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TIME TRANSFORMATIONS FOR DELAY DIFFERENTIAL EQUATIONS

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Abstract. We study changes of variable, called time transformations, which reduce a delay differential equation (DDE) with a variable non-vanishing delay and an unbounded lag function to another DDE with a constant delay. By using this reduction, we can easily obtain a superconvergent integration of the original equation, even in the case of a non-strictly-increasing lag function, and study the type of decay to zero of solutions of scalar linear non-autonomous equations with a strictly increasing lag function.

1. **Introduction.** We consider delay differential equations (DDEs) of the form

$$\begin{cases} y'(t) = f(t, y(t), y(t-\tau(t))), \ t \geq t_0, \\ y(t) = g(t), \ t \leq t_0, \end{cases} \tag{1}$$

where the dependence on the past is contained in the non-local term $y(t-\tau(t))$ and the *delay* $\tau(t)$ depends on t. However, in the following, we use the so-called *lag function* θ defined by

$$\theta(t) := t - \tau(t)$$

instead of the delay function τ and so the non-local term in (1) is written as $y(\theta(t))$.

By a change of variable $t = \alpha(s)$ – which we call *time transformation* (by viewing the variable t as time) – we can reduce the DDE (1) to a DDE with a constant delay. The price we pay is a more complicated right-hand side of the equation, e.g. constant coefficient equations become variable coefficients equations. On the other hand, we bring back the equation into a well-known setting where we can resort to known analytic techniques for studying the qualitative behaviour of the solution as well as to standard numerical techniques for computing it.

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As an example, consider the constant coefficient DDE with proportional delay,

$$\begin{cases} y'(t) = ay(t) + by(qt), \ t > 0, \\ y(0) = y_0 \end{cases} \tag{2}$$

where $q \in (0,1)$. This equation was introduced in [10] where it arose in the modelling of the dynamics of the pantograph collecting the current for an electric locomotive; it is now known as the *pantograph equation*, after the paper [5]. The change of variable $t = e^s$ reduces it to the variable coefficient DDE with constant delay

$$\begin{cases} z'(s) = ae^s z(s) + be^s z(s-\tau), \ s > -\infty, \\ z(-\infty) = y_0, \end{cases}$$

where $z(s) = y(t)$, $\tau = \log q^{-1}$ and the initial condition is shifted to $-\infty$. This transformation is well-known in the literature and was considered in several theoretical or numerical papers: see [6], [7], [8] and [9].

The plan of the present paper is the following. In Section 2, we study the reduction of a DDE (1) with a non-vanishing delay and an unbounded lag function to a constant delay equation. Sections 3 and 4 contain two possible applications of such reductions. In Section 3, we present an approach based on time transformations for the superconvergent integration of DDEs with a non-strictly-increasing lag function. We stress that in literature superconverge results in the integration of DDEs are given only for strictly increasing lag functions (see [1] and [11]). In Section 4, we analyze the type of decay to zero of solutions of scalar linear non-autonomous DDEs with a strictly increasing lag function by a technique based on time transformations. Conclusions are drawn in Section 5.

2. **Reduction of DDEs.** Let us consider the DDE

$$\begin{cases} y'(t) = f(t, y(t), y(\theta(t))), \ t \in [t_0, \overline{t}), \\ y(t) = g(t), \ t \le t_0, \end{cases} \tag{3}$$

where $t_0 \in \mathbb{R}$, $\overline{t} \in (t_0, +\infty]$, $f : [t_0, \overline{t}) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $\theta : [t_0, \overline{t}) \to \mathbb{R}$ is such that $\theta(t) \le t$ for all $t \in [t_0, \overline{t})$, and $g : (-\infty, t_0] \to \mathbb{R}^d$. We assume that the functions f, θ and g are sufficiently regular and that the initial-value problem (3) possesses a unique solution.

We show how the DDE (3) can be reduced, by a time transformation, to a DDE with another lag function.

Let $s_0 \in \mathbb{R}$, $\overline{s} \in (s_0, +\infty]$ and $\kappa : [s_0, \overline{s}) \to \mathbb{R}$ such that $\kappa(s) \le s$ for all $s \in [s_0, \overline{s})$. Moreover, let $\alpha : (-\infty, \overline{s}) \to \mathbb{R}$ be such that:

(i) α is continuous and the right derivative $\alpha'(s)$ exists for all $s \in [s_0, \overline{s})$;

(ii) $\alpha(s) \le t_0$ for all $s \le s_0$, $\alpha(s) \ge t_0$ for all $s \in [s_0, \overline{s})$ and $\lim_{s \uparrow \overline{s}} \alpha(s) = \overline{t}$;

(iii) $\theta(\alpha(s)) = \alpha(\kappa(s))$ for all $s \in [s_0, \overline{s})$.

Note that (ii) implies $\alpha(s_0) = t_0$.

The function α is a time transformation reducing (3) to the DDE

$$\begin{cases} z'(s) = f(\alpha(s), z(s), z(\kappa(s)))\,\alpha'(s), \ s \in [s_0, \overline{s}), \\ z(s) = g(\alpha(s)), \ s \le s_0, \end{cases} \tag{4}$$

with lag function κ. As with the original DDE (3), we assume that there exists a unique solution of (4).

The link between the solution y of (3) and the solution z of (4) is given by

$$z(s) = y(\alpha(s)),\ s < \overline{s}. \tag{5}$$

In fact, for $s_0 \leq s < \overline{s}$, we have $\alpha(s) \geq t_0$ by (ii) and then

$$\begin{aligned} z'(s) &= y'(\alpha(s))\,\alpha'(s) \\ &= f(\alpha(s), y(\alpha(s)), y(\theta(\alpha(s))))\,\alpha'(s) \\ &= f(\alpha(s), y(\alpha(s)), y(\alpha(\kappa(s))))\,\alpha'(s) \\ &= f(\alpha(s), z(s), z(\kappa(s)))\,\alpha'(s) \end{aligned}$$

by (iii). On the other hand, for $s \leq s_0$, we have $\alpha(s) \leq t_0$ by (ii) and then

$$z(s) = y(\alpha(s)) = g(\alpha(s)).$$

Therefore, we can solve (4) and then *"reconstruct"* the solution of (3) by means of (5).

Note that we do not pretend that the time transformation α is strictly increasing. If this holds, then the function z essentialy coincides with the function y since it is obtained from y by the time scaling $s = \alpha^{-1}(t)$. If the transformation α is not strictly increasing, then there exists some value $t \in [t_0, \overline{t})$ of the transformation which is assumed more times, and so, the "reconstruction" of the solution y by (5) may show more than once in a neighbourhood of t.

We give two examples of strictly increasing time transformations reducing to a constant delay equation.

Example 2.1. *Let $q \in (0,1)$, $s_0 \in \mathbb{R}$ and $\tau > 0$. The DDE with proportional delay of parameter q,*

$$\begin{cases} y'(t) = f(t, y(t), y(qt)),\ t \geq t_0, \\ y(t) = g(t),\ t \leq t_0, \end{cases}$$

where $t_0 > 0$, can be reduced to the DDE with constant delay τ

$$\begin{cases} z'(s) = f(\alpha(s), z(s), z(s-\tau))\,\alpha'(s),\ s \geq s_0, \\ z(s) = g(\alpha(s)),\ s \leq s_0. \end{cases}$$

by the time transformation

$$\alpha(s) = t_0 e^{\frac{\log q^{-1}}{\tau}(s-s_0)},\ s \in \mathbb{R}.$$

Example 2.2. *Let $a > 0$, $p \in (0,1)$, $s_0 \in \mathbb{R}$ and $\tau > 0$. The DDE with a non-linear lag function,*

$$\begin{cases} y'(t) = f(t, y(t), y(at^p)),\ t \geq t_0, \\ y(t) = g(t),\ t \leq t_0, \end{cases}$$

where $t_0 > a^* := a^{\frac{1}{1-p}}$, can be reduced to the DDE with constat delay τ,

$$\begin{cases} z'(s) = f(\alpha(s), z(s), z(s-\tau))\,\alpha'(s),\ s \geq s_0, \\ z(s) = g(\alpha(s)),\ s \leq s_0. \end{cases}$$

by the time transformation

$$\alpha(s) = a^* e^{\log \frac{t_0}{a^*} \cdot e^{\frac{\log p^{-1}}{\tau}(s-s_0)}},\ s \in \mathbb{R}.$$

In the sequel, we will study time transformations reducing to a constant delay equation, i.e.

$$\kappa(s) = s - \tau,\ s \in [s_0, \overline{s}),$$

in (4) for some $\tau > 0$.

We assume that $\overline{t} = +\infty$, θ is unbounded and there exists $\tau^* > 0$ such that $\theta(t) \le t - \tau^*$ for all $t \in [t_0, \overline{t})$ (i.e. the DDE (3) has a *non-vanishing delay*).

Let $\{\xi_k\}_{k \ge 0}$ be the sequence such that $\xi_0 := t_0$ and, for each $k \ge 0$, ξ_{k+1} is the minimum root of odd multiplicity of the equation

$$\theta(\xi) = \xi_k, \tag{6}$$

i.e. ξ_{k+1} is the first point where the curve $\theta = \theta(t)$ crosses the horizontal line $\theta = \xi_k$. The points $\xi_0, \xi_1, \xi_2, ...$ are known as *principal discontinuity points* of the DDE (3) (see [2, p. 26]).

Note that the sequence $\{\xi_k\}_{k \ge 0}$ is strictly increasing and $\lim_{k \to \infty} \xi_k = +\infty$. Moreover,

$$\theta(\xi) \le \xi_k \quad \text{if} \quad \xi \le \xi_{k+1},\ k \ge 0. \tag{7}$$

We will, in the first instance, assume that:

(A) for any $k \ge 0$, ξ_{k+1} is the unique root of odd multiplicity of the equation (6) .

This means that ξ_{k+1} is the unique point where the curve $\theta = \theta(t)$ crosses the horizontal line $\theta = \xi_k$.

Note that strictly increasing lag functions satisfy (A). The case where the assumption (A) is not fulfilled will be adressed in Subsection 2.2.

By (A) we have

$$\theta(\xi) \ge \xi_k \quad \text{if} \quad \xi \ge \xi_{k+1},\ k \ge 0 \tag{8}$$

(compare with (7)).

We now show how to construct, under the assumption (A), a time transformation reducing the original DDE (3) to a DDE with a constant delay τ. Two distinct constructions are presented. The first, called *backward construction*, is defined for any lag function satisfying (A): however, in Subsection 2.2, it will also be applied a to lag function for which (A) does not hold. The construction uses images of the function θ and proceeds, for the construction of the transformation $\alpha(s)$, in the negative direction of the $s-$axis by using the property (iii),

$$\alpha(s - \tau) = \theta(\alpha(s)),\ s \ge s_0.$$

The second, called *forward construction*, is defined only for a strictly increasing lag function. It uses counterimages of θ and proceeds in the positive direction of the $s-$axis by using

$$\alpha(s) = \theta^{-1}(\alpha(s - \tau)),\ s \ge s_0.$$

We stress that the backward construction will be used in Section 4 for developing superconvergent integrations of DDEs with a non-strictly-increasing lag function whereas the forward construction will be used in Section 5 for studying the rate of decay to zero of solutions of linear scalar non-autonomous DDEs with a strictly increasing lag function.

2.1. **The backward construction.** Let us assume that we want to "reconstruct" the solution y of (3) in $\left[t_0, \xi_{\overline{k}}\right]$ for some index $\overline{k} > 1$.

We arbitrarily fix $\tau > 0$, $S \in \mathbb{R}$ and a strictly increasing differentiable function $\omega : [S-\tau, S] \to \mathbb{R}$ such that $\omega(S-\tau) = \xi_{\overline{k}-1}$ and $\omega(S) = \xi_{\overline{k}}$.

Then, recursively define $\alpha : \left[S - \left(\overline{k}+1\right)\tau, S\right] \to \mathbb{R}$ by

$$\alpha(s) := \omega(s), \ s \in [S-\tau, S], \tag{9a}$$

$$\alpha(s) := \theta(\alpha(s+\tau)), \ s \in [S-(k+1)\tau, S-k\tau), \ k = 1, 2, ..., \overline{k}. \tag{9b}$$

This yields

$$\alpha(s) = \theta^k(\omega(s+k\tau)), \ s \in [S-(k+1)\tau, S-k\tau), \ 0 \le k \le \overline{k}, \tag{10}$$

where θ^k denotes the $k-$th iterate of θ.

Proposition 1. *The function α is continuous, right-differentiable with right derivative recursively given by*

$$\alpha'(s) = \omega'(s), \ s \in [S-\tau, S), \tag{11a}$$

$$\alpha'(s) = \theta'(\alpha(s+\tau))\alpha'(s+\tau), \ s \in [S-(k+1)\tau, S-k\tau), \tag{11b}$$

$$k = 1, 2, ..., \overline{k},$$

and it satisfies

$$\alpha(s) \le \xi_{\overline{k}-k} \text{ if } s \le S - k\tau \tag{12}$$

$$\alpha(s) \ge \xi_{\overline{k}-k} \text{ if } s \ge S - k\tau, \ 0 \le k \le \overline{k}. \tag{13}$$

Moreover:

a) α is of class C^p, p positive integer, on every interval $[S-(k+1)\tau, S-k\tau]$, $0 \le k \le \overline{k}$, if ω is of class C^p and θ is of class C^p on $\left[t_0, \xi_{\overline{k}}\right]$;

b) α is strictly increasing in $[S-k\tau, S]$, $2 \le k \le \overline{k}+1$, if and only if θ is strictly increasing in $\left[\xi_{\overline{k}-k+1}, \xi_{\overline{k}}\right]$.

Proof. As for the continuity of α, it is sufficient to prove that

$$\lim_{s \uparrow (S-k\tau)} \alpha(s) = \alpha(S-k\tau), \ k = 1, ..., \overline{k}.$$

For $k = 1$, we have

$$\lim_{s \uparrow (S-\tau)} \alpha(s) = \lim_{s \uparrow (S-\tau)} \theta(\omega(s+\tau)) = \theta\left(\xi_{\overline{k}}\right) = \xi_{\overline{k}-1} = \alpha(S-\tau),$$

and, for $k = 1, 2, ..., \overline{k}-1$, if

$$\lim_{s \uparrow (S-k\tau)} \alpha(s) = \alpha(S-k\tau),$$

then

$$\lim_{s \uparrow (S-(k+1)\tau)} \alpha(s) = \lim_{s \uparrow (S-(k+1)\tau)} \theta(\alpha(s+\tau)) = \theta(\alpha(S-k\tau)) = \alpha(S-(k+1)\tau).$$

By (9), we obtain that α is right-differentiable on $\left[S - \left(\overline{k}+1\right)\tau, S\right)$ and the equalities (11) hold.

The properties (12) and (13) follow by

$$\alpha(s) \le \xi_{\overline{k}-k} \text{ if } s \in [S-(k+1)\tau, S-k\tau)$$

$$\alpha(s) \ge \xi_{\overline{k}-k} \text{ if } s \in [S-k\tau, S-(k-1)\tau), \ 0 \le k \le \overline{k}.$$

These inequalities hold since

$$\xi_{\overline{k}-1} \leq \alpha(s) = \omega(s) \leq \xi_{\overline{k}} \text{ if } s \in [S-\tau, S]$$

and, for $k = 1, 2, ..., \overline{k}$, if

$$\alpha(s) \leq \xi_{\overline{k}-(k-1)} \text{ if } s \in [S-k\tau, S-(k-1)\tau)$$

then

$$\alpha(s) = \theta(\alpha(s+\tau)) \leq \xi_{\overline{k}-k} \text{ if } s \in [S-(k+1)\tau, S-k\tau)$$

by (7), and if

$$\alpha(s) \geq \xi_{\overline{k}-k} \text{ if } s \in [S-k\tau, S-(k-1)\tau)$$

then

$$\alpha(s) = \theta(\alpha(s+\tau)) \geq \xi_{\overline{k}-(k+1)} \text{ if } s \in [S-(k+1)\tau, S-k\tau)$$

by (8).

Moreover, if ω is of class C^p and θ is of class C^p on $\left[t_0, \xi_{\overline{k}}\right]$, then α is of class C^p on $[S-\tau, S]$ and, by assuming α to be of class C^p on $[S-k\tau, S-(k-1)\tau]$, $k = 1, 2, ..., \overline{k}$, we can take, on $[S-(k+1)\tau, S-k\tau]$, the derivatives of α up to the order p and these derivatives turn out to be continuous. This proves the property a).

Finally, we prove the property b). Let us assume that θ is strictly increasing in $\left[\xi_{\overline{k}-k+1}, \xi_{\overline{k}}\right]$, $2 \leq k \leq \overline{k}+1$, and prove that α is strictly increasing in $[S-k\tau, S]$. If α is strictly increasing on $[S-l\tau, S-(l-1)\tau)$ with $l = 1, ..., k-1$, then it is strictly increasing on $[S-(l+1)\tau, S-l\tau)$: in fact, for $s_1, s_2 \in [S-(l+1)\tau, S-l\tau)$ such that $s_1 < s_2$, we have $\alpha(s_1+\tau) < \alpha(s_2+\tau)$ and

$$\alpha(s_1+\tau), \alpha(s_2+\tau) \in \left[\xi_{\overline{k}-l}, \xi_{\overline{k}-(l-1)}\right]$$

by (12) and (13); thus

$$\alpha(s_1) = \theta(\alpha(s_1+\tau)) < \theta(\alpha(s_2+\tau)) = \alpha(s_2).$$

Since α coincides with the strictly increasing function ω on $[S-\tau, S]$, it is strictly increasing on every interval $[S-(l+1)\tau, S-l\tau)$, $0 \leq l \leq k-1$. By continuity of α, we have that α is strictly increasing on $[S-k\tau, S]$.

Vice versa, if θ is not strictly increasing on $\left[\xi_{\overline{k}-k+1}, \xi_{\overline{k}}\right]$, $2 \leq k \leq \overline{k}+1$, then there are $t_1, t_2 \in \left[\xi_{\overline{k}-k+1}, \xi_{\overline{k}}\right]$, with $t_1 < t_2$, such that $\theta(t_1) \geq \theta(t_2)$. By the continuity of α, $\alpha(S-(k-1)\tau) = \xi_{\overline{k}-k+1}$ and $\alpha(S) = \xi_{\overline{k}}$, there are $s_1, s_2 \in [S-(k-1)\tau, S]$ with $s_1 < s_2$ such that $\alpha(s_1) = t_1$ and $\alpha(s_2) = t_2$. Hence

$$\alpha(s_1-\tau) = \theta(\alpha(s_1)) = \theta(t_1) \geq \theta(t_2) = \theta(\alpha(s_2)) = \alpha(s_2-\tau),$$

and then α is not strictly increasing on $[S-k\tau, S]$. □

The function α is a time transformation reducing the DDE

$$\begin{cases} y'(t) = f(t, y(t), y(\theta(t))), \ t \in \left[t_0, \xi_{\overline{k}}\right), \\ y(t) = g(t), \ t \leq t_0, \end{cases} \tag{14}$$

to the DDE with constant delay

$$\begin{cases} z'(s) = f(\alpha(s), z(s), z(s-\tau))\,\alpha'(s), \ s \in \left[S-\overline{k}\tau, S\right), \\ z(s) = g(\alpha(s)), \ s \leq S-\overline{k}\tau. \end{cases} \tag{15}$$

In fact, α satisfies the conditions (i), (ii) and (iii) at page 3, for the lag function $\kappa(s) = s-\tau$, $s \leq s_0$.

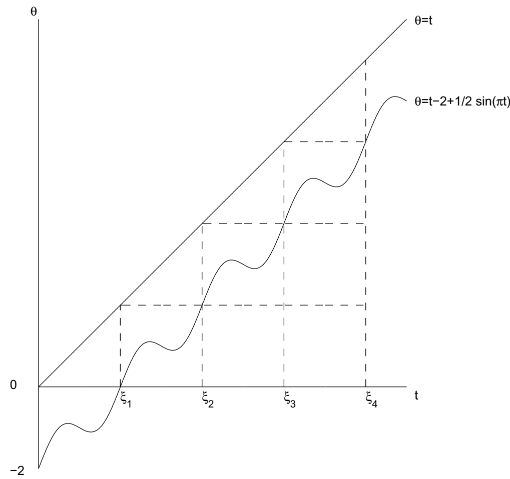

FIGURE 1. Lag function and principal discontinuity points for the DDE (17).

Hence, we can solve (15) and then recover the solution of (14) by

$$z(s) = y(\alpha(s)),\ s \in \left[S - \overline{k}\tau, S\right].$$

Since α is continuous, $\alpha\left(S - \overline{k}\tau\right) = t_0$ and $\alpha(S) = \xi_{\overline{k}}$, all the values $y(t)$, $t \in \left[t_0, \xi_{\overline{k}}\right]$, are obtained when s runs in $[S - k\tau, S]$. However, when θ is not strictly increasing on $\left[\xi_1, \xi_{\overline{k}}\right]$, some values of y are obtained more times.

Our next example illustrates a backward construction. We consider the following choice of the parameters S, τ and ω:

$$S = 0,\ \tau = 1,\ \omega(s) = \xi_{\overline{k}} + s\left(\xi_{\overline{k}} - \xi_{\overline{k}-1}\right),\ s \in [-1, 0]. \tag{16}$$

Example 2.3. *Consider the DDE*

$$\begin{cases} y'(t) = f\left(t, y(t), y\left(t - 2 + \frac{1}{2}\sin \pi t\right)\right),\ t \geq 0, \\ y(t) = g(t),\ t \leq 0. \end{cases} \tag{17}$$

The lag function is shown in Figure 1 and satisfies the assumption (A). The principal discontinuity points are

$$\xi_k = 2k,\ k \geq 0.$$

For $\overline{k} = 3$, the function α is recursively given by

$$\begin{aligned} \omega(s) &= 2\left(\overline{k} + s\right),\ s \in [-1, 0] \\ \alpha(s) &= \alpha(s+1) - 2 + \frac{1}{2}\sin(\pi\alpha(s+1)),\ s \in [-(k+1), -k), \\ k &= 1, 2, ..., \overline{k}. \end{aligned}$$

and is shown in Figure 2.

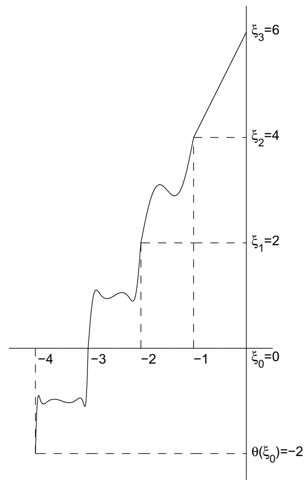

FIGURE 2. Graph of the function α for the DDE (17) when $\overline{k} = 3$ and the parameters S, τ and ω are given in (16).

2.2. **The backward construction for a lag function not satisfying the assumption (A).** In this subsection, we drop the assumption (A),

(A) for any $k \geq 0$, ξ_{k+1} is the unique root of odd multiplicity of the equation (6).

Let us introduce the sets $\mathcal{D}_k$, $k \geq 0$, recursively defined by

$$\begin{aligned} \mathcal{D}_0 &= \{t_0\} \\ \mathcal{D}_{k+1} &= \{\xi \in [\, t_0, +\infty\,) : \text{ there exists } \eta \in \mathcal{D}_k \text{ such that } \xi \\ &\quad\ \text{is a root of odd multiplicity of the equation } \theta\,(\xi) = \eta\}\,, \\ k &= 0, 1, 2, \ldots \quad . \end{aligned}$$

The points in the union of the sets $\mathcal{D}_k$, $k \geq 0$, are called (*primary*) *discontinuity points* (see [2, p. 21]). We have

$$\xi_k \in \mathcal{D}_k,\ k \geq 0,$$

and, if (A) holds,

$$\mathcal{D}_k = \{\xi_k\}\,,\ k \geq 0.$$

The backward construction given in (9) can be also applied to a lag function not satifying the assumption (A), but the recursion (9b) is accomplished only if $\alpha\,(s+\tau) \geq t_0$ since $\theta\,(t)$ is defined only for $t \geq t_0$.

We obtain again a continuous function α such that, for $0 \leq k \leq \overline{k}$,

$$\begin{aligned} \alpha\,(s) &\leq \xi_{\overline{k}-k},\ s \leq S - k\tau, \\ \alpha\,(S - k\tau) &= \xi_{\overline{k}-k} \end{aligned} \tag{18}$$

and then $\alpha\,(s) \leq t_0$ for $s \leq s_0 = S - \overline{k}\tau$ and $\alpha\,(s_0) = t_0$.

On the other hand, it is not longer true that $\alpha\,(s) \geq t_0$ for $s \geq s_0$ (recall conditions (i), (ii) and (iii) at page 3). In fact, for $s \in [S - (k+1)\,\tau, S - k\tau)$, $0 \leq k \leq \overline{k} - 1$, s is a root of odd multiplicity of the equation $\alpha\,(x) = t_0$ if and only if there exists $\eta \in \mathcal{D}_k$ such that $s + k\tau$ is a root of the equation $\omega(x) = \eta$. Hence, there exists such a point s in the interval $[S - (k+1)\tau, S - k\tau)$ if and only if

$$\mathcal{D}_k \cap \left[\xi_{\overline{k}-1}, \xi_{\overline{k}}\right) \neq \varnothing.$$

We conclude that all the points s where $\alpha(s)$ crosses t_0 can be determined by the discontinuity points in $\left[\xi_{\overline{k}-1}, \xi_{\overline{k}}\right)$.

Since $\alpha(s) \geq t_0$, $s \geq s_0$, is not longer true, $\alpha(s)$ is not defined for all $s \in [s_0 - \tau, S]$ (recall that the recursion (9b) is accomplished only if $\alpha(s+\tau) \geq t_0$).

Let us introduce the subsets of $[s_0 - \tau, S]$:

$$\begin{aligned} A &= \{s \in [s_0 - \tau, S] : \alpha(s) \text{ is defined and } \alpha(s) \geq t_0\}, \\ B &= \{s \in [s_0 - \tau, S] : \alpha(s) \text{ is defined and } \alpha(s) \leq t_0\}, \\ C &= \{s \in [s_0 - \tau, S] : \alpha(s) \text{ is not defined}\}. \end{aligned}$$

Note that

$$\begin{aligned} A &= \bigcup_{k>0} (B + k\tau) \cap [s_0 - \tau, S] \qquad (19) \\ \overline{C} &= \bigcup_{k>0} (B - k\tau) \cap [s_0 - \tau, S]. \end{aligned}$$

where $\overline{C}$ is the closure of C. Moreover, the set B is a union of intervals whose ends are the points s where $\alpha(s)$ crosses t_0 (which separate A and B) or shifts of $-\tau$ of such points (which separate B and C). The sets A and C are also unions of intervals. Finally, note that

$$A \cap \overline{C} = \emptyset. \qquad (20)$$

The reduced constant delay equation now takes the form

$$\begin{cases} z'(s) = f(\alpha(s), z(s), z(s-\tau))\,\alpha'(s), \ s \in A, \\ z(s) = g(\alpha(s)), \ s \in B. \end{cases} \qquad (21)$$

Next example illustrates a backward construction for a lag function not satisfying the assumption (A).

Example 2.4. *Let us consider the DDE*

$$\begin{cases} y'(t) = f(t, y(t), y(\theta(t))), \ t \geq 0, \\ y(t) = g(t), \ t \leq 0. \end{cases} \qquad (22)$$

where

$$\theta(t) = t^3 - \frac{9}{2}t^2 + 6t - \frac{9}{5}, \ t \in [0, 3].$$

The lag function is shown in Figure 3 and it does not satisfy the assumption (A). The principal discontinuity points in $[0, 3]$ *are given in the table below:*

k	ξ_k
0	0
1	0.4199
2	0.6146
3	0.7773
4	2.5329
5	2.9715

.

The discontinuity points in $[t_0, \xi_5)$ *are collected in the sets:*

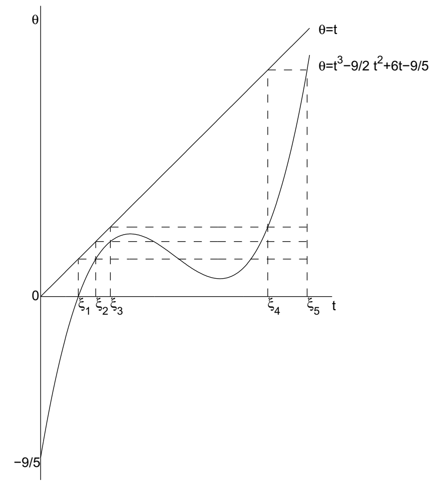

FIGURE 3. Lag function and principal discontinuity points for the DDE (22).

$$\begin{aligned}
\mathcal{D}_0 &= \{t_0\} \\
\mathcal{D}_1 &= \{\xi_1\} \\
\mathcal{D}_2 &= \{\xi_2, \eta_2, \mu_2\} \\
\mathcal{D}_3 &= \{\xi_3, \eta_3, \mu_3, \eta_{23}, \mu_{23}\} \\
\mathcal{D}_4 \cap [t_0, \xi_5) &= \{\xi_4, \eta_{34}, \mu_{34}\}
\end{aligned}$$

where

- η_2 *and* μ_2, *with* $\eta_2 < \mu_2$, *are the roots of* $\theta(\xi) = \xi_1$ *different from* ξ_2;
- η_3 *and* μ_3, *with* $\eta_3 < \mu_3$, *are the roots of* $\theta(\xi) = \xi_2$ *different from* ξ_3;
- η_{23}, μ_{23}, η_{34} *and* μ_{34} *are the roots of* $\theta(\xi) = \eta_2$, $\theta(\xi) = \mu_2$, $\theta(\xi) = \eta_3$ *and* $\theta(\xi) = \mu_3$, *respectively.*

We choose $\overline{k} = 5$ *and the parameters in (16) so that* $S = 0$, $s_0 = -5$ *and* α *is a straight line on* $[-1, 0]$. *The function* α *defined by (9) is displayed in Figure 4.*

Since

$$\begin{aligned}
\mathcal{D}_0 \cap [\xi_4, \xi_5) &= \varnothing \\
\mathcal{D}_1 \cap [\xi_4, \xi_5) &= \varnothing \\
\mathcal{D}_2 \cap [\xi_4, \xi_5) &= \varnothing \\
\mathcal{D}_3 \cap [\xi_4, \xi_5) &= \{\eta_{23}, \mu_{23}\} \\
\mathcal{D}_4 \cap [\xi_4, \xi_5) &= \{\xi_4, \eta_{34}, \mu_{34}\},
\end{aligned}$$

$\alpha(s)$ *crosses the point* t_0 *in the interval* $[-4, -3)$ *at the points* σ_1 *and* σ_2, *with* $\sigma_2 < \sigma_1$, *such that*

$$\begin{aligned}
\omega(\sigma_1 + 3) &= \mu_{23} \\
\omega(\sigma_2 + 3) &= \eta_{23}
\end{aligned}$$

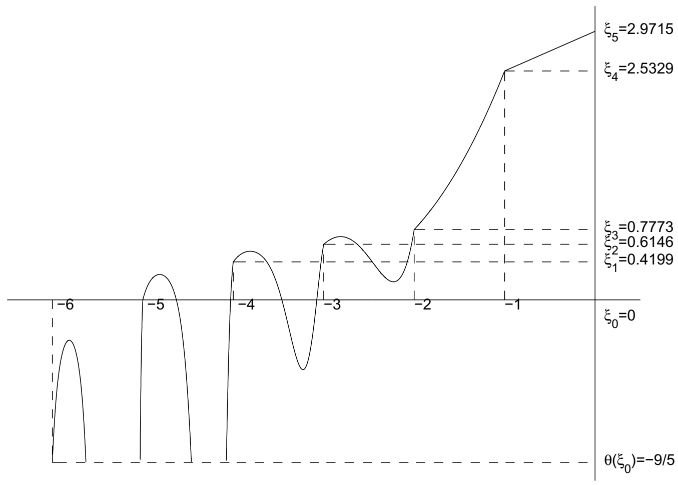

FIGURE 4. Graph of the function α for the DDE (22) when $\overline{k}=5$ and the parameters S, τ and ω are given in (16).

and in the interval $[-5,-4)$ *at the points* $\sigma_5=-5$, σ_4 *and* σ_3, *with* $\sigma_5<\sigma_4<\sigma_3$, *such that*

$$\begin{aligned}\omega(\sigma_3+4) &= \mu_{23}\\ \omega(\sigma_4+4) &= \eta_{23}\\ \omega(\sigma_5+4) &= \xi_5.\end{aligned}$$

The sets B *and* C *are given by*

$$\begin{aligned}B &= [-6,\sigma_4-1]\cup[\sigma_3-1,-5]\cup[\sigma_4,\sigma_2-1]\cup[\sigma_1-1,\sigma_3]\cup[\sigma_1,\sigma_2]\\ C &= (\sigma_4-1,\sigma_3-1)\cup(\sigma_2-1,\sigma_1-1).\end{aligned}$$

2.3. **The forward construction.** Now, let us assume that

$$\theta'(t)>0,\ t\geq t_0, \tag{23}$$

holds for the unbounded lag function θ. So, θ is strictly increasing and has a strictly increasing inverse $\vartheta:[\theta(t_0),+\infty)\to[t_0,+\infty)$ that is differentiable with derivative

$$\vartheta'(u)=\frac{1}{\theta'(\vartheta(u))},\ u\in[\theta(t_0),+\infty).$$

Similarly to the backward construction, we arbitrarily fix $\tau>0$, $S\in\mathbb{R}$ and a continuosly differentiable function $\omega:[S-\tau,S]\to\mathbb{R}$ such that

$$\omega'(s)>0 \text{ for } s\in[S-\tau,S],\ \omega(S-\tau)=\theta(t_0) \text{ and } \omega(S)=t_0. \tag{24}$$

Then we define $\alpha:[S-\tau,+\infty)\to\mathbb{R}$ by

$$\begin{aligned}\alpha(s) &:= \omega(s),\ s\in[S-\tau,S), &(25)\\ \alpha(s) &:= \vartheta(\alpha(s-\tau)),\ s\in[S+k\tau,S+(k+1)\tau), &(26)\\ k &= 0,1,2,\dots.\end{aligned}$$

This yields

$$\alpha(s) = \vartheta^k(\omega(s - k\tau)), \ s \in [S + (k-1)\tau, S + k\tau), \ k \geq 0. \tag{27}$$

Proposition 2. *The function α is continuous, right-differentiable with positive right derivative recursively given by*

$$\begin{aligned} \alpha'(s) &= \omega'(s), \ s \in [S - \tau, S), && (28) \\ \alpha'(s) &= \vartheta'(\alpha(s-\tau))\alpha'(s-\tau), \ s \in [S + k\tau, S + (k+1)\tau), && (29) \\ k &= 0, 1, 2, \dots, \end{aligned}$$

and it satisfies

$$\alpha(S + k\tau) = \xi_k, \ k \geq 0.$$

The proof of Proposition 2 is straightforward. Note that α is strictly increasing and maps $[S - \tau, +\infty)$ into $[\theta(t_0), +\infty)$.

The function α is a time transformation reducing the DDE

$$\begin{cases} y'(t) = f(t, y(t), y(\theta(t))), \ t \geq t_0, \\ y(t) = g(t), \ t \leq t_0, \end{cases}$$

to the DDE with constant delay

$$\begin{cases} z'(s) = f(\alpha(s), z(s), z(s-\tau))\alpha'(s), \ s \geq S, \\ z(s) = g(\alpha(s)), \ s \leq S. \end{cases}$$

Hence, we can solve the DDE with constant delay and then reconstruct the solution y in $[t_0, +\infty)$ by

$$z(s) = y(\alpha(s)), \ s \in [S, +\infty).$$

Since α is strictly increasing, when s runs in $[S, +\infty)$ all the values of y in $[t_0, +\infty)$ are attained one single time.

3. **Superconvergent integration.** In this section, we illustrate an application of the reduction to a constant delay equation which concerns superconvergent integrations of a DDE with a non-vanishing delay and an unbounded lag function.

In the context of the numerical integration of DDEs, a method is said to be *superconvergent* if it attains at the mesh points an order of convergence higher than $\min\{q+1, p\}$, where q is the uniform order and p is the discrete order (see [2, p. 156, Theorem 6.2.1]).

It is well known (see [4, p. 341, Theorem 17.1]) that a Runge-Kutta (RK) method of order p is superconvergent up to the order p, when it is applied to a constant delay DDE with stepsize given by a submultiple of the delay and past values are approximated by stage values. The explanation for this result is that the mesh point approximations are the same as when the RK method is applied to the ordinary differential equation of Bellman's method of steps (see [3]).

In [11], the superconvergence up to the order p is also obtained for a continuous Runge-Kutta (CRK) method, which is a natural continuous extension (NCE) of an RK method of order p, when it is applied on a *constrained mesh* to a DDE with a variable delay and a strictly increasing lag function. For the particular case of a collocation method, this result was first proved in [1]. On the other hand, there are no results in the literature concerning superconvergence in the case of a non-strictly-increasing lag function.

The reduction to DDEs with constant delay offers an alternative approach for the superconvergent integration. This approach has the advantage that it does not require an interpolant between mesh points and works for a non-strictly-increasing lag function. The reduction is accomplished by a time transformation obtained by using the backward construction.

We begin by considering the case where the assumption (A) holds.

As illustrated in Subsection 2.1, for fixed but otherwise arbitrary principal discontinuity point $\xi_{\overline{k}}$, we can reduce the DDE (14) to the constant delay DDE (15).

The equation (15) can be now integrated by a $\nu-$stage RK method (A, b, c) by using a stepsize $h = \frac{\tau}{m}$, m a positive integer. The method proceeds along the mesh $\{s_n\}_{n=-m,...,0,...,\overline{N}}$, where $s_n = s_0 + nh$ and $\overline{N} = \overline{k}m$, and it yields a sequence $\{z_n\}_{n=0,1,...,N}$, where z_n is an approximation of $z(s_n) = y(\alpha(s_n))$, given by

$$\begin{aligned} z_{n+1} &= z_n + h \sum_{i=1}^{\nu} b_i f\left(\alpha\left(s_{n+1}^i\right), Z_{n+1}^i, Z_{n+1-m}^i\right) \alpha'\left(s_{n+1}^i\right),\ n = 0, 1, ..., \overline{N} - 1, \\ z_0 &= g\left(\alpha\left(s_0\right)\right), \end{aligned}$$

where $s_{n+1}^i = s_n + c_i h$, $i = 1, ..., \nu$. The stage values Z_{n+1}^i, $i = 1, ..., \nu$ and $n = -m, ...0, ..., \overline{N} - 1$, are given by

$$\begin{aligned} Z_{n+1}^i &= z_n + h \sum_{j=1}^{\nu} a_{ij} f\left(\alpha\left(s_{n+1}^j\right), Z_{n+1}^j, Z_{n+1-m}^j\right) \alpha'\left(s_{n+1}^j\right),\ n \geq 0, \\ Z_{n+1}^i &= g\left(\alpha\left(s_{n+1}^i\right)\right),\ n < 0, \end{aligned}$$

where, even for $n < 0$, $s_{n+1}^i = s_{n+1} + c_i h$.

Note that, by (9) and (11), the values $\alpha\left(s_{n+1}^i\right)$ and $\alpha'\left(s_{n+1}^i\right)$, $i = 1, ..., \nu$ and $n = -m, ...0, ..., \overline{N} - 1$, are computed by

$$\begin{aligned} \alpha\left(s_{n+1}^i\right) &= \omega\left(s_{n+1}^i\right),\ i = 1, ..., \nu,\ n = \overline{N} - m, \overline{N} - m + 1, ..., \overline{N} - 1 \\ \alpha\left(s_{n+1}^i\right) &= \theta\left(\alpha\left(s_{n+1+m}^i\right)\right),\ i = 1, ..., \nu,\ n = -m, ..., 0, ..., \overline{N} - m - 1 \end{aligned}$$

and

$$\begin{aligned} \alpha'\left(s_{n+1}^i\right) &= \omega'\left(s_{n+1}^i\right),\ i = 1, ..., \nu,\ n = \overline{N} - m, \overline{N} - m + 1, ..., \overline{N} - 1 \\ \alpha'\left(s_{n+1}^i\right) &= \theta'\left(\alpha\left(s_{n+1+m}^i\right)\right) \alpha'\left(s_{n+1+m}^i\right),\ i = 1, ..., \nu, \\ n &= -m, ..., 0, ..., \overline{N} - m - 1, \end{aligned}$$

respectively.

By the superconvergence of the RK method when applied to equations with constant delay, we obtain

$$\max_{n=0,1,...,\overline{N}} |z_n - y(\alpha(s_n))| = O(h^p),$$

where p is the order of the RK method.

Note that in this type of superconvergent integration, we are using in the interval $\left[\theta(t_0), \xi_{\overline{k}}\right]$ the constrained mesh $\{\alpha(s_n)\}_{n=-m,...,0,...,\overline{N}}$ for approximating the solution y of (14). In case of a strictly increasing lag function θ, such a mesh coincides with the constrained mesh used in the classical superconvergent integration by NCEs of RK methods.

Now, we analyze the case of a lag function not satisfying the assumption (A).

As illustrated in Subsection 2.2, we can reduce the DDE (14) to the constant delay DDE (21). From that subsection, we recall that the union of all iterated shifts of τ of the set B, where $\alpha(s) \leq t_0$ holds, yields the set A, where $\alpha(s) \geq t_0$ holds (see (19)). Hence, we can construct a mesh on $\left[S - \overline{k}\tau, S\right]$ as follows: we start by defining an arbitrary mesh on the set B and then, by successive shifts of τ, we fill by mesh points the set A. The DDE (21) is then solved on such a mesh by a RK method of order p. We proceed by solving the DDE successively on the intervals $[S-(k+1)\tau, S-k\tau]$, $k = \overline{k}-1, \overline{k}-2, ..., 1, 0$. Thus

$$z(s) = z_k(s), \ s \in [S-(k+1)\tau, S-k\tau], \ 0 \leq k \leq \overline{k},$$

where

$$z_0(s) = g(\alpha(s)), \ s \in \left[S - \left(\overline{k}+1\right)\tau, S - \overline{k}\tau\right]$$

and, for $1 \leq k \leq \overline{k}$, z_k is the solution of the ordinary differential equation

$$\left\{ \begin{array}{l} z_k'(s) = f(\alpha(s), z_k(s), z_{k-1}(s-\tau))\,\alpha'(s), \ s \in A_k, \\ z_k(s) = g(\alpha(s)), \ s \in B_k, \\ z_k(S-k\tau) = z_{k-1}(S-k\tau) \end{array} \right. \tag{30}$$

where

$$\begin{array}{rcl} A_k & = & A \cap [S-(k+1)\tau, S-k\tau] \\ B_k & = & B \cap [S-(k+1)\tau, S-k\tau]. \end{array}$$

Since the sets A, B and C are unions of intervals (recall Subsection 2.2), the interval $[S-(k+1)\tau, \ \ S-k\tau]$ is a union of subintervals which are subsets of A, B or C. By (20), two such subintervals which are subsets of A and C, respectively, are not consecutive. Moreover, since $\alpha(S-(k+1)\tau) = \xi_{\overline{k}-k+1}$ and $\alpha(S-k\tau) = \xi_{\overline{k}-k}$ hold (recall (18)), the first and the last subintervals are subsets of A. We solve the differential equation on the subintervals which are subsets of A (the initial value is $z_{k-1}(S-k\tau)$ for the first subinterval and $g(t_0)$ for the others) whereas, in the subintervals which are subset of B the function z_k is known and in the subintervals which are subsets C it is not defined.

Since the argument of the Bellman method can be repeated for the integration of the equations (30), it follows that the approximations of the solution of (21) at the mesh points are uniformly $O(h^p)$, where h is the maximum stepsize of the mesh.

4. **Decay to zero of solutions.** In this section, as another application of the reduction to constant delay equations, we study the type of decay to zero of solutions of linear scalar non-autonomous DDEs with a strictly increasing lag function.

Let us consider the linear scalar non-autonomous DDE

$$\left\{ \begin{array}{l} y'(t) = \lambda(t)\,y(t) + \mu(t)\,y(\theta(t)), \ t \geq t_0, \\ y(t) = g(t), \ t \leq t_0, \end{array} \right. \tag{31}$$

where we assume that the delay is non-vanishing, the lag function is unbounded and (23) holds.

It is well known that if

$$\inf_{t \geq t_0} \{-\lambda(t)\} > 0 \text{ and } \sup_{t \geq t_0} \frac{|\mu(t)|}{-\lambda(t)} < 1,$$

then $y(t) \to 0$ as $t \to +\infty$, for every initial function g (see [2, Theorem 9.2.2, p. 256]). Moreover, it is known that if there exists $\tau^{**} > 0$ such that $\theta(t) \geq t - \tau^{**}$ for

$t \geq t_0$, that holds in the case of a constant delay, the decay to zero is of exponential type, i.e.

$$y(t) = O\left(e^{-c(t-t_0)}\right),\ t \to +\infty, \tag{32}$$

for some $c > 0$ called the *decay constant*.

The following lemma gives a very precise estimate of the decay constant c in (32), in case of a constant delay.

Lemma 4.1. *Let us consider a DDE (31) with constant delay $\tau > 0$. If*

$$L = \inf_{t \geq t_0} \{-\lambda(t)\} > 0 \text{ and } R = \sup_{t \geq t_0} \frac{|\mu(t)|}{-\lambda(t)} < 1,$$

then

$$|y(t)| \leq e^{-\frac{G^{-1}(\tau L;R)}{\tau}\cdot(t-t_0)} \max_{t_0-\tau \leq \xi \leq t} \left| e^{\frac{G^{-1}(\tau L;R)}{\tau}\cdot(\xi-t_0)} g(\xi) \right|,\ t \geq t_0 - \tau, \tag{33}$$

where $G^{-1}(\cdot;R)$ is the inverse of the strictly increasing function

$$G(x;R) = \frac{x}{1-e^x R},\ x \in \left(0, \log R^{-1}\right),$$

mapping the interval $\left(0, \log R^{-1}\right)$ into $(0, +\infty)$.

Moreover, for any $L > 0$ and $R \in [0,1)$, there exists an equation (31) with constant delay τ such that

$$\inf_{t \geq t_0} \{-\lambda(t)\} = L \text{ and } \sup_{t \geq t_0} \frac{|\mu(t)|}{-\lambda(t)} = R,$$

whose solution satisfies (33) with equality: such an equation has

$$\lambda(t) = -L, \quad \mu(t) = LR,\ t \geq t_0,$$

and its solution is

$$y(t) = e^{-\frac{G^{-1}(\tau L;R)}{\tau}\cdot(t-t_0)},\ t \geq t_0 - \tau.$$

Proof. Set $t_n = t_0 + n\tau$, $n = -1, 0, 1, \ldots$. We prove by induction that if $c > 0$ satisfies

$$c \leq (1 - e^{c\tau} R)\, L, \tag{34}$$

then, for every $n = 0, 1, 2, \ldots$, we have

$$|y(t)| \leq e^{-c(t-t_0)} \max_{t_0-\tau \leq \xi \leq t_0} \left| e^{c(\xi-t_0)} g(\xi) \right|,\ t_{n-1} \leq t \leq t_n.$$

The case $n = 0$ is trivial.

For a given $n = 0, 1, 2, \ldots$, if

$$|y(t)| \leq e^{-c(t-t_0)} \max_{t_0-\tau \leq \xi \leq t_0} \left| e^{c(\xi-t_0)} g(\xi) \right|,\ t_{n-1} \leq t \leq t_n,$$

then, for $h \in [0, \tau]$,

$$y(t_n + h) = e^{\int_0^h \lambda(t_n+\sigma)d\sigma} y(t_n) + \int_0^h e^{\int_u^h \lambda(t_n+\sigma)d\sigma} \mu(t_n+u)\, y(t_n+u-\tau)\, du$$

and hence

$$\begin{aligned}|y(t_n+h)| &\leq \left[e^{\int_0^h \lambda(t_n+\sigma)d\sigma} + e^{c\tau}R\int_0^h e^{\int_u^h \lambda(t_n+\sigma)d\sigma}\left(-\lambda(t_n+u)\right)e^{-cu}du\right] \\ &\quad \cdot e^{-c(t_n-t_0)}\max_{t_0-\tau\leq\xi\leq t_0}\left|e^{c(\xi-t_0)}g(\xi)\right|.\end{aligned}$$

Integration by parts yields

$$\int_0^h e^{\int_u^h \lambda(t_n+\sigma)d\sigma}\left(-\lambda(t_n+u)\right)e^{-cu}du = e^{-ch} - e^{\int_0^h \lambda(t_n+\sigma)d\sigma} + c\int_0^h e^{\int_u^h \lambda(t_n+\sigma)ds}e^{-cu}du,$$

and so

$$\begin{aligned}|y(t_n+h)| &\leq \left[(1-e^{c\tau}R)e^{\int_0^h \lambda(t_n+\sigma)d\sigma} + e^{c\tau}Re^{-ch}\right. \\ &\quad \left. + ce^{c\tau}R\int_0^h e^{\int_u^h \lambda(t_n+\sigma)d\sigma}e^{-cu}du\right] \\ &\quad \cdot e^{-c(t_n-t_0)}\max_{t_0-\tau\leq\xi\leq t_0}\left|e^{c(\xi-t_0)}g(\xi)\right| \\ &\leq \left[(1-e^{c\tau}R)e^{-Lh} + e^{c\tau}Re^{-ch} + ce^{c\tau}R\int_0^h e^{-L(h-u)}e^{-cu}du\right] \\ &\quad \cdot e^{-c(t_n-t_0)}\max_{t_0-\tau\leq\xi\leq t_0}\left|e^{c(\xi-t_0)}g(\xi)\right|.\end{aligned}$$

Hence,

$$|y(t)| \leq e^{-c(t-t_0)}\max_{t_0-\tau\leq\xi\leq t_0}\left|e^{c(\xi-t_0)}g(\xi)\right|,\ t_n\leq t\leq t_{n+1},$$

if

$$(1-e^{c\tau}R)e^{-Lh} + e^{c\tau}Re^{-ch} + ce^{c\tau}R\int_0^h e^{-L(h-u)}e^{-cu}du \leq e^{-ch},\ h\in[0,\tau]. \quad (35)$$

By rearranging (35), we obtain

$$1-e^{c\tau}R + ce^{c\tau}R\int_0^h e^{(L-c)s}ds \leq (1-e^{c\tau}R)e^{(L-c)h},\ h\in[0,\tau],$$

and, by deriving both sides with respect to h, we see that (34) implies (35).

Now, (34) can be restated as

$$G(\tau c;R) \leq \tau L$$

and

$$c = \frac{G^{-1}(\tau L;R)}{\tau}$$

turns out to be the best choice for c.

The second part of Lemma 4.1 follows by observing that $-\frac{G^{-1}(\tau L;R)}{\tau}$ is the rightmost characteristic root of the constant coefficient equation

$$y'(t) = -Ly(t) + LRy(t-\tau), \ t \geq t_0.$$

□

Now, by a time transformation reducing to a constant delay equation and by previous Lemma 36, we study the decay to zero of solutions of (31).

The forward construction of Subsection 2.3, with $\tau = 1$ and $S = s_0 = 0$, defines a time transformation α, which has a positive right derivative and maps $[-1, +\infty)$ into $\to [\theta(t_0), +\infty)$, for any given function $\omega = \alpha|_{[-1,0]}$ that is continuosly differentiable and satisfies (24).

The next theorem gives a very precise estimate on the type of decay to zero for a class of equations (31) determined by a time transformation.

Theorem 4.2. *Let us consider a DDE (31) with a non-vanishing delay and an unbounded lag function satisfying (23) and let α be a time transformation obtained by the forward construction with $\tau = 1$ and $S = 0$. If*

$$L = \inf_{t \geq t_0} \left\{ \alpha'\left(\alpha^{-1}(t)\right)(-\lambda(t)) \right\} > 0 \text{ and } R = \sup_{t \geq t_0} \frac{|\mu(t)|}{-\lambda(t)} < 1,$$

we have

$$|y(t)| \leq e^{-G^{-1}(L;R)\cdot\alpha^{-1}(t)} \cdot \max_{\theta(t_0) \leq \xi \leq t_0} \left| e^{G^{-1}(L;R)\cdot\alpha^{-1}(\xi)} g(\xi) \right|, \ t \geq \theta(t_0), \tag{36}$$

where G^{-1} is defined in Lemma 4.1.

Moreover, for any $L > 0$ and $R \in [0,1)$, there exists an equation (31) such that

$$\inf_{t \geq t_0} \left\{ \alpha'\left(\alpha^{-1}(t)\right) \cdot (-\lambda(t)) \right\} = L \text{ and } \sup_{t \geq t_0} \frac{|\mu(t)|}{-\lambda(t)} = R,$$

whose solution satisfies (36) with equality.

Proof. The time transformation α reduces (31) to the constant delay equation

$$\begin{cases} z'(s) = \alpha'(s)\lambda(\alpha(s))z(s) + \alpha'(s)\mu(\alpha(s))z(s-1), \ s \geq 0, \\ z(s) = g(\alpha(s)), \ -1 \leq s \leq 0. \end{cases} \tag{37}$$

Since

$$\inf_{s \geq 0} \{-\alpha'(s)\lambda(\alpha(s))\} = \inf_{t \geq t_0} \left\{ \alpha'\left(\alpha^{-1}(t)\right)(-\lambda(t)) \right\} = L$$

and

$$\sup_{s \geq 0} \frac{|\alpha'(s)\mu(\alpha(s))|}{-\alpha'(s)\lambda(\alpha(s))} = \sup_{t \geq t_0} \frac{|\mu(t)|}{-\lambda(t)} = R,$$

Lemma 4.1 yields an exponential decay to zero for the reduced equation (37): we have

$$|z(s)| \leq e^{-G^{-1}(L;R)\cdot s} \max_{-1 \leq \xi \leq 0} \left| e^{G^{-1}(L;R)\cdot \xi} g(\alpha(\xi)) \right|, \ s \geq -1.$$

Hence, (36) holds for the original equation (31).

Now, we prove the second part of the theorem. For $L > 0$ and $R \in [0,1)$, let us consider the equation (31) given by

$$\lambda(t) = -\frac{L}{\alpha'\left(\alpha^{-1}(t)\right)}, \quad \mu(t) = \frac{LR}{\alpha'\left(\alpha^{-1}(t)\right)}, \ t \geq t_0.$$

The reduced equation (37) takes the form

$$z'(s) = -Lz(s) + LRz(s-1), \ s \geq 0$$

and has

$$z(s) = e^{-G^{-1}(L;R)s}, \ s \geq -1,$$

as a solution. Then, the original equation has

$$y(t) = e^{-G^{-1}(L;R)\cdot\alpha^{-1}(t)}, \ t \geq \theta(t_0),$$

as a solution. □

Note that for $t = \xi_k$, $k \geq 0$, the inequality (36) yields

$$|y(\xi_k)| \leq e^{-G^{-1}(L;R)k} \cdot \max_{\theta(t_0)\leq\xi\leq t_0} \left| e^{G^{-1}(L;R)\cdot\alpha^{-1}(\xi)} g(\xi) \right|.$$

Moreover, for a lag function θ such that

$$\theta'(t) \leq 1, \ t \geq t_0,$$

the condition

$$\inf_{t\geq t_0} \left\{ \alpha'\left(\alpha^{-1}(t)\right)(-\lambda(t)) \right\} > 0 \tag{38}$$

holds if the function λ satisfies $\inf_{t\geq t_0}\{-\lambda(t)\} > 0$. However, we stress that the condition (38) covers also cases where $\lambda(t)$ vanishes as $t \to \infty$.

We now apply Theorem 4.2 to the proportional delay case (see Example 2.1) and to the case of a non linear lag function $\theta(t) = at^p$ with $a > 0$ and $p \in (0,1)$ (see Example 2.2).

Proposition 3. *Let*

$$\theta(t) = qt, \ t \geq t_0 > 0, \tag{39}$$

in (31), where $q \in (0,1)$. If

$$\mathcal{L} = \inf_{t\geq t_0}\{t \cdot (-\lambda(t))\} > 0 \text{ and } R = \sup_{t\geq t_0} \frac{|\mu(t)|}{-\lambda(t)} < 1,$$

then

$$|y(t)| \leq \left(\frac{t}{t_0}\right)^{-\frac{G^{-1}\left(\log q^{-1}\cdot\mathcal{L};R\right)}{\log q^{-1}}} \cdot \max_{qt_0\leq\xi\leq t_0} \left| \left(\frac{\xi}{t_0}\right)^{\frac{G^{-1}\left(\log q^{-1}\cdot\mathcal{L};R\right)}{\log q^{-1}}} g(\xi) \right|, \ t \geq qt_0. \tag{40}$$

Moreover, for any $\mathcal{L} > 0$ and $R \in [0,1)$, there exists an equation (31) with lag function (39) such that

$$\inf_{t\geq t_0}\{t\cdot(-\lambda(t))\} = \mathcal{L} \text{ and } \sup_{t\geq t_0}\frac{|\mu(t)|}{-\lambda(t)} = R$$

whose solution satisfies (40) with equality.

Proof. The time transformation

$$\alpha(s) = t_0 e^{\log q^{-1} \cdot s},\ s \geq -1,$$

which corresponds, in the forward construction, to the choice

$$\omega(s) = t_0 e^{\log q^{-1} \cdot s},\ s \in [-1, 0],$$

yields

$$\alpha^{-1}(t) = \frac{\log \frac{t}{t_0}}{\log q^{-1}},\ t \geq t_0.$$

Moreover, we have

$$\alpha'(s) = \log q^{-1} \cdot t_0 e^{\log q^{-1} \cdot s} = \log q^{-1} \cdot \alpha(s),\ s \geq -1,$$

and so

$$L = \inf_{t \geq t_0} \left\{ \alpha'\left(\alpha^{-1}(t)\right)(-\lambda(t)) \right\} = \log q^{-1} \cdot \mathcal{L}.$$

So, (36) reads as (40).

The second part of the proof follows by the second part of Theorem 4.2. ☐

Proposition 4. *Let*

$$\theta(t) = at^p,\ t \geq t_0 > a^* := a^{\frac{1}{1-p}}, \tag{41}$$

in (31), where $a > 0$ and $p \in (0, 1)$. If

$$\mathcal{L} = \inf_{t \geq t_0} \left\{ t \log \frac{t}{a^*} \cdot (-\lambda(t)) \right\} > 0 \qquad \text{and} \qquad R = \sup_{t \geq t_0} \frac{|\mu(t)|}{-\lambda(t)} < 1,$$

then

$$\begin{aligned} |y(t)| \quad &\leq \quad \left(\frac{\log \frac{t}{a^*}}{\log \frac{t_0}{a^*}} \right)^{-\frac{G^{-1}\left(\log p^{-1} \cdot \mathcal{L}; R\right)}{\log p^{-1}}} \\ &\quad \cdot \max_{at_0^p \leq \xi \leq t_0} \left| \left(\frac{\log \frac{\xi}{a^*}}{\log \frac{t_0}{a^*}} \right)^{-\frac{G^{-1}\left(\log p^{-1} \cdot \mathcal{L}; R\right)}{\log p^{-1}}} g(\xi) \right|,\ t \geq at_0^p. \end{aligned} \tag{42}$$

Moreover, for any $\mathcal{L} > 0$ and $R \in [0, 1)$, there exists an equation (31) with lag function (41) such that

$$\inf_{t \geq t_0} \left\{ t \log \frac{t}{a^*} \cdot (-\lambda(t)) \right\} = \mathcal{L} \ \text{and} \ \sup_{t \geq t_0} \frac{|\mu(t)|}{-\lambda(t)} = R$$

whose solution satisfies (42) with equality.

Proof. Let us consider the time transformation

$$\alpha(s) = a^* e^{\log \frac{t_0}{a^*} \cdot e^{\log p^{-1} \cdot s}},\ s \geq -1,$$

for which we have

$$\alpha^{-1}(t) = \frac{\log \frac{\log \frac{t}{a^*}}{\log \frac{t_0}{a^*}}}{\log p^{-1}},\ t \geq t_0,$$

and

$$\alpha'(s) = \log p^{-1} \cdot \log \frac{t_0}{a^*} \cdot e^{\log p^{-1} s} \cdot a^* e^{\log \frac{t_0}{a^*} \cdot e^{\log p^{-1} \cdot s}}$$
$$= \log p^{-1} \cdot \log \frac{\alpha(s)}{a^*} \cdot \alpha(s), \ s \geq -1.$$

Hence,

$$L = \inf_{t \geq t_0} \left\{ \alpha' \left(\alpha^{-1}(t) \right) (-\lambda(t)) \right\} = \log p^{-1} \cdot \mathcal{L}$$

and (36) becomes (42).

The second part follows by the second part of Theorem 4.2. □

4.1. **Asympotic order of solutions.** Theorem 4.2 can be used as a starting point for an investigation of the asymptotic order of vanishing solutions of DDEs (31). In this subsection, we study the asymptotic order in the following two cases:

1) $\lim_{t \to +\infty} \theta'(t) = q$, where $q \in (0,1)$.
2) $\lim_{t \to +\infty} \frac{\theta'(t)}{t^l} = b$, where $l \in (-1,0)$ and $b > 0$.

To this aim, we give three lemmas whose proofs are trivial.

Lemma 4.3. *Let $\beta : [-1, +\infty) \to \mathbb{R}$ be such that*

$$\beta(s) \leq h\beta(s-1), \ s \geq 0 \quad (\beta(s) \geq h\beta(s-1), \ s \geq 0)$$

where $h > 0$. Then

$$\beta(s) \leq M e^{\log h \cdot s}, \ s \geq -1 \quad (\beta(s) \geq M e^{\log h \cdot s}, \ s \geq -1)$$

where

$$M = \sup_{s \in [-1,0)} \left\{ \beta(s) e^{-\log h \cdot s} \right\} \qquad (M = \inf_{s \in [-1,0)} \left\{ \beta(s) e^{-\log h \cdot s} \right\}).$$

Lemma 4.4. *Let $\beta : [-1, +\infty) \to \mathbb{R}$ be such that*

$$\beta(s) \leq h\beta(s-1)^r, \ s \geq 0 \quad (\beta(s) \geq h\beta(s-1)^r, \ s \geq 0)$$

where $h > 0$, $r > 0$ and $r \neq 1$. Then

$$\beta(s) \leq h^* e^{M e^{\log r \cdot s}}, \ s \geq -1 \quad (\beta(s) \geq h^* e^{M e^{\log r \cdot s}}, \ s \geq -1)$$

where $h^* = h^{\frac{1}{1-r}}$ and

$$M = \sup_{s \in [-1,0)} \left\{ \log \frac{\beta(s)}{h^*} \cdot e^{-\log r \cdot s} \right\} \quad (M = \inf_{s \in [-1,0)} \left\{ \log \frac{\beta(s)}{h^*} \cdot e^{-\log r \cdot s} \right\}).$$

Lemma 4.5. *Let $\beta, \gamma : [-1, +\infty) \to \mathbb{R}$ be such that*

$$\beta(s) \geq a e^{b e^{cs}}, \ s \geq -1,$$

and

$$\gamma(s) \geq h\beta(s-1)^r \cdot \gamma(s-1), \ s \geq 0,$$

where $a, b, c, h, r > 0$ with $e^{-c}(r+1) = 1$ and $he^{-c}a^r = 1$. Then

$$\gamma(s) \geq M' b e^{cs} a e^{b e^{cs}}, \ s \geq -1,$$

where

$$M' = \inf_{s \in [-1,0)} \left\{ \frac{\gamma(s)}{b e^{cs} a e^{b e^{cs}}} \right\}.$$

The following theorem deals with the case 1).

Theorem 4.6. *Let us consider a DDE (31) such that*

$$\lim_{t\to+\infty}\theta'(t)=q$$

where $q\in(0,1)$. *If there exist* $\overline{t}>0$, $C>0$ *and* $r>-1$ *such that*

$$-\lambda(t)\geq Ct^{r},\ t\geq\overline{t},$$

and

$$R=\limsup_{t\to+\infty}\frac{|\mu(t)|}{-\lambda(t)}<1,$$

then

$$y(t)=o\left(t^{-c}\right),\ t\to+\infty,$$

for any $c\in\left(0,\frac{\log R^{-1}}{\log q^{-1}}\right)$.

Proof. Let $\varepsilon\in(0,\min\{q,1-q\})$ and let t_0 be a new initial point for (31) such that $t_0\geq\varepsilon^{-1}$,

$$\theta'(t)\leq q_2\quad\text{and}\quad q_1t\leq\theta(t)\leq q_2t,\quad t\geq t_0,$$

where $q_1=q-\varepsilon$ and $q_2=q+\varepsilon$ (the second inequality follows by L'Hôpital's rule). Hence, for the inverse ϑ of θ, we obtain

$$\vartheta(u)\leq q_1^{-1}u,\ u\geq\theta(t_0),\tag{43}$$

and

$$\vartheta'(u)=\frac{1}{\theta'(\vartheta(u))}\geq q_2^{-1},\ u\geq\theta(t_0).\tag{44}$$

For an arbitrary function ω, Lemma 4.3 with (26) and (43) yields

$$\alpha(s)\leq Me^{\log q_1^{-1}\cdot s},\ s\geq-1,$$

where

$$M=\sup_{s\in[-1,0]}\left\{\omega(s)e^{-\log q_1^{-1}\cdot s}\right\},$$

and then

$$\alpha^{-1}(t)\geq\frac{\log\frac{t}{M}}{\log q_1^{-1}},\ t\geq\theta(t_0).$$

On the other hand, Lemma 4.3 with (29) and (44) gives

$$\alpha'(s)\geq M'e^{\log q_2^{-1}\cdot s},\ s\geq-1,$$

where

$$M'=\min_{s\in[-1,0]}\left\{\omega'(s)e^{-\log q_2^{-1}\cdot s}\right\},$$

and so

$$\alpha'\left(\alpha^{-1}(t)\right)\geq M'\left(\frac{t}{M}\right)^{\frac{\log q_2^{-1}}{\log q_1^{-1}}},\ t\geq t_0.$$

By (36), we obtain

$$|y(t)|\leq\left(\frac{t}{M}\right)^{-\frac{G^{-1}(l;r)}{\log q_1^{-1}}}\max_{\theta(t_0)\leq\xi\leq t_0}\left|\left(\frac{\xi}{M}\right)^{\frac{G^{-1}(l;r)}{\log q_1^{-1}}}y(\xi)\right|,\ t\geq\theta(t_0),\tag{45}$$

where

$$l=\frac{M'}{M^{\frac{\log q_2^{-1}}{\log q_1^{-1}}}}\cdot\inf_{t\geq t_0}\left\{t^{\frac{\log q_2^{-1}}{\log q_1^{-1}}}\cdot(-\lambda(t))\right\},\ r=\sup_{t\geq t_0}\frac{|\mu(t)|}{-\lambda(t)}.$$

By taking

$$\omega(s)=t_0e^{\log\frac{t_0}{\theta(t_0)}\cdot s},\ s\in[-1,0],$$

we obtain

$$M=\frac{\theta(t_0)}{q_1},\ M'=\log\frac{t_0}{\theta(t_0)}\cdot\frac{\theta(t_0)}{q_2}.$$

With this choice of the function ω, by letting $\varepsilon\to 0$, we have $t_0\to+\infty$, $l\to+\infty$, $r\to R$ and

$$\frac{G^{-1}(l;r)}{\log q_1^{-1}}\to\frac{\log R^{-1}}{\log q^{-1}}.$$

Hence, for any $c\in\left(0,\frac{\log R^{-1}}{\log q^{-1}}\right)$, there exist $\varepsilon>0$ and an initial point t_0 such that

$$c<\frac{G^{-1}(l;r)}{\log q_1^{-1}}$$

and then, by (45),

$$\frac{|y(t)|}{t^{-c}}\leq t^{c-\frac{G^{-1}(l;r)}{\log q_1^{-1}}}\cdot M^{\frac{G^{-1}(l;r)}{\log q_1^{-1}}}\cdot\max_{\theta(t_0)\leq\xi\leq t_0}\left|\left(\frac{\xi}{M}\right)^{\frac{G^{-1}(l;r)}{\log q_1^{-1}}}y(\xi)\right|\to 0,\ t\to+\infty.$$

□

We now consider the case 2).

Theorem 4.7. *Let us consider a DDE (31) such that*

$$\lim_{t\to+\infty}\frac{\theta'(t)}{t^l}=b,\tag{46}$$

where $l\in(-1,0)$ *and* $b>0$. *If there exist* $\overline{t}>0$, $C>0$ *and* $r>-1$ *such that*

$$-\lambda(t)\geq Ct^r,\ t\geq\overline{t},$$

and

$$R=\limsup_{t\to+\infty}\frac{|\mu(t)|}{-\lambda(t)}<1,$$

then

$$y(t)=o\left((\log t)^{-c}\right),\ t\to+\infty,$$

for any $c\in\left(0,\frac{\log R^{-1}}{\log(l+1)^{-1}}\right)$.

Proof. We rewrite the limit (46) as

$$\lim_{t\to+\infty}\frac{\theta'(t)}{t^{p-1}}=ap,$$

where $p=l+1$ and $a=\frac{b}{p}$.

Let $\varepsilon>0$ and let t_0 be a new initial point such that $t_0\geq\varepsilon^{-1}$ and

$$\theta'(t)\leq a_2pt^{p-1}\ \text{ and }\ a_1t^p\leq\theta(t)\leq a_2t^p,\ t\geq t_0.$$

where $a_1 = a - \varepsilon$ and $a_2 = a + \varepsilon$ (the second inequality follows by L'Hôpital's rule). Hence,

$$a_2^{-\frac{1}{p}} u^{\frac{1}{p}} \leq \vartheta(u) \leq a_1^{-\frac{1}{p}} u^{\frac{1}{p}}, \; u \geq \theta(t_0), \tag{47}$$

and

$$\vartheta'(u) = \frac{1}{\theta'(\vartheta(u))} \geq a_2^{-1} p^{-1} (\vartheta(u))^{1-p} \geq a_2^{-\frac{1}{p}} p^{-1} u^{\frac{1-p}{p}}, \; u \geq \theta(t_0). \tag{48}$$

For an arbitrary function ω, Lemma 4.4 with (26) and (47) yields

$$a_2^* e^{M_2 e^{\log p^{-1} \cdot s}} \leq \alpha(s) \leq a_1^* e^{M_1 e^{\log p^{-1} \cdot s}}, \; s \geq -1, \tag{49}$$

where $a_1^* = a_1^{\frac{1}{1-p}}$,

$$M_1 = \sup_{s \in [-1,0)} \left\{ \log \frac{\omega(s)}{a_1^*} \cdot e^{-\log p^{-1} \cdot s} \right\},$$

$a_2^* = a_2^{\frac{1}{1-p}}$ and

$$M_2 = \inf_{s \in [-1,0)} \left\{ \log \frac{\omega(s)}{a_1^*} \cdot e^{-\log p^{-1} \cdot s} \right\}.$$

Thus,

$$\alpha^{-1}(t) \geq \frac{\log \frac{\log \frac{t}{a_1^*}}{M_1}}{\log p^{-1}}, \; t \geq t_0.$$

On the other hand, Lemma 4.5 with (29), (48) and (49) gives

$$\alpha'(s) \geq M' M_2 e^{\log p^{-1} \cdot s} a_2^* e^{M_2 e^{\log p^{-1} \cdot s}}, \; s \geq -1,$$

where

$$M' = \inf_{s \in [-1,0)} \left\{ \frac{\omega'(s)}{M_2 e^{\log p^{-1} \cdot s} a_2^* e^{M_2 e^{\log p^{-1} \cdot s}}} \right\},$$

and we conclude that

$$\alpha'\left(\alpha^{-1}(t)\right) \geq M' \frac{M_2}{M_1} \log \frac{t}{a_1^*} \cdot a_2^* \left(\frac{t}{a_1^*}\right)^{\frac{M_2}{M_1}}, \; t \geq t_0.$$

By (36), we have

$$|y(t)| \leq \left(\frac{\log \frac{t}{a_1^*}}{M_1}\right)^{-\frac{G^{-1}(l;r)}{\log p^{-1}}} \max_{\theta(t_0) \leq \xi \leq t_0} \left| \left(\frac{\log \frac{\xi}{a_1^*}}{M_1}\right)^{\frac{G^{-1}(l;r)}{\log p^{-1}}} g(\xi) \right|, \; t \geq t_0,$$

with

$$l = M' \frac{M_2}{M_1} \frac{a_2^*}{(a_1^*)^{\frac{M_2}{M_1}}} \cdot \inf_{t \geq t_0} \left\{ \log \frac{t}{a_1^*} \cdot t^{\frac{M_2}{M_1}} (-\lambda(t)) \right\}, \; r = \sup_{t \geq t_0} \frac{|\mu(t)|}{-\lambda(t)}.$$

The choice

$$\omega(s) = a_1^* e^{\log \frac{t_0}{a_1^*} \cdot e^{\log \frac{\log \frac{t_0}{a_1^*}}{\log \frac{\theta(t_0)}{a_1^*}} \cdot s}}, \; s \in [-1, 0],$$

yields

$$M_1 = p^{-1} \log \frac{\theta(t_0)}{a_1^*}, \; M_2 = \log \frac{t_0}{a_1^*}, \; M' = \frac{a_1^*}{a_2^*} \log \frac{\log \frac{t_0}{a_1^*}}{\log \frac{\theta(t_0)}{a_1^*}}.$$

and then

$$G^{-1}(l;r) \to \log R^{-1},\ \varepsilon \to 0.$$

Hence, for any $c \in \left(0, \frac{\log R^{-1}}{\log p^{-1}}\right)$, there exist $\varepsilon > 0$ and an initial point t_0 such that

$$c < \frac{G^{-1}(l;r)}{\log p^{-1}},$$

and thus we obtain

$$\begin{aligned} \frac{|y(t)|}{(\log t)^{-c}} \quad \le \quad & (\log t)^{c-\frac{G^{-1}(l;r)}{\log p^{-1}}} \cdot \left(\frac{\log \frac{t}{a_1^*}}{\log t}\right)^{-\frac{G^{-1}(l;r)}{\log p^{-1}}} \\ & \cdot (M_1)^{\frac{G^{-1}(l;r)}{\log p^{-1}}} \cdot \max_{\theta(t_0)\le\xi\le t_0} \left| \left(\frac{\log \frac{\xi}{a_1^*}}{M_1}\right)^{\frac{G^{-1}(l;r)}{\log p^{-1}}} g(\xi) \right| \\ & \to 0,\ t \to +\infty. \end{aligned}$$

□

5. **Conclusions.** In the previous sections we have introduced the concept of time transformation. This change of variable allows for the reduction of a given DDE with a non-vanishing-delay and an unbounded lag function to another DDE with constant delay. We have presented two applications of this reduction: an easy approach to the superconvergent integration of equations with a non-strictly-increasing lag function and a study of the type of decay to zero for solutions of scalar linear nonautonomous equations with a strictly increasing lag function. In this paper, we have not dealt with *state-dependent* equations. In a sequel to the present paper we shall show that this time transformation can also be used to reduce a state-dependent DDE to one with a prescribed simple lag function that is independent of the solution. This has major implications in, e.g., the superconvergence analysis and the computation of the breaking points, for state-dependent DDEs.

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