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On the singular values and eigenvalues of the Fox–Li and related operators

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Abstract. The Fox–Li operator is a convolution operator over a finite interval with a special highly oscillatory kernel. It plays an important role in laser engineering. However, the mathematical analysis of its spectrum is still rather incomplete. In this expository paper we survey part of the state of the art, and our emphasis is on showing how standard Wiener–Hopf theory can be used to obtain insight into the behaviour of the singular values of the Fox–Li operator. In addition, several approximations to the spectrum of the Fox–Li operator are discussed and results on the singular values and eigenvalues of certain related operators are derived.

Contents

1. Introduction

The Fox–Li operator is

$$(\mathcal{F}_\omega f)(x) := \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega(x-y)^2} f(y)\,\mathrm{d}y, \quad x \in (-1,1),$$

where ω is a positive real number [12]. This is a bounded linear operator on $L^2(-1,1)$, and its spectrum, $\sigma(\mathcal{F}_\omega)$, has important applications in laser and

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maser engineering; see [15] and Section 60 of [26]. Unfortunately, little is rigorously known about $\sigma(\mathcal{F}_\omega)$. The operator $\mathcal{F}_\omega$ is obviously compact. Hence $\sigma(\mathcal{F}_\omega)$ consists of the origin and an at most countable number of eigenvalues accumulating at most at the origin. Computation in Figure 1 seems to indicate that they lie on a spiral, commencing near the point $\sqrt{\pi/\omega}\,\mathrm{e}^{\mathrm{i}\pi/4}$ and rotating clockwise to the origin, except that, strenuous efforts notwithstanding, the precise shape of this spiral is yet unknown.

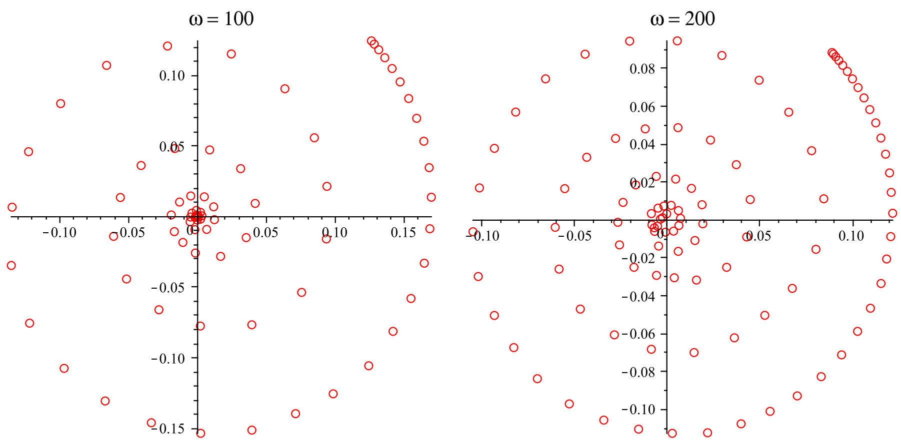

FIGURE 1. Fox–Li eigenvalues for $\omega = 100$ and $\omega = 200$.

Indeed, rigorous results on the spectrum of the Fox–Li operator are fairly sparse. Henry Landau [20] studied the behaviour of the ε-pseudospectrum

$$\sigma_\varepsilon(\mathcal{F}_\omega) := \{\lambda \in \mathbb{C} : \|(\mathcal{F}_\omega - \lambda I)^{-1}\| \geq 1/\varepsilon\}$$

as $\omega \to \infty$ (and "invented" the notion of the pseudospectrum on this occasion). We also recommend Sections 6 and 60 of Trefethen and Embree's book [26] for a nice introduction into this subject. L. A. Vainshtein employed arguments from physics to arrive at the conclusion that the eigenvalues of $\mathcal{F}_\omega$ are

$$\lambda_n \approx \sqrt{\frac{\pi}{\omega}}\,\mathrm{e}^{\mathrm{i}\pi/4} \exp\left(-\frac{\zeta(1/2)\pi^{3/2}}{16\sqrt{2}\,\omega^{3/2}}n^2 - \mathrm{i}\frac{\pi^2}{16\omega}n^2\right),$$

where $\zeta(1/2)$ is the value of Riemann's zeta function at $1/2$, Michael Berry and his collaborators have written a number of papers on physical aspects of the spectrum and its applications in laser theory [3], [4], [5], and, in a recent paper, three of us have analysed several efficient numerical methods for the determination of $\sigma(\mathcal{F}_\omega)$ and, with greater generality, of spectra of integral operators with high oscillation [11].

Much more is known on the set $s(\mathcal{F}_\omega)$ of the singular values of $\mathcal{F}_\omega$, that is, the set of the positive square roots of the points in $\sigma(\mathcal{F}_\omega\mathcal{F}_\omega^*)$. Here are two rigorous results.

Theorem 1.1. *We have $s(\mathcal{F}_\omega) \subset [0, \sqrt{\pi/\omega}\,)$ for every $\omega > 0$.*

To describe the finer behaviour of $s(\mathcal{F}_\omega)$ as $\omega \to \infty$, it is convenient to pass to the scaled sets $\omega s^2(\mathcal{F}_\omega) := \{\omega s_j^2 : s_j \in s(\mathcal{F}_\omega)\}$. The following result was conjectured by Slepian [24] and proved in [21].

Theorem 1.2 (Landau and Widom)**.** *As $\omega \to \infty$, the sets $\omega s^2(\mathcal{F}_\omega)$ converge to the line segment $[0, \pi]$ in the Hausdorff metric, and, for each $\varepsilon \in (0, \pi/2)$,*

$$
\begin{aligned}
|\omega s^2(\mathcal{F}_\omega) \cap (\pi - \varepsilon, \pi)| &= \frac{4\omega}{\pi} + \frac{\log(2\omega)}{\pi^2} \log \frac{\varepsilon}{\pi - \varepsilon} + o(\log \omega), \\
|\omega s^2(\mathcal{F}_\omega) \cap (\varepsilon, \pi - \varepsilon)| &= \frac{2\log(2\omega)}{\pi^2} \log \frac{\pi - \varepsilon}{\varepsilon} + o(\log \omega), \\
|\omega s^2(\mathcal{F}_\omega) \cap (0, \varepsilon)| &= \infty,
\end{aligned}
$$

where $|E|$ denotes the number of points in E, with multiplicities counted.

Unlike $\sigma(\mathcal{F}_\omega)$, the set $s(\mathcal{F}_\omega)$ consists of points on the nonnegative real half-line. The spiral goes away! However, Theorems 1.1 and 1.2 replace the spiral by a different, arguably just as striking, feature: although $s(\mathcal{F}_\omega)$ is all the time contained in $[0, \sqrt{\pi/\omega}\,)$ and fills this segment more and more densely as $\omega \to \infty$, about $4\omega/\pi$ singular values cluster near the right endpoint, while the overwhelming rest of them is concentrated near the left endpoint. Of course, this phenomenon, illustrated for different values of ω in Figure 2, is not too much a surprise for those who are familiar with Toeplitz and Wiener–Hopf operators with piecewise continuous symbols; see, for instance, [21], [30] or Example 5.15 of [7]. Anyway, $s(\mathcal{F}_\omega)$ provides us at least with a poor shadow of the spirals shown in Figure 1.

Theorems 1.1 and 1.2 are known. We nevertheless thought it could be worth stating them explicitly and citing the mathematics behind them. One piece of that mathematics is the switch between convolution by the kernel $k(\omega t)$ over $(-1, 1)$ and convolution by the kernel $\omega^{-1}k(t)$ over $(-\omega, \omega)$. At the first glance, this might look like a triviality, but the classics, including Grenander and Szegő [17], Widom [28], [29], [31] and Gohberg and Feldman [16], have demonstrated that the right switch at the right time and the right place may lead to remarkable insight; see also [33]. For example, in this way we may pass from highly oscillatory kernels on $(-1, 1)$ to non-oscillating kernels on $(-\omega, \omega) \simeq (0, 2\omega)$ and thus to truncated Wiener–Hopf operators. The spectral theory of pure Wiener–Hopf operators, that is, of convolutions over $(0, \infty)$, is simpler than that of truncated Wiener–Hopf operators. As we are the closer to a pure Wiener–Hopf operator the larger ω is, it follows that, perhaps counter-intuitively and in a delicate sense, higher oscillations are simpler to treat than lower oscillations.

Another piece of mathematics that is relevant in this connection is Szegő limit theorems, and in particular Landau and Widom's extension of such theorems to a second order asymptotic formula for Wiener–Hopf operators generated by the characteristic function of an interval. It turns out that the

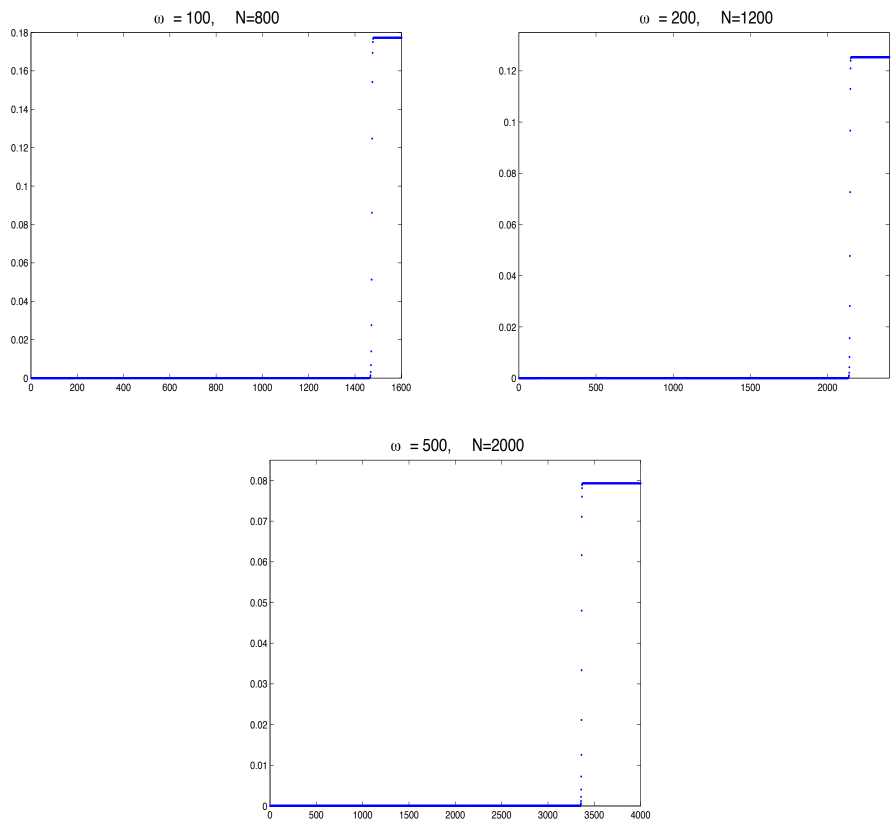

FIGURE 2. Fox–Li singular values for different values of ω, approximated as eigenvalues of a $(2N+1)\times(2N+1)$ matrix.

operator $\mathcal{F}_\omega\mathcal{F}_\omega^*$ is unitarily equivalent to just such a Wiener–Hopf operator. Moreover, the operator $\mathcal{F}_\omega\mathcal{F}_\omega^*$ is at least as important as its coiner $\mathcal{F}_\omega$. For instance, $\mathcal{F}_\omega\mathcal{F}_\omega^*$ is a crucial actor in random matrix theory. There one is interested in the determinants $\det(I-\lambda\mathcal{F}_\omega\mathcal{F}_\omega^*)$ (note that $\mathcal{F}_\omega\mathcal{F}_\omega^*$ is a trace class operator), and the study of these determinants has evolved into results of remarkable depth; see, for example, [13], [14], [19].

The paper is organized as follows. In Section 2 we record some known results on Wiener–Hopf operators, which are then employed in Section 3 to describe the behaviour of the singular values and eigenvalues of fairly general convolution operators with highly oscillatory kernels. Theorems 1.1 and 1.2 will also be derived there. Section 4 contains some attempts on explaining where the spiral in $\sigma(\mathcal{F}_\omega)$ might come from and what its precise shape might be.

2. Wiener–Hopf operators

We denote by $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the Fourier–Plancherel transform,

$$(Ff)(\xi) := \int_{-\infty}^{\infty} f(t)\mathrm{e}^{\mathrm{i}\xi t}\,\mathrm{d}t, \quad \xi \in \mathbb{R},$$

and frequently write $\hat{f}$ for Ff. Let $a \in L^\infty(\mathbb{R})$. Then the multiplication operator $M(a) : f \mapsto af$ is bounded on the space $L^2(\mathbb{R})$. The convolution operator $C(a) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined as $C(a)f = F^{-1}M(a)Ff$. The Wiener–Hopf operator $W(a)$ generated by a is the compression of $C(a)$ to $L^2(0,\infty)$, that is, the operator

$$W(a) := P_+C(a) \,|\, L^2(0,\infty),$$

where P_+ stands for the orthogonal projection of $L^2(\mathbb{R})$ onto $L^2(0,\infty)$. Finally, for $\tau > 0$, we denote by $W_\tau(a)$ the compression of $W(a)$ to $L^2(0,\tau)$,

$$W_\tau(a) := P_\tau W(a) \,|\, L^2(0,\tau),$$

where $P_\tau : L^2(0,\infty) \to L^2(0,\tau)$ is again the orthogonal projection. If $a = \hat{k}$ for some $k \in L^1(\mathbb{R}) \cup L^2(\mathbb{R})$, we have

$$\begin{aligned}
(C(\hat{k})f)(x) &= \int_{-\infty}^{\infty} k(x-y)f(y)\,\mathrm{d}y, \quad x \in \mathbb{R},\\
(W(\hat{k})f)(x) &= \int_{0}^{\infty} k(x-y)f(y)\,\mathrm{d}y, \quad x \in (0,\infty),\\
(W_\tau(\hat{k})f)(x) &= \int_{0}^{\tau} k(x-y)f(y)\,\mathrm{d}y, \quad x \in (0,\tau).
\end{aligned}$$

The relevant function a in the Fox–Li setting is

$$a(\xi) := \sqrt{\pi}\,\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\mathrm{i}\xi^2/4}, \quad \xi \in \mathbb{R}. \tag{2.1}$$

For this function, $C(a)$ is the bounded operator given by

$$(C(a)f)(x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(x-y)^2} f(y)\,\mathrm{d}y, \quad x \in \mathbb{R}, \tag{2.2}$$

and letting

$$a_\omega(\xi) := \sqrt{\pi/\omega}\,\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\mathrm{i}\xi^2/(4\omega)}, \quad \xi \in \mathbb{R}, \tag{2.3}$$

we get the bounded operator

$$(C(a_\omega)f)(x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega(x-y)^2} f(y)\,\mathrm{d}y, \quad x \in \mathbb{R}, \tag{2.4}$$

In contrast to this, the operator $\mathcal{L}_\omega$ that is formally defined by

$$(\mathcal{L}_\omega f)(x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega|x-y|} f(y)\,\mathrm{d}y, \quad x \in \mathbb{R}, \tag{2.5}$$

is not bounded on $L^2(\mathbb{R})$. However, the compression of the last operator to L^2 over a finite interval is obviously compact.

It is well-known that $\sigma(C(a))$ is equal to $\mathcal{R}(a)$, the essential range of a. Note also that $C(a)$, $W(a)$, $W_\tau(a)$ are self-adjoint if a is real-valued.

Theorem 2.1 (Hartman and Wintner)**.** *If $a \in L^\infty(\mathbb{R})$ is real-valued, then $\sigma(W(a))$ equals* $\operatorname{conv}\mathcal{R}(a)$, *the convex hull of $\mathcal{R}(a)$.*

The analogue of this theorem for Toeplitz matrices appeared first in [18]. A full proof is also in Theorem 1.27 of [7] or Section 2.36 of [8]. The easiest way to pass from Toeplitz matrices to Wiener–Hopf operators is to employ the trick of Section 9.5(e) of [8].

Theorem 2.2. *If $a \in L^\infty(\mathbb{R})$ is real-valued, then $\sigma(W_\tau(a)) \subset \sigma(W(a))$ for every $\tau > 0$, and $\sigma(W_\tau(a))$ converges to $\sigma(W(a))$ in the Hausdorff metric as $\tau \to \infty$.*

This was established in [9]. Combining the last two theorems, we arrive at the conclusion that if $a \in L^\infty(\mathbb{R})$ is real-valued, then $\sigma(W_\tau(a)) \subset \operatorname{conv}\mathcal{R}(a)$ for every $\tau > 0$ and $\sigma(W_\tau(a)) \to \operatorname{conv}\mathcal{R}(a)$ in the Hausdorff metric.

Theorem 2.3. *If $a \in L^\infty(\mathbb{R})$ is real-valued and $\mathcal{R}(a)$ is not a singleton, then an endpoint of the line segment* $\operatorname{conv}\mathcal{R}(a)$ *cannot be an eigenvalue of $W_\tau(a)$.*

This is well-known. The short proof is as follows. It suffices to show that if $a \ge 0$ almost everywhere and $a > 0$ on a set of positive measure, then 0 is not an eigenvalue of $W_\tau(a)$. Assume the contrary, that is, let $W_\tau(a)f = 0$ for some $f \in L^2(0,\tau)$ with $\|f\| = 1$. Then

$$\begin{aligned} 0 &= (W_\tau(a)f, f) = (P_\tau P_+ F^{-1} M(a) F f, f) \\ &= (M(a)Ff, Ff) = \int_{-\infty}^{\infty} a(\xi)|\hat{f}(\xi)|^2 \,\mathrm{d}\xi. \end{aligned}$$

The Fourier transform of the compactly supported function f cannot vanish on a set of positive measure. Consequently, since $a > 0$ on a set of positive measure, we have

$$\int_{-\infty}^{\infty} a(\xi)|\hat{f}(\xi)|^2 \,\mathrm{d}\xi > 0,$$

which is a contradiction.

Theorem 2.4 (Szegő's First Limit Theorem)**.** *Let $a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ be a real-valued function and let $\varphi \in C(\mathbb{R})$ be a function such that $\varphi(x)/x$ has a finite limit as $x \to 0$. Then $\varphi(W_\tau(a))$ is a trace class operator for every $\tau > 0$, $\varphi \circ a$ belongs to $L^1(\mathbb{R})$, and*

$$\lim_{\tau\to\infty} \frac{\operatorname{tr}\varphi(W_\tau(a))}{\tau} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(a(\xi))\,\mathrm{d}\xi.$$

This theorem is proved in Section 8.6 of [17]. The following theorem, which was established in [21] and a second proof of which is also in [30], remarkably sharpens Theorem 2.4 for a special but important class of functions a. We denote by $\chi_{(\alpha,\beta)}$ the characteristic function of the interval (α,β).

Theorem 2.5 (Landau and Widom). *Let $\gamma > 0$ be a real number and (α, β) be a finite interval. Then for every $\varphi \in C^\infty(\mathbb{R})$ satisfying $\varphi(0) = 0$,*

$$\operatorname{tr}\varphi(W_\tau(\gamma\chi_{(\alpha,\beta)})) = \tau\,\frac{\varphi(\gamma)(\beta-\alpha)}{2\pi} + \frac{\log\tau}{\pi^2}\int_0^\gamma \frac{\gamma\varphi(x) - x\varphi(\gamma)}{x(\gamma - x)}\,\mathrm{d}x + O(1).$$

Let $\dot{\mathbb{R}}$ denote the one-point compactification of $\mathbb{R}$ and let $a \in C(\dot{\mathbb{R}})$. Then the spectrum of $W(a)$ is the union of the range $\mathcal{R}(a) = a(\mathbb{R}) \cup \{a(\infty)\}$ and all points in $\mathbb{C} \setminus \mathcal{R}(a)$ whose winding number with respect to the continuous and closed curve $\mathcal{R}(a)$ is nonzero (see, e.g., Theorem VII.3.6 of [16] or Theorem 2.42 plus Section 9.5(e) of [8]). It may happen that $\sigma(W(a)) = \mathcal{R}(a)$. This is of course the case if the function a is real-valued. We also encounter this situation if

$$a(\xi) = \hat{k}(\xi) = \int_{-\infty}^{\infty} k(t)\mathrm{e}^{\mathrm{i}\xi t}\,\mathrm{d}t$$

with a kernel $k \in L^1(\mathbb{R})$ which is even, $k(t) = k(-t)$ for all t. In the last case, $a(\xi)$ traces out a curve from the origin to $a(0)$ as ξ moves from $-\infty$ to 0 and then $a(\xi)$ goes back to the origin in the reverse direction along the same curve when ξ moves further from 0 to ∞. Thus, all points outside this curve have winding number zero.

Theorem 2.6. *Let $a \in C(\dot{\mathbb{R}}) \cap L^1(\mathbb{R})$ and suppose $\mathcal{R}(a)$ has no interior points and $\sigma(W(a)) = \mathcal{R}(a)$. Then $\sigma(W_\tau(a)) \to \mathcal{R}(a)$ in the Hausdorff metric as $\tau \to \infty$. Furthermore, if $\varphi : \mathbb{C} \to \mathbb{C}$ is a continuous function such that $\varphi(z)/z$ has a finite limit as $z \to 0$, then $\varphi \circ a$ is in $L^1(\mathbb{R})$ and*

$$\lim_{\tau\to\infty} \frac{\operatorname{tr}\varphi(W_\tau(a))}{\tau} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(a(\xi))\,\mathrm{d}\xi.$$

This theorem is the continuous analogue of results by Widom [32] and Tilli [25] (see also Example 5.39 of [7]). Some comments are in order.

First of all note that if λ is not in $\mathcal{R}(a)$, then $W_\tau(a) - \lambda I = W_\tau(a - \lambda)$ is invertible for all sufficiently large τ and the norms of the inverses are uniformly bounded (see, e.g., Theorem 9.40 of [8]). This in conjunction with the compactness of $\mathcal{R}(a)$ implies that if $\varepsilon > 0$ is given then $\sigma(W_\tau(a))$ is contained in the ε-neighbourhood of $\mathcal{R}(a)$ for all $\tau > \tau_0(\varepsilon)$. The second part of the theorem implies that in fact every point of $\mathcal{R}(a)$ is a limit point of a family $\{\lambda_\tau\}_{\tau>0}$ with $\lambda_\tau \in \sigma(W_\tau(a))$. Consequently, $\sigma(W_\tau(a)) \to \mathcal{R}(a)$ in the Hausdorff metric.

Secondly, let φ be as in the theorem. Then $\operatorname{tr}\varphi(W_\tau(a))$ may simply be interpreted as $\sum_j \varphi(\lambda_j)$ where $\{\lambda_j\}$ is the (at most countable) family of eigenvalues of $W_\tau(a)$, counted according to their algebraic multiplicity. To prove the second part of the theorem, one can proceed as in [25]. We only remark that our assumption ensures that $\varphi(z) = zh(z)$ with a continuous function $h : \mathbb{C} \to \mathbb{C}$, that, by Runge's theorem, h can be approximated uniformly on $\mathcal{R}(a)$ by rational functions r_n with prescribed poles in the bounded components of $\mathbb{C} \setminus \mathcal{R}(a)$, and that the second part of the theorem is easy to prove for $\varphi(z) = zr_n(z)$.

We finally turn to the continuous analogue of the Avram–Parter theorem, which says that we may drop real-valuedness in Theorem 2.4 when passing from eigenvalues to singular values or equivalently, when replacing $W_\tau(a)$ by $|W_\tau(a)| := (W_\tau(a)W_\tau(a)^*)^{1/2}$. Notice that the eigenvalues of $|W_\tau(a)|$ are just the singular values of $W_\tau(a)$.

Theorem 2.7 (Avram and Parter)**.** *Let $a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and let φ be a continuous function on $[0,\infty)$ such that $\varphi(x)/x$ has a finite limit as $x \to 0$. Then $\varphi(|W_\tau(a)|)$ is a trace class operator for every $\tau > 0$, $\varphi \circ |a|$ is a function in $L^1(\mathbb{R})$, and*

$$\lim_{\tau\to\infty} \frac{\operatorname{tr} \varphi(|W_\tau(a)|)}{\tau} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(|a(\xi)|)\, d\xi.$$

The discrete version of this theorem is due to Avram [2] and Parter [22]. The proofs given in Section 5.6 of [7] or in Section 4 of [6] for the Toeplitz case can be easily adapted to the Wiener–Hopf case.

3. Highly oscillatory convolution-type problems

An extremely fortunate peculiarity of the Fox–Li operator $\mathcal{F}_\omega$ is that $\mathcal{F}_\omega \mathcal{F}_\omega^*$ is also unitarily equivalent to a convolution operator over $(-1,1)$. Here is the precise result.

Lemma 3.1. *Let V be the unitary operator*

$$V : L^2(-1,1) \to L^2(-1,1), \quad (Vf)(x) := \mathrm{e}^{-\mathrm{i}\omega x^2} f(x).$$

Then

$$(V\mathcal{F}_\omega\mathcal{F}_\omega^* V^* f)(x) = \int_{-1}^{1} \frac{\sin(2\omega(x-y))}{\omega(x-y)}\, f(y)\,\mathrm{d}y, \quad x \in (-1,1).$$

Proof. Straightforward computation. □

Lemma 3.1 puts us into the following general context. Let a be a function in $L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Then $a \in L^2(\mathbb{R})$ and hence there is a function $k \in L^2(\mathbb{R})$ such that $a = \hat{k}$. Since $\hat{k} \in L^1(\mathbb{R})$, we also know that k is continuous and $k(\pm\infty) = 0$. For $\omega > 0$, we set

$$k_\omega(t) := k(\omega t)$$

and consider the compression of the convolution operator $C(\hat{k}_\omega)$ to $L^2(-1,1)$:

$$(C_{(-1,1)}(\hat{k}_\omega)f)(x) := \int_{-1}^{1} k(\omega(x-y)) f(y)\,\mathrm{d}y, \quad x \in (-1,1).$$

Lemma 3.1 just says that $\mathcal{F}_\omega\mathcal{F}_\omega^*$ is unitarily equivalent to $C_{(-1,1)}(\hat{k}_\omega)$ with

$$k(t) = \frac{\sin(2t)}{t},$$

in which case

$$a(\xi) = \hat{k}(\xi) = \int_{-\infty}^{\infty} \frac{\sin(2t)}{t} \, \mathrm{e}^{\mathrm{i}\xi t} \, \mathrm{d}t = \pi \chi_{(-2,2)}(\xi). \tag{3.1}$$

Lemma 3.2. *Let U be the unitary operator*

$$U : L^2(-1,1) \to L^2(0,\tau), \quad (Uf)(t) := \sqrt{\frac{2}{\tau}} \, f\left(\frac{2t-\tau}{\tau}\right).$$

Then

$$U C_{(-1,1)}(\hat{k}_\omega) U^* = \frac{2}{\tau} \, W_\tau(\hat{k}_{2\omega/\tau}).$$

Proof. Taking into account that U^* is given by

$$U^* : L^2(0,\tau) \to L^2(-1,1), \quad (U^* g)(x) = \sqrt{\frac{\tau}{2}} \, g\left(\frac{\tau x + \tau}{2}\right),$$

this can again be verified by direct computation. □

Theorem 3.3. *Let $a \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ be real-valued. Then the spectrum of $\omega C_{(-1,1)}(\hat{k}_\omega)$ is contained in $\operatorname{conv} \mathcal{R}(\hat{k})$ for every $\omega > 0$ and converges to $\operatorname{conv} \mathcal{R}(\hat{k})$ in the Hausdorff metric as $\omega \to \infty$. Moreover, if $\varphi \in C(\mathbb{R})$ and $\varphi(x)/x$ has a finite limit as $x \to 0$, then*

$$\lim_{\omega \to \infty} \frac{\operatorname{tr} \varphi(\omega C_{(-1,1)}(\hat{k}_\omega))}{2\omega} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\hat{k}(\xi)) \, \mathrm{d}\xi.$$

Proof. Using Lemma 3.2 with $\tau = 2\omega$, we observe that the spectrum of the operator $\omega C_{(-1,1)}(\hat{k}_\omega)$ coincides with the spectrum of $W_{2\omega}(\hat{k})$. All assertions are therefore immediate consequences of Theorems 2.1, 2.2 and 2.4. □

Proofs of Theorems 1.1 and 1.2. From Lemmas 3.1 and 3.2 and (3.1) we infer that the operator $\omega \mathcal{F}_\omega \mathcal{F}_\omega^* = \omega C_{(-1,1)}(\hat{k}_\omega)$ is unitarily equivalent to the operator

$$\omega \, \frac{2}{2\omega} \, W_{2\omega}(\hat{k}) = W_{2\omega}(\pi \chi_{(-2,2)}).$$

Theorem 3.3 now implies that $\omega s^2(\mathcal{F}_\omega)$ is a subset of $\operatorname{conv} \mathcal{R}(\pi \chi_{(-2,2)}) = [0,\pi]$ for all $\omega > 0$ and converges to $[0,\pi]$ in the Hausdorff metric as $\omega \to \infty$. The point π cannot belong to $\omega s^2(\mathcal{F}_\omega)$, since otherwise it would be an eigenvalue of $W_{2\omega}(\pi \chi_{(-2,2)})$, contradicting Theorem 2.3. This proves Theorem 1.1 and the first part of Theorem 1.2.

To prove the second part of Theorem 1.2, let

$$N_{(\alpha,\beta)} := |\omega s^2(\mathcal{F}_\omega) \cap (\alpha,\beta)|$$

for $0 \le \alpha < \beta \le \pi$ and notice that $N_{(\alpha,\beta)} = \operatorname{tr} \chi_{(\alpha,\beta)}(W_{2\omega}(\pi \chi_{(-2,2)}))$. We first consider $(\alpha,\beta) = (\pi - \varepsilon, \pi)$. Choose a function $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi(x) = 0$ for $x < \pi - \varepsilon - \eta$, $\varphi(x)$ increases from 0 to 1 for $\pi - \varepsilon - \eta < x < \pi - \varepsilon$, and $\varphi(x) = 1$ for $x > \pi - \varepsilon$. Since $\chi_{(\pi-\varepsilon,\pi)} \le \varphi$, we have

$$N_{(\pi-\varepsilon,\pi)} \le \operatorname{tr} \varphi(W_{2\omega}(\pi \chi_{(-2,2)})),$$

and Theorem 2.5 tells us that the right-hand side is

$$2\omega\,\frac{4}{2\pi}+\frac{\log(2\omega)}{\pi^2}\int_0^{\pi}\frac{\pi\varphi(x)-x}{x(\pi-x)}\,\mathrm{d}x+O(1).$$

The $O(1)$ does not exceed some constant $C_\eta<\infty$ and the integral equals

$$\int_0^{\pi-\varepsilon-\eta}\frac{-x}{x(\pi-x)}\,\mathrm{d}x+\int_{\pi-\varepsilon-\eta}^{\pi-\varepsilon}\frac{\pi\varphi(x)-x}{x(\pi-x)}\,\mathrm{d}x+\int_{\pi-\varepsilon}^{\pi}\frac{\pi-x}{x(\pi-x)}\,\mathrm{d}x,$$

which, for $\eta\to 0$, converges to

$$-\int_0^{\pi-\varepsilon}\frac{\mathrm{d}x}{\pi-x}+\int_{\pi-\varepsilon}^{\pi}\frac{\mathrm{d}x}{x}=\log\frac{\varepsilon}{\pi-\varepsilon}.$$

Thus, if $\delta>0$ is given, there is an $\eta>0$ and a constant $C_\eta<\infty$ such that

$$N_{(\pi-\varepsilon,\pi)}\le\frac{4\omega}{\pi}+\frac{\log(2\omega)}{\pi^2}\log\frac{\varepsilon}{\pi-\varepsilon}+\delta\,\frac{\log(2\omega)}{\pi^2}+C_\eta$$

for all $\omega>0$. Clearly, $C_\eta<(\delta/\pi^2)\log(2\omega)$ whenever $\omega>\omega_1(\delta)$, and for these ω we then have

$$N_{(\pi-\varepsilon,\pi)}\le\frac{4\omega}{\pi}+\frac{\log(2\omega)}{\pi^2}\log\frac{\varepsilon}{\pi-\varepsilon}+\frac{2\delta}{\pi^2}\log(2\omega).$$

Approximating the function $\chi_{(\pi-\varepsilon,\pi)}$ by a C^∞ function ψ from below, one can show analogously that given $\delta>0$, there is some $\omega_2(\delta)$ such that

$$N_{(\pi-\varepsilon,\pi)}\ge\frac{4\omega}{\pi}+\frac{\log(2\omega)}{\pi^2}\log\frac{\varepsilon}{\pi-\varepsilon}-\frac{2\delta}{\pi^2}\log(2\omega)$$

for all $\omega>\omega_2(\delta)$. Combining the last two estimates and taking into account that $o(\log(2\omega))=o(\log\omega)$, we obtain that

$$N_{(\pi-\varepsilon,\pi)}=\frac{4\omega}{\pi}+\frac{\log(2\omega)}{\pi^2}\log\frac{\varepsilon}{\pi-\varepsilon}+o(\log\omega),$$

as asserted. To prove the result for $N_{(\varepsilon,\pi-\varepsilon)}$ we may proceed similarly. Employing Theorem 2.5 with $\varphi(\pi)=0$ we get

$$\begin{aligned}N_{(\varepsilon,\pi-\varepsilon)}&=\frac{\log(2\omega)}{\pi^2}\int_{\varepsilon}^{\pi-\varepsilon}\frac{\pi}{x(\pi-x)}\,\mathrm{d}x+o(\log\omega)\\&=\frac{\log(2\omega)}{\pi^2}\cdot 2\,\log\frac{\pi-\varepsilon}{\varepsilon}+o(\log\omega),\end{aligned}$$

again as desired.

Finally, by Theorem 2.3, the operator $W_{2\omega}(\pi\chi_{(-2,2)})$ is injective. Consequently, so also is $\omega\mathcal{F}_\omega\mathcal{F}_\omega^*$, which implies that $\omega\mathcal{F}_\omega\mathcal{F}_\omega^*$ has dense and thus infinite-dimensional range. It follows that $\omega s^2(\mathcal{F}_\omega)\cap(0,\pi)$ is an infinite set. As this set has only $4\omega/\pi+o(\omega)$ points in $[\varepsilon,\pi)$, we arrive at the conclusion that infinitely many points must lie in $(0,\varepsilon)$. $\square$

Let us return to the general context of Theorem 3.3. Fix two real numbers $\alpha < \beta$ and suppose that $0 \notin [\alpha, \beta]$. Invoking Theorem 2.4 and including the characteristic function $\chi_{(\alpha,\beta)}$ between two continuous functions ψ and φ as in the preceding proof, one obtains that if the measure of the set $\{\xi : \hat{k}(\xi) = \alpha\} \cup \{\xi : \hat{k}(\xi) = \beta\}$ is zero then

$$\lim_{\omega\to\infty} \frac{|\sigma(\omega C_{(-1,1)}(\hat{k}_\omega)) \cap (\alpha,\beta)|}{2\omega} = \frac{1}{2\pi} \operatorname{mes} \{\xi : \hat{k}(\xi) \in (\alpha,\beta)\}, \tag{3.2}$$

$\operatorname{mes} E$ denoting the (Lebesgue) measure of E.

Example 3.4. Let $k(t) = \mathrm{e}^{-t^2}$, in which case

$$a(\xi) = \hat{k}(\xi) = \sqrt{\pi}\, \mathrm{e}^{-\xi^2/4}.$$

Theorems 3.3 and 2.3 along with (3.2) imply that then all eigenvalues of $\omega C_{(-1,1)}(\hat{k}_\omega)$ are contained in $[0, \sqrt{\pi})$, that they fill $[0, \sqrt{\pi}]$ densely as ω goes to ∞, and that the number of eigenvalues of $C_{(-1,1)}(\hat{k}_\omega)$ in $(\alpha/\omega, \beta/\omega)$ is

$$\frac{\omega}{\pi} \operatorname{mes} \{\xi : \alpha < \sqrt{\pi}\, \mathrm{e}^{-\xi^2/4} < \beta\} + o(\omega).$$

To have another example, take $k(t) = (1 - \cos t)/t^2$. Then

$$a(\xi) = \hat{k}(\xi) = \pi(1 - |\xi|)_+.$$

Hence, the eigenvalues of $\omega C_{(-1,1)}(\hat{k}_\omega)$ fill the segment $[0, \pi]$ densely and if $0 < \alpha < \beta \le \pi$, the number of eigenvalues of $C_{(-1,1)}(\hat{k}_\omega)$ belonging to $(\alpha/\omega, \beta/\omega)$ is

$$\frac{\omega}{\pi} \operatorname{mes} \{\xi : \alpha < \pi(1 - |\xi|)_+ < \beta\} + o(\omega) = \frac{2(\beta - \alpha)}{\pi^2}\, \omega + o(\omega). \qquad \square$$

The kernel k occurring in Theorem 3.3 satisfies $k(t) = \overline{k(-t)}$ for all t and is not necessarily in $L^1(\mathbb{R})$. The following result addresses eigenvalues for kernels in $L^1(\mathbb{R})$ for which $k(t) = k(-t)$. Notice that neither the former nor the latter assumptions are in force for the Fox–Li kernel $k(t) = \mathrm{e}^{\mathrm{i}t^2}$.

Theorem 3.5. *Let $k \in L^1(\mathbb{R})$ and suppose $k(t) = k(-t)$ for all t and $\mathcal{R}(\hat{k})$ has no interior points. Then $\omega\sigma(C_{(-1,1)}(\hat{k}_\omega))$ converges to $\mathcal{R}(\hat{k})$ in the Hausdorff metric as $\omega \to \infty$ and if $\varphi : \mathbb{C} \to \mathbb{C}$ is a continuous function such that $\varphi(z)/z$ has a finite limit as $z \to 0$, then $\varphi \circ a$ is in $L^1(\mathbb{R})$ and*

$$\lim_{\omega\to\infty} \frac{1}{2\omega} \sum_j \varphi(\omega\lambda_j) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\hat{k}(\xi))\, \mathrm{d}\xi,$$

the sum over the λ_j in $\sigma(C_{(-1,1)}(\hat{k}_\omega))$.

Proof. Lemma 3.2 with $\tau = 2\omega$ shows that $\omega C_{(-1,1)}(\hat{k}_\omega)$ is unitarily equivalent to $W_{2\omega}(\hat{k})$. As $\hat{k}(\xi) = \hat{k}(-\xi)$ for all ξ, it follows that $\sigma(W(\hat{k})) = \mathcal{R}(\hat{k})$. The assertion is therefore an immediate consequence of Theorem 2.6. $\square$

Herewith a result on the singular values for arbitrary kernels in $L^1(\mathbb{R})$.

Theorem 3.6. *Let $k \in L^1(\mathbb{R})$. Then $\omega s(C_{(-1,1)}(\hat{k}_\omega)) \subset \mathcal{R}(|\hat{k}|)$ for every $\omega > 0$ and $\omega s(C_{(-1,1)}(\hat{k}_\omega))$ converges to $\mathcal{R}(|\hat{k}|)$ in the Hausdorff metric as $\omega \to \infty$. Moreover, if φ is a continuous function on $[0,\infty)$ such that $\varphi(x)/x$ has a finite limit as $x \to 0$ then $\varphi \circ \hat{k}$ is in $L^1(\mathbb{R})$ and*

$$\lim_{\omega\to\infty} \frac{1}{2\omega} \sum_j \varphi(\omega s_j) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(|\hat{k}(\xi)|)\,\mathrm{d}\xi, \tag{3.3}$$

the sum over the s_j in $s(C_{(-1,1)}(\hat{k}_\omega))$.

Proof. Once more by Lemma 3.2 with $\tau = 2\omega$,

$$\omega^2 U C_{(-1,1)}(\hat{k}_\omega) C_{(-1,1)}(\hat{k}_\omega)^* U^* = W_{2\omega}(\hat{k}) W_{2\omega}(\hat{k})^*,$$

whence $\omega |C_{(-1,1)}(\hat{k}_\omega)| = |W_{2\omega}(\hat{k})|$. First of all, this shows that

$$\omega s(C_{(-1,1)}(\hat{k}_\omega)) \subset [0, \|W_\tau(\hat{k})\|] \subset [0, \max|\hat{k}|] = \mathcal{R}(|\hat{k}|), \tag{3.4}$$

and secondly, using Theorem 2.7 we obtain (3.3). Finally, (3.3) and (3.4) together imply the convergence of $\omega s(C_{(-1,1)}(\hat{k}_\omega))$ to $\mathcal{R}(\hat{k})$ in the Hausdorff metric. $\square$

Example 3.7. Let $k(t) = \mathrm{e}^{\mathrm{i}t^2}\mu(t)$ where μ is in $L^1(\mathbb{R})$ and $\mu(t) = \mu(-t)$ for all t. In that case the preceding two theorems are applicable and describe the eigenvalues and singular values of the operator

$$(\mathcal{F}_{\omega,\mu} f)(x) := \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega(x-y)^2} \mu(\sqrt{\omega}(x-y)) f(y)\,\mathrm{d}y, \quad x \in (-1,1),$$

which is just $C_{(-1,1)}(k_{\sqrt{\omega}})$. Take, for instance, $\mu_\varepsilon(t) = \mathrm{e}^{-\varepsilon t^2}$ with a fixed $\varepsilon > 0$, that is, consider

$$(\mathcal{F}_{\omega,\varepsilon} f)(x) := (\mathcal{F}_{\omega,\mu_\varepsilon} f)(x) = \int_{-1}^{1} \mathrm{e}^{(\mathrm{i}-\varepsilon)\omega(x-y)^2} f(y)\,\mathrm{d}y, \quad x \in (-1,1).$$

We have

$$\hat{k}(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{(\mathrm{i}-\varepsilon)t^2} \mathrm{e}^{i\xi t}\,\mathrm{d}t = \sqrt{\frac{\pi}{\varepsilon - \mathrm{i}}} \exp\left(-\frac{\xi^2}{4(\varepsilon - \mathrm{i})}\right), \tag{3.5}$$

which may also be written in the form

$$\hat{k}(\xi) = \sqrt{\frac{\pi(\varepsilon + \mathrm{i})}{1+\varepsilon^2}} \exp\left(-\frac{\varepsilon\xi^2}{4(1+\varepsilon^2)}\right) \exp\left(-\mathrm{i}\frac{\xi^2}{4(1+\varepsilon^2)}\right).$$

Thus, $\mathcal{R}(\hat{k})$ is a spiral commencing at $\sqrt{\pi(\varepsilon+\mathrm{i})/(1+\varepsilon^2)}$ (which is about $\sqrt{\pi}\,\mathrm{e}^{\mathrm{i}\pi/4}$ if $\varepsilon > 0$ is small) and rotating clockwise into the origin. Theorem 3.5 tells that when $\omega \to \infty$, then the set of the eigenvalues of $\sqrt{\omega}\mathcal{F}_{\omega,\varepsilon}$

converges in the Hausdorff metric to this spiral. The spiral has the parametric representation

$$z = \sqrt{\frac{\pi(\varepsilon+\mathrm{i})}{1+\varepsilon^2}}\,\mathrm{e}^{-(\mathrm{i}+\varepsilon)\varphi}, \quad 0 \le \varphi < \infty,$$

and the number of eigenvalues of the scaled operator $\sqrt{\omega}\mathcal{F}_{\omega,\varepsilon}$ that lie near the beginning arc of the spiral given by $0 \le \varphi < x$ is

$$\frac{\sqrt{\omega}}{\pi}\,\mathrm{mes}\left\{\xi \in \mathbb{R} : 0 \le \frac{\xi^2}{4(1+\varepsilon^2)} < x\right\} + o(\sqrt{\omega}).$$

On the other hand, Theorem 3.6 says that the singular values of $\sqrt{\omega}\mathcal{F}_{\omega,\varepsilon}$ densely fill the segment $[0, \max|\hat{k}|] = [0, \pi^{1/2}(1+\varepsilon^2)^{-1/4}]$ and that the number of singular values in $(\alpha/\sqrt{\omega}, \beta/\sqrt{\omega})$ is

$$\frac{\sqrt{\omega}}{\pi}\,\mathrm{mes}\left\{\xi \in \mathbb{R} : \alpha < \frac{\sqrt{\pi}}{\sqrt[4]{1+\varepsilon^2}}\,\exp\left(-\frac{\varepsilon\xi^2}{4(1+\varepsilon^2)}\right) < \beta\right\} + o(\sqrt{\omega}).$$

Figure 3 demonstrates how, for growing ω, the spectrum lies increasingly near to the spiral $\hat{k}$. □

Example 3.8. Fix $\varepsilon > 0$ and consider the operator

$$(\mathcal{M}_{\omega,\varepsilon} f)(x) := \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\omega|x-y|}\mathrm{e}^{-\varepsilon\omega|x-y|} f(y)\,\mathrm{d}y, \quad x \in (-1,1).$$

This is some kind of regularization of the compression of operator (2.5) to $L^2(-1,1)$. Clearly, $\mathcal{M}_{\omega,\varepsilon} = C_{(-1,1)}(\hat{k}_\omega)$ with $k(t) = \mathrm{e}^{(\mathrm{i}-\varepsilon)|t|}$. We have

$$\hat{k}(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{(\mathrm{i}-\varepsilon)|t|}\mathrm{e}^{i\xi t}\,\mathrm{d}t = \frac{2(\varepsilon-\mathrm{i})}{\xi^2 + (\varepsilon-\mathrm{i})^2},$$

and Theorems 3.5 and 3.6 imply that $\omega\sigma(\mathcal{M}_{\omega,\varepsilon})$ and $\omega s(\mathcal{M}_{\omega,\varepsilon})$ converge in the Hausdorff metric to $\mathcal{R}(\hat{k})$ and $\mathcal{R}(|\hat{k}|)$, respectively, as $\omega \to \infty$. It is readily seen that $\mathcal{R}(|\hat{k}|) = [0, m_\varepsilon]$ where

$$m_\varepsilon = \begin{cases} \varepsilon^{-1}(1+\varepsilon^2)^{1/2} & \text{for} \quad 0 < \varepsilon \le 1, \\ 2(1+\varepsilon^2)^{-1/2} & \text{for} \quad 1 \le \varepsilon < \infty. \end{cases}$$

To determine the range of $\hat{k}$, consider the Möbius transform

$$\gamma(z) := \frac{2(\varepsilon-\mathrm{i})}{z + (\varepsilon-\mathrm{i})^2}.$$

The set $\gamma(\mathbb{R})$ is a circle passing through $\gamma(\infty) = 0$ and $\gamma(1-\varepsilon^2) = 1/\varepsilon + \mathrm{i}$, while $\gamma(1-\varepsilon^2+\mathrm{i}\mathbb{R})$ is the straight line through $\gamma(\infty) = 0$, $\gamma(1-\varepsilon^2) = 1/\varepsilon + \mathrm{i}$ and $\gamma(-(\varepsilon-\mathrm{i})^2) = \infty$. As $\mathbb{R}$ and $1-\varepsilon^2+\mathrm{i}\mathbb{R}$ intersect at a right angle, so also must $\gamma(\mathbb{R})$ and $\gamma(1-\varepsilon^2+\mathrm{i}\mathbb{R})$. Consequently, the line segment $[0, 1/\varepsilon + \mathrm{i}]$ is a diameter of the circle $\gamma(\mathbb{R})$, which shows that

$$\gamma(\mathbb{R}) = \left\{z \in \mathbb{C} : \left|z - \frac{1}{2}\left(\frac{1}{\varepsilon} + \mathrm{i}\right)\right| = \frac{1}{2}\sqrt{\frac{1}{\varepsilon^2} + 1}\right\}. \tag{3.6}$$

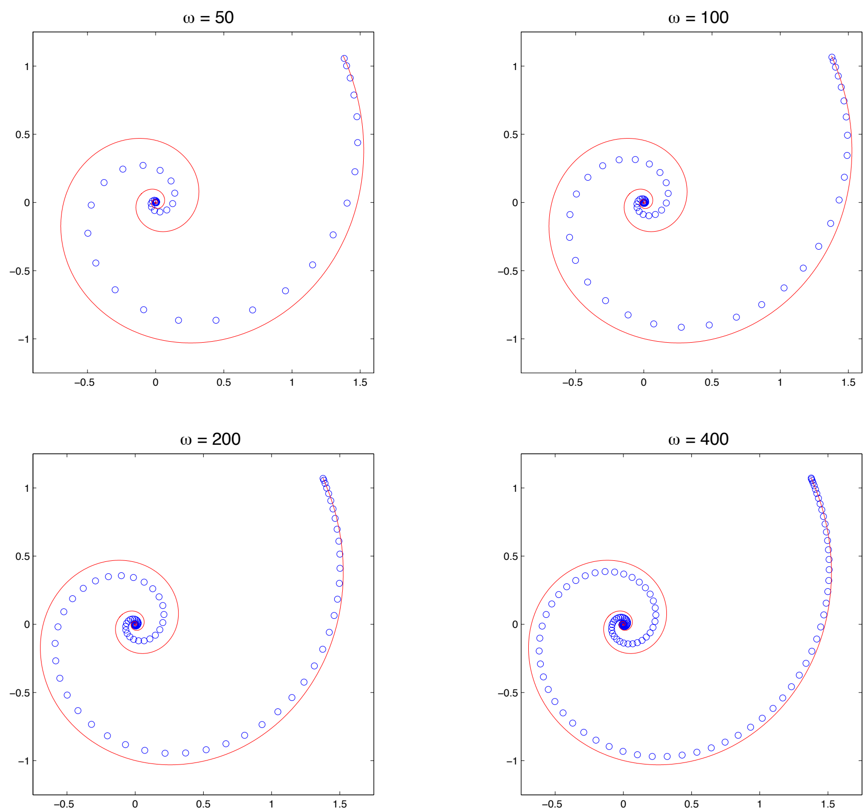

FIGURE 3. Spectra of $\mathcal{F}_{\omega,\varepsilon}$ from Example 3.7 for $\varepsilon = 1/4$ and different values of ω, as well as the spiral $\hat{k}$.

The range $\mathcal{R}(\hat{k})$ is $\gamma([0,\infty])$, and a moment's thought reveals that this is the arc of $\gamma(\mathbb{R})$ that is described in the clock-wise direction from $\gamma(0) = 2/(\varepsilon - \mathrm{i})$ to $\gamma(\infty) = 0$. Figure 4 illustrates how the eigenvalues approximate the circle (3.6) and that their distribution mimics the values of $\hat{k}$ at equally spaced points. Notice that the convergence is very slow for small $\varepsilon > 0$.

As in Example 3.8, the limit passage $\varepsilon \to 0$ does not yield anything for $\mathcal{M}_\omega := \mathcal{M}_{\omega,0}$, the compression of operator (2.5) to $L^2(-1,1)$. However, the case $\varepsilon = 0$ was treated in [10] on the basis of pure asymptotic expansions, and that paper virtually completely explains the asymptotic behaviour of the eigenvalues of $\mathcal{M}_\omega$. We refer in this connection also to [11]. On the other hand, apart from the scaling, the lower right picture of Figure 4 nicely resembles the numerical data for the operator $\mathcal{M}_\omega$ shown in papers [11, Figure 1.2] and [10, Figure 2]. $\square$

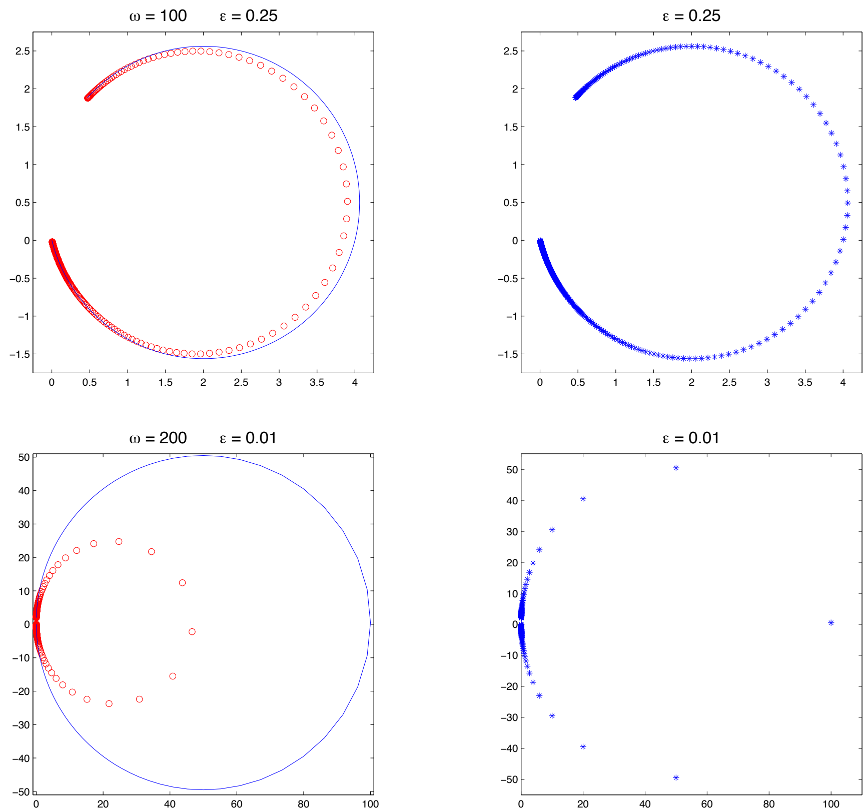

FIGURE 4. Spectra of $\mathcal{M}_{\omega,\varepsilon}$ from Example 3.8 for different values of ε and ω (left), as well as the values of the Fourier transform $\hat{k}(\xi)$ for $\xi = j/100$ with $j = 0, 1, \dots, 5000$ (right).

4. Attempts on the Fox–Li spectrum itself

Inasmuch as the singular values of $\mathcal{F}_\omega$ or the eigenvalues of the operator $\mathcal{F}_{\omega,\varepsilon}$ of Example 3.7 are interesting, the real prize is the spectrum of the Fox–Li operator $\mathcal{F}_\omega$.

Staying within Wiener–Hopf operators. We know from (2.4) that $\mathcal{F}_\omega$ equals $C_{(-1,1)}(a_\omega)$ with a_ω given by (2.3). To get a large truncated Wiener-Hopf operator, we employ Lemma 3.2 with $\tau = 2\sqrt{\omega}$ and see that $\mathcal{F}_\omega$ is unitarily equivalent to

$$\frac{1}{\sqrt{\omega}}\, W_{2\sqrt{\omega}}(a) \quad \text{with} \quad a(\xi) = \sqrt{\pi}\, \mathrm{e}^{\mathrm{i}\pi/4} \mathrm{e}^{-\mathrm{i}\xi^2/4}.$$

(Note that this and also (2.1) formally result from (3.5) with $\varepsilon = 0$.) However, because a is neither in $C(\dot{\mathbb{R}})$ nor in $L^1(\mathbb{R})$, Theorem 2.6 is not applicable.

The convolution operator generated by a has the kernel $\ell(t) := \mathrm{e}^{\mathrm{i}t^2}$. In Example 3.7 we saved matters by passing from $\ell(t)$ to $\ell(t)\mathrm{e}^{-\varepsilon t^2}$ for $\varepsilon > 0$, but this operation changed the operator and thus also its spectral characteristics dramatically. Another strategy is to consider

$$\ell^{[\omega]}(t) := \chi_{(-2\sqrt{\omega},2\sqrt{\omega})}(t)\,\mathrm{e}^{\mathrm{i}t^2}, \tag{4.1}$$

which a function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that

$$\ell^{[\omega]}_{\sqrt{\omega}}(t) := \ell^{[\omega]}(\sqrt{\omega}t) = \chi_{(-2,2)}(t)\mathrm{e}^{\mathrm{i}\omega t^2}$$

and which allows us to write

$$\mathcal{F}_\omega = C_{(-1,1)}\left(\hat{\ell}^{[\omega]}_{\sqrt{\omega}}\right).$$

Now Lemma 3.2 with $\tau = 2\sqrt{\omega}$ yields

$$U\mathcal{F}_\omega U^* = \frac{1}{\sqrt{\omega}}\,W_{2\sqrt{\omega}}(\hat{\ell}^{[\omega]}) \tag{4.2}$$

with

$$\hat{\ell}^{[\omega]}(\xi) = \int_{-2\sqrt{\omega}}^{2\sqrt{\omega}} \mathrm{e}^{\mathrm{i}t^2}\mathrm{e}^{\mathrm{i}\xi t}\,\mathrm{d}t. \tag{4.3}$$

Consequently,

$$\sqrt{\omega}\,\sigma(\mathcal{F}_\omega) = \sigma(W_{2\sqrt{\omega}}(\hat{\ell}^{[\omega]})). \tag{4.4}$$

But what is the spectrum on the right of (4.4)? Note that both the truncation interval and the generating function of the Wiener–Hopf operator depend on the parameter ω.

Fix ω and consider the Wiener–Hopf operator $W(\hat{\ell}^{[\omega]})$. From (4.3) we see that $\hat{\ell}^{[\omega]}$ is an analytic and even function. Consequently, $\mathcal{R}(\hat{\ell}^{[\omega]})$ has no interior points and $\sigma(W(\hat{\ell}^{[\omega]}) = \mathcal{R}(\hat{\ell}^{[\omega]})$. Theorem 2.6 therefore implies that the eigenvalues of $W_\tau(\hat{\ell}^{[\omega]})$ are asymptotically distributed (in a well-defined sense) along the curve $\mathcal{R}(\hat{\ell}^{[\omega]})$ as $\tau \to \infty$.

The problem is that in our case $\tau = 2\sqrt{\omega}$ is not independent of ω. So let us, flying in the face of mathematical rigour, keep the dependence of the generating function on ω but assume that if ω is very large then convolution over $(0, 2\sqrt{\omega})$ may be replaced by convolution over $(0,\infty)$. This amounts to the replacement

$$\sigma(W_{2\sqrt{\omega}}(\hat{\ell}^{[\omega]})) \approx \sigma(W(\hat{\ell}^{[\omega]})) = \mathcal{R}(\hat{\ell}^{[\omega]}) \tag{4.5}$$

and thus to saying that $\sigma(\mathcal{F}_\omega) \approx (1/\sqrt{\omega})\mathcal{R}(\hat{\ell}^{[\omega]})$: cf. Figure 5. However, we emphasize once again that we cannot muster any rigorous argument that would justify the replacement (4.5).

Turning to Toeplitz matrices. The following approach seems to be equally unsuccessful theoretically but provides at least a better chance for tackling the problem numerically. Namely, we fix ω and discretize $\mathcal{F}_\omega$ at

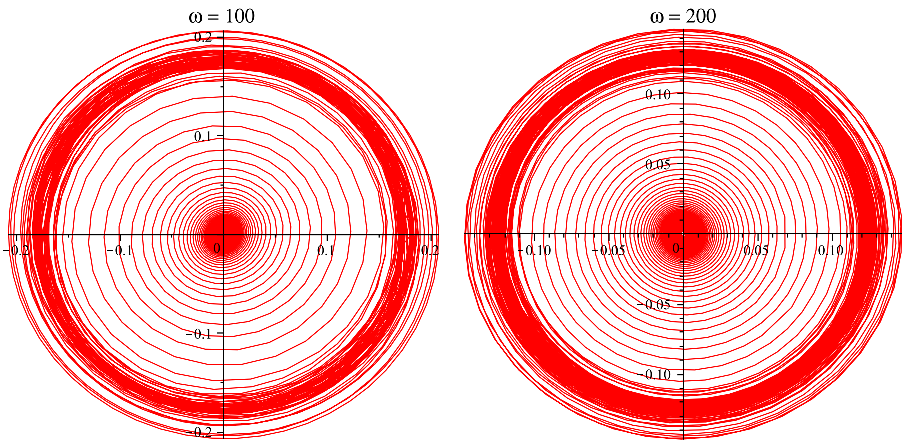

FIGURE 5. The spirals $(1/\sqrt{\omega})\mathcal{R}(\hat{\ell}^{[\omega]})$ for $\omega = 100$ and $\omega = 200$.

$2N+1$ equidistant points, whereby $\mathcal{F}_\omega f = \lambda f$ is approximated by the algebraic eigenvalue problem

$$B^{[N]} f^{[N]} = \lambda^{[N]} f^{[N]},$$

where

$$B^{[N]} := (v_{j-k}^{[N]})_{j,k=-N}^{N} \quad \text{with} \quad v_n^{[N]} := \frac{1}{N} \mathrm{e}^{\mathrm{i}\omega n^2/N^2}.$$

Thus, Toeplitz matrices enter the scene.[1] Given a function v in L^1 over the unit circle $\mathbb{T}$ with Fourier coefficients

$$v_n := \frac{1}{2\pi} \int_0^{2\pi} v(\mathrm{e}^{i\theta}) \mathrm{e}^{-\mathrm{i}n\theta}\, d\theta, \quad n \in \mathbb{Z},$$

let $T(v)$ and $T_N(v)$ denote the infinite Toeplitz matrix $(v_{j-k})_{j,k=0}^{\infty}$ and the $(2N+1) \times (2N+1)$ Toeplitz matrix $(v_{j-k})_{j,k=-N}^{N}$, respectively. Note that $T(v)$ induces a bounded operator on $\ell^2(\mathbb{Z}_+)$ if and only if $v \in L^\infty(\mathbb{T})$. We may now write

$$B^{[N]} = T_N(v^{[N]}) \tag{4.6}$$

where

$$v^{[N]}(\mathrm{e}^{\mathrm{i}\theta}) := \sum_{n=-2N}^{2N} v_n^{[N]} \mathrm{e}^{in\theta} = \frac{1}{N} \sum_{n=-2N}^{2N} \mathrm{e}^{\mathrm{i}\omega n^2/N^2} \mathrm{e}^{in\theta}. \tag{4.7}$$

Clearly, (4.6) is just the discrete analogue of (4.2) while (4.7) corresponds to (4.1). This time we don't have a perfect counterpart of (4.4), that is,

[1]Better quality of approximation follows once we half each $B_{\pm N,k}^{[N]}$, a procedure that corresponds to discretizing the integral with the compound trapezoidal rule. However, once we do so, the Toeplitz structure is lost.

the equality $\sigma(\mathcal{F}_\omega) = \sigma(T_N(v^{[N]}))$. However, since $\mathcal{F}_\omega$ is compact, standard approximation arguments reveal that

$$\sigma(T_N(v^{[N]})) \to \sigma(\mathcal{F}_\omega) \text{ in the Hausdorff metric as } N \to \infty. \tag{4.8}$$

This might be a reasonable basis for approximating $\sigma(\mathcal{F}_\omega)$ numerically. (Note that Figure 1 was produced in just this manner, by computing the eigenvalues of $T_N(v^{[N]})$ for really large N. This brute force approach to eigenvalue approximation, which is justified by the compactness of the Fox–Li operator, can be much improved by using the methodology of [11].) But as both the order and the generating function of the Toeplitz matrices $T_N(v^{[N]})$ vary with N, a theoretical prediction of the limit of $\sigma(T_N(v^{[N]}))$ is difficult.

The function $v^{[N]}(\mathrm{e}^{i\theta})$ is again an analytic and even function of $\theta \in [-\pi, \pi]$ and hence $\sigma(T(v^{[N]})) = v^{[N]}(\mathbb{T})$, that is, we may have recourse to the discrete version of Theorem 2.6. The analogue of (4.5) is the replacement

$$\sigma(T_N(v^{[N]})) \approx \sigma(T(v^{[N]})) = v^{[N]}(\mathbb{T}) \tag{4.9}$$

and thus the approximation $\sigma(\mathcal{F}_\omega) \approx v^{[N]}(\mathbb{T})$, but as in the case of Wiener–Hopf operators, we do not know any rigorous justification for (4.9).

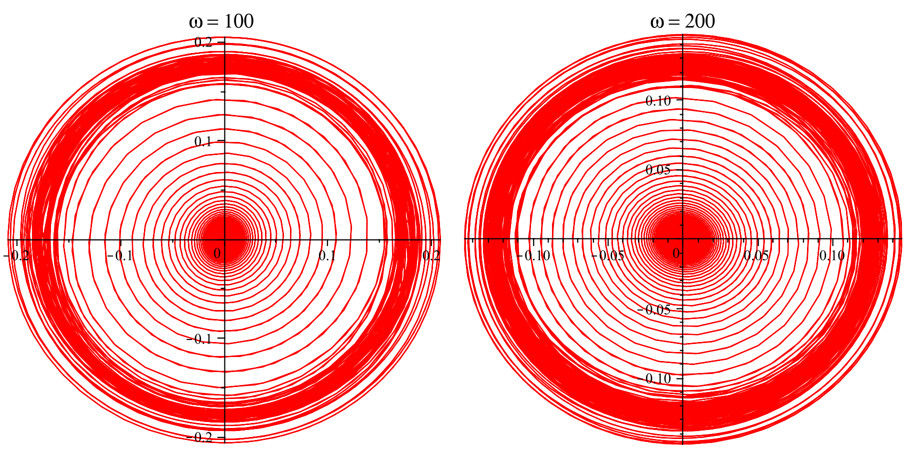

FIGURE 6. The spirals $v^{[N]}(\mathbb{T})$ for $N = 500$, $\omega = 100$ and $\omega = 200$.

Figure 6 displays the spirals $v^{[N]}(\mathbb{T})$ for two different values of ω. Note the uncanny similarity of Figures 5 and 6. This is striking enough to call for an explanation. Commencing from (4.7) with $\theta = \sqrt{\omega}\,\xi/N$, we approximate

$$\begin{aligned} v^{[N]}(\mathrm{e}^{\mathrm{i}\theta}) &= \frac{1}{N}\sum_{n=-2N}^{2N} \mathrm{e}^{\mathrm{i}\omega n^2/N^2}\mathrm{e}^{\mathrm{i}\sqrt{\omega}\,\xi n/N} = \int_{-2}^{2} \mathrm{e}^{\mathrm{i}\omega x^2}\mathrm{e}^{\mathrm{i}\sqrt{\omega}\,\xi x}\,\mathrm{d}x + O(1/N) \\ &= \frac{1}{\sqrt{\omega}}\int_{-2\sqrt{\omega}}^{2\sqrt{\omega}} \mathrm{e}^{\mathrm{i}t^2}\mathrm{e}^{\mathrm{i}\xi t}\,\mathrm{d}t + O(1/N) = \frac{1}{\sqrt{\omega}}\,\hat{\ell}^{[\omega]}(\xi) + O(1/N), \end{aligned}$$

and this explains the similarity of Figures 5 and 6. Insofar as the Fox–Li spectrum is concerned, comparison with Figure 1 shows that these two figures are equally wrong and that, consequently, the replacements (4.5) and (4.9) indeed lead us astray.

Incidentally, the integral in (4.3) can be computed explicitly: after some elementary algebra we have

$$\hat{\ell}^{[\omega]}(\xi) = \frac{\pi^{\frac{1}{2}} \mathrm{e}^{-\frac{1}{4}\mathrm{i}\xi^2}}{2(-\mathrm{i}\omega)^{\frac{1}{2}}} \left[\operatorname{erf}\left(2(-\mathrm{i}\omega)^{\frac{1}{2}} + \tfrac{1}{2}(-\mathrm{i})^{\frac{1}{2}}\xi\right) + \operatorname{erf}\left(2(-\mathrm{i}\omega)^{\frac{1}{2}} - \tfrac{1}{2}(-\mathrm{i})^{\frac{1}{2}}\xi\right)\right].$$

The asymptotic estimate

$$\operatorname{erf} z = 1 - \mathrm{e}^{-z^2}/(\pi^{\frac{1}{2}} z) + O(z^{-3}),$$

which is valid for $|\arg z| < \frac{3}{4}\pi$ [1, p. 298], easily shows that

$$\hat{\ell}^{[\omega]}(\xi) \approx \frac{(\mathrm{i}\pi)^{\frac{1}{2}} \mathrm{e}^{-\frac{1}{4}\mathrm{i}\xi^2}}{\omega^{\frac{1}{2}}} - \frac{\mathrm{i}\mathrm{e}^{4\mathrm{i}\omega}}{\omega^{\frac{1}{2}}} \left(\frac{\mathrm{e}^{-2\mathrm{i}\omega^{\frac{1}{2}}\xi}}{4\omega^{\frac{1}{2}} - \xi} + \frac{\mathrm{e}^{2\mathrm{i}\omega^{\frac{1}{2}}\xi}}{4\omega^{\frac{1}{2}} + \xi}\right) + O(\omega^{-2})$$

for $4\sqrt{\omega} > |\xi|$ and

$$\hat{\ell}^{[\omega]}(\xi) \approx \frac{\mathrm{i}\mathrm{e}^{4\mathrm{i}\omega}}{\omega^{\frac{1}{2}}} \left(\frac{\mathrm{e}^{-2\mathrm{i}\omega^{\frac{1}{2}}\xi}}{\xi - 4\omega^{\frac{1}{2}}} - \frac{\mathrm{e}^{2\mathrm{i}\omega^{\frac{1}{2}}\xi}}{\xi + 4\omega^{\frac{1}{2}}}\right) + O(\xi^{-2})$$

for $|\xi| > 4\sqrt{\omega}$. This explains the two regimes observed in the spiral in Figures 5 and 6: an extended rotation with roughly equal amplitude as long as $|\xi| < 4\omega^{\frac{1}{2}}$, followed by attenuation.

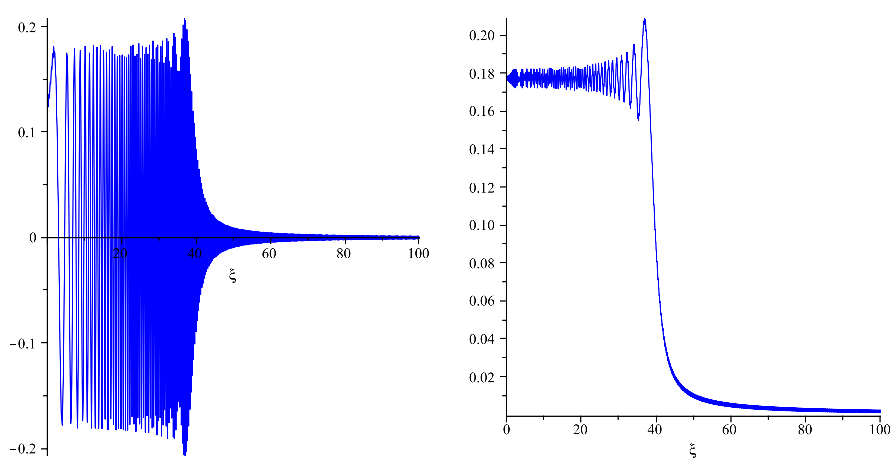

FIGURE 7. The real part and the absolute value, respectively, of the spiral $(1/\sqrt{\omega})\mathcal{R}(\hat{\ell}^{[\omega]})$ from Figure 4 for $\omega = 100$.

In Figure 7 we display the real part and the absolute value, respectively, of the spiral $(1/\sqrt{\omega})\mathcal{R}(\hat{\ell}^{[\omega]})$ for $\omega = 100$ and $\xi \geq 0$. Note that the maximum of the absolute value is attained at $4\sqrt{\omega} = 40$ and it neatly separates the two regimes which we have just described.

Is theta-three the power broker behind the scene? Another interesting observation, so far without any obvious implications for the spectrum of the Fox–Li operator, is the close connection of the function $v^{[N]}$ with the Jacobi theta function θ_3. Recalling the definition of $v^{[N]}$, we have

$$v^{[N]}(\mathrm{e}^{2\mathrm{i}\alpha}) = \frac{1}{N}\left[1 + 2\sum_{k=1}^{2N} q_N^{k^2}\cos(2\alpha k)\right] \quad \text{with} \quad q_N = \mathrm{e}^{\mathrm{i}\omega/N^2}.$$

Compare this with the standard definition (for which, see, e.g., p. 314 of [23]):

$$\theta_3(\alpha, q) := 1 + 2\sum_{k=1}^{\infty} q^{k^2}\cos(2\alpha k).$$

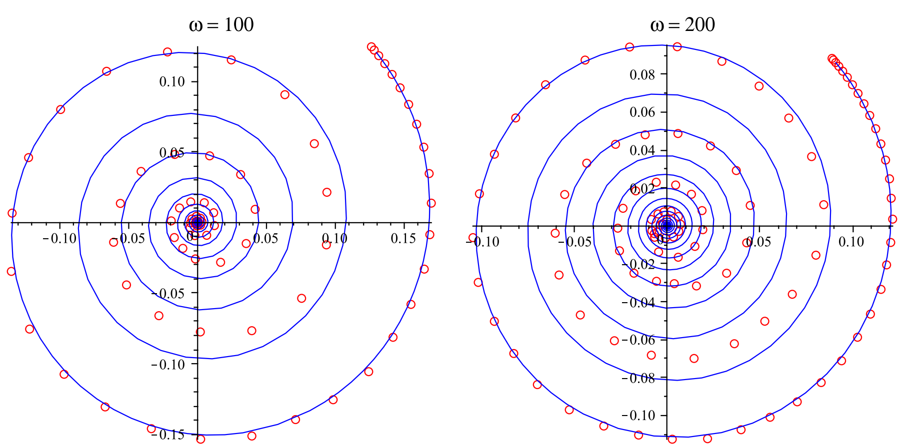

FIGURE 8. Attenuated theta spirals, superimposed on the spectra, for $\omega = 100$ and $\omega = 200$.

The snag is that absolute convergence of the series requires $|q| < 1$, while $|q_N| = 1$. What makes $v^{[N]}$ stay nice when $N \to \infty$ is the normalizing factor $1/N$.

Yet, there appears to be a connection between the theta function and $\sigma(\mathcal{F}_\omega)$, and this is confirmed by our numerical experimentation. Thus, we consider sequences $\mathbf{q} = \{q_{N,\omega}\}_{N=1}^{\infty}$ such that $|q_{N,\omega}| < 1$ for all N and

$$\lim_{N\to\infty} \frac{q_{N,\omega}}{q_N} = \lim_{N\to\infty} \frac{q_{N,\omega}}{\mathrm{e}^{\mathrm{i}\omega/N^2}} = 1,$$

and examine the quotient

$$\frac{\theta_3(\alpha, q_{N,\omega})}{N} \quad \text{for} \quad N \gg 1, \quad -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}.$$

(It is enough, by symmetry, to restrict α to $[0, \pi/2]$.) Everything now depends on the specific choice of the sequence $\mathbf{q}$: in our experience, we need to attenuate $|q_N|$ by exactly the right amount to obtain a good fit with the Fox–Li spiral. After a large number of trials, we have used

$$q_{N,\omega} = 1 - \frac{\omega^{1/2}}{2^{1/2}N^2},$$

and this results in Figure 8, where we have superimposed the theta function curve on the eigenvalues of $\mathcal{F}_\omega$ for $\omega = 100$ and $\omega = 200$. Although the match is far from perfect, in particular in the intermediate regime along the spiral, and we can provide neither rigorous proof nor intuitive explanation, there is enough in the figure to indicate that, at the very last, we might be on the right track in seeking the explicit form for the spectral spiral of $\sigma(\mathcal{F}_\omega)$.

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