HIERARCHICAL APPROACH VERSUS TRADITIONAL APPROACH TO INSTRUCTION IN EIGHTH GRADE MATHEMATICS

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By

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ABSTRACT

Purpose of the Study

The main purpose of this study was to compare the achievement and retention of Grade VIII students taught a unit on solving equations using Gagné's Hierarchical Approach with a group using the Traditional Textbook Sequence.

Procedures

The investigation was carried out in three Grade VIII classes in one school in Avondale, Newfoundland. The sample consisted of 76 students. These students were randomly assigned to two groups of the same size. One group was randomly assigned the Textbook Approach and the other group the Hierarchical Approach.

The Textbook Group learned how to solve first degree, one variable equations by studying a programed booklet developed by following the exact sequence of the textbook that was used at this grade level. The Hierarchical Group studied a programed booklet prepared by following the ideas of Robert Gagné. This approach consisted of performing a task analysis on the terminal objective, setting up a learning hierarchy for these skills, and sequencing these tasks in the programed booklet according to the learning hierarchy. A prerequisite test would be given the Hierarchical Group and students would begin the instruction at the point where the prerequisite skills were missing.

The students received two - forty minute periods of instruction per day. It took ten days for the students to complete the programed booklets. During the study, students remained in their own classrooms and worked entirely on their own with a minimal of teacher guidance. Students were not aware that they had been divided into groups and were taking part in a study. From the exterior, both instruction booklets appeared to be the same.

Students were given a posttest one day after the completion of the instruction and a retention test was given two weeks later. These tests were alternate forms of each other. The statistical technique of the analysis of covariance was used to determine if the difference in achievement and retention between the two groups were significant. The level of significance was set at .05.

Conclusions

1. The Hierarchical Approach to instruction produced significantly better achievement results than the Textbook Approach.
2. The Hierarchical Approach to instruction produced significantly better retention results than the Textbook Approach.
3. The Hierarchical Approach to instruction would be one method of reducing underachievement in our mathematics classrooms.

ACKNOWLEDGEMENTS

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CHAPTER I

INTRODUCTION

Rationale for the Study

The problem of teaching students who do not achieve in relationship to their capabilities is a complex one. Educators quite often attribute their failure to laziness, immaturity, low I. Q., personality and emotional factors, or family environment. There is no doubt that all these factors do influence a student's achievement, but as Callahan and Robinson (1973) point out:

> Often teachers tend to overlook the contribution of poor instruction and deficient abilities in prerequisite skills to underachievement (Callahan and Robinson, 1973, p. 578).

All too often in teaching, instruction is given and learning is supposed to take place without sufficient thought given to the prerequisites or the sequencing. Bassler and Kolb (1971) compare this type of a teacher to a carpenter who tries to build a house without a blueprint. They say:

> He begins nailing lumber together, saws off ends that seem to be sticking out, tacks on roofing to exposed boards, puts plaster on rough surfaces, and generally goes through the motions of building a house. But without any kind of plan, the finished product is likely to be a monstrosity (Bassler and Kolb, 1971, p. 61).

Just as the carpenter needs a blueprint to build a house properly, teachers need a blueprint in order to teach effectively. One great danger is that teachers sometimes consider the textbook to be the ideal blueprint. This is indeed a false assumption, since, if one were to examine the presentation of material in some of our mathematics text-books, one would soon discover that many of the essential steps in the acquisition of this knowledge are omitted or misplaced. Teachers should develop their own blueprints of the instructional process, since it is really their responsibility to ensure that their students know, or learn, the needed prerequisites for any topic.

Many researchers have shown the importance of students knowing, or learning, the prerequisite skills before higher order skills are attempted. (Gagné and Paradise, 1961; Gagné, Mayer, Gerstens and Paradise, 1962; Wiegand, 1969; Miller, 1969; Brown, 1970; Okey and Gagné, 1970; Peyton, 1971; Russell, 1972; Phillips and Kane, 1973; Callahan and Robinson, 1973.)

The investigator believes that many of the difficulties students have in mathematics are caused by their lack of required prerequisite skills and by the fact that these "gaps" are ignored in the instructional process. Robert Gagné (1967) has the problem well in focus when he says:

> If learning at any level is to occur with greatest facility, careful attention must be paid to the prerequisites of such learning. It will be difficult for a child to learn the principles of geometry unless he has previously acquired the concepts of line, angle, triangle, intersection, and so on (Gagné, 1967, p. 202).

Purpose of the Study

This study will attempt to answer the following question. Would students achieve better and retain more mathematics if teachers followed a Hierarchical Approach to instruction, instead of the Traditional Textbook Sequence?

The Hierarchical Approach consists of determining the prerequisite capabilities for the learning task by constructing a "learning hierarchy" for each terminal objective. The instructional process would start at the point in the hierarchy where the student knew all the prerequisites. This approach should effectively eliminate any "gaps" in the learning process. It is based on Gagné's theory of learning hierarchies.

Gagnê's Learning Hierarchy Theory

In a report of a study of mathematics learning, Gagné (1962), applied the term "learning hierarchy" to refer to an ordered collection of specific intellectual capabilities. Gagné hypothesized that any intellectual skill can be analyzed into a hierarchy of subordinate, intellectual skills. These subordinate, intellectual skills (behaviors) are arranged in a sequence such that all the necessary prerequisite behaviors are listed below a terminal behavior. The learning hierarchy would resemble the illustration in Figure 1.

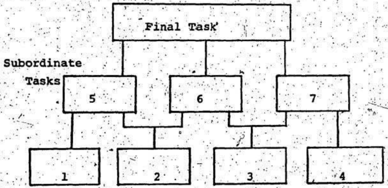


FIGURE 1. TYPICAL LEARNING HIERARCHY

It is important to note that tasks 1 and 2 are prerequisite for tasks 5; 2 and 3 are prerequisite for 6; 3 and 4 are prerequisite for 7; 5, 6, and 7 are prerequisites for the final task. The acquisition of all subordinate behaviors listed in the learning hierarchy is hypothesized to be required for the acquisition of the terminal behavior of the learning hierarchy.

Gagné's definition of learning hierarchies has been accepted by many researchers in this field. (Walbesser and Eisenberg 1972; Kane 1971; Briggs 1968; White 1974; and AAAS Commission on Science Education, 1968)

Gagné (1970), Walbesser and Eisenberg (1972), White (1974), and others agree that in order to construct a learning hierarchy one must start at the terminal objective and work backwards to determine what the prerequisite learning must be. One would continually ask the question, "what should the learner already know in order to learn this new capability?" In essence, one would perform a task analysis of the terminal objective and then sequence the subtasks in a hierarchy.

Transfer is very important in Gagné's theory of learning hierarchies. As Strauss (1972) aptly puts it:

> The key to Gagné's theory is positive transfer. It is employed to describe how a child ascends a learning hierarchy (Strauss, 1972, p. 86).

Positive transfer occurs when an individual: (1) learns capacity X, which enables him to perform a new capacity Y, which he could not perform before he learned X, or (2) learns Y more readily as a result of having learned X. Gagné (1970) believes that if a child has a store of prerequisite knowledge, he will transfer it horizontally (at the same capacity level) and vertically (to the next highest level in the learning hierarchy). If on the other hand, he does not possess these prerequisites, he cannot transfer it and he cannot move to the next highest level on the hierarchy.

Mathematics and Learning Hierarchy Theory

The crucial point to remember is that mathematics is a very structured subject. In order to learn effectively and meaningfully, students should have some structure of the subject or topic in their cognitive domain. Bruner (1963) looks at structure this way:

> Grasping the structure of a subject is understanding it in a way that permits many other things to be related to it meaningfully. To learn structure, in short, is to learn how things are related (Bruner, 1963, p. 7).

Using Gagné's method of breaking the subject matter into learning hierarchies would be an excellent way of exposing students to the relationships that exist between different segments of the mathematics curriculum. As Gagné (1970) says:

> Learning hierarchies are the best way to describe the "structure" of any topic, course, or discipline. They describe the intellectual skills the individual needs to possess in order to perform intellectual operations with that subject - to learn about it, to think about it, to solve problems in it. (Gagné, 1970, p. 245).

Hypotheses

The null hypothesis for this study was stated as follows:

H_O: There will be no significant difference ($p = .05$) in achievement and retention of a class of students taught a mathematics topic following Gagné's Hierarchical Approach and a class using the Traditional Textbook Sequence.

The alternate hypothesis was stated as follows:

H_A: There will be a significant difference ($p = .05$) in achievement and retention of a class of students taught a mathematics topic following Gagné's Hierarchical Approach and a class using the Traditional Textbook Sequence.

For the purpose of this study, achievement was defined as the result obtained on a posttest administered at the end of the instruction period and retention was determined by a retention test administered two weeks later.

CHAPTER II

REVIEW of RELATED RESEARCH AND LITERATURE

Gagné's Initial Studies

A number of studies pertaining to the construction and testing of learning sequences have been conducted by the University of Maryland Project in conjunction with Robert Gagné. In the first of these studies reported by Gagné and Paradise (1961), the investigators analyzed a final behavior represented by constructing solutions to linear algebraic equations. The procedure identified three immediate subordinate behaviors. The analysis was then repeated on each of the three subordinate behaviors and yielded a learning hierarchy of twenty-two behaviors subordinate to the terminal behavior. The study was designed to test the hypothesis that the acquisition of a terminal behavior depends upon the attainment of a hierarchy of subordinate behaviors. The results of the study supported the hypothesis.

The hypothesis of the Gagné and Paradise study was also investigated with different instructional materials in a later study. Gagné, Mayor, Gerstens, and Paradise (1962) reported a study to test the hypothesis that a final behavior of adding integers depends upon the attainment of a hierarchy of subordinate behaviors. The results of this experiment provide additional support for the conclusion that acquisition of each behavior in a hierarchy is dependent upon the previous mastery of the subordinate, relevent behaviors.

A third study by Gagné and others (1963) continued the experiments concerning the sequencing of knowledge. As in the previous studies, the initial step consisted of defining final behaviors and using the analysis procedures described previously to identify a hierarchy of subordinate capabilities. The study was designed to investigate two hypotheses: (1) the attainment of each behavior in the hierarchy upon positive transfer of training from the lower level capabilities, and (2) such transfer required high recallability of all the next lower subordinate tasks. The experimental data supported the conclusions of the preceding studies, namely, the attainment of any behavior in a learning hierarchy depends upon the achievement of the relevant supporting behaviors.

Studies Supporting Hierarchical Sequencing

Wiegand (1969) conducted a study of subordinate skills in a science problem. A learning hierarchy was constructed indicating hypothesized prerequisite capabilities for this task. The experiment confirmed the hypothesis that learning of initially missing subordinate skills produced marked positive transfer in the learning of a complex problem solving task in science.

Miller (1969) conducted a study using eight program sequences on matrix arithmetic. The results showed that scrambled sequences worked as well as the logical sequences for definitions and addition of matrices. However, in sequences where subjects were forced to learn matrix multiplication before learning definitions and matrix addition, they performed significantly worse than those who learned needed definitions and matrix addition first. Miller concluded that mastery of individual tasks in a hierarchy can be accompolished in several ways, including a scrambled programmed sequence. However, a logical sequence still appears to be the best in terms of overall effectiveness and effeciency.

Niedermeyer, Brown, and Sulzen (1969) compared three learning sequences (logical, scrambled, and reverse frame orders) for a topic in grade nine mathematics. Sixteen grade nine algebra students in each of the three sequence groups, plus a control group, served as the subjects. While the logical order group was the only sequence group to perform significantly better than the controls on both a test of concepts and a problem solving test, none of the three sequence groups differed significantly from each other on posttest performance. Logical group students did, however, make significantly fewer program errors. They also tended to consider the program "interesting" whereas scrambled and reverse order groups felt "neutral" about the program.

Brown (1970) found that logical sequencing facilitated learning of programmed mathematical materials. He concluded that when a sequence involves tasks that are complex problem solving behaviors, ordering is an important factor in learning.

Okey and Gagné (1970) conducted an interesting study in science teaching. An initial instructional program on solving solubility product problems was studied by a group of 49 chemistry students. Following instruction, performance of these students was measured on a criterion test and on 15 skills identified as subordinate to the final task. Performance on these subordinate skills was used to locate specific skills failed by a substantial number of students. Gagné's cumulative learning model served as the basis for identifying the subordinate skills and for predicting instruction needed to overcome defeciencies. Twenty frames were added to the original program in accordance with the learning hierarchy. A group of 57 students then studied this revised program. Analysis of covariance showed significant differences favoring the group using the revised program on a posttest performance measure.

Peyton (1971) investigated the Gagné conjecture concerning the ordering of conditions within a learning hierarchy. He concluded that achievement at each level of the hierarchy did mediate to achievement at the next level in the hierarchy from the lowest level up to, and including, problem solving.

Russell (1972) constructed and validated learning hierarchies composed of concepts in non-metric geometry and tried to determine the feasibility of developing a complete program for mathematics methods courses using learning hierarchies for each unit of study. He concluded that the hierarchical method is a very effective method of teaching elementary geometry in mathematics methods courses.

Phillips and Kane (1973) conducted an experiment whereby fourth graders were taught addition of rational numbers using seven different sequences: Logical, Guttman, Random, Item Difficulty, Correlation, Textbook, and a sequence developed by AAAS Commission on Science Education. The differential effects of sequence on achievement, transfer, retention, and time to complete the program was investigated using analysis of variance. No overall significant differences were found at the .05 level. However, the F ratio of 2.12 for the analysis of variance on retention was very near the critical value of 2.15. The experimenters suggest two possible sources of error in their experiment: (1) Teachers may have given more help and time than they were instructed to, (2) Examination of the responses revealed that many students did not write their answers in lowest terms, thereby getting the wrong answer but knowing how to solve the problem.

Uprichard (1973) reported a very practical study on the effect of sequence in the acquisition of three set relations by pre-schoolers. His experiment showed that sequence does make a difference. He concluded that the most efficient instructional sequence appropriate for pre-schoolers in learning "equivalence", "greater than", "less than", appears to be "equivalence", "greater than", "less than".

Callahan and Robinson (1973) studied the effect of using a hierarchical approach with underachievers in mathematics. The researchers indicated that this method worked very well and it reduced underachievement.

Studies Rejecting Rigorous Methods Of Content Sequencing

Roe, Case, and Roe (1962) conducted a comparative study of sequencing using a 71 item program on elementary probability. One group of students received a logically ordered form of the program, and one received a random version of it. A criterion test was administered to each subject immediately upon completion of the program. No significant differences were reported on time required for learning, errors during learning, criterion test score, or time required for criterion test.

Levin and Baker (1963) reported a study in which a 60 item geometry program for second graders was scrambled within 20 item blocks. The results showed no significant differences in measure of acquisition, retention, or transfer between those who worked through the logical program and those who completed the scrambled program.

Merrill (1965) hypothesized that learning and retention of a hierarchical task are facilitated by mastering each successive part of the material before proceeding to the next step. The results of his study did not support the hypothesis.

Miller (1965) conducted a study in which a 98 frame program on topics in ratio and proportion was presented in logical and random sequences to seventh graders. The author reported substantial differences in error rates which supported the interdependency of the frames. The results, however, indicated that the scrambling of frames had little, if any, effect upon learning from the program.

Payne et al. (1967) designed a study to examine the effects of scrambling upon the learning of three programs. The three programs were ranked by trained, independent observers from low to fairly high in logical interdependence. It was hypothesized that the effect of scrambling would be greatest for those programs dealing with tasks having the most logical development. The results of both immediate and delayed retention tests did not confirm this hypothesis.

Niedermeyer (1968) expressed concern over design and methodological weaknesses of the studies cited above. He claimed that the items in the logical sequence were not hierarchical in structure. He also points out that many of the subjects already knew a considerable amount of material presented in learning sequence. Thus, any meaningful assesment of sequence effect on learning was difficult to obtain.

Pyatte (1969) indicated a major problem with studies comparing logical and random ordered sequences. It is difficult to determine if the logical sequence is actually logical and the random sequence "random". He suggests that even in some of the random sequences many of the subtasks remain in the hypothesized hierarchical ordering.

Summary

In summary, it appears that mastery of individual subtasks in a hierarchy can be achieved in several ways, including learning from randomly programmed sequences. However, as Miller (1969) pointed out, logical sequencing into a hierarchical structure still appears to be the best in terms of overall effeciency and effectiveness. Several of the studies reviewed here suggest that varying sequences of instruction does not make any difference in the effectiveness of the instruction. But as Kane (1971) says, "many of these studies are plagued with design problems (p. 9)". Briggs (1968) expressed a very healthy attitude towards research when he says:

> Consider the positive results which were found from some of the experiments reviewed, continued research is believed worthwhile on the topic of course, or task structure, as it relates to the sequencing of instruction (Briggs, 1968, p. 118).

CHAPTER III

DESIGN AND PROCEDURES

This chapter describes the manner in which the investigation was conducted. It includes a description of the population and sample used in the study, the experimental design, the instructional approaches, the control of variables, the collection of data, and the limitations of the study.

Population and Sample

The population for the study consisted of eighth grade students who had not been taught a formal method for solving equations of the type ($ax + b = c$) (where a, b, and c are integers and $a \neq 0$) previous to the study.

The sample consisted of 76 eighth grade male students at Roncalli High School. Roncalli High School is situated in Avondale, Newfoundland and is under the jurisdiction of the Roman Catholic School Board for Conception Bay Center. It has an enrollment of 525 students in grades seven to eleven; all boys in grades seven to nine, and both boys and girls in grades ten and eleven. The sample used was the entire grade eight population of the school.

The particular school was chosen for the study because it happened to be the school where the investigator taught. The principal and teachers of this school were very co-operative and helpful during the study. Two of the mathematics teachers in the school helped carry out the investigation by supervising the classes during the instructional period.

The Instructional Unit

A unit of work on solving equations at the grade eight level was selected for the study. The terminal objective for the unit was stated as follows: Given any equation of the type $ax + b = c$ (where a, b, and c are integers and $a \neq 0$), the student will be able to solve it showing each step in the process.

At first glance, this seems to be a fairly simple objective but when one examines it more closely, it becomes obvious that there are numereous prerequisites which the student must know, or learn, in order to understand the principle of solving first degree, one variable equations. The concepts of solution, equivalent equations, identity elements, and inverse elements, along with commutative principle, associative principle, zero principle, closure, and the basic operations of integers must all be grasped before one can meaningfully learn how to solve equations.

It is the investigator's belief that this section of work is not properly presented in some of our textbooks. The presentation, unless altered by the teacher, leaves "gaps" in the learning process and robs the student of the opportunity to understand the mathematics involved.

Two programmed instructional units were developed by the investigator to teach this unit on solving equations. One program was developed using the traditional textbook sequence whereas the other used the hierarchical sequence. These instructional approaches are described fully in the next two sections of this chapter.

The Textbook Approach To Instruction

The Textbook Approach followed the traditional methodology which is prevalent in many of our mathematics textbooks and which is adopted by many mathematics teachers without even considering other instructional approaches. Figure 2 shows what the investigator considers to be the Textbook or Conventional Model of Instruction.

In developing the instructional booklet for the Textbook Group, the following points acted as guidelines: (1) The program followed the exact sequence of the textbooks used at that grade level. (2) No attempt was made in the unit to fill any "gaps" that might exist in the textbook presentation. (3) Exercises of the same type as in the traditional textbook were used in the instructional unit. (4) No mechanism was built into the program to ensure that the student knew the prerequisite capabilities before he proceeded to higher level capabilities. In essence, the investigator tried to ensure that the instructional unit exposed the student to the same material as the textbook, and in the same sequence as the textbook. A copy of this

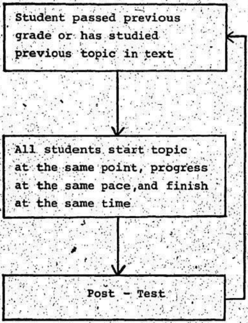


Figure 2. Textbook or Conventional Model of Instruction

instructional booklet is included in the accompanying booklet "Materials Developed For The Study".

The Hierarchical Approach To Instruction

The Hierarchical Approach involved the use of Gagné's ideas in sequencing the instructional material. Figure 3 gives an overview of this approach.

A learning hierarchy was constructed for the terminal objective by using Gagné's approach. The reasonableness of the hypothesized hierarchy was checked by experienced mathematics teachers and by subject matter experts. All necessary revisions were made. The revised hierarchy is shown in Appendix A.

All the intermediate objectives, or the objectives of the subtasks, were stated in behavioral terms. These objectives can be found in Appendix B. These intermediate objectives cover the entire hierarchy.

Programed instructional lessons were constructed for the intermediate objectives. A copy of these instructional lessons is included in the accompanying booklet "Material Developed For The Study". A check-point, or mini-quiz, was included at the end of each lesson to ensure that each student reached an acceptable level of performance on each of the subtasks before he proceeded. If a student's performance on a particular lesson was unacceptable, then, he had to repeat that lesson before he was permitted to proceed to the next lesson.

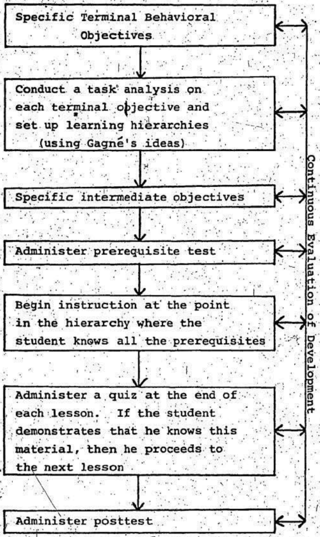


Figure 3. Hierarchical Approach To Instruction

A prerequisite test based on the intermediate objective plus the terminal objective determined the level of performance of each student with respect to the hierarchy. Instruction began at the point where the student had mastered all the prerequisite capabilities. The main emphasis in this approach was that students learned the prerequisites before they proceeded to higher level capabilities, thereby ensuring that there were no "gaps" in the learning process.

Experimental Design

A prerequisite test was given to the entire population, (76 eighth grade, male students), two days before the instruction was scheduled to begin. This prerequisite test covered all the prerequisite skills as well as the terminal objective. A copy of this test is included in Appendix C. Since no student demonstrated on this test that he could already perform the terminal objective, no one was dropped from the initial population.

The students were then randomly assigned to two groups of the same size, (38 students per group). One group was randomly assigned the Hierarchical Approach and the other group the Textbook Approach.

Three classes of students took part in the study. Each class included both Hierarchical Group students and Textbook Group students. The Hierarchical Group used the booklets prepared following the learning hierarchy and the Textbook

Group used the booklet prepared following the textbook sequence. Instruction began on March 29, 1976 and continued for ten days with the students receiving two - forty minute periods of instruction per day. During the study students worked entirely on their own.

Students were given a posttest one day after the completion of the instruction and a retention test was given two weeks later. The posttest and retention test were alternate forms of the same exam.

Controlling Variables

The investigator made every effort to control all possible independent variables and to eliminate any extraneous variables.

The teacher variable was eliminated by using programed booklets. Teachers, who supervised the classes during the experiment, were given specific instructions as to how much help the students should be given. Teachers were requested not to give students extra instruction or individual help. They were told to help students only if instructions were not clear, or if the print was not clear, or if there was some confusion as to the procedure to be followed.

Throughout the study,students worked independently and the booklets were collected at the end of each period. During the ten day period,when the study was taking place, the students studied no mathematics other than work from the programed booklets. Students remained in their own classrooms for the duration of the study and were not aware that they had been divided into groups and were taking part in a study. From the exterior both instruction booklets appeared to be the same. During the two week period following the completion of the instruction and before the retention test was given, students did not study any mathematics since this was their Easter holiday break.

Data Collected

A posttest was given to both groups immediately following the completion of the programed booklets. A copy of the posttest is included in Appendix B. The mean and standard deviation was calculated for each group. An analysis of covariance was used to see if the difference on the posttest means was significant ($p = .05$). The prerequisite test results were used as a covariate.

In order to compare retention, a test was given to both groups two weeks after the completion of the topic. This retention test was an alternate form of the posttest. A copy of the retention test is included in Appendix B. The mean and standard deviation were calculated for each group. Two analysis of covariances were carried out on these results to determine if the difference in means was significant ($p = .05$). The prerequisite test results were used as a covariate in one analysis, whereas the posttest results were used in the other analysis.

Limitations

The study had several obvious limitations. The school where the experiment was conducted was not selected randomly. It happened to be the school where the investigator taught. The experiment involved a relatively small sample, 76 males. The fact that the experiment was carried out with males only,may indicate that we can not generalize to females. The unit of work used was fairly small and so there might be some danger in generalizing the results to a larger segment of work, or to a complete course. It might also be argued by some that the hierarchy which was used is not truly hierarchical in nature and therefore there is no real comparison.

CHAPTER IV

ANALYSIS OF DATA

This chapter presents the findings of the study. The results of each of the analysis used in testing the hypothesis are given.

Prerequisite Test Results

A prerequisite test based on the learning hierarchy was given to both groups before any instruction was started. The results of this test appear in Table 1.

TABLE 1

Prerequisite Test Results

Group	Mean	Standard Deviation
Textbook (N= 38)	33.47%	11.02
Hierarchical (N = 38)	27.87%	9.68

The mean for the Textbook Group was 6.6% better than that for the Hierarchical Group. A t-test was used to determine if this difference was significant at the .05 level. The t-test yielded a value of .31. Since a t value of 2.00 would be necessary for a significant difference at the .05 level, it was concluded that there was no significant difference in the prerequisite test means.

Posttest Results

At the end of the instruction period both groups were given a posttest which tested the students performance on the terminal objective. The results of this test appear in Table 2.

TABLE 2

Posttest Results

Group	Mean	Standard Deviation
Textbook (N = 38)	21.84%	28.18
Hierarchical (N = 38)	49.21%	29.32

At first glance,it seemed that the instruction had a negative effect on the Textbook Group. They obtained a 33.47% average on the prerequisite test compared to a 21.84% average on the posttest. But when one examines the nature of the two tests, the reason for this discrepancy can easily be seen. The prerequisite test was based mainly on the prerequisite skills,whereas the posttest was based entirely on the terminal objective. Many of the Textbook Group knew how to perform these simpler prerequisite skills but they could not perform the higher level capability of solving first degree, one variable equations

An analysis of covariance was carried out on the posttest results using the prerequisite test results as a covariate. The results of this analysis are given in Table 3.

TABLE 3

Analysis of Covariance of the Posttest Results Using the Prerequisite Test Results as a Covariate

Source of Variance	df	ss'	ms'	F
Between	2	64418	32209	$F_{2,73} = 53.83$*
Within	73	43677	598.3	

* $P < .05$

The analysis showed that the Hierarchical Group posttest results were significantly better than the Textbook Group results. An F value of 3.13 would be needed for significance at the .05 level. The analysis yielded a value of 53.83. Even at the .01 level of significance, the Hierarchical Group results were significantly better than the Textbook Group results.

Retention Test Results

Approximately two weeks after the instruction had ended, both groups were given a retention test. This test was an alternate form of the posttest. The results of this test appear in Table 4.

TABLE 4

Retention Test Results

Group	Mean	Standard Deviation
Textbook (N = 38)	14.32%	21.69
Hierarchical (N = 38)	31.61%	21.69

The results of the retention test were very disappointing. Students in both groups retained far less than the researcher had anticipated. There may have been many reasons for this poor performance on the retention test. One of the most important reasons was the timing of the exam. Students were given the exam on the first day of classes after their Easter break. They had no prior notice of the exam. The teachers who supervised the exam noted that some of the students were not over enthusiastic about writing an exam at this time. The implication being that they might not have given it their best effort. The fact that many students scored low on the exam should not affect our overall findings since both groups were treated exactly alike.

Two analyses of covariances were carried out on the retention test results. One analysis used the prerequisite test results as a covariate whereas the other used the posttest results as the covariate. The results of the first analysis are given in Table 5.

TABLE 5

Analysis of Covariance of Retention Test Results Using the Prerequisite Test Results as a Covariate

Source of Variance	df	ss'	ms'	F
Between	2	54906	27453	
Within	73	42994	588.96	$F_{2,73} = 46.6$*

* $p < .05$

The analysis showed that even when the initial differences in prerequisite capabilities were eliminated, the Hierarchical Group performed significantly better than the Textbook Group on the retention test. An F value of 3.13 would be needed for significance at the .05 level. The analysis yielded an F value of 46.6. Even at the .01 level of significance, the Hierarchical Group did significantly better than the Textbook Group.

The investigator interprets this analysis as saying that the hierarchical students retained more and performed significantly better on the retention test because they learned more during the instruction period.

The second analysis of covariance used the posttest results as a covariate. The results of this analysis is given in Table 6.

TABLE 6

Analysis of Covariance of Retention Test Results Using the Posttest Results as a Covariate

Source of Variance	df	ss'	ms'	F
Between	2	54906	27453	$F_{2,73} = 46.6^{*}$
Within	73	42994	588.96	

$^{*}P < .05$

This analysis showed that even when the investigator eliminated the differences that existed in the two groups at the end of the instructional period, the Hierarchical Group did significantly better than the Textbook Group on the retention test. An F value of 3.13 would be needed for significance at the .05 level. The analysis yielded an F value of 37.36. Even at the .01 level of significance, the Hierarchical Group did significantly better than the Textbook Group.

Since in the second analysis the differences in achievement between the two groups at the end of the instruction period were eliminated by using the posttest results as a covariate, it cannot be assumed that the only reason the hierarchical students retained more is because they learned more. The investigator interprets the second analysis as saying that the Hierarchical Group retained more because they learned differently. It is the investigator's view that the Hierarchical Approach gave students the opportunity to see the interrelationships that exist within the topic; to study a topic in mathematics without being at the disadvantage of not knowing the necessary prerequisites. In short, the learning process was more meaningful for the hierarchical students.

In summary, these two analyses of the retention test results indicate that the Hierarchical Group did significantly better than the Textbook Group for two reasons: firstly, they knew more mathematics at the end of the instruction period, and secondly, they learned the mathematics differently; they learned with understanding.

Alternate Hypothesis Accepted

The findings of this study led to the acceptance of the alternate hypothesis and to the rejection of the null hypothesis as stated in Chapter I. The alternate hypothesis was stated as follows:

> H_A: There will be a significant difference ($P = .05$) in achievement and retention of a class of students taught a mathematics topic following Gagné's Hierarchical Approach and a class using the traditional Textbook Sequence.

The analysis of the results did indeed show that the Hierarchical Group achieved significantly better and retained significantly more than the Textbook Group.

Some Observations on the Results

The researcher believes that the main reason the Hierarchical Group performed significantly better than the Textbook Group on both the achievement and retention tests was that the Hierarchical Approach produced a much better instructional program than the Textbook Approach. The process of performing a task analysis on the terminal objective and then arranging the subtasks in a learning hierarchy ensured that all the prerequisite capabilities were included in the hierarchical instruction booklet. No such assurance existed about the textbook instruction booklet.

The Hierarchical Approach provided students with the opportunity to fill any "gaps" that might exist in their knowledge of the prerequisite capabilities before they proceeded to higher level capabilities. The Textbook Approach did not include these differential starting points. Teachers who supervised the classes during the study observed that many students in the Textbook Group experienced severe problems with some of the prerequisite skills such as the basic operations of integers.

Another important factor which made the two approaches different was the inclusion of the "check-points" in the hierarchical booklets. These check-points were mini-quizzes which were placed at the end of each lesson.

These check-points forced students to demonstrate their knowledge of the prerequisite capabilities before they proceeded. If a student did not get the check-point questions correct, then he had to repeat that section. The fact that the hierarchical students knew that they had to get the check-point questions correct before they moved forward , might have acted as a motivational factor for these students.

The researcher believes that the inclusion of these check-points in the Hierarchical Group booklet had a positive effect upon the results of these students. The investigator further contends that the Textbook Group booklet, even with its many "gaps" and poorer sequencing, could be greatly improved by including these check-points at the end of each lesson. Obviously, this could not be done in this study, since the Traditional Textbook Approach does not include the use of these quizzes to ensure that the prerequisites are known before the student proceeds to higher level capabilities.

CHAPTER V

SUMMARY AND CONCLUSIONS

This chapter includes a summary of the study, conclusions that were drawn from the analysis of data and from teacher observations, implications for mathematics teaching, and recommendations for further investigations.

Summary

Teachers are continually faced with the problems of students who do not achieve in relationship to their capabilities. Many reasons are often given to account for this underachievement; laziness, low I. Q., personality and emotional factors, family background, and a host of others. There is one reason which is most often overlooked. This reason is poor instruction caused by improper sequencing of the subject material, "gaps" in the presentation of the material and "gaps" in the students prerequisite capabilities.

In this study the researcher has examined one method of overcoming this poor instruction. This method is that of using Gagné's ideas concerning task analysis and learning hierarchies to ensure that the subject material is sequenced properly without any "gaps".

The main purpose of this study was to compare the achievement and retention of grade eight students taught a topic on solving equations using Gagné's Hierarchical

Approach with a group using the Traditional Textbook Sequence.

The population for the study, all chosen from the same school, consisted of 76 eighth grade students who had not been taught a formal method for solving equations of the type $ax + b = c$ (where a, b, and c are integers and $a \neq o$) previous to the study. The students were randomly assigned to two groups of the same size. One group was randomly assigned the Hierarchical Approach and the other the Textbook Approach.

The Textbook Group learned how to solve equations by studying a programmed booklet developed by following the sequence of the textbook presently used at that grade level. No attempt was made to fill any "gaps" that might exist in the textbook presentation or to ensure that lower level capabilities were known before the students progressed to higher level capabilities.

The Hierarchical Group studied a programmed booklet prepared following the ideas of Robert Gagné. This approach involved finding out all the prerequisite skills by performing a task analysis on the terminal objective, setting up a learning hierarchy for these skills and sequencing these tasks in the programmed booklet according to the learning hierarchy. Instruction began at the point in the hierarchy where the student knew all the prerequisites. A student was not permitted to progress to a higher level capability until he demonstrated that he could perform the lower level prerequisites. The main emphasis in this approach was to eliminate "gaps" in the learning process.

Instruction began on March 29, 1976 and continued for 10 days with the students receiving two forty minute periods of instruction per day. During the study, students remained in their own classrooms and worked entirely on their own with a minimal of teacher guidance. Students were not aware that they had been divided into groups and were taking part in a study. From the exterior, both instructional booklets appeared the same.

Students were given a posttest one day after the completion of the instruction and a retention test was given two weeks later. The statistical technique of the analysis of covariance was used to determine if the differences in achievement and retention were significant. The level of significance was set at .05.

Conclusions

Based upon the statistical analysis of the data gathered in the investigation and the observations of the teachers who supervised the classes during the study, the following conclusions were drawn:

1. The Hierarchical Approach to instruction produced significantly better achievement results than the Textbook Approach. This conclusion was based upon an analysis of covariance of the post-test results using the prerequisite test results as a covariate. The analysis of covariance test

showed significance at the .05 level.

2. The Hierarchical Approach to instruction produced significantly better retention results than the Textbook Approach. Two analyses of covariances were carried out on the retention test results; one analysis used the prerequisite test results as the covariate, whereas the other used the post-test results as a covariate. Both analyses showed significance at the .05 level. Based upon the results of these two analyses, the investigator concluded that the hierarchical students retained more for two reasons: (a) The students learned more during the instructional period. (b) The learning was more meaningful for the hierarchical students.
3. Since the investigator believes that this study indicated that underachievement in many cases is linked to poor instruction, caused by improper sequencing and "gaps" in th e learning process, it follows that the Hierarchical Approach to instruction would be one method of reducing under-achievement in mathematics classrooms.
4. Students taught to solve equations using the Hierarchical Approach experienced less difficulties than those taught the same topic using the Text-book Approach. This conclusion was based upon the observations of teachers who supervised the classes during the study.

5. Within the limitations of this study, there is much support for the use of the Hierarchical Approach to instruction in the mathematics classroom.

Implications For Mathematics Teaching

This study has shown that Gagné's Hierarchical Approach to instruction is a very worthwhile approach for mathematics teachers to adopt and use in their classrooms. It has shown that this approach can help students achieve better results and retain more mathematics.

The investigator believes that the experience of performing a task analysis and setting up a learning hierarchy for a particular objective would be an excellent one for any teacher. This exercise could bring many teachers to the realization that there is much more involved in learning a simple objective than a superficial examination reveals. Many teachers would soon realize that quite often they assume their students know too much because not enough attention is paid to the prerequisite skills.

The Hierarchical Approach to instruction provides an excellent means of determining what prerequisite skills a student needs in order to learn a specific concept, principle, etc. The importance of mapping the sequence of learning is mainly just this; it enables one to avoid the mistakes that arise from omitting essential steps in the acquisition of knowledge. Without such a plan, omissions of this sort are unfortunately easy to make. Following a preplanned sequence, then, and thus avoiding the omission of prerequisite capabilities along any route of learning, appears to be a highly important procedure to adopt in achieving effectiveness for instruction.

Many critics would oppose the use of the Hierarchical Approach on the grounds that it is too time consuming. The investigator realizes that the process of performing a task analysis and setting up a learning hierarchy is very time consuming, especially when one is a novice in this area. The solution to this problem would be for groups of teachers to work together and share the instructional units which they develop. The investigator believes that the time spent developing these instructional sequences would be well invested and would pay off in terms of improved student achievement and retention.

Recommendations For Further Research

In consideration of the results of this investigation, the following recommendations for further research are suggested:

1. It is recommended that similar studies be undertaken in different schools, at different grade levels, and to include female as well as male students.
2. The investigator recommends that the present study be replicated using teachers to teach the topic instead of using programed booklets. The

investigator does not know what effect the programed booklets had on the results obtained. The use of programed booklets may have introduced other variables such as reading level, willingness or unwillingness of students to work with a minimal of teacher guidance and boredom.

3. A study should be undertaken to verify (using statistical techniques) the validity of the hierarchy which the investigator developed in this study. The investigator did not use statistical techniques to validate the hierarchy. Many excellent statistical tecniques have been developed for this purpose (Walbesser, 1972; White, 1974).

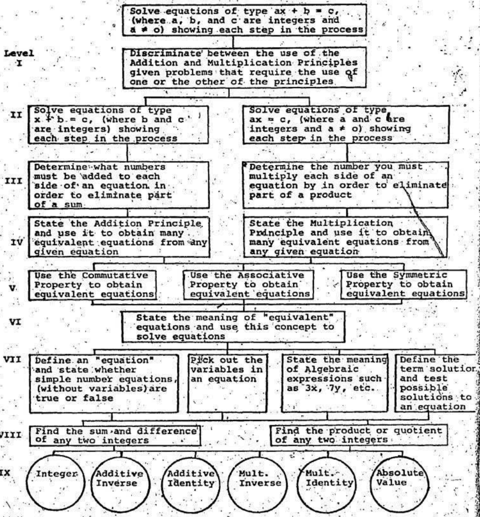
4. Further research should be carried out on the use of "check-points" in learning sequences. The investigator would be especially interested in determining the effect these "check-points" have on achievement and retention. Check-points are questions at the end of each lesson which the student _must_ know before he proceeds to the next section. The study could use two groups, both using the same sequence, one with the "check-points" and one without them.

5. Studies should be undertaken to determine if the material in our mathematics textbooks, at different grade levels is indeed arranged hierarchically.
6. Additional research should be carried out on the textbook presentation of material to determine if any of the necessary prerequisite skills are omitted and to determine the effect these omissions have on student achievement and retention.

Appendix A

Learning hierarchy for the objective:
Given any equation of the type $ax + b = c$, (where a, b, and c are integers and $a \neq o$), the student should be able to solve it showing each step in the process.

Learning Hierarchy For The Terminal Objective



Appendix B

Intermediate objectives for the terminal objective: Given any equation of the type $ax + b = c$ (where a, b, and c are integers and $a \neq 0$) the student should be able to solve it showing each step in the process.

Intermediate Objectives

Level IX:

(1) Given several numbers, the students should be able to pick out those which are integers.

(2) Given any integer, the student should be able to state its additive inverse.

(3) Given any integer, the student should be able to state its multiplicative inverse.

(4) Given any integer, the student should be able to state its absolute value.

Level VIII:

(1) Given any two integers, the student should be able to find their sum.

(2) Given any two integers, the student should be able to find their difference.

(3) Given any two integers, the student should be able to find their products.

(4) Given any two integers, the student should be able to find their quotient.

Level VII:

(1) Given several number sentences, the student should be able to pick out the equations.

(2) Given a simple number equation, (without variables), the student should be able to state whether it is true or false.

Level VII:

(3) Given an equation of the form $ax + b = c$, (where a, b and c are integers and $a \neq o$) the student should be able to state the variable in the equation.

(4) Given an algebraic expression such as 7x or 5y, the student should be able to state its meaning.

(5) Given the term "solution" the student should be be able to define it.

(6) Given an equation and some possible solutions, the student should be able to determine which of these possible solutions are true solutions.

Level VI:

(1) Given the term "equivalent equations", the student should be able to define it.

(2) Given several equivalent equations and the solution of one of these equations, the student should be able to state the solution of the other equations.

(3) Given two equations and the solution of one of these equations, the student should be able to determine if the second equation is equivalent to the first.

(4) Given two equivalent equations, one of which whose solution is obvious, the student should be able to find the solution of the other equation.

Level V:

(1) Given any equation of the type $ax + b = c$, the student should be able to use the commutative property of addition to write an equation equivalent to it.

(2) Given any equation of the form $(ax + b) + d = c + d$, (where a, b, c, and d are integers and $a \neq 0$) the student should be able to use associative property of addition to write an equation equivalent to it.

(3) Given any equation of the form $ax + b = c$, the student should be able to use the symmetric property to write an equation equivalent to it.

Level IV:

(1) The student should be able to state the addition principle.

(2) Given an equation, the student should be able to obtain an equivalent equation by adding the same number to each side of the equation.

(3) Given two equations such as $3x = 6$ and $3x + 7 = 6 + 7$ (the second equation was obtained by using the addition principle), the student should be able to determine if the equations are equivalent.

(4) The student should be able to state the multiplication principle.

(5) Given an equation, the student should be able to obtain an equivalent equation by multiplying each side of the equation by the same number.

Level IV:

(6) Given two equations such as $2x = 8$ and $6x = 24$, (the second equation was obtained by using the multiplicative principle), the student should be able to determine if the equations are equivalent.

Level III:

(1) Given an equation of the type $x + b = c$, the student should be able to state the number which must be added to each side in order to solve it.

(2) Given any equation of the type $ax = c$ (where a and c are integers and $a \neq 0$), the student should be able to state the number that each side of the equation has to be multiplied by in order to solve it.

Level II.

(1) Given any equation of the type $x + b = c$, (where b and c are integers), the student should be able to solve it, showing each step in the process.

(2) Given any equation of the type $ax = c$, (where a and c are integers and $a \neq 0$), the student should be able to solve it, showing each step in the process.

Level I:

(1) Given equations of the form $ax = c$ and of the form $x + b = c$, the student should be able to determine whether to use the addition or the

Level I:

(1) multiplication principle and then solve the equation showing each step in the process.

Appendix C

Tests Used During Study

Time: 40 minutes

Name_______________

Homeroom:_______________

Prerequisite Test

MATHEMATICS

GRADE VIII

This test is not the same as an ordinary test. The purpose of it is to find out how much you know about solving equations before you begin to study this topic. You are not expected to be able to do all the questions on this test. Answers as many questions as you are able. Don't worry about those questions which you cannot answer, since you will soon learn much more about solving equations.

PLACE THE ANSWER TO EACH QUESTION IN THE SPACE AT THE RIGHT

1. Which of the following are integers? 1.__________

   -6, $3/2$, $.76$, 36

2. What is the additive inverse (opposite) of -20? 2.__________

3. What is the multiplicative inverse (reciprocal) of 8? 3.__________

4. $|-17| =$? 4.__________

5. $-6 + {}^{-}8 =$? 5__________

6. Find the sum of 18 and -29 6__________

7. $-16 + 16 =$? 7__________

8. $3 \cdot -6 =$? 8.__________

9. Find the product of -10 and -3. 9.__________

10. $-24 - -8 =$ .... 10 ________

11. A number sentence with an = sign is called a (n) .... 11 ________

12. What is the variable in the equation $6y - 19 = 3$? 12 ________

13. The expression $3x$ means which one of the following:

(a) 3 plus x (c) 3 divided by x

(b) 3 multiplied by x (d) none of the above 13 ________

14. Is -2 the solution of the equation $6x + 7 = -5$? 14 ________

15. Equations which have the same solution are called .... equations. 15 ________

16. Use the commutative property of addition to write an equation which is equivalent to the equation: $-6 + x = 10$. 16 ________

17. Use the associative property of addition to write an equation which is equivalent to the equation $(x + 3) + {}^{-}3 = 6 + {}^{-}3$ 17 ________

18. Use the symmetric property of addition to write an equation which is equivalent to the equation $6 = x + 7$. 18 ________

19. Do the equations $6x = 70$ and $6x + 10 = 70 + 10$ have the same solution? 19 ________

20. Do the equations $2x = 8$ and $6 \cdot 2x = 6.8$ have the same solution? 20. ________

SOLVE EACH OF THE FOLLOWING EQUATIONS. SHOW YOUR WORKINGS IN THE SPACE PROVIDED BELOW EACH PROBLEM:

(1) $x + 6 = {}^{-}5$

(4) $-9y + 7 = 70$

(2) $-7a = {}^{-}42$

(5) $6x + {}^{-}13 = 35$

(3) $-20 + x = {}^{-}8$

Time: 30 minutes

Name: ____________________

Grade: ____________________

POST-TEST

MATHEMATICS

GRADE VIII

Use the Addition and Multiplication principles to solve each of the following equations. The answers alone will **not** be sufficient. You **must** show each step in the solution process. (Do all your work on the paper provided).

(1) $a + 13 = {}^{-}16$

(2) $26y = 221$

(3) $22 = {}^{-}14 - 4x$

(4) $11x + 127 = 72$

(5) $-13 = 6p - 10$

(6) ${}^{-}16 + {}^{-}19y = {}^{-}16$

(7) $3 - 4x = 0$

(8) $-41a - 176 = 275$

(9) $74 = 14x - 17$

(10) $-27y + 73 = {}^{-}179$

Time: 30 minutes

Name: ____________________

Grade: ____________________

RETENTION TEST

MATHEMATICS

GRADE VIII

Use the Addition and Multiplication principles to solve each of the following equations. The answers alone will not be sufficient. You must show each step in the solution process. (Do all your work on the paper provided).

(1) $x - 8 = {}^{-}7$

(2) $18a = {}^{-}153$

(3) ${}^{-}16 = 13 - 3p$

(4) $13x + {}^{-}57 = 86$

(5) $-19 = 6y - 16$

(6) $13 + {}^{-}21x = 13$

(7) $7 - 3a = 0$

(8) ${}^{-}32x - 187 = 197$

(9) $93 = 15x - 17$

(10) $127y + 73 = {}^{-}188$

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TO THE READER

This booklet contains all the materials that were developed by the investigator for this study. It includes:

(i) An overview of the study.

(ii) Procedures to be followed by teachers supervising the classes during the study.

(iii) Procedures to be followed by students during the study.

(iv) A copy of the Hierarchical Instruction booklet.

(v) A copy of the Textbook Instruction booklet.

(vi) Tests used during the study.

(vii) Teacher observation sheets.

To The Teacher

Purpose of Study

This study will attempt to answer the following question. Would students achieve better and retain more mathematics if teachers followed a hierarchical approach to instruction, instead of the traditional textbook sequence?

The Hierarchical Approach

The hierarchical approach would consist of determining the prerequisite capabilities for the learning task by constructing a "learning hierarchy" for each terminal objective.

The learning hierarchy which is followed in the hierarchical approach is shown on the next page.

The instructional process would start at the point in the hierarchy where the student knew all the prerequisites. This approach should effectively eliminate any "gaps" in the learning process. It is based on Gagné's theory of learning hierarchies.

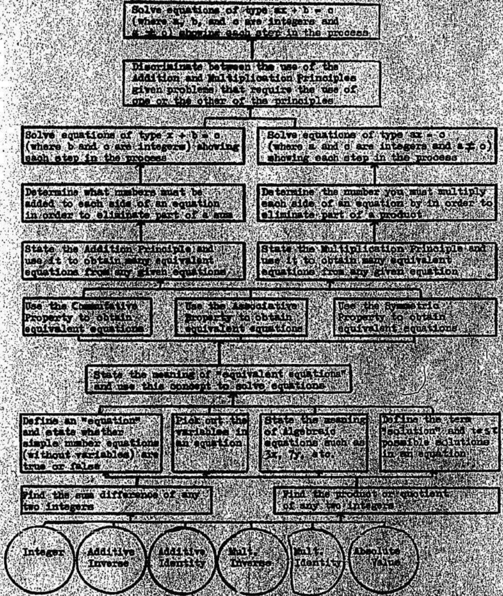
If you are interested in reading my complete internship proposal, I would be only too happy to make it available.

Overview of Project

All the grade eight students at Roncalli will be given a pre-test on March 26. Anyone who can already perform the terminal objective will be dropped from the population. The remaining students will be randomly assigned to two groups of the same size. One group will be assigned the Hierarchical Approach and the other the Textbook Approach. Students will remain in their own classrooms throughout the study. The hierarchical group will use the booklet prepared following the learning hierarchy, and the textbook group will use the booklet prepared following the textbook. Instruction will begin on March 29 and will continue for 1 - 1/2 weeks. Students will be given a post-test when they have finished the booklet and they will be given a retention test two weeks after the instruction ends.

COLOURED

Learning Hierarchies for the objective: Given any equation of the type $ax + b = c$ (where a, b, and c are integers and $a \neq 0$), the student should be able to solve it showing each step in the process.



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The following procedures should be followed during the instruction period.

1. Don't give students the impression that they are taking part in a study. They will not know that they have been divided into two groups, or that two different instructional approaches are being used. We will tell students that we are using a new method of instruction for the next week or so and there is no real reason they should suspect otherwise.

2. Students should not be given extra instruction or individual help. Help the student only if instructions are not clear, or if the print is not clear, or there is some confusion as to the procedure to be followed.

3. Ensure that students work on their own. There is to be absolutely no communication between students.

4. The Hierarchical Approach booklets contain check-points These are really mini-quizzes. Students are directed to check with the teacher before proceeding after the check-point. A student should have at least two-thirds of the check-point questions correct in order to proceed to the next section. If the student has less than two - thirds correct , direct him to go back and read that section again before proceeding. The answers to the check points are included in your booklet.

5. If any difficulties arise during the instruction period, please consult me.

6. Note (write down) any worthwhile observations that you may notice during the study. For example, the students working on one approach seem to encounter less difficulty than those working on the other approach. (A few pages have been provided at the end of this booklet for this purpose).

7. Collect the booklets at the end of each period and remove them from the classroom.

8. Browse through this handout, and if you have any questions, please feel free to direct them to me.

Thank you for your co-operation.

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NAME: ____________________

HOMEROOM: ____________________

PRE-TEST

MATHEMATICS

GRADE VIII

This test is not the same as an ordinary test. The purpose of it is to find out how much you know about solving equations before you begin to study this topic. You are not expected to be able to do all the questions on this test. Answers as many questions as you are able. Don't worry about those questions which you cannot answer, since you will soon learn much more about solving equations.

PLACE THE ANSWER TO EACH QUESTIONS IN THE SPACE AT THE RIGHT.

1. Which of the following are integers?
   -6, 3/2, .76, 36 — 1.__________
2. What is the additive inverse (opposite) of -20? — 2.__________
3. What is the multiplicative inverse (reciprocal) of 8? — 3.__________
4. $|-17| = \underline{\ ?\ }$ — 4.__________
5. $-6 + {}^{-}8 = \underline{\ ?\ }$ — 5.__________
6. Find the sum of 18 and -29. — 6.__________
7. $-16 + 16 = \underline{\ ?\ }$ — 7.__________
8. $3 \cdot -6 = \underline{\ ?\ }$ — 8.__________
9. Find the product of -10 and -3. — 9.__________
10. $-24 \div -8 = \underline{\ ?\ }$ — 10.__________
11. A number sentence with an "=" sign is called a(n).. — 11.__________
12. What is the variable in the equation $6y - 19 = 3$? — 12.__________
13. The expression 3x means which one of the following:
    (a) 3 plus x (c) 3 divided by x
    (b) 3 multiplied by x (d) none of the above — 13.__________
14. Is -2 the solution of the equation $6x + 7 = -5$? — 14.__________
15. Equations which have the same solution are called ..... equations. — 15.__________

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16. Use the commutative property of addition to write an equation which is equivalent to the equation: $-6 + x = 10$. 16.________________

17. Use the associative property of addition to write an equation which is equivalent to the equation $(x + 3) + {}^{-}3 = 6 + {}^{-}3$ 17.________________

18. Use the symmetric property of addition to write an equation which is equivalent to the equation $6 = x + 7$ 18.________________

19. Do the equations $6x = 70$ and $6x + 10 = 70 + 10$ have the same solution? 19.________________

20. Do the equations $2x = 8$ and $6 \cdot 2x = 6.8$ have the same solution? 20.________________

SOLVE EACH OF THE FOLLOWING EQUATIONS. SHOW YOUR WORKINGS IN THE SPACE PROVIDED BELOW EACH PROBLEM:

(1) $x + 6 = {}^{-}15$

(2) $-7a = {}^{-}42$

(3) $-20 + x = {}^{-}8$

(4) $-9y + 7 = 70$

(5) $6x + {}^{-}13 = 35$

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Time: 30 minutes.

NAME: ____________________

GRADE: ____________________

POST-TEST

MATHEMATICS

GRADE VIII

Use the Addition and Multiplication principles to solve each of the following equations. The answers alone will <u>not</u> be sufficient. You <u>must</u> show each step in the solution process. (Do all your work on the paper provided).

(1) $a + 13 = {}^{-}16$

(2) $26y = 221$

(3) $22 = {}^{-}14 - 4x$

(4) $11x + 127 = 72$

(5) $-13 = 6p - 10$

(6) ${}^{-}16 + {}^{-}19y = {}^{-}16$

(7) $3 - 4x = 0$

(8) $-41a - 176 = 275$

(9) $74 = 14x - 17$

(10) $-27y + 73 = {}^{-}179.$

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Time: 30 minutes

NAME: ____________________

GRADE: ____________________

RETENTION TEST

MATHEMATICS

GRADE VIII

Use the Addition and Multiplication principles to solve each of the following equations. The answers alone will <u>not</u> be sufficient. You <u>must</u> show each step in the solution process. (Do all your work on the paper provided).

(1) $x - 8 = {}^{-}17$

(2) $18a = {}^{-}153$

(3) ${}^{-}16 = 13 - 3p$

(4) $13x + {}^{-}57 = 86$

(5) ${}^{-}19 = 6y - 16$

(6) $13 + {}^{-}21x = 13$

(7) $7 - 3a = 0$

(8) ${}^{-}32x - 187 = 197$

(9) $93 = 15x - 17$

(10) $-27y + 73 = {}^{-}188$

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Booklet For The Hierarchical Approach

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TO THE STUDENT

This booklet is very different from an ordinary textbook; it cannot be read in the way other books are read. You must read the material carefully, study the examples, complete all the exercises, check your own answers and proceed through the booklet at your own rate. The success of this booklet depends upon how carefully you read the material, answer the exercises, etc. When you finish this booklet you should be able to solve equations such as $-3x + 16 = -17$ and $22 = -14 - 4x$. A test will be given at the end of this booklet to see how much you have learned.

Take a few minutes and look through this booklet. Note the red lines. There is a red line below each exercise or set of exercises. The answers to the exercises are below the red lines. The answers are provided so that you may check your own work. If you get some of the answers wrong, please read the section again so that you may see why they are wrong. Note also that a few areas have been blocked in green. The purpose of this is to draw your attention to the importance of these ideas.

The following procedures should be followed in using this booklet:

(1) Work on your own; there is to be no talking to other students.

(2) Use the piece of cardboard provided to cover everything below the red lines. When you have finished the work above the red line, lower the cardboard to the next red line and check your answers to the exercises.

(3) Read the material carefully and study all the examples thoroughly.

(4) Place your answers to the exercises in the blank spaces provided after each exercise.

(5) If you get some of the exercises wrong, go back and read the section again so that you can see where you went wrong.

(6) Your booklet contains check points. These are really mini-quizes and it is essential that you get most of the questions in each check point correct before you go on to the next section. You must check with your teacher at the end of each check point.

(7) These booklets will be collected at the end of each period.

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This section on the <u>Basic Operations of Integers</u> is only included in some of the Hierarchical Approach Booklets. The pre-test determines if it is included

SECTION 1 ADDITION OF INTEGERS

The set of integers includes the whole numbers and their additive inverses (opposites). It consists of ______ $^-6$, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, ____________.

Each integer has an additive inverse (opposite). The additive inverse of 3 is -3. The additive inverse of -5 is 5.

Exercise: State the additive inverse of each of the following numbers.

(a) 6 ______

(b) -3 ______

(c) 20 ______

(d) -40 ______

Answers: (a) -6 (b) 3 (c) $^-20$ (d) 40

The sum of an integer and its additive inverse is 0. For example $3 + {}^-3 = 0$, $-10 + 10 = 0$.

Exercise: Give the answer to each of the following addition problems.

(a) $9 + {}^-9 =$ ______

(b) $-21 + 21 =$ ______

(c) $16 + {}^-16 =$ ______

Answers: (a) 0, (b) 0 (c) 0

Zero is a very important integer. It is given a special name, the Additive Identity. Adding zero to any number does not change it. That is, $6 + 0 = 6$, $-3 + 0 = {}^-3$ $-100 + 0 = {}^-100$.

Exercise: Give the answers to each of the following addition problems.

(a) $21 + 0 =$ ______

(b) $-17 + 0 =$ ______

(c) $0 + {}^-5 =$ ______

Answers: (a) 21 (b) -17 (c) -5

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The sum of two <u>positive</u> integers is always a <u>positive</u> integer. Example; $3 + 5 = 8$

<u>Exercise:</u> (a) $6 + 8 =$ ________
(b) $7 + 9 =$ ________
(c) $15 + 19 =$ ________

<u>Answers:</u> (a) 14
(b) 16
(c) 34

We have agreed that the sum of two <u>negatives</u> integers is a <u>negative</u> integer. For example ${}^-3 + {}^-2 = {}^-5$. This can easily be seen in physical terms, since a loss of 3 plus a loss of 2 would result in a loss of 5.

<u>REMEMBER;</u> Negative + Negative = Negative

<u>Exercise:</u> Add
(a) $-6 + {}^-7 =$ ________
(b) $-10 + {}^-20 =$ ________
(c) $-15 + {}^-8 =$ ________
(d) $-7 + {}^-42 =$ ________
(e) $-50 + {}^-18 =$ ________

<u>Answers:</u> (a) -13
(b) -30
(c) -23
(d) -49
(e) -68

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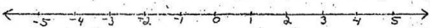
You may have noticed that, to this point, in our discussion of the addition of integers we have restricted ourselves to the addition of two positive integers or the addition of two negative integers. You have seen when we add a positive to a positive we obtain a positive number and when we add a negative to a negative we obtain a negative number. When we add two integers, however, one of which is negative and the other positive, the sum may be positive or it may be negative, depending upon the numbers added.

In order to understand how to add a positive and a negative number we should first understand the idea of <u>Absolute Value</u>. The absolute value of a number refers to the number of units the number is from the origin on a number line.



The symbol $|\ |$ is read "Absolute Value".
The number 3 is 3 units from the origin so we say the absolute value of 3 is 3 or $|3| = 3$.
The number -10 is 10 units from the origin so we say the absolute value of -10 is 10 or $|-10| = 10$.

<u>Exercise:</u>

(a) $|-6| =$ ________
(b) $|20| =$ ________
(c) $|-56| =$ ________
(d) $|87| =$ ________

Answers: (a) 6
(b) 20
(c) 56
(d) 87

Now lets consider the sum of a positive and a negative integer and see how we employ the idea of absolute value.

Example: Consider the sum of $3 + {}^-4$. We can write -4 as $-3 + {}^-1$ and we get $3 + (-3 + {}^-1)$. Then if we use the associative property to regroup we get $(3 + {}^-3) + {}^-1$. The problem is easy from this point onward.

$$(3 + {}^-3) -1$$
$$0 + {}^-1$$
$$-1$$

Therefore: $3 + {}^-4 = {}^-1$

Example: Consider the ~~sum of~~ $12 + {}^-7$. We can write 12 as $5 + 7$ and we get $(5 + 7) + -7$. Then if we use the associative property to regroup, we get $5 + (7 + {}^-7)$. The problem is easy from this point onward.

$$5 + (7 + {}^-7)$$
$$5 + 0$$
$$5$$

Therefore: $12 + {}^-7 = 5$

Question: From the two examples, can you suggest a short cut that we could use to get the sum of a positive and a negative integer. __________________________

____

Answer: The short cut would be:

Step One: Get the difference of the two numbers.

Step Two: Give the answer a positive sign if the positive number has the greater absolute value, or give it a negative sign if the negative number has the greater absolute value.

Example: In order to find the sum of $-6 + 4$ wee see that the difference of the two numbers is 2, and the answer will have a negative sign since the number with the greater absolute value is negative.

So: $-6 + 4 = -2$

Example: In finding the sum of $10 + {}^{-}2$ we see that the difference of the two numbers is 8, and the answer will have a positive sign since the number with the greater absolute value is positive.

So: $10 + {}^{-}2 = 8$.

Exercises: Find the following sums:

(a) $6 + {}^{-}3 =$ ________

(b) $-10 + 3 =$ ________

(c) $20 + {}^{-}25 =$ ________

(d) $10 + {}^{-}9 =$ ________

(e) $-126 + 105 =$ ________

Answers: (a) 3

(b) -7

(c) -5

(d) 1

(e) -21

Subtraction of integers is very simple once we know how to add integers. The basic idea to remember is that subtracting is the same as adding the additive inverse (opposite).

Examples: (a) $6 - 4 = 6 + {}^{-}4 = 2$

(b) $-3 - {}^{-}2 = {}^{-}3 + 2 = {}^{-}1$

(c) $10 - {}^{-}7 = 10 + 7 = 17$

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Complete each of the following problems. Part (a) has been done as an example.

Exercises: (a) $7 - 4 = \underline{7 + {}^-4} = \underline{\ 3\ }$

(b) $-6 - 7 =$ ______ = ______

(c) $10 - {}^-3 =$ ______ = ______

(d) $-18 - 20 =$ ______ = ______

(e) $12 - 6 =$ ______ = ______

Answers: (a) $7 + {}^-4 = 3$

(b) $-6 + {}^-7 = -13$

(c) $10 + 3 = 13$

(d) $-18 + {}^-20 = -38$

(e) $12 + {}^-6 = 6$

Check Point: (1) $-7 + {}^-11 =$ ______

(2) $16 - {}^-4 =$ ______

(3) $8 + {}^-8 =$ ______

(4) $15 + -16 =$ ______

(5) $-8 - 6 =$ ______

Please check with your teacher before you proceed.

SECTION 11 MULTIPLICATION OF INTEGERS

Each integer has a multiplicative inverse. Another name for the multiplicative inverse is reciprocal. The multiplicative inverse of 2 is 1/2. The multiplicative inverse of -4 is $-1/4$.

Exercise: Give the multiplicative inverse (reciprocal) of each of the following numbers.

(a) 6 ______

(b) -7 ______

(c) -26 ______

(d) 15 ______

(e) -120 ______

Answers: (a) 1/6 (b) -1/7 (c) -1/26 (d) 1/15 (e) -1/120

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The product of a number and its multiplicative inverse is always 1. For example $3 \cdot 1/3 = 1$, $-7 \cdot -1/7 = 1$, and $26 \cdot 1/26 = 1$. The number 1 occupies a very important place in the system of integers. It is called the <u>Multiplicative Identity</u>. The product of any number and 1 is always that number. For example: $6 \cdot 1 = 6$, and $-10 \cdot 1 = {}^{-}10$.

Exercises: (a) The multiplicative inverse of -7 is ________

(b) $-7 \cdot {}^{-}1/7 =$ ________

(c) $-20 \cdot 1 =$ ________

(d) $8 \cdot 1 =$ ________

(e) $1/10 \cdot 10 =$ ________

Answers: (a) $-1/7$

(b) 1

(c) -20

(d) 8

(e) 1

The <u>product of two positive integers is a positive integer</u>. For example: $6 \cdot 3 = 18$, $7 \cdot 10 = 70$

Exercises: Find the following products.

(a) $9 \cdot 11 =$ ________

(b) $13 \cdot 14 =$ ________

(c) $16 \cdot 8 =$ ________

Answers: (a) 99

(b) 182

(c) 128

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It can easily be shown that the product of a positive and a negative integer is a negative integer.

Suppose we wanted to find the product of 2 and -6

$$2 \cdot {}^{-}6 = {}^{-}6 + {}^{-}6 = -12$$

$$\text{So: } 2 \cdot {}^{-}6 = -12$$

Suppose we wanted to find the product of 4 and -3.

$$4 \cdot {}^{-}3 = {}^{-}3 + {}^{-}3 + {}^{-}3 + {}^{-}3 = -12$$

$$\text{So: } 4 \cdot {}^{-}3 = -12$$

Remember:	Positive · Negative = Negative
	Negative · Positive = Negative

Exercises: Find the following products.

(a) $2 \cdot {}^{-}4 =$ ______

(b) $-8 \cdot 3 =$ ______

(c) $15 \cdot -6 =$ ______

(d) $-10 \cdot 8 =$ ______

(e) $12 \cdot -3 =$ ______

Answers: (a) -8

(b) -24

(c) -90

(d) -80

(e) -36

The product of two <u>negative</u> numbers is a <u>positive</u> number. This can be easily seen in a practical situation. Suppose the temperature drops 4 degrees each day. Three days ago the temperature would be 12 degrees higher than it would be today. This can be represented mathematically by the equation $-4 \cdot {}^{-}3 = 12$.

Remember: Negative · Negative = Positive

Exercises: Find the following products

(a) $-6 \cdot -3 =$ ______

(b) $-10 \cdot -5 =$ ______

(c) $-8 \cdot -3 =$ ______

(d) $-12 \cdot -6 =$ ______

(e) $-11 \cdot -7 =$ ______

Answers:

(a) 18

(b) 50

(c) 24

(d) 72

(e) 77

The division of integers is very simple once we know how to multiply integers. The same basic rules apply. These rules are:

Positive ÷ Positive = Positive
Positive ÷ Negative = Negative
Negative ÷ Positive = Negative
Negative ÷ Negative = Positive

Exercises: (a) $^{-}6 \div 2 =$ ______

(b) $^{-}8 \div {}^{-}4 =$ ______

(c) $10 \div {}^{-}5 =$ ______

(d) $25 \div 5 =$ ______

(e) $-54 \div -6 =$ ______

Answers: (a) -3, (b) 2 (c) -2 (d) 5 (e) 9

Check Point:

(a) $-4 \cdot {}^{-}3 =$ ______

(b) $-10 \cdot 9 =$ ______

(c) $-6 \cdot {}^{-}1/6 =$ ______

(d) $30 \div -5 =$ ______

(e) $15 \cdot -6 =$ ______

This section is included in all
the Hierarchical Approach Booklets

SECTION 1 SOLVING EQUATIONS

A number sentence containing an equal sign, =, is called an Equation. 12 + 9 = 21 is an example of an equation. It is a true equation, since 12 plus 9 is equal to 21.

The equation 8 + 9 = 21 is a false equation since 8 plus 9 is not equal to 21.

Exercise: State whether each of the following equations is true or false. Part (a) has been done as an example.

(a) 6 + 20 = 26 True
(b) 8 + 7 = 14 ________
(c) 9 − 3 = 6 ________
(d) 7 ÷ 1/2 = 14 ________
(e) 8 x 4·5 = 34 ________

Answer: (a) True (b) False (c) True (d) True (e) Fals

Most equations contain expressions such as 3x, 7y, 9a, etc. Remember that 3x means 3 multiplied by x; 7y means 7 multiplied by y, and 9a means 9 multiplied by a. The expressions 3x, 7y, and 9a are products.

Exercise: State the meaning of each of the following expressions. Part (a) has been done as an example

(a) 7a means 7 multiplied by a
(b) 10x means ________
(c) 3p means ________

Answers: (a) 7 multiplied by a
(b) 10 multiplied by x
(c) 3 multiplied by p

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The letter in the equation is referred to as a variable. In the equation $2x = 10$, x is the varaible. In the equation $a + 5 = 7$, a is the variable.

Exercise: State the variables in each of the following equations. Part (a) has been done as an example.

(a) $6y - 3 = 13$ ____y____

(b) $3x + 4 = 5$ ________

(c) $7a = 21$ ________

(d) $5p = 20$ ________

Answers: (a) y (b) x (c) a (d) p

We solve an equation by finding a value for the variable which will make the equation true. That is, we find a value for the variable which will make the left side of the equation equal to the right side of the equation.

Example: The solution of the equation $x + 4 = 6$, is $x = 2$. This is the only value which will make the equation true.

We can easily check to see if 2 is the solution by replacing the variable in the equation by 2 and observing if the left side of the equation is equal to the right-hand side.

Check: $x + 4 = 6$

$2 + 4 = 6$ (Replace the x by 2)

$6 = 6$ (The left side is equal to the right side, so 2 is the solution)

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Exercise: Is 3 the solution of the equation $x + 4 = 6$?

_____________ (show workings in space below)

Answer: (a) No

$x + 4 = 6$

$3 + 4 = 6$

$7 = 6$ (This is not true, so 3 is not the solution)

Exercise: Is 5 the solution of the equation $3x = 15$?

_____________ (show workings in space below)

Answer: (b) Yes

$3x = 15$

$3 \cdot 5 = 15$

$15 = 15$ (This is true, so 5 is the solution of the equation)

Check Point: Answer each of the following questions.

(1) ~~Is the equation $6 + 7 = 14$ a true equation?~~ _____________

(2) State the variable in the equation $3x - 7 = 14$. _____________

(3) Is 25 the solution of the equation $x - 7 = 18$? _____________

Please check with your teacher before you proceed.

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SECTION 2 EQUIVALENT EQUATIONS

The equations $x + 4 = 6$ and $x + 2 = 4$ have the same solution set. The solution is $x = 2$. Equations which have the same solution are said to be equivalent. Another equation which is equivalent to the two above is $x + 1 = 3$. The solution of this equation must also be 2.

Exercise: The solution of the equation $x + 3 = 10$ is 7. Is the equation $x + 1 = 8$ equivalent to the equation $x + 3 = 10$?
__________ Why? ____________________

Answer: Yes, the equation $x + 1 = 8$ also has the solution of 7.

Exercise: The solution of the equation $5x = 20$ is 4. What is the solution of any equation which is equivalent to the equation $5x = 20$? __________

Answer: 4

Equivalent equations play a very important role in solving equations, since if two equations are equivalent, then they have the same solution. In order to solve an equation we write a chain of equivalent equations until we find one whose solution is obvious.

Example: Suppose you were asked to solve the equation $7x + 12 = 117$ and you were told that the equation $x + 1 = 16$ is equivalent to the equation $7x + 12 = 117$. We can easily see that the solution of the equation $x + 1 = 16$ is 15, and since the equation $7x + 12 = 117$ is equivalent to $x + 1 = 16$, it must also have a solution of 15.

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We can check to see if 15 is the correct solution for the equation $7x + 12 = 117$ by replacing x in the equation by 15.

Check: $7x + 12 = 117$

$7 \cdot 15 + 12 = 117$ (we replace x by 15)

$105 + 12 = 117$

$117 = 117$ (since the left side is equal to the right side, 15 is the correct solution)

Exercise: Solve the equation $3x - 2 = 4$, given that the equation $x + 5 = 7$ is equivalent to it. ______

Answer: 2

Exercise: Solve the equation $2x - 3 = 9$, given that the equation $2x = 12$ is equivalent to it. ______

Answer: 6

Exercise: If the equations $6x - 4 = 14$, $6x = 18$, and $x = 3$ are all equivalent, then their solution is ______

Answer: 3

CHECK POINT: Answer each of the following questions.

1. What name is given to equations which have the same solution? ______
2. Are the equations $x - 7 = 4$ and $x = 11$ equivalent? ______ Why? ______
3. The equations $6x - 3 = 27$ and $x + 3 = 8$ are equivalent equations. Find the solution of the equation $6x - 3 = 27$. ______

Please check with your teacher before you proceed.

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SECTION 111 OBTAINING EQUIVALENT EQUATIONS

The equations $x + 6 = 4$ and $6 + x = 4$ are equivalent. They both have the solution $^{-}2$. Notice that the only difference in the two equations is that the order of the terms on the left side has been rearranged. The commutative property of addition states that we can change the order without changing the answer.

> Note: The commutative property of addition allows us to rearrange the order on either side of the equation without changing the value of the variable. For example, the equation $10 + x = {}^{-}6$ is equivalent to the equation $x + 10 = -6$.

Exercise: Use the commutative property to write an equation which is equivalent to each of the equations below. Part (a) has been done as an example.

(a) $^{-}3 + x = 4$ $\underline{x + {}^{-}3 = 4}$

(b) $10 + y = 15$ ________

(c) $12 = {}^{-}7 + x$ ________

Answer: (a) $x + {}^{-}3 = 4$ (b) $y + 10 = 15$ (c) $12 = x + {}^{-}7$

The equations $(x + 3) + 5 = 10$, and $x + (3 + 5) = 10$ are equivalent since the system of integers is associative. In other words we can change the grouping on either side of an equation without changing the value of the variable.

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Exercise: Use the associative property to write an equation which is equivalent to each of the following below. Part (a) has been done as an example.

(a) $(x + 6) + {}^{-}2 = 10$ $x + (6 + {}^{-}2) = 10$

(b) $(x + 7) + 6 = 15$ ______________

(c) $6 + (7 + x) = 5$ ______________

Answers: (a) $x + (6 + {}^{-}2) = 10$

(b) $x + (7 + 6) = 15$

(c) $(6 + 7) + x = 5$

The equations $10 = x + 3$, and $x + 3 = 10$ are equivalent. Both equations have a solution of 7. Any equation can be <u>completely reversed</u> without changing it's solution. The property which allows us to do this is called the <u>symmetric property</u>.

Exercise: Use the symmetric property to write an equivalent equation for each of the equations below.

(a) $10 = x + 6$ ______________

(b) ${}^{-}13 = x + {}^{-}7$ ______________

(c) $0 = x + 4$ ______________

Answers: (a) $x + 6 = 10$, (b) $x + {}^{-}7 = {}^{-}13$, (c) $x + 4 = 0$

CHECK POINT: Answer each of the following questions.

(1) Use the commutative property to write an equation which is equivalent to the equation $x + 6 = 10$.

(2) Use the symmetric property to write an equation which is equivalent to the equation $7 = x + 3$. ______________

(3) Use the associative property to write an equation which is equivalent to the equation $(x + 6) + 5 = 20$: ______________

Please check with your teacher before you proceed.

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SECTION IV ADDITION PRINCIPLE

Suppose we take the equation $x + 3 = 5$. The solution of this equation is 2. If we added 4 to each side of this equation, we would obtain a new equation $x + 7 = 9$. The solution of this equation is also 2, since $2 + 7 = 9$. This means that the new equation which we obtained by adding 4 to each side of the first equation is <u>equivalent</u> to the original equation.

Exercise: Suppose you take the equation $x + 7 = 13$. It's solution is 6.

(a) Write a new equation by adding 3 to each side of the given equation. ____________

Simplify it. ____________

(b) Is the solution of this equation also 6? ____________

(c) Is the equation $x + 7 = 13$ equivalent to the equation $x + 10 = 16$.

Answers: (a) $(x + 7) + 3 = 13 + 3$, $x + 10 = 16$

(b) yes

(c) Yes

Exercise: Suppose you take the equation $2x = 10$. It's solution is 5.

(a) Write a new equation by adding $^-3$ to each side of the given equation. ____________ Simplify it. ____________

(b) Is the solution of this new equation also 5? ______

(c) Is the equation $2x = 10$ equivalent to the equation $2x + {}^-3 = 7$? ____________

Answers: (a) $2x + {}^-3 = 10 + {}^-3$, $2x + {}^-3 = 7$

(b) Yes

(c) Yes

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Exercise: Can you state a general principle that seems to apply when you add a number to both sides of an equation? ______________________________

____

Answer: You obtain an equation which is equivalent to the original equation.

(Remember equivalent equations have the same solution.)

> The ADDITION PRINCIPLE states that you obtain an equivalent equation when you add the same number to both sides of an equation. That is, if $a = b$, then $a + c = b + c$.

The addition principle is extremely important to the solution of equations, since it gives us another method of obtaining equivalent equations. (Remember equivalent equations have the same solution).

CHECK POINT: Answer the questions below:

(1) If we add the same number to both sides of an equation we obtain an ______________________ equation.

(2) Equivalent equations have the same ______________________

(3) Are the equations $10x = 42$ and $10x + 15 = 42 + 15$ equivalent?

____

Please check with your teacher before you proceed.

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SECTION V OBTAINING EQUIVALENT EQUATIONS

We have learned that we can use the ADDITION PRINCIPLE to obtain an equivalent equation (an equation with the same solution).

Exercise: Suppose we take the equation $x + 6 = 7$.

(a) Obtain an equivalent equation by adding 3 to each side of the equation $x + 6 = 7$. ____________________

(b) Obtain an equivalent equation by adding 4 to each side of the equation $x + 6 = 7$. ____________________

(c) Obtain an equivalent equation by adding -6 to each side of the equation $x + 6 = 7$. ____________________

(d) Which of the equations obtained in parts a, b, or c is the easiest to solve? ____________________

Answers: (a) $x + 9 = 10$ (b) $x + 10 = 11$ (c) $x = 1$ (d) The one obtained in part C

Exercise: Suppose we take the equation $x + {}^{-}4 = 6$

(a) Obtain an equivalent equation by adding 2 to each side of the equation $x + {}^{-}4 = 6$ ____________________

(b) Obtain an equivalent equation by adding 4 to each side of the equation $x + {}^{-}4 = 6$. ____________________

(c) Obtain an equivalent equation by adding -4 to each side of the equation $x + {}^{-}4 = 6$ ____________________

(d) Which of the equations obtained in parts a, b, or c is the easiest to solve? ____________________

Answers: (a) $x + {}^{-}2 = 8$ (b) $x = 10$ (c) $x + {}^{-}8 = 2$ (d) The one obtained in part B

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So we have seen that we obtained the simplest equivalent equation for the equation $x + 6 = 7$ when we added -6 to each side, and we obtained the simplest equivalent equation for the equation $x + {}^-4 = 6$ when we added 4 to each side. Can you see the relationship between the number added to each side of the equation and the number on the left side of the equation? If you can see this relationship, state it. ______________

____

Obviously the number on the left side and the number added are additive inverses of each other. That is, -6 is the additive inverse of 6 and 4 is the additive inverse of -4.

> In order to eliminate (get rid of) part of a sum in an equation, we can add the additive inverse of that part to each side of the equation. The new equation which we obtain is a simpler equivalent equation.

Example: To solve the equation $x + 6 = 7$ we must eliminate the 6 from the left side of the equation. We do this by adding the additive inverse of 6, which is -6 to each side of the equation. The equation which we obtain is a simpler equivalent equation.

Exercise: State the number we would have to add to each side of these equations in order to solve it. Part (a) is done as an example.

(a) $x + {}^-3 = 10$	3	(d) $x + {}^-6 = 70$	______
(b) $y + 71 = 60$	______	(e) $x - 5 = 6$	______
(c) $a + 7 = 4$	______	(f) $16 = a + {}^-17$	______

Answers: (a) 3 (b) -71 (c) -7 (d) 6 (e) 5 (f) 17

Check Point:

1. If we are going to solve the equation $x + 9 = 11$, which of the following numbers should be added to each side?
(a) 9 (b) 7 (c) -9 (d) 0 ______________

2. If we are going to solve the equation $x + {}^-6 = 5$, which of the following numbers should be added to each side?
(a) 6 (b) -6 (c) 9 (d) 5 ______________

3. What number should be added to each side of the equation [illegible] $= 10$ in order to solve it? ______________

Please check with your Teacher before you proceed.

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SECTION VI SOLVING EQUATION OF FORM $x + b = c$

When we talk about equations of the form $x + b = c$, it is understood that x is a variable, and a and b represent specific numbers. Some examples of this type of equation are $x + 6 = 10$, $x + {}^{-}3 = {}^{-}7$, and $x + 10 = {}^{-}17$.

The basic idea involved in solving any equation is to write a chain of equivalent equations until we find an equation whose solution is obvious.

In order to solve equations such as $x + 6 = 10$, $x + {}^{-}3 = {}^{-}7$, and $x + 10 = {}^{-}17$, we must eliminate the number which is added to the x. We get rid of this number by adding its additive inverse to each side of the equation. The equation which we obtain is a simpler equivalent equation.

Example: Solve the equation $x + {}^{-}7 = 5$

$x + {}^{-}7 = 5$

$(x + {}^{-}7) + 7 = 5 + 7$ (Using the Addition Principle we add 7 to each side of the equation)

$x + ({}^{-}7 + 7) = 12$ (Using the Associative Property we regroup the left side)

$x + 0 = 12$ (All of these equations are equivalent, that is, they have the same solution)

$x = 12$

Check

$x + {}^{-}7 = 5$

$12 + {}^{-}7 = 5$ (We replace the x by 12)

$5 = 5$ (Since the left side of the equation is equal to the right side, 12 is the correct solution)

Question: In the example above, why did we add 7 to each side?

Answer : Because 7 is the additive inverse of -7.

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Exercises: Solve each of the following equations.
Do your workings in the space provided below each problem.

Exercise 1: $x+3=10$

Answer: $x = 7$

Exercise 2: $x - 4 = 7$

Answer: $x = 11$

Exercise 3: $x + 3 = {}^{-}8$

Answer: $x = -11$

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Exercise 4: $x + {}^{-}8 = -15$

Answer: $x = -7$

Exercise 5: $10 + x = -7$

Answer: $x = -17$

Exercise 6: $6 = x - 5$

Answer: $x = 11$

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Exercise 7: $a + {}^{-}20 = 7$

Answer: $a = 27$

Exercise 8: $y - 15 = -3$

Answer: $y = 12$

Exercise 9: $x - {}^{-}3 = 7$

Answer: $x = 4$

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Exercise 10: $-18 = -5 + x$

Answer: $x = -13$

Check Point: Solve each of the following equations showing each step in the process.

1. $x + 7 = 10$

2. $x + 6 = -15$

3. $-10 + x = -6$

Please check with your teacher before you proceed.

SECTION VII MULTIPLICATION PRINCIPLE

Suppose we take the equation $2x=6$. The solution of this equation is 3. If we multiplied each side of this equation by 4, we would obtain a new equation $8x=24$. The solution of this new equation is also 3, since $8 \cdot 3=24$. This means that the new equation which we obtained by multiplying each side by 4 is equivalent to the original equation.

Exercises:

Suppose you take the equation $3x=12$. It's solution is 4.

(a) Write a new equation by multiplying each side of the given equation by 2. ______

(b) Is the solution of the new equation also 4? ______

(c) Is the equation $3x=12$ equivalent to the equation $6x=24$?

______

Answers: (a) $6x=24$
(b) Yes
(c) Yes

Exercise: Suppose you take the equation $x+1=4$. It's solution is 3.

(a) Write a new equation by multiplying each side of the given equation by 5. ______

(b) Is the solution of the new equation also 3? ______

(c) Is the equation $x+1=4$ equivalent to the equation $5x+5=20$?

______

Answers: (a) $5x+5=20$
(b) Yes
(c) Yes

Exercise: Can you state a general principle that seems to apply when you multiply each side of an equation by the same number. ______

______

Answer: You obtain an equation which is equivalent to the original equation.

THE MULTIPLICATION PRINCIPLE

> The Multiplication Principle states that you obtain an equivalent equation when you multiply both sides of an equation by the same number. That is, if $a=b$, then $a \cdot c = b \cdot c$

The Multiplication Principle is extremely important in the solution of equations since it gives us another method of obtaining equivalent equations. (Remember equivalent equations have the same solution.)

Check Point: Answer each of the following questions.

(1) In your own words state the Multiplication Principle.

____

____.

(2) If you multiply each side of an equation by the same number you obtain an ______________________ equation.

(3) Is the equation $-3x=24$ equivalent to the equation $x=-8$?

______________________.

SECTION VIII: OBTAINING EQUIVALENT EQUATIONS

We have learned that we can use the Multiplication Principle to obtain an equivalent equation. (Equivalent equations are equations which have the same solution.)

Exercise: Suppose we take the equation $3x=12$

(a) Obtain an equivalent equation by multiplying each side of the equation $3x=12$ by 2. ______________________

(b) Obtain an equivalent equation by multiplying each side of the equation $3x=12$ by 1/3. ______________________

(c) Obtain an equivalent equation by multiplying each side of the equation $3x=12$ by 1/4. ______________________

(d) Which of the equations obtained in parts a, b, or c is the

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easiest to solve. ______________________.

Answer: (a) $6x=24$
(b) $x=4$
(c) $3/4x=3$
(d) The one obtained in part (b).

Exercise: Suppose we take the equation $-2x=10$.

(a) Obtain a new equation by multiplying the equation $-2x=10$ by 3. ______________________.

(b) Obtain a new equation by multiplying the equation $-2x=10$ by 1/2. ______________________.

(c) Obtain a new equation by multiplying the equation $-2x=10$ by $-1/2$. ______________________.

(d) Which of the equations obtained in parts a, b, or c is the easiest to solve. ______________________.

Answers: (a) $-6x=30$
(b) $-x=5$
(c) $x=-5$
(d) The one obtained in part (c).

So we have seen that we obtain the simplest equivalent equation for the equation $3x=12$ when we multiplied each side of the equation by 1/3 and we obtained the simplest equivalent equation for the equation $-2x=10$ by multiplying each side of the equation by $-1/2$. Can you see the relationship between the number you multiply each side of the equation by and the number before the x term. If so, state it. ______________________

______________________.

Obviously the number we multiplied by and the number before the x term are multiplicative inverses(Reciprocals) of each other. That is 1/3 is the multiplicative inverse of 3 and $-1/2$ is the multiplicative inverse of -2.

> In order to eliminate <u>part of a product</u> in an equation we multiply each side of the equation by the Multiplicative Inverse of that part. The new equation which is obtained is a simpler equivalent equation.

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Example:

To solve the equation 3x=21 we must eliminate the 3 from the left hand side of the equation. We do this by multiplying each side of the equation by the multiplicative inverse (Reciprocal) of 3, which is 1/3. The equation which we obtain is a simpler equivalent equation.

Exercise: State the number we would have to multiply each side of each of these equations by in order to solve it. Part (a) is done as an example.

(a) 4x=16 ___1/4___
(b) -2x=10 __________
(c) 3x=9 __________
(d) -6x=-17 __________
(e) 27x=-107 __________

Answers: (a) 1/4
(b) -1/2
(c) 1/3
(d) -1/6
(e) 1/27

Check Point: Answer each of the following questions:

1. If you are going to solve the equation 8x=102 which of the following numbers should we multiply each side of the equation by.
   (a) 1/4 (b) 1/8 (c) -1/8 (d) 8 __________

2. If you are going to solve the equation -6x=72 which of the following numbers should we multiply each side of the equation by.
   (a) -1/6 (b) 1/6 (c) 1/3 (d) -1/3 __________

3. If you are going to solve the equation 2/3x=16 which of the following numbers should we multiply each side of the equation by.

   (a) 2/3 (b) 3/2 (c) -3/2 (d) -2/3 __________

Please check with your teacher before you proceed.

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SECTION 1X SOLVING EQUATIONS OF TYPE $a \cdot x = c$

The basic idea involved in solving any equation is to write a chain of equivalent equations until we find one whose solution is obvious.

If we are going to solve equations such as $3x = 12$, $-5x = 20$, etc., we must eliminate (get rid of) the number before the x term. We eliminate this number by multiplying each side of the equation by the reciprocal of this number. The equation which we obtain is a simplier equivalent equation.

Exercise: Solve: $6x = 42$

$1/6 \cdot (6x) = 1/6 \cdot 42$ (Using the Multiplication Principle, we multiply each side of the equation by 1/6)

$(1/6 \cdot 6) \cdot x = \frac{42}{6}$ (Using the Associative Property, we regroup the left side)

$1 \cdot x = 7$
$x = 7$ (All of the equations obtained in the solution of the problem are equivalent, that is, they have the same solution)

Check: $6x = 42$
$6 \cdot 7 = 42$ (We replace the x by 6)

$42 = 42$ (Since the left hand side of the equation is equal to the right hand side, 6 is the correct solution)

Exercise: In the example above, why did we multiply by 1/6 ?

Answer: Since 1/6 is the reciprocal of 6.

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Exercises: Solve each of the following equations.
(Do your workings in the space provided below each problem)

Exercise 1: $5x = 20$

Answer: $x = 4$

Exercise 2: $-3a = 18$

Answer: $a = -6$

Exercise 3: $7x = -49$

Answer: $x = -7$

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Exercise 4: $-4y = -24$

Answer: $y = 6$

Exercise 5: $8x = 20$

Answer: $x = 2.5$

Exercise 6: $-7y = 0$

Answer: $y = 0$

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Exercise 7: $-25 = 5a$

Answer: $a = -5$

Exercise 8: $-6x = -120$

Answer: $x = 20$

Exercise 9: $100 = 25y$

Answer: $y = 4$

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Exercise 10: $-16x = 56$

Answer: $x = -3.5$

Check Point : Solve each of the following equations showing each step in the process.

(1) $9x = 30$

(2) $-6y = -30$

(3) $10a = 52$

Please check with your teacher before you proceed.

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SECTION X: SOLVING MIXED EQUATIONS

So far we have learned to solve two types of equations: Type 1 is of the form $x + b = c$, for example $x + 3 = 10$, $x + -6 = -17$ and $6 -x = 14$. Type 2 is of the form $ax = c$, for example $3x = 9$, $-4x = -20$, and $-7x = 35$. The basic difference in solving the two types of equations is that in Type 1 we have to get rid of part of a sum whereas in Type 2 we want to get rid of part of a product.

SUMMARY:

When we want to get rid of part of a sum we use the addition principle and add the additive inverse of that part to each side of the equation.

When we want to get rid of part of a product we use the multiplication principle and multiply each side by the reciprocal (multiplication inverse) of that part.

Examples:

1. To solve $3x = 21$ we need to eliminate the 3 (part of a product) so we use the multiplication principle and multiply each side by the reciprocal of 3 which is 1/3.

2. To solve $x + -4 = 6$ we need to eliminate the -4 (part of a sum) so we use the addition principle and add the additive inverse of -4 (which is 4) to each side .

Exercises: Solve each of the following equations
(Do your workings in the space provided below each probl

Exercise 1: $x + 6 = 21$

Answer: $x = 15$

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Exercise 2: $4y = 28$

Answer: $y = 7$

Exercise 3: $-3y = -30$

Answer: $y = 10$

Exercise 4: $x-7 = -20$

Answer: $x = -13$

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Exercise 5: $-8x = 20$

Answer: $x = -2.5$

Exercise 6: $-10 + a = -4$

Answer: $a = 6$

Exercise 7: $-36 = -6y$

Answer: $y = 6$

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Exercise 8: $-150x = 0$

Answer: $x = 0$

Exercise 9: $20 = -x + 6$

Answer: $x = -14$

Exercise 10: $x + {}^{-}17 = 70$

Answer: $x = 87$

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<u>Check Point:</u> Solve each of the following equations showing each step in the process.

(1) $-4x = -36$

(2) $x + {}^{-}16 = 10$

(3) $-8 + x = 0$

Check with your teacher before you proceed.

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SECTION XI SOLVING EQUATIONS OF TYPE $ax + b = c$

In solving an equation such as $3x + 6 = 15$ we have two numbers to eliminate from the left-side of the equation. We have the 6 which is part of a sum and the 3 which is part of a product.

First we eliminate the 6 by adding -6 to each side of the equation.

$$3x + (6 + {}^{-}6) = 15 + {}^{-}6$$
$$3x + 0 = 9$$
$$3x = 9 \qquad \text{(Note this equation is equivalent to } 3x + 6 = 15\text{)}$$

Next we eliminate the 3 in this new equation by multiplying each side by 1/3.

$$3x = 9$$
$$3x \cdot 1/3 = 9 \cdot 1/3$$
$$(3 \cdot 1/3) \cdot x = 9 \cdot 1/3$$
$$1 \cdot x = 3$$
$$x = 3$$

> Note: We used both the addition and the multiplication principles to solve the equation above. We first used the addition principle to eliminate the 6 and then we used the multiplication principle to eliminate the 3. Remember that throughout the whole problem we were obtaining equations which are equivalent to the equation $3x + 6 = 15$.

Example: Solve: $-2x + -6 = 30$

$$-2x + -6 = 30$$
$$-2x + (-6 + 6) = 30 + 6 \quad \leftarrow \text{(We are using the addition principle to eliminate the -6.)}$$
$$-2x + 0 = 36$$
$$-2x = 36$$
$$-2x \cdot -1/2 = 36 \cdot -1/2 \quad \leftarrow \text{(We are using the multiplication principle to eliminate the -2)}$$
$$(-2 \cdot -1/2) \cdot x = 36 \cdot -1/2$$
$$1 \cdot x = -18$$
$$x = -18$$

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We can check and see if -18 is the correct solution of the equation $-2x + {}^-6 = 30$ by replacing the variable x by -18, simplifying the left hand side of the equation and observing if the left hand side is equal to the right hand side.

Check:

$$-2x + {}^-6 = 30$$
$$-2 \cdot {}^-18 + {}^-6 = 30$$
$$36 + {}^-6 = 30$$
$$30 = 30$$

Exercises: Solve each of the following equations showing each step in the process.

1. $2x + 8 = 20$

Answer: $x = 6$

2. $-3x - 20 = -2$

Answer: $x = {}^-6$

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Exercise 3: $6x - 5 = 31$

Answer: $x = 6$

Exercise 4: $8x - 72 = 0$

Answer: $x = 9$

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Exercise 5: $12y + 12 = 156$

Answer: $y = 12$

Exercise 6: $6x + 5 = 17$

Answer: $x = 2$

Exercise 7: $4y + 7 = -5$

Answer: $y = -3$

Exercise 8: $-5 + 3y = 7$

Answer: $y = 4$

Exercise 9: $9 - 2x = 15$

Answer: $x = -3$

Exercise 10: $-20 = 3x + 40$

Answer: $x = -20$

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Exercise 11: $^{-}6p + 13 = {}^{-}59$

Answer: $p = 12$

Exercise 12: $22 = {}^{-}14 - 4x$

Answer: $x = {}^{-}9$

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Exercise 13: $6a + 14 = {}^{-}19$

Answer: $a = -5.5$

Exercise 14: $13x - 23 = 120$

Answer: $x = 11$

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<u>Check Point</u>: Solve each of the following equations showing each step in the process.

(1) $6x + 15 = 57$

(2) $-4y - 13 = 77$

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(3) $-16 + 3a = -34$

Please check with your teacher.

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Booklet For The Textbook Approach

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TO THE STUDENT

This booklet is very different from an ordinary textbook; it cannot be read in the way other books are read. You must read the material carefully, study the examples, complete all the exercises, check your own answers and proceed through the booklet at your own rate. The success of this booklet depends upon how carefully you read the material, answer the exercises, etc. When you finish this booklet you should be able to solve equations such as $-3x + 16 = -17$ and $22 = -14 - 4x$. A test will be given at the end of this booklet to see how much you have learned.

Take a few minutes and look through this booklet. Note the red lines. There is a red line below each exercise or set of exercises. The answers to the exercises are below the red lines. The answers are provided so that you may check your own work. If you get some of the answers wrong, please read the section again so that you may see why they are wrong. Note also that a few areas have been blocked in green. The purpose of this is to draw your attention to the importance of these ideas.

The following procedures should be followed in using this booklet:

(1) Work on your own; there is to be no talking to other students.

(2) Use the piece of cardboard provided to cover everything below the red line. When you have finished the work above the red line, lower the cardboard to the next red line and check your answers to the exercises.

(3) Read the material carefully and study all the examples thoroughly.

(4) Place your answers to the exercises in the blank spaces provided after each exercise.

(5) If you get some of the exercises wrong, go back and read the section again so that you can see where you went wrong.

(6) These booklets will be collected at the end of each period.

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SECTION 1 SOLUTIONS OF EQUATIONS

A number sencence with = is called an equation. some equations are tuue and some are false.

Exercise: Which are true and which are false

(a) 3 + 2 = 5 ________

(b) 5 − 7 = 4 ________

(c) 4 + 5 = 7 ________

Answers: (a) True

(b) False

(c) False

Exercise: Some equations with variables are neither true nor false. Find numbers that make these true.

(a) 3 + 4 = x ________

(b) 4 + y = 5 ________

(c) t + 2 = 4 ________

(d) 7 + r = 9 ________

(e) n + 6 = 11 ________

(f) 3 + 9 = b ________

Answers:

(a) 7 (d) 2

(b) 1 (e) 5

(c) 2 (f) 12

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A number that makes an equation true is called a <u>solution</u> of the equation.

Exercise: Find solutions of these equations

(a) $10 - 2 = x$ __________

(b) $8 - 10 = y$ __________

(c) $3 + x = 5$ __________

Answers:

(a) 8

(b) −2

(c) 2

To <u>solve</u> an equation, we find all it's solutions.

Exercise: Solve:

(a) $5 - 10 = x$ __________ (d) $x = x \cdot x$ __________

(b) $4 + y = 8$ __________ (e) $5 \cdot y = 0$ __________

(c) $x + x = 6$ __________ (f) $-4 \cdot y = 0$ __________

Answers:

(a) −5 (d) 0 and 1

(b) 4 (e) 0

(c) 3 (f) 0

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Exercise: Solve:- Use your imagination

(1) $5 + 4 = x$ ______	(8) $3 + t = {}^-1$ ______
(2) $4 - 6 = y$ ______	(9) $5 + w = 3$ ______
(3) $3 - {}^-2 = t$ ______	(10) $x + x = 10$ ______
(4) $-5 - {}^-3 = y$ ______	(11) $y + y = {}^-8$ ______
(5) $3 + x = 6$ ______	(12) $t + t = 0$ ______
(6) $5 + y = 7$ ______	(13) $y \cdot y = y$ ______
(7) $-2 + y = 5$ ______	(14) $x + x = x$ ______

Answers:

(1) 9	(6) 2	(11) -4
(2) -2	(7) 7	(12) 0
(3) -6	(8) -4	(13) 0 and 1
(4) 15	(9) -2	(14) 0
(5) 3	(10) 5	

Remember that addition and subtraction are opposite operations. For the equation $x + 3 = 5$, there are two related sentences, $5-x=3$ and $5-3 = x$.

Exercise: Write two related sentences for $x + 4 = 7$

______ ______

Answer: $7 - 4 = x$, $7 - x = 4$

Exercise: Which is the easiest to solve

(a) $x + 4 = 7$

(b) $7 - 4 = x$ __________

(c) $7 - x = 4$

Answer: Part b, $7 - 4 = x$

Exercise: (a) For $-5 + x = 7$, write two related sentences

__________ __________

(b) Choose the one with x alone on one side. Solve it. __________ __________

(c) Is the solution also a solution of $-5 + x = 7$?

____

Answers: (a) $7 - {}^-5 = x$, $7 - x = {}^-5$

(b) $7 - {}^-5 = x$; solution is 12

(c) Yes

Exercise: (a) For $y + 2 = -5$, write a related sentence with y alone on one side . __________

(b) Solve your related sentence. __________

(c) Is the solution also a solution of $y + 2 = -5$?

____

Answer: (a) $y = -5 - 2$

(b) -7

(c) Yes

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We can solve an equation like $x + 3 = {}^{-}7$ this way:

(a) Write a related sentence with x alone on one side.

(b) Solve the related sentence.

(c) Check the solution in the original equation.

Exercise: <u>Solve</u>

(1) $x + {}^{-}2 = 6$ ________

(2) $x + {}^{-}3 = 5$ ________

(3) $y + {}^{-}3 = 0$ ________

(4) $y + {}^{-}4 = 0$ ________

(5) $x + {}^{-}2 = {}^{-}3$ ________

(6) $x + {}^{-}3 = {}^{-}2$ ________

(7) $t + 2 = {}^{-}5$ ________

(8) $t + 3 = {}^{-}4$ ________

(9) $w + {}^{-}4 = {}^{-}6$ ________

(10) $w + {}^{-}3 = {}^{-}5$ ________

(11) ${}^{-}3 + x = {}^{-}1$ ________

(12) ${}^{-}4 + x = {}^{-}2$ ________

(13) $y + {}^{-}5 = {}^{-}5$ ________

(14) $y + {}^{-}6 = {}^{-}6$ ________

(15) $2 = x + 1$ ________

(16) $4 = x + 2$ ________

(17) $5 = x + {}^{-}1$ ________

(18) $7 = x + {}^{-}2$ ________

(19) $7 = 3 + y$ ________

(20) $5 = 3 + y$ ________

(21) ${}^{-}2 = 1 + y$ ________

(22) ${}^{-}3 = 2 + y$ ________

(23) ${}^{-}5 = t + 3$ ________

(24) ${}^{-}7 = t + 4$ ________

(25) ${}^{-}2 = x + {}^{-}3$ ________

(26) ${}^{-}2 = w + {}^{-}2$ ________

(27) ${}^{-}10 = x + {}^{-}5$ ________

Answers:

(1) 8	(8) -7	(15) 1	(22) -5
(2) 8	(9) -2	(16) 2	(23) -8
(3) 3	(10) -2	(17) 6	(24) -11
(4) 4	(11) 2	(18) 9	(25) 1
(5) -1	(12) 2	(19) 4	(26) 0
(6) 1	(13) 0	(20) 2	(27) -5
(7) -7	(14) 0	(21) -3	

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SECTION 111 SOLVING EQUATIONS OF TYPE $a \cdot x = b$

Remember that multiplication and division are opposite operations. For the equation $3 \cdot x = 6$, there are two related sentences, $\frac{6}{x} = 3$ and $\frac{6}{3} = x$

Exercise: Write two related sentences for $4 \cdot x = 12$ ________ ________

Answer: $\frac{12}{4} = x$; $\frac{12}{x} = 4$

Exercise: Which is the easiest to solve?
(a) $4 \cdot x = 12$ (b) $\frac{12}{x} = 4$ (c) $\frac{12}{4} = x$ ________

Answer: Part c , $\frac{12}{4} = x$

Exercise: (a) For $5 \cdot x = 20$, write two related sentences

________ ________ ________

(b) Choose the related sentence with x alone on one side. Solve it. ________ ________ ________

(c) Is this solution also a solution of ~~$3 \cdot y = 21$~~? $5 \cdot x = 20$

____

Answers: (a) $20 \div 5 = x$, $20 \div x = 5$
(b) $20 \div 5 = x$, 4
(c) yes

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Exercise: (a) For $3 \cdot y = 21$, write a related sentence with y alone on one side. ________

(b) Solve your related sentence. ________

(c) Is the solution also a solution of $3 \cdot y = 21$?

____

Answers: (a) $y = 21 \div 3$

(b) 7

(c) Yes

> We can solve an equation like $4 \cdot x = 16$ this way;
> (a) Write a related sentence with x alone on one side.
> (b) Solve the related sentence.
> (c) Check the solution in the original equation.

Exercise: Solve

(1) $3 \cdot x = 9$ ________
(2) $4 \cdot x = 16$ ________
(3) $7 \cdot y = 14$ ________
(4) $8 \cdot y = 24$ ________
(5) $2 \cdot x = {}^{-}8$ ________
(6) $3 \cdot x = {}^{-}12$ ________
(7) ${}^{-}2 \cdot x = {}^{-}8$ ________
(8) ${}^{-}4 \cdot x = {}^{-}12$ ________
(9) ${}^{-}5 \cdot y = 15$ ________
(10) ${}^{-}3 \cdot y = 12$ ________
(11) $14 \cdot t = 14$ ________
(12) $12 \cdot t = {}^{-}12$ ________
(13) $6 \cdot y = {}^{-}6$ ________
(14) ${}^{-}5 \cdot y = 5$ ________
(15) $32 = 4 \cdot x$ ________
(16) $35 = 5 \cdot x$ ________
(17) $25 = 5 \cdot w$ ________
(18) $30 = 6 \cdot w$ ________
(19) ${}^{-}12 = {}^{-}3 \cdot x$ ________
(20) ${}^{-}15 = {}^{-}3 \cdot x$ ________
(21) $14 = 7 \cdot u$ ________
(22) $20 = 5 \cdot u$ ________
(23) ${}^{-}30 = 6 \cdot y$ ________
(24) ${}^{-}35 = 5 \cdot y$ ________
(25) $10 \cdot y = 70$ ________
(26) ${}^{-}1 \cdot y = 5$ ________
(27) ${}^{-}1 \cdot x = {}^{-}7$ ________

Answers:

(1) 3	(7) 4	(13) -1	(19) 4	(25) 7
(2) 4	(8) 3	(14) -1	(20) 5	(26) -5
(3) 2	(9) -3	(15) 8	(21) 2	(27) 7
(4) 3	(10) -4	(16) 7	(22) 4	
(5) -4	(11) 1	(17) 5	(23) -5	
(6) -4	(12) -1	(18) 5	(24) -7	

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SECTION 1V THE ADDITION PRINCIPLE

Exercise:

(a) Is this sentence true? $3 + 1 = 4$ ________

(b) Add 2 to $3 + 1$ ________. Add 2 to 4 ________
Is $(3+1) + 2 = 4+2$ true? ________

Answers:

(a) Yes

(b) 6, 6, Yes

Exercise:

(a) Using the number sentnece $3 + 1 = 4$, add 6 to $3 + 1$ and add 6 to 4 ________

(b) Is the sentance $(3+1) + 6 = 4+6$ true? ________

Answers:

(a) $10 = 10$

(b) Yes

Exercise:

(a) Using the number sentence $3 + 1 = 4$, add -6 to $3 + 1$ and add -6 to 4 ________

(b) Is $(3+1) + {}^{-}6 = 4 + {}^{-}6$ true? ________

Answers:

(a) $-2 = -2$

(b) Yes

Exercise:

(a) Is this sentnece true? $9 = 10 + {}^{-}1$ ________

(b) Write the sentence you obtain when you add 2 is each side of $9 = 10 + {}^{-}1$ ________

(c) Simplify: $9 + 2 = (10+{}^{-}1)+ 2$ ________

(d) Is $9 + 2 = (10 + {}^{-}1) +2$ a true sentnece? ________

Answers:

(a) Yes

(b) $9 + 2 = (10 + {}^{-}1) + 2$

(c) $11 = 11$

(d) Yes

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Exercise: Do you see a pattern in the exercises in this section? Try to describe it.

Answer: If you add the same number to both sides of a true equation, you obtain a true equation.

> If an equation $a = b$ is true, we get another true equation when we add any number c
>
> $$a + c = b + c$$
>
> We call this the <u>addition principle</u>

In exercises 1 - 20 add the given number to get a new equation. Then simplify both sides of the new equation. Number 1 has been done as an example.

Exercise 1: $4 + 2 = 6$
Add 3

$(4+2)+3 = 6+3$
$6 + 3 = 9$
$9 = 9$

Exercise 2: $5 = 7+{}^{-}2$
Add -1

Answer:
$5+ {}^{-}1 = (7 + {}^{-}2)+ {}^{-}1$
$5+ {}^{-}1 = 5 + {}^{-}1$
$4 = 4$

Exercise:3: $8= 9 -1$
Add 2

Answer:
$8 + 2 = (9 -1) + 2$
$8 + 2 = 8 + 2$
$10 = 10$

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Exercise 4: $5 \cdot 2 = 10$

Add $^{-}1$

Answer: $5 \cdot 2 + {}^{-}1 = 10 + {}^{-}1$

$10 + {}^{-}1 = 10 + {}^{-}1$

$9 = 9$

Exercise 5: $6 = {}^{-}3 \cdot {}^{-}2$

Add 3

Answer: $6 + 3 = ({}^{-}3 \cdot {}^{-}2) + 3$

$6 + 3 = 6 + 3$

$9 = 9$

Exercise 6: $13 = 6 + 7$

Add $^{-}12$

Answer: $13 + {}^{-}12 = (6+7) + {}^{-}12$

$13 + {}^{-}12 = 13 + {}^{-}12$

$1 = 1$

Exercise 7: $x + 4 = 9$

Add $^{-}4$

Answer: $(x+4) + {}^{-}4 = 9 + {}^{-}4$

$x + (4 + {}^{-}4) = 5$

$x + 0 = 5$

$x = 5$

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Exercise 8: $y + 3 = 10$

Add -3

Answer:

$$(y+3) + {}^-3 = 10 + {}^-3$$
$$y + (3+ {}^-3) = 10 \div {}^-3$$
$$y + 0 = 7$$
$$y = 7$$

Exercise 9: $t + 4 = 8$

Add $^-4$

Answer:

$$(t+4) + {}^-4 = 8 + {}^-4$$
$$t + (4 + {}^-4) = 8 + {}^-4$$
$$t + 0 = 4$$
$$t = 4$$

Exercise 10: $x + {}^-5 = 7$

Add 5

Answer:

$$(x+ {}^-5) +5 = 7 + 5$$
$$x + (-5 +5) = 7 + 5$$
$$x + 0 = 12$$
$$x = 12$$

Exercise 11: $y + {}^-7 = 2$

Add 7

Answer:

$$(y\neq 7) + 7 = 2 + 7$$
$$y + (-7 +7) = 2 + 7$$
$$y + o = 9$$
$$y = 9$$

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Exercise 12: $w + 12 = -3$

Add -12

Answer: $(w+12) + {}^{-}12 = {}^{-}3 + {}^{-}12$

$w + (12 + {}^{-}12) = -3 + {}^{-}12$

$w + 0 = -15$

$w = -15$

Exercise 13: $3 + x = 7$

Add -3

Answer: $-3 + (3+x) = 7 + {}^{-}3$

$(-3 + 3) + x = 7 + {}^{-}3$

$0 + x = 4$

$x = 4$

Exercise14: $-5 + y = 2$

Add 5

Answer: $5 + (-5 + y) = 2+5$

$(5 + {}^{-}5) + y = 2 + 5$

$0 + y = 7$

$y = 7$

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Exercise 15: $10 + u = -3$

Add -10

Answer:

$-10 + (10 + u) = -10 + {}^-3$

$(-10 + 10) + u = -10 + {}^-3$

$0 + u = -13$

$u = -13$

Exercise 16: $8 = x + 5$

Add -5

Answer:

$-5 + 8 = (x+5) + {}^-5$

$-5 + 8 = x + (5 + {}^-5)$

$-5 + 8 = x + 0$

$3 = x$

Exercise 17: $13 = y + {}^-4$

Add 4

Answer:

$13 + 4 = (y + {}^-4) + 4$

$13 + 4 = y + ({}^-4 + 4)$

$13 + 4 = y + 0$

$17 = y$

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Exercise 18: $-12 = t + {}^{-}4$

Add 4

Answer:

$-12 + 4 = (t + {}^{-}4) + 4$

$-12 + 4 = t + (-4 + 4)$

$-12 + 4 = t + 0$

$-8 = t$

Exercise 19: $7 = 5 + x$

Add $^{-}5$

Answer:

$-5 + 7 = -5 + (5+x)$

$-5 + 7 = (-5 + 5) + x$

$-5 + 7 = 0 + x$

$2 = x$

Exercise 20: $-7 = -3 + y$

Add 3

Answer:

$3 + {}^{-}7 = 3 + (-3 + y)$

$3 + {}^{-}7 = (3 + {}^{-}3) + y$

$-4 = 0 + y$

$-4 = y$

SECTION V USING THE ADDITION PRINCIPLE

Exercise: Think about $x + 2 = 11$

(a) What is the inverse of 2 ? ____________

(b) Use the addition principle. Add -2 to each side of the equation. ____________

____

(c) What is the solution of your new equation?

____

(d) Is it also a solution of $x + 2 = 11$?

____

Answer:

(a) $^{-}2$

(b) $(x + 2) + {}^{-}2 = 11 + {}^{-}2$

(c) 9

(d) Yes

Exercise: Think about $^{-}12 = y + {}^{-}3$

(a) What is the inverse of $^{-}3$? ____________

(b) Use the addition principle. Add 3 to each side of the equation. ____________

(c) What is the solution of your new equation?

____

(d) Is it also a solution of the original equation?

____

Answer:

(a) 3

(b) $^{-}12 + 3 = (y + {}^{-}3) + 3$

(c) $^{-}9$

(d) Yes

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Exercise: Think about $4 + x = {}^{-}9$

(a) What should be added to get x alone? ____________

(b) Add that number. ____________

(c) What is the solution of your new equation?

____

(d) Is it also a solution of $4 + x = {}^{-}9$? ____________

Answers:

(a) -4

(b) $(4 + x) + {}^{-}4 = -9 + {}^{-}4$

(c) -13

(d) Yes

> We can solve an equation like $x + 7 = 12$ this way:
>
> (a) Use the addition principle to get the variable alone on one side of the equation.
>
> (b) Check the solution in the original equation.

Example: $x + {}^{-}5 = 2$

$(x + {}^{-}5) + 5 = 2 + 5$

$x = 7$

Check $x + {}^{-}5 = 2$

$7 + {}^{-}5 = 2$

$2 = 2$

Thus 7 is the solution of $x + {}^{-}5 = 2$

Exercise: Solve: Use the addition principle. Do your workings in the space provided below each problem.

(1) $x + 3 = 12$

Answer: 9

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Exercise 2: $y + {}^-5 = 8$

Answer: 13

Exercise 3: $t + {}^-2 = -9$

Answer: -7

Exercise 4: $x + {}^-8 = {}^-3$

Answer: 5

Exercise 5: $y + {}^-7 = 0$

Answer: 7

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Exercise 6: $u + {}^{-}17 = {}^{-}3$

Answer: 14

Exercise 7: $3 + x = 13$

Answer: I0

Exercise 8: $-2 + y = 12$

Answer: 14

Exercise 9: $6 + w = -5$

Answer: -11

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Exercise 10: $-5 + x = -31$

y

Answer: -26

Exercise: 11: $-23 + y = 4$

Answer: 27

Exercise 12: $47 + u = -12$

Answer: -59

Exercise 13: $14 = x + 2$

Answer: 12

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Exercise 14: $12 = y + {}^{-}7$

Answer: 19

Exercise 15: $17 = u + {}^{-}17$

Answer: 34

Exercise 16: $-19 = x + {}^{-}3$

Answer: -16

Exercise 17: $-2 = y + {}^{-}27$

Answer: 25

COLOURED

Exercise 18: $4 = u + {}^{-}25$

Answer: 29

Exercise 19: $11 = {}^{-}2 + x$

Answer: 13

Exercise 20: $-6 = {}^{-}23 + y$

Answer: 17

Exercise 21: $-1 = w + {}^{-}24$

Answer: 23

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Exercise 22: $33 = w + {}^{-}11$

Answer: 44

Exercise 23: $-9 = r + {}^{-}18$

Answer: 9

Exercise 24: $15 = {}^{-}25 + w$

Answer: 40

Exercise 25: $8 + x = 18$

Answer: 10

Exercise 26: $-15 = 25 + y$

Answer: -40

SECTION VI THE MULTIPLICATION PRINCIPLE

Exercise: (a) Is this sentence true? $3 \cdot 5 = 15$ ______
(b) Multiply $3 \cdot 5$ by 2 ______ Multiply 15 by 2 ______
(c) Is $(3 \cdot 5) \cdot 2 = 15 \cdot 2$ true? ______

Answers: (a) Yes
(b) 30, 30
(c) Yes

Exercise: (a) Using the number sentence $3 \cdot 5 = 15$ multiply each side by 4 ______
(b) Is the sentence $4 \cdot (3 \cdot 5) = 4 \cdot 15$ true ? ______

Answers: (a) $60 = 60$
(b) Yes

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Exercise: (a) Using the number sentence $3 \cdot 5 = 15$ multiply each side by o ________

(b) Is the sentence $0 \cdot (3 \cdot 5) = o \cdot 15$ true? ________

Answers: (a) $0 = 0$

(b) Yes

Exercise: (a) Is the sentence true? $2 \cdot 3 = 6$ ________

(b) Multiply each side of this equation by 1/2 ________

(c) Is the sentence $\frac{1}{2} \cdot (2 \cdot 3) = 1/2 \cdot 6$ true? ________

Answers: (a) Yes

(b) $3 = 3$

(c) Yes

Exercise: (a) Multiply each side of the number sentence $2 \cdot 3 = 6$ by 1/3 ________

(b) Is $1/3 \cdot (2 \cdot 3) = 1/3 \cdot 6$ true? ________

Answers: (a) $2 = 2$

(b) Yes

Exercise: Do you see a pattern in the exercises in this section? Try to describe it. ________

Answer: If you multiply each side of a true equation by the same number you obtain a true equation.

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> If an equation a=b is true, we get another true equation when we multiply by any number c
>
> $$a \cdot c = b \cdot c$$
>
> We call this the multiplication principle.

In exercise 1-20 multiply by the given number to get a new equation. Then simplify both sides of the new equation. Number 1 has been done as an example.

Exercise 1: $4 \cdot 3 = 12$
Multiply by 4

$4 \cdot (4 \cdot 3) = 4 \cdot 12$
$4 \cdot 12 = 4 \cdot 12$
$48 = 48$

Exercise 2: $2 \cdot 5 = 10$
Multiply by 3

Answer: $3 \cdot (2 \cdot 5) = 3 \cdot 10$
$3 \cdot 10 = 3 \cdot 10$
$30 = 30$

Exercise 3: $8 = 2 \cdot 4$
Multiply by 2

Answer: $2 \cdot 8 = 2 \cdot (2 \cdot 4)$
$16 = 2 \cdot 8$
$16 = 16$

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Exercise 4: $-2 \cdot 6 = {}^-12$

Multiply by 3

Answer: $3 \cdot (-2 \cdot 6) = {}^-12 \cdot 3$

$3 \cdot {}^-12 = {}^-12 \cdot 3$

$-36 = -36$

Exercise 5: $-9 = -3 \cdot 3$

Multiply by 2

Answer: $2 \cdot (-9) = 2 \cdot (-3 \cdot 3)$

$-18 = 2 \cdot -9$

$-18 = -18$

Exercise 6: $-4 = 2 \cdot -2$

Multiply by 3

Answer: $3 \cdot (-4) = 3(2 \cdot {}^-2)$

$-12 = 3 \cdot {}^-4$

$-12 = -12$

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Exercise 7: $3 \cdot 4 = 12$
Multiply by 1/3

Answer: $1/3 \cdot (3 \cdot 4) = 1/3 \cdot 12$
$(1/3 \cdot 3) \cdot 4 = 1/3 \cdot 12$
$4 = 4$

Exercise 8: $3 \cdot 4 = 12$
Multiply by 1/4

Answer: $1/4 \cdot (3 \cdot 4) = 1/4 \cdot 12$
$1/4 \cdot 12 = 1/4 \cdot 12$
$3 = 3$

Exercise 9: $2 \cdot {}^{-}5 = -10$
Multiply by 1/2

Answer: $1/2\ (2 \cdot {}^{-}5) = 1/2 \cdot (-10)$
$1/2 \cdot -10 = 1/2 \cdot -10$
$-5 = -5$

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Exercise 10: $2 \cdot -5 = -10$

Multiply by $-1/5$

Answer: $-1/5 \cdot (2 \cdot -5) = -1/5\ (-10)$

$-1/5 \cdot (-10) = -1/5 \cdot (-10)$

$2 = 2$

Exercise 11: $10 = -2 \cdot -5$

Multiply by $-1/2$

Answer: $-1/2 \cdot (10) = -1/2\ (-2 \cdot -5)$

$-1/2 \cdot 10 = -1/2 \cdot 10$

$-5 = -5$

Exercise 12: $10 = -2 \cdot -5$

Multiply by $-1/5$

Answer: $-1/5 \cdot (10) = -1/5 \cdot (-2 \cdot -5)$

$-1/5 \cdot 10 = -1/5 \cdot 10$

$-2 = -2$

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Exercise 13: $3 \cdot x = 21$

Multiply by 1/3

Answer: $1/3 \cdot (3 \cdot x) = 1/3 \cdot (21)$

$(1/3 \cdot 3) \cdot x = 21/3$

$x = 7$

Exercise 14: $5 \cdot x = 50$

Multiply by 1/5

Answer: $1/5 \cdot (5 \cdot x) = 1/5 \cdot 50$

$(1/5 \cdot 5) \cdot x = 50/5$

$x = 10$

Exercise 15: $7 \cdot y = 42$

Multiply by 1/7

Answer: $1/7 \cdot (7y) = 1/7 \cdot 42$

$(1/7 \cdot 7) \cdot y = 42/7$

$y = 6$

Exercise 16: $-3 \cdot x = 33$

Multiply by $-\frac{1}{3}$

______

______

______

Answer:

$-1/3 \cdot (-3 \cdot x) = -1/3 \cdot 33$

$(-1/3 \cdot -3) \cdot x = -33/3$

$x = -11$

Exercise 17: $-5 \cdot y = -35$

Multiply by $-1/5$

______

______

______

Answer:

$-1/5 \cdot (-5 \cdot y) = -1/5 \cdot (-35)$

$(-1/5 \cdot -5) \cdot y = 35/5$

$y = 7$

Exercise 18: $-12 \cdot w = 48$

Multiply by $-1/12$

______

______

______

Answer:

$-1/12 \cdot (-12 \cdot w) = -1/12 \cdot 48$

$(-1/12 \cdot -12) \cdot w = -48/12$

$w = -4$

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Exercise 19: $75 = 1/5 \cdot x$
Multiply by 5

Answer: $5 \cdot 75 = 5 \cdot (1/5 \cdot x)$
$5 \cdot 75 = (5 \cdot 1/5) \cdot x$
$375 = x$

Exercise 20: $36 = -1/9 \cdot y$
Multiply by -9

Answer: $36 \cdot -9 = (-1/9 \cdot y) \cdot -9$
$-324 = (-1/9 \cdot -9)\ y$
$-324 = y$

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SECTION V11 USING THE MULTIPLICATION PRINCIPLE

Exercise: Think about $3 \cdot x = 15$

(a) What is the reciprocal of 3? ____________

(b) Use the multiplication principle. Multiply by 1/3 ____________

(c) What is the solution of your new equation?

____

(d) Is it also a solution of $3 \cdot x = 15$? ____________

Answers:

(a) 1/3

(b) $1/3 \cdot (3 \cdot x) = 15 \cdot 1/3$

(c) 5

(d) Yes

Exercise: Think about $1/5 \cdot y = -30$

(a) What is the reciprocal of 1/5 ? ____________

(b) Use the multiplication principle. Multiply by 5. ____________

(c) What is the solution of the new equation?

____

(d) Is it also a solution of the original equation?

____

Answers:

(a) 5

(b) $5 \cdot (1/5 \cdot y) = 5 \cdot -30$

(c) -150

(d) Yes

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Exercise: Let's try $-4 \cdot x = 20$

(a) What is the reciprocal of -4 ? ____________

(b) Use the multiplication principle. Multiply by $-1/4$. ____________

(c) What is the solution of your new equation?

____

(d) Is it also a solution of the original equation?

____

Answers:

(a) $-1/4$

(b) $-1/4 \cdot (-4 \cdot x) = -1/4 \cdot 20$

(c) -5

(d) Yes

We can solve an equation like $4 \cdot x = 16$ this way:
(a) Use the multiplication principle to get the variable alone on one side of the equation.
(b) Check the solution in the original equation.

Example:

Solve: $7x = 42$
$1/7 \cdot (7x) = 1/7 \cdot 42$
$x = 6$

Check: $7x = 42$
$7 \cdot 6 = 42$
$42 = 42$

Exercise 1: Solve: Use the multiplication principle
(Do your workings in the space provided below each problem)

$3 \cdot x = 12$

Answer: 4

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Exercise 2: $-5 \cdot y = 40$

Answer: -8

Exercise 3: $-2 \cdot t = -28$

Answer: 14

Exercise 4: $-8 \cdot x = 64$

Answer: -8

Exercise 5: $-6 \cdot y = 60$

Answer: -10

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Exercise 6: $9 \cdot w = -36$

Answer: -4

Exercise 7: $-3 \cdot x = -9$

Answer: 3

Exercise 8: $-15 \cdot y = -45$

Answer: 3

Exercise 9: $12 \cdot u = -48$

Answer: -4

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Exercise 10: $4 \cdot x = 3$

Answer: $3/4$

Exercise 11: $100 = -10 \cdot n$

Answer: -10

Exercise 12: $-64 = 8 \cdot y$

Answer: -8

Exercise 13: $15 = -3 \cdot x$

Answer: -5

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Exercise 14: $-15 = 3 \cdot y$

Answer: -5

Exercise 15: $-20 = -4 \cdot u$

Answer: 5

Exercise 16: $7 = 4 \cdot x$

Answer: $7/4$

Exercise 17: $-4 \cdot x = 16$

Answer: -4

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Exercise 18: $25 = -5 \cdot y$

Answer: -5

Exercise 19: $-36 = 6 \cdot t$

Answer: -6

Exercise 20: $-7 \cdot x = -49$

Answer: 7

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SECTION V111 USING THE PRINCIPLES TOGETHER

Now that we know how to use the addition and multiplication principle, let's see how to use them together.

Exercise:

Let's think about the equation $3 \cdot x + 5 = 14$.

(a) Using the addition principle, add -5 to each side of the equation. ____________________

(b) Now use the multiplication principle and multiply each side of the new equation by 1/3. __________________________

(c) What is the solution of your last equation? ______________

(d) Is this solution also a solution of the original equation? ______________.

Answers:

(a) $3 \cdot x = 9$

(b) $x = 3$

(c) 3

(d) Yes

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Exercise: Let's solve $-4 + 4 \cdot y = 8$

(a) Using the addition principle add 4 to each side of the equation. ____________________

(b) Using the multiplication principle multiply each side of the equation by 1/4. ____________________

(c) What is the solution of your last equation?

____

(d) Is this solution also a solution of the original equation? ____________________

Answers: (a) $4 \cdot y = 12$
(b) $y = 3$
(c) 3
(d) Yes

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We can solve an equation like $3 \cdot x + 2 = 11$ this way:

(a) First use the addition principle. Add -2.
(b) Then use the multiplication principle. Multiply by $1/3$
(c) Check the solution in the original equation.

Example: Solve:

$$6x + 7 = 19$$
$$6x + 7 + {}^-7 = 19 + {}^-7$$
$$6x = 12$$
$$1/6 \cdot 6x = 1/6 \cdot 12$$
$$x = 2$$

Check:

$$6x + 7 = 19$$
$$6 \cdot 2 + 7 = 19$$
$$12 + 7 = 19$$
$$19 = 19$$

Exercises:

Solve: Use the addition and the multiplication principles. Do your workings in the space provided below each problem.

(1) $6 \cdot x + 5 = 17$

Answer: 2

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Exercise 2. $4 \cdot y + 7 = {}^-5$

Answer: $^-3$

Exercise 3. $2 \cdot t + 6 = 10$

Answer: 2

Exercise 4: $3 \cdot y + {}^-5 = 7$

Answer: 4

Exercise 5: $7 \cdot x + {}^-5 = 9$

Answer: 2

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Exercise 6: $7 \cdot w + 3 = 38$

Answer: 5

Exercise 7: $13 = 3 \cdot x + {}^{-}2$

Answer: 5

Exercise 8: $-12 = 4 + 2 \cdot y$

Answer: -8

Exercise 9: $-4 = -1 + 3 \cdot y$

Answer: -1

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Exercise 10: $4 \cdot x - 3 = 5$

Answer: 2

Exercise 11: $-5 \cdot y + 5 = 10$

Answer: -1

Exercise 12: $-7 \cdot t + 1 = 15$

Answer: -2

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Teacher Observations