FINITE ELEMENT FORMULATION OF VISCOUS SEA ICE MODEL

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FINITE, ELEMENT FORMULATION

OF

VISCOUS SEA ICE MODEL

by

Satti Vijaya Bhaskara Reddy, B.E.

A Thesis submitted in partial fulfillment of the requirements for the degree of Master of Engineering

Department of Engineering and Applied Science Memorial University of Newfoundland

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St. John's

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Newfoundland

The steady drift of sea ice in the Arctic Basin has been calculated by using the Finite Element Method, assuming that the ice is a viscous material. The Arctic Basin is discretized into 9 x 13 square elements of size 250 km by 250 km. The constitutive relations proposed by Glen (1970) have been used in the modeling. The momentum and continuity equations are solved for the dce velocity due to wind stress ocean current and ocean tilt. The drift has been calculated for four seasons in a span of two years by using proper viscosities in summer and winter. The vector addition of ice velocities due to wind, ocean current and tilt are plotted as combined velocities for Sul four seasons Summer 1, Winter 1, Aumer 2 and Winter 2.

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The results of the finite Element Method of gnalysis are compared with that of the Fourier Transform Method (Fibler, 1976) and it is found that the reduits obtained by both methods agree with each other in the middle of the Basin. A comparison between slip and no slip boundary condition is also presented Boundary elements are used and an iterative procedure is bollowed to incorporate the nonlinear boundary conditions.

A non-stoady drift of pack ice in the Strait of Belle Isle, which any set of the strain strains and currents have been calculated by the Finite Element Method formulation. It is observed that the pack ice velocities are harmonic as are the water velocities but with a time lag which depends upon viscosity parameters of ice.

Various velocity fields have been plotted and all patterns of calculated velocities are observed to be realistic.

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LIST OF SYMBOLS USED

Empirical constant relating to air-ice Ekman layer. Empirical constant relating to water-ice Ekman layer. Force due to variations in internal ice stress.

Coriolis parameter.

Long term geostrophic currents.

g Gravitational acceleration.

H Sea surface height.

[K] Matrix derived from the energy dissipation in icewater system.

Unit vector perpendicular to x-y plane.

 $L_{1,1}L_{2,L_3}$ Natural co-ordinates for a triangular element. $l_{1,1}$ Length parameter.

Wave length of forcing function.

[M] Mass matrix.

Mass of pack ice per unit area.

P Atmospheric pressure.

{Q} Load vector.

{q} Generalised velocity vector.

S , Boundary line.

s Variable length along boundary.

T Ocean tilt.

Velocity component in x direction.

Velocity component in y direction.

Ice velocity vector.

- Tw Water stress on ice.
- T, Air stress on ice.
- 6 Ekman angle in water.
- Ekman angle in air.
- p Density of air.
- oii Internal stress components in ice.
- eii Strain-rate components.
- $\dot{\gamma}_{xy}$ Strain-rate due to shear.
- Ω. * Rotation of earth in radians per second.
- λ. Latitude.
- ζ Bulk viscosity.
- Shear viscosity.
 - Frequency.

CHAPTER 1

The factors affecting sea ice formation are said nity of water, wand, currents and the intensity of cooling. There are different types of ice in sea water. Fresh needle-like crystal's closing together form a uniform sheet called young ice. The other types are paneake ice, fast ice of shores and pack ice. The Arctic pack ice covers large areas of the Arctic Ocean. The thickness of ice varies from zero to few meters. The pack ice sheets are observed. In hundreds of meters across, and with patches of leads. Leads are open water or thin sheets of ice in the Arctic region. Pack ice is found during the greater part of the year in most portions of the marginal sens. Baffin Bay and the western part of the jabrader Sea, the Canadian Archipelage, Davis Strait and the East Greenland Coast.

Pack ice moves on the surface of sea water due to wind ocean currents, body forces and interaction of weighboring ice. As it moves, it may heap up, lead ice into pressure ridges and hummocks in the form of rubble. The drift of ice is not generally noticeable to a person standing on it even though he might have moved hundreds of kilometers in a few months. The drift of ice in the Arctic Basin has been discussed by Mansen (1902) and Sverdrup (1928), and it has been an important research field in recent years due to the recent discovery of hydrocarbon deposits in the Beaufort Sea. Antarctic ice drift was studied by Brennecke (1921) and the deviation angle between drift of ice and wind direction was found to be 34°. to the left. The Mansen and Sverdrup studies of the deviation angle between the wind direction and the Arctic ice drift direction is 30° to the right. Nansen explained this devia-

On the suggestion of Nansen, Ekman (1905) found that the effect of wind would be up to some depth in sea water. He observed that the velocities of different layers would deviate more and more as we go deeper and deeper in the water, but the magnitudes of velocity of different layers. reduce exponentially with depth.

There are permanent drift tracks in the Arctic Basin which have been observed in recent years more closely than ever before (Figure 1.1). There is sufficient data available to acquire a rather accurate picture of the long term mean drift. The important drift tracks of ice in the Arctic Basin are a gyre in the Behufort Sea and a transpolar drift stream. The drift pattern is very much wind driven. The small divergence and the difference in the import and export of ice have little effect on the Basin. Many scientists have proposed different models to predict the pack ice drift in the Arctic Basin. The idea behind mathematical model for is to find out whether some realistic flow fields could be calculated, and more importantly, to see whether the pressure field within the ice cover can be interpreted sensibly in the light of haven icd conditions in different areas of the basin.

There are two aspects of sea ice modeling which are not well understood, and different theories on ice drift have been proposed due to limited knowledge and data. The first aspect deals with the mechanical properties of ice in small scale, and the second aspect deals with the behavior of pack ice on a large scale. Considerable literature exists on the research in these two fields.

Tcc is a crystalline material with hexagonal symmetry. Its characteristics, such as thickness, quality, and type of grain structure, are subject to variability with weather. Elasticity, plasticity and all mechanical properties are sensitive to temperature. Brine volume in sea ice has a noticeable influence on its properties. Statistics and probability theory are entering all fields of engineering, and their use in ice studies is indispensable. Several years of observation are necessary to establish useful information regarding ice properties on a gmall or large scale.

In view of the above mentioned factors, sea ice has been modeled as a viscous material by many scientists (Campbell, 1965, Glen,1970, Hibler,1974), as a plastic material by Coon (1972), as an elastic-plastic material by Cooa, et al (1974) and as an inviscid fluid by Rothrock (1973).

A finite element formulation of the viscous sea icemodel is presented here. Glen (1970) proposed the constitutive relations, and Hibler and Tucker (1976) proposed that the viscosity parameters in the viscous sea ice model vary according to the seasons. In the finite element analysis, a set of two linear differential equations with variable viscosity parameters are solved to obtain velocities of pack ice which is driven by wind, ocean current and ocean tilt. The results obtained by the finite element method are, in good agreement with those obtained by other methods. The advantage of the finite element method is that it is capable of solving the differential equations over an irregular domain. Realistic boundary conditions can be incorporated by introducing the boundary elements whose properties change according to the conditions whether ice is pushing into or pulling away from the boundary. The results of computations are presented for the steady state ice drift in the Arctic Basin and the transient ice velocity of pack ice in the strait of Belle Isle which is driven by strong tidal streams.

CHAPTER 2 MECHANICAL BEHAVIOR OF PACK ICE

2.1 KINEMATICS

The velocity field of pack'ice varies in space and in time. But velocities alone are poor indicators of ice behavior because the mechanical behavior of pack ice is related to velocity differences between floes, or the velocity gradients may be well defined in a continuum, but in a real ice cover they are not. Abrupt changes may take place at floe boundaries. To measure the response of ice to real weather systems, which may vary over a periods of about a day, the measurements should be arranged over periods of lees than a day. Temporal averages are of not much use. If spacial averages are to be useful, there must be a continuum length of scale 1₀ which is large compared to floe size 1₁ and small compared to the smallest space scale 1₂ of the driving forces (Maykut, Thorndike and Untersteiner, 1972), that is, 1₂ is to exist, such that

 $l_i < l_c < l_f$ (2.1)

1 your dial to

The lengths 1 and 1 are not well defined, but in the Arctic Ocean rigid pieces of ice larger than 50 km across are rare, and atmospheric pressure systems in the Arctic Ocean are typically 1000 km to 2000 km across.

2

Nye (1973a) proposed to smooth the velocity field eliminating variations shorter than 1 and then defining strain rates. But this idea is not practical. To obtain correct size of gauge length 1, two methods have been adopted. In the first method, strain rates were simultaneously measured on a gauge of 100 km and on a gauge of 20 km and less. If both these gauge lengths are within the continuum scales, the measured strain rates would correlate perfectly. These measurements showed that the net dilatation over a 25 day period (Hibler, et al, 1974) was strongly dependent on the gauge length. It was clear that one did not obtain the same estimate of strain rate by measuring over these different gauge lengths. In the second method, Nye and Thomas (1974) showed that the variance of strain rate measurements decreases as the gauge length 1, increases, and this effect was observed upto a gauge length of 100 km. From these two methods, it appears that the continuum length scale is 100 km or larger,

From different data (Colony and Kash, 1975, Dunbar and Wittman, 1963), we now have a good idea of typical velocities, vorticities and strain rates in the central Arctic Ocean. Pack ice moves at speeds of 10 km/day, occasionally reaching twice that speed. The actual path followed by a point in the ice is several times the net distance it travels over a period of months.

Instantaneous strain rates/ can be as large as several

percent per day, but the long term averages are considerably smaller. During the 40 day period documented by Thorndike (1974), the mean rates of divergence and shear (defined as the sum and difference of principal strain rates) were 0.001 and 0.002 per day, respectively. These numbers reflect the tendency of ice to deform more in shear than in divergence or convergence. The vorticity (defined as the vertical component of curl <u>u</u>, where <u>u</u> is the velocity), behaves much the same as the rate of shear.

From most of these observations, one can infer that there is much difference between 'intra' and 'interfloe' strain, and that the strain which contributes to the large scale circulation of pack ice is caused by the relative movement of pieces of icearch not by the continuous deformation of pieces themselves.

2.2 THE STATE VARIABLES OF ICE COVER

The study of the properties of pack ice is complicated as it depends on changes in weather. The ice thickness may increase due to rigging or rafting or thermodynamic changes in weather. Some open leads may form due to rigging and rafting. Thus the continuously varying characteristics such as compactness and thickness of ice must be considered in the model of sea ice. There are different approaches followed by different scientists to include these effects in their models.

In one approach, the compactness C has been defined as the fraction of the area covered by ice. The remaining fraction (1-C) is covered by open water. Drogaitsev (1956) and Nikiforov (1957) assumed that the ice covered area C within the total area B is conserved $\left(\frac{dC}{dt} \in 0\right)$. This means that only the area covered by open water (B-C) can change. Doronin (1970) modified the above equation by introducing a source term in the compactness equation to allow for thermodynamic changes in C. In the case of ridging, one can include a sink term in the compactness equation.

A different but more complete approach is to introduce terms of ice thickness distribution, g(h), which is defined as the fraction of the area covered by ice in the thickness band h_1 to h_2 and is given by $\int_{h_1}^{h_2} g(h) dh$ (Maykut and Thorndyke, 1973). AIDJEX scientists (Coon, et al. 1974) used a function Ψ as a redistribution function in the thickness distribution. This function describes the creation of open water and the transfer of ice from one thickness to another by rating and ridging. They incorporated the above concept in the elastic-plastic model for sea ice.

In the viscous sea ice model, pack ice is treated as a viscous material which resists changes in area of ice cover by the bulk viscosity modulus and the shear deformation by shear viscosity modulus. These viscosity parameters vary according to the season.

2.3 MOMENTUM EQUATION

The model of pack ice will have a momentum equation for the steady state condition in the following form:

 $-\mathrm{mf} \underline{k} \times \underline{u} + \underline{\tau}_{w} + \underline{\tau}_{a} + \underline{F} + \underline{G} + \underline{T} = 0 \quad (2.2)$

where <u>u</u> is the ice velocity whose components are <u>u</u> and <u>v</u> in the <u>x</u> and <u>y</u> directions, f the Coriclis parameter which is equal to twice the product of earth's rate of rotation and sine of latitude, <u>k</u> a unit vector normal to <u>x</u>-y plane, <u>m</u> the ice mass per unit area, <u>F</u> the force due to variations in the internal ice stress ($\mathbf{F}_{1} = \sigma_{ij,j}$), <u>Iw</u> and <u>Im</u> the water and air stresses on the ice, and <u>B</u> and <u>T</u> the effect of long term geostrophic currents and ocean tilt on the ice motion. The components of current and air stresses are given by a modified Ekman layer theory (Rothrock, 1975).

$$\begin{split} \tau_{\mathbf{x}_{\mathbf{x}}} &= \mathbf{B} \left(\cos\phi \ \mathbf{U}_{\mathbf{g}} - \sin\phi \ \mathbf{V}_{\mathbf{g}} \right) \\ \\ \tau_{\mathbf{x}_{\mathbf{y}}} &= \mathbf{B} \left(\sin\phi \ \mathbf{U}_{\mathbf{g}} + \cos\phi \ \mathbf{V}_{\mathbf{g}} \right) \\ \\ \mathbf{G}_{\mathbf{x}} + \tau_{\mathbf{w}_{\mathbf{x}}} &= \mathbf{D} \left(-\cos\theta \ (\mathbf{u} - \mathbf{U}_{\mathbf{w}}) + \sin\theta \ (\mathbf{v} - \mathbf{V}_{\mathbf{w}}) \right) \end{split}$$
 (2.3)

 $G_y + \tau_{w_v} = -D \{ \sin \theta (u - U_w) + \cos \theta (v - V_w) \}$

where ϕ and θ are Ekman angles in the air and water, respectively, Ug and Vg are the x and y components of the geostrophic wind, and Ug and Vg are the x and y components of the geostrophic ocean flow beneath the Ekman layer. In equation (2.3), the G components are those components linear in U_w and V_w. In the case when G is neglected, the sea ice is effectively considered to be moving across a stagnant ocean. The geostrophic current flow is computed from ocean height H as follows:

 $\underline{\mathbf{U}}_{\mathbf{w}} = \mathbf{g} \ \mathbf{f}^{-1} \ \underline{\mathbf{k}} \times \underline{\nabla} \mathbf{H} \quad \boldsymbol{\clubsuit}$

where g is the gravitational acceleration. The filt component is given by

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(2.5

(2.6)

The geostrophic wind is related to the surface atmospheric pressure P by the following expression:

 $\underline{\mathbf{U}}_{\mathbf{g}} = (\boldsymbol{\rho}_{\mathbf{a}} \cdot \mathbf{f})^{-1} \underline{\mathbf{k}} \times \underline{\nabla} \mathbf{P}$

where ρ_{o} is the density of air.

T = -m g VH

As the earth rotates about its axis (7.29x10⁻⁵ radians/sec), a velocity vector on it is continuously changing relative to space. This change in velocity per gecond is the Coriolis acceleration and it is perpendicular to both the velocity vector and rotation vector of the earth. As pack the has considerable mass, we cannot neglect the Coriolis force which is given by, the first term in the momentum equation (2.2).

For the steady state condition, we have neglected the acceleration term on the right hand side of equation (2.2). This was discussed by Buynitskiy (1851). Rothrock (1970) showed that the acceleration of pack ice is of the order of 10^{-5} cm/sec² as a typical value in the Arctic Basin on the time scale of one day. But when two floes collide, the time scale is very much. shorter than one day and these small scale accelerations are almost certainly significant, in which case we have to include an <u>mu</u> term on the right hand side of equation (2.2).

The deformation behavior of pack can be treated as that of a highly damped material. The wind drags the ice over a relatively stationary ocean, opposed mostly by water drag and resistance of the ice cover to deformation. The wind usually changes so slowly that the ice offers little inertial resistance to it. The work of the wind is dissipated partly by work against internal stress and partly by work on the ocean. The kinetic energy of the ice cover and its changes are relatively unimportant in the Arctic Basin.

2.4 CONSTITUTIVE RELATIONS

In order to integrate the momentum equation (2.2), we need constitutive relations which relate the stress tensor to other variables in the problem. In understanding ice dynamics, this has been the greatest single stumbling block. Different scientists have used different constitutive relations treating pack ice as a viscous material (Campbell, 1965), an incompressible and inviscid fluid (Rothrock, 1973), s cavitating fluid (Nikiforov, et al, 1967, Doronin, 1970), and an elastic-plastic material (Coon, et al, 1974).

Glen (1970) discussed various constitutive relations for a viscous model and proposed a constitutive relation stating that the internal ice stress \underline{P} is related to stress and strain rates in the ice. The following are the relations, proposed by Glen:

 $\sigma_{i,i} = 2 \eta \dot{\epsilon}_{i,i} + (\zeta - \eta) \dot{\epsilon}_{kk} \delta_{i,i}$ (2.7)

$$F_{i} = \sigma_{i+1} = \eta \, \nabla^{2} u_{i} + \zeta \, \nabla_{i} (\nabla_{k} \, u_{k})$$
(2.8)

where n and ζ are shear and bulk viscosities characterising the ice cover on the geophysical scale, and $\dot{\epsilon}_{ij}$ are the strain rate components given below

 $\dot{\varepsilon}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ (2.9)

Substituting equations (2.7 to 2.9) into

13

equation (§.2), a system of two coupled second order. Hinear partial differential equations is obtained for the drift rate u(x,y) and v(x,y).

AND AND

Bibler and Tucker (1976) estimated the values of bulk and shear viscosities by correlating the observed and predicted drift rates, and they found that the viscosity values giving the best fit between the observed and predicted values show Appronounced seasonal variation that correlates well with ice growth rate. They suggested the following empirical linear relationship between viscosity and ice growth rate which yields predictions in reasonable agreement with both long (yearly) and short (monthly) term observed drift rates:

 $\ln(\eta) = \ln(\zeta) = 0.92 \langle G \rangle + 24.73$ (2.10)

where <G> is the 60 day averaged growth rate which varies according to season. The range of best fit values of viscosity is from '10¹⁹ to 4x10¹¹ kg/sec.

CHAPTER ::

FINITE ELEMENT METHOD FORMULATION

3.1 INTRODUCTION

The Finite Element Method is one of the popular numerical techniques which is finding a place in a wide variety of englneering problems: The approximate solution is obtained at a few selected points (nodes) of a domain.

In a continuum problem of any dimension the field variable (whether it is pressure, temperature, velocity, stress or some other quantity) possesses an infinite number of values, becausé it is a function of each generic point, in the body or solution : region. Consequently the problem is one with an infinite number of unknowns. In the finite element method, the solution region is discretized into elements. Element boundaries are connected with specified points called nodes or nodal points. Now we express the unknown field variable in terms of assumed approximate functions called interpolation functions within each element. These interpolation functions are defined in terms of the values of the field variables at the nodal points. Thus, the problem is reduced to one of a finite number of unknowns of the field variable at the nodal points. individual element properties can be formulated by one of the following four approaches: the direct approach, the variational approach, the weighted residuals approach and the energy balance approach. The choice of these approaches depends on the type of the problem.

In the present problem, we use the variational approach. In the chassical variational formulation, the problem is to find the unknown function of functions, which make stationary a functional or functionals, subjected to some given boundary conditions.

One of the important

advantages of the variational approach is that it implicitly imposes the natural boundary conditions. In the finite element method, we explicitly impose geometric boundary conditions. Thus, in this method both types of boundary conditions are satisfied. The finite element method can be applied to problems with complicated boundary conditions and irregular shaped boundaries. 3.2 THE FINITE ELEMENT FORMULATION

Carlos and

The momentum equation is written with the inertial term in the following form:

 $-\mathbf{m} \mathbf{f} \mathbf{\underline{k}} \times \mathbf{\underline{u}} + \mathbf{\underline{f}}_{\mathbf{\underline{u}}} + \mathbf{\underline{f}}_{\mathbf{\underline{n}}} + \mathbf{\underline{F}} + \mathbf{\underline{G}} + \mathbf{\underline{T}} - \mathbf{\underline{m}} \mathbf{\underline{u}} = \mathbf{0} \qquad (2.2a)$

where 'mu' is the change of momentum per unit area. In solving the momentum equation, we take ice velocity components as unknowns. A compatible velocity field is assumed in order to have a continuum which does not overlap or has holes in it during deformation. Strain rates are obtained by spacial differentiation of velocity components. Stress components can be obtained in terms of strain rate 'components by the use of constitutive relations. Thus, the set of two equations (2.2a) can be expressed in terms of velocity components u, and v:

For the finite element formulation, we need a functional as explained in section (3.1), and this is obtained by multiplying the momentum equations (2.2a) by variations of u and a and integrating the sum of products over the domain. Thus, we have

δΠ = [[δu . δv] 2mQsinl. -2mQsin)

 $+ \begin{bmatrix} -D \cos \theta & D \sin \theta \\ -D \sin \theta & -D \cos \theta \end{bmatrix} \begin{bmatrix} u - U_{y} \\ v - V_{y} \end{bmatrix} + \begin{bmatrix} B \cos \theta & -B \sin \theta \\ B \sin \theta & B \cos \theta \end{bmatrix} \begin{bmatrix} U_{g} \\ v_{g} \end{bmatrix}$ $+ \begin{bmatrix} q_{xy, x} + \tau_{xy, y} + T_{x} \\ \sigma_{xy, x} + \sigma_{yy, y} + T_{y} \end{bmatrix} = \begin{bmatrix} n \\ 0 \end{bmatrix} \begin{bmatrix} U_{g} \\ B \sin \theta & B \cos \theta \end{bmatrix} \begin{bmatrix} U_{g} \\ v_{g} \end{bmatrix}$

where Ω is the earth's rate of rotation, λ is the latitude, and σ_{ij} is the two dimensional stress tensor whose components are the vertically integrated quantities across the thickness of pack ice. The units of σ_{ij} , are force per unit length.

Integrating some of the terms in the above equation (3.1) by parts, we can express the functional in the following form by using the constitutive relations (2.7) and the equation (2.9):

$$= \int_{2\pi i 0}^{1} \frac{1}{2} \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} -D \cos \theta & D \sin \theta \\ +2m \Omega \sin \lambda \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

 $\begin{array}{c|c} + \begin{bmatrix} \mathbf{v} & \mathbf{v} \end{bmatrix} \begin{bmatrix} \mathbf{D} \cos \theta & -\mathbf{D} \sin \theta \end{bmatrix} \begin{bmatrix} \mathbf{U}_w \\ \mathbf{w} \end{bmatrix} \\ \begin{bmatrix} \mathbf{D} \sin \theta & \mathbf{D} \cos \theta \end{bmatrix} \begin{bmatrix} \mathbf{V}_w \end{bmatrix}$

+
$$\begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \begin{bmatrix} \mathbf{T}_{\mathbf{x}} \\ \\ \\ \mathbf{T}_{\mathbf{y}} \end{bmatrix}_{\mathbf{x}}$$

$$\mathbf{a} = \mathbf{v} \begin{bmatrix} \mathbf{m} & \mathbf{0} \\ \mathbf{0} & \mathbf{m} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$$

$$d\mathbf{A} + \begin{bmatrix} \mathbf{u} \\ \mathbf{S}_{T} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{n} \end{bmatrix} d\mathbf{z}$$

$$\mathbf{A} + \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{v} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{v} \end{bmatrix} d\mathbf{z}$$

$$\mathbf{v} = \begin{bmatrix} \mathbf{v} \\ \mathbf{v} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{v} \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} \mathbf{v} \\ \mathbf{v} \end{bmatrix}$$

where u_n , u_t , σ_{nn} and τ_{nt} are the velocity and stress components rolative to the coordinate system <u>n</u> and <u>t</u> which are the unit outward normal and unit tangent vectors at the boundary, S_t is that part of the boundary where boundary forces are prescribed (see Figure 4-20).

In the above functional (3.2), we note that there are some terms which represent the rate of work done by external forces, and there are other terms which represent the rate of energy dissipation. When the functional is stationary (1.6.5 M=0), the incremental rate of work done by external forces is equal to the incremental rate of energy dissipation which takes place in the form of deformation of pack ice, drag due to water and the change of kinetic energy.

For the finite element formulation; the functional (3.2) over the whole domain can be written as the sum of the functional over finite size elements into which the whole domain is discretized. Thus, we have

where Π_{e} represents the functional (3.2) in which the integration is performed over element area and boundary.

Ι = Σ Π.

A bilinear interpolation function is assumed for each of the velocity components in a triangular element. The derivation of element matrices, defining the element properties, is given in the Appendix A. These element matrices are derived from the functional (3.2). After assembling the element matrices, we can represent the first variation of functional (3.2) in the following form:

 $\delta \Pi = \delta \{q\}^{\dagger} \left[-[K]\{q\} - [M]\{\dot{q}\} + \{Q\} \right] = 0$ (3.3)

where [K] is the matrix derived from the energy dissipation due to water drag and deformation of pack ice, [M] the mass matrix derived from the kinetic energy expression. (Q) the

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generalised force vector representing the forces caused by wind, currents and ocean tilt, and (q) the generalised modal velocity vector of the system.

If δ{q} is to be arbitrary, equation (3.3) requires that

[M]{q} + [K]{q} = {Q}

The above equations are solved to obtain the nodal velocities of the system. For computations of pack ice drift in the Arctic Basin, only the steady state velocities are calculated due to the small contribution of the inertial term in relation to the other terms in the momentum equation (2.2). Hence, we solve the following equation for steady state pack ice drift.

 $[K]{q} = {Q}$

The finite element program, written by Desai and Abel (1973), was modified to compute velocity of pack ice using the formulation given above. In this program, four triangular elements are assembled to make a general quadrilateral element by elminating the internal node by the static condensation process (Desai and Abel, 1973). The resulting algebraic system of equations after assembling the element properties give rise to an unsymmetric banded matrix. A new subroutine was written to eavier a system of algebraic equations with

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(3.4)

(3.5)

unsymmetric banded matrix by the method of decomposition of the matrix into lower and upper triangular matrices. Several other steps were incorporated in the program in order to compute consistent generalised forces at the nodes which are desorbled in Appendix A.

Runge-Kutta mehtod of integration of the first order equation (3.4) is also incorporated in the program in order to compute time dependent velocity of pack ice. For the first order system in time, there is a characteristic time for the system which is an indicator of the response time of the system, and it depends-upon the inertial mass and the energy dissipating property of the system. When the viscosity parameters are large, as they are for the pack ice in the Arotic Basin, the response time is/small, and we may as well solve the steady state velocity of pack ice using equation. (3.5). For the locesly compacted ice, the viscosity parameters may be small, and we may integrate the equation (3.4) using a time step equal to a fraction of the characteristic time of the system.

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CHAPTER 4

RESULTS OF COMPUTATION

4.1 INTRODUCTION

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In this chapter, we present the results of finite element analysis of the ice drift in the Arctic Ocean and in the Strait of Belle Isle using viscous sea ice model described in previous chapters.

In order to check the finite element computer program, some hypothetical atmospheric pressure variations are assumed and the resulting wind stress (equations 2.3 and 2.6) is imposed on the pack ice. The results obtained by finite element method were compared with the results obtained by Fourier Transform method. The agreement was found to be excellent. The program was then used to compute velocity of pack ice due to forces caused by wind, currents and ocean tilt. The velocities of pack ice obtained from each forcing function were superposed to obtain the overall ice velocity. We present the results of our calculations for a two year period using different values or viscosities in summer and winter.

The finite element method allows one to prescribe any kind of kinematic or force type boundary condition on any irregular boundary. But the boundary conditions are different in the cases when the ice cover is pushed against the shore and when it is receding from the shore. A special boundary element is incorporated and an iterative scheme is followed to take care of this special boundary condition.

Finally, we present the results of some of our calculations on the motion of ice in the Strait of Belle Isle which is driven back and forth by strong tidal streams and currents.

4.2 HYPOTHETICAL ATMOSPHERIC PRESSURE VARIATIONS

In order to check the finite element computer program, we calculate the pack ice velocities due to wind stresses only, and we compare the results with the exact solution for a particular case when the atmospheric pressure variation is given by the following function.

 $P(x,y) = P_0 \sin \frac{\pi x}{\xi} \sin \frac{\pi y}{\xi} \qquad (4.1)$

where t = 2000 km and $P_0 = 1000 \text{ Pa}(N/m^2)$

This pressure variation creates a clockwise wind velocity field around the centre of 2000 km by 2000 km square area as shown in Figure 4.1. The geostrophic wind velocity components are obtained by using equation (2.5) as follows

$$\begin{split} \mathbf{U}_{\mathbf{g}} &= -\frac{1}{p}\frac{\mathbf{s}\mathbf{p}}{\mathbf{s}\mathbf{y}} = -\frac{\pi}{\lambda}\frac{\mathbf{P}_{\mathbf{p}}}{\mathbf{s}\mathbf{p}}\sin\frac{\pi \mathbf{x}}{\lambda}\cos\frac{\pi \mathbf{y}}{\lambda}\\ \mathbf{v}_{\mathbf{g}} &= \frac{1}{p}\frac{\mathbf{s}\mathbf{p}}{\mathbf{s}\mathbf{s}\mathbf{x}} = -\frac{\pi}{\lambda}\frac{\mathbf{P}_{\mathbf{p}}}{\mathbf{p}}\cos\frac{\pi \mathbf{x}}{\lambda}\sin\frac{\pi \mathbf{y}}{\lambda}. \end{split} \tag{4.2}$$

where ρ (density of air) = 1.3 kg m⁻³ and f (Coriolis. parameter) = 1.46 × 10⁻⁴ s⁻¹.

The components of air and water stresses are obtained from equations (2.3) using the following values of different parameters:

> B = 0.01462 kg s⁻¹ m⁻² D = 0.59 kg s⁻¹ m⁻² ϕ = 6 = 30⁰

The mass of pack ice per unit area is calculated from the average thickness of ice in the Arctic Basin:

 $m = 3.0 \times 10^3 \text{ kg m}^{-2}$

The exact solution of pack ice drift over a stagmant ocean due to the atmospheric pressure variations given in equation (4.1) is available using the Fourier Transform method in which the solution is repeated every 4000 km over an infinite domain in the x and y directions.

For the finite element analysis, the 2000 km by 2000 km square area is discretized winto. Bx8=64 square elements of size 250 km by 250 km. While using finite element method for this case, the values of ice velocity obtained from the exact solution are prescribed only at the boundary of a 2000 km by 2000 km square area to generate a solution inside the area.

The results of the finite element analysis are compared with the exact solution using two sets of values of shear and bulk viscosities:

 $\eta = \zeta = 2 \times 10^{10} \text{ kg s}^{-1}$ $\eta = \zeta = 4 \times 10^{11} \text{ kg s}^{+1}$

In both cases, an excellent agreement is obtained between the exact solution and the results of finite element analysis. The velocity fields of pack ice are presented in Figure 4.2a and Figure 4.2b. One can observe from Figures 4.2m and b that there is convergence when low values of viscosity are used, and divergence of pack ice occurs due to high values of viscosity. This is due to the Coriolis force which makes the pack ice converge for low values of viscosity. For high values of viscosity, pack ice offers greater resistance to Coriolis force, and it diverges under the influence of wind stresses. Dependence of convergence or divergence upon the values of shear and bulk viscosities was first pointed out by Hibler (1974).

This check on the results of finite element analysis confirms the correct implementation of finite element program,

4.3 PACK ICE DRIFT IN ARCTIC OCEAN

The external forces which make the pick²ice move are mainly caused by the geostrophic wind, geostrophic current and ocean tilt. The Coriolis force and the drag force also influence the movement of pack ice and these forces are dependent upon the ice velocity. Since the system of differential equations governing the motion of pack ice are linear, we can solve for the ice velocities due to each forcing function separately and add the ice velocity components obtained from different types of forces to obtain the combined velocity of pack ice.

The Arctic Basin is discretized into 9x13 = 117 square elements of size 250 km by 250 km as shown in Figure 4.3. For the boundary, conditions, the velocity of ice normal to boundary and the tangential shear stresses at the boundary are preseribed to be equal to zero. This means that the ice is allowed to elip at the boundary without any frictional resistance from the shore. The values of parameters used in the computations are same as those in the previous section except for the values of shear and bulk viscosities which are taken equal to $4x10^{19}$ kg s⁻¹ for summer senson and $3x10^{11}$ kg s⁻¹ for summer senson.

The wind stress acting on the pack ice can be computed from the atmospheric pressure data using equations (2.3) and (2.6). Four sets of atmospheric pressure data for the Arctic Baais were obtained from the Cold Regions Research and Engineering Laboratory, Kanover, N.R., (Hibler, 1976), Each set of data is 180 day averaged atmospheric pressure at points on a 250 km by 250 km grid system in the Arcic Basin. Thus, the computations were performed using atmospheric pressure data, given in Tables 4.1 to 4.4, for four seasons Summer 1. Winter 1. Summer 2 and Winter 2 and using coresponding values of viscosity. Figure 4. shows the velocity field due to wind only, in Summer 1.

The forces caused by the geostrophic currents and the ocean tilt can be calculated from the ocean height data using equations (2.4) and (2.5). The ocean height data at points on the same 250 km by 250 km grid water was also obtained from the Cold Regions Research and Engineering Laboratory (Hibler, 1976). This data was generated from the long term Coachman's (1974).ocean height values, which is presented in Table 3.5. Two sets of pack ice velocities caused by ocean current and tilt were computed using different values of viscosity for summer and winter. Figures 4.5 and 4.6 show the velocity fields caused by ocean current and tilt in Summer 1.

In order to obtain the velocity field due to the combined forces of wind, currents and ocean tilt, we add all the velocity fields for each of the four seasons considered here. The combined velocity fields are shown in Figures 4.7 to 4.10 for Summer 1, Winter 1, Summer 2 and Winter 2, respectively.

The results of finite element analysis are compared with that of Fourier Transform method (Hibler, 1976.) and this comparison is shown in Figure 4.11. We note that the results obtained from both methods agree with each other in the middle of the basin. The differences near the boundary are due to the fact that different boundary conditions have been assumed for solution by the two methods. In the Fourier Transform method, the solution repeats itself every 4000km in the x and y direction. Slip type boundary conditions have been assumed for the solution by finite element method. We can also note from Figure 4.11 that the influence of different boundary conditions extends upto a distance of approximately 750km. A comparison of pack ice drift velocity for Winter 1 pressure data is presented in Figure 4.12, when the slip and no-slip boundary conditions are used. It, can be noted that the effect of boundary condition extends to a distance of 750km.

4 BOUNDARY ELEMENT

The boundary conditions that can be imposed in the finite element solution are of two types: either the velocity or the stress in the normal and the tangential direction of the boundary can be prescribed. These boundary conditions may not be valid when the pack ice edge is moving away from the shore because the pack ice can not offer resistance in tension. The imposition of boundary condition must take care of the condition whether the pack ice edge is retreating from the shore or pushing against the shore. On the one hand, when the pack ice boundary is moving away from the shore, the boundary should be free to move without any stress imposed on it, on the other hand, when the pack ice boundary is pushing against the shore, the velocity normal to the shore must be zero and the tangential stress at the boundary must not be zero (preferably proportional to the normal pressure 'according to Coulomb vield criteria)

The above boundary conditions cannot be expressed linearly in terms of unknown velocity components of spack ice. Hence, an iterative procedure is followed to incorporate these non-linear boundary conditions with the help of boundary elements whose material properties (i.e., viscosities) can be changed as the iterative solution progresses. These boundary elements are long and not so wide elements placed at the boundary as shown in Figure 4.13. The velocity at the boundary nodes are made equal to zero. The material properties of these elements are first taken to be the same as those in the domain. The model velocities of pack ice are obtained and the stresses in the boundary elements are calculated. If a tensile stress exists in the direction normal to the boundary, the element properties of the element are reduced to a low value. It should be reduced to zero but this is not done in order to avoid difficulties in numerical calculations. This procedure of obtaining the solution and checking the stress in the boundary element to change its material properties is repeated a number of times until we get the same solution as in the previous iterative step.

In Figure 4.13, the boundary elements have a width of 25km and length of 250km. The computer program was modified to incorporate the above iterative procedure. The results of computation of pack ice velocities due to wind stress in Summer 1 is presented in Figure 4.14. Only two iterations were necessary to obtain this result. The velocity field in Figure 4.14 can be compared with the velocity field in Figure 4.4, in which a slip boundary condition is imposed. These calculations were not repeated for all cases of driving forces due to wind, currents and tilt, because we only want to demonstrate the feasibility of such an iterative solution incorporating non-linear boundary condition.

4.5 STRAIT OF BELLE ISLE

The strait of Belle Isle lies between Newfoundland and Labrador (see Figure 4.15). The tidal streams in the strait move the pack ice to and fro, and the reversal in the direction of movement takes place every 5 or 6 hours. The Faculty of Engineering at Memorial University of Newfoundland has accumulated extensive data on the movement of ice in the form of time lapse photographs taken at Pointe Amour, Labrador, during the year 1974-75. It can be observed from these photographs that the pack ice reverses its direction, and some sort of eddys in pack ice are formed at the time of reversal of velocity.

An attempt is made to model the movement of pack ice due to water movement in the strait by the finite element method. It is reported in the Labrador and Hudson Bay Pilot (1965), issued by the Canadian Hydrographic Service, that the tidal streams attain an average velocity of 2 to 3 knots. The west-going stream reaches its maximum rate about 2 hours 35 minutes before high water at Harrington Marbour and the eastgoing stream at about the same interval before low water there.

The water movement actually experienced are the resultant of these tidal streams and of currents whose direction and rate are largely determined by meteorological conditions. The most unusual distribution of currents is an east-going current across the whole of the western entrance down to a depth of 20 fathoms (44 m) and a west-going current at greater depths. Owing to the meteorological effects, the flow on any day may be either prodominantly west-going or predominantly east-going, for the currents will quite commonly be strong enough to overcome the weaker of the two streams. The strongest east-going flows occur with the passage of a low pressure region over northern Quebee and the resultant of the current and the east-going tidal stream may have rates up to 3½ knots. The strongest west-going flows occur when an atmospheric pressure depression lies to the south of the strait and then the resultant current and west-going tidal stream may reach a rate of up to 5 knots on the northern side of the western entrance.

In order to model the movement of pack ice in the strait, we assume that the movement of water has a sinusoidal variation with a peak velocity of 34 knots and a period of 12.5 hours. The interference of currents due to meteorological conditions is not taken into account because this is a first study on the movement of pack ice in the strait.

The strait of Belle Isle is discretized into finite size elements as shown in Figure 4.16. The forces acting on the pack ice are only due to the current. The Ekman angle in water is taken to be zero for this case. The system equations (3.4) are first order differential equations with respect to/time. The integration is performed according to the 2nd order, Runge-Kutta method with a time step of 300 to 800 seconds. The initial conditions are assumed to be zero. There is a characteristic time associated with the first order system and the integration procedure remains stable as long as the integrating time step is smaller than the characteristic time.

The results of computation are presented in Figures 4.17 and 4.18. In Figures 4.17 the velocity of pack ice at the narrowest section is plotted at different time instants. The velocity remains constant throughout the section except near the boundary where a boundary layer effect exists. At the time of the velocity reversal, the velocity of pack ice near the shores reverses earlier than the main body of pack ice. This non-uniform pack ice motion can be observed in the time lapse photographs of ice in the strait. After a short time, the whole section starts to move together in the reversed direction. In Figures 4.18. the velocity of pack ice at the middle point of the narrowest section is plotted against time. Plots of water velocity with respect to time are also shown in Figures 4.18. It can be observed that the pack ice velocity lags behind the water velocity by a constant time. This time lag can be explained with the help of a simple model.

A mass m (kg) is attached to a linear damper of coefficient c (kg s⁻¹) (see Figure 4.19), such that the free end of the damper has a harmonic velocity of amplitude U and frequency s. The resultant steady state velocity of the mass m can be obtained by solving the following countion:

35.

$v + v = U \sin \omega t$

where $\tau = \frac{m}{C}(s)$, the characteristic time of the system. The solution of the above equation can be written in the following form:

 $v(t) = V \sin(\omega t - \phi)$

where $V = (1+\tau^2 \omega^2)^{\frac{1}{2}} U$

tan o = Tw

 $\phi = T \omega$ (T is time lag).

For small values of ϕ , we can conclude that the time lag T is equal to the characteristic time τ .

If we wary the values of bulk and shear viscosities in our model, we are essentially varying the damping coefficient c in the simplified model. As c increases, the characteristic time decreases. This trend is observed in the computation of pack ice velocity when the values of viscosity are changed as shown in. Figures 4.18. If very large values of viscosity are used, the characteristic time would be very small, and it may not be worthwhile to integrate the first order aystem equation (3.4). In that case, the time varying term in equation (3.4) may be ignored due to smallness of inertial forces in comparison to the dissipative forces in the system. It is for this reason that steady state velocities are

(4.3)

calculated in the Arctic Basin where the values of viscosity are very large.

From the time lapse photographs of the pack ice motion in the strait of Belle Isle, it is possible to determine the time instants when the reversal of ice motion took place. As monifored earlier, the movement of water in the strait is correlated to tide levels at Harington Harbor. We assume that the water velocity in the strait is harmonic with respect to time. With the help of the 'Labrador and Hudson Bay Pilot, (1965)' and the 'Canadian Tide and Current Tables (1974, 1975, 1976)', we have been able to determine the time instants of water velocity reversals in the strait on those windless days when the pack ice motion was photographed. It is found that the time lag between the reversals of water velocity and that of the pack ice's approximately 18 to 29 minutes when there is no wind.

The theoretical model can be made to similate the motion of pack ice in the strait by properly choosing the values of viscosity. Computations are performed for three different values of viscosity and the following table gives the characteristic time corresponding to each value of viscosity used.

η,ς (kg s ⁻¹).	T (seconds
0.6×10 ⁵	1980
1,2×10 ⁵	1140
2.4×105	390

The approach of parameterizing the model is to choose those values of bulk and shear viscosities which give theoretical time lag equal to the actual time lag.

The above model describes the motion of pack ice in the struit for a sinuscidal variations of water velocity. For the determination of pack ice motion due to different meteorological conditions, one must first model the movement of water through the strait taking many factors into account, e.g., effect of the Moon and the Sun, atmospheric pressure, bottom topography of the strait, etc.

CHAPTER 5

DISCUSSION AND CONCLUSION

A number of investigators have employed viscous sea ice drift models over several years (Doronin, 1970; Laikhtman, 1962; Campbell, 1965; Hibler, 1974). In all those models, the stress transmitted through the sea ice cover on the geophysical scale is considered to be a single valued function, usually linear, of the strain rate tensor. In order to make use of a viscous type model, it is necessary to have some way of parameterizing the viscosity as a function of the state of ice. Doronin (1970) used a relationship between compactness and viscosity. A simpler approach is to parameterize the viscosity to a first approximation as a function of the thermodynamics of the ice cover; in particular the growth rate of ice as used by Hibler (1976b).

The finite element formulation of a viscous sea ice mdoel has been accomplished in this thesis for both steady and nonsteady drift velocity of pack ide. The working of a finite element computer program is checked by comparing its solution for a hypothetical pressure variations with the exact solution obtained by Fourier Transform method (Hibler, 1976b). Four sets of pack ice velocity field in the Arctic Basin are computed and plotted using six monthly averaged atmospheric pressure data and long term ocean height.

The slip type boundary condition is not strictly valid when pack ice pushes against the shore. Then pack ice recedes from the shore, free (is. zero stress) boundary condition must be imposed. This non-linear behavior of boundary conditions has been incorporated in the iterative proceeding to introducing boundary elements whose properties change according to the state of stress in the element.

Then the forces acting on the pack ice change rapidly and the response time for the ice to those forces is large, non-steady ice velocity has to be computed. This has been done for the to and fro motion of pack ice in the strait of Belle fale due to tidal streams and currents. For a harmonic variation of tidal stream velocity with respect to time, the pack ice velocity is also harmonic with a time lag with respect to the water velocity. The time lag in the model can be adjusted by varying the values of bulk and shear viscosities, and it can be matched to the actual time lag which has been determined in a few instances with the help of time lapse photographs of ice motion and the tide tables.

SCOPE FOR FURTHER STUDY

This work has been the first step towards the finite element formulation of the viscous model for sea ice. More detailed computations of the pack ice velocity field can be accomplished with better atmospheric pressure data and by taking non-linear boundary conditions into account over an irregular domain.

The pack ice drift in the strait of Belle Isle is strongly influenced by the water movement in the strait which is the resultant of the tidal streams and the currents affected by the Noon, the Sun and the meteorological condtions. Further studies on the pack ice drift in the strait should include all the factors influencing the water movement. This would also shed some light on the question whether there is any net import or export of energy from the Gulf of St.Lawrence through the strait of Belle Isle.

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APPENDIX A

TRIANGULAR ELEMENT FORMULATION

A brief discussion on the development of a triangular finite element is presented here. Figure A-I shows a typical triangular element with nodes 1, 2, 3 "numbered in anticlockwise direction. The following are the interpolation functions for the velocity components within the triangular element:

A.1)

We can write the above equations in the following form:

 $u = \alpha_1 + \tilde{\alpha}_2 x + \alpha_3 y$ $v = \alpha_4 + \alpha_5 x^3 + \alpha_5 y$

 $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\$

In order to express velocity components in terms of the values of velocity components at the nodes, we eliminate the coefficients a's by the use of the following equations:

u, (A.3) α1 ú, az u, α. v. **y**1 α. v. V2 as У 3

where u_i and v_i (i=1,3) are the nodal velocities in the x and y directions, and x_i and y_i are the coordinates of the nodal points 1,2 and 3. By inverting the above set of equations and substituting for the coefficients a's in equation (4.2), we get

$\begin{bmatrix} u \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} 1 & x & y & 0 & 0 \end{bmatrix}$	2A23 2A31 2A12 0 0 0]	[u1]
v 0001xy	b ₁ b ₂ b ₃ 0 0 0	u2
	a ₁ a ₂ a ₃ 0 0 0	us
	0 0 0 2A23 2A31 2A12	v1
	0 0 0 bi b2 b3	V2
	0 0 0 81 82 83	v,
	(A.4)	

 $\begin{array}{l} a_{1j} = x_1y_j - y_1x_j \\ a_1 = x_1 - x_2 \quad a_2 = x_1 - x_1 \\ b_1 = y_2 - y_3 \quad b_2 = y_3 - y_1 \quad b_3 = y_1 - y_2 \\ 2A = a_3b_2 - a_2b_3 = a_1b_3 - a_3b_1 = a_2b_1 - a_1b_2 \end{array}$

It is convenient to express equation (A.4) in terms of the natural coordinates of the triangle as follows: $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} L_1 & L_2 & L_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_1 & L_2 & L_3 \end{bmatrix}$ [u1] = [N]{q_}. (A.5)

 $L_1 = \frac{1}{2A}(2A_{23} + b_1x + a_1y_2)$ $L_2 = \frac{1}{2A}(2A_{31} + b_2x + a_2y)$ $L_3 = \frac{1}{2A}(2A_{12} + b_3x + a_3y)$

 $L_1 + L_2 + L_3 = 1$

{q_}}is the generalised modalryelocity vector. There is a convenient formula for integrating the natural coordinates over the area of a triangular element. and it is given below:

> $\begin{bmatrix} L_1^p & L_1^q & L_1^r & dA & = & \frac{p! & q! & r!}{(p+q+r+2)!} & 2A \end{bmatrix}$ (A.6)

By differentiating equation (A.5), we obtain the following expression for the strain rate components:

$$\hat{z} = \begin{bmatrix} u_{1,\chi} \\ v_{1,y} \\ u_{1,y} + v_{1,\chi} \end{bmatrix} = \frac{1}{22} \begin{bmatrix} b_{1} & b_{2} & b_{1} & 0 & 0 & 0 \\ 0 & 0 & a_{1} & a_{2} & a_{3} \\ a_{1} & a_{2} & a_{4} & b_{1} & b_{2} & b_{3} \end{bmatrix} \begin{bmatrix} u_{1} \\ u_{2} \\ u_{3} \\ v_{1} \\ v_{2} \\ v_{3} \end{bmatrix} = \begin{bmatrix} B \end{bmatrix} (q_{1})$$

$$(A.7)$$

There are four types of expressions in the functional (3.2) and we shall deal with them one by one.

Type 1.

The energy dissipation due to water drag, the rate of work done by Coriolis force and the rate of change of kinatic energy can be expressed by an expression of the following form:

$$\left\| \frac{1}{2} \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \right\| d\mathbf{A}$$

Substituting equation (A.5) into the above expression, we have ,

$$\begin{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 & v_1 & v_2 & v_1 \end{bmatrix} \begin{bmatrix} u_1 & 0 \\ L_2 & 0 \\ U_3 & 0 \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \end{bmatrix} \begin{bmatrix} L_1 & L_2 & L_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_1 & L_2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 & 0 \\ 0 & 0 & 0 & L_1 & L_2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 & u_3 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 & u_3 & u_3 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 & u_3 & u_3 \\ 0 & U_1 & u_3 & u_3 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 & u_3 & u_3 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 & u_3 & u_3 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 & u_3 & u_3 & u_3 \\ 0 & U_1 & u_2 & u_3 & u_3 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 & u_3 & u_3 & u_3 \\ 0 & U_1 & u_2 & u_3 & u_3 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 & u_3 & u_3 & u_3 \\ 0 & U_1 & u_2 & u_3 & u_3 & u_3 \\ 0 & U_1 & u_2 & u_3 & u_3 & u_3 \end{bmatrix} \begin{bmatrix} u_1 & u_1 & u_1 & u_2 & u_3 & u_3 & u_3 & u_3 & u_3 \\ 0 & U_1 & u_2 & u_3 & u_3$$

 $= \frac{1}{2} \left[\mathbf{q}_{\mathbf{e}} \right]^{\dagger} \left[\mathbf{K}_{\mathbf{p}} \right] \left[\mathbf{q}_{\mathbf{e}} \right]$ (A.9)

 $\text{here } [K_p] = A \begin{bmatrix} P_{1,1}(S) & P_{1,2}(S) \\ P_{2,1}(S) & P_{1,2}(S) \end{bmatrix} \quad \text{and } [S] = \begin{bmatrix} \frac{1}{2} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

The mass matrix can be derived from the above term and it is written as follows:

Type 2.

If the forces due to wind, ocean currents and ocean tilt are known, we can group the second, third and fourth terms in the functional as follows:

$$\int \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} F_{\mathbf{x}}(\mathbf{x}, \mathbf{y}) \\ F_{\mathbf{y}}(\mathbf{x}, \mathbf{y}) \end{bmatrix} d\mathbf{A}$$
 (A.11)

The functions $F_{x}(x,y)$ and $F_{y}(x,y)$ can be obtained from the atmospheric pressure data and the ocean height data by the use of equations (2.3 to 2.6). We assume that F_{x} and F_{y} vary linearly with respect to x and y within the triangular element, and we can write

$$\begin{bmatrix} \mathbf{P}_{\mathbf{x}} \\ \mathbf{P}_{\mathbf{y}} \\ \mathbf{y} \\ \mathbf{y} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{1} & \mathbf{L}_{2} & \mathbf{L}_{3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L}_{1} & \mathbf{L}_{2} & \mathbf{L}_{3} \\ \mathbf{I}_{\mathbf{x}} \\ \mathbf{I}_{\mathbf{x}} \\ \mathbf{F}_{\mathbf{x}} \\ \mathbf{F}_{\mathbf{x}} \\ \mathbf{F}_{\mathbf{y}} \end{bmatrix}$$
 (A.12)

where F_{x1} and F_{y1} (1=1 to 3) are the values of F_x and F_y at the nodes of the element. Now we can write the expression (A.11) in the following manner:

1 u2 u3 V1 V2 V3][L ₁ 0]	$\begin{bmatrix} L_1 & L_2 & L_1 & 0 & 0 & 0 \end{bmatrix}$	Fxi
At Stragel	L2 0	0 0 0 L ₁ L ₂ L ₃	F'X2
	L3 0	요즘 수요가 봐.	F
	0 L1	승규는 아파 가지 않는	F.
	0 L2		F v2
영화 영화 영화	0 L3	같은 그는 것을 같다.	F

-	{q_}}	-{(2)
	e		е.

e {Q _e } = A	$\frac{F_{x1}}{6} + \frac{F_{x2}}{12} + \frac{F_{x3}}{12}$	
	$\frac{F_{x1}}{12} + \frac{F_{x2}}{6} + \frac{F_{x3}}{12}$	
	$\frac{F_{x1}}{12} + \frac{F_{x2}}{12} + \frac{F_{x3}}{6}$	
	$\frac{F_{y1}}{6} + \frac{F_{y2}}{12} + \frac{F_{y3}}{12}$	
<u>\</u>	$\frac{F_{y1}}{12} + \frac{F_{y2}}{6} + \frac{F_{y3}}{12}$	1.2.2
	$\frac{F_{y1}}{12} + \frac{F_{y2}}{12} + \frac{F_{y3}}{6}$	1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1. 1

The rate of energy dissipation into the material deformation is expressed by the following expression in the functional (3.2):

$$\frac{1}{2} \iint \begin{bmatrix} u_{1,\mathbf{x}} & v_{1,\mathbf{y}} & u_{1,\mathbf{y}} + v_{1,\mathbf{x}} \end{bmatrix} \begin{bmatrix} z+n & z-n & 0\\ z-n & z+n & 0\\ 0 & 0 & n \end{bmatrix} \begin{bmatrix} u_{1,\mathbf{x}} \\ v_{1,\mathbf{y}} \\ u_{1,\mathbf{y}} + v_{1,\mathbf{x}} \end{bmatrix} dA$$
(A.14)

 $= \frac{1}{2} \iint \{ \stackrel{\bullet}{\epsilon} \}^{t} [R] \{ \stackrel{\bullet}{\epsilon} \} dA$

 $= \frac{1}{2} \left[\left\{ \left\{ q_{e} \right\}^{t} \left[B \right]^{t} \left[B \right] \left[B \right] \left\{ q_{e} \right\}^{t} \right] \right] dA$

 $= \frac{1}{2} \{q_{e}\}^{t} [K_{e}] \{q_{e}\}$

where $[K_0] = \frac{1}{4A} \begin{bmatrix} b_1 & 0 & a_1 \\ b_2 & 0 & a_1 \\ b_3 & 0 & a_3 \\ 0 & a_1 & b_1 \\ 0 & a_2 & b_2 \\ 0 & a_2 & b_2 \\ 0 & a_3 & b_1 \end{bmatrix} \begin{bmatrix} c+\eta & c-\eta & 0 \\ c+\eta & 0 & 0$

In the derivation of above expressions, we have made use of equations (A.7). The matrix [R] represents the constitutive 4 relations in terms of moduli of viscosity.

Type 4.

The life integral term in the functional (3.2) represents the rate of work done by the prescribed boundary stresses, and it can be presented in the following form:

$$\begin{bmatrix} \mathbf{u}_{n} & \mathbf{u}_{t} \\ \mathbf{s}_{t} \\ \mathbf{r}_{nt} \end{bmatrix} \begin{bmatrix} \mathbf{d}_{n} \\ \mathbf{s}_{t} \\ \mathbf{y}_{t} \end{bmatrix} \begin{bmatrix} \mathbf{d}_{n} \\ \mathbf{s}_{t} \end{bmatrix} \begin{bmatrix} \mathbf{d}_{n} \\ \mathbf{y}_{t} \end{bmatrix} \begin{bmatrix} \mathbf{d}_{n} \\ \mathbf{y}_{t} \end{bmatrix} \begin{bmatrix} \mathbf{d}_{n} \\ \mathbf{y}_{t} \end{bmatrix}$$

where $X = \sigma_{nn} \cos \theta + \tau_{nt} \sin \theta$ $Y = \sigma_{nn} \sin \theta + \tau_{n+} \cos \theta$

and θ is the angle between the outward normal and x-axis.

Suppose that the element (e) forms part of the boundary curve S_i as shown in Figure A-2 and the nodes i and j lie on the boundary where the stressess are prescribed. We can evaluate the boundary integral in equation (A.15) as sum of integrals over segments, such as i-j in Figure A-2. We can express the velocity components at the boundary in terms of values of velocities, $(u_i, u_j, v_i \text{ and } v_j)$ at -nodes i and j as follows:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (1 - \frac{g}{1_{1,j}}) & \frac{g}{1_{1,j}} & 0 & 0 \\ 0 & 0 & (1 - \frac{g}{1_{1,j}}) & \frac{g}{1_{1,j}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_1 \\ v_1 \\ v_1 \\ v_1 \end{bmatrix}$$
(A.1)

where s and l_{ij} are the arc-lengths defined in Figure A-2. If we assume that X and Y are also linearly varying along the boundary segment i-j, we can specify X and Y in terms

of their nodal values $(X_{j}, X_{j}, Y_{j} \text{ and } Y_{j})$ in the following form:

$$\begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} = \begin{bmatrix} (\mathbf{1} - \frac{\mathbf{B}}{\mathbf{1} \mathbf{1} \mathbf{j}}) & \frac{\mathbf{B}}{\mathbf{1} \mathbf{1} \mathbf{j}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\mathbf{1} - \frac{\mathbf{B}}{\mathbf{1} \mathbf{1} \mathbf{j}}) & \frac{\mathbf{B}}{\mathbf{1} \mathbf{1} \mathbf{j}} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{\mathbf{1}} \\ \mathbf{X}_{\mathbf{j}} \\ \mathbf{Y}_{\mathbf{j}} \\ \mathbf{Y}_{\mathbf{j}} \end{bmatrix}$$

Substituting equations (A.16) and (A.17) into equation (A.15), we have

 $= {\left(\mathbf{q}_{\mathbf{b}} \right)^{\mathsf{t}} \left(\mathbf{Q}_{\mathbf{b}} \right)$

where $\{\mathbf{q}_{\mathbf{b}}\}$ and $\{\mathbf{Q}_{\mathbf{b}}\}$ are the generalised velocity and force vectors at the boundary.

The functional is equation (3.2) can be evaluated over each element of the discretized system according to the procedure gives above for the four types of expressions in the functional. This leads to the development of element matrices and generalised force vectors which are assembled in the usual

manner to give rise to an expression of the functional π for the whole system.

$$\begin{split} \mathbf{n} &= \Sigma \ \mathbf{n}_{e} = \Sigma \ \left[\frac{1}{2} (\mathbf{q}_{e})^{\mathsf{T}} [\mathbf{x}_{p}] (\mathbf{q}_{e}) - \frac{1}{2} (\mathbf{q}_{e})^{\mathsf{T}} [\mathbf{x}_{e}] (\mathbf{q}_{e}) \right] \\ &- \frac{1}{2} (\mathbf{q}_{e})^{\mathsf{T}} (\mathbf{x}_{e}) (\mathbf{q}_{e}) + (\mathbf{q}_{e})^{\mathsf{T}} (\mathbf{q}_{e}) + (\mathbf{q}_{e})^{\mathsf{T}} (\mathbf{q}_{e}) \right] \end{split}$$

 $\vec{\Pi} = -\frac{1}{2} (q)^{t} [K] (q) - \frac{1}{2} (q)^{t} [M] (q) + (q)^{t} (Q)$ (A.19)

where the matrices [K] and [M] and the vectors (q) and (Q) refer to the entire discretized system after assembling the element matrices and vectors according to the usual procedure of assembly in the finite element method.

The first variation of the functional given in equation (A.19) is written as follows:.

$$\delta \Pi = \delta \{q\}^{t} - [K] \{q\} - [M] \{q\} + \{Q\} = 0 \quad (A.20)$$

If 6{q} is to be arbitrary, the equation (A.20) gives the following system equations:

$$[M](q) + [K](q) = \{Q\}$$
 (A.21)

The above first order differential equation with respect to time can be integrated by any one of the numerical integration procedure.

or





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Figure 4.1 Wind velocity field due to hypothetical atmospheric pressure variation. Scale: 7.5 m s⁻¹ per grid length.

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igure 4.2a Pack ice velocity field for $\zeta = \eta = 2.0 \times 10^{10} \text{ kg s}^{-1}$ Scale: 0.08 m s⁻¹ per grid length


Figure 4.2b Pack ice velocity field for $\zeta = n = 4.0^{\circ} \times 10^{11} \text{ kg s}^{-1}$ Scale: 0.08 m s⁻¹ per grid length

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61 Pack ice velocity field due to wind stresses only in = 4.0 × 10¹⁰ kg s⁻¹ per grid length. -3 E Scale: 0.08 Summer 1. Figure a Martin States Sheet



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ocean currents 4.0 × 1010 to wind per grid length. . 5 and ocean tilt in Summer 1. Scale: 0.08 m s⁻¹ per grid 1 Pack ice velocity field due Figure 4

Pack ice velocity field due to wind, ocean currents and ocean tilt in Winter 1. (c = n = 3.0 × 10¹¹ kg. Scale: 0.08 m s⁻¹ per grid length igure

ocean currents × 1010 0 Pack ice velocity field due to wind grid length. 1 and ocean tilt in Summer 2. Der Scale: 0.08 m 1000 6.1 Figure SAL AND AND

Pack ice velocity field due to wind, ocean currents 1 = 3.0 × 10¹¹-kg per grid length and ocean tilt in Winter 2. (5 = Scale: 0.08 m 1.10 Pigure STRUCTURE STRUCTURE Ser Acon 17



50 domain. (C=n=3×10¹¹ kg Effect of slip and no slip boundary conditions o slip boundary condition grid length. the Jer. the solution within ٦, 0.025 condition Scale: oundary Figure 4.12 lip











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1012.42 1011.64 1011.18 1012.25 1012.36 1012.34 1012.21 1011.10 1012.83 1012.63 1012.33 1012.40 1012.45 1012.53 1012.64 1012.57 1011.78 1612.36 1012.62 1012.49 1012.67 1012.74 1012.54 1012.70 1012.14 1012.91 1012.88 1012.42 1011.29 1 (millibars) 1012.58 1012.58 1011.79 1012.98 111.6101 1012.98 1012.26 1012.27 1012.43 1012.64 1012.79 1012.86 1012.90 1012.62 1012.54 1012.12 1011.99 1011.84 1011.85 1011.69 1012.19 +1.6101 SC.CIOI 1012.40 1012.92 1012.74 1012.85 Pressure in Summer 1013.10 1014.56 1014.10 1013.57 1013.38 1012.69 20"1101 1013.79 1013.16 1012.91 1013.18 1013.13 1012.82 1012.39 1013.84 1013.10 1013.08 1013.18 1013.52 1013.38 1012.99 1012.98 1013.66 1013.36 1013.15 1013.14 1014.04 1013.93 1014.07 1013.48 1013.89 1914-81 1014-26 1012.45 1013.60 1013.79 Atmospheric 1015.33 1014.60 1015.58 1014.76 1014.95 1014.38 1012.19 1014.51 1015.06 1014.31 1014.49 1014.57 1414.68 1014.28 1014.57 1015.32 1014.94 1014.45° 1015.53 . 1015.35 1015.57 1012.33 1015.45 Table 4:1 1012.34 1012.36 1015.12 1015.94 . 1015.77 1016.05 1015.40 1015.75 1015.83 1015.51 1014.33 1413.12 1012.89 17.4101 * 24.2191 1015.47 1015.64 1015.45 1014.06 1012.76 1014.67 C\$**5101 1015.99 1015.05 10.5101

Winter'1 (millibars) Pressure in Atmospheric Table 4.2

1016.52 1016.40 1017-11 1018.88 1017.26 1018.52 1017.57 1018.14 1014.11 1018.43 1016.38 1016.87 1018.04 1019.28 06.8101 1019-63 16.9101 26.9101 1020.04 020.23

1019.35 1019.34 1019.15 1019.07 1019.10 1018.99 1019.31 1019.00 81.9101 1019.37 1019-68 11.4101 26.9101 1020.08 1020.20 1020.22 #C-0201 1020.50 14.020 1020.47

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1014-50

11.1101 1010.21 1010.71 1011.07 1010.94 1012.13 1011.46 1014.12 1012.89 1015.44

2 (millibars) Pressure in Summer Atmospheric Table 4.3

1011.95 1012.02 1011.40 1012.32 1012.56 1012.73 1012.51 1012.45 1012.52 1012.59 1012.59 1012.37 1012.20 1012.56 1012.26 1012.17 1012.07 1011.67 1011.52 1011.07 1012.02 1011.50 1011.17 1012.07 1011.88 1012.44 1013.15 1013.15 1013.00 1012.63 1012.38 1012.11 1012.06 1011.91 1012.55 1012.35 1012.11 1012.06 1011.90 1010.55 014.90 1014.90 1010.86 1010.76 1010.54 1014.49 1010.36 1010.75 1010.75 1011.84 1012.14 1012.76 1012.91 1013.09 1013.00 1012.77 1012.47 1012.07 1011,78 1012.42 1012.24 1012.91 1013.07* 1012.92 1012.86 1012.74 1012.53 .1012.21 1012.13 1012.23 1011.10 1011.07 1012.66 1012.93 1012.87 1012.89 1012.78 1013.20 1913.01 1012.77 1012.64 1012.58 1012.56 1012.38 1012.27 1012.09 1012.03 1012.00 1012.31 1012.66 1012.94 1011.97 1012.24 1012.65 1012.92 1011.92 3012.30 1012.66 1012.89 1412.08 1012.41 1012.73 1012.86 1013.11 1013.27 1013.08 1012.91 1013.35 1013.14 1012.92 1011.23 1011.50 1011.61 1011.46 1012.69 1012.70 1012.84 1013.25 CO.1101 011.79 14-210 TU.1101 1011.62 1011.67 1011.47 1012.32 1012.09 1012.05 91.1101 48-1101

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1014.79 1017.46 1018.82 1019.15 1018.34 1014.29 1012.89 1011.88 46.1101 1011.38 1018.52 1015.51 . 1016.93 1016.11 Atmospheric Pressure in Winter 2 (millibars) 1015.65 1015.70 1019-68 1017.60 1918.22 1011.13 1010.29 1018.32 1018.51 1018.38 1018.3101 1017.57 1017.41 1014.46 1012.74 1010-27 1009.88 1017.70 1010.39 1009.40 1015.64 1014.25 1013.60 1012.81 1008.64 1008.91 1008.42 1008.61 49.1101 1017.90 1015.96 24-7101 1016.78 1014.60 1012.46 1011.19 1010-01 1006.07 1016.36 1008.53 1016.57 1017.01 1017.96 1017.80 1015.17 1014.14 1013.97 1011.57 1000.57 1008.91 1010.60 1008.97 11.7.111 1012.35 1011.16 1009.67 1017.25 1016.48 1017.39 1017.18 1017.64 1016.29 1015.18. 1015.47 1015.46 1012.16 1010.99 1010.13 1009.15 1008.64 1017.61. 1016.14 1010.53 1010.68 1009.62 1016.79 1017.44 1014.12 1012.95 1009.82 1009.20 1011.97 1017.61 1017.24 1015.61 1013.13 1012.65 1017.46 101+.16 1012.53 1011.71 Table 4.4 1012-94 1011-85 50"LINE 1017.04 1015.64 Trtetot 1413.9U d1010.75 1417.61 20.0101 1013.13 1013.07 1016.74 +L.CIOI 1012.40 017.40 11.410 1012.94 1012.49 21.1111 #610 M 014.28 1012.57 1012.39 26-110

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able 4.5. Ocean Heights (cms)

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36.00	35.00	34.00	00.65	32.50	32.00	30.00	28.00	25.00	20.00	19.00	18.00	15.00	13.00
	36.00	36.00	35.00	35.00	34.00	32.00	30.00	25.00	22.00	19.00	18.00	14.00	13.00
	38.00	40.00	•1.00	42.00	40.00	36.90	33.00	30.50	26.00	22.00	10.00	14.00	13.00
35.00	00*46	53.00	52.00	48.00		+2.00	34.50	00.66	30.00	24.00	30.00	14.00	15.00
37.00	43.00	00.00	60.00	53.00	52.00.		00.04	36.00	32.00	37.00	23.00	19.00	15.00
		\$6.00	00.14	55.00	00.02	44.00	45.00	42.00	36.00	00.15	35.00	19.00	12.00
00-65	00.64	54.40	10.00	45.00		42.00	•3.00	42.00	34.00		00.01		11.00
	40.00	42.00		01.46	37.00	36.00	34.00	00-95	00-66	30.00	33.40	30.00	20.00
	1	1				100		1.1.1			1.1.1		

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