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THRESHOLD DYNAMICS IN A TIME-DELAYED PERIODIC SIS EPIDEMIC MODEL

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Abstract. The global dynamics of a periodic SIS epidemic model with maturation delay is investigated. We first obtain sufficient conditions for the single population growth equation to admit a globally attractive positive periodic solution. Then we introduce the basic reproduction ratio $\mathcal{R}_0$ for the epidemic model, and show that the disease dies out when $\mathcal{R}_0 < 1$, and the disease remains endemic when $\mathcal{R}_0 > 1$. Numerical simulations are also provided to confirm our analytic results.

1. **Introduction.** Many mathematical models for the spread of infectious diseases are described by autonomous systems of differential equations (see, e.g., [2, 4]). However, certain diseases admit seasonal behavior and it is now well understood that seasonal fluctuations are often the primary factors responsible for recurrent epidemic cycles. Periodic changes in social interactions can alter the contact rate for some directly transmitted contagious infections. For example, in the case of childhood infectious disease, the contact rates vary seasonally according to the school schedule [5]. Fluctuations of birth rates are also evidenced in the works of population dynamics [8, 19]. Periodic vaccination strategies are often used to control diseases [6]. We further refer to two surveys [1, 7] and references therein for seasonal fluctuations in epidemic models. It thus becomes natural to model these diseases by periodically forced nonlinear systems.

A central concept in the study of the spread of communicable diseases is the basic reproduction number, denoted by $\mathcal{R}_0$, which is defined as the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual (see, e.g., [2, 4]). In many cases, one may expect that such a disease can invade the susceptible population if $\mathcal{R}_0 > 1$. Thus, we need to reduce $\mathcal{R}_0$ to be less than 1 in order to eradicate a disease. For a large class of autonomous compartmental epidemic models, the explicit formula for $\mathcal{R}_0$ was obtained in [16]. This work has been extended recently to the periodic case in [17].

The purpose of this paper is to obtain a threshold type result on the global dynamics for a periodic SIS epidemic model with maturation delay. The model is presented in the next section and a single species growth model is analyzed with three types of birth rate functions. In section 4, we introduce the basic reproduction

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ratio and show that it acts as a threshold parameter for the uniform persistence and global extinction of the disease. The last two sections give some numerical simulations and concluding remarks.

2. **The model.**

Many epidemiological models are formulated so that the infectious disease spreads in a population which either is a fixed closed population or has a fixed size with balancing inflows and outflows due to births and deaths or migration. However, it is generally accepted in ecology that the sizes of plant and animal populations are influenced and constrained by foraging, predation, competition and limited resources. In [3], Cooke *et al.* considered the variable population size and derived a time-delayed SIS epidemic model:

$$\begin{cases} S'(t) &= B(N(t-\tau))N(t-\tau)e^{-d_1\tau} - dS(t) - \frac{\beta S(t)I(t)}{N(t)} + \gamma I(t), \\ I'(t) &= \frac{\beta S(t)I(t)}{N(t)} - (d+d_2+\gamma)I(t), \end{cases}$$

where I is the number of the infective population, S is the number of the susceptible population and $N(t) = S(t)+I(t)$. Here $d > 0$ is the death rate constant at the adult stage, $B(N)$ is a birth rate function, τ is the maturation time, $d_2 \geq 0$ is the disease induced death rate, $\gamma > 0$ is the recovery rate ($\frac{1}{\gamma}$ is the average infective time), and d_1 is the death rate constant for the juvenile stage. The standard incidence function is used with $\beta\frac{I}{N}$ giving the average number of adequate contacts with infectives of one susceptible per unit time. Typical examples of birth rate functions $B(N)$ in the biological literature are:

(B1) $B_1(N) = \frac{p}{q+N^n}$, with $p, q, n > 0$ and $\frac{p}{q} > d$.
(B2) $B_2(N) = \frac{A}{N} + c$, with $A > 0$, $d > c > 0$.
(B3) $B_3(N) = be^{-aN}$, with $a > 0$, $b > d$.

Their model was obtained under the following assumptions:

(1) Transmission of disease is assumed to occur due to contact between susceptibles and infectives.
(2) There is no vertical transmission.
(3) The disease confers no immunity, thus upon recovery an infective individual returns to the susceptible class (hence the name SIS model).

This type of model is appropriate for some bacterial infections. For a fatal disease, the recovery rate constant is set to zero, giving an SI model.

Let $B(t, N)$ and $d(t)$, respectively, be the time-dependent birth and death rates of the population at the adult state, and $d_1(t)$ be the death rate of the population at the juvenile stage. Assume that the maturation delay is $\tau > 0$. It then follows that the rate of entry into the adult stage is

$$B(t-\tau, N(t-\tau))N(t-\tau)e^{-\int_{t-\tau}^{t} d_1(s)ds}.$$

Thus, we obtain the following nonautonomous SIS model:

$$\begin{cases} S'(t) = B(t-\tau, N(t-\tau))N(t-\tau)e^{-\int_{t-\tau}^{t} d_1(s)ds} - d(t)S(t) - \dfrac{\beta(t)S(t)I(t)}{N(t)} \\ \qquad + \gamma(t)I(t), \\ I'(t) = \dfrac{\beta(t)S(t)I(t)}{N(t)} - (d(t)+d_2(t)+\gamma(t))I(t), \end{cases} \tag{2.1}$$

where $N(t) = S(t) + I(t)$, $B(t, N)$, $\beta(t)$, $d(t)$, $d_1(t)$, $d_2(t)$ and $\gamma(t)$ are nonnegative. To incorporate seasonal effects, we further assume that all these functions are T-periodic in t for some $T > 0$. It is easy to see that the function

$$\alpha(t) := e^{-\int_{t-\tau}^{t} d_1(s)ds}$$

is also T-periodic in t. Thus, model (2.1) is a periodic time-delayed differential system. We should point out that the model (2.1) with $B(t, N) = \frac{a}{N} + c$ and $d(t)$, $d_1(t)$, $d_2(t)$ and $\gamma(t)$ being constants was studied in [21]. Here we investigate the global dynamics of (2.1) with the general forms of birth rate functions.

We assume that $B(\cdot, \cdot) \in C^1(\mathbb{R}\times(0, +\infty), \mathbb{R}_+)$ and $B(t, N)N$ admits a continuous extension $G(t, N)$ from $\mathbb{R} \times (0, +\infty)$ to $\mathbb{R} \times \mathbb{R}_+$. It then follows that for any $\phi \in C([-\tau, 0], \mathbb{R}^2_+)$, there is a unique local solution $(S(t, \phi), I(t, \phi))$ of system (2.1) with $(S(\theta, \phi), I(\theta, \phi)) = \phi(\theta)$, $\forall\theta \in [-\tau, 0]$ (see, e.g., [10, Theorem 2.3]). Further, we have $(S(t, \phi), I(t, \phi)) \geq 0$ in its maximal interval of existence according to [13, Theorem 5.2.1]. It is also easy to see that if $\phi = (\phi_1, \phi_2) \in C([-\tau, 0], \mathbb{R}^2_+)$ with $\phi_2(0) > 0$, then $I(t, \phi) > 0$ and $S(t, \phi) > 0$ for all $t > 0$ in its maximal interval of existence. For any function $x : [-\tau, \sigma) \to \mathbb{R}^m$, $\sigma > 0$, we define $x_t \in C([-\tau, 0], \mathbb{R}^m)$ by $x_t(\theta) = x(t + \theta)$, $\forall\theta \in [-\tau, 0]$. In what follows, we write $\hat{x}$ for the element of $C([-\tau, 0], \mathbb{R}^m)$ satisfying $\hat{x}(\theta) = x$ for all $\theta \in [-\tau, 0]$.

3. **A single population growth model.**

In this section, we consider the single-species population growth model:

$$N'(t) = \alpha(t)B(t - \tau, N(t - \tau))N(t - \tau) - d(t)N(t) \triangleq F(t, N(t), N(t - \tau)), \quad (3.1)$$

where $\alpha(t) = e^{-\int_{t-\tau}^{t} d_1(s)ds}$. We will establish four sets of sufficient conditions under which system (3.1) admits a globally attractive positive T-periodic solution, and hence, the single population stabilizes eventually at an oscillating state.

For any $\phi \in C([-\tau, 0], \mathbb{R}_+)$, there is a unique local solution $N(t, \phi)$ of (3.1) with $N(\theta, \phi) = \phi(\theta)$, $\forall\theta \in [-\tau, 0]$ (see, e.g., [10, Theorem 2.3]). Moreover, we have $N(t, \phi) \geq 0$ in its maximal interval of existence according to [13, Theorem 5.2.1].

Consider the linear equation with time delay τ:

$$u'(t) = a(t)u(t) + b(t)u(t - \tau), \quad (3.2)$$

where $a(t)$, $b(t)$ are T-periodic and continuous, $b(t) > 0, \forall t \geq 0$.

For any $\phi \in C([-\tau, 0], \mathbb{R})$, let $u(t, \phi)$ be the unique solution of (3.2) satisfying $u_0 = \phi$. Let $\tilde{P}$ be the Poincaré map associated with (3.2) on $C([-\tau, 0], \mathbb{R})$, that is, $\tilde{P}(\phi) = u_T(\phi)$. The following result comes from [18, Proposition 2.1].

Lemma 3.1. *Let $r(\tilde{P})$ be the spectral radius of $\tilde{P}$. Then $r = r(\tilde{P})$ is a positive eigenvalue of $\tilde{P}$ with a positive eigenfunction. Moreover, $u(t) = v_0(t)e^{\frac{t}{T}ln(r)}$ is a solution of (3.2), where $v_0(t)$ is T-periodic and $v_0(t) > 0$, $\forall t \geq 0$. If $\tau = kT$ for some integer $k \geq 0$, then $r - 1$ has the same sign as $\int_0^T(a(t) + b(t))dt$.*

Note that the condition $r(\tilde{P}) < 1$ ($r(\tilde{P}) > 1$) implies that the zero solution of (3.2) is stable (unstable). Thus, Lemma 3.1 implies that in the case where the time delay is an integer multiple of the time period, the stability of zero solution of (3.2) is equivalent to that of zero solution of the linear periodic ordinary differential equation $u'(t) = (a(t) + b(t))u(t)$.

3.1. **A general periodic form of $B_1(N)$.** Assume that

(H1) $B(\cdot,\cdot) \in C^1(\mathbb{R}\times(0,+\infty),\mathbb{R}_+)$ with $\frac{\partial B(t,N)}{\partial N} < 0$, $\forall N \in (0,+\infty)$, $\frac{d(t)}{\alpha(t)} > B(t-\tau,\infty)$; and there exists $G(\cdot,\cdot) \in C(\mathbb{R}\times\mathbb{R}_+,\mathbb{R}_+)$ such that $G(t,N) = B(t,N)N$, $\forall t \in \mathbb{R}$, $N > 0$.

(H2) $G(t,0) = 0$ and r_1=r$(P_1) > 1$, where r(P_1) is the spectral radius of P_1, and P_1 is the Poincaré map of the following linear equation

$$N'(t) = \alpha(t)B(t-\tau,0)N(t-\tau) - d(t)N(t). \tag{3.3}$$

(H3) $\frac{\partial G(t,N)}{\partial N} > 0$, $\forall N \in \mathbb{R}_+$, $t \in \mathbb{R}$.

It then follows that the periodic function $F(t,v_1,v_2)$ has the following properties:

(1) $F(t,0,0) = 0$, $F(t,0,v_2) \ge 0$, $\frac{\partial F(t,v_1,v_2)}{\partial v_2} > 0$, $\forall v_1, v_2 \ge 0$.
(2) F is strictly subhomogeneous, i.e., for any $\lambda \in (0,1)$, $\forall v_1, v_2 > 0$, $F(t,\lambda v_1,\lambda v_2) > \lambda F(t,v_1,v_2)$.
(3) There exists a positive number $h_0 > 0$ such that $F(t,h_0,h_0) \le 0$.

The following result is a straightforward consequence of [18, Theorem 2.1].

Theorem 3.2. *Assume (H1)-(H3) hold. Then equation (3.1) admits a globally attractive positive T-periodic solution $N^*(t)$ in $C([-\tau,0],\mathbb{R}_+)\setminus\{\hat{0}\}$.*

3.2. **A general periodic form of $B_2(N)$.** Assume that

(A1) $B(\cdot,\cdot) \in C^1(\mathbb{R}\times(0,+\infty),\mathbb{R}_+)$ with $\frac{\partial B(t,N)}{\partial N} < 0$, $\forall N > 0$, $t \in \mathbb{R}$, and $\frac{d(t)}{\alpha(t)} > B(t-\tau,\infty)$ for all $t \in \mathbb{R}$.

(A2) There exists $G(\cdot,\cdot) \in C(\mathbb{R}\times\mathbb{R}_+,\mathbb{R}_+)$ such that $G(t,N) = B(t,N)N$, $\forall t \in \mathbb{R}$, $N > 0$, and $G(t,0) > 0$, $\forall t \in \mathbb{R}$.

(A3) $\frac{\partial G(t,N)}{\partial N} > 0$, $\forall N \in \mathbb{R}_+$, $t \in \mathbb{R}$.

Theorem 3.3. *Assume (A1)-(A3) hold. Then equation (3.1) admits a globally attractive positive T-periodic solution $N^*(t)$ in $C([-\tau,0],\mathbb{R}_+)\setminus\{\hat{0}\}$.*

Proof. From (A2), we have $F(t,0,0) > 0$ and there is $h_0 > 0$ such that $F(t,h,h) \le 0$ for all $h > h_0$. It then follows from [13, Remark 5.2.1] that $[\hat{0},\hat{h}]$ is positively invariant. Thus, for any $\phi \ge \hat{0}$, we can find some $h_\phi > h_0$ such that $\phi \le \hat{h}_\phi$, and hence $N(t,\phi)$ exists for all $t \ge 0$. Define the Poincaré map $P_2 : C([-\tau,0],\mathbb{R}_+) \to C([-\tau,0],\mathbb{R}_+)$ by $P_2(\phi) = N_T(\phi)$. Thus, [13, Theorem 5.1.1 and Corollary 5.3.5] imply that P_2 is monotone and $P_2^{n_0}$ is strongly monotone when $n_0T \ge 2\tau$. By the theory of delay differential equation (see, e.g., [10, Theorem 3.6.1]), $P_2^{n_0}$ is compact. Moreover, we note that $F(t,u,v)$ is strictly subhomogeneous in (u,v). Using the similar arguments as in [20, Theorem 3.3], we can deduce that P_2 is strictly subhomogeneous in the sense $P_2(\alpha\phi) > \alpha P_2(\phi)$ for $\phi \gg \hat{0}$ and $0 < \alpha < 1$. Thus, $P_2^{n_0}$ is also strictly subhomogeneous.

Note that $\hat{0} \le P_2(\hat{0})$. We claim $\hat{0} < P_2(\hat{0})$. Suppose not, then $P_2(\hat{0}) = \hat{0}$, hence, $N(T+\theta,\hat{0}) = 0$ for all $\theta \in [-\tau,0]$ and $N'(T,\hat{0}) = 0$. However, $N'(T,\hat{0})$=$G(T-\tau,0)\alpha(t) > 0$, a contradiction. Consequently, $\hat{0} < P_2(\hat{0})$. Thus,

$$\hat{0} < P_2(\hat{0}) \le P_2^2(\hat{0}) \le \cdots \le P_2^{n_0}(\hat{0}) \ll P_2^{n_0+1}(\hat{0}) \le \cdots.$$

Therefore, for any $\phi_1 \in \omega_{n_0}(\hat{0})$, we have $\phi_1 \ge P_2^{n_0+1}(\hat{0}) \gg 0$, where $\omega_{n_0}(\phi)$ denotes the omega-limit set of ϕ under $P_2^{n_0}$. Moreover, $\forall \phi \ge \hat{0}$ and $\forall \psi \in \omega_{n_0}(\phi)$, we have $\psi \ge P_2^{n_0+1}(\hat{0}) \gg 0$ from the monotonicity of $P_2^{n_0}$.

By [19, Theorem 2.3.2] as applied to $P_2^{n_0}$, there exists a $\phi_0 \gg \hat{0}$ with $P_2^{n_0}(\phi_0) = \phi_0$ such that $\phi_0 = \omega_{n_0}(\varphi)$ for all of $\varphi \geq \hat{0}$. Regarding (3.1) as an n_0T-periodic system, we then see that (3.1) admits a globally attractive positive n_0T-periodic solution $N(t, \phi_0)$. It remains to prove that $N(t, \phi_0)$ is T-periodic, that is, ϕ_0 is a fixed point of P_2. Since

$$\hat{0} < P_2(\hat{0}) \leq P_2^2(\hat{0}) \leq \cdots \leq P_2^{n_0}(\hat{0}) \ll P_2^{n_0+1}(\hat{0}) \leq \cdots$$

and $P_2^{nn_0}(\hat{0}) \to \phi_0$ as $n \to \infty$, it easily follows that $P_2^n(\hat{0}) \to \phi_0$ as $n \to \infty$, and hence, ϕ_0 is the fixed point of P_2. Therefore, $N(t, \phi_0)$ is a globally attractive T-periodic solution for (3.1) in $C([-\tau, 0], \mathbb{R}_+) \setminus \{\hat{0}\}$. □

3.3. **A general periodic form of** $B_3(N)$**.** In this subsection, we take $B(t, N) = p(t)e^{-q(t)N}$ and assume that

(S1) $p(t)$, $q(t)$, $d(t)$, $d_1(t)$ are nonnegative and T-periodic in t, and $p(t) > 0$, $q(t) > 0$ for all $t \in \mathbb{R}$;
(S2) r=r$(P_3) > 1$, where r(P_3) is the spectral radius of P_3, and P_3 is the Poincaré map of the following linear equation

$$N'(t) = \alpha(t)p(t-\tau)N(t-\tau) - d(t)N(t). \tag{3.4}$$

Note that

$$\begin{aligned} N'(t) &= \alpha(t)p(t-\tau)e^{-q(t-\tau)N(t-\tau)}N(t-\tau) - d(t)N(t) \\ &\leq \alpha(t)\frac{p(t-\tau)}{q(t-\tau)}e^{-1} - d(t)N(t). \end{aligned}$$

Consider the periodic ordinary differential equation

$$\bar{U}'(t) = \alpha(t)\frac{p(t-\tau)}{q(t-\tau)}e^{-1} - d(t)\bar{U}(t). \tag{3.5}$$

It then follows that equation (3.5) has a unique periodic solution

$$\bar{U}^*(t) = e^{-\int_0^t d(s)ds} \times \left[\int_0^t \alpha(w)\frac{p(w-\tau)}{q(w-\tau)}e^{-1}e^{\int_0^w d(s)ds}dw + \frac{\int_0^T \alpha(w)\frac{p(w-\tau)}{q(w-\tau)}e^{-1}e^{\int_0^w d(s)ds}dw}{e^{\int_0^T d(s)ds} - 1}\right]$$

and $\bar{U}^*(t)$ is globally asymptotically attractive for (3.5) with $\bar{U}(0) \geq 0$. By the comparison theorem, we have $N(t, \phi) \leq \bar{U}(t, \phi(0))$ for all t in its maximal interval of existence, where $\bar{U}(t, \phi(0))$ is the solution of (3.5) with $\bar{U}(0) = \phi(0)$. Since $\lim_{t\to\infty}(\bar{U}(t, \phi(0)) - \bar{U}^*(t)) = 0$, the solution for (3.1) exists globally, and the periodic solution semiflow for (3.1) is point dissipative.

In addition to (S1)-(S2), we further assume that

(S3) $\bar{U}^*(t) \leq \frac{1}{q(t)}$.

Then the following result holds.

Theorem 3.4. *Assume (S1)-(S3) hold. Then (3.1) admits a globally attractive positive T-periodic solution $N^*(t)$ in $C([-\tau, 0], \mathbb{R}_+) \setminus \{\hat{0}\}$.*

Proof. Let P_4 is the Poincaré map associated with (3.1). It then follows that $\omega(\psi) \subseteq [\hat{0}, \bar{U}_0^*]$ for any $\psi \in C([-\tau, 0], \mathbb{R}_+)$, where $\omega(\psi)$ is the omega limit set of $\psi \geq \hat{0}$ for P_4 and $\bar{U}_0^* \in C([-\tau, 0], \mathbb{R}_+)$ with $\bar{U}_0^*(\theta) = \bar{U}^*(\theta)$, $\forall \theta \in [-\tau, 0]$. Furthermore, for each $\phi \in [\hat{0}, \bar{U}_0^*]$, we have $\phi(0) \leq \bar{U}^*(0)$, and hence, $N(t,\phi) \leq \bar{U}^*(t)$ for all $t \geq 0$, which implies that $[\hat{0}, \bar{U}_0^*]$ is positively invariant for P_4.

For a positive $\varepsilon > 0$, let r_ε be the spectral radius of

$$v'(t) = (\alpha(t)p(t-\tau) - \varepsilon_0)v(t-\tau) - d(t)v(t). \tag{3.6}$$

Since $r(P_3) > 1$, we can choose ε_0 small enough such that $r_{\varepsilon_0} > 1$ and $\alpha(t)p(t-\tau) - \varepsilon_0 > 0$. From Lemma 3.1, (3.6) admits a solution $v^*(t) = e^{\frac{t}{T} ln r_{\varepsilon_0}} u_0(t)$, where $u_0(t)$ is positive and T-periodic. Hence, $v^*(t) \to \infty$.

For $\varepsilon_0 > 0$, we choose a sufficiently small positive number δ_0, such that

$$\alpha(t)p(t-\tau)e^{-q(t-\tau)N} \geq \alpha(t)p(t-\tau) - \varepsilon_0,\ \forall t \geq 0,\ 0 \leq N < \delta_0.$$

Since $\lim\limits_{\phi\to 0} N_t(\phi) \to 0$ uniformly for $t \in [0,T]$, there exists $\delta_1 > 0$ such that

$$\|N_t(\phi)\| \leq \delta_0,\ \forall t \in [0,T],\ \|\phi\| \leq \delta_1.$$

We first claim that $\limsup\limits_{n\to\infty} \|P_4^n \psi\| \geq \delta_1$ for all of $\psi \in [\hat{0}, \bar{U}_0^*] \setminus \{\hat{0}\}$. Suppose not, and $\limsup\limits_{n\to\infty} \|P_4^n \phi\| < \delta_1$ for some $\phi \in [\hat{0}, \bar{U}_0^*] \setminus \{\hat{0}\}$, then there exists an integer $N_1 \geq 1$ such that $\|P_4^n \phi\| < \delta_1, \forall n \geq N_1$. For any $t - \tau \geq N_1 T$, we have $t = nT + t'$ with $n \geq N_1$, $t' \in [0,T]$ and $\|N_t(\phi)\| = \|N_{t'}(P_4^n \phi)\| \leq \delta_0$. Then

$$\begin{aligned} N'(t,\phi) &\geq \alpha(t)p(t-\tau)e^{-q(t-\tau)N(t-\tau,\phi)}N(t-\tau,\phi) - d(t)N(t,\phi) \\ &\geq (\alpha(t)p(t-\tau) - \varepsilon_0)N(t-\tau,\phi) - d(t)N(t,\phi). \end{aligned}$$

Since $N(t,\phi) > 0$, $\forall t > \tau$, we can choose a small number $k > 0$ such that $N(t) > kv^*(t)$, $\forall t \in [N_2 T, N_2 T + \tau]$, where $N_2 > N_1$ and $N_2 T > \tau$. By the comparison theorem (see e.g. [13, Theorem 5.1.1]), we have $N(t,\phi) > kN^*(t)$, $\forall t \geq N_2 T + \tau$. Thus, $\lim\limits_{t\to\infty} N(t,\phi) = \infty$, a contradiction.

Let $X = [\hat{0}, \bar{U}_0^*]$ and $X_0 = \{\phi \in X : \phi(0) > 0\}$, define $\partial X_0 = X \setminus X_0$. Note that P_4 is point dissipative, asymptotically smooth and the orbits of bounded sets are bounded. It then follows from [9, Theorem 2.4.6] that P_4 admits a global attractor $A \in X$. It is clear that $M_\partial := \{\phi \in \partial X_0 : P_4^n(\phi) \in \partial X_0, \forall n \geq 0\} = \{\hat{0}\}$ and $\Omega(M_\partial) := \cup_{\phi\in M_\partial} \omega(\phi) = \{\hat{0}\}$, where $\omega(\phi)$ is the ω-limit set of ϕ with respect to P_4. In view of the above claim, $\{\hat{0}\}$ is isolated in X and $W^s(\hat{0}) \cap X_0 = \varnothing$ where $W^s(\hat{0})$ is the stable set of $\hat{0}$ for P_4. Moreover, for each $\psi \in \partial X_0$ and $\psi \neq \hat{0}$, there exists a $t_0 \in [0,\tau]$ such that $N(t_0,\psi) > 0$, where $N(t,\psi)$ is the solution of equation (3.1) through ψ. Hence, $N(t,\psi) > 0$ for all $t \geq t_0$, which implies that $P_4^n(\psi) \in X_0$ for $nT > \tau$. Therefore, $\omega(\phi) \neq \{\hat{0}\}$ and there is no cycle in ∂X_0 from $\hat{0}$ to $\hat{0}$. By the acyclicity theorem on uniform persistence for maps (see [19, Theorem 1.3.1 and Lemma 1.3.1]), it follows that $P_4 : C([-\tau,0],\mathbb{R}_+) \to C([-\tau,0],\mathbb{R}_+)$ is uniformly persistent with respect to X_0. Note that P_4 is an α-contraction for an equivalent norm in $C([-\tau,0],\mathbb{R}_+)$ (see [9, Theorem 4.1.1]). Moreover, P_4 is point dissipative and P_4^n is compact for $nT > \tau$. Thus, [12, Theorem 4.5] implies that $P_4 : X_0 \to X_0$ admits a global attractor A_0 in X_0. Since for every $\phi \in A_0$, $N(t,\phi) > 0$ for all $t \geq 0$, it follows from the invariance of A_0 for P_4 that $A_0 \subset int(C([-\tau,0],\mathbb{R}_+))$. Consequently, for any $\psi \in X \setminus \{\hat{0}\}$, we have $\omega(\psi) \subset A_0 \subset int(C([-\tau,0],\mathbb{R}_+))$.

Define

$$E(t,u,v) := \alpha(t)p(t-\tau)ve^{-q(t-\tau)v} - d(t)u.$$

For any $\phi \in [\hat{0}, \bar{U}_0^*]$, we have

$$\begin{aligned}&\frac{\partial E}{\partial v}(t, N(t), N(t-\tau,\phi))\\ &= (1 - q(t-\tau)N(t-\tau,\phi))\alpha(t)p(t-\tau)e^{-q(t-\tau)N(t-\tau,\phi)}\\ &> (1 - q(t-\tau)U^*(t-\tau))\alpha(t)p(t-\tau)e^{-q(t-\tau)N(t-\tau,\phi)} \geq 0.\end{aligned}$$

It then follows that $P_4^{n_0}$ is strongly monotone in $[\hat{0}, \bar{U}_0^*]$ when $n_0 T \geq 2\tau$. Note that $E(t, N(t), N(t-\tau))$ is strictly subhomogeneous. Using the same argument as in [20, Theorem 3.3], we can deduce that P_4 is strictly subhomogeneous. Thus, $P_4^{n_0}$ is also strictly subhomogeneous. It then follows from [19, Theorem 2.3.2], as applied to $P_4^{n_0} : U = [\hat{0}, \bar{U}_0^*] \to U$, that $P_4^{n_0}$ has a fixed point $\phi_0 \gg 0$ in $[\hat{0}, \bar{U}_0^*]$ such that every nonempty compact invariant set of $P_4^{n_0}$ is in $int(C([-\tau, 0], \mathbb{R}_+))$. Since for each $\psi \in C([-\tau, 0], \mathbb{R}_+) \setminus \{\hat{0}\}$, $\omega(\psi)$ is a nonempty compact invariant set of $P_4^{n_0}$ in $[\hat{0}, \bar{U}_0^*]$ and $\omega(\psi) \subset int(C([-\tau, 0], \mathbb{R}_+))$, it follows that $\omega(\psi) = \phi_0$, and hence, $P_4(\phi_0) = \phi_0$. Therefore, $N(t, \phi_0)$ is a globally attractive T-periodic solution for (3.1) in $C([-\tau, 0], \mathbb{R}_+) \setminus \{\hat{0}\}$. □

Assume that

(S3)$'$ $\max\limits_{0\leq t\leq T}\{\alpha(t)p(t-\tau)e^{-2}\} < \min\limits_{0\leq t\leq T}\{\frac{1}{\tau e^{1+\tau d(t)}}\}$.

Then, we have the following result.

Theorem 3.5. *Assume that (S1),(S2) and (S3)$'$ hold. Then (3.1) admits a globally attractive positive T-periodic solution in $C([-\tau, 0], \mathbb{R}_+) \setminus \{\hat{0}\}$.*

Proof. Note that

$$E(t, u_2, v_2) - E(t, u_1, v_1) \geq -d(t)(u_2 - u_1) - p(t-\tau)e^{-2}\alpha(t)(v_2 - v_1).$$

We use the exponential ordering introduced in [14] to prove this theorem. For some $\mu \geq 0$, we define

$$\tilde{K_\mu} = \{\phi \in C([-\tau, 0], \mathbb{R}_+) : \phi \geq 0 \text{ and } \phi(\mathrm{s})\mathrm{e}^{\mu\mathrm{s}} \text{ is nondecreasing on } [-\tau, 0]\},$$

and $K_\mu = \tilde{K_\mu} \cap C_L$ where C_L is the Banach space of Lipschitz functions on $[-\tau, 0]$ with the norm $\|\phi\|_{Lip} := |\phi| + \sup\{|\frac{\phi(s)-\phi(t)}{s-t}| : s \neq t, s, t \in [-\tau, 0]\}$.

Denote the exponential ordering defined by K_μ as $\leq_\mu$. Then if $\phi <_\mu \psi$, we have

$$e^{-\mu\tau}[\psi(-\tau) - \phi(-\tau)] \leq \psi(0) - \phi(0), \quad i.e., \quad \psi(-\tau) - \phi(-\tau) \leq e^{\mu\tau}[\psi(0) - \phi(0)].$$

Therefore,

$$\begin{aligned}&\mu(\psi(0) - \phi(0)) + E(t, N(t,\psi), N(t-\tau,\psi)) - E(t, N(t,\phi), N(t-\tau,\phi))\\ &= \mu(\psi(0) - \phi(0)) + E(t, \psi(0), \psi(-\tau)) - E(t, \phi(0), \phi(-\tau))\\ &\geq \mu(\psi(0) - \phi(0)) - d(t)(\psi(0) - \phi(0)) - \alpha(t)p(t-\tau)e^{-2}(\psi(-\tau) - \phi(-\tau))\\ &\geq [\mu - d(t) - \alpha(t)p(t-\tau)e^{-2}e^{\mu\tau}](\psi(0) - \phi(0)).\end{aligned}$$

Since

$$\max_{0\leq t\leq T}\{\alpha(t)p(t-\tau)e^{-2}\} < \min_{0\leq t\leq T}\{\frac{1}{\tau e^{1+\tau d(t)}}\} \quad \text{and} \quad \psi(0) - \phi(0) > 0,$$

there is some $\mu > 0$ such that

$$\mu - d(t) - \alpha(t)p(t-\tau)e^{-2}e^{\mu\tau} > 0,$$

and hence,

$$\mu(\psi(0)-\phi(0))+E(t,\psi(0),\psi(-\tau))-E(t,\phi(0),\phi(-\tau))>0.$$

For every $\phi\geq\hat{0}$, we have $N(t,\phi)\geq 0$ and there exists $M_\phi>0$ such that $\phi\ll_\mu \hat{M}_\phi$ and $E(t,M_\phi,M_\phi)<0$. Thus $N(t,\phi)\leq M_\phi$, $N(t,\phi)$ exists for all $t\geq 0$. By [13, Theorem 6.2.3], $P_5^{n_0}$ is strongly monotone in the ordered space (C_L,K_μ) for $n_0T\geq\tau$, where P_5 is the Poincaré map of (3.1).

If $\phi\gg_\mu 0$ in K_μ, then $N(t,\phi)>0$ for all $t>-\tau$. For $0<\lambda<1$, let $W(t)=N(t,\lambda\phi)-\lambda N(t,\phi)$, then W(0)=0. Since

$$\begin{aligned} W'(0) &= N'(0,\lambda\phi)-\lambda N'(0,\phi)\\ &= \alpha(0)p(-\tau)e^{-q(-\tau)\lambda\phi(-\tau)}\lambda\phi(-\tau)-\lambda\alpha(0)p(-\tau)e^{-q(-\tau)\phi(-\tau)}\phi(-\tau)>0, \end{aligned}$$

we have $W(t)>0$ for all sufficiently small $t>0$. We further claim $W(t)>0$ for all $t>0$. Suppose not. Then there is $t_0>0$ such that $W(t_0)=0$, $W(t)>0$ for $t<t_0$, and $\left.\frac{dW(t)}{dt}\right|_{t_0}\leq 0$. Since $\lambda\phi\ll_\mu\phi$, $N(t_0-\tau,\lambda\phi)<N(t_0-\tau,\phi)$. Then we have

$$\begin{aligned} &\left.\frac{dW(t)}{dt}\right|_{t_0}=E(t_0,N(t_0,\lambda\phi),N(t_0-\tau,\lambda\phi))-\lambda E(t_0,N(t_0,\phi),N(t_0-\tau,\phi))\\ &=\alpha(t_0)p(t_0-\tau)e^{-q(t_0-\tau)N(t_0-\tau,\lambda\phi)}N(t_0-\tau,\lambda\phi)-d(t_0)N(t_0,\lambda\phi)\\ &\quad-[\alpha(t_0)p(t_0-\tau)e^{-q(t_0-\tau)N(t_0-\tau,\phi)}\lambda N(t_0-\tau,\phi)-\lambda d(t_0)N(t_0,\phi)]\\ &>\alpha(t_0)p(t_0-\tau)e^{-q(t_0-\tau)N(t_0-\tau,\lambda\phi)}\lambda N(t_0-\tau,\phi)\\ &\quad-\alpha(t_0)p(t_0-\tau)e^{-q(t_0-\tau)N(t_0-\tau,\phi)}\lambda N(t_0-\tau,\phi)\\ &=\alpha(t_0)p(t_0-\tau)[e^{-q(t_0-\tau)N(t_0-\tau,\lambda\phi)}-e^{-q(t_0-\tau)N(t_0-\tau,\phi)}]\lambda N(t_0-\tau,\phi)>0, \end{aligned}$$

a contradiction. This proves that $W(t)>0$ for all $t>0$.

For every $\phi\gg_\mu 0$, let $Z(t)=[N(t,\lambda\phi)-\lambda N(t,\phi)]'+\mu[N(t,\lambda\phi)-\lambda N(t,\phi)]$. Then $Z(0)=W'(0)>0$, hence for sufficiently small $t>0$, $Z(t)>0$. We claim that $Z(t)>0$ for all $t>0$. Suppose not. Thus, there is $t_0>0$ such that $Z(t_0)=0$ and $Z(t)>0$ for $t<t_0$. It then follows that

$$\begin{aligned} &Z(t_0)=\alpha(t_0)p(t_0-\tau)e^{-q(t_0-\tau)N(t_0-\tau,\lambda\phi)}N(t_0-\tau,\lambda\phi)+\mu[N(t_0,\lambda\phi)-\lambda N(t_0,\phi)]\\ &\quad-[\lambda\alpha(t_0)p(t_0-\tau)e^{-q(t_0-\tau)N(t_0-\tau,\phi)}N(t_0-\tau,\phi)-\lambda d(t_0)N(t_0,\phi)]-d(t_0)N(t_0,\lambda\phi)\\ &>\alpha(t_0)p(t_0-\tau)e^{-q(t_0-\tau)N(t_0-\tau,\lambda\phi)}N(t_0-\tau,\lambda\phi)+[\mu-d(t_0)][N(t_0,\lambda\phi)-\lambda N(t_0,\phi)]\\ &\quad-\alpha(t_0)p(t_0-\tau)e^{-q(t_0-\tau)\lambda N(t_0-\tau,\phi)}\lambda N(t_0-\tau,\phi)\\ &\geq-\alpha(t_0)p(t_0-\tau)e^{-2}[N(t_0-\tau,\lambda\phi)-\lambda N(t_0-\tau,\phi)]\\ &\quad+[\mu-d(t_0)][N(t_0,\lambda\phi)-\lambda N(t_0,\phi)]. \end{aligned}$$

Since $Z(t)>0$ for all $t<t_0$, we have $N(t_0-\tau,\lambda\phi)-\lambda N(t_0-\tau,\phi)\leq e^{\mu\tau}\,[N(t_0,\lambda\phi)-\lambda N(t_0,\phi)]$, and hence

$$Z(t_0)>[-\alpha(t_0)p(t_0-\tau)e^{-2}e^{\mu t}+\mu-d(t_0)][N(t_0,\lambda\phi)-\lambda N(t_0,\phi)]>0,$$

a contradiction. Thus, $Z(t)>0$ for all $t>0$. It then follows from [13, Theorem 6.2.3] that $N_t(\lambda\phi)\gg_\mu\lambda N_t(\phi)$ for $t>\tau$ and $P_5^{n_0}(\alpha\phi)\gg_\mu\alpha P_5^{n_0}(\phi)$ in K_μ for $n_0T>\tau$.

Since for every $\phi\in C([-\tau,0],\mathbb{R}_+)\setminus\{\hat{0}\}$ and $t>0$, we have

$$\begin{aligned} &[N(t,\phi)]'+\mu N(t,\phi)\\ &=\alpha(t)p(t-\tau)e^{-q(t-\tau)N(t-\tau,\phi)}N(t-\tau,\phi)-d(t)N(t,\phi)+\mu N(t,\phi)\\ &>[\mu-d(t)]N(t,\phi)\geq 0, \end{aligned}$$

and hence, $P_5^{n_0}(\phi) \in int(K_\mu)$ for $n_0 T > \tau$. By using $P_5^{n_0}(\phi)$ if necessary, we may therefore assume that $\phi \in int(K_\mu)$ to study the asymptotic behavior of $\phi > 0$ under $P_5^{n_0}$.

For any $\beta \geq 1$, choose $V_\beta = [\hat{0}, \beta \hat{h_0}]_{K_\mu}$ where h_0 is determined such that $p(t-\tau)e^{-q(t-\tau)h}\alpha(t) < d(t)$ always holds for all $t \geq 0$ and $h > h_0$. Then V_β is positively invariant. First note that when $n_0 T > \tau$, $P_5^{n_0}$ is order-compact in the sense that $P_5^{n_0}([u, v]_{K_\mu})$ is precompact for all of $u <_{K_\mu} v$. Moreover, $P_5^{n_0}$ is strictly subhomogeneous and strongly monotone with respect to the exponential ordering.

By the continuity and differentiability of solutions with respect to initial values, it follows that the P_5 is differentiable at zero, and $DP_5(0) = P_3$, where P_3 is the Poincaré map of the linear equation of (3.4). Clearly, $P_3^{n_0}$ is compact. Moreover, $P_3^{n_0}$ is strongly positive for the exponential ordering K_μ. Furthermore, $D(P_5^{n_0}(\hat{0})) = (DP_5(\hat{0}))^{n_0}$ and $r\{D(P_5^{n_0}(\hat{0}))\} = r\{(DP_5(\hat{0}))\}^{n_0} = [r(P_3)]^{n_0}$. By [19, Theorem 2.3.4], $P_5^{n_0}$ has a unique positive fixed point ϕ_0 in V_β, and ϕ_0 is globally asymptotically stable with respect to $V_\beta \setminus \{\hat{0}\}$. This implies that $\omega_{n_0}(\phi) = \phi_0$ for all $\phi \in V_\beta$, where $\omega_{n_0}(\phi)$ is the ω-limit set of ϕ associated with $P_5^{n_0}$.

By the arbitrariness of β, it then follows that (3.1) admits a globally attractive, positive $n_0 T$-periodic solution $N(t, \phi_0)$ in $C([-\tau, 0], \mathbb{R}_+) \setminus \{\hat{0}\}$. It remains to prove that $N(t, \phi_0)$ is also T-periodic. For $\phi > \hat{0}$, since $P_5^{nn_0}(\phi) \to \phi_0$ as $n \to \infty$, it then follows that $P_5(P_5^{nn_0}(\phi)) \to P_5(\phi_0)$ as $n \to \infty$. On the other hand, $P_5(P_5^{nn_0}(\phi)) = P_5^{nn_0}(P_5(\phi)) \to \phi_0$ as $n \to \infty$. Thus, $P_5(\phi_0) = \phi_0$, and $N(t, \phi_0)$ is a globally attractive T-periodic solution for (3.1) in $C([-\tau, 0], \mathbb{R}_+) \setminus \{\hat{0}\}$. ☐

4. **Threshold dynamics.** We now assume that a disease is invading the population, and the population is divided into susceptible and infective classes. The disease transmission is modeled by system (2.1). In this section, we will study the global dynamics of system (2.1). Let

$$M := C([-\tau, 0], \mathbb{R}_+^2),\ M_0 := \{(\phi_1, \phi_2) \in M : \phi_2(0) > 0\} \text{ and } \partial M_0 := M \backslash M_0.$$

Clearly, M_0 is an open set relative to M. Note that $(N^*(t), 0)$ is the disease-free periodic solution of (2.1). By linearizing (2.1) at $(N^*(t), 0)$, we obtain the following equation for the infective population variable I:

$$I'(t) = \beta(t)I(t) - (d(t) + d_2(t) + \gamma(t))I(t). \tag{4.1}$$

Let C_T be the ordered Banach space of all T-periodic functions from $\mathbb{R}$ to $\mathbb{R}$, which is equipped with the maximum norm $\|\cdot\|$ and the positive cone $C_T^+ := \{\phi \in C_T : \phi(t) \geq 0,\ \forall t \in \mathbb{R}\}$. According to the theory developed in [17] with $F(t) = \beta(t)$ and $V(t) = d(t) + d_2(t) + \gamma(t)$, we define the next infection operator $L : C_T \to C_T$ by

$$(L\phi)(t) = \int_0^\infty Y(t, t-a)\beta(t-a)\phi(t-a)da, \quad \forall t \in \mathbb{R}, \quad \phi \in C_T,$$

where $Y(t, s) = e^{-\int_s^t V(u)du} = e^{-\int_s^t (d(u)+d_2(u)+\gamma(u))du}$, $t \geq s$. Then the basic reproduction ratio is defined as $\mathcal{R}_0 := \rho(L)$, the spectral radius of L. By [17, Lemma 2.2], it follows that

$$\mathcal{R}_0 = \frac{\int_0^T \beta(t)dt}{\int_0^T (d(t) + d_2(t) + \gamma(t))dt}.$$

Note that in section 3, we have obtained four sets of sufficient conditions for system (3.1) to have a globally attractive positive T-periodic solution $N^*(t)$ (see

Theorems 3.1-3.4). We are now in a position to prove the threshold type result on the global dynamics of (2.1) in terms of $\mathcal{R}_0$.

Theorem 4.1. *Assume that (3.1) has a globally attractive positive T-periodic solution $N^*(t)$ in $C([-\tau,0],R_+)\setminus\{\hat{0}\}$, and that there is an L such that $B(t-\tau,N)\alpha(t) < d(t)$, $\forall N > L$, $t > 0$. Let $G(t,N) = B(t,N)N$ satisfy one of the following conditions:*

(C1) $G(t,0)\equiv 0$ and $r = r(\check{P}) > 1$, where $r(\check{P})$ is the spectral radius of $\check{P}$ and $\check{P}$ is the Poincaré map of the following linear equation:

$$N'(t) = \alpha(t)B(t-\tau,0)N(t-\tau) - (d(t)+d_2(t))N(t).$$

(C2) $G(t,0) > 0$ for all $t \geq 0$.

Then the following statements are valid:

(a) If $\mathcal{R}_0 < 1$, then any solution $(S(t,\phi),I(t,\phi))$ of system (2.1) with $\phi \in M_0$ satisfies $\lim_{t\to\infty}(S(t,\phi)-N^*(t)) = 0$ and $\lim_{t\to\infty} I(t,\phi) = 0$.

(b) If $\mathcal{R}_0 > 1$, system (2.1) has a positive T-periodic solution in M_0, and there is an $\eta > 0$ such that any solution $(S(t,\phi),I(t,\phi))$ of system (2.1) with $\phi \in M_0$ satisfies $\liminf_{t\to\infty} S(t,\phi) \geq \eta$ and $\liminf_{t\to\infty} I(t,\phi) \geq \eta$.

Proof. Let $(S(t,\phi),I(t,\phi))$ be the unique solution of (2.1) with $(S(\theta,\phi),I(\theta,\phi)) = \phi(\theta)$, $\forall\theta \in [-\tau,0]$. Since $N(t,\phi) = S(t,\phi)+I(t,\phi) \geq 0$ in the maximal interval of existence, N(t) satisfies the differential inequality

$$N'(t) \leq \alpha(t)B(t-\tau,N(t-\tau))N(t-\tau) - d(t)N(t).$$

For $\phi \in M$, there is a $M_\phi > L$ and $\hat{M}_\phi > \phi$ such that $B(t-\tau,M_\phi)\alpha(t) \leq d(t)$. By [13, Theorem 5.2.1], $N(t,\phi)$ is uniformly bounded. Since $S(t,\phi) \leq N(t,\phi)$ and $I(t,\phi) \leq N(t,\phi)$, it follows that each solution $(S(t,\phi),I(t,\phi))$ exists globally on $[0,\infty)$, and solutions of (2.1) is uniformly bounded in M. Define $\Phi(t)\phi = (S_t(\phi),I_t(\phi))$, $t \geq 0$, $\phi \in M$. Then $\Phi(t)$ is a T-periodic semiflow on M. We have following claims:

Claim 1. There is some $\delta_1 > 0$ such that $\limsup_{n\to\infty}\|\Phi(nT)\phi\| \geq \delta_1$ for all $\phi \in M_0$.

In the case where (C1) holds, for a positive $\varepsilon > 0$, let r_ε be the spectral radius of

$$u'(t) = (\alpha(t)B(t-\tau,0^+)-\varepsilon)u(t-\tau) - (d(t)+d_2(t))u(t). \tag{4.2}$$

Since r$(\check{P}) > 1$, we can choose ε small enough such that $r_\varepsilon > 1$ and $B(t,0^+)-\varepsilon > 0$ for all $t \geq 0$. From Lemma 3.1, (4.2) admits a solution $u^*(t) = e^{\frac{t}{T} ln r_\varepsilon}u_0(t)$, where $u_0(t)$ is positive and T-periodic. Hence $u^*(t) \to \infty$.

For $\varepsilon > 0$, we can choose a sufficiently small positive number δ_0, such that

$$\alpha(t)B(t-\tau,N) \geq \alpha(t)B(t-\tau,0^+) - \varepsilon,\ \forall t \geq 0,\ 0 \leq N < \delta_0.$$

Since $\lim_{\phi\to 0} N_t(\phi) \to 0$ uniformly for $t \in [0,T]$, there exists $\delta_1 > 0$ such that

$$\|N_t(\phi)\| \leq \delta_0,\ \forall t \in [0,T],\ \|\phi\| \leq \delta_1.$$

Suppose, by contradiction, that $\limsup_{n\to\infty}\|\Phi(nT)\phi\| < \delta_1$ for some $\phi \in M_0$. Then there exists an integer $N_1 \geq 1$ such that $\|\Phi(nT)\phi\| < \delta_1$, $\forall n \geq N_1$. For any $t-\tau \geq N_1T$, we have $t = nT+t'$ with $n \geq N_1$, $t' \in [0,T]$ and $\|\Phi(t)\phi\| = \|\Phi(t')\Phi(nT)\phi\| \leq \delta_0$. Then,

$$\begin{aligned} N'(t) &\geq \alpha(t)B(t-\tau,N(t-\tau))N(t-\tau) - (d(t)+d_2(t))N(t) \\ &\geq (\alpha(t)B(t-\tau,0^+)-\varepsilon)N(t-\tau) - (d(t)+d_2(t))N(t). \end{aligned}$$

Since $N(t,\phi) = S(t,\phi) + I(t,\phi) > 0$, $\forall t > 0$, $\forall \phi \in M_0$, we can choose a small number $k > 0$ such that $N(t,\phi) > ku^*(t), \forall t \in [N_1T, N_1T+\tau]$. By the comparison theorem [13, Theorem 5.1.1], we have $N(t,\phi) > ku^*(t), \forall t \geq N_1T$, and hence, $\lim_{t\to\infty} N(t,\phi) = \infty$, a contradiction to the uniform boundedness of $N(t,\phi)$.

In the case where (C2) holds, we can choose ε small enough such that

$$\min_{t\geq 0}\{\alpha(t)B(t-\tau,0^+) - \varepsilon\} > \max_{t\geq 0}\{d(t)+d_2(t)\}.$$

For $\varepsilon > 0$, we can choose a sufficiently small positive number δ_0, such that

$$\alpha(t)B(t-\tau,N) \geq \alpha(t)B(t-\tau,0^+) - \varepsilon,\ \forall t \geq 0,\ 0 \leq N < \delta_0.$$

Since $\lim_{\phi\to 0} N_t(\phi) \to 0$ uniformly for $t \in [0,T]$, there exists $\delta_1 > 0$ such that

$$\|N_t(\phi)\| \leq \delta_0,\ \forall t \in [0,T],\ \|\phi\| \leq \delta_1.$$

Suppose, by contradiction, that $\limsup_{n\to\infty} \|\Phi(nT)\phi\| < \delta_1$ for some $\phi \in M_0$. Then there exists an integer $N_1 \geq 1$ such that $\|\Phi(nT)\phi\| < \delta_1$, $\forall n \geq N_1$. For any $t - \tau \geq N_1T$, we have $t = nT + t'$ with $n \geq N_1$, $t' \in [0,T]$ and $\|\Phi(t)\phi\| = \|\Phi(t')\Phi(nT)\phi\| \leq \delta_0$. Thus

$$\begin{aligned}
N'(t) &\geq \alpha(t)B(t-\tau,N(t-\tau))N(t-\tau) - (d(t)+d_2(t))N(t)\\
&\geq (\alpha(t)B(t-\tau,0^+) - \varepsilon)N(t-\tau) - (d(t)+d_2(t))N(t)\\
&> \min_{t\geq 0}\{\alpha(t)B(t-\tau,0^+) - \varepsilon\}N(t-\tau) - \max_{t\geq 0}\{d(t)+d_2(t)\}N(t).
\end{aligned}$$

Since

$$\min_{t\geq 0}\{\alpha(t)B(t-\tau,0^+) - \varepsilon\} > \max_{t\geq 0}\{d(t)+d_2(t)\},$$

it follows from [13, Theorem 5.1.1] that there is a solution $u^*(t) = e^{st}u$ with $s > 0$ and $u > 0$ for the following equation:

$$u(t) = \min_{t\geq 0}\{\alpha(t)B(t-\tau,0^+) - \varepsilon\}u(t-\tau) - \max_{t\geq 0}\{d(t)+d_2(t)\}u(t). \tag{4.3}$$

Hence, $u^*(t) \to \infty$ as $t \to \infty$. Since $N(t,\phi) = S(t,\phi) + I(t,\phi) > 0$, $\forall t > 0$, $\phi \in M_0$, we can choose a small number $k > 0$ such that $N(t,\phi) > ku^*(t)$, $\forall t \in [N_1T, N_1T+\tau]$. By the comparison theorem [13, Theorem 5.1.1], we have $N(t,\phi) > ku^*(t)$, $\forall t \geq N_1T + \tau$. Thus $\lim_{t\to\infty} N(t,\phi) = \infty$, also a contradiction. This completes the proof of claim 1.

In the case where $\mathcal{R}_0 < 1$, we have $\int_0^T \beta(t)dt < \int_0^T (d(t)+d_2(t)+\gamma(t))dt$. If $I(0) > 0$, then $N(t) \geq I(t) > 0$, $\forall t \geq 0$ and hence, we have

$$I'(t) \leq (\beta(t) - (d(t)+d_2(t)+\gamma(t)))I(t),\ \forall t \geq 0.$$

Then

$$I(t) \leq I(0)e^{\int_0^t \beta(s)-(d(s)+d_2(s)+\gamma(s))ds}\quad \forall t \geq 0,$$

and hence, $\lim_{t\to\infty} I(t) = 0$. Therefore, system (2.1) is asymptotic to the following periodic time-delayed equation:

$$N'(t) = B(t-\tau,N(t-\tau))N(t-\tau)\alpha(t) - d(t)N(t), \tag{4.4}$$

which is the same as (3.1). Note that $N^*(t)$ is a global attractive solution of (3.1). Next, we use the theory of internally chain transitive sets (see e.g., [11, 19]) to prove $\lim_{t\to\infty}(S(t) - N^*(t)) = 0$.

In fact, if we denote the Poincaré map $P := \Phi(T) : M \to M$, then $P^n(\phi) = \Phi(nT)\phi, \forall n \geq 0, \phi \in M$. Let $\phi = (\phi_1,\phi_2) \in M \setminus \{\hat{0}\}$ and $\omega = \omega(\phi)$ be the omega limit set of $\{P^n(\phi)\}$. Since $I(t,\phi) \to 0$ as $t \to \infty$, there holds $\omega = \bar{\omega} \times \{\hat{0}\}$. We first claim that $\bar{\omega} \neq \{\hat{0}\}$. Assume not, i.e., $\bar{\omega} = \{\hat{0}\}$, then $\lim_{n\to\infty}(S_{nT}(\phi), I_{nT}(\phi)) = \lim_{n\to\infty}\Phi(nT)\phi = (\hat{0},\hat{0})$, which contradicts claim 1. It is easy to see that $P^n|_\omega(\phi,\hat{0}) = (\bar{P}^n(\phi),\hat{0})$ where $\bar{P}$ is the periodic solution semiflow of (3.1). By [19, Lemma 1.2.1], ω is an internally chain transitive set for P, and hence, $\bar{\omega}$ is an internally chain transitive set for $\bar{P}$. Define $N_0^* \in C([-\tau,0],\mathbb{R}_+)$ by $N_0^*(\theta) = N^*(\theta)$, $\forall \theta \in [-\tau,0]$. Since $\bar{\omega} \neq \{\hat{0}\}$ and N_0^* is a globally stable fixed point for $\bar{P}$ in $C([-\tau,0],\mathbb{R}_+) \setminus \{\hat{0}\}$, we have $\bar{\omega} \cap W^s(N_0^*) \neq \varnothing$, where $W^s(N_0^*)$ is the stable set of N_0^*. By [19, Theorem 1.2.1], we then get $\bar{\omega} = N_0^*$. This proves $\omega = (N_0^*,\hat{0})$, and hence, $\lim_{t\to\infty}((S(t,\phi),I(t,\phi)) - (N^*(t),0)) = 0$.

In the case where $\mathcal{R}_0 > 1$, we have $\int_0^T \beta(t)dt > \int_0^T (d(t)+d_2(t)+\gamma(t))dt$. Fix a number $\eta_0 \in (\frac{1}{\mathcal{R}_0},1)$, since

$$\lim_{(I(t),N(t))\to(0,N^*(t))} \frac{N(t)-I(t)}{N(t)} = 1 > \eta_0,$$

there exists $\eta_1 > 0$, such that

$$\frac{N(t)-I(t)}{N(t)} > \eta_0,\ \forall 0 \le I(t) \le \eta_1,\ |N(t)-N^*(t)| \le 2\eta_1.$$

Since $\lim_{\phi\to(N_0^*,\hat{0})} \Phi(t)\phi = (N_0^*,\hat{0})$ uniformly for $t \in [0,T]$, there exists $\eta_2 > 0$ such that $\|\Phi(t)\phi - (N_0^*,0)\| \le \eta_1, \forall t \in [0,T]$, $\|\phi - (N_0^*,0)\| \le \eta_2$. Then we have the following claim:

Claim 2. $\limsup_{n\to\infty} \|\Phi(nT)\phi - (N_0^*,\hat{0})\| \ge \eta_2$ for all $\phi \in M_0$.

Suppose, by contradiction, that $\limsup_{n\to\infty} \|\Phi(nT)\phi - (N_0^*,\hat{0})\| < \eta_2$ for some $\phi \in M_0$. Then there exists an integer $N_2 \ge 1$ such that $\|\Phi(nT)\phi - (N_0^*,0)\| < \eta_2, \forall n \ge N_2$. For any $t \ge N_2T$, we have $t = nT + t'$ with $n \ge N_2$ and $t' \in [0,T]$. Thus, we have

$$\|\Phi(t)\phi - (N_0^*,0)\| = \|\Phi(t')(\Phi(nT)\phi) - (N_0^*,\hat{0})\| \le \eta_1,\ \forall t \ge N_2T.$$

Therefore, $I(t)$ satisfies the following differential inequality

$$I'(t) \ge (\beta(t)\eta_0 - (d(t)+d_2(t)+\gamma(t)))I(t),\ \forall t \ge N_2T.$$

By the comparison theorem, it follows that

$$I(t) \ge I(N_2T)e^{\int_{N_2T}^t (\beta(s)\eta_0 - (d(s)+d_2(s)+\gamma(s))ds}.$$

Since $\mathcal{R}_0 > 1$ and $\eta_0 \in (\frac{1}{\mathcal{R}_0},1)$, we have $\lim_{t\to\infty} I(t) = \infty$, a contradiction.

In the case where $G(t,0) \equiv 0$, we choose

$$M_1 = (\hat{0},\hat{0}) \quad \text{and} \quad M_2 = (N_0^*,\hat{0}).$$

It then follows that M_1 and M_2 are disjoint, compact and isolated invariant set for P in ∂M_0, and $\tilde{A}_\partial := \bigcup_{\phi\in\partial M_0}\omega(\phi) = \{M_1,M_2\}$. Further, no subset of M_1, M_2 forms a cycle in ∂M_0. In view of two claims above, we see that M_1 and M_2 are isolated invariant sets for P in M, and $W^s(M_i) \cap M_0 = \emptyset$, $i = 1,2$, where $W^s(M_i)$ is the stable sets of M_i for P.

In the case where $G(t,0) > 0$ for all $t \ge 0$, M_2 is the only compact invariant set for P in ∂M_0, and hence we only choose i=2 in the above argument.

By the acyclicity theorem on uniform persistence for maps (see [19, Theorem 1.3.1

and Remark 1.3.1]), it follows that $P : M \to M$ is uniformly persistent with respect to M_0. Thus, [19, Theorem 3.1.1] implies that the periodic semiflow $\Phi(t) : M \to M$ is also uniformly persistent with respect to M_0. According to [21, Theorem 3.1], system (2.1) has an T-periodic solution $(S^*(t), I^*(t))$ with $(S_t^*, I_t^*) \in M_0$ for all $t \geq 0$. Clearly, $S_t^* > 0$ and $I_t^* > 0$ for all $t > 0$.

It follows from [12, Theorem 4.5], with $\rho(x) = d(x, \partial M_0)$, that P: $M_0 \to M_0$ has a compact global attractor A_0. Since $A_0 = P(A_0) = \Phi(T)A_0$, it follows that $\phi_1(0) > 0$ and $\phi_2(0) > 0$ for all $\phi \in A_0$. Let $B_0 := \bigcup_{t\in[0,T]} \Phi(t)A_0$. We have $B_0 \subset M_0$ and $\lim_{t\to\infty} d(\Phi(t)\phi, B_0) = 0$ for all $\phi \in M_0$. Define a continuous function $p : M \to \mathbb{R}_+$ by

$$p(\phi) = \min(\phi_1(0), \phi_2(0)), \quad \forall \phi = (\phi_1, \phi_2) \in M.$$

Since B_0 is a compact subset of M_0, we have $\inf_{\phi\in B_0} p(\phi) = \min_{\phi\in B_0} p(\phi) > 0$. Consequently, there exists $\eta > 0$ such that

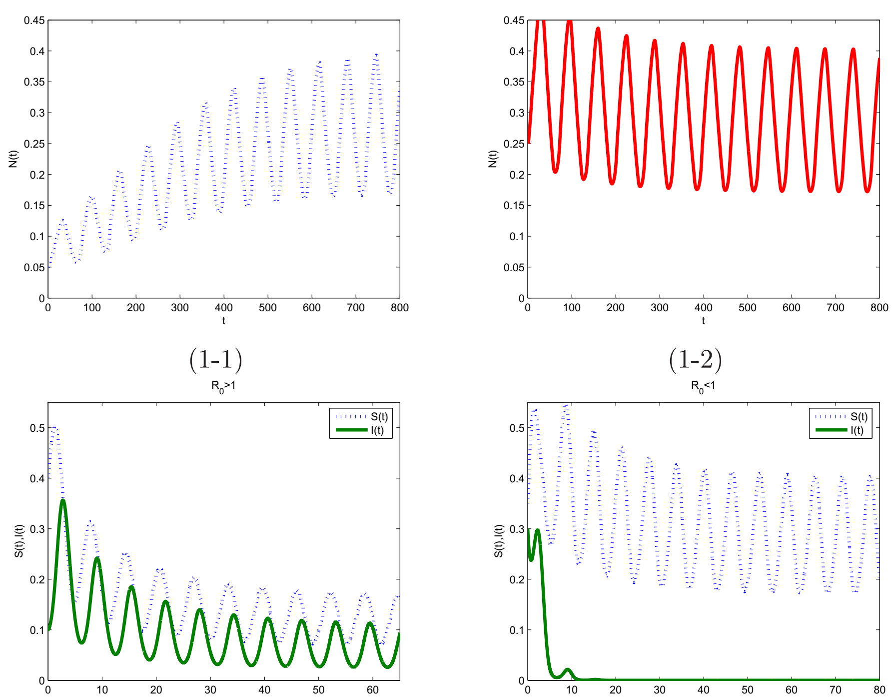
$$\liminf_{t\to\infty} \min(S(t,\phi), I(t,\phi)) = \liminf_{t\to\infty} p(\Phi(t)\phi) \geq \eta, \quad \forall \phi \in M_0.$$

This completes the proof. □

5. **Numerical simulations.** In this section, we use specific birth functions to verify our results in the previous two sections by numerical simulations.

Example 1. In this example, we choose $B(t,N)N = N\frac{2(1+\cos(t))}{1+N}$, $d(t) = 0.5$, $d_1(t) = 1$, $\tau = 1$. Then $\alpha(t) = e^{-1}$ and the equation (3.1) becomes

$$N'(t) = N(t-1)\frac{2(1+\cos(t-1))}{1+N(t-1)}e^{-1} - \frac{1}{2}N(t).$$



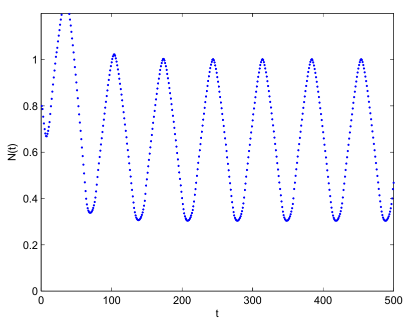
(1-1) (1-2)

(1-3) (1-4)

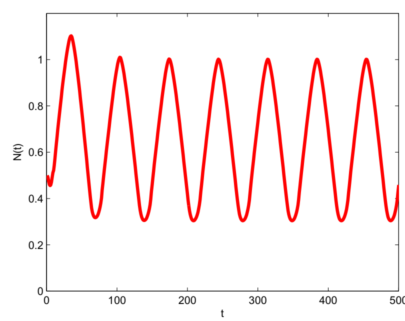
It is easy to see that (H1)-(H3) in Theorem 3.2 hold for this equation, our numerical simulations in Fig. (1-1) and Fig. (1-2) show that there is a globally asymptotically attractive positive periodic solution $N^*(t)$. Moreover, if we choose $d_2(t) = \frac{1}{10}$, $\beta(t) = 1 + \sin(t)$ and $\gamma(t) = \frac{1}{10}$, then $\mathcal{R}_0 = \frac{10}{7} > 1$. Thus, we have Fig. (1-3), which shows that the disease is uniform persistence and there is a positive periodic solution when $\mathcal{R}_0 > 1$. On the contrary, if we choose $d_2(t) = \frac{1}{5}(1+\sin(t))$, $\beta(t) = \frac{1}{2}(1 + 3\sin(t))$ and $\gamma(t) = \frac{1}{5}$, then $\mathcal{R}_0 = \frac{5}{9} < 1$. We have Fig. (1-4) for this case. For other initial data, we have similar simulations, which may suggest that every solution converges to the disease-free periodic solution.

Example 2. In this example, we choose $B(t,N)N = 0.8 + N$, $d(t) = 1$, $d_1(t) = 1 + \sin(t)$, $\tau = 1$. Then $\alpha(t) = e^{-1+\cos(t)-\cos(t-1)}$ and the equation (3.1) becomes

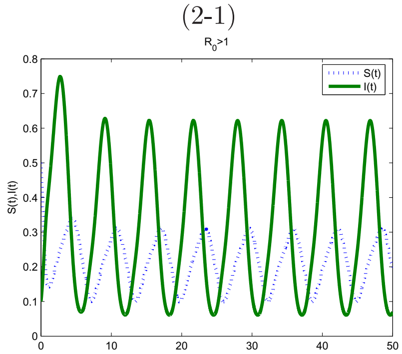
$$N'(t) = (0.8 + N(t-1))e^{-1+\cos(t)-\cos(t-1)} - N(t).$$



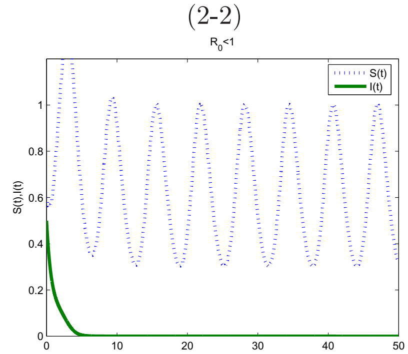
(2-1)



(2-2)



(2-3)



(2-4)

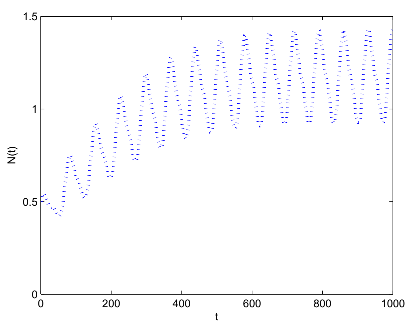
It is easy to see that (A1)-(A3) in Theorem 3.3 hold for this equation, our numerical simulations in Fig. (2-1) and Fig. (2-2) show that there is a globally asymptotically attractive positive periodic solution $N^*(t)$. Moreover, if we choose $d_2(t) = \frac{1}{5}$, $\beta(t) = 4(1 + \sin(t))$ and $\gamma(t) = \frac{1}{5}$, then $\mathcal{R}_0 = \frac{20}{7} > 1$. Thus, we have Fig. (2-3), which shows that the disease is uniform persistence and there is a positive periodic solution when $\mathcal{R}_0 > 1$. On the other hand, if we choose $d_2(t) = \frac{1}{5}$, $\beta(t) = \frac{1}{2}(1 + \sin(t))$ and $\gamma(t) = \frac{1}{5}$, then $\mathcal{R}_0 = \frac{5}{14} < 1$. We have Fig. (2-4) for this case. For other initial data, we have similar simulations, which may suggest every solution converges to the disease-free periodic solution.

Example 3. In this example, we choose $B(t,N)N = 1.2N(1+\sin(t))e^{-\frac{1}{2}N}$, $d(t) = \frac{1}{5}$, $d_1(t) = 1+\sin(t)$, $\tau = 4$. Then $\alpha(t) = e^{-4+\cos(t)-\cos(t-4)}$ and the equation (3.1) becomes

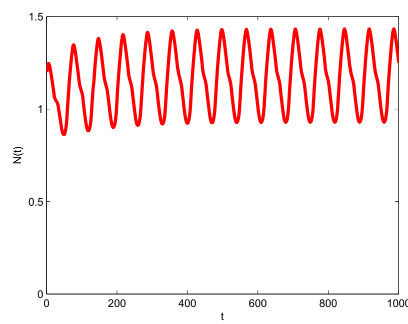
$$N'(t) = \frac{6}{5}N(t-4)e^{-4+\cos(t)-\cos(t-4)}(1+\sin(t-4))e^{-\frac{1}{2}N(t-4)} - \frac{1}{5}N(t).$$

It is easy to see that (S1)and (S2) hold for this equation. In this case, $\frac{1}{q(t)} = 2$ and (3.5) becomes

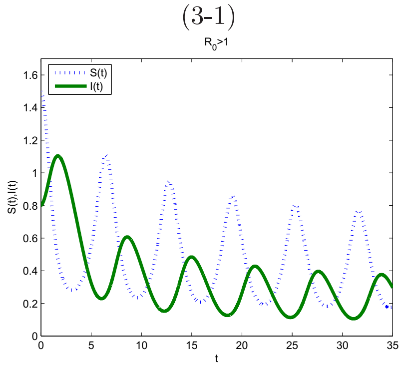
$$\begin{aligned} \bar{U}'(t) &= \alpha(t)\frac{p(t-\tau)}{q(t-\tau)}e^{-1} - d(t)\bar{U}(t) \\ &= e^{-4+\cos(t)-\cos(t-4)}\frac{1.2(1+\sin(t-4))}{\frac{1}{2}}e^{-1} - \frac{1}{5}\bar{U}(t)(t) \\ &\le e^{-3}\times 4.8 - \frac{1}{5}\bar{U}(t). \end{aligned}$$



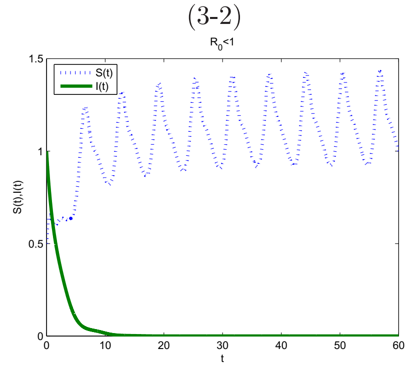
(3-1)



(3-2)



(3-3)

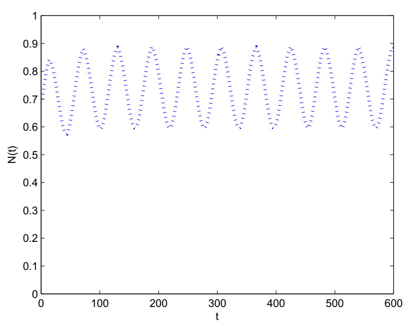


(3-4)

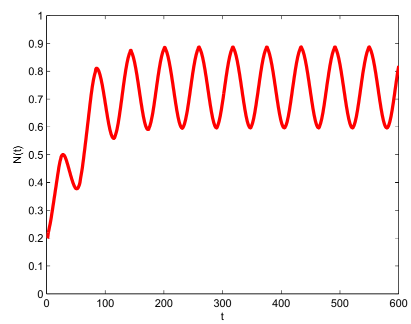
Hence, $\bar{U}^*(t) \le \frac{5\times 4.8}{e^3} \le 2$, and (S3) holds. Our numerical simulations in Fig. (3-1) and Fig. (3-2) show that there is a globally asymptotically attractive positive periodic solution $N^*(t)$. Moreover, if we choose $d_2(t) = \frac{1}{5}$, $\beta(t) = 1+\sin(t)$ and $\gamma(t) = \frac{1}{5}$, then $\mathcal{R}_0 = \frac{5}{3} > 1$. Then, we have Fig. (3-3), which shows that the disease is uniform persistence and there is a positive periodic solution when $\mathcal{R}_0 > 1$. On the other hand, if we choose $d_2(t) = \frac{1}{5}$, $\beta(t) = 0.2(1+\sin(t))$ and $\gamma(t) = \frac{1}{5}$, then $\mathcal{R}_0 = \frac{1}{3} < 1$. We have Fig. (3-4) for this case. For other initial data, we have similar simulations, which may imply that every solution converges to the disease-free periodic state.

Example 4. In this example, if we choose $d(t) = 0.2$, $d_1(t) = 1+0.2\sin(t)$, $\tau = 0.1$ and $B(t,N)N = N(1+\cos(t))e^{-2N}$, then $\alpha(t) = e^{-0.1+0.2(\cos(t)-\cos(t-0.1))}$ and the equation (3.1) becomes

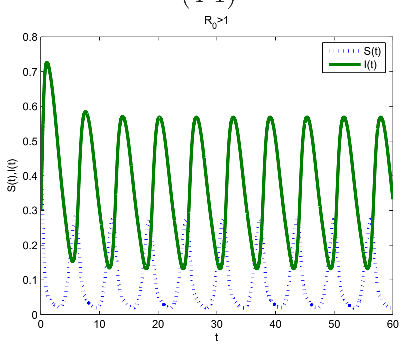
$$N'(t) = N(t-0.1)(1+\cos(t-0.1))e^{-2N(t-0.1)}e^{-\frac{1}{10}+\frac{1}{5}(\cos(t)-\cos(t-0.1))} - \frac{1}{5}N(t).$$



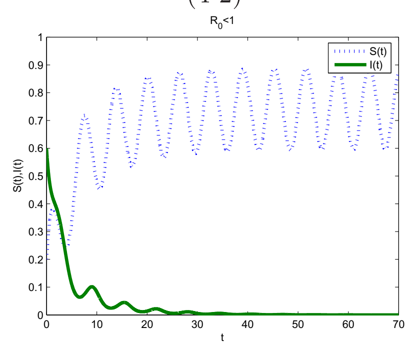
(4-1)



(4-2)



(4-3)



(4-4)

It is easy to see that (S1), (S2) and (S3)$'$ hold for this equation, our numerical simulations in Fig. (4-1) and Fig. (4-2) show that there is a globally asymptotically attractive positive periodic solution $N^*(t)$. Moreover, if we choose $d_2(t) = \frac{1}{5}$, $\beta(t) = 4(1+\sin(t))$ and $\gamma(t) = \frac{1}{5}$, then $\mathcal{R}_0 = \frac{20}{3} > 1$. Thus, we have Fig. (4-3), which shows that the disease is uniform persistence and there is a positive periodic solution when $\mathcal{R}_0 > 1$. On the other hand, if we choose $d_2(t) = \frac{1}{5}$, $\beta(t) = \frac{1}{2}(1+\sin(t))$ and $\gamma(t) = \frac{1}{5}$, then $\mathcal{R}_0 = \frac{5}{6} < 1$. We have Fig. (4-4) for this case. For other initial data, we have similar simulations, which may suggest that every solution converges to the disease-free periodic solution.

6. **Concluding remarks.** In this paper, we first consider a time-delayed periodic single-species population model and obtain four sets of conditions to ensure that the population will stabilize eventually at an oscillating state. When the disease invades the population and susceptibles contact infectives under the standard incidence law, we find an explicit formula for $\mathcal{R}_0$ in the form of the division of the average

contact rate and the mass of the average disease induced death rate, disease recovery rate and death rate. Furthermore, we show that there exists an endemic periodic solution and the disease remains endemic when $\mathcal{R}_0 > 1$, and the disease dies out when $\mathcal{R}_0 < 1$. In order to eradicate such a disease, we should decrease the average contact rate, or increase the average disease recovery rate to make $\mathcal{R}_0 < 1$.

As discussed in [21], we remark that in the case $d_2(t) \equiv 0$, $N(t)$ satisfies equation (3.1), and hence

$$\lim_{t\to\infty}(N(t) - N^*(t)) = 0.$$

Note that I(t) satisfies the following nonautonomous equation

$$I'(t) = \frac{\beta(t)(N(t) - I(t))I(t)}{N(t)} - (d(t) + \gamma(t))I(t), \tag{6.1}$$

which is asymptotic to the following periodic equation

$$I'(t) = \frac{\beta(t)(N^*(t) - I(t))I(t)}{N^*(t)} - (d(t) + \gamma(t))I(t). \tag{6.2}$$

If $\mathcal{R}_0 > 1$, i.e., $\int_0^T(\beta(t) - d(t) - \gamma(t))dt > 0$, and $\beta(t) > 0$, $\forall t \in [0, T]$, then it follows from [19, Theorem 5.2.1] that equation (6.2) admits a unique positive T-periodic solution $I^*(t)$, which is globally asymptotically stable in $\mathbb{R}_+ \setminus \{0\}$. It then follows from the theory of asymptotically periodic system (see [19, Section 3.2]) that $\lim_{t\to\infty}(I(t) - I^*(t)) = 0$. This implies that system (2.1) has a globally attractive positive T-periodic solution $(N^*(t) - I^*(t), I^*(t))$.

By applying the perturbation theory of a globally stable fixed point (see [15, Theorem 2.2]) and the theorem on uniform persistence uniform in parameters (see [19, Theorem 1.4.2]) to the Poincaré map of system (2.1), we can further show that if $\mathcal{R}_0 > 1$, $\beta(t) > 0$, $\forall t \in [0, T]$, and $\|d_2(\cdot)\| := \max_{0\le t\le T}|d_2(t)|$ is sufficiently small, system (2.1) has a globally attractive positive T-periodic solution $(\bar{S}(t), \bar{I}(t))$. On the other side, our numerical results (for example, see Figs.(1-3), (2-3), (3-3) and (4-3)) suggest that in the case where $\mathcal{R}_0 > 1$, every solution with nontrivial initial data is asymptotic to a periodic solution, while these periodic solutions may be different. This implies that there may be no uniqueness of positive T-periodic solution for some $d_2(t) \geq 0$. It is worthy to study the uniqueness, multiplicity, and stability of positive solution of (2.1) in the case where $\mathcal{R}_0 > 1$. We leave this challenging problem for further investigation.

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