1), and now λ is a point lying above the contour. The integrals in (C.39) may be evaluated, for example, by closing the contour in the lower half-plane. The analysis parallels that of the preceeding pages, and the final result is

$$\frac{d}{d\lambda}\left\{\log\left[r_{+}(\lambda)\right]\right\} = \frac{1}{2(\lambda+i\alpha)} - \frac{1}{\lambda+i\tau}$$

$$+ \frac{\nu\lambda(\pi/2+\phi)}{\pi(\lambda^2+\tau^2)\sqrt{\lambda^2+\alpha^2}} + \frac{(\lambda\pi/2-\psi/\tau)}{\pi(\lambda^2+\tau^2)} \qquad \text{(C. 40)}.$$

wherein again ϕ and ψ are as given in equations (C. 32) and (C. 21), respectively. This gives us equation (4. 43) as desired.

APPENDIX D: Derivation of Equations (4.48) & (4.49)

Equation (4.42) tells us that

$$\frac{d}{d\lambda}\left[\log[r_{-}(\lambda)]\right] = \frac{-1}{2(\lambda - i\alpha)} + \frac{1}{\lambda - i\tau} - \frac{\nu\lambda(\pi/2 - \phi)}{\pi(\lambda^2+\tau^2)\sqrt{\lambda^2 + \alpha^2}} - \frac{(\lambda\pi/2 + \psi i\tau)}{\pi(\lambda^2+\tau^2)} \qquad (D.1).$$

This equation may be broken down to give

$$\frac{d}{d\lambda}\left[\log[r_{-}(\lambda)]\right] = -\frac{1}{2(\lambda - i\alpha)} + \frac{1}{(\lambda - i\tau)} - \frac{\nu}{4(\lambda + i\tau)\sqrt{\lambda^2 + \alpha^2}} - \frac{\nu}{4(\lambda - i\tau)\sqrt{\lambda^2 + \alpha^2}} - \frac{\lambda}{2(\lambda^2+\tau^2)} + \frac{\psi}{2\pi(\lambda + i\tau)} - \frac{\psi}{2\pi(\lambda - i\tau)} + \frac{\nu\lambda\phi}{\pi(\lambda^2+\tau^2)\sqrt{\lambda^2 + \alpha^2}}$$

which may be formally integrated as

$$\log[r_{-}(\lambda)] = -(1/2)\log(\lambda - i\alpha) + \log(\lambda - i\tau) - \frac{1}{4}\log(\lambda^2+\tau^2) + \frac{\psi}{2\pi}\log\left[\frac{(\lambda + i\tau)}{(\lambda - i\tau)}\right] - \frac{\nu}{4}(P_1 + P_2) + \frac{\nu}{\pi}P_3 \qquad (D.2),$$

wherein

$$P_1 = \int_{i}^{\lambda} \frac{d\xi}{(\xi + i\tau)\sqrt{\xi^2 + \alpha^2}} \qquad (D.3),$$

$$P_2 = \int^{\lambda} \frac{d\xi}{(\xi - i\tau)\sqrt{\xi^2+\alpha^2}} \qquad \text{(D. 4)},$$

and

$$P_3 = \int^{\lambda} d\xi \frac{\xi\phi}{(\xi^2 + \tau^2)\sqrt{\xi^2+\alpha^2}} \qquad \text{(D. 5)}.$$

The symbol ϕ is defined through equation (C. 32) as

$$\sin\phi = \frac{i\xi}{\alpha} \qquad \text{(D. 6)}.$$

The integrands in the expressions for P_1 and P_2 are identical, respectively, to those in equations (C. 12) and (C. 14) defining L_1 and L_2. Considering equation (D. 3) for P_1, the change of variable

$$\xi = i\alpha\sin\theta \qquad \text{(D. 7)}$$

gives

$$P_1 = \frac{1}{\alpha}\int^{\Omega} \frac{d\theta}{\sin\theta + \sin\psi} \qquad \text{(D. 8)},$$

wherein

$$\Omega = \sin^{-1}(\lambda/i\alpha) \qquad \text{(D. 9)},$$

and

$$\sin\psi = \tau/\alpha \qquad \text{(D. 10)},$$

as in equation (C. 21). Hence, from Appendix C we realise that

$$P_1 = \frac{1}{\alpha\cos\psi}\left[\log\left[\frac{\sin\left[\frac{\theta+\psi}{2}\right]}{\cos\left[\frac{\theta-\psi}{2}\right]}\right]\right]^{\Omega} \qquad \text{(D. 11)}.$$

Using standard trigonometrical relations we may rewrite this last expression as

$$P_1 = \frac{1}{2\alpha\cos\psi}\left[\log\left\{\frac{1 - \cos(\theta+\psi)}{1 + \cos(\theta-\psi)}\right\}\right]^{\theta=\Omega}$$

which becomes, upon evaluating the limit

$$P_1 = \frac{1}{2\alpha\cos\psi}\log\left\{\frac{1 - \cos\Omega\cos\psi + \sin\Omega\sin\psi}{1 + \cos\Omega\cos\psi + \sin\Omega\sin\psi}\right\} \qquad \text{(D.12)}.$$

From equations (D.9) and (D.10), respectively, we find that

$$\cos\Omega = \frac{\sqrt{\lambda^2 + \alpha^2}}{\alpha} \qquad \text{(D.13)},$$

and

$$\cos\psi = -\frac{\nu}{\alpha} \qquad \text{(D.14)},$$

so that equation (D.12) becomes

$$P_1 = -\frac{1}{2\nu}\log\left[\frac{i\alpha^2 + \lambda\tau + i\nu\sqrt{\lambda^2 + \alpha^2}}{i\alpha^2 + \lambda\tau - i\nu\sqrt{\lambda^2 + \alpha^2}}\right] \qquad \text{(D.15)}.$$

We shall now consider equation (D.4) for P_2. Making the same change of variable as in equation (D.7) we obtain

$$P_2 = \frac{1}{\alpha}\int^{\Omega}\frac{d\theta}{\sin\theta - \sin\psi} \qquad \text{(D.16)},$$

wherein Ω and ψ are again given, respectively, by equations (D.9) and (D.10). Once again, from Appendix C we find that

$$P_2 = -\frac{1}{\alpha\cos\psi}\left[\log\left[\frac{\cos\left[\frac{\theta+\psi}{2}\right]}{\sin\left[\frac{\theta-\psi}{2}\right]}\right]\right]^{\theta=\Omega}$$

which may be written as

$$P_2 = -\frac{1}{2\alpha\cos\psi}\left[\log\left\{\frac{1 + \cos(\theta+\psi)}{1 - \cos(\theta-\psi)}\right\}\right]^{\theta=\Omega}$$

or in an expanded form as

$$P_2 = -\frac{1}{2\alpha\cos\psi}\left[\log\left\{\frac{1 + \cos\theta\cos\psi - \sin\theta\sin\psi}{1 - \cos\theta\cos\psi - \sin\theta\sin\psi}\right\}\right]^{\theta=\Omega}.$$

Evaluating this expression at the limit, with the aid of equations (D.9), (D.10), (D.13), and (D.14), tells us that

$$P_2 = \frac{1}{2\nu}\log\left\{\frac{i\alpha^2 - \lambda\tau - i\nu\sqrt{\lambda^2 + \alpha^2}}{i\alpha^2 - \lambda\tau + i\nu\sqrt{\lambda^2 + \alpha^2}}\right\} \qquad \text{(D.17)}.$$

Hence, from equations (D.15) and (D.17) we find that

$$P_1 + P_2 = \frac{1}{2\nu}\log\left\{\frac{i\alpha^2 + \lambda\tau - i\nu\sqrt{\lambda^2 + \alpha^2}}{i\alpha^2 + \lambda\tau + i\nu\sqrt{\lambda^2 + \alpha^2}}\right\}$$

$$+ \frac{1}{2\nu}\log\left\{\frac{i\alpha^2 - \lambda\tau - i\nu\sqrt{\lambda^2 + \alpha^2}}{i\alpha^2 - \lambda\tau + i\nu\sqrt{\lambda^2 + \alpha^2}}\right\}.$$

This last expression may be simplified in stages to give

$$P_1 + P_2 = \frac{1}{2\nu}\log\left\{\frac{(\lambda^2+\alpha^2+\nu^2) - 2\nu\sqrt{\lambda^2 + \alpha^2}}{(\lambda^2+\alpha^2+\nu^2) + 2\nu\sqrt{\lambda^2 + \alpha^2}}\right\},$$

and finally

$$P_1 + P_2 = \frac{1}{\nu}\log\left[\frac{\sqrt{\lambda^2+\alpha^2}-\nu}{\sqrt{\lambda^2+\alpha^2}+\nu}\right] \quad (D.18).$$

which may be alternately written

$$P_1 + P_2 = \frac{-1}{\nu}\log(\lambda^2+\tau^2) + \frac{2}{\nu}\log\left[\sqrt{\lambda^2+\alpha^2}-\nu\right] \quad (D.19).$$

Using this expression for $P_1 + P_2$ in equation (D.2) yields

$$\log[r_-(\lambda)] = -(1/2)\log(\lambda-i\alpha) + \log(\lambda-i\tau)$$

$$+\frac{\nu}{2\pi}\log\left[\frac{(\lambda+i\tau)}{(\lambda-i\tau)}\right] - (1/2)\log[\sqrt{\lambda^2+\alpha^2}-\nu]$$

$$+\frac{\nu}{\pi}P_3 \quad (D.20).$$

Let us consider equation (D.5), which gave us that

$$P_3 = \int^{\lambda} d\xi \, \frac{\xi\sin^{-1}(i\xi/\alpha)}{(\xi^2+\tau^2)\sqrt{\xi^2+\alpha^2}}.$$

On noting that

$$\frac{d}{d\xi}\log\left[\frac{\sqrt{\xi^2+\alpha^2}+\nu}{\sqrt{\xi^2+\alpha^2}-\nu}\right] = \frac{-2\nu\xi}{(\xi^2+\tau^2)\sqrt{\xi^2+\alpha^2}} \quad (D.21).$$

and therefore that

$$\frac{d}{d\xi}\left\{\sin^{-1}(i\xi/\alpha)\log\left[\frac{\sqrt{\xi^2+\alpha^2}+\nu}{\sqrt{\xi^2+\alpha^2}-\nu}\right]\right\}$$

$$= -\frac{2\nu\xi\sin^{-1}(i\xi/\alpha)}{(\xi^2+\tau^2)\sqrt{\xi^2+\alpha^2}} + \frac{i}{\sqrt{\xi^2+\alpha^2}}\log\left[\frac{\sqrt{\xi^2+\alpha^2}+\nu}{\sqrt{\xi^2+\alpha^2}-\nu}\right] \quad (D.22).$$

our expression for P_3 becomes

$$P_3 = -\frac{\sin^{-1}(i\lambda/\alpha)}{2\nu}\log\left\{\frac{\sqrt{\lambda^2+\alpha^2}+\nu}{\sqrt{\lambda^2+\alpha^2}-\nu}\right\}$$

$$+\frac{i}{2\nu}\int^{\lambda}\frac{d\xi}{\sqrt{\xi^2+\alpha^2}}\log\left\{\frac{\sqrt{\xi^2+\alpha^2}+\nu}{\sqrt{\xi^2+\alpha^2}-\nu}\right\} \qquad \text{(D. 23).}$$

We may thusly rewrite equation (D. 20):

$$\log[r_-(\lambda)] = -(1/2)\log(\lambda-i\alpha) + \log(\lambda+i\tau)$$

$$+\frac{\nu}{2\pi}\log\left[\frac{(\lambda+i\tau)}{(\lambda-i\tau)}\right] - (1/2)\log\left[\sqrt{\lambda^2+\alpha^2}-\nu\right] \qquad \text{(D. 24).}$$

$$+\frac{\sin^{-1}(i\lambda/\alpha)}{2\pi}\log\left[\frac{\sqrt{\lambda^2+\alpha^2}-\nu}{\sqrt{\lambda^2+\alpha^2}+\nu}\right] + \frac{i}{2\pi}\int^{\lambda}\frac{d\xi}{\sqrt{\xi^2+\alpha^2}}\log\left(\frac{\sqrt{\xi^2+\alpha^2}+\nu}{\sqrt{\xi^2+\alpha^2}-\nu}\right)$$

hence equation (4. 48).

We shall now focus our attention on an expression for $\log[r_+(\lambda)]$. From equation (4. 43) we have

$$\frac{d}{d\lambda}\left\{\log[r_+(\lambda)]\right\} = \frac{1}{2(\lambda+i\alpha)} - \frac{1}{(\lambda+i\tau)}$$

$$+\frac{\nu\lambda(\pi/2+\phi)}{\pi(\lambda^2+\tau^2)\sqrt{\lambda^2+\alpha^2}} + \frac{(\lambda\pi/2 + \nu i\tau)}{\pi(\lambda^2+\tau^2)}.$$

This equation may be broken down to give

$$\frac{d}{d\lambda}\left\{\log[r_+(\lambda)]\right\} = \frac{1}{2(\lambda+i\alpha)} - \frac{1}{(\lambda+i\tau)}$$

$$+ \frac{\nu}{4(\lambda + i\tau)\sqrt{\lambda^2 + \alpha^2}} + \frac{\nu}{4(\lambda - i\tau)\sqrt{\lambda^2 + \alpha^2}}$$

$$+ \frac{\lambda}{2(\lambda^2 + \tau^2)} + \frac{\psi}{2\pi(\lambda + i\tau)} - \frac{\psi}{2\pi(\lambda - i\tau)}$$

$$+ \frac{\nu\lambda\phi}{\pi(\lambda^2 + \tau^2)\sqrt{\lambda^2 + \alpha^2}} ,$$

which may be formally integrated to give

$$\log[r_+(\lambda)] = (1/2)\log(\lambda + i\alpha) - \log(\lambda + i\tau) + \frac{1}{4}\log(\lambda^2 + \tau^2) + \frac{\psi}{2\pi}\log\left[\frac{(\lambda + i\tau)}{(\lambda - i\tau)}\right] + \frac{\nu}{4}[P_1 + P_2] + \frac{\nu}{\pi}P_3 \qquad (D.25),$$

wherein P_1, P_2, and P_3 are defined, respectively, in equations (D.3) through (D.5). By substituting from equation (D.19) and (D.23) for the quantities $P_1 + P_2$ and P_3, respectively, we obtain

$$\log[r_+(\lambda)] = (1/2)\log(\lambda + i\alpha) - \log(\lambda + i\tau) + \frac{\psi}{2\pi}\log\left[\frac{(\lambda + i\tau)}{(\lambda - i\tau)}\right] + (1/2)\log\left[\sqrt{\lambda^2 + \alpha^2} - \nu\right] \qquad (D.26),$$
$$+ \frac{\sin^{-1}(i\lambda/\alpha)}{2\pi}\log\left[\frac{\sqrt{\lambda^2 + \alpha^2} - \nu}{\sqrt{\lambda^2 + \alpha^2} + \nu}\right] + \frac{i}{2\pi}\int^{\lambda}\frac{d\xi}{\sqrt{\xi^2 + \alpha^2}}\log\left[\frac{\sqrt{\xi^2 + \alpha^2} + \nu}{\sqrt{\xi^2 + \alpha^2} - \nu}\right]$$

hence equation (4.49).

APPENDIX E: Derivation of Equations (4.51) and (4.54)

From equation (4.50) we have the approximation to be made as given by

$$R(\xi) = \frac{\sqrt{\xi^2+\alpha^2} + \nu}{\sqrt{\xi^2+\alpha^2}}$$

$$\approx R_1(\xi) = \frac{(\xi^2 + \tau^2)(\xi^2 + b^2)}{\sqrt{\xi^2+\alpha^2}\sqrt{\xi^2+a^2}(\xi^2 + c^2)} \qquad (E.1).$$

wherein a, b, and c shall be chosen so that the inverse transform of both $R(\xi)$ and $R_1(\xi)$ share the same singular behaviour at the origin, area, and second moment.

The area and first two moments of the inverse transform of $R(\xi)$ are proportional to, respectively, R'(0), R'(0), and R''(0), wherein the prime ' denotes differentiation with respect to ξ [see Papoulis [19]]. From equation (E.1) we find that

$$R(0) = \frac{\alpha + \nu}{\alpha} \qquad (E.2),$$

$$R'(0) = 0 \qquad (E.3),$$

and

$$R''(0) = \frac{-\nu}{\alpha^3} \qquad (E.4).$$

Similarly, it is easily shown that

$$R_1(0) = \frac{\tau^2 b^2}{\alpha a c^2} \qquad (E.5).$$

$$R_1'(0) = 0 \qquad \text{(E. 6)}.$$

and

$$R_1'' = \frac{2b^2}{\alpha ac^2} + \frac{2\tau^2}{\alpha ac^2} - \frac{\tau^2 b^2}{\alpha^3 ac^2}$$

$$- \frac{\tau^2 b^2}{\alpha a^3 c^2} - \frac{2\tau^2 b^2}{\alpha ac^4} \qquad \text{(E. 7)}$$

Requiring that the inverse transforms of $R(\xi)$ and $R_1(\xi)$ have the same area demands, from equations (E. 2) and (E. 5), that

$$\frac{\tau^2 b^2}{\alpha ac^2} = \frac{\alpha + \nu}{\alpha} \qquad \text{(E. 8)}.$$

Bearing in mind that we are trying to approximate $R(\xi)$ by a function $R_1(\xi)$ which has a relatively simple Wiener-Hopf decomposition, we shall choose

$$a = \alpha \qquad \text{(E. 9)}.$$

so that, with the aid of equation (4.39), equation (E. 8) becomes

$$\frac{b^2}{c^2} = \frac{\alpha}{(\alpha - \nu)}$$

that is

$$b^2 = \frac{c^2\alpha}{(\alpha - \nu)} \qquad \text{(E. 10)}.$$

Equality of the second moments of the inverse transforms of $R(\xi)$ and $R_1(\xi)$ allows us to write, from equations (E. 4) and (E. 5), with the aid of equations (E. 9), (E. 10), and (4.39),

$$\frac{-\nu}{\alpha} = \frac{2}{\alpha(\alpha - \nu)} + \frac{2(\alpha^2 - \nu^2)}{\alpha^2 c^2} - \frac{(\alpha + \nu)}{\alpha^3}$$

$$- \frac{(\alpha+\nu)}{\alpha^3} - \frac{2(\alpha+\nu)}{\alpha c^2} .$$

This last equation is readily solved for c^2 to yield

$$c^2 = 2\alpha(\alpha-\nu) \qquad \text{(E. 11)},$$

and so, from equation (E. 10),

$$b^2 = 2\alpha^2 \qquad \text{(E. 12)}.$$

Our approximation to $R(\xi)$ as given in equation (E. 1) thus becomes

$$R(\xi) = \frac{\sqrt{\xi^2+\alpha^2} + \nu}{\sqrt{\xi^2+\alpha^2}}$$

$$\simeq R_1(\xi) = \frac{(\xi^2 + \tau^2)(\xi^2 + b^2)}{(\xi^2 + \alpha^2)(\xi^2 + c^2)} \qquad \text{(E. 13)},$$

wherein b^2 and c^2 are as given in equations (E. 11) and (E. 12), respectively. This gives us equation (4. 51).

It must now be shown that the inverse Fourier transforms of both $R(\xi)$ and $R_1(\xi)$ share the same singular behaviour [at the origin] in the x-domain. From (E. 1) and (E. 13) we have that

$$F^{-1}\left\{R(\xi)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{i\xi x}\left[\frac{\sqrt{\xi^2+\alpha^2} + \nu}{\sqrt{\xi^2+\alpha^2}}\right] \qquad \text{(E. 14)},$$

and

$$F^{-1}\left\{R_1(\xi)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{i\xi x}\frac{(\xi^2 + \tau^2)(\xi^2 + b^2)}{(\xi^2 + \alpha^2)(\xi^2 + c^2)} \qquad \text{(E. 15)},$$

wherein the symbol 'F^{-1}' is taken to mean the inverse Fourier transform operator for the $x-\xi$ variable pair.

We observe that the singular behaviour of both the $F^{-1}[R(\xi)]$ and $F^{-1}[R_1(\xi)]$ functions as x approaches zero is determined by the integrand behaviour for large $|\xi|$ in equation (E.14) and (E.15), respectively [see, for example, [3]]. Both integrands approach unity for large values of $|\xi|$, so that $F^{-1}[R(\xi)]$ and $F^{-1}[R_1(\xi)]$ do indeed share the same singular behaviour at the origin in the x-domain.

From equation (4.25),

$$S(\xi) = \frac{\sqrt{\xi^2+\alpha^2}}{\sqrt{\xi^2+\alpha^2} + \mu}$$

we see that the expressions for $S(\xi)$ and $R(\xi)$ are quite similar. In fact, from equation (4.28) we have

$$S(\xi) = \frac{1}{R(\xi)}\Big]_{\nu\to\mu} .$$

Thus, from the preceeding analysis we are led to the approximation

$$S(\xi) = \frac{\sqrt{\xi^2+\alpha^2}}{\sqrt{\xi^2+\alpha^2} + \mu}$$

$$\approx S_1(\xi) = \frac{(\xi^2 + \alpha^2)(\xi^2 + d^2)}{(\xi^2 + a^2)(\xi^2 + b^2)} \qquad \text{(E.16)},$$

wherein d^2 is given by

$$d^2 = 2\alpha(\alpha - \mu) \qquad \text{(E.17)},$$

and σ^2 is defined by equation (F.10). The function $S_1(\xi)$ has been chosen so that both its inverse tranform along with that of $S(\xi)$ share the same singular behaviour at the origin, area, and second moment. Equation (4.54) has thus been derived,

In order to effect the Wiener-Hopf decomposition of the product of $A_1(\xi)$ and $S_1(\xi)$ (which will be the approximation to equation (4.23)) we shall have to investigate the analytic structure of the terms

$$(\xi^2 + c^2) = (\xi^2 + 2\alpha(\alpha-\nu)) \qquad \text{(E.18)},$$

and

$$(\xi^2 + d^2) = (\xi^2 + 2\alpha(\alpha-\mu)) \qquad \text{(E.19)},$$

appearing in equations (E.13) and (E.16), respectively. It is clear that the term $(\xi^2 + b^2)$ will vanish from the product of $A_1(\xi)$ with $S_1(\xi)$. Factoring equations (E.18) and (E.19) leads to

$$(\xi^2 + c^2) = (\xi + ic)(\xi - ic) \qquad \text{(E.20)},$$

$$(\xi^2 + d^2) = (\xi + id)(\xi - id) \qquad \text{(E.21)},$$

wherein

$$c = \sqrt{2\alpha(\alpha-\nu)} \qquad \text{(E.22)},$$

and

$$d = \sqrt{2\alpha(\alpha-\mu)} \qquad \text{(E.23)},$$

and also wherein we've employed our usual convention for the choice of the branch of the multivalued square root function. Now, α lies in the first quadrant in the complex plane, relatively close to the real axis. From the Appendix B, we know that $(\alpha-\nu)$ wil lie either in the first or fourth quadrant, close to the real axis. Hence, from our choice of branch of the root function, we see that c wil lie either in the first or fourth quadrant, and so ic and −ic will be entirely in the upper or lower complex half-planes, respectively. A similar argument holds for id and −id. These conclusions may be summarised diagrammatically (approximately) as shown in diagram E.1.

APPENDIX F: Derivation of Equations (4.62) and (4.64)

From equation (3.132) we have

$$\mathbf{E}_s = \left[T_{sE}(\mathbf{J}_s)\right] * K_{00} \qquad \text{(F.1)}.$$

wherein, from equations (3.80) and (3.111)

$$T_{sE} = \frac{1}{i\omega\epsilon_0}[\nabla(\nabla\cdot\ ) + k^2] \qquad \text{(F.2)},$$

and

$$K_{00} = \frac{e^{-ikr}}{4\pi r} \qquad \text{(F.3)},$$

respectively. For our current source $\mathbf{J}_s$ we shall take an elementary [infinitesimal] dipole located at a height h above the ground and at a distance x_0 to the left of the vertical interface at $x=0$. For this scenario we may write

$$\mathbf{J}_s = I\,dl\,\delta(x+x_0)\delta(y)\delta(z-h)\mathbf{a}_z \qquad \text{(F.4)}.$$

Hence

$$\nabla\cdot\mathbf{J}_s = I\,dl\,\delta(x+x_0)\delta(y)\delta'(z-h) \ .$$

and so

$$\nabla(\nabla\cdot\mathbf{J}_s) = I\,dl\left\{\delta'(x+x_0)\delta(y)\delta'(z-h)\mathbf{a}_x + \delta(x+x_0)\delta'(y)\delta'(z-h)\mathbf{a}_y\right.$$

$$\left. + \delta(x+x_0)\delta(y)\delta''(z-h)\mathbf{a}_z\right\} .$$

Using this last result in equations (F.2) and (F.1) reveals that

$$E_{s_z} = \frac{I\,dl}{i\omega\epsilon_0}\left\{\delta(x+x_0)\delta(y)\delta''(z-h)\right.$$

$$+ k^2 \delta(x+x_0)\delta(y)\delta(z-h) \Big\} * K_{00} \quad \text{(F.5).}$$

Shifting the differentiation and performing the convolution in this last equation allows us to write

$$E_{s_z} = \frac{I\,dl}{i\omega\epsilon_0}\left\{ \frac{\partial^2}{\partial z^2}\frac{e^{-ikr'}}{4\pi r'} + k^2 \frac{e^{-ikr'}}{4\pi r'} \right\} \quad \text{(F.6),}$$

wherein

$$r' = \left\{ (x+x_0)^2 + y^2 + (z-h)^2 \right\}^{1/2} \quad \text{(F.7).}$$

Carrying out the differentiation results in

$$E_{s_z} = \frac{I\,dl}{4\pi i\omega\epsilon_0}\left[\frac{3(z-h)^2}{r'^4} + \frac{3ik(z-h)^2}{r'^3} - \frac{1}{r'^2} - \frac{k^2 z'^2}{r'^2} - \frac{ik}{r'} + k^2 \right] \frac{e^{ikr'}}{r'} \quad \text{(F.8).}$$

If we now assume that

$$\left. \begin{array}{l} r' \gg (z-h) \\ r' \gg 1 \end{array} \right\} \quad \text{(F.9),}$$

i.e. the "far-field" approximation, then equation (F.8) becomes

$$E_{s_z} \approx \frac{I\,dl\,k^2}{4\pi i\omega\epsilon_0}\frac{e^{-ikr'}}{r'} \quad \text{(F.10),}$$

hence equation (4.61).

Taking a three-dimensional Fourier transform of equation (F.10) allows us to write approximately

$$\overline{\overline{\overline{E}}}_{s_z} = \frac{I\,dl\,k^2}{4\pi i\omega\epsilon_0}\,\overline{\overline{\overline{\frac{e^{-ikr'}}{r'}}}} \quad \text{(F.11).}$$

We know from equation (3.112) that the function $e^{-ikr'}/r'$ satisfies the equation

$$\nabla^2\left[\frac{e^{-ikr'}}{r'}\right] + k^2\left[\frac{e^{-ikr'}}{r'}\right] = -4\pi\delta(x+x_0)\delta(y)\delta(z-h) \qquad (F.12).$$

Taking a three-dimensional Fourier transform of equation (F.12) gives

$$\overline{\overline{\overline{\frac{e^{-ikr'}}{r'}}}} = \frac{4\pi e^{-i\zeta h}\, e^{+i\xi x_0}}{\zeta^2 + \chi^2} \qquad (F.13),$$

wherein ζ is the z Fourier transform variable, and χ^2 is given by

$$\chi^2 = \xi^2 + \eta^2 - k^2 \qquad (F.14).$$

Hence, applying an inverse z-ζ Fourier transform to equation (F.13) results in [see Appendix A]

$$\overline{\overline{\frac{e^{-ikr'}}{r'}}} = \frac{2\pi e^{i\xi x_0}\, e^{-|z-h|u_0}}{u_0} \qquad (F.15),$$

wherein u_0 is as defined in equation (3.172). Equation (F.11) tells us thus that, approximately,

$$\overline{\overline{E_{s_z}}} = \frac{I\,dl\,k^2}{i\omega\epsilon_0}\, e^{i\xi x_0}\, \frac{e^{(z-h)u_0}}{2u_0} \qquad (F.16),$$

hence equation (4.63).

APPENDIX G: Derivation of Equation (4.67)

From equation (4.66) we have

$$\overline{[h_2 D(x,y)]_-} = \frac{\overline{D(x,y)}}{2} + \frac{1}{2\pi i}\frac{1}{\xi} * \overline{D(x,y)} \qquad (G.1),$$

wherein, from equation (4.65) we have

$$\overline{D(x,y)} = -i\omega\mu_0 I\, dl\, \frac{e^{i\xi x_0}\, e^{-hu_0}}{\sqrt{\xi^2+\alpha^2}} \qquad (G.2)$$

Taking the limit as the antenna height h approaches zero in equations (G.1) and (G.2) implies that

$$\lim_{h\to 0} \overline{[h_2 D(x,y)]_-} = -\frac{i\omega\mu_0 I\, dl\, e^{i\xi x_0}}{2(\sqrt{\xi^2+\alpha^2}+\mu)}$$

$$-\frac{\mu_0 \omega I\, dl}{2\pi}\,\frac{1}{\xi} * \frac{e^{i\xi x_0}}{\sqrt{\xi^2+\alpha^2}+\mu} \qquad (G.3).$$

We shall define the convolution appearing in equation (G.3) as follows

$$M(\xi) = \frac{1}{\xi} * \frac{e^{i\xi x_0}}{\sqrt{\xi^2+\alpha^2}+\mu}$$

$$= \int_{-\infty}^{\infty} \frac{d\lambda}{(\xi-\lambda)}\, \frac{e^{i\lambda x_0}}{\sqrt{\lambda^2+\alpha^2}+\mu} \qquad (G.4),$$

wherein the integral is to be interpreted in the Cauchy Principal Value sense. Closing the integration contour in the upper half-plane and referring to diagram G.1 we find that

$$M(\xi) + \int_{\Gamma} + \int_{\substack{\text{branch}\\\text{cut}}} = 2\pi i \times \text{residue } @ \ \lambda = i\sigma \tag{G.5}$$

The residue of the integrand at $\lambda = i\sigma$ is given by

$$\lim_{\lambda \to i\sigma} (\lambda - i\sigma) \frac{e^{i\lambda x_0}\left[\sqrt{\lambda^2 + \alpha^2} - \mu\right]}{(\xi - \lambda)(\lambda^2 + \sigma^2)}$$

which is equal to

$$\frac{\mu i e^{-\sigma x_0}}{(\xi - i\sigma)\sigma}$$

so that equation (G.5) becomes

$$M(\xi) + \int_{\Gamma} + \int_{\substack{\text{branch}\\\text{cut}}} = \frac{-2\pi\mu e^{-\sigma x_0}}{(\xi - i\sigma)\sigma} \tag{G.6}$$

In equation (G.6) consider

$$\int_{\Gamma} = \int_{\Gamma} \frac{d\lambda}{(\xi - \lambda)} \frac{e^{i\lambda x_0}}{\sqrt{\lambda^2 + \alpha^2} + \mu}$$

In the limit as the contour Γ shrinks onto the point ξ, bearing in mind the orientation of the traverse of the contour, we find that

$$\int_{\Gamma} \frac{d\lambda}{(\xi - \lambda)} \frac{e^{i\lambda x_0}}{\sqrt{\lambda^2 + \alpha^2} + \mu} = \pi i \frac{e^{i\xi x_0}}{\sqrt{\xi^2 + \alpha^2} + \mu} \tag{G.7}$$

Equation (G.6) becomes

$$M(\xi) = \frac{-2\pi\mu e^{-\sigma x_0}}{(\xi - i\sigma)\sigma} - \frac{\pi i e^{i\xi x_0}}{\sqrt{\xi^2 + \alpha^2} + \mu} - \int_{\substack{\text{branch}\\\text{cut}}} \tag{G.8}$$

To evaluate the integral around the branch cut we realise that

$$\int_{\substack{branch\\cut}} = \int_{i\infty}^{i\alpha} \frac{d\lambda}{(\xi - \lambda)} \frac{e^{i\lambda x_0}}{-\sqrt{\lambda^2 + \alpha^2} + \mu} + \int_{i\alpha}^{i\infty} \frac{d\lambda}{(\xi - \lambda)} \frac{e^{i\lambda x_0}}{\sqrt{\lambda^2 + \alpha^2} + \mu} \, ,$$

the right-hand side of which may be simplified to show that, say,

$$I = \int_{\substack{branch\\cut}} = -2 \int_{i\alpha}^{i\infty} \frac{d\lambda}{(\xi - \lambda)} \frac{e^{i\lambda x_0} \sqrt{\lambda^2 + \alpha^2}}{\lambda^2 + \alpha^2 - \mu^2} \qquad \text{(G.9)}.$$

By making the change of variable

$$\lambda = i\alpha \sin\theta \qquad \text{(G.10)},$$

equation (G.9) becomes

$$I = -\frac{2}{\alpha} \int_{\pi/2}^{\pi/2 + i\infty} d\theta \, \frac{e^{-\alpha x_0 \sin\theta} \cos^2\theta}{(\sin\theta - A)(\cos^2\theta - B^2)} \qquad \text{(G.11)},$$

wherein

$$A = \xi / i\alpha \qquad \text{(G.12)},$$

and

$$B^2 = \mu^2/\alpha^2 \qquad \text{(G.13)}.$$

Making the change of variable

$$\psi = \theta - \pi/2 \qquad \text{(G.14)}$$

forces us to rewrite equation (G.11) as

$$I = -\frac{2}{\alpha} \int_0^{i\infty} d\psi \, \frac{e^{-\alpha x_0 \cos\psi} \sin^2\psi}{(\cos\psi - A)(\sin^2\psi - B^2)} \qquad \text{(G.15)}.$$

The further change of variable

$$iz = \psi \quad (G.16)$$

wherein z is **NOT** to be confused with our Cartesian spatial variable, yields

$$I = \frac{2i}{\alpha}\int_0^\infty dz \frac{\sinh^2 z \, e^{-\alpha x_0 \cosh z}}{(\cosh z - A)(\sinh^2 z + B^2)} \quad (G.17).$$

We shall now approximately evaluate this integral expression for I by the method of steepest descents. The saddle-point of the integrand, which is given by the solution of

$$\frac{d}{dz}\left\{-\alpha x_0 \cosh z\right\} = 0 ,$$

is found to be the point z = 0, whereat the exponent takes on the value $-\alpha x_0$.

The steepest descent path is that curve on which

$$\mathrm{Im}\left\{-\alpha \cosh z\right\} = CONSTANT ,$$

wherein the constant is determined by the value of the function within the brackets at the saddle point z = 0. Therefore, we find that the equation of the steepest descent path is given by

$$\mathrm{Im}\left\{-\alpha x_0 \cosh z\right\} = \mathrm{Im}\left\{-\alpha x_0\right\} \quad (G.18).$$

Since x_0 is real, the steepest descent path is clearly influenced by the value of (complex) α. A careful analysis reveals that the correct steepest descent paths for various α values are approximated by the curves shown in diagram G.2. It is to be remembered that α lies entirely in the first quadrant in the complex plane [see Appendix B].

By closing the integration path, as given in equation (G.17), as is shown in diagram G.3 we see that the contribution from the vertical portion of the contour will vanish in the limit. Note that S.D.P. stands for "steepest descent path".

The integral in equation (G.17) becomes

$$I = \frac{2i}{\alpha} e^{-x_0 \alpha} \int_{S.D.P.} dz \frac{\sinh^2 z\, e^{-x_0[-\alpha+\alpha\cosh z]}}{(\cosh z - A)(\sinh^2 z + B^2)} \qquad (G.19).$$

The quantity in brackets ( ) in the exponent is seen to be real and non-negative on the steepest descent path. We shall now make the assumption

$$\sinh^2 z + B^2 = \sinh^2 z \qquad (G.20).$$

In view of equation (G.13) and Appendix B this is a reasonable assumption, especially since we shall only be considered with the first order term in the asymptotic expansion of the integral in equation (G.19). Note that this assumption is equivalent to a consideration of $\overline{\overline{[h_2 D(x,y)]}}_-$ for a dipole when medium 1 is highly conducting. Equation (G.19) thus becomes approximately

$$I = \frac{2i}{\alpha} e^{-x_0 \alpha} \int_{S.D.P.} dz \frac{e^{-x_0[-\alpha+\alpha\cosh z]}}{\cosh z - A} .$$

Making the change of variable

$$-\alpha + \alpha\cosh z = s \qquad (G.21)$$

we obtain

$$I = \frac{-2Ie^{-x_0 \alpha}}{\alpha\sqrt{2\alpha}(1-A)} \int_0^\infty \frac{ds}{\sqrt{s}} \left[1 + \frac{s}{2\alpha}\right]^{-1/2} \left[1 + \frac{s}{\alpha(1-A)}\right]^{-1} e^{-x_0 s} \qquad (G.22).$$

Expanding the terms in brackets in the integrand and retaining O(s) terms only we find that

$$I = \frac{-2Ie^{-x_0 \alpha}}{\alpha\sqrt{2\alpha}(1-A)} \int_0^\infty \frac{ds}{\sqrt{s}}\, e^{-x_0 s} \left\{1 + Cs + \cdots\right\} \qquad (G.23).$$

wherein the constant C is given by

$$C = -\frac{(5-A)}{\alpha(1-A)} \qquad (G.24).$$

Integration of equation (G.23) gives

$$I = \frac{-2Ie^{-x_0 \alpha}}{\alpha\sqrt{2\alpha}(1-A)} \left[\frac{\sqrt{\pi}}{\sqrt{x}} + O(x_0^{-3/2})\right]$$

and, assuming that the interface is some distance away from the source (i.e. $|x_0| \gg 1$) we may write, to first order,

$$I \sim \frac{-\sqrt{2\pi}\, I e^{-x_0 \alpha}}{\alpha^{3/2}(1-A)x_0^{1/2}} \qquad (G.25).$$

Substituting from equation (G.12) tells us that

$$I \sim \frac{-\sqrt{2\pi} e^{-x_0 \alpha}}{\sqrt{\alpha}(\xi - I\alpha)\sqrt{x_0}} \qquad (G.26).$$

Equation (G.8) is thus approximated by

$$M(\xi) = \frac{-2\pi\mu e^{-\sigma x_0}}{\sigma(\xi - i\sigma)} - \frac{\pi i\, e^{i\xi x_0}}{\sqrt{\xi^2+\alpha^2}+\mu} + \frac{\sqrt{2\pi}\, e^{-x_0\alpha}}{\sqrt{x_0\alpha}\,(\xi - i\alpha)} \qquad (G.27),$$

and in turn equation (G.3) by

$$\lim_{h\to 0} \overline{[h_2 D(x,y)]}_- = \frac{\mu_0 \omega I\, dl}{2\pi}\left[\frac{2\pi\mu e^{-\sigma x_0}}{\sigma(\xi - i\sigma)} - \frac{\sqrt{2\pi}\, e^{-x_0\alpha}}{\sqrt{x_0\alpha}\,(\xi - i\alpha)}\right] \qquad (G.28),$$

hence equation (4.67).

APPENDIX H: Derivation of equation (4.73)

From equations (4.71) and (4.72) we have

$$\overline{\overline{[h_2 E_z^+]_-}} = 2\mu\mu_0 \omega l\, dl\, \frac{e^{-\sigma x_0}(\xi - ic)(\sigma + c)}{(\xi - i\tau)(\xi - id)(\sigma+\tau)(\sigma+d)}$$

$$- \frac{\mu_0 \omega i dl}{\sqrt{2\pi x_0}} \cdot \frac{e^{-x_0 \alpha}(\xi - i\sigma)(\xi - ic)(\alpha+\sigma)(\alpha+c)}{(\xi - i\alpha)(\xi - i\tau)(\xi - id)(\alpha+\tau)(\alpha+d)\sqrt{\alpha}} \qquad \text{(H.1)},$$

and

$$\begin{aligned} \alpha^2 &= \eta^2 - k^2 \\ \sigma^2 &= \alpha^2 - \mu^2 \\ \tau^2 &= \alpha^2 - \nu^2 \\ c^2 &= 2\alpha(\alpha - \nu) \\ d^2 &= 2\alpha(\alpha - \mu) \end{aligned} \qquad \text{(H.2)},$$

respectively. Applying a two dimensional inverse Fourier transform to equation (H.1) results in

$$h_2 E_z^+ = \frac{2\mu\mu_0 \omega i dl}{(2\pi)^2} \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\xi\, e^{i\eta y} e^{i\xi x}\, \frac{e^{-\sigma x_0}(\xi - ic)(\sigma + c)}{(\xi - i\tau)(\xi - id)(\sigma+\tau)(\sigma+d)}$$

$$- \frac{\mu_0 \omega i dl}{(2\pi)^2 \sqrt{2\pi x_0}} \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} d\xi\, e^{i\eta y} e^{i\xi x}$$

$$\times \frac{e^{-\alpha x_0}(\xi - i\sigma)(\xi - ic)(\alpha+\sigma)(\alpha+c)}{(\xi - \alpha)(\xi - i\tau)(\xi - id)(\alpha+\tau)(\alpha+d)\sqrt{\alpha}} \qquad \text{(H.3)}.$$

By virtue of the left-hand side of this equation, we're interested in evaluating the integrals for values of $x > 0$. Each integration with respect to the transform variable ξ may be effected by closing the integration contour in the upper half complex ξ- plane. In the first integral we capture the poles at $i\tau$ and at id; in the second integral we enclose the poles at $i\alpha$, $i\tau$, and id. We thus obtain,

$$h_2 E_z^+ = \frac{4\pi i \mu\mu_0 \omega I\, dl}{(2\pi)^2} \int_{-\infty}^{\infty} d\eta\, \frac{e^{i\eta y - \sigma x_0}(\sigma+c)}{(\sigma+\tau)(\sigma+d)} \left\{ \frac{e^{-\tau x}(\tau-c)}{(\tau-d)} + \frac{e^{-dx}(d-c)}{(d-\tau)} \right\}$$

$$- \frac{2\pi i \mu_0 \omega I\, dl}{(2\pi)^2 \sqrt{2\pi x_0}} \int_{-\infty}^{\infty} d\eta\, \frac{e^{i\eta y - \alpha x_0}(\alpha+\sigma)(\alpha+c)}{(\alpha+\tau)(\alpha+d)\sqrt{\alpha}} \qquad \text{(H.4)}.$$

$$\times \left\{ e^{-\alpha x}\frac{(\alpha-\sigma)(\alpha-c)}{(\alpha-\tau)(\alpha-d)} + e^{-\tau x}\frac{(\tau-\sigma)(\tau-c)}{(\tau-\alpha)(\tau-d)} + e^{-dx}\frac{(d-\sigma)(d-c)}{(d-\alpha)(d-\tau)} \right\}$$

In view of the quantities defined in equation (H.2), the equation (H.4) giving us $h_2 E_z^+$ is somewhat horrendous. We are now in a position when judicious care must be taken if approximations are to be made with the ultimate aim of obtaining physically meaningful results.

We recall from the discussion preceeding equation (4.51) that approximations were to be made, of the functions $R(\xi)$ and $S(\xi)$, whose inverse transforms retained the same singularity, area, and second moment as the original functions. We did not want spurious zeros or singularities to creep into the approximations, especially those lying near the real ξ-axis that would gain an equal footing with or even overshadow the acceptable physical results.

In our approximation of $\overline{V(\xi)}$ by $\overline{V_1(\xi)}$ in equation (4.58) we admitted new poles and zeros into the approximation with the introduction, respectively, of the terms $(\xi^2 + c^2)$ and $(\xi^2 + \tilde{d}^2)$. In view of our declared intentions in the preceeding paragraph it would seem careless to attach too much significance to the effect of these terms in the final solution.

Indeed, consider the term e^{-dx} occurring in both of the integrals in equation (H.4). Subsequent analysis will bear out the fact that the significant contribution in both integrals comes from that portion of the integration contour near $\eta = 0$, i.e. $\alpha = k' \sim O(1)$ [see Appendix B]. Expanding d in a power series, from equation (H.2), tells us that to first order in μ

$$d \approx \sqrt{2}\alpha - \frac{\mu}{\sqrt{2}} .$$

If we now assume that $|\mu| \ll 1$ we find that

$$e^{-dx} \approx e^{-\sqrt{2}\alpha x}$$

and so if x is even, say, 10 metres away from the interface we should be justified in neglecting the contribution of e^{-dx} in comparison with $e^{-\alpha x}$ or $e^{-\sigma x}$. In any event, for fixed μ we readily see that the term $e^{\mu x/\sqrt{2}}$ only strengthens our neglect as x increases [recall that μ has a negative real part]. It is thus reasonable to approximate equation (H.4) by

$$h_2E_z^+ = \frac{4\pi i\mu\mu_0\omega I\, dl}{(2\pi)^2}\int_{-\infty}^{\infty} d\eta\; e^{i\eta y - \sigma x_0 - \tau x}\,\frac{(\sigma+c)(\tau-c)}{(\sigma+\tau)(\sigma+d)(\tau-d)}$$

$$- \frac{2\pi i \mu_0 \omega I\, dl}{(2\pi)^2 \sqrt{2\pi x_0}} \int_{-\infty}^{\infty} d\eta\, \frac{e^{i\eta y - \alpha x_0}}{\sqrt{\alpha}} \tag{H.5}$$

$$\times \left\{ e^{-\alpha x} \frac{\mu^2 (2\alpha\nu - \alpha^2)}{\nu^2 (2\alpha\mu - \alpha^2)} + e^{-\tau x} \frac{(\alpha+\sigma)(\alpha+c)(\tau-\sigma)(\tau-c)}{(\alpha+\tau)(\alpha+d)(\tau-\alpha)(\tau-d)} \right\}$$

Once again, bearing in mind equation (H.2), the terms in the integrals in equation (H.5) are still exceedingly complicated. We shall now focus our attention on obtaining the most significant terms in an asymptotic expansion for $h_2 E_z^+$. To this end we shall make the further approximations

$$\sigma^2 \approx \alpha^2$$
$$\tau^2 \approx \alpha^2 \tag{H.6}$$

which are consistent with those made in deriving equation (4.67) [see Appendix G]. Note that the physical differences between the two lower media is still retained in the terms c and d, for example. These last approximations are equivalent to neglecting $O(\mu^2)$ and $O(\nu^2)$ terms. Equation (H.5) thus becomes approximately

$$h_2 E_z^+ = \frac{4\pi i \mu\mu_0 \omega I\, dl}{(2\pi)^2} \int_{-\infty}^{\infty} d\eta\, e^{i\eta y - \alpha(x+x_0)} \frac{(2\alpha\nu - \alpha^2)}{\alpha(2\alpha\mu - \alpha^2)} \tag{H.7}$$

$$- \frac{2\pi i \mu_0 \omega I\, dl}{(2\pi)^2 \sqrt{2\pi x_0}} \left[1 + \frac{\mu^2}{\nu^2} \right] \int_{-\infty}^{\infty} d\eta\, e^{i\eta y - \alpha(x+x_0)} \frac{(2\alpha\nu - \alpha^2)}{\sqrt{\alpha}(2\alpha\mu - \alpha^2)}$$

wherein, as in equation (H.5), use has been made of the relations in equation (H.2). This last equation is equivalent to

$$h_2 E_z^+ = \frac{4\pi i \mu\mu_0 \omega I\, dl}{(2\pi)^2} \left[2\nu + \frac{\partial}{\partial(x+x_0)} \right] I_1$$

$$- \frac{2\pi i \mu_0 \omega I\, dl}{(2pl)^2 \sqrt{2\pi x_0}} \left[1 + \frac{\mu^2}{\nu^2}\right] \left\{2\nu + \frac{\partial}{\partial(x + x_0)}\right\} I_2 \qquad (H.8)$$

wherein

$$I_1 = \int_{-\infty}^{\infty} d\eta\, \frac{e^{i\eta y - \alpha(x + x_0)}}{\alpha(2\mu - \alpha)} \qquad (H.9)$$

and

$$I_2 = \int_{-\infty}^{\infty} d\eta\, \frac{e^{i\eta y - \alpha(x + x_0)}}{\sqrt{\alpha(2\mu - \alpha)}} \qquad (H.10)$$

By defining the new variables or constants

$$\begin{aligned} x + x_0 &= \rho \cos(\theta) \\ y &= \rho \sin(\theta) \\ \eta &= k \sin(\psi) \\ \frac{2\mu i}{k} &= \cos(\psi_1) \end{aligned} \qquad (H.11)$$

we find that

$$I_1 = \frac{1}{2k} \int_C d\psi\, \frac{e^{-ik\rho \cos(\psi + \theta)}}{\cos\left[\frac{\psi + \psi_1}{2}\right] \cos\left[\frac{\psi - \psi_1}{2}\right]} \qquad (H.12)$$

and

$$I_2 = \frac{e^{i\pi/4}}{2\sqrt{k}} \int_C d\psi\, \frac{\sqrt{\cos(\psi)}\, e^{-ik\rho \cos(\psi + \theta)}}{\cos\left[\frac{\psi + \psi_1}{2}\right] \cos\left[\frac{\psi - \psi_1}{2}\right]} \qquad (H.13)$$

wherein C is the integration contour as shown in diagram H.1 [note that we've tacitly taken the limit as Im(k) → 0_]. Also in diagram H.1 we've indicated the approximate position of the poles of both integrands. The spatial variables defined in equation (H. 11) are shown in diagram H. 2.

The integrals occurring in equations (H. 12) and (H. 13) are somewhat similar to others in the literature (e.g. see Wait [28]). Unfortunately, as they stand now the integrals are intractable and we shall thus resort to the classical steepest-descent method for their asymptotic evaluation.

The saddle point of either integrand in equations (H. 12) and (H. 13) is found to be the point $\psi_0 = -\theta$. Their common steepest descent path, which passes through the saddle point ψ_0 and satisfies the equation

$$Im\,[-ik\cos(\psi + \theta)] = CONSTANT$$

is found to be

$$\cos(\psi_R + \theta)\cosh(\psi_{Im}) = 1 \qquad (H. 14).$$

We now deform the integration contour in both integrals so that it passes through the saddle point $[\psi_0 = -\theta]$ and lies on the steepest descent path S as given by equation (H. 14). Provided θ is not approaching $\pm\pi/2$ this last contour deformation will not be affected by the poles of either integrand as shown in diagram H. 1. We shall henceforth restrict our attention to the evaluation of the integrals in equations (H. 12) and (H. 13) for the case of small θ [see diagram H. 3].

We now effect the change of variable

$$\cos(\psi + \theta) = 1 - i\tau^2 \qquad (H. 15)$$

whereupon the integrals I_1 and I_2 reduce to

$$I_1 = \frac{2\sqrt{2}\, e^{-i\pi/4}\, e^{-ikp}}{k} \tag{H.16}$$

$$\times \int_0^\infty d\tau \, \frac{e^{-kp\tau^2}}{\sqrt{1-i\tau^2/2}} \, \frac{\cos(\psi_1) + \cos(\theta) - i\tau^2\cos(\theta)}{\left[\tau^2+2i\cos^2\left[\frac{\psi_1+\theta}{2}\right]\right]\left[\tau^2+2i\cos^2\left[\frac{\psi_1-\theta}{2}\right]\right]}$$

and

$$I_2 = \frac{2\sqrt{2}\, e^{-ikp}}{\sqrt{k}} \tag{H.17}$$

$$\times \int_0^\infty d\tau \, \frac{\sqrt{G(\tau)}\; e^{-kp\tau^2}}{\sqrt{1-i\tau^2/2}} \, \frac{\cos(\psi_1) + \cos(\theta) - i\tau^2\cos(\theta)}{\left[\tau^2+2i\cos^2\left[\frac{\psi_1+\theta}{2}\right]\right]\left[\tau^2+2i\cos^2\left[\frac{\psi_1-\theta}{2}\right]\right]}$$

wherein $G(\tau)$ is given by

$$G(\tau) = \cos\theta\,(1-i\tau^2) + \sqrt{2}\, e^{i\pi/4}\, \tau \sin\theta \sqrt{1-i\tau^2/2} \tag{H.18}$$

In order to evaluate the integrals in (H.16) and (H.17) we shall need the following restrictions

$$\theta = 0 \;, \quad |\mu| \ll 1 \tag{H.19}$$

In light of the restrictions in (H.19), and bearing in mind equations (H.11) and (H.18), we see that

$$\begin{aligned} G(\tau) &\approx (1 - i\tau^2) \\ \cos(\psi_1) + \cos(\theta) - i\tau^2\cos(\theta) &\approx 2(1-i\tau^2/2) \end{aligned} \tag{H.20}$$

Equations (H.16) and (H.17) thus become, approximately

$$I_1 = \frac{4\sqrt{2}e^{-i\pi/4}\, e^{-ikp}}{k} \int_0^\infty d\tau \, \frac{\sqrt{1-i\tau^2/2}\; e^{-kp\tau^2}}{\left[\tau^2 + 2i\cos^2(\psi_1/2)\right]^2} \,,$$

and

$$I_2 = \frac{4\sqrt{2}\, e^{-ik\rho}}{\sqrt{k}} \int_0^\infty d\tau \frac{\sqrt{1-i\tau^2/2}\,\sqrt{1-i\tau^2}\, e^{-k\rho\tau^2}}{\left[\tau^2 + 2i\cos^2(\psi_1/2)\right]^2}$$

respectively. The significant portions of these integrals come from the vicinity of the saddle point [i.e. $\tau = 0$]. Expanding the various terms in the numerator we see that, to first order,

$$I_1 \approx \frac{4\sqrt{2}\, e^{-i\pi/4}\, e^{-ik\rho}}{k} \int_0^\infty d\tau \frac{e^{-k\rho\tau^2}}{\left[\tau^2+2i\cos^2(\psi_1/2)\right]^2} \quad \text{(H.21)}$$

and

$$I_2 \approx \frac{4\sqrt{2}\, e^{-ik\rho}}{\sqrt{k}} \int_0^\infty d\tau \frac{e^{-k\rho\tau^2}}{\left[\tau^2+2i\cos^2(\psi_1/2)\right]^2} \quad \text{(H.22)}$$

Consider the integral common to both equations (H.21) and (H.22); say

$$K = \int_0^\infty d\tau \frac{e^{-k\rho\tau^2}}{\left[\tau^2+c^2\right]^2} \quad \text{(H.23)}$$

wherein

$$c^2 = 2i\cos^2(\psi_1/2) \quad \text{(H.24)}$$

Since we have that

$$\frac{\partial^2}{\partial\rho^2}\left[e^{-k\rho c^2} K\right] = k^2 e^{-k\rho c^2} \int_0^\infty d\tau\, e^{-k\rho\tau^2}$$

$$= \frac{k\sqrt{k\pi}}{2} \frac{e^{-k\rho c^2}}{\sqrt{\rho}}$$

we readily see that

$$\frac{\partial}{\partial \rho}\left\{e^{-k\rho c^2} K\right\} = \frac{k\pi}{2c} \, erfc(c\sqrt{\rho k}) ,$$

wherein "erfc" denotes the complementary error function. Therefore

$$K = e^{k\rho c^2} \frac{k\pi}{2c} \int_{\rho}^{\infty} d\tau \, erfc(c\sqrt{\tau k}) \qquad (H.25).$$

Equations (H.21) and (H.22) now become

$$I_1 = -e^{k\rho c^2} \frac{2\sqrt{2}\pi e^{-i\pi/4} e^{-ik\rho}}{c} \int_{\rho}^{\infty} d\tau \, erfc(c\sqrt{\tau k}) \qquad (H.26),$$

and

$$I_2 = -e^{k\rho c^2} \frac{2\sqrt{2k}\pi e^{-ik\rho}}{c} \int_{\rho}^{\infty} d\tau \, erfc(c\sqrt{\tau k}) \qquad (H.27).$$

so that from equation (H.8) we find that approximately

$$h_2 E_z^+ = \frac{-\sqrt{2}\, e^{i\pi/4} \mu\mu_0 \omega I \, dl}{c} \left\{2\nu + \frac{\partial}{\partial \rho}\right\}$$

$$\times \left\{e^{k\rho c^2} e^{-ik\rho} \int_{\rho}^{\infty} d\tau \, erfc(c\sqrt{\tau k})\right\}$$

$$- \frac{i\mu_0 \omega I \, dl \sqrt{k}}{c\sqrt{\pi x_0}} \left[1 + \frac{\mu^2}{\nu^2}\right] \left\{2\nu + \frac{\partial}{\partial \rho}\right\}$$

$$\times \left\{e^{k\rho c^2} e^{-ik\rho} \int_{\rho}^{\infty} d\tau \, erfc(c\sqrt{\tau k})\right\} \qquad (H.28).$$

If we denote

$$K_1 = \frac{\sqrt{2}\, e^{i\pi/4} \mu \mu_0 \omega I\, dl}{c} \quad \text{(H. 29)},$$

and

$$K_2 = \frac{i \mu_0 \omega I\, dl}{c \sqrt{\pi}} \left[1 + \frac{\mu^2}{\nu^2} \right] \quad \text{(H. 30)},$$

and observe from equations (H. 24) and (H. 11) that

$$\frac{\partial}{\partial \rho} \left[e^{-ik\rho(1+ic^2)} \right] = \frac{\partial}{\partial \rho} \left[e^{ik\rho \cos(\psi_1)} \right]$$

$$= ik \cos(\psi_1) \left[e^{k\rho c^2} e^{-ik\rho} \right]$$

$$= -2\mu \left[e^{k\rho c^2} e^{-ik\rho} \right] \quad \text{(H. 31)},$$

then equation (H. 28) may be rewritten

$$h_2 E_z^+ = -2(K_1 + \frac{K_2}{\sqrt{x_0}})(\nu - \mu)\, e^{k\rho c^2} e^{-ik\rho} A$$

$$- (K_1 + \frac{K_2}{\sqrt{x_0}})\, e^{k\rho c^2} e^{-ik\rho} \frac{\partial A}{\partial \rho} \quad \text{(H. 32)},$$

wherein

$$A = \int_{\rho}^{\infty} d\tau\, \mathrm{erfc}(c \sqrt{\tau k}) \quad \text{(H. 33)}.$$

The integral A may be evaluated exactly using integration by parts. The result is

$$A = -\rho \, erfc [c\sqrt{\rho k}] + \frac{\sqrt{\rho}\, e^{-c^2 \rho k}}{c\sqrt{k\pi}} + \frac{erfc[c\sqrt{\rho k}]}{2c^2 k} \qquad (H.34).$$

Observing that

$$\frac{\partial}{\partial \rho} erfc [c\sqrt{\rho k}] = -\frac{c\sqrt{k}}{\sqrt{\pi}} \frac{e^{-c^2 \rho k}}{\sqrt{\rho}} \qquad (H.35).$$

we may substitute from equation (H.34) for A into equation (H.32) to find

$$h_2 E_z^+ = (K_1 + \frac{K_2}{\sqrt{\kappa_0}}) e^{k\rho c^2} e^{-ik\rho} erfc [c\sqrt{\rho k}]$$

$$-2(K_1 + \frac{K_2}{\sqrt{\kappa_0}})(\nu - \mu) e^{k\rho c^2} e^{-ik\rho}$$

$$\times \left[-\rho \, erfc [c\sqrt{\rho k}] + \frac{\sqrt{\rho}\, e^{-c^2 \rho k}}{c\sqrt{k\pi}} + \frac{erfc[c\sqrt{\rho k}]}{2c^2 k} \right] \qquad (H.36).$$

In order that we may gain somewhat of a comparison with the classical results, we shall employ an asymptotic expansion of the complementary error function appearing in equation (H.36). From equations (H.24), (H.11), and our previous assumptions on $|\mu|$ we [illegible] that

$$|c| \approx 1$$

so that the argument of the complementary error function is certainly such that

$$|c\sqrt{\rho k}| \gg 1$$

(Recall that $\rho = (x + x_0)\cos\theta = x + x_0$ since $\theta = 0$. Therefore $\rho > x_0$ by virtue of the h_2 function appearing on the left-hand side of equation (H.36)). Taking the expansion [see Abramowitz & Stegun [1]]

$$erfc[c\sqrt{\rho k}] \approx \frac{e^{-c^2\rho k}}{c\sqrt{\rho k\pi}}\left[1 - \frac{1}{2c^2\rho k} + \frac{1\cdot 3}{(2c^2\rho k)^2} - \cdots\right]$$

and substituting into equation (H.36) we find, after some simplification, that

$$h_2E_z^+ \approx (K_1 + \frac{K_2}{\sqrt{x_0}})\,\frac{e^{-ik\rho}}{c\sqrt{\rho k\pi}}\left[1 - \frac{1}{2c^2\rho k} + \cdots\right]$$

$$- 2(K_1 + \frac{K_2}{\sqrt{x_0}})\,\frac{e^{-ik\rho}}{c^3k\sqrt{\pi\rho k}}\,(\nu-\mu)\left[1 - \frac{1}{c^2\rho k} + \cdots\right] \qquad \text{(H.37)}.$$

From equations (H.29) and (H.30) we may substitute for K_1 and K_2 in the expression (H.37). Retaining the most significant terms we find that approximately

$$h_2E_z^+ = \frac{\vec{k}^2 I\,dl}{2\pi i\omega\epsilon_0}\,\frac{e^{-ik\rho}}{\rho}$$

$$\times\left\{\left(\frac{1+2\nu i/k}{1+2\mu i/k}\right)\left[c_2\left(\sqrt{\rho} - \frac{1}{2c_1^2k\sqrt{\rho}}\right) - c_3\frac{\sqrt{\rho}}{\sqrt{x_0}}\left(1 - \frac{1}{2c_1^2k\rho}\right)\right]\right.$$

$$\left.- \frac{(\mu - \nu)}{c_1^4k^2\sqrt{\rho}}\left[c_2 - \frac{c_3}{\sqrt{x_0}}\right]\right\} \qquad \text{(H.38)},$$

wherein

$$c_1 = c = i[1 + 2\mu i/k] \qquad (H.39),$$

$$c_2 = \frac{2\sqrt{2}\, e^{i\pi/4} \mu \sqrt{\pi}}{\sqrt{k}(1 + 2\mu i/k)} \qquad (H.40),$$

and

$$c_3 = \frac{2(1 + \mu^2/\nu^2)}{i(1 + 2\mu i/k)} \qquad (H.41),$$

hence equation (4.73).

APPENDIX I: DIAGRAMS

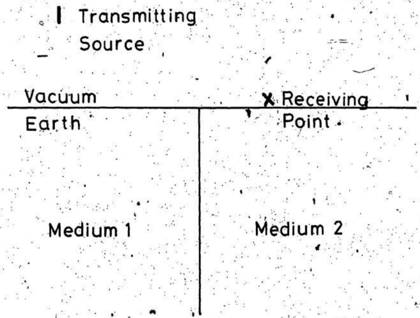

Diagram 1.1 Geometry of the mixed-path problem.

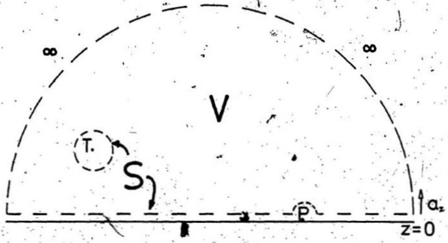

Diagram 2.1 The surface S and the volume V used in Green's Integral Theorem.

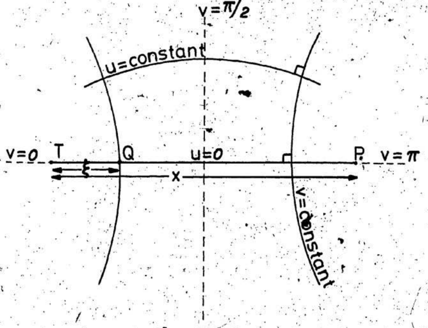

Diagram 2.2 Integration region used in equations (2.12).

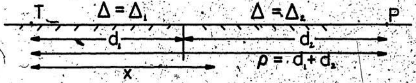

Diagram 2.3 Bremmer's physical geometry in the case of two adjacent homogeneous media.

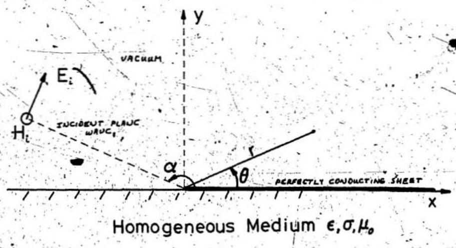

Diagram 2.4 Physical geometry of Clemmow's initial model.

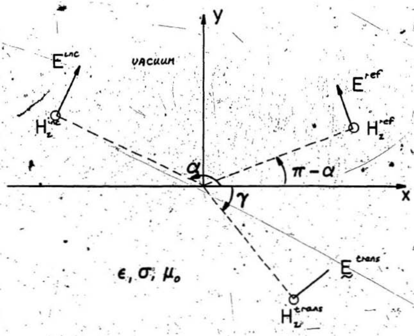

Diagram 2.5 Resultant propagation if no infinitely conducting sheet were present.

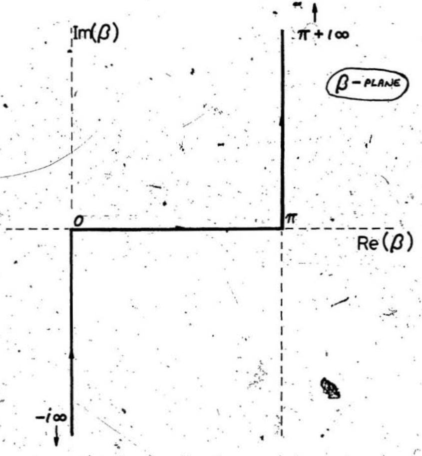

Diagram 2.6 Integration contour representing a general superposition of homogeneous and inhomogeneous plane waves.

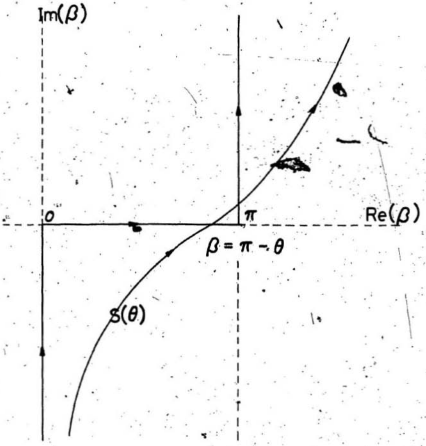

Diagram 2.7 Clemmow's steepest descent integration path $S(\theta)$.

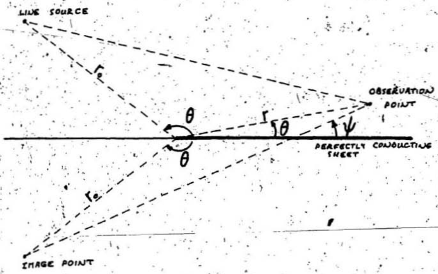

Diagram 2.8 Clemmow's geometry for the source and observation points.

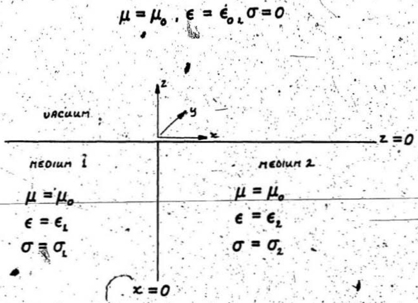

Diagram 3. 1 Physical geometry used in this work.

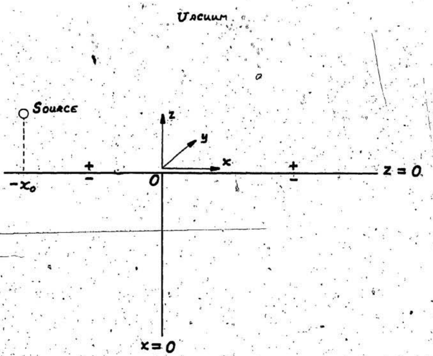

Diagram 3.2 Scenario showing source location and various "surface" subscripts on field quantities.

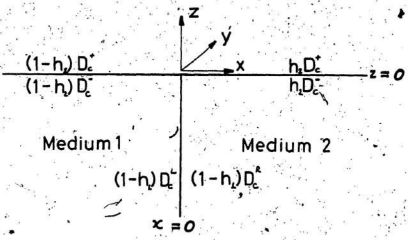

Diagram 3.3 Various displacement current field vector surface supports.

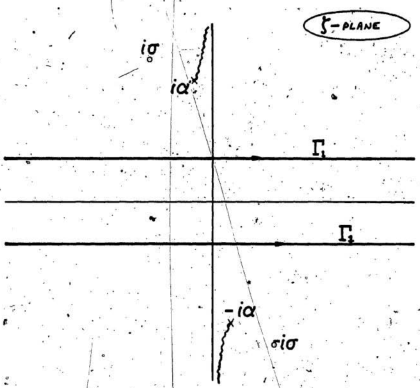

Diagram 4.1 Integration contours used in equations (4.31) & (4.32).

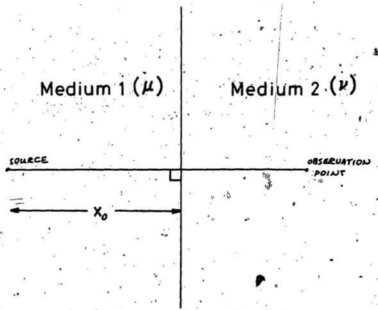

Diagram 4.2 Geometrical quantities used in equation (4.73).

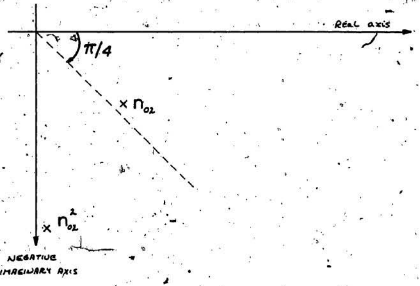

Diagram. B. 1 Location of the refractive index in the complex plane.

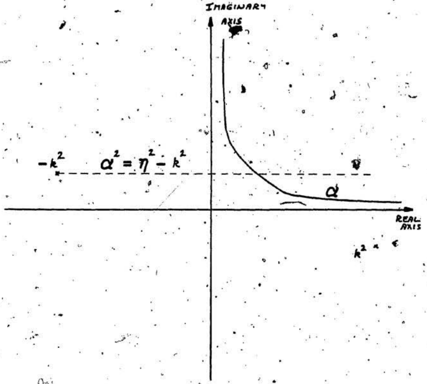

Diagram B.2 Location of α in the complex plane.

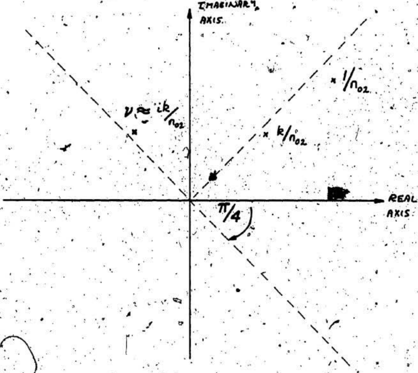

Diagram B.3 Location of ν in the complex plane.

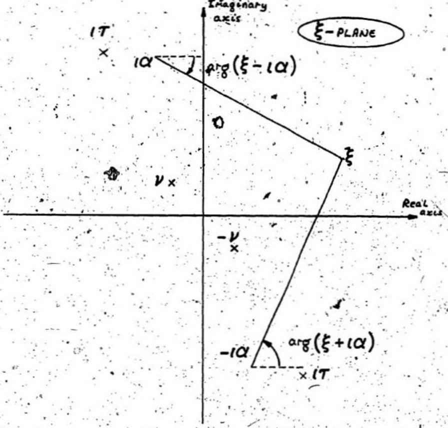

Diagram B. 4 Analytical structure of $R(\xi)$ [not to scale].

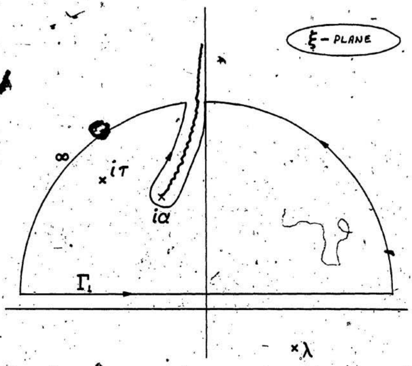

Diagram C.1 Integration contour used in equation (C.14).

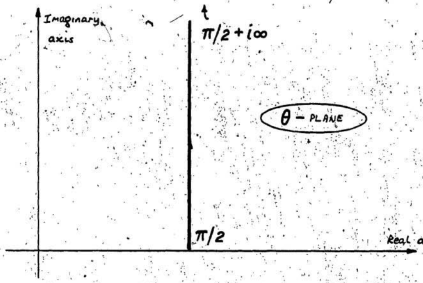

Diagram C.2. Integration contour used in equation (C.20).

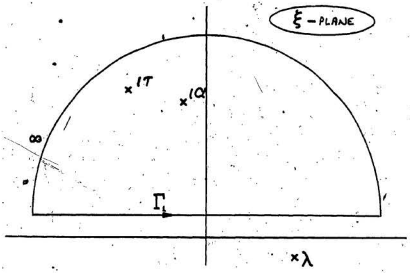

Diagram C.3 Integration contour used in equations (C.3) & (C.4).

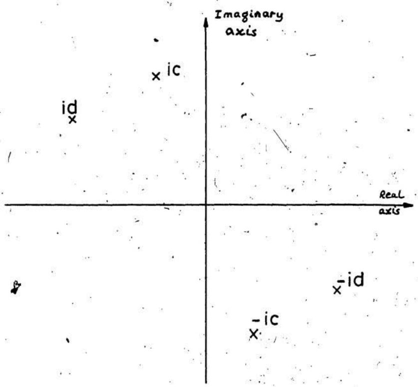

Diagram E.1 Location of "constant" appearing in $A_1(\xi)$ in the complex plane.

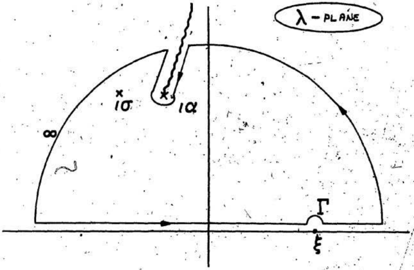

Diagram G.1 Integration contour used in equation (G.5).

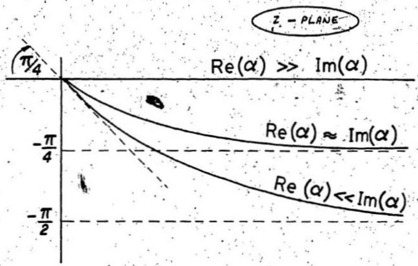

Diagram G.2 Steepest descent paths for various values of α.

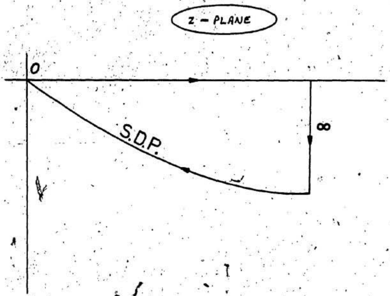

Diagram G.3 Integration contour used in equation (G.19).

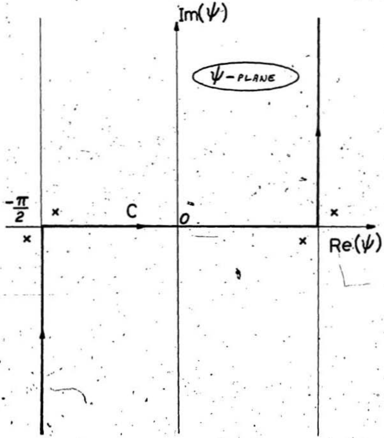

Diagram H.1 Integration contour used in equations (H.12) & (H.13).

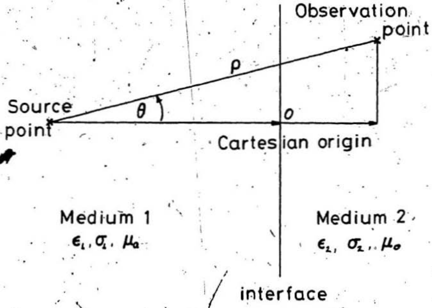

Diagram H.2 Spatial variables defined in equation (H.11).

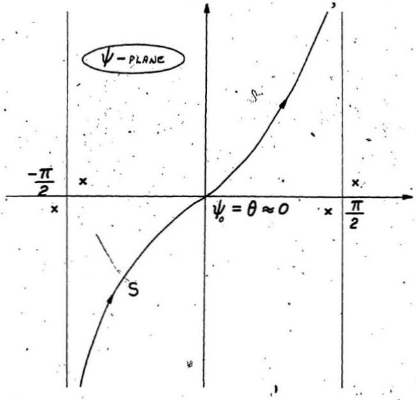

Diagram H.3 Steepest descent integration contour.