# AN INVESTIGATION INTO THE VAN HIELE LEVELS OF THINKING IN THE GEOMETRY COMPONENT OF ACADEMIC AND ADVANCED MATHEMATICS



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# AN INVESTIGATION INTO THE VAN HIELE LEVELS OF THINKING IN THE GEOMETRY COMPONENT OF ACADEMIC AND ADVANCED MATHEMATICS

By

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ABSTRACT

The purpose of this study was to examine the van Hiele level of thinking of students enrolled in academic and advanced mathematics courses, to determine their readiness for deductive reasoning and to make comparisons between these groups and a group tested in the United States.

A sample of 17 schools was randomly selected from both urban and rural settings in the province of Newfoundland and Labrador. A sample of 561 students was chosen and administered a modified version of the van Hiele Level Test in October 1985 and again in October 1986.

An analysis of the results indicated that at the beginning of level II the majority of students, 57.1 percent using the 3 of 5 criteria and 77.8 percent using the stricter 4 of 5 criteria, were not at a sufficient van Hiele level to begin the study of deductive geometry. Analysis of the posttest given in level III indicated an increase in the van Hiele level over level II. However, a large percentage of students, 38.5 percent using the 3 of 5 criteria and 64.1 percent using the 4 of 5 criteria, were still below the necessary van Hiele level.

Comparisons of the advanced and academic groups favored the advanced groups in both level II and III. They were at higher van Hiele levels than their academic counterparts. A comparison of van Hiele levels with a group tested by Usiskin in the United States favored the Newfoundland student for both the academic and advanced programs.

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# CHAPTER I

# THE PROBLEM

## Introduction

The teaching of deductive reasoning in geometry has been the center of controversy for many years. Reeve (1930) advised teachers that "informal geometry represents about all the geometry that many of their students are capable of understanding" (p. 14). Freudenthal (1973) summarized the situation faced by teachers in a typical geometry classroom when he stated that: "There are students who will never build deductive systems of their own or even rebuild those of others, though they must still learn mathematics" (p. 403).

Geometry students have their own preferences. In a survey, (Usiskin, 1980) students were asked what they liked and disliked most about geometry. There were a wide range of answers to what they liked most about geometry but to what they disliked most there was only one strong reply; deductive proof. This negative attitude towards one aspect of the geometry program has generated renewed discussion about what geometry should be taught, at what grade levels should it be taught and to whom should it be taught. Much of the research now being conducted stems from the work of the van Hieles.

Pierre van Hiele and his wife, Dina van Hiele-Geldof, both high school teachers in the Netherlands, were concerned about the level of difficulty being experienced by their students. They believed that geometry involved thinking at a high level and that their students had not yet reached that level because of a lack of experience in thinking at lower levels (Geddes, 1984).

A study of the van Hiele theory of different thought levels has resulted in educators rethinking the junior and senior high school geometry programs. In an attempt to have schools in the United States fit better into the van Hiele theory, Shaughnessy and Burger (1985) have proposed that all students in the United States study geometry without proof for at least one-half year.

Piaget's levels of thinking, which have their basis in the maturation of the child (Adler, 1971), have given way to the van Hiele theory of thinking levels based on "instruction rather than biological maturation" (Geddes, 1984, p. 3). This change in theories has created courses based on the inductive approach to geometry at the junior high level. The problem of what approach is needed in the senior high school has invoked a great deal of discussion because of students' failure to master the deductive aspects of the present course. Freudenthal (1973) believed "...deductivity was not taught by reinvention as Socrates did, but that is was imposed on the learner" (p. 402). He supported the van Hiele view that instruction and experience are essential in the development of deductive skills. The van Hieles saw the problem as the result of the breakdown in communication between the teacher using language appropriate to a higher level of reasoning and the student using a lower level and thus being unable to understand the teacher (Geddes, 1984).

Allendoerfer (1969), Haitt (1979), Hoffer (1981), Sherard (1981), and Shaughnessy and Burger (1985) have indicated that there is much more to geometry than proof and that too much time is being devoted to writing proof. This has resulted in a diversification of the curriculum, moving away from its emphasis on proof and expanding the use of other types of geometry, such as transformational geometry. This shift in emphasis was not intended to downplay the importance of proof. Senk (1982) stressed the significance of proof in geometry when she wrote:

> An understanding of the concept of proof and the facility to write proofs are fundamental to success in the study of higher mathematics. (p. 1)

The development of deductive reasoning through the teaching of geometry was accepted as one of the goals of mathematics education in Newfoundland (Course description for Academic Mathematics 2203, 1982). However, the problems and frustrations experienced by students made its inclusion in the curriculum a topic of much concern. A major study in the United States, (Usiskin, 1982), and two Newfoundland studies (Boone, 1984 and Taaffe, 1983), have demonstrated that the majority of students in grade nine and level I are at a low van Hiele level and are not ready for deductive geometry. Usiskin also demonstrated that student achievement at the senior high level depended on their entry level from junior high.

The Newfoundland senior high schools use a credit system spread over a three year period, level I, II and III which roughly correspond with grades 10, 11 and 12 in other provinces.

The Newfoundland mathematics curriculum is divided into three streams. The practical courses were designed and are offered to students who have achieved minimal success with mathematics and who expect to have a career in which practical mathematics skills will be more useful than the algebra and geometry skills taught in the academic and advanced courses.

The academic program was designed for the 60 to 80 percent of students who have experienced moderate success in mathematics. This is a diversified group, containing a variety of ability levels within a class. Geometry makes up approximately 50 percent of their mathematics instructional time. The text series for the course is Math Is/Geometry 1203, Math Is/Geometry 2203 and Math Is/Geometry and Trigonometry 3203 (Ebos, F.; Tuck, B.; Hatcher, G; Drost, D., 1984). Deductive reasoning and proof writing were first introduced in Academic 1203.

The advanced mathematics program was designed for those with higher ability levels, the top 15 to 25 percent of the mathematics students. The advanced program required approximately 50 percent of instructional time in grade 9, Advanced Mathematics 1201 and Advanced Mathematics 2201 to be spent on geometry, with particular emphasis on deductive reasoning and proof writing. The text was Geometry (Moise and Downs, 1975).

Fifty percent of the students' final grades in level III mathematics courses is determined by a provincial examination in June. The emphasis on proof has been reduced on the final examinations to allow for the difficulties being experienced by students. However, its continued inclusion in both the academic and advanced programs demonstrates the importance placed on deductive proof in the field of mathematics in Newfoundland.

In summary, the problems faced by educators in the teaching of deductive proof are not new to the 1980's. What is new is the attention being focused on the problem of how best to develop thinking abilities. The consideration being given to deductive thinking and the research to find better ways to develop deductive abilities center on the work of the van Hieles. Research has changed the curriculum placing the emphasis on inductive geometry at the junior high level. These changes and the continued application of the van Hiele theory will have an impact on the senior high curriculum.

## Purpose of the Study

A goal of education is to achieve maximum learning for each student in each of the subject areas presented. If maximum learning is to be achieved in geometry, the level of instruction must be matched with the level of students' understanding (Geddes, 1984).

The purpose of this study considered students' readiness to reason deductively and write proofs. The researcher looked at which program, academic or advanced was most effective in raising the van Hiele levels of the students. The students who participated in the study had at least one year of geometry experience with deductive reasoning.

The questions to be considered in this study are:

1. Are students at the beginning of Academic Mathematics 2203 and Advanced Mathematics 2201 prepared for deductive proof?
2. Are students enrolled in Advanced Mathematics 2201 better prepared for deductive proof than those in Academic Mathematics 2203?
3. Are students of Academic Mathematics 3203 and Advanced Mathematics 3201 prepared for deductive proof?
4. Did the students make gains in their van Hiele levels from pretest to posttest?

5. How do students at the end of Academic Mathematics 2203 and Advanced Mathematics 2201 compare with those in the United States at the end of one year of geometry?

## Significance of the Study

A goal of the educational system is "... to stimulate student to be creative, to have ideas of their own, to be more than just memorizing machine" (Allendoerfer, 1969, p. 166). The importance of developing the thinking capacity of students is one of the reasons why geometry is a major component of high school Mathematics. Schlanch, when asked why geometry was taught, replied:

> It gives the student an outlook upon a great field of human thought... it gives him valuable habits of thinking and ideals of methods which have transfer value and which help him in orderly and systematic thinking. (Reeve, 1930, p. 134)

Deductive geometry provides the opportunity to extend the thinking levels of a student if the proper instructional experiences are presented to assist in the student's understanding.

The van Hieles proposed a theory that placed a student's level of thinking into one of five general levels. The lowest level involves the use of simple recognition and the highest involves the ability to understand the nature of Mathematics.

The van Hieles believed deductive reasoning was a higher thinking level than simple recognition or comprehension (Wirszup, 1976). They believed that transition from lower levels of thinking to the higher level of deductive reasoning could be achieved if enough of the symbols needed for the new higher level had been accumulated (Geddes, 1984).

Buck (1968) believed the learning process followed some logical order:

> The human being is not born skilled in reasoning. He learns to reason by reasoning, step by step, from a satisfiable hypothesis to an acceptable conclusion (p. 467).

The van Hieles believed previous learning experiences were an essential component in the development of higher levels were of thinking. The learning process and the gains in the thinking levels are not a continuous flow but consist of many discontinuities.

> The discontinuities are ... jumps in the learning curve, [and] these jumps reveal the presence of levels. The learning process has stopped; later on it will start itself once again. In the meantime, the pupil seems to have "matured." The teacher does not succeed in further explanation of the subject. He and ... the other students who have reached the new level seem to speak a language which cannot be understood by the pupils who have not yet reached the new level. They might accept the explanation of the teacher, but the subject taught will not sink into their minds. The pupil himself feels helpless; perhaps he can imitate certain actions, but he has no view of his own activity until he has reached the new level. At that time the learning process will take on a more continuous character. Routines will be formed and an algorithmic skill will be acquired as the prerequisites to a new jump which may lead to a still higher level (Wirszup, 1976, p. 79).

These discontinuities and the different thinking levels of students within the same mathematics class question the teaching of deductive reasoning to students who are not yet ready. The intent of this study was to collect data to determine the appropriateness of the geometry courses taught in Newfoundland high schools. Are the student's van Hiele levels of thinking sufficient to meet the requirements of the course? The results of this study provide information on the number of students who have achieved at least the third van Hiele level and are thus capable of either working with deductive or ready for the introduction of deductive proof.

There has been some doubt expressed about the value of unit two in the geometry component of Academic Mathematics 2203 since it reviews the deductive proofs previously studied in Academic Mathematics 1203. The analysis of data on van Hiele levels of students enrolled in the academic courses will help determine the appropriateness of the inclusion of unit two on deductive reasoning.

## Limitations of the Study

The study was conducted on a sample of Newfoundland and Labrador Academic Mathematics 2203 and Advanced Mathematics 2201 students using a van Hiele level test designed by the CDASSG Project (Cognition Development and Achievement in Secondary School Geometry; Usiskin, 1982) found in Appendix A. Students enrolled in the practical stream were not tested, restricting the sample, and consequently made comparisons to other general groups, such as the Usiskin group in the United States, difficult. (Usiskin, 1982)

# CHAPTER II

# REVIEW OF RELATED RESEARCH

## Introduction

This chapter reviews the literature on the van Hiele theory. It provides a short historical account of the development of the theory and some of its recent uses. Aspects of the theory are described in detail. A summary of research projects are presented and the implications of the van Hiele theory are discussed.

## The History of the van Hiele Theory

The van Hiele theory is the result of work done by two high schools teachers in the Netherlands. Pierre M. van Hiele authored a dissertation on the role of intuition in the teaching of geometry and his wife, Dina van Hiele-Geldof completed her doctoral thesis on didactics in geometry. Dina died shortly after the completion of her dissertation leaving her husband, Pierre, to present their ideas to the Mathematics Education community. In the years 1958-59 he wrote several papers, including The Thought of the Child and Geometry, in which he discussed five levels of thought development in Geometry.

Mathematics educators, methodologists and psychologists at the Soviet Academy of Pedagogical Sciences became interested and began researching the levels proposed by van Hiele. The work done by Stolyar (1965) and Pyshkalo (1968) (cited in Wirszup, 1976) confirmed the validity of van Hiele's level theory and they began applying these new ideas in the Soviet mathematics curriculum.

Much of the work of the van Hieles and the success of the Soviet programs went unnoticed in the West. In 1973, Freudenthal publicized the work of the van Hieles in his book Mathematics as an Educational Task, bringing the theory to the attention of Western Europe. Wirszup believed the breakthroughs being made in geometry education in the Soviet Union were a direct result of the application of the van Hiele level theory. The achievements of the Russians did not go unnoticed. Coxford (1978) saw the potential of the van Hiele theory in the development of the entire geometry curriculum. Coxford proposed a structured curriculum in which the level dictated the activity:

> The van Hiele levels of thought provide a structure within which a geometric curriculum can be developed throughout the school period. The levels suggest the type of activity we should provide for the learners so that their knowledge will develop. (p. 327)

He suggested that more research should be done in the United States in the area of cognitive development to determine how the van Hiele theory might be used to improve the curriculum.

The writings of Wirszup and Coxford have resulted in several research projects in the United States dealing with the van Hiele levels. These will be discussed later.

## The van Hiele Theory

The van Hieles approached the problems being experienced by their geometry students from two directions. Pierre formulated the psychological principles while Dina focused on the didactic experiment as a technique to raise the student's thought level (Hoffer, 1982).

The van Hiele level theory has three aspects: "1. the existence of levels 2. properties of the levels, and 3. movement from one level to the next" (Usiskin, 1982, p. 4).

The Existence of Levels.

Pierre van Hiele's work in developing levels of thought in geometry was a direct result of earlier work done by Piaget. Van Hiele noticed that the problems presented to children in Piaget studies often required a knowledge of vocabulary or properties that were above the child's level of thinking (Hoffer, 1982). He noted that the learning of geometry was a discontinuous process which suggested the existence of levels (Geddes, 1982).

Van Hiele as translated in Fuys (1984) described the levels of his theory which have been renumbered for easier comparison to work being done today.

> At the <u>Base Level</u> (Level 1) of geometry, figures are judged by their appearance. A child recognizes a rectangle by its form and rectangle seems different to him than a square. When one has shown a six-year-old child what a rhombus is, what a rectangle is, what a square is, what a parallelogram is he is capable of reproducing these figures without error on a geoboard of Gettagno, even in difficult arrangements. We have used the geoboard in our research so that the child will not be bothered by the difficulties resulting from figures. At the Base Level, a child does not recognize a parallelogram in the shape of a rhombus. At the level, the rhombus is not a parallelogram, a the rhombus seems to him a completely different thing.
>
> At the <u>Second Level</u> of geometry, the figures are bearers of their properties. That a figure is a rectangle means that it has four right angles, diagonals are equal, and opposite sides are equal. Figures are recognized by their properties. If one tells us that the figure drawn on a blackboard has four right angles, it is a rectangle even if the

figure is drawn badly. But at this level properties are not yet ordered, so that a square is not necessarily identified as being a rectangle.

At the Third Level properties are ordered. They are deduced one from another: one property precedes or follows another property. At this level the intrinsic meaning of deduction is not understood by the students. The square is recognized as being a rectangle because at this level definitions of figure come into play.

At the Fourth Levels, thinking is concerned with the meaning of deduction, with the converse of a theorem, with necessary and sufficient conditions (p. 245-246).

At the fifth level, which is generally impossible to achieve in general education, is an analysis of the nature of a Mathematician activity (p. 250).

The fifth level of the theory is an extension of the fourth, or deductive level and is very difficult to determine using conventional test methods because of the high level of thinking involved.

Hoffer (1981) has done extensive work using the van Hiele level theory. He has simplified the levels and assigned names to each.

Level 1: Recognition. The student learns some vocabulary and recognizes a shape as a whole.

Level 2: Analysis. The student analysis properties of figures.

Level 3: Ordering. The student logically orders figures and understands interrelationships between figures and the importance of accurate definitions.

Level 4: Deduction. The student understands the significance of deduction and the role of postulates, theorems and proof.

Level 5: Rigor. The student understands the importance of precision in dealing with foundations and interrelationships between structure (p. 13-14).

Hoffer has also pointed out that level five was rarely achieved by high school students and was often omitted from discussions of the van Hiele levels theory.

The existance of levels has implications for the geometry curriculum. Hoffer, like Wirszup, believed that deductive geometry in high school required at least the ordering thought level and that many of those who were unsuccessful were at a lower thought level.

Properties of the Levels

Van Hiele (1958) identified some of the properties associated with the levels. Usiskin has assigned a name to each of the levels.

> Property 1. Adjacency. At each level there appears in an extrinsic way that which was intrinsic at the preceding level.
>
> Property 2. Distinction: Each level has its own linguistic symbols and its own system of relations connecting these signs.
>
> Property 3. Separation: Two people who reason at two different levels cannot understood each other.
>
> Property 4. Attainment: The maturation which leads to a higher happens in a special way ... the phases include inquiry, direct orientation, explicitation, free orientation and integration (p. 246).
>
> Usiskin (1982) has included an additional property to the theory.
>
> Property 5. Fixed Sequence: A student cannot be at van Hiele level n without having gone through level n-1 (p. 5).

Phases of Learning

In the statement of the level theory van Hiele was optimistic that if students were provided with sufficient instructional time and geometric experiences they would increase their thought levels by passing through the learning phases. Hoffer (1982) described the necessary phases but noted there was no set order through which a student must pass.

> Phase 1. Inquiry. The teacher engages the students in conversations about the objects of the study to be pursued.. Teacher learns how the students interpret the words and gives the students some understanding of what topic is to be studied. Questions are raised and observations made that use the vocabulary and objects of the topic and set the stage for further study.
>
> Phase 2. Directed Orientation. The teacher carefully sequences activities for student exploration by which students begin to realize what direction the study is taking, and they become familiar with the characteristic structures. Many of the activities in this phase are one-step tasks which elicit specific responses.
>
> Phase 3. Explicating. The students with minimal prompting by the teacher and building on previous experiences refine their use of the vocabulary and express their opinions about the inherent structures of the study. During the phase, the students begin to form the system of relations of the study.
>
> Phase 4. Free Orientation. The students now encounter multi-step tasks or different ways. They gain experience in finding their own way or resolving the tasks. By orienting themselves, many of the relations between the objects of the study become explicit to the students.
>
> Phase 5. Integration. The students now review the methods at their disposal and form an overview. The objects and relations are unified and internalized into a new domain of thought. The teacher aides this process by providing global surveys of what the students already know being careful not to present new or discordant ideas. (p. 5)

The five phases of learning must be completed before a new thought level can be attained. Van Hiele, as translated in Fuys (1984) stated:

> At the completion of this fifth phase a new level of thought is attained. The student has at his disposal a system of relations which are related to the whole of the domain explored. (p. 247)

## Research on the van Hiele Theory

The study of geometry by children has challenged educators for years. The question arising from childrens study of geometry is why was it that so many children who master most school subjects have little success in their study of geometry? To answer this question researchers in the Soviet Union began an intensive study of all aspects of the curriculum.

In reports of Soviet research, Pyshkalo (1968) and Stolyar (1965) concluded that only 10 to 15 percent of students who finished fifth grade reached the second van Hiele level. However when students in an experimental grade two class, based on a curriculum proposed by van Hiele, were tested 75 percent had achieved or surpassed the second level. This rate was higher than the 50 percent level of students in a regular seventh grade program.

Much of the research in the Soviet Union concentrated on the movement from one level to the next in children in grades one through eight. Researchers concluded "the most important factor in the improvement of curricula and teaching methods lies in establishing a single sequence in the formation of mathematical concepts." (Psyhkalo, 1968 cited in Wirszup, 1976, p. 91)

Boltyanskii (cited in Wirszup, 1976) concluded that geometry should not be isolated as a separate lesson for students in the early grades; it should be integrated into the curriculum.

The Soviets set as their goal the achievement of level 1 in grade 1 and the attainment of level 2 in grades two and three.

Coxford, like the Soviets, saw the potential of the van Hiele level theory and suggested that research in the United States should collect data in a variety of areas including:

1. Carefully documented longitudinal case studies of children.

2. The gathering of data by age sampling to compare cognitive structures and developmental stages.

3. An analysis of the effects of instruction on cognitive structures (Hoffer, 1982, p. 14)

Major research projects such as the Oregon Project, the Brooklyn Project and the Chicago Project were begun in the United States to gather data on the van Hiele theory.

The Oregon Project: Assessing Children's Development in Geometry (Burger, 1982)

> The study described here is an investigation of children's reasoning processes in geometry and of the usefulness of the van Hiele levels in describing their reasoning. (Burger, 1982, p. 1).

The project was sponsored by the National Science Foundation and continued from September 1979 through February 1982. Researchers interviewed and taped over 70 students in grades one to twelve using two 45-minute sessions per student. The project developed two sets of tasks, one on triangles, the other on quadrilaterals, that involved the use of drawing, sorting, identification familiar and mystery figures, and establishing the logical equivalence of several geometrical definitions (Burger, 1982).

As a result of Burger's work, a set of indicators were established that could be used to identify the van Hiele level of a student, regardless of age. The levels had now been written in terms of observed behaviours.

The Brooklyn Project: Geometric Thinking Among Adolescents in Inner City Schools (Geddes, 1982)

> The general purpose ... is to determine whether the van Hiele model provides a reasonable structure for describing and understanding geometry learning as it takes place in the context of formal schooling. (Geddes, 1982, p. 2)

The project was sponsored by the National Science Foundation and continued from December, 1979 through January, 1982. Four instructional modules were developed and presented to 40 inner-city adolescents in eight 45-minute sessions. One conclusion of the study was that modules utilizing concrete objects appeared to offer instructional advantages. The project also conducted a review of several textbooks series. They found that the texts were inconsistent with the van Hiele theory. Students at the junior high level who required level three thinking abilities has received only a level one background, making understanding of the materials difficult at best (Geddes, 1982).

The Chicago Project: Cognitive Development and Achievement in Secondary School Geometry (Usiskin, 1982)

> The fundamental purpose of this project is to test the ability of the van Hiele theory to describe and predict the performance of students in secondary geometry. (Usiskin, 1982, p. 8)

The project was sponsored by the National Institution of Education and continued from July, 1980 to June, 1982. It was the most comprehensive of all the projects, with a sample of 2699 students enrolled in a one year geometry course in 13 schools in the United States.

Students were tested in the first week of school using a van Hiele Level Test and an Entering Geometry Test. Near the end of the school year, the same students were tested using the same van Hiele

Level Test, a Proof Test and a Comprehensive Assessment Program Geometry Test. The van Hiele Level Test usdd was constructed by project members using the description of behaviours expected at each level by the van Hieles.

The van Hiele Level Test classified students using a 3 of 5 questions correct at each level or a stricter 4 of 5 correct at each level. Using the 3 of 5 criteria it was possible to classify 70 percent of the students tested and using the 4 of 5 criteria, 88 percent. Analysis of the results indicates that the entering van Hiele Level of the majority of students, 54 percent using 3 of 5 and 81 percent using the 4 of 5, were at or below the base recognition level (p. 100).

The results of the posttest, using the van Hiele Level Test, showed some gains. However 16 percent and 35 percent, using the 3 of 5 and 4 of 5 respectively, were at or below the recognition level (p. 105).

Project researchers compared van Hiele levels and geometry achievement. They found a strong relationship between the two, suggesting the van Hiele test was a good predictor of success in proof, which requires a van Hiele level of at least four.

The low van Hiele levels of students enrolled in the one-year geometry course implied one of the reasons 47 percent of all students chose not to take the geometry. The low levels clearly demonstrated the truth of Wirszup's and Hoffer's claims, that proof was inappropriate for a large number of students.

Usiskin was able to compare students who failed in geometry proof with their entering van Hiele levels, 71 percent of those who failed had entrance levels at or below recognition. To add further to the student placement problems, a text analysis found that some of the texts used were inappropriate for the average class.

The low van Hiele levels and a lack of entering geometry knowledge indicated that students were not learning even the simplest geometry concepts at the junior high school levels.

Other, less extensive studies using the van Hiele levels have been undertaken. Mayberry (1983) reported a study of undergraduate preservice teachers in which she found that 70% of the responses of students who had completed a high school geometry course were below the level four needed to understand deductive reasoning.

Senk (1982), using the data collected from the proof section of the CSASSG project, concluded that even after a full year course of geometry with proof only about half the students could do anything more than simple proofs.

Two recent Newfoundland studies of geometry have used the van Hiele levels. Taaffe (1983) studied the relationship between van Hiele levels and proof writing abilities. He found that students with high van Hiele levels had increased chances of writing correct proofs. He also found a slight difference between students in advanced and academic programs, with the advanced students being slightly better in proof writing.

Taaffe's concluded that even after a year of study, which included deductive reasoning, only 9.5 percent of the level I mathematics students in his sample were at the deductive level on the van Hiele scale. In a second Newfoundland study, Boone (1984) found that the vast majority of grade nine students were at van Hiele levels below that required for deductive reasoning. He also studied the effect of two different text series on the van Hiele level and found that text selection was a factor in determining the van Hiele level.

Implications of the van Hiele Theory

A major implication of the van Hiele theory is that it permits educators to test the present thinking level, of students and to assign materials that will allow them to reach the next thought level. It may now be possible to maximize students' learning of a student using appropriate course materials based on their current van Hiele level. Hoffer (1982) saw the significance of the van Hiele theory when he stated:

> ... it provides us with a peephole through which we can use our mathematical eye to view children interaction with mathematics. (p. 19)

As a means of reducing some of the problems encountered by students in the past, the use of the van Hiele level theory should have a significant impact on teaching methods, materials and the general curriculum.

The van Hiele theory demonstrates that before students can deduce on their own they must first move through the lower levels of thinking and that this can only be achieved by providing the instructional time and geometric experiences necessary. Usiskin (1982) and Taaffe (1984) found that the majority of students entering a geometry course that required a thought level of three or more had thought levels of one or two. Thus, many of the students who enrolled in proof-oriented courses experienced little or no success with proof. Usiskin (1982) saw the need for change. He stated: "This study confirms the need for systematic geometry instruction before high school if we desire greater geometry knowledge and proof-writing success among students" (p. 89).

The writers of text materials should be aware of the van Hiele levels of their students. They must insure that the level of the text does not exceed the level of the student but, at the same time, provide experiences that will raise the thought level of the student. Van Hiele, as translated in Fuys (1984), described what he thought should be the structure of a geometry course. The first part of the course, the aspect of geometry stage, ought to allow for the attainment of the first two levels of thought. One would use a collection of concrete geometric figures and materials with which students would themselves make models of the figures. The second part of the course, the essence of geometry, would allow for the attainment of the third level of thought. Students begin to learn relationships by again manipulating the learning materials. Next, the discernment of geometry stage develops the fourth level of thought, in which students would begin using the theorems in an orderly fashion. This would reveal the ideas behind and the links between the various theorems. If the course could be continued to the fifth level, usually not possible for high schools, the students would begin to analyze the nature of mathematics.

The geometry course structure proposed by van Hiele need not be confined to a single year of geometry study but should be integrated into the K-12 mathematics program. Curriculum developers and textbook authors can no longer focus on a particular grade or school level, but should consider the entire mathematics curriculum from K-12 to prevent possible gaps in the learning curve or the omission of a van Hiele level. Students move through each level in an ordered fashion; missing one level would make the attainment of the next level of thought impossible. The curriculum should reflect this upward development of thought levels if the goal of deductive geometry is to be achieved by a high percentage of students.

Teachers should not allow the establishment of communication barriers. They should approach each student with a vocabulary that is suited to the level of that particular student. Geddes (1982) wrote:

> Many failures in teaching geometry result from language barriers... the teacher using a language of a higher level than is understood by the student. (p. 5)

The curriculum in the Newfoundland junior high schools was changed to follow more closely the thought development process put forth by the van Hieles. The senior high program was changed to include more and difficult types of geometry without consideration of the thought levels possessed by the students in the senior high. Taaffe's work indicates that the vast majority of students in level I are not yet ready for proof. Are the students in level II and III ready for deductive reasoning?

## Summary

The van Hieles, in the later 1950's, developed a theory of thought levels in geometry. They proposed that a student's level of thought fit into one of five levels and that a student could move through the levels if certain criteria were met.

Since then, researchers in the Soviet Union and the United States have verified the existence of these thought levels. They have found that a large number of students were operating on too low a van Hiele thought level to perform sophisticated behaviours such as proof. The major implication of the theory was that students can only learn material that are at the appropriate learning level.

"Geometry serves as a vehicle for stimulating and exercising general thinking skills and problem solving abilities" (Sherard, 1981, p. 21). Since problem solving has been made the focus of mathematics (Course description for Academic Mathematics 2203, 1982) then the geometry program should be made as effective as possible to insure that students learn and reach their highest possible van Hiele thought level.

The research to date, including the two Newfoundland studies, Taaffe (1983) and Boone (1984) have shown that students up to grade 10 are not prepared to study deductive proof. The van Hiele theory suggested that the level of these students can be raised to the appropriate levels if the proper materials were presented in earlier courses. The researcher in this study will collect data on older high school students who have already studied deductive proof as part of their earlier mathematics courses in an attempt to determine not only their level of readiness for deductive proof but to determine if the study of geometry in level I and II has been effective in raising the students thinking level. The work of Taaffe and Boone will be extended to present a fuller picture of the present state of geometry in the senior high schools of Newfoundland.

## CHAPTER III

## THE EXPERIMENTAL DESIGN

### Introduction

The major purpose of this study was to investigate the deductive reasoning readiness of second and third level academic and advanced students. In addition different groups were compared to determine if one group was better prepared for deductive reasoning than the other.

### Population

The population for this study consisted of all Newfoundland students enrolled in Academic Mathematics 2203 or Advanced Mathematics 2201.

### Sampling Procedures

A sample of 561 students was chosen from 17 randomly selected schools in both rural and urban settings. Students who had changed Mathematics programs, advanced to academic for example, were not included in the sample. The schools chosen offered a variety of Mathematics streams including those which offered no advanced program. Students enrolled in the practical stream were not tested, restricting the sample and making cross group references with other general groups such as Usiskin's more difficult.

### Test Instrument

The modified van Hiele level test used in this study was developed by the CDASSG project in Chicago. The original test contained 25 multiple choice questions with five questions on each of the five van Hiele levels. The modified version used for this study contained the 20 multiple choice items designed to test the first four levels of the van Hiele theory, since the fifth level is very seldom achieved by high school students."

Usiskin (1982) tested the reliability of his test design and discussed the results:

> The van Hiele test, for purposes of reliability, is considered as five 5-items tests. The K-R formula 20 reliabilities (Horst modification number in parentheses) for the five parts in Fall are .31 (.36), .44 (.48), .49 (.60), .13 (.13), and .10 (.11), and in the Spring are .39 (.43), .55 (.59), .56 (.59), .30 (.31) and .26 (.27). One reason for the low reliabilities is the small number of items; similar tests at each level 25 items long would have reliabilities .74, .82, .88, .69 and .65 in Spring. The low reliabilities at levels 4 and 5 may be a by product of the lack of specification of the van Hiele theory at these levels. (p. 29)

The van Hiele test has been administered twice in Newfoundland. The results of the Taaffe (1983) study compares favorably with the overall results of the Usiskin (1982) study. For student placement into the van Hiele model Taaffe reported 83 and 93% on the 3 of 5 and 4 of 5 criteria respectively while Usiskin reported 87 and 86%. Taaffe's mean van Hiele scores also compared favourably with Usiskins: Taaffee 2.22 and 1.45 on the 3 of 5 and 4 of 5 respectively, while Usiskin reported 2.55 and 1.79.

The answer sheet required the students name and school so that pre and posttest comparisons could be made. Students were also asked to check the following:

1. What Mathematics course are you enrolled in this year?
   - Academic Mathematics 3203
   - Academic Mathematics 2203

- Advanced Mathematics 3201
- Academic Mathematics 2201

2. What Mathematics course did you take last year?
   - Academic Mathematics 1203
   - Advanced Mathematics 2201
   - Academic Mathematics 2202
   - Advanced Mathematics 1201
   - Consumer Mathematics 1202
   - Vocational Mathematics 2202

Test Administration

The test papers and answer sheets were forwarded to the program coordinators of the boards from which schools had been selected. The coordinators were given the following instructions.

1. Tests were written in early October.

2. Deliver test papers and answer sheets in a sealed enclosure to the schools involved the day before testing with instructions to the teacher that the enclosures not be opened before testing.

3. Provide the teachers with a list of regulations for the students: (a) Students are not to write on the test paper, only on the answer sheet provided. (b) All test papers and answer sheets are to be collected at the end of the exam. (c) The exam shall last for 35 minutes. (d) No aids such as calculators are permitted.

4. Collect and return all answer sheets to the experimenter and retain all extra papers at the board office for later use.

5. Provide notice of the follow-up exam, to be given the following October, and at that time follow the same procedures as before.

A list of the names of the students writing the pretest was sent to all schools so that the same group would write the posttest one year later.

Analysis

The study considered five major questions.

Question 1. Are students at the beginning of Academic Mathematics 2203 and Advanced Mathematics 2201 prepared for deductive proof?

This question was tested by administering a modified version of the van Hiele level test to all students in the sample in October. The students were then classified using either the 3 of 5 criteria or the stricter 4 of 5 criteria. The results were then displayed in both number and percent form in tables.

Usiskin (1982) discussed the possibility of Type I or Type II error. "The choice of criterion, given the nature of this test, is based upon whether one wishes to reduce Type I or Type II error. Recall that Type I error refers to a decision made (in this case a student meeting a criterion) when it should not have been made.

P (3 of 5 correct by random guessing) = 0.05792

P (4 of 5 correct by random guessing) = 0.00672

Therefore, the 4 of 5 criterion avoids about 5 percent of cases in which a Type I error may be expected to manifest itself... The 3 of 5 criterion avoids about 7 percent of cases in which Type II may be expected to appear" (p. 23-24).

Question 2. Are students enrolled in Advaned Mathematics 2201 better prepared for deductive proof than those in Academic Mathematics 2203?

This questions was tested using the following hypothesis.

Hypotheses: The students' van Hiele levels of thinking and their course enrollment are independent.

This was tested using the chi-square test for independence of the van Hiele level and the course enrollment. Tables were constructed for both criteria and the 0.05 level of significance was applied.

Question 3. Are students of Academic Mathematics 3203 and Advanced 3201 prepared for deductive reasoning?

This question was tested by administering a modified version of the van Hiele level test to the students in the sample the following October. Students were classified into van Hiele levels using either the 3 of 5 or the stricter 4 of 5 criteria and the results were displayed both in number and percent form in tables.

Question 4. Did students make gains in their van Hiele levels from pretest to posttest?

This question was tested using the folloiwng hypothesis.

Hypothesis: (a) The distribution of levels from pretest to posttest in the academic course enrollment were independent.

(b) The distribution of levels from pretest to posttest and the advanced course enrollment were independent.

This was tested using the chi-square test for independence of the van Hiele levels achieved on the pretest and posttest and the course enrollment. Tables were constructed using both the 3 of 5 and 4 of 5 criteria.

Question 5. How do students a the end of Academic Mathematics 2203 and 3203 and Advanced Mathematics 2201 and 3201 compare with those in the United States at the end of one year of geometry?

This question was tested using the following hypothesis.

Hypothesis: There was no significant difference in van Hiele levels of Academic and Advanced Mathematics students at the end of the courses in Newfoundland and students having completed one year of geometry in the United States.

This was tested using a chi-square test for homogeneity of van Hiele levels in students in Newfoundland and in the United States.

CHAPTER IV

# THE RESULTS OF THE STUDY

## RESEARCH QUESTIONS AND RESULTS

In this chapter the results of the testing are presented. The van Hiele levels of students at the beginning of level II and level III were examined. The relationship between course and van Hiele level was analyzed and comparisons were made between Newfoundland students and students in the United States.

The sample for this study was chosen from all the students enrolled in Academic Mathematics 2203 and Advanced Mathematics 2201 in the 1985-86 school year. The students writing the pretest who continued into the next level of mathematics were tested a second time in October of the 1986-87 school year. The sample was restricted to the academic and advanced groups because the students in the practical program study had very little deductive geometry since the course was intended for the lower ability students.

Newfoundland students have three choices in mathematics study, practical, academic and advanced. The majority of students choose the middle stream academic program with smaller numbers choosing practical or advanced. Furthermore, many of the smaller schools in the Province offer only the academic program since it meets the needs of the majority of students while others offer a choice between academic and practical because of the small numbers of students targeted for the advanced courses and the inability to provide a teacher for such a small number of students.

A breakdown of the sample used for the pretest and posttest is provided in Table I. The sample in the pretest consisted of 561

TABLE I

Sample Breakdown by Course on the Pretest and Posttest

| Testing | Pretest | | Posttest | |
|---|---|---|---|---|
| Course | # | % | # | % |
| Academic | 391 | 69.8 | 279 | 67.2 |
| Advanced | 170 | 30.2 | 136 | 32.8 |
| Total | 561 | 100 | 415 | 100 |

students of which 69.8 percent studied Academic Mathematics 2203 and 30.2 percent studied Advanced Mathematics 2201. The sample used in the posttest contained 74 percent (415) of the original students of whom 67.2 percent were enrolled in Academic Mathematics 3203 and 32.8 percent in the Advanced Mathematics 3201 program. The decline in the sample size was due to student migration and the dropping or changing of courses from one year to the next. The number of advanced students in the sample made up 30.2 percent of the sample and was higher than the Provincial averages of 18.9 percent of level I students to a low of 12.5 percent for level III students who are enrolled in the Advanced Mathematics Program. The higher percentage was selected to ensure sufficient numbers of students to make comparisons with other groups possible.

Question 1

Are students at the beginning of Academic Mathematics 2203 and Advanced Mathematics 2201 prepared for deductive reasoning?

The measure of students' levels of readiness was assessed by administering a modified version of the van Hiele Geometry Test to 391 students in Academic Mathematics 2203 and 170 students in Advanced Mathematics 2201.

Students were classified using the results of this testing based on the van Hiele levels of thought: Recognition, Analysis, Ordering, or Deduction.

Two criteria were established: 3 of 5 correct on each level of the test to reduce the chance of Type II Error or a stricter 4 of 5 correct which reduces the change of a Type I Error. No student could attain a level of n without having meet the criteria for each of the levels below n. Students not satisfying this criteria were classified as nofits.

The number and percentage of students in Academic 2203 and Advanced 2201 and their van Hiele levels is shown in Table II. It was possible to classify 86.4 percent of the students tested using the 3 of 5 criteria.

The number and percentage of students at each van Hiele level by course studied using the stricter 4 of 5 is displayed in Table III. It was possible to classify 90.7 percent of the students using this criteria.

The study of deductive geometry requires an entry level of at least three on the van Hiele scale. Students were tested and their van Hiele levels were recorded in Table II and III. An analysis of the results indicates that the third level or higher on the van Hiele scale had only been achieved by 22.2 percent and 42.9 percent of students when the 4 of 5 and 3 of 5 criteria were applied respectively. A breakdown by course reveals that 15.3 percent of the academic and 37.6 percent of the advanced students were at or above the third level and were ready for materials involving deductive geometry.

Consequently, it can be concluded that the majority of students are not ready for deductive geometry at the beginning of level II, regardless of the testing criteria chosen.

Question 2

Are the students enrolled in Advanced Mathematics 2201 differently prepared than the students in Academic Mathematics 2203?

TABLE II

Van Hiele Levels Achieved on the Pretest

3 of 5 Criterion

| Testing / Course | Academic 2203 | | Advanced 2201 | | Total | |
|---|---|---|---|---|---|---|
| | # | % | # | % | # | % |
| Recognition or Below | 75 | 19.2 | 6 | 3.5 | 81 | 14.4 |
| Analysis | 128 | 32.7 | 35 | 20.6 | 163 | 29.1 |
| Ordering | 94 | 24.0 | 42 | 24.7 | 136 | 24.2 |
| Deduction | 41 | 10.5 | 64 | 37.6 | 105 | 18.7 |
| Nofit | 53 | 13.6 | 23 | 13.6 | 76 | 13.5 |
| Total | 391 | 100 | 170 | 100 | 561 | 100 |

TABLE III

Van Hiele Levels Achieved on the Pretest

4 of 5 Criterion

| Course Level | Academic 2203 | | Advanced 2201 | | Total | |
|---|---|---|---|---|---|---|
| | # | % | # | % | # | % |
| Recognition or Below | 174 | 44.5 | 39 | 22.9 | 213 | 37.9 |
| Analysis | 122 | 31.7 | 50 | 29.4 | 172 | 30.6 |
| Ordering | 48 | 12.3 | 36 | 21.2 | 84 | 15.0 |
| Deduction | 12 | 3.1 | 28 | 16.5 | 40 | 7.2 |
| Nofit | 35 | 9.0 | 17 | 10.0 | 52 | 9.3 |
| Total | 391 | 69.8 | 170 | 30.2 | 561 | 100 |

Hypothesis: The van Hiele level of thinking and the course enrollment are independent.

The hypothesis was tested using the chi-square test of independence on van Hiele level and course. The level of significance was 0.05 and the degrees of freedom 5. A chi-square value of at least 11.07 was necessary to reject the null hypothesis.

The contingency table for the 3 of 5 criteria was constructed (see Table IV). A chi-square value of 71.9 was found, resulting in the rejection of the hypothesis. The advanced students were at higher levels than the academic students. The contingency table for the 4 of 5 criteria was constructed (see Table V). A chi-square value of 50.9 was found causing the rejection of the null hypothesis.

Consequently, it can be concluded that there is a significant difference in the van Hiele level of students in Academic Mathematics 2203 and Advanced Mathematics 2201.

Question 3

Are students at the beginning of level III who study Academic Mathematics 3203 or Advanced Mathematics 3201 ready for deductive reasoning?

The answer to this question was obtained by administering a modified version of the van Hiele Geometry test to the students previously tested in the Fall of 1985. The posttest sample was only 74 percent of the original due to student migration, drop outs, changing program of study and failure to advance to the next course level.

Table VI contains the number and percentage of students in Academic Mathematics 3203 and Advanced Mathematics 3201 and the van Hiele levels achieved using the 3 of 5 criteria. It was possible to classify 91.1 percent of the students tested. An analysis of the

TABLE IV.

Contingency Table for the 3 of 5 Criteria to Test Independence by Course

| Course / Level | Academic 2203 | Advanced 2201 |
|---|---|---|
| Recognition or Below | 75 | 6 |
| Analysis | 128 | 35 |
| Ordering | 94 | 42 |
| Deduction | 41 | 64 |
| Nofit | 53 | 23 |

$\chi^2 = 71.9$ $P < 0.05$

TABLE V

Contingency Table for the 4 of 5 Criteria to Test Independence by Course

| Course / Level | Academic 2203 | Advanced 2201 |
|---|---|---|
| Recognition or Below | 174 | 39 |
| Analysis | 122 | 50 |
| Ordering | 48 | 36 |
| Deduction | 12 | 28 |
| Nofit | 35 | 17 |

$\chi^2 = 50.9 \qquad P < 0.05$

TABLE VI

Van Hiele Levels Achieved on the Posttest

3 of 5 Criterion

| Course / Level | Academic 3203 | | Advanced 3201 | | Total | |
|---|---|---|---|---|---|---|
| | # | % | # | % | # | % |
| Recognition or Below | 26 | 9.3 | 0 | 0.0 | 26 | 6.2 |
| Analysis | 80 | 28.7 | 17 | 12.5 | 97 | 23.4 |
| Ordering | 88 | 31.6 | 48 | 35.3 | 136 | 32.8 |
| Deduction | 53 | 19.0 | 66 | 48.5 | 119 | 28.7 |
| Nofit | 32 | 11.5 | 5 | 3.7 | 37 | 8.9 |
| Total | 279 | 100 | 136 | 100 | 415 | 100 |

overall results indicates that 61.5 percent of the students tested had achieved the ordering level or above. A breakdown by course reveals a large difference in achievement of the ordering level or above between the academic (50.6 percent) and the advanced groups (83.8 percent).

The numbers and percentages of students at each of the van Hiele levels using the 4 of 5 criteria are shown in Table VII. It was possible to classify 89.4 percent of students tested using the stricter criteria. A total of 35.9 percent of students tested were at the ordering or deductive levels. A breakdown by course shows only 24 percent of the academic students had achieved at least the ordering level whereas 60.3 percent of the advanced students had reached that level. Closer examination of Table VII also revealed a large number of students who were at or below the recognition level of thought, 31.6 percent for students in Academic Mathematics 3203 and a smaller 8.8 percent for students in Advanced Mathematics 3201.

The results from this part of the study indicate the majority (61.5 percent) of students beginning Academic Mathematics 3203 and Advanced Mathematics 3201 are at an appropriate van Hiele level to study deductive reasoning in geometry. However, there is still a large percentage (28.5) of students who are below the ordering level and who are poor candidates for deductive geometry.

Question 4

Did the distribution of levels of students change from pretest to posttest?

Hypothesis: The scores on the pretest and posttest are independent.

The hypothesis was tested using the chi-square test for independence on the van Hiele level and course for the pretest and

TABLE VII

Van Hiele Levels Achieved on the Posttest

4 of 5 Criterion

| Course / Level | Academic 3203 | | Advanced 3201 | | Total | |
|---|---|---|---|---|---|---|
| | # | % | # | % | # | % |
| Recognition or Below | 88 | 31.6 | 12 | 8.8 | 100 | 21.4 |
| Analysis | 96 | 34.4 | 26 | 19.1 | 122 | 29.4 |
| Ordering | 51 | 18.3 | 48 | 35.3 | 99 | 23.9 |
| Deduction | 16 | 5.7 | 34 | 25.0 | 50 | 12.0 |
| Nofit | 28 | 10.0 | 16 | 11.8 | 44 | 10.6 |
| Total | 279 | 100 | 136 | 100 | 415 | 100 |

posttest. Both the 3 of 5 and 4 of 5 criteria were tested at the 0.05 level. A chi-square of at least 11.07 was necessary to reject the null hypothesis.

The contingency tables used for the chi-square test with academic and advanced students using both the 3 of 5 and 4 of 5 criteria, were constructed, Tables VIII-XI. The chi-square test produced a result greater than 11.07 in each case, rejecting the hypothesis. The distribution of the van Hiele level depended on the level of course studied for both the academic and advanced groups, regardless of the criteria chosen.

The distribution of mean scores for the pretest to posttest using both the 3 of 5 and 4 of 5 criteria are shown in Table XII. Both groups in the sample improved their van Hiele levels from pretest to posttest. The academic group increased from 2.29 to 2.6 using the 3 of 5 and from 1.6 to 1.9 on the 4 of 5 criteria. The advanced group increased from 3.1 to 3.37 on the 3 of 5 and from 2.3 to 2.86 using the 4 of 5 criteria. However, the mean for the academic group was below the level three necessary to deal effectively with materials requiring deductive thought.

Question 5

How do students who have completed Academic Mathematics 3203 and Advanced Mathematics 3201 compare with students in the United States who have completed a one year course in geometry?

Hypothesis: There is no significant difference in van Hiele level of students at the beginning of Academic Mathematics 3203 and Advanced Mathematics 3201 and students having completed a one year course in the United States.

TABLE VIII

Contingency Table for the 3 of 5 Criteria to Test Independence

Academic Mathematics 2203 versus Academic Mathematics 3203

| Course Level | Academic 2203 | Academic 3203 |
|---|---|---|
| | # | # |
| Recognition or Below | 75 | 26 |
| Analysis | 128 | 80 |
| Ordering | 94 | 88 |
| Deduction | 41 | 53 |
| Nofit | 53 | 32 |

$\chi^2 = 19.3 \qquad P < 0.05$

TABLE IX

Contingency Table for the 3 of 5 Criteria to Test Independence

Advanced Mathematics 2201 versus Advanced Mathematics 3201

| Course / Level | Advanced 2203 | Advanced 3203 |
|---|---|---|
| | # | # |
| Recognition or Below[1] | 6 | 0 |
| Analysis | 35 | 17 |
| Ordering | 42 | 48 |
| Deduction | 64 | 66 |
| Nofit | 23 | 5 |

$\chi^2 = 20.5$ $P < 0.05$

TABLE X

Contingency Table for the 4 of 5 Criteria to Test Independence

Academic Mathematics 2203 versus Academic Mathematics 3203

| Course Level | Academic 2203 | Academic 3203 |
|---|---|---|
| | # | # |
| Recognition or Below | 174 | 88 |
| Analysis | 122 | 96 |
| Ordering | 48 | 51 |
| Deduction | 12 | 16 |
| Nofit | 35 | 28 |

$\chi^2 = 14.4 \qquad P < 0.05$

TABLE XI

Contingency Table for the 4 of 5 Criteria to Test Independence

Advanced Mathematics 2201 versus Advanced Mathematics 3201

| Course Level | Advanced 2201 | Academic 3201 |
|---|---|---|
| | # | # |
| Recognition or Below | 39 | 12 |
| Analysis | 50 | 26 |
| Ordering | 36 | 48 |
| Deduction | 28 | 34 |
| Nofit | 17 | 16 |

$\chi^2 = 20.7 \quad P < 0.05$

TABLE XII

Distribution of Mean Scores for Pretest and Posttest

| | 3 of 5 | | 4 of 5 | |
|---|---|---|---|---|
| Course / Testing | Academic 2203 | Advanced 2201 | Academic 2203 | Advanced 2201 |
| Prettest | 2.29 | 3.1 | 1.6 | 2.3 |
| Posttest | 2.6 | 3.37 | 1.9 | 2.86 |

The hypothesis was tested using the chi-square test for homogeneity of the van Hiele levels of Newfoundland and the United States. Both the 3 of 5 and 4 of 5 criteria were tested using $p<0.05$ level of significance.

The contingency Tables XIII-XVI for the academic and advanced groups compared to the United States group as tested by the CDASSG Project were constructed. Chi-square values of for all tables exceeded 11.07 rejecting the hypothesis. There is a significant difference in the van Hiele level of Newfoundland students studying Academic 3203 and Advanced 3201 and students in the United States who have completed a one year course in Geometry.

Newfoundland students, in particular those in Advanced Mathematics, are at a higher van Hiele level and are better prepared for deductive geometry, than students in the United States who have completed a one year course in Geometry.

TABLE XIII

Contingency Table for the 3 of 5 Criteria to Test Independence

Newfoundland versus United States

Academic Mathematics

| Course Level | Academic 3203 | United States |
|---|---|---|
| | # | # |
| Recognition or Below | 26 | 323 |
| Analysis | 80 | 470 |
| Ordering | 88 | 630 |
| Deduction | 53 | 365 |
| Nofit | 32 | 269 |

$\chi^2 = 11.13$ $P < 0.05$

TABLE XIV

Contingency Table for the 4 of 5 Criteria to Test Independence

Newfoundland versus United States

Academic Mathematics

| Course Level | Academic 3203 | United States |
|---|---|---|
| | # | # |
| Recognition or Below | 88 | 732 |
| Analysis | 96 | 513 |
| Ordering | 51 | 413 |
| Deduction | 16 | 113 |
| Nofit | 28 | 286 |

$\chi^2 = 12.77 \quad P < 0.05$

TABLE XV

Contingency Table for the 3 of 5 Criteria to Test Independence

Newfoundland versus United States

Advanced Mathematics

| Course Level | Advanced 3201 | United States |
|---|---|---|
| | # | # |
| Recognition or Below | 0 | 323 |
| Analysis | 17 | 470 |
| Ordering | 48 | 630 |
| Deduction | 66 | 365 |
| Nofit | 5 | 269 |

$\chi^2 = 99 \quad P < 0.05$

TABLE XVI

Contingency Table for the 4 of 5 Criteria to Test Independence

Newfoundland versus United States

Advanced Mathematics

| Course Level | Advanced 3201 | United States |
|---|---|---|
| | # | # |
| Recognition or Below | 12 | 732 |
| Analysis | 26 | 513 |
| Ordering | 48 | 413 |
| Deduction | 34 | 113 |
| Nofit | 16 | 286 |

$\chi^2 = 115.8$ $P < 0.05$

## SUMMARY

The results demonstrate clearly that the majority of level II students have not attained the ordering level on the van Hiele scale and are not ready for deductive reasoning. The advanced students are much better but a large percentage of these students are at too low a van Hiele level to be successful. The level achieved does depend on the course studied, with the advanced group scoring higher in both level II and III. Some gains were made from level II to III, by both groups with the gain being dependent on the course studied. The Newfoundland and United States comparison favored the Newfoundland group, in particular those studying the advanced course.

# Chapter V

## Discussion and Implications

### Summary

The goal of this study was to evaluate the appropriateness of the present Newfoundland geometry curriculum by comparing students' van Hiele levels with that required by the level II and III geometry curriculum. A comparison was also made between the academic and advanced groups in an effort to determine which course was most effective in raising the van Hiele level of the students. Finally, a comparison was made between Newfoundland students and students tested by the CDASSG Project in the United States.

A modified version of the van Hiele Geometry test was administered to a sample of 561 academic and advanced students in early October, 1985 of these, 391 were retested in the Fall of 1986. The posttest sample was only 74 percent of the original number due to student migration, high school drop-outs, changing courses or failure to advance to the next grade level in mathematics.

### Conclusions

The majority of students tested enrolled in Academic Mathematic 2203 and Advanced Mathematics 2201 were not at an appropriate van Hiele level to begin the study of deductive geometry. One year later many students in Academic Mathematics 3203 were still below the level three necessary to begin the study of deductive geometry. The advanced students tested were better prepared for deductive reasoning than those in the academic program but many fell short of the level three needed for deductive reasoning.

Both the academic and advanced mathematics programs were successful in raising the van Hiele level of the students. The distribution of levels from the pretest to posttest were dependent on course; the levels of the advanced students increased to higher levels than those of the academic students. A comparison of students in Newfoundland and the United States students showed a significant difference in the van Hiele level of the two student groups. The academic students in Newfoundland scored slightly higher than their counterparts in the United States and the advanced group scored considerably higher than the American students.

## Discussion

Reeve (1930) was quoted in the introduction of this study: "Informal Geometry represents about all the Geometry that many of their students are capable of understanding" (p. 14). The results of this study support this statement. The majority of Academic 2203 students fell far below the level three needed to begin study of proof, using the 3 of 5 over 65 percent and using the 4 of 5 over 84 percent did not achieve level three. One year later 49.4 percent and 76 percent using the 3 of 5 and 4 of 5 criteria of the academic students were at a level less than 3, an indication of their lack of readiness to study deductive geometry.

A large number of Academic Mathematics 2203 and Academic Mathematics 3203 students, 44.5 percent 31.6 percent respectively were at or below the recognition level when the 4 of 5 criteria was applied. Even an optimistic Dina van Hiele reported that a total of 70 lessons were necessary to raise a student from level one to level three. Because geometry constitutes about 50 percent of the academic program, it is unlikely that these students will achieve the deductive level of thought since most of the year should be spent getting these student to the level three needed to begin the study of deductive materials. The validity of teaching proof in level I and II was questioned because so many students were below the level necessary to begin such work.

Why were the students so poorly prepared for geometry on a higher van Hiele level? Probable conclusions drawn from the research would be that the students have not moved through the levels properly or that the level of the material presented was inappropriate to more the students through the levels. Since students can not attain level n without first having achieved level n-1, the advancement in level stopped. The students were unable to understand the higher level materials because of the absence of the lower skill levels.

The results for the advanced students were better than the academic for both level II and III. In Advanced Mathematics 2201, 62.3 percent and 37.7 percent achieved at least level three using the 3 of 5 and 4 of 5 criteria respectively. In the posttest, 83.8 percent and 60.3 percent, using the 3 of 5 and 4 of 5, achieved at least the third level. However, some students, 22.9 percent of the Advanced Mathematics 2201 and 8.8 percent of the Advanced Mathematics 3201, were at or below the recognition level when the 4 of 5 criteria was applied, numbers that can not be ignored in a proof-orientated program.

The higher levels achieved by the advanced students could be related to the type of student choosing the advanced program, since it was intended for the upper 15 to 25 percent of mathematics students. Another difference in the two streams was the exposure to deductive reasoning, with the advanced students having spent a larger portion of their time on deductive items. A third possible reason was that the advanced program presented material in an order that allowed students to move more quickly through the van Hiele levels without missing any of the lower levels.

The gap between the levels of advanced and academic students could have been larger if schools not offering the advanced program had been removed from the sample. Some of the higher level academic students in these schools would have chosen the advanced course if it was offered, thus further decreasing the percentage of academic students at or above the third van Hiele level.

The majority of Newfoundland students were not at the first van Hiele level, but the number was large enough to lend some support to Wirszup's claim that "the majority of students are at the first level of development in geometry while the course they take demands the fourth level of thought" (cited in Usiskin, 1982, p. 37).

Boone (1984) and Taaffe (1985) concluded that the van Hiele levels of grade 9 and level I students were less than the level three necessary for the introduction of deductive reasoning. The low van Hiele levels caused Boone to question the content of the junior high school geometry program. The course content was modified to downplay the role of deductive proof and include other types of geometry considered lower in terms of the van Hiele level required. The number of students in level II and III who have van Hiele levels at or below recognition reinforced the need for an assessment of the materials presented at the junior high level. Could it be that material presented in the previous grades has not been suited to the students' van Hiele level and has kept them from advancing on the van Hiele scale?

The results of the testing done on the level III students provided information about the students who have almost completed the Newfoundland mathematics program. The students' low van Hiele levels were a result of the geometry taught or not taught, over the last 13 years of school. The low levels of many students showed a program that has failed to meet the needs of a large percentage of students. The idea of what geometry should be taught and how it should be taught were a topic of much discussion in the late 60's and 70's. Given that the geometry program used in Newfoundland was changed to include different types and levels of geometry, can the low van Hiele level be explained? Is the Geometry being taught to all students? Are the ideas presented at the appropriate grade level or the appropriate van Hiele level? The answers to those questions have serious implications for the senior high school program. Academic students study proof in levels I, II and III but the majority are below the van Hiele level three, using the 4 of 5 criteria, and are consequently unable to begin to develop deductive thought patterns. The question of suitability of the present academic course should be considered. One of two paths may be followed, either change the course content removing the higher level items or design a program that raises the levels of the students to a point where they can begin working towards at the deductive level.

The proposed 15 percent practical, 70 academic and 15 percent advanced breakdown may require re-evaluation. Students' van Hiele levels for the academic stream were lower than needed to study deductive reasoning. Some of the students in the academic program may be misplaced in the sections involving deductive proof, but may be able to understand the other topics in the course. A switch to the practical stream by a large number of students is unacceptable, but a rearrangement of some of the deductive materials in the academic stream may be in order. The smaller number of advanced students at low van Hiele levels suggests some form of testing may be appropriate to determine the students best suited for the deductive reasoning in the advanced program.

The distribution of van Hiele levels was compared from pretest to posttest. Students in both the academic and advanced programs had increased their levels over the pretest levels. Some of the gains made can be attributed to the fact that some students at the lower van Hiele levels dropped to the next level of mathematics program in level III, advanced to academic for example, leaving a higher proportion of higher van Hiele level students enrolled in the course for the posttest. Part of the increase may be due to experiences provided in the courses themselves, the academic students study proofs very similar to those studied in level I and look at proof from a coordinate viewpoint, thus providing students a second or third opportunity to see proofs missed at an earlier level. The increase in van Hiele levels for the advanced students may be attributed to the type of proof studied and the variety of deductive experiences presented in the course.

A comparison between Newfoundland students with students having completed a one year course in geometry in the United States showed a significant difference in favour of the Newfoundland students. The difference in the advanced group was expected since these students were drawn from the high achievers whereas the United States study covers a sample of all students enrolled in geometry. The results for the academic group was slightly higher than their counterparts in the United States, an indication of some success in the Newfoundland mathematics program when compared to the United States program. This difference may be due to the time period in which the geometry is presented to students, three years instead of one. The content of the curriculum may account for some of the differences. The inclusion of different topics over the past years may have increased the levels of students compared to that their counterparts in the United States.

The differences between the United States and Newfoundland students were more pronounced when the percentages of students studying geometry were considered. In the United States only 53 percent of all students take geometry in high school where as in Newfoundland 80 to 85 percent of students study either the academic or advanced programs. The students choosing not to take geometry in the United States may be the low van Hiele level students who have experience difficulty at the junior high level. If they had been included in the sample the results favouring Newfoundland may have been even higher.

The results of this study were not unlike those of Usiskin who found a large proportion of students studying deductive Geometry were below the third van Hiele level. These findings verify the properties of the van Hiele levels which predict difficulties in increasing the thought levels unless certain phases were present. These include, inquiry, direct orientation, explicitation, free orientation and integration.

The difficulties in communicating geometry to a student on any given level may result in a failure to increase levels. A mixed class or even an advanced class may have students of all van Hiele levels present. Thus, material presented at level three would not be understood by students whose thought levels were below the level of the presentation. The breakdown in communication was a result of materials in the curriculum not being structured to meet the level of the students. Coxford (1978) believed the level suggested the activity needed to develop the concept (p. 327).

The students in level II and III had higher van Hiele levels than those tested in grade 9 and level I by Boone (1984) and Taaffe (1983). The students tested increased their mean van Hiele levels in each year. An indication that the geometry curriculum has had at least limited success for the academic groups and a higher rate of success for the advanced groups.

Recommendations for Further Study

An analysis of the results of the data collected in this study does not answer all the questions that need to be asked. The discussion of the results indicates a large number of students have not achieved the third van Hiele level. The question of why these students have reached this level has yet to be answered.

It has been shown that many students increase their van Hiele level as they progress through the grades while others remain at or below the recognition level throughout high school. Another question that needs answering is; What are the characteristics of students who improve their van Hiele levels compared to those who do not?

The discussion of the results has indicated a significant difference in the van Hiele levels of students in the advanced program compared to those in the academic program. The question that arises from this is; Are the increases in van Hiele level of the advanced program a result of the curriculum or the type of student enrolled in the program?

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APPENDIX A

Van Hiele Geometry Test

# VAN HIELE GEOMETRY TEST*

## Directions

Do not open this test booklet until you are told to do so.

This test contains 20 questions. It is not expected that you know everything on this test.

When you are told to begin:

1. Read each question carefully.
2. Decide upon the answer you think is correct. There is only one correct answer to each question. Cross out the letter corresponding to your answer on your answer sheet.
3. Use the space provided on the answer sheet for figuring or drawing. Do not mark on this test booklet.
4. If you want to change an answer, completely erase the first answer.
5. You will have 35 minutes for this test.

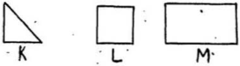
Wait until your teacher says that you may begin.

*This test is based on the work of P.M. van Hiele.

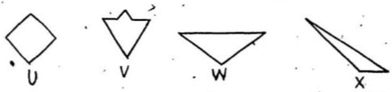
# VAN HIELE GEOMETRY TEST

1. Which of these are squares?

   (A) K only
   (B) L only
   (C) M only
   (D) L and M only
   (E) All are squares.

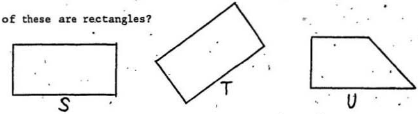



2. Which of these are triangles?



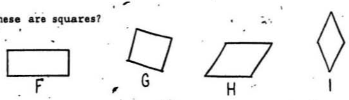

   (A) None of these are triangles.
   (B) V only
   (C) W only
   (D) W and X only
   (E) V and W only

3. Which of these are rectangles?



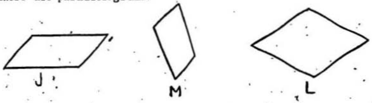

   (A) S only
   (B) T only
   (C) S and T only
   (D) S and U only
   (E) All are rectangles.

4. Which of these are squares?




(A) None of these are squares.
(B) G only
(C) F and G only
(D) G and I only
(E) All are squares.

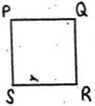
5. Which of these are parallelograms?




(A) J only
(B) L only
(C) J and M only
(D) None of these are parallelograms.

6. PQRS is a square.

Which relationship is true in all squares?



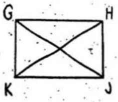

(A) $\overline{PS}$ and $\overline{RS}$ have the same length.

(B) $\overline{QS}$ and $\overline{PR}$ are perpendicular.

(C) $\overline{PS}$ and $\overline{QR}$ are perpendicular.

(D) $\overline{PS}$ and $\overline{QS}$ have the same length.

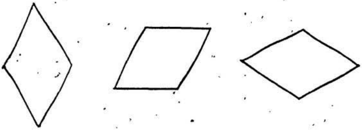
7. In a rectangle GHJK, $\overline{GH}$ and $\overline{HK}$ are the <u>diagonal</u>.



Which of (A) - (D) is not true in every rectangle?

(A) There are four right angles.
(B) There are four sides.
(C) The diagonals have the same length.
(D) The opposite sides have the same length.
(E) All of (A) - (D) are true in every rectangle.

8. A rhombus is a 4 - sided figure with all sides of the same length. Here are three examples.

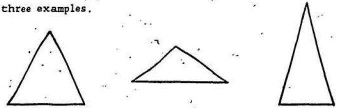


Which of (A) - (D) is not true in every rhombus?

(A) The two diagonals have the same length.
(B) Each diagonal bisects two angles of the rhombus.
(C) The two diagonals are perpendicular.
(D) The opposite angles have the same measure.
(E) All of (A) - (D) are true in every rhombus.

9. An isosceles triangle is a triangle with two sides of equal length.

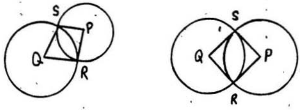
Here are three examples.



Which of (A) - (D) is true in every isosceles triangle?

(A) The three sides must have the same length.
(B) One side must have twice the length of another side.
(C) There must be at least two angles with the same measure.
(D) The three angles must have the same measure.
(E) None of (A) - (D) is true in every isosceles triangle.

10. Two circles with centers P and Q intersect at and S to form a 4-sided figure PRQS. Here are two examples.




Which of (A) - (D) is <u>not</u> always true?

(A) PRQS will have two pairs of sides of equal length.
(B) PRQS will have at least two angles of equal measure.

(C) The lines $\overline{PQ}$ and $\overline{RS}$ will be perpendicular.
(D) Angles P and Q will have the same measure.
(E) All of (A) - (D) are true.

11. Here are two statements,

Statement 1: Figure F is a rectangle.
Statement 2: Figure F is a triangle.

Which is correct?

(A) If 1 is true, then 2 is true.
(B) If 1 is false, then 2 is true.
(C) 1 and 2 cannot both be true.
(D) 1 and 2 cannot be false.
(E) None of (A) - (D) is correct.

12. Here are two statements.

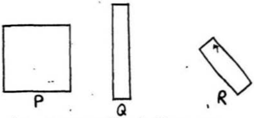
Statement S: $\triangle$ABC has three sides of the same length.
Statement T: IN $\triangle$ ABC, $\angle$B and $\angle$C have the same measure.

Which is correct?

(A) Statements S and T cannot both be true.
(B) If S is true, then T is true.
(C) If T is true, then S is true.
(D) If S is false, then T is false.
(E) None of (A) - (D) is correct.

13. Which of these can be called rectangles?

(A) All can.
(B) Q only
(C) R only
(D) P and Q only
(E) Q and R only



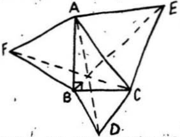

14. Which is true?

(A) All properties of rectangles are properties of all squares.
(B) All properties of squares are properties of all rectangles.
(C) All properties of rectangles are properties of all parallelograms.
(D) All properties of squares are properties of all parallelograms.
(E) None of (A) - (D) is true.

15. What do all rectangles have that some parallelograms do not have?

(A) Opposite sides equal.
(B) Diagonals equal
(C) Opposite sides parallel
(D) Opposite angles equal
(E) None of (A) - (D)

16. Here is a right triangle ABC. Equilateral triangles ACE, ABF, and BCD have been constructed on the sides of ABC.




From this information, one can prove that AD, BE and CF have a point in common. What would this proof tell you

(A) Only in this triangle drawn can we be sure that AD, BE and CF have a point in common.
(B) In some but not all right triangles, AD, BE and CF have a point in common.
(C) In any right triangle, AD, BE and CF have a point in common.
(D) In any triangle, AD, BE and CF have a point in common.
(E) In any equilateral triangle, AD, BE and CF have a point in common.

17. Here are three properties of a figure.
    Property D: It has diagonals of equal length.
    Property S: It is a square.
    Property R: It is a rectangle.

    Which is true?

    (A) D implies S which implies R.
    (B) D implies R which implies S.
    (C) S implies R which implies D.
    (D) R implies D which implies S.
    (E) R implies S which implies D.

18. Here are two statements.

    I. If a figure is a rectangle, its diagonals bisect each other.

    II. If the diagonals of a figure bisect each other, the figure is a rectangle.

    Which is correct:

    (A) To prove I is true, it is enough to prove that II is true.
    (B) To prove II is true, it is enough to prove that I is true.
    (C) To prove II is true, it is enough to find one rectangle whose diagonals bisect each other.
    (D) To prove II is false, it is enough to find one non-rectangle whose diagonals bisect each other.
    (E) None of (A)-(D) is correct.

19. In geometry:

    (A) Every term can be defined and every true statement can be proved true.
    (B) Every term can be defined but it is necessary to assume that certain statements are true.
    (C) Some terms must be left undefined but every true statement can be proved true.
    (D) Some terms must be left undefined and it is necessary to have some statements which are assumed true.
    (E) None of (A)-(D) is correct.

20. Examine these three sentences.

(1) Two lines perpendicular to the same line are parallel.

(2) A line that is perpendicular to one of two parallel lines is perpendicular to the other.

(3) If two lines are equidistant, then they are parallel.

In the figure below, it is given that lines m and p are perpendicular and lines n and p are perpendicular. Which of the above sentences could be the reason that line m is parallel to line n?

(A) (1) only
(B) (2) only
(C) (3) only
(D) Either (1) or (2)
(E) Either (2) or (3)