Timescales for radiation belt electron acceleration and loss due to resonant wave-particle interactions:

1. Theory

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[1] Radiation belt electrons can interact with various modes of plasma wave in their drift orbits about the Earth, including whistler-mode chorus outside the plasmasphere, and both whistler-mode hiss and electromagnetic ion cyclotron waves inside the plasmasphere. Electrons undergo gyroresonant diffusion in their interactions with these waves. To determine the timescales for electron momentum diffusion and pitch angle diffusion, we develop bounce-averaged quasi-linear resonant diffusion coefficients for field-aligned electromagnetic waves in a hydrogen or multi-ion (H+, He+, O+) plasma. We assume that the Earth’s magnetic field is dipolar and that the wave frequency spectrum is Gaussian. Evaluation of the diffusion coefficients requires the solution of a sixth-order polynomial equation for the resonant wave frequencies in the case of a multi-ion (H+, He+, O+) plasma, compared to the solution of a fourth-order polynomial equation for a hydrogen plasma. In some cases, diffusion coefficients for field-aligned waves can provide a valuable approximation for diffusion rates for oblique waves calculated using higher-order resonances. Bounce-averaged diffusion coefficients for field-aligned waves can be evaluated generally in minimal CPU time and can therefore be profitably incorporated into comprehensive kinetic radiation belt codes.


1. Introduction

[2] The Earth’s outer electron radiation belt (3 < L < 7) is highly variable, particularly during magnetic storms or other geomagnetically disturbed periods [e.g., Paulikas and Blake, 1979; Baker et al., 1986, 1994, 1997; Li et al., 1997; Reeves et al., 1998; Obara et al., 2000; Miyoshi et al., 2003; Miyoshi and Kataoka, 2005]. Modeling the dynamics of energetic radiation belt electrons is of much current interest in space physics. In part, motivation for this interest is that relativistic (>1 MeV) electrons, which are generated during the recovery phase of some storms, pose a serious potential hazard to orbiting satellites [Baker et al., 1998; Baker, 2001, 2002]. Relativistic electrons are also precipitated into the Earth’s atmosphere where they change the electrical and chemical properties of the stratosphere and mesosphere [e.g., Lastovicka, 1996; Callis et al., 1998]. By providing such a coupling between the magnetosphere and middle atmosphere, relativistic electron precipitation could therefore constitute an important link between solar activity and global climate variability.

[3] Wave-particle interactions play a fundamental role in radiation belt electron dynamics; see the reviews by Gendrin [2001], Horne [2002], and Thorne et al. [2005a]. Gyroresonant interactions are associated with high-frequency waves in the range 0.1 < \Omega_o < 0.8 |\Omega_e|, where \Omega_o is the oxygen ion gyrofrequency and |\Omega_e| is the electron gyrofrequency. Waves in this frequency range include whistler-mode chorus, plasmaspheric hiss, and electromagnetic ion cyclotron (EMIC) waves. Drift-resonant interactions are associated with ULF waves at lower frequencies of a few mHz. Energy diffusion due to cyclotron resonance with chorus waves is a viable mechanism for generating relativistic electrons (>1 MeV) in the outer zone during storms [Summers et al., 1998, 2002, 2004a, 2004b; Roth et al., 1999; Summers and Ma, 2000; Horne et al., 2005a; Thorne et al., 2005a; Varotsou et al., 2005]. Radial (cross-L) diffusion driven by enhanced ULF waves has also been cited as an effective mechanism for generating stormtime relativistic electrons [Hudson et al., 2001; Elkington et al., 2003]. While radial diffusion is effective for energizing electrons outside geosynchronous orbit, there is evidence that an additional local acceleration mechanism (e.g., VLF chorus diffusion) is required to explain the observed relativistic electron flux increases in the region 3 < L < 5 [e.g., O’Brien et al., 2003; Miyoshi et al., 2003, 2004; Shprits and Thorne, 2004; Iles et al., 2006]. Both Horne et al. [2005b] and Shprits et al.

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[2006a] demonstrated that ULF wave-driven radial diffusion could not explain the formation of the new radiation belt in the “slot” region following the 2003 Halloween storm. Horne et al. [2005b] and Shprits et al. [2006a] found that energy diffusion driven by whistler-mode waves could explain the gradual buildup of electron fluxes to energies of several MeV in this slot region normally devoid of energetic electrons. Gyroresonant interactions with whistler-mode chorus also provide a process for electron loss from the radiation belts. Electron cyclotron resonance with chorus can cause pitch angle scattering into the loss cone and subsequent loss to the atmosphere [Horne and Thorne, 2003; Summers et al., 2005; Thorne et al., 2005b]. EMIC waves and plasmaspheric hiss can also contribute to the scattering loss of radiation belt electrons [Summers and Thorne, 2003; Albert, 2003; Summers et al., 2005; Meredith et al., 2006]. Magnetopause shadowing, or other processes including radial diffusion, may additionally contribute to the loss of radiation belt electrons. A broad discussion of the issues of radiation belt electron transport, acceleration, and loss is provided collectively in the papers by Li and Temerin [2001], Friedel et al. [2002], O’Brien et al. [2003], Green and Kivelson [2004], Green et al. [2005], and Thorne et al. [2005a].

It is clear from the literature that further work is needed to quantify electron acceleration and loss processes in the radiation belts. The aim of the present work and the companion paper [Summers et al., 2007] is to apply quasi-linear diffusion theory to quantify cyclotron resonant interactions of radiation belt electrons with whistler-mode chorus, plasmaspheric hiss, and EMIC waves. We use quasi-linear theory to investigate the average properties of the cyclotron resonant diffusion process. Nonlinear effects including phenomena such as phase trapping are therefore not included and are beyond the scope of our investigation. Summers [2005] derived readily computable formulae for the quasi-linear diffusion coefficients for cyclotron resonance with field-aligned (R-mode or L-mode) electromagnetic waves. In order to determine the rates of momentum diffusion and pitch angle diffusion appropriate to the Earth’s assumed dipole magnetic field, we carry out bounce-averaging of the diffusion coefficients due to Summers [2005], and we assume a Gaussian frequency spectrum for the waves. Whereas Summers [2005] presented detailed results only for a hydrogen plasma, here we consider the more general case of a multi-ion (H+, He+, O+) plasma. Local diffusion coefficients are given in section 2. In section 3 we plot R-mode and L-mode wave dispersion curves, with corresponding minimum resonant energy curves. We also plot in section 3 examples of “resonance regions” which are regions in (kinetic energy, pitch angle)-space over which gyroresonance can take place. Evaluation of resonant quasi-linear diffusion coefficients requires calculation of the appropriate resonant wave frequencies. We discuss the determination of the resonant frequencies for R-mode or L-mode waves in a multi-ion (H+, He+, O+) plasma in section 4. The bounce-averaged diffusion coefficients are derived in section 5. In section 6 we briefly discuss our results. In the accompanying paper [Summers et al., 2007] we apply the results presented herein to evaluate radiation belt electron timescales for acceleration and loss due to gyroresonance with VLF chorus, ELF plasmaspheric hiss, and EMIC waves.

2. Quasi-Linear Diffusion Coefficients

We assume an infinite, homogeneous, collisionless plasma immersed in a uniform, static magnetic field \( \mathbf{B}_0 \). In the presence of superposed electromagnetic waves. We use quasi-linear diffusion theory to describe the effects of the waves on the particles in terms of a kinetic equation for the gyrophase-averaged phase-space density \( \Phi \). Ensemble-averaging of the wave fields is carried out. The general, relativistic quasi-linear diffusion equation for \( \Phi \), in the limit of gyroresonant diffusion, can be written in the form [e.g., Melrose, 1980],

\[
\frac{\partial \Phi}{\partial t} = \frac{1}{\sin \alpha} \frac{\partial}{\partial \alpha} \left( D_{\alpha\alpha} \sin \alpha \frac{\partial \Phi}{\partial \alpha} \right) + \frac{1}{\sin \alpha} \frac{\partial}{\partial \alpha} \left( D_{\alpha p} \sin \alpha \frac{\partial \Phi}{\partial p} \right) + \frac{1}{p^2} \frac{\partial}{\partial p} \left( p^2 D_{pp} \frac{\partial \Phi}{\partial p} \right) + \frac{1}{p} \frac{\partial}{\partial p} \left( p^2 D_{p0} \frac{\partial \Phi}{\partial \omega} \right),
\]

where \( D_{\alpha\alpha} \), \( D_{\alpha p} \), and \( D_{pp} \) are the diffusion or Fokker-Planck coefficients which depend on the properties of the waves; \( p = \gamma m_s v \) is the momentum of the particle of species \( \sigma \), rest mass \( m_\sigma \) and speed \( \gamma = (1 - v^2/c^2)^{-1/2} \) is the Lorentz factor (\( c \) is the speed of light); \( \alpha \) is the pitch angle of the particle, and \( t \) denotes time. In the present study we treat only the special case of electromagnetic waves propagating parallel or antiparallel to the background magnetic field \( \mathbf{B}_0 \). We assume that the R-mode (\( s = 1 \)) and L-mode (\( s = -1 \)) waves each have the Gaussian spectral density,

\[
\tilde{W}_s(\omega) = \frac{|\Delta B_s|^2}{8\pi} \frac{1}{\Delta \omega} \left[ 1 - \frac{1}{\Delta \omega} e^{-\left(\frac{2\omega}{\Delta \omega}\right)^2} \right],
\]

with

\[
\rho = \frac{\sqrt{\pi}}{2} \left[ \text{erf} \left( \frac{\omega_m - \omega}{\Delta \omega} \right) + \text{erf} \left( \frac{\omega - \omega_m}{\Delta \omega} \right) \right],
\]

where \( \omega_s \) is the lower frequency limit, \( \omega_s \) is the upper frequency limit, \( \omega_m \) is the frequency of maximum wave power, \( \Delta \omega \) is a measure of the bandwidth, and \( \text{erf} \) is the error function. The wave spectral density (2) has been normalized so that

\[
\frac{|\Delta B_s|^2}{8\pi} = \int_{\omega_s}^{\omega_s} \tilde{W}_s(\omega)d\omega,
\]

where \( |\Delta B_s| \) is the mean wave amplitude.

Following the study of Summers [2005], we can now express the diffusion coefficients for the particle species \( \sigma \) as follows:

\[
D_{\alpha\alpha} = \frac{\pi}{2} \frac{\Omega_{\sigma}^2}{\rho} \frac{1}{\Omega_{\sigma}} \left( E + 1 \right)^2 \sum_j \sum_i \int \frac{R \left( 1 - \frac{\gamma_\alpha \cos \alpha}{\gamma_\beta \cos \beta} \right) |ds/dy|}{\delta x |\delta \cos \alpha - ds/dy|} e^{-\left(\frac{\gamma_\alpha \omega_m}{\gamma_\beta \omega_m}\right)^2},
\]
where we have introduced the dimensionless variables,

\[ x = \frac{\omega}{|c|}, \quad y = \frac{ck}{|c|} \]

and \( E \) is the dimensionless particle kinetic energy given by

\[ E = E_0/(m_e c^2) = \gamma - 1; \quad \beta = \frac{\gamma c}{E} = \left[ (E + 2)/(E + 1) \right]^{1/2} \]

\( \Omega_e \) is the cold-plasma electron gyrofrequency, where \( e \) is the unit charge; \( \Omega_e = \epsilon B_0 / (m_e c) \) is the nonrelativistic electron gyrofrequency where \( \epsilon \) is the particle charge; \( R = \Delta B_1 / B_0 \) is the ratio of the energy density of the turbulent magnetic field to that of the background field, i.e., the relative wave power; \( \omega_n = \omega / \Omega_e \), \( \delta X = \delta \omega / |\Omega_e| \), and the derivative \( dx/dy = d\omega / dy \) is determined from the appropriate dispersion relation. In (5)–(7) the summations are carried over the wave modes specified by \( s = 1 \) (R-mode) and \( s = -1 \) (L-mode), and over the resonant roots \( \omega_j, k_j \) corresponding to each wave mode. As well as satisfying the relevant dispersion equation, the resonant frequency \( \omega_j \) and corresponding resonant wave number \( k_j \) satisfy the Doppler gyroresonance condition, which, for electrons or protons, can be expressed as

\[ \frac{x}{\beta \cos \alpha} + \frac{a}{\gamma} = y \]

where

\[ a = \frac{s \lambda}{\gamma} \]

Here, \( \lambda = -1 \) corresponds to electrons and \( \lambda = +1 \) to protons, with \( \epsilon = m_e / m_p \), where \( m_e \) is the electron rest mass and \( m_p \) the proton rest mass. As in the work of Summers [2005], we restrict the wave frequency \( \omega_j \) to be always positive, and we take a positive wave number \( (k_j > 0) \) to represent a forward propagating wave and negative wave number \( (k_j < 0) \) to represent a backward wave.

The classical theory of waves in a uniform cold plasma is given, for instance, by Stix [1992]. The dispersion relations for R-mode \((s = 1)\) or L-mode \((s = -1)\) waves propagating parallel or antiparallel to a uniform magnetic field in a cold, multi-ion (H, He, O) plasma are

\[ \frac{y^2}{x^2} = 1 + \frac{1}{\alpha^*} \left( \frac{1 - \epsilon \eta_1}{s - x + se} - \frac{\epsilon \eta_2}{4x + se} - \frac{\epsilon \eta_3}{16x + se} \right) \]

where the parameter \( \alpha^* \), defined by (12), plays an important role in cold-plasma theory and can be written as \( \alpha^* = \frac{\Omega_e^2}{\omega_p e^2} \). The electron plasma frequency where \( N_0 \) is the electron number density; \( \eta_1 = N_1 / N_0 \), \( \eta_2 = N_2 / N_0 \), and \( \eta_3 = N_3 / N_0 \), where \( N_1, N_2 \), and \( N_3 \) denote the hydrogen \((H^+)\), helium \((He^+)\), and oxygen \((O^+)\) ion number densities, respectively. The fractional ion compositions satisfy \( \eta_1 + \eta_2 + \eta_3 = 1 \), by charge neutrality. In the special case of a hydrogen plasma \((\eta_1 = 1, \eta_2 = \eta_3 = 0)\), equations (11) reduce to the form,

\[ \frac{y^2}{x^2} = 1 - \frac{b}{(x-s)(x+se)} \]

where

\[ b = (1 + \epsilon) / \alpha^* \]

The parameter \( \alpha^* \) is a cold-plasma parameter; \( \omega_p e^2 = (4\pi N_0 e^2 / m_e)^{1/2} \) is the electron plasma frequency where \( N_0 \) is the electron number density; \( \eta_1 = N_1 / N_0 \), \( \eta_2 = N_2 / N_0 \), and \( \eta_3 = N_3 / N_0 \), where \( N_1, N_2 \), and \( N_3 \) denote the hydrogen \((H^+)\), helium \((He^+)\), and oxygen \((O^+)\) ion number densities, respectively. The fractional ion compositions satisfy \( \eta_1 + \eta_2 + \eta_3 = 1 \), by charge neutrality. In the special case of a hydrogen plasma \((\eta_1 = 1, \eta_2 = \eta_3 = 0)\), equations (11) reduce to the form,

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outside the plasmasphere, and we adopt a saturated density variation, \( N_0 \propto L^{-4} \), inside the plasmasphere.

The resonant roots \( x_j, y_j \) occurring in formulae (5)–(7) are solutions of the simultaneous equations (9) and (11) in the case of a multi-ion (H\(^+\), He\(^+\), O\(^+\)) plasma, or equations (9) and (13) for a hydrogen plasma. We defer discussion of the calculation of the roots \( x_j, y_j \) until section 4.

It is possible that a singularity can occur in the diffusion coefficients (5)–(7) if the condition \( \beta \cos \alpha = dx/dy \) is satisfied at a resonance. This is equivalent to the condition that the parallel speed of the particle equals the group speed of the wave. A point at which such a singularity occurs is called a critical point [see Summers, 2005].

The reader is referred to Summers [2005] for a discussion of critical values in the case of a hydrogen plasma. We do not discuss critical values further in the present paper.

3. Dispersion Curves, Minimum Resonant Energy Curves, and \((\mathbf{E}_k, \alpha)\) Resonance Regions

The dispersion relations for R-mode \((s = 1)\) or L-mode \((s = -1)\) waves are given by equation (13) for a hydrogen plasma and by (11) for a multi-ion (H\(^+\), He\(^+\), O\(^+\)) plasma. We plot a selection of dispersion curves in the top of Figures 3–6 for a range of values of the parameter \( \alpha^* = \frac{\Omega_e^2}{\omega_{pe}^2} \).

Figure 2. Equatorial profiles of the parameter \( \alpha^* \) for a dipole geomagnetic field, with the plasmapause boundary located at \( L = 3 \). Assumed density \( N_0 \) profiles inside \((2 \leq L < 3)\) and outside \((3 < L \leq 8)\) the plasmasphere are specified in the top and bottom.

[10] It should be noted that the definitions of \( D_{\alpha\alpha} \) and \( D_{\alpha p} = D_{p\alpha} \) used by Summers [2005] and also in the present paper differ, for instance, from those of Lyons [1974] and Horne et al. [2005a]. Specifically, the diffusion rates \( D_{\alpha\alpha} \) and \( D_{\alpha p} \) herein correspond respectively to the results \( D_{\alpha\alpha}/p^2 \) and \( D_{\alpha p}/p^2 \) given by Lyons [1974] and Horne et al. [2005a].

Figure 3. R-mode dispersion curves for a hydrogen plasma for specified values of \( \alpha^* = \frac{\Omega_e^2}{\omega_{pe}^2} \) (top). Corresponding minimum resonant energy profiles for electrons (middle) and protons (bottom).
In Figure 3 we plot dispersion curves for R-mode waves in a hydrogen plasma. In Figure 4, comparison of the dispersion curves for R-mode waves in hydrogen and multi-ion (H⁺, He⁺, O⁺) plasmas demonstrates that inclusion of additional ions has negligible effect, except possibly at very small frequencies. For L-mode waves, however, inclusion of additional ions changes the basic dispersion properties. Specifically, L-mode waves propagate in a single band in a hydrogen plasma (Figure 5), but in three separate bands in a multi-ion (H⁺, He⁺, O⁺) plasma (Figure 6). In Figure 6 we show L-mode dispersion curves for three different sets of values of the ion composition (η₁, η₂, η₃) to illustrate that the fractional ion composition can sensitively influence the dispersion relation. As shown in Figures 3–6, in all cases, the dispersion curves are strongly influenced by the value of the cold plasma parameter \( \alpha^* = \Omega_e^2/\omega_{pe}^2 \).

[12] Detailed examination of the gyroresonance condition (9) reveals that for a given value of \( \alpha^* \) and for waves of a particular mode and frequency, there exists a minimum value of the particle kinetic energy \( E_{\text{min}} \) for which gyroresonant wave-particle interaction can take place. Here we develop formulae for \( E_{\text{min}} \) for both electrons and protons, each in resonance with either R-mode or L-mode waves. If \( v_\parallel, v_\perp \) denote the components of particle velocity parallel and perpendicular to \( B_0 \), where \( v^2 = v_\parallel^2 + v_\perp^2 \), then the gyroresonance condition (9), expressed in terms of \( v_\parallel \) and \( v_\perp \), is represented by an ellipse, the “resonance ellipse,” in \((v_\parallel, v_\perp)\)-space. From the resonance ellipse, it is straightforward to deduce that the minimum particle energy for which gyroresonant wave-particle interaction can take place is found by setting \( v_\perp = 0 \) in the resonance condition. We find

![Figure 4](image-url)  
**Figure 4.** Comparison of R-mode dispersion curves in a hydrogen plasma and multi-ion (H⁺, He⁺, O⁺) plasma for specified \( \alpha^* \)-values (top). Corresponding minimum resonant energy profiles for electrons (middle) and protons (bottom). Fractional ion number densities are \( \eta_1 = 0.75, \eta_2 = 0.2, \eta_3 = 0.05 \).

![Figure 5](image-url)  
**Figure 5.** As in Figure 3, but for L-mode waves.
that the minimum resonant energy for electrons ($\sigma = e$) or protons ($\sigma = p$) can be written

$$E_{\text{min}} = \frac{(E_k)^{\text{min}}}{m_{\sigma}c^2} = \gamma_{\text{min}} - 1 = \left[1 - \left(\frac{v_{\min}}{c}\right)^2\right]^{-1/2} - 1, \tag{16}$$

where

$$\frac{v_{\min}}{c} = \begin{cases} \frac{xy - s(1 + y^2 - x^2)^{1/2}}{1 + y^2}, & \sigma = e \\ \frac{xy + e(s^2 + y^2 - x^2)^{1/2}}{s^2 + y^2}, & \sigma = p \end{cases}$$

corresponding to R-mode ($s = 1$) or L-mode ($s = -1$) waves. In each of Figures 3–6, corresponding to the given dispersion curves in the top, we plot the minimum resonant energy curves for electrons in the middle and for protons in the bottom. Figures 3–6 demonstrate that the minimum resonant energy $E_{\text{min}}$ increases as the value of the parameter $\alpha' = \Omega_{\perp}^2/\omega_{pe}^2$ increases. Thus since $\alpha' \propto B_0^2/N_0$, the value of $E_{\text{min}}$ increases as the background magnetic field $B_0$ increases or as the electron number density $N_0$ decreases. Clearly, $E_{\text{min}}$ depends critically on the wave frequency and the local values of $B_0$ and $N_0$. Figure 6 also demonstrates that for L-mode waves in a multi-ion plasma the value of $E_{\text{min}}$ can be sensitively dependent on the fractional ion number densities ($\eta_1, \eta_2, \eta_3$). A detailed examination of the variation of minimum resonant energy with fractional ion composition is given by Summers and Thorne [2003].

[13] Formulae (5)–(7) are exact, closed-form expressions for the rates of resonant diffusion $D_{\sigma\sigma}, D_{\sigma p}/p$, and $D_{pp}/p^2$, that can be evaluated as functions of the particle kinetic energy $E_k$ and pitch angle $\alpha$. Input parameters to be specified are $|\Omega_{\|}|, |\Omega_{\perp}|, R, \alpha', x_m, \delta x$, and $\rho$. For a given wave mode and set of input parameters, it is useful to identify the regions in the $(E_k, \alpha)$ - plane over which...
gyroresonance can take place. We plot examples of such \((E_k', \alpha)\) “resonance regions” in Figure 7. In the top we consider electrons interacting with R-mode or L-mode waves and, in the bottom, protons interacting with R-mode or L-mode waves. The diffusion coefficients (5)–(7) exist within the shaded regions shown in Figure 7 and are zero outside the shaded regions. The extent of the resonance region is generally determined by the particle species, the wave mode, the frequency band \(\omega_1 < \omega < \omega_2\), and the value of the parameter \(\alpha^* = \frac{W e^2}{\omega_{pe}^2}\). From the gyroresonance condition (9) we find that the boundary of the \((E_k', \alpha)\) resonance region generally includes the curves,

\[
\cos \alpha = \frac{\phi_1 + s \lambda/(E + 1)}{\beta \kappa_1}, \quad \cos \alpha = \frac{\phi_2 + s \lambda/(E + 1)}{\beta \kappa_2},
\]

where \(\beta = [E(E + 2)]^{1/2}/(E + 1)\), \(\phi_1 = \omega_1/|\Omega_e|\), \(\kappa_1 = c k_1/|\Omega_e|\), \(\phi_2 = \omega_2/|\Omega_e|\), and \(\kappa_2 = c k_2/|\Omega_e|\).

4. Resonant Wave Frequencies

[14] The wave frequencies \(x = \omega/|\Omega_e|\) and wave numbers \(y = ck/|\Omega_e|\) that are resonant with a particle of given kinetic energy \(E\) and pitch angle \(\alpha\) are given by the simultaneous solution of the resonance condition (9) and the dispersion equations (11), for a multi-ion (H\(^+\), He\(^+\), O\(^+\)) plasma.

![Figure 7](image1.png)

Figure 7. Examples of \((E_k, \alpha)\) resonance regions for the specified wave-particle interactions and the given parameter values.

![Figure 8](image2.png)

Figure 8. \((E_k, \alpha)\) resonance regions for R-mode/electron interaction. Sensitivity of the resonance region to changes in the value of \(\alpha^* = \frac{W e^2}{\omega_{pe}^2}\) for a fixed frequency band, is tested in the top; sensitivity to a change in the frequency band, for a fixed \(\alpha^*\), is tested in the bottom.
where
\[ A_1 = [s(84 \varepsilon - 64) + 128 \alpha + \xi^2 s(84 - 64 \varepsilon)]/[64(1 - \xi^2)], \]
\[ A_2 = [s(21 \varepsilon - 84) + as(168 \varepsilon - 128) + 64 a^2 + \xi^2 s(84 - 21 \varepsilon + A\xi^2/\alpha^2)]/[64(1 - \xi^2)], \]
\[ A_3 = [s\varepsilon^2(\varepsilon - 21) + a\varepsilon^2(42 \varepsilon - 168) + \alpha^2 s(84 - 64) + \xi^2 s(21 - \varepsilon) + B\xi^2/\alpha^2]]/[64(1 - \xi^2)], \]
\[ A_4 = [\alpha s \varepsilon^2(24 - 42) + \alpha^2 s(21 \varepsilon - 84) - \varepsilon^2(1 - \xi^2) + C\xi^2/\alpha^2)]/[64(1 - \xi^2)], \]
\[ A_5 = a^2 [a s(\varepsilon - 21) - 2\varepsilon] [64(1 - \xi^2)], \]
\[ A_6 = -\alpha^2 \varepsilon^2/[64(1 - \xi^2)], \]
\[ A_7 = 64 + \varepsilon [64 \eta_1 + 16 \eta_2 + 4 \eta_3], \]
\[ B = s \varepsilon [64 - (164 - 20 \varepsilon) \eta_1 + (16 - 17 \varepsilon) \eta_2 + (4 - 5 \varepsilon) \eta_3], \]
\[ C = \beta \cos \alpha, \beta = [E(\varepsilon + 2)]^{1/2}(E + 1), \]
\[ a = s \lambda_1, \gamma = E + 1, \]
and \(\alpha^*\) is defined by (12). The corresponding wave numbers \(\gamma\) can be determined from (9). In the special case of a hydrogen plasma \((\eta_1 = 1, \eta_2 = 0, \eta_3 = 0)\), equation (18) can be shown to reduce to a quartic equation, as given by equation (A1) of Summers [2005]. This quartic equation alternately follows by eliminating the wave number \(\gamma\) from equations (9) and (13).

[15] Since we assume that the wave frequency \(\omega\) is positive, we seek real resonant roots \(x > 0\) in the appropriate frequency range. For \(\cos \alpha > 0\), a forward propagating wave corresponds to \(\gamma > 0\), and a backward propagating wave to \(\gamma < 0\). We identify the following four cases: (1) R-mode waves interacting with electrons \((s = 1, \lambda = -1)\), (2) L-mode waves interacting with protons \((s = -1, \lambda = \varepsilon)\), (3) R-mode waves interacting with protons \((s = 1, \lambda = \varepsilon)\), (4) L-mode waves interacting with electrons \((s = -1, \lambda = -1)\). For a hydrogen plasma, the roots of interest are in the range \(0 < x < 1\) for R-mode waves, and \(0 < x < \varepsilon\) for L-mode waves. In the cases 1, 2, and 3, there is a maximum of three relevant roots, while in case 4 there is one relevant root.

5. Bounce-Averaging

[16] Equations (5)–(7), which have been derived for a uniform background magnetic field, give values for the local rates of diffusion, i.e., at a given point in space. In order to apply (5)–(7) to a magnetic mirror geometry such as the Earth’s magnetic field, the diffusion coefficients must be bounce-averaged, i.e., averaged over particle bounce-orbits. The Earth’s dipole magnetic field \(B\) is given by
\[ B = B_{eq} f(\lambda), \]
where \(B_{eq}\) is the equatorial magnetic field, and \(\lambda\) is the magnetic latitude. Constancy of the (first) adiabatic invariant associated with the gyration of a particle about a field line gives
\[ \sin^2 \alpha = \frac{\sin^2 \alpha_{eq}}{B_{eq}}, \]
where \(\alpha_{eq}\) is the particle pitch angle at any point along a field line, and \(\alpha_{eq}\) is the equatorial pitch angle. Equations (19) and (21) imply that
\[ \sin^2 \alpha = f(\lambda) \sin^2 \alpha_{eq}. \]

In order to carry out bounce-averaging of the diffusion coefficients over a particle bounce-orbit, the local diffusion coefficients must first be related to the equivalent equatorial coefficients [Roberts, 1969; Lyons et al., 1972; Lyons, 1974]. This is achieved by multiplying \(D_{w eq}, D_{\alpha p}\), and \(D_{pp}\), respectively, by \(\partial \alpha_{eq}/\partial \alpha\), \(\partial \alpha_{eq}/\partial \alpha\), and 1, where, from (21),
\[ \frac{\partial \alpha_{eq}}{\partial \alpha} = \frac{\tan \alpha_{eq}}{\tan \alpha}. \]

The bounce-averaged values of the diffusion coefficients (5)–(7) therefore take the form,
\[ \langle D_{\alpha w} \rangle = D_{\alpha w} (\alpha_{eq}) = \frac{1}{\tau_B} \int_0^\tau D_{\alpha w} (\alpha) \left( \frac{\partial \alpha_{eq}}{\partial \alpha} \right)^2 d\tau, \]
\[ \langle D_{\alpha p} \rangle = D_{\alpha p} (\alpha_{eq}) = \frac{1}{\tau_B} \int_0^\tau D_{\alpha p} (\alpha) \left( \frac{\partial \alpha_{eq}}{\partial \alpha} \right) d\tau, \]
\[ \langle D_{pp} \rangle = D_{pp} (\alpha_{eq}) = \frac{1}{\tau_B} \int_0^\tau D_{pp} (\alpha) d\tau, \]
where \(\tau_B\) is the bounce-period of the particle; \(\tau_B\) can be written in the approximate form,
\[ \tau_B = \frac{4 R_0 S(\alpha_{eq})}{v_0}. \]
\[ S(\alpha_{eq}) = 1.3 - 0.56 \sin \alpha_{eq}. \] (29)

\[ \langle D_{ao} \rangle = \frac{1}{S(\alpha_{eq})} \int_{0}^{\lambda_m} D_{ao}(\alpha) \frac{\cos \alpha \cos^2 \lambda}{\cos^2 \alpha_{eq}} \, d\lambda, \] \hspace{1cm} (30)

\[ \langle D_{ap} \rangle = \langle D_{po} \rangle = \frac{1}{S(\alpha_{eq})} \int_{0}^{\lambda_m} D_{ap}(\alpha) \frac{\sin \alpha \cos^2 \lambda}{\sin \alpha_{eq} \cos \alpha_{eq}} \, d\lambda, \] \hspace{1cm} (31)

\[ \langle D_{pp} \rangle = \frac{1}{S(\alpha_{eq})} \int_{0}^{\lambda_m} D_{pp}(\alpha) \frac{\sin^2 \alpha \cos^2 \lambda}{\sin^2 \alpha_{eq} \cos \alpha} \, d\lambda, \] \hspace{1cm} (32)

where \( \lambda_m \) is the latitude of the mirror point of the particle, given by

\[ X^6 + (3 \sin^4 \alpha_{eq})X - 4 \sin^4 \alpha_{eq} = 0, \] (33)

with \( X = \cos^2 \lambda_m \). Equation (33) is obtained by setting \( \alpha = 90 \text{ deg} \) in equation (22).

[18] In results (30)–(32), \( \alpha \) is regarded as a function of \( \alpha_{eq} \) and \( \lambda \), as given by relation (22); thus the bounce-averaged diffusion coefficients \( \langle D_{ao} \rangle, \langle D_{ap} \rangle, \) and \( \langle D_{pp} \rangle \) are functions of \( \alpha_{eq} \). To calculate the integrals in (30)–(32), for a given value of \( \alpha_{eq} \), the local diffusion coefficients \( D_{ao}, D_{ap}, \) and \( D_{pp} \) must first be evaluated at a set of specified \( \lambda \)-values in the range \( 0 < \lambda < \lambda_m \). This requires the calculation of the resonant roots (the appropriate solutions of the polynomial equation (18)) at each of the specified \( \lambda \)-values. The integrals in (30)–(32) can then be evaluated by a standard numerical quadrature technique.

[19] In Figure 9 we plot values of the particle mirror latitude \( \lambda_m \) as a function of \( \alpha_{eq} \), as determined by equation (33). We also show in Figure 9 the variation of the function \( f \) with magnetic latitude \( \lambda \), as given by equation (20).

6. Discussion

[20] 1. We have presented formulae for the bounce-averaged quasi-linear (momentum, mixed, and pitch angle) diffusion coefficients for cyclotron resonance with field-aligned electromagnetic waves in a hydrogen or multi-ion (H\(^+\), He\(^+\), O\(^+\)) plasma. The formulae are exact, tractable, and fully relativistic. The wave frequency spectrum is assumed to be Gaussian. Evaluation of the diffusion coefficients requires the solution of a sixth-order polynomial equation for the resonant wave frequencies in the case of a multi-ion (H\(^+\), He\(^+\), O\(^+\)) plasma, compared to the solution of a fourth-order polynomial equation for a hydrogen plasma. In general, the diffusion rates depend on the wave spectral properties, including the wave amplitude, frequency band, MLT distribution and latitudinal distribution, in addition to the background magnetic field \( B_0 \) and the electron number density \( N_0 \).

[21] 2. In the companion paper [Summers et al., 2007], we apply the results derived here to the interaction of radiation belt electrons with (1) R-mode VLF chorus waves outside the plasmasphere, (2) R-mode ELF hiss inside the plasmasphere, and (3) L-mode EMIC waves inside the plasmasphere. Timescales for electron acceleration or momentum diffusion can be found by using (32) to evaluate \( \langle D_{pp} \rangle/\beta^2 \) as a function of equatorial pitch angle, for an electron of a given energy, at a particular L shell. To estimate the electron loss timescale \( \tau_{\text{loss}} \) for a specific wave mode, we determine the quantity \( \tau_{\text{loss}} = 1/\langle D_{ao} \rangle \), where
($D_{eq}$) is the bounce-averaged pitch angle diffusion coefficient (given by (30)) for the specified wave band, evaluated at $\alpha_{eq} = (\alpha_L)_{eq}$ where $(\alpha_L)_{eq}$ is the equatorial loss cone angle given by

$$\sin(\alpha_{eq}) = \left[ L^2 (4L - 3) \right]^{-1/4}.$$  (34)

[23] While the diffusion coefficients presented in this paper have been derived on the assumption of field-aligned wave propagation, this assumption need not unduly restrict the applicability of our results. In some cases, first-order-harmonic diffusion rates can give a valuable approximation to diffusion rates for oblique waves calculated using higher-order resonances. For instance, first-order-harmonic pitch angle diffusion rates for EMIC waves calculated by Summers and Thorne [2003] agree well with corresponding higher-order calculations by Albert [2003] for oblique waves with wave-normal angle of 25. In another example, Thorne et al. [2005b] found that MeV electron scattering rates near the loss cone, due to chorus, agreed within a factor of 2 with calculations from a diffusion code employing $\pm 5$ harmonic resonances. This approximate agreement is acceptable for the determination of electron lifetimes since in the study of Thorne et al. [2005b] there is a larger uncertainty in the observed lifetimes and in the properties of the waves. In an associated study, Shprits et al. [2006b] compared bounce-averaged energy and pitch angle diffusion coefficients for field-aligned chorus waves with calculations for oblique waves that take account of higher-order resonances. Shprits et al. [2006b] found that errors associated with the neglect of higher-order scattering are smaller than the inaccuracies associated with uncertainties in the input values of the plasma density and latitudinal distribution of the waves.

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