Properties of the Steiner Triple Systems of Order 19

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Abstract

Properties of the 11 084 874 829 Steiner triple systems of order 19 are examined. In particular, there is exactly one 5-sparse, but no 6-sparse, STS(19); there is exactly one uniform STS(19); there are exactly two STS(19) with no almost parallel classes; all STS(19) have chromatic number 3; all have chromatic index 10, except for 4 075 designs with chromatic index 11 and two with chromatic index 12; all are 3-resolvable; and there are exactly two 3-existentially closed STS(19).

Keywords: automorphism, chromatic index, chromatic number, configuration, cycle structure, existential closure, independent set, partial parallel class, rank, Steiner triple system of order 19.

1 Introduction

A *Steiner triple system* (STS) is a pair $(X, \mathcal{B})$, where X is a finite set of *points* and $\mathcal{B}$ is a collection of 3-subsets of points, called *blocks* or *triples*, with the property that every 2-subset of points occurs in exactly one block. The size of the point set, $v := |X|$, is the *order* of the design, and an STS of order v is commonly denoted by STS(v). Steiner triple systems form perhaps the most fundamental family of combinatorial designs; it is well known that they exist exactly for orders $v \equiv 1, 3 \pmod 6$ [31].

Two STS(v) are *isomorphic* if there is a bijection between their point sets that maps blocks onto blocks. Denoting the number of isomorphism classes of STS(v) by $N(v)$, we have $N(3) = 1$, $N(7) = 1$, $N(9) = 1$, $N(13) = 2$ and $N(15) = 80$. Indeed, due to their relatively small number, the STSs up to order 15 have been studied in detail and are rather well understood. An extensive study of their properties was carried out by Mathon, Phelps and Rosa in the early 1980s [35].

For the next admissible parameter, we have $N(19) = 11\,084\,874\,829$, obtained in [26]. Of course, this huge number prohibits a discussion of each individual design. Because the designs are publicly available in compressed form [28], however, examination of some of their properties can be easily automated. Computing resources set a strict limit on what is feasible: one CPU year permits 2.8 milliseconds on average for each design.

Many properties of interest can nonetheless be treated. In Section 2, results, mainly of a computational nature, are presented. They show, amongst other things, that there is exactly one 5-sparse, but no 6-sparse, STS(19); that there is one uniform STS(19); that there are two STS(19) with no almost parallel classes; that all STS(19) have chromatic number 3; that all have chromatic index 10, except for 4 075 designs with chromatic index 11 and two with chromatic index 12; that all STS(19) are 3-resolvable; and that there are two 3-existentially closed STS(19). Some tables from the original classification [26] are repeated for completeness. In Section 3, some properties that remain open are mentioned, and the computational resources needed in the current work are briefly discussed.

Table 1: Automorphism group order

$\|\mathrm{Aut}\|$	#	$\|\mathrm{Aut}\|$	#	$\|\mathrm{Aut}\|$	#	$\|\mathrm{Aut}\|$	#
1	11 084 710 071	8	101	19	1	96	1
2	149 522	9	19	24	11	108	1
3	12 728	12	37	32	3	144	1
4	2 121	16	13	54	2	171	1
6	182	18	11	57	2	432	1

2 Properties

2.1 Automorphisms

The automorphisms and automorphism groups of the STS(19) were studied in [6, 26]; we reproduce the results here (with a correction in our Table 2).

Representing an automorphism as a permutation of the points, the nonidentity automorphisms can be divided into two types based on their order. The automorphisms of prime order have six cycle types

$$19^1, \quad 1^1 2^9, \quad 1^1 3^6, \quad 1^3 2^8, \quad 1^7 2^6, \quad 1^7 3^4,$$

and the automorphisms of composite order have nine cycle types

$$1^1 9^2, \quad 1^1 6^3, \quad 1^1 3^2 6^2, \quad 1^1 2^1 4^4, \quad 1^1 2^1 8^2, \quad 1^3 8^2, \quad 1^3 4^4, \quad 1^3 2^2 6^2, \quad 1^3 2^2 4^3.$$

Table 1 gives the order of the automorphism group for each isomorphism class. Tables 2 and 3 partition the possible orders of the automorphism groups into classes based on the types of prime and composite automorphisms that occur in the group. Compared with [26], Table 2 has been corrected by transposing the classes 18c and 18d, and the classes 12a and 12b (this correction is incorporated in the table reproduced in [4]).

A list of the 104 STS(19) having an automorphism group of order at least 9 is given in compact notation in the supplement to [6]. Cyclic STS(19) were first enumerated in [1] and 2-rotational ones (automorphism cycle type $1^1 9^2$) in [38]; these systems are listed in [35]. The 184 reverse STS(19) (automorphism cycle type $1^1 2^9$), together with their automorphism groups, were determined in [10].

In this paper, certain STS(19) are identified as follows: A1–A4 are the cyclic systems as listed in [35]; B1–B10 are the 2-rotational STS(19) as listed in [35]; and S1–S7 are the sporadic STS(19) listed in the Appendix. In addition, an STS(19) can be identified by the order of its automorphism group when this is unique (the listings in [6] are useful for retrieving such designs). Design A4, with an automorphism group of order 171, is both cyclic and 2-rotational and is therefore also listed as B8 in [35]; it is the *Netto triple system* [39]. A reader interested in copies of STS(19) that are not included among the sporadic examples here will apparently need to carry out some computational work, perhaps utilizing the catalogue from [28]—the authors of the current work are glad to provide consultancy for such an endeavour.

Table 2: Automorphisms (prime order)

Order	Class	19^1	$1^1 2^9$	$1^1 3^6$	$1^3 2^8$	$1^7 2^6$	$1^7 3^4$	#
432				*	*	*	*	1
171		*		*				1
144				*	*	*		1
108				*	*	*	*	1
96				*	*	*		1
57		*		*				2
54				*		*	*	2
32					*	*		3
24				*	*	*		11
19		*						1
18	a		*	*				1
	b			*	*		*	2
	c			*		*	*	6
	d			*		*		2
16					*	*		13
12	a			*	*	*		8
	b			*	*			7
	c			*		*		12
	d				*	*	*	10
9				*				19
8	a				*	*		84
	b				*			17
6	a		*	*				14
	b			*	*			14
	c			*		*		116
	d				*		*	10
	e					*	*	28
4	a				*	*		839
	b				*			662
	c					*		620
3	a			*				12 664
	b						*	64
2	a		*					169
	b				*			78 961
	c					*		70 392
#		4	184	12 885	80 645	72 150	124	164 758

Table 3: Automorphisms (composite order)

Class	$1^1 9^2$	$1^1 6^3$	$1^1 3^2 6^2$	$1^1 2^1 4^4$	$1^1 2^1 8^2$	$1^3 8^2$	$1^3 4^4$	$1^3 2^2 6^2$	$1^3 2^2 4^3$	#
432			*			*	*	*		1
171	*									1
144			*	*	*		*			1
108			*					*		1
96			*				*			1
57										2
54			*							2
32				*			*			3
24			*							11
19										1
18a		*								1
18b								*		2
18c			*							6
18d			*							2
16				*	*		*			5
16				*						6
16						*	*			1
16							*			1
12a			*							8
12b										7
12c										12
12d								*		10
9	*									9
9										10
8a							*			2
										82
8b				*			*			5
					*		*			10
						*	*			2
6a		*								14
6b										14
6c			*							104
										12
6d								*		10
6e										28
4a										839
4b				*						498
							*			153
										11
4c									*	48
										572
#	10	15	137	518	16	4	185	24	48	

Table 4: Number of subsystems

STS(7)	STS(9)	#	STS(7)	STS(9)	#
0	0	10 997 902 498	3	1	45
0	1	270 784	4	0	2 449
1	0	86 101 058	4	1	25
1	1	12 956	6	0	75
2	0	572 471	6	1	5
2	1	641	12	0	2
3	0	11 819	12	1	1

2.2 Subsystems and Ranks

A *subsystem* in an STS is a subset of blocks that forms an STS on a subset of the points. A subsystem in an STS(v) has order at most $(v-1)/2$; hence a subsystem in an STS(19) has order 3, 7 or 9. Moreover, the intersection of two subsystems is a subsystem. It follows that each STS(19) has at most one subsystem of order 9, with equality for 284 457 isomorphism classes [42]. The number of subsystems of each order in each isomorphism class was determined in [29] and these results are collected in Table 4. The STS(19) with 12 subsystems of order 7 and 1 subsystem of order 9 is the system having an automorphism group of order 432, and the other two STS(19) with 12 subsystems of order 7 are the systems having automorphism groups of orders 108 and 144.

The *rank* of an STS is the linear rank of its point–block incidence matrix over GF(2). In this setting, a nonempty set of points is (linearly) dependent if every block intersects the set in an even number of points. Counting the point–block incidences in a dependent set in two different ways, one finds that a dependent set necessarily consists of $(v+1)/2$ points so that its complement is the point set of a subsystem of order $(v-1)/2$. An in-depth study of the rank of STSs has been carried out in [11].

In particular, for $v = 19$ there is at most one dependent set, with equality if and only if there exists a subsystem of order 9. It follows that the rank of an STS(19) is 18 if there exists a subsystem of order 9 (284 457 isomorphism classes) and 19 otherwise (11 084 590 372 isomorphism classes).

The rank over GF(2) gives the dimension of the binary code generated by the (rows or columns of) the incidence matrix. The code generated by the rows of a point–block incidence matrix is the *point code* of the STS. There exist nonisomorphic STS(19) that have equivalent point codes [27].

2.3 Small Configurations

A *configuration* $\mathcal{C}$ in an STS $(X, \mathcal{B})$ is a subset of blocks $\mathcal{C} \subseteq \mathcal{B}$. Small configurations in STSs have been studied extensively; see [8, Chapter 13], [17] and [19]. The number of any configuration of size at most 3 is a function of the order of the STS. We address small configurations with some particular properties.

A configuration $\mathcal{C}$ with $|\mathcal{C}| = \ell$ and $| \cup_{C \in \mathcal{C}} C| = k$ is a (k, ℓ)-*configuration*. A configuration is *even* if each of its points occurs in an even number of blocks. If no point of a configuration occurs in exactly one block, then the configuration is *full*.

The only even (and only full) configuration of size 4 is the *Pasch configuration*, the $(6, 4)$-configuration depicted in Figure 1. The numbers of Pasch configurations in the STS(19) were tabulated in [26]; for completeness, we repeat the result in Table 5.

Table 5: Number of Pasches

Pasch	#	Pasch	#	Pasch	#	Pasch	#
0	2 591	17	954 710 609	34	2 190 166	51	366
1	35 758	18	845 596 671	35	1 301 951	52	482
2	263 646	19	716 603 299	36	775 233	53	78
3	1 315 161	20	583 321 976	37	452 306	54	278
4	4 958 687	21	457 755 898	38	267 642	55	69
5	15 095 372	22	347 324 307	39	152 122	56	137
6	38 481 050	23	255 589 428	40	92 056	57	24
7	84 328 984	24	182 938 899	41	51 019	58	104
8	162 045 054	25	127 614 183	42	31 587	59	6
9	276 886 518	26	87 003 115	43	16 974	60	41
10	426 050 673	27	58 052 942	44	11 827	62	47
11	596 271 997	28	38 010 203	45	6 008	64	3
12	765 958 741	29	24 457 073	46	4 629	66	18
13	910 510 124	30	15 492 114	47	2 151	70	5
14	1 008 615 673	31	9 663 499	48	2 099	78	2
15	1 047 850 033	32	5 956 712	49	724	84	3
16	1 027 129 335	33	3 623 356	50	991		

Three STS(19) with 84 Pasch configurations were found in [23]. Indeed, 84 is the maximum possible number of Pasch configurations and the list of such STS(19) in [23] is complete. The three systems are those having automorphism groups of order 108, 144 and 432, also encountered in Section 2.2.

Replacing the blocks of a Pasch configuration, say $\mathcal{P} = \{\{a, b, c\}, \{a, y, z\}, \{x, b, z\}, \{x, y, c\}\}$, by the blocks of $\mathcal{P}' = \{\{x, y, z\}, \{x, b, c\}, \{a, y, c\}, \{a, b, z\}\}$ transforms an STS into another STS. This operation is a *Pasch switch*. All but one of the 80 isomorphism classes of STS(15) contain at least one Pasch configuration. Any one of these can be transformed to any other by some sequence of Pasch switches [16, 22]. A natural question is whether the same is true for the STS(19), that is, if each STS(19) containing at least one Pasch configuration can be transformed to any other such design via Pasch switches. The answer is in the negative.

In [21] the concept of *twin Steiner triple systems* was introduced. These are two STSs each of which contains precisely one Pasch configuration that when switched produces the other system. If in addition the twin systems are isomorphic we have *identical twins*. In

[20] nine pairs of twin STS(19) are given. By examining all STS(19) containing a single Pasch configuration, we have established that there are in total 126 pairs of twins, but no identical twins.

We also consider STSs that contain precisely two Pasch configurations, say $\mathcal{P}$ and $\mathcal{Q}$, such that when $\mathcal{P}$ (respectively $\mathcal{Q}$) is switched what is obtained is an STS containing just one Pasch configuration $\mathcal{P}'$ (respectively $\mathcal{Q}'$). There are precisely 9 such systems. In every case the two single Pasch systems obtained by the Pasch switches are nonisomorphic. One such system is S1 (in the Appendix).

For size 6, there are two even configurations, known as the *grid* and the *prism* (or *double triangle*); these $(9,6)$-configurations are depicted in Figure 1.

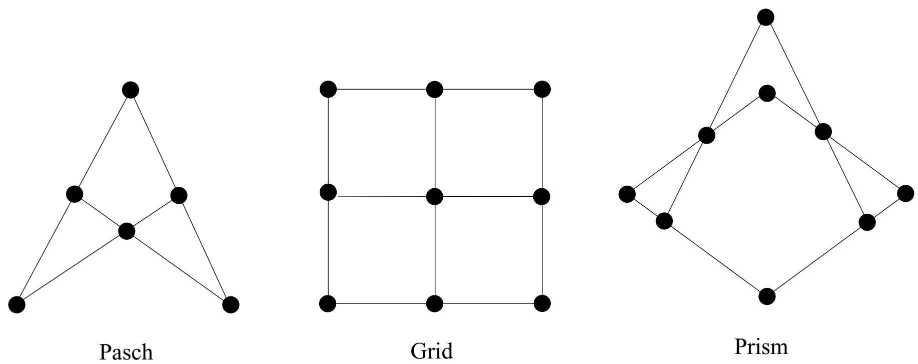

Figure 1: The even configurations of size at most 6

Every STS contains an even configuration of size at most 8, see [15]. However, no STS(19) missing either a grid or a prism was known. Indeed, a complete enumeration of grids and prisms establishes that there is no such STS(19). The distribution of the numbers of grids is shown in Table 9 and that for prisms in Table 10. The smallest number of grids in an STS(19) is 21 (design S4) and the largest is 384 (the STS(19) with automorphism group order 432). The smallest number of prisms is 171 (design A4) and the largest is 1 152 (the designs with automorphism group orders 108, 144 and 432). In particular, then, every STS(19) contains both even $(9,6)$-configurations.

An STS is *k-sparse* if it does not contain any $(n+2,n)$-configuration for any $4 \leqslant n \leqslant k$. In studying k-sparse systems it suffices to focus on full configurations, because an $(n+2,n)$-configuration that is not full contains an $(n+1,n-1)$-configuration. Because k-sparse STS(19) with $k \geqslant 4$ are anti-Pasch, one could simply check the 2 591 anti-Pasch STS(19). A more extensive tabulation of small $(n+2,n)$-configurations was carried out in this work.

There is one full $(7,5)$-configuration (the *mitre*) and two full $(8,6)$-configurations, known as the *hexagon* (or 6-*cycle*) and the *crown*. These are drawn in Figure 2, and their numbers are presented in Tables 11, 12 and 13.

The existence of a 5-sparse STS(19) was known [7]. By Table 11 there are exactly four nonisomorphic anti-mitre STS(19). Moreover, by Tables 12 and 13 there is a unique STS(19) with no hexagon and exactly four with no crown. Considering the intersections

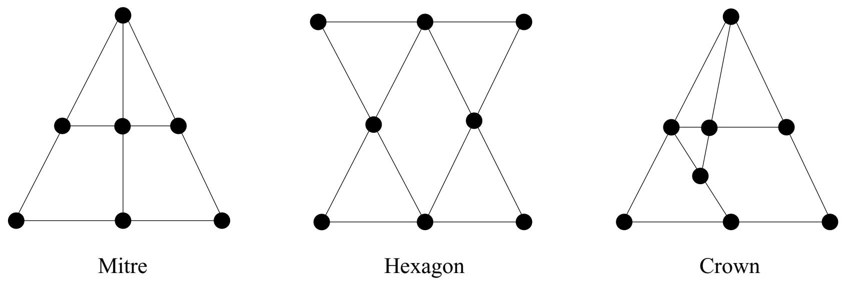

Figure 2: The full $(7,5)$- and $(8,6)$-configurations

of the classes of STS(19) with these properties, and the anti-Pasch ones, only two STS(19) are in more than one of the classes: one has no Pasch and no mitre, and one has no Pasch and no crown.

Theorem 1. *The numbers of* 4*-sparse,* 5*-sparse and* 6*-sparse* STS(19) *are* 2 591*,* 1 *and* 0*, respectively.*

The unique 5-sparse—that is, anti-Pasch and anti-mitre—STS(19) is A4. The unique STS(19) having no Pasch and no crown is A2, and the unique STS(19) with no hexagon is S5. The other three anti-mitre systems are B4, S6 and A3, and the other three anti-crown systems are those with automorphism group orders 108, 144 and 432. The largest number of mitres, hexagons and crowns in an STS(19) is 144 (for the three STS(19) with automorphism group orders 108, 144 and 432), 171 (for A4) and 314 (for S7), respectively.

2.4 Cycle Structure and Uniform Systems

Any two distinct points $x, y \in X$ of an STS determine a *cycle graph* in the following way. The points x, y occur in a unique block $\{x, y, z\}$. The cycle graph has one vertex for each point in $X \setminus \{x, y, z\}$ and an edge between two vertices if and only if the corresponding points occur together with x or y in a block.

A cycle graph of an STS is 2-regular and consists of a set of cycles of even length. Hence they can be specified as integer partitions of $v-3$ using even integers greater than or equal to 4. For $v = 19$, the possible partitions are $l_1 = 4+4+4+4$, $l_2 = 4+4+8$, $l_3 = 4+6+6$, $l_4 = 4+12$, $l_5 = 6+10$, $l_6 = 8+8$ and $l_7 = 16$. The cycle vector of an STS is a tuple showing the distribution of the cycle graphs; for STS(19) we have $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ with $\sum_{i=1}^{7} a_i = \binom{19}{2} = 171$, where a_i denotes the number of occurrences of the partition l_i.

The cycle vector $(0, 0, 0, 0, 0, 0, 171)$ is of particular interest; an STS all of whose cycle graphs consist of a single cycle is *perfect*. It is known [25] that there is no perfect STS(19). A more general family consists of the STSs with $a_i = \binom{v}{2}$ for some i; such STSs are *uniform*. Uniform STS(19) are known to exist [39].

An extensive investigation of the cycle vectors of STS(19) was carried out. The results are summarized in Table 6, where the designs are grouped according to the support of the cycle vector, that is, $\{i : a_i \neq 0\}$. Only 28 out of 128 possible combinations of cycle graphs are actually realised.

Table 6: Combinations of cycle graphs

Type	#	Type	#	Type	#
5	1	3567	125	24567	75 786 636
57	5	4567	5 009 893	34567	174 351 058
134	3	12347	39	123457	51 146
347	1	12457	56	123467	15
357	1	12467	1	124567	8 658 874
457	17	13457	89	134567	11 039 468
567	2 585	13467	2	234567	8 685 731 027
1347	5	14567	135 588	1234567	2 124 060 807
2457	255	23457	46 863		
3457	259	23567	10		

The main observation from Table 6 is the following.

Theorem 2. *There is exactly one uniform* STS(19).

The following conclusions can also be drawn from Table 6. The anti-Pasch systems are one with cycle graph 5; five with cycle graphs 5 and 7; and 2 585 with cycle graphs 5, 6 and 7. The unique 6-cycle-free system has cycle graphs 1, 2, 4, 6 and 7. The numbers of k-cycle-free systems for $k = 4, 6, 8, 10, 12$ and 16 are 2 591, 1, 381, 66, 2 727 and 4, respectively. The unique uniform STS(19) is the 5-sparse system A4 of Theorem 1.

2.5 Independent Sets

An *independent set* $I \subseteq X$ in a Steiner triple system $(X, \mathcal{B})$ is a set of points with the property that no block of $\mathcal{B}$ is contained in I. A *maximum independent set* is an independent set of maximum size. There exists an STS(19) that contains a maximum independent set of size m if and only if $m \in \{7, 8, 9, 10\}$, and $m = 10$ arises precisely when the design contains a subsystem of order 9; see [8, Chapter 17]. The following theorem collects the results of a complete determination.

Theorem 3. *The numbers of* STS(19) *with maximum independent set size* 7*,* 8*,* 9 *and* 10 *are* 2*,* 10 133 102 887*,* 951 487 483 *and* 284 457*, respectively.*

The two systems that have maximum independent set of size 7 are the (cyclic) systems A2 and A4.

2.6 Chromatic Number

A *colouring* of a Steiner triple system $(X, \mathcal{B})$ is a partition of X into independent sets. A partition of X into k independent sets is a *k-colouring.* The *chromatic number* of an STS is the smallest integer k such that the STS has a k-colouring, and corresponding colourings are *optimal.* Designs with a unique optimal colouring have been termed *uniquely colourable* [41]. A colouring is *equitable* if the cardinalities of the colour classes differ by at most one. An STS is *k-balanced* if every k-colouring is equitable.

No STS(v) with $v > 3$ is 2-chromatic [40]. Moreover, every STS(19) is 4-colourable [13, Theorem 6.1]; see also [24, Theorem 5]. Consequently, the chromatic number of any STS(19) is either 3 or 4. No STS(19) with chromatic number 4 was known; indeed as we see next, none exists. An exhaustive search establishes the following.

Theorem 4. *Every* STS(19) *is* 3*-chromatic. More specifically,*

(*i*) *every* STS(19) *has a* 3*-colouring with colour class sizes* $(7, 7, 5)$ *and*

(*ii*) *every* STS(19) *except for designs* A2 *and* A4 *has a* 3*-colouring with colour class sizes* $(8, 6, 5)$.

Next we show that Theorem 4 completes the determination of the combinations of 3-colouring patterns that can occur in an STS(19). For a given 3-colouring of an STS(19), let the colour classes be (C_1, C_2, C_3). Let $c_i = |C_i|$ for $1 \leqslant i \leqslant 3$. Without loss of generality suppose that $c_1 \geqslant c_2 \geqslant c_3$, and denote the pattern of colour class sizes by the corresponding integer triple (c_1, c_2, c_3). Informally, we refer to the colour classes C_1, C_2, C_3 as red, yellow and blue. It is shown in [12, Section 2.4] and [13] that any 3-colouring of an STS(19) must have one of the six patterns

$$(7, 6, 6),\ (7, 7, 5),\ (8, 6, 5),\ (8, 7, 4),\ (9, 5, 5),\ (9, 6, 4),$$

and that certain reductions are possible.

Lemma 1. *An* STS(19) *that has a* 3*-colouring with colour class sizes*

(*i*) $(7, 7, 5)$ *also has one with sizes* $(7, 6, 6)$,

(*ii*) $(8, 6, 5)$ *either has one with sizes* $(7, 7, 5)$ *or one with sizes* $(7, 6, 6)$,

(*iii*) $(8, 7, 4)$ *also has one with sizes* $(7, 7, 5)$,

(*iv*) $(9, 5, 5)$ *either has one with sizes* $(9, 6, 4)$ *or one with sizes* $(8, 6, 5)$,

(*v*) $(9, 6, 4)$ *also has one with sizes* $(8, 6, 5)$,

(*vi*) $(8, 7, 4)$ *also has one with sizes* $(8, 6, 5)$,

(*vii*) $(9, 5, 5)$ *also has one with sizes* $(8, 6, 5)$,

(*viii*) $(9, 6, 4)$ *also has one with sizes* $(9, 5, 5)$,

(*ix*) $(9, 6, 4)$ *also has one with sizes* $(8, 7, 4)$.

Proof. For (*i*)–(*v*), see [12, Section 2.4] or [13, Section 4]. It remains only to prove (*vi*)–(*ix*).

Let x_{ijk}, $1 \leqslant i \leqslant j \leqslant k$, denote the number of blocks containing points belonging to colour classes C_i, C_j and C_k, with appropriate multiplicities. Thus, for example, x_{122} is the number of blocks that contain a red point and two yellow points. Write x for x_{223}. As in the proof of [12, Theorem 2.4.1] we can construct the following table by a straightforward computation.

(c_1, c_2, c_3)	x_{122}	x_{133}	x_{112}	x_{113}	x_{223}	x_{233}	x_{123}
(7, 6, 6)	$15 - x$	x	$3 + x$	$18 - x$	x	$15 - x$	6
(7, 7, 5)	$21 - x$	$x - 5$	$1 + x$	$20 - x$	x	$15 - x$	5
(8, 6, 5)	$15 - x$	$x - 3$	$7 + x$	$21 - x$	x	$13 - x$	4
(8, 7, 4)	$21 - x$	$x - 7$	$6 + x$	$22 - x$	x	$13 - x$	2
(9, 5, 5)	$10 - x$	$x - 2$	$12 + x$	$24 - x$	x	$12 - x$	1
(9, 6, 4)	$15 - x$	$x - 6$	$12 + x$	$24 - x$	x	$12 - x$	0

Suppose we have an $(8, 7, 4)$ 3-colouring of an STS(19). Then $x \geqslant 7$ since $x_{133} = x - 7 \geqslant 0$. Moreover, $x_{233} = 13 - x \leqslant 6$. Therefore we can find a yellow point to change to blue without creating a blue-blue-blue block. This proves (*vi*).

Suppose we have a $(9, 5, 5)$ 3-colouring. Since $x_{122} + x_{133} = 8 < 9$ we can find a red point to be changed to either yellow or blue. This proves (*vii*).

Suppose we have a $(9, 6, 4)$ 3-colouring. If $x_{233} < 6$, we can change a yellow point to blue. So we may assume that $x_{233} = 6$. Then $x_{133} = x_{123} = 0$. Hence each blue point occurs exactly three times in the yellow-blue-blue blocks and paired with three yellow points. So each blue point must occur paired with three yellow points in yellow-yellow-blue blocks. This is impossible; hence (*viii*) is proved.

Again, suppose we have a $(9, 6, 4)$ 3-colouring. If $x_{122} < 9$, we can change a red point to yellow. Otherwise $x_{122} \geqslant 9$. This forces $x = x_{223} = x_{233} = 6$ and $x_{133} = x_{123} = 0$, which is impossible by the same argument as in the proof of (*viii*). This proves (*ix*). ☐

The main result of this section is a straightforward consequence of Theorem 4 and Lemma 1.

Theorem 5. *Any* STS(19) *is* 3*-colourable with one of the following six combinations of* 3*-colouring patterns:*

$$\begin{aligned}
\mathcal{C}_1 &= \{(7,6,6),(7,7,5)\},\\
\mathcal{C}_2 &= \{(7,6,6),(7,7,5),(8,6,5)\},\\
\mathcal{C}_3 &= \{(7,6,6),(7,7,5),(8,6,5),(8,7,4)\},\\
\mathcal{C}_4 &= \{(7,6,6),(7,7,5),(8,6,5),(9,5,5)\},\\
\mathcal{C}_5 &= \{(7,6,6),(7,7,5),(8,6,5),(8,7,4),(9,5,5)\},\\
\mathcal{C}_6 &= \{(7,6,6),(7,7,5),(8,6,5),(8,7,4),(9,5,5),(9,6,4)\}.
\end{aligned}$$

The first combination in Theorem 5, $\{(7,6,6),(7,7,5)\}$, occurs in only two STS(19), both of which are cyclic; in fact these are the two exceptions of Theorem 4(ii), systems A2 and A4. The other two cyclic STS(19), A1 and A3, have the colouring pattern combination $\{(7,6,6),(7,7,5),(8,6,5)\}$. It is easy to find examples exhibiting each of the remaining combinations.

We are now able to answer the open problem of whether there exists a 3-balanced STS(19) [13, Problem 1]. By [13, Theorem 4.1] and Theorems 4 and 5 we immediately get the following.

Corollary 1. *Every* STS(19) *is* 3*-chromatic and has an equitable* 3*-colouring. There exists no* 3*-balanced* STS(19).

In a separate computation we obtained the frequency of occurrence of each combination of 3-colouring patterns. We also obtained information concerning the size of maximum independent sets. Our results are presented in Table 7 in the form of a two-way frequency table of maximum independent set size against combinations of 3-colouring patterns $\mathcal{C}_i$ as defined in Theorem 5. The cell in row $\mathcal{C}_i$, column j gives the number of STS(19) that have 3-colouring pattern combination $\mathcal{C}_i$ and maximum independent set size j. Observe that the total count for size 10 is in agreement with [42], and it is worth pointing out that the zero entries in rows $\mathcal{C}_2$ to $\mathcal{C}_6$ can be deduced by elementary arguments without the need for any extensive computation. In particular, it is not difficult to show that an independent set of size 10 excludes the possibility of a (9,5,5) 3-colouring.

Table 7: Colourings and maximum independent sets

Colouring	7	8	9	10	Total
$\mathcal{C}_1$	2	0	0	0	2
$\mathcal{C}_2$	0	53 680 512	2 650 830	1 241	56 332 583
$\mathcal{C}_3$	0	10 079 422 375	421 936 849	283 216	10 501 642 440
$\mathcal{C}_4$	0	0	2 912 144	0	2 912 144
$\mathcal{C}_5$	0	0	464 995 662	0	464 995 662
$\mathcal{C}_6$	0	0	58 991 998	0	58 991 998
Total	2	10 133 102 887	951 487 483	284 457	11 084 874 829

2.7 Almost Parallel Classes

A set of nonintersecting blocks that do not contain all points of the design is a *partial parallel class*, and a partial parallel class with $\lfloor v/3 \rfloor$ blocks is an *almost parallel class.* Consequently, six nonintersecting blocks of an STS(19) form an almost parallel class. For each STS(19) we determined the total number of almost parallel classes in the following way.

For each STS(19), the point to be missed by the almost parallel class is specified, after which the problem of finding the almost parallel classes can be formulated as instances of the exact cover problem. In the exact cover problem, a set U and a collection $\mathcal{S}$ of subsets of U are given, and one wants to determine (one or all) partitions of U using sets from $\mathcal{S}$. To solve instances of the exact cover problem, the libexact software [30], which implements ideas from work by Knuth [32], was utilized. The results are presented in Table 8.

There is a conjecture that for all $v \equiv 1, 3 \pmod 6$, $v \geqslant 15$, there exists an STS(v) whose largest partial parallel class has fewer than $\lfloor v/3 \rfloor$ blocks [4, Conjecture 2.86], [8, Conjectures 19.4 and 19.5], [41, Section 3.1]. The results in the current work are in accordance with this conjecture.

In fact, Lo Faro already showed that every STS(19) has a partial parallel class with five blocks [33] and, constructively, that there indeed exists an STS(19) with no almost parallel class [34]. The current work shows that there are exactly two STS(19) with no almost parallel classes. These are A4 and the unique design with automorphism group of order 432. The largest number of almost parallel classes, 182, arises in S3.

A set of blocks of a design with the property that each point occurs in exactly α of these blocks is an *α-parallel class*. A partition of all blocks into α-parallel classes is an *α-resolution*, and a design that admits an α-resolution is *α-resolvable*. A Steiner triple system whose order v is not divisible by 3 cannot have a (1-)parallel class, but may have a 3-parallel class. The existence of Steiner triple systems of order at least 7 without a 3-parallel class is an open problem [8, p. 419].

A complete search demonstrates that every STS(19) not only has a 3-parallel class, but a 3-resolution. It is, however, not always the case that every 3-parallel class can be extended to a 3-resolution. That is, some STS(19) contain a 6-parallel class that is *nonseparable*, in that it does not further partition into two 3-parallel classes. Using [3], the largest α for which an STS(v) contains a nonseparable α-parallel class is 3, 1, 3, 5 and 6 for $v = 7$, 9, 13, 15 and 19, respectively.

2.8 Chromatic Index

While the chromatic number concerns colouring points, the chromatic index concerns colouring blocks. More precisely, the *chromatic index* of an STS is the smallest number of colours that can be used to colour the blocks so that no two intersecting blocks receive the same colour.

An STS(v) is resolvable if and only if its chromatic index is $(v-1)/2$. Since 19 is not divisible by 3, there is no resolvable STS(19), and the smallest possible chromatic index for such a design is $\lceil 57/6 \rceil = 10$.

By elementary counting, an STS(19) with chromatic index 10 must have at least 7 disjoint almost parallel classes. Moreover, the chromatic index of an STS(19) with no almost parallel classes is at least $\lceil 57/5 \rceil = 12$. We now describe the computational approach used to show that 10, 11 and 12 are the only possible chromatic indices for an STS(19).

Exact algorithms and greedy algorithms for finding the chromatic index and upper bounds on the chromatic index of STSs were presented in the early 1980s [2, 5]. Now

Table 8: Number of almost parallel classes

APC	#	APC	#	APC	#	APC	#
0	2	79	764 738	110	526 902 725	141	43 290
36	1	80	1 224 282	111	495 595 995	142	25 609
40	1	81	1 924 007	112	458 547 878	143	14 838
48	5	82	2 974 055	113	417 254 801	144	8 604
50	1	83	4 513 033	114	373 408 256	145	4 827
51	1	84	6 737 331	115	328 678 489	146	2 907
52	2	85	9 882 490	116	284 606 260	147	1 581
54	5	86	14 239 039	117	242 381 171	148	1 028
56	14	87	20 170 633	118	203 039 046	149	522
57	6	88	28 071 379	119	167 316 900	150	386
58	16	89	38 411 235	120	135 654 277	151	210
59	6	90	51 637 134	121	108 190 905	152	173
60	31	91	68 231 490	122	84 895 844	153	75
61	27	92	88 611 342	123	65 517 542	154	85
62	58	93	113 110 188	124	49 778 191	155	32
63	65	94	141 933 285	125	37 203 375	156	53
64	158	95	175 017 943	126	27 381 347	157	6
65	225	96	212 214 494	127	19 807 367	158	22
66	476	97	252 843 760	128	14 108 068	159	6
67	774	98	296 203 531	129	9 891 578	160	24
68	1 606	99	341 097 019	130	6 829 506	162	5
69	2 801	100	386 153 551	131	4 633 657	164	12
70	5 363	101	429 813 668	132	3 105 171	166	3
71	9 930	102	470 269 272	133	2 044 697	167	1
72	18 098	103	505 968 628	134	1 327 796	168	1
73	32 270	104	535 235 668	135	847 519	172	4
74	56 959	105	556 712 827	136	536 040	174	4
75	98 415	106	569 489 811	137	332 998	180	1
76	168 833	107	572 707 805	138	203 608	182	1
77	284 405	108	566 389 062	139	123 411		
78	470 557	109	550 847 618	140	74 672		

modern algorithms for finding colourings and chromatic numbers of graphs can be used to determine the chromatic number of the line graph of the design, which equals the chromatic index of the design.

To find a 10-colouring, the algorithm starts by finding sets of 7 disjoint almost parallel classes. To do this, for each STS(19), all almost parallel classes are first found (as in Section 2.7). Using these, sets of 7 disjoint ones are obtained by an algorithm for finding cliques in graphs (form one vertex for each almost parallel class and place edges between disjoint classes). The Cliquer software [37] can be utilized to find the cliques. The final step is an exhaustive search for three partial parallel classes to partition the remaining $57 - 7 \cdot 6 = 15$ blocks.

A more general exhaustive search algorithm was applied to instances with chromatic index greater than 10. The final result is as follows.

Theorem 6. *The numbers of* STS(19) *that have chromatic index* 10*,* 11 *and* 12 *are* 11 084 870 752*,* 4 075 *and* 2*, respectively.*

Consequently, exactly the two STS(19) with no almost parallel classes (see Section 2.7) have chromatic index 12. Our results are consistent with the observation that no STS(v) with $v > 7$ and chromatic index exceeding the minimum chromatic index by more than 2 is known to exist [8, pp. 366–367], [41, p. 411].

2.9 Existential Closure

The *block intersection graph* of an STS has one vertex for each block and an edge between two vertices exactly when the corresponding blocks intersect. A graph $G = (V, E)$ is n*-existentially closed* if for every n-element subset $S \subseteq V$ of vertices and for every subset $T \subseteq S$, there exists a vertex $x \notin S$ that is adjacent to every vertex in T and nonadjacent to every vertex in $S \setminus T$.

In [14] n-existentially closed block intersection graphs of STSs are studied. The block intersection graph of an STS(v) is 2-existentially closed if and only if $v \geqslant 13$, it cannot be 4-existentially closed [36, Theorem 1] for any v, and the only possible orders for which it can be 3-existentially closed are 19 and 21. In fact, two STS(19) possess 3-existentially closed block intersection graphs [14].

The following result from [14, Theorem 4.1] helps in designing an algorithm for determining whether the block intersection graph of an STS is 3-existentially closed.

Theorem 7. *The block intersection graph of an* STS(v) *is* 3*-existentially closed if and only if*

(i) *the* STS(v) *contains no subsystem* STS(7)*,*

(ii) *the* STS(v) *contains no subsystem* STS(9)*,*

(iii) *for every set of three nonintersecting blocks, if* $v < 19$ *there exists a block that intersects none of the three, and if* $v \geqslant 19$ *there exists a block that intersects all three.*

No STS(19) other than those discovered in [14] is 3-existentially closed.

Theorem 8. *The number of* 3*-existentially closed* STS(19) *is* 2*.*

The two 3-existentially closed STS(19) are A3 and S2.

3 Conclusions

The main aim of the current work has been to compute all kinds of properties of STS(19) and collect them in a single place. However, it is impossible to accomplish this task in an exhaustive manner, so we omit discussion of properties that (1) we do not consider to have large general interest, (2) we are not able to present in a compact manner, or (3) we simply are not able to compute at the present time.

For example, we consider various kinds of colouring problems, such as those studied in [9, 18], to be of the first type. Any properties that have been used as invariants for STSs cannot, by definition, be tabulated in a compact way and are of the second type; examples of this type include various forms of so-called trains.

The third type of problems contain some very interesting open problems, including those of determining intersection numbers of STSs, maximal sets of disjoint STSs, and whether all STSs are derived. Further information on these problems can be found in [4, 8]. For example, just determining whether a single STS is derived remains a major challenge.

The problems were addressed using three different computational environments (in Canada, Finland and Great Britain), so we do not try to give exact details about the computations. The computational resources needed partition the problems roughly into three groups: those taking days or at most a couple of weeks ("easy"), those taking up to a couple of years ("intermediate") and those taking up to ten years ("hard"). These CPU times are roughly the times needed for one core of a "contemporary microprocessor".

The intermediate calculations were those of determining subconfigurations (10 CPU weeks), determining the almost parallel classes (1.5 CPU years), constructing the frequency table of maximum independent set size against 3-colouring pattern combination (12 CPU weeks), showing existence of 3-parallel classes (7 CPU months) and searching for 3-existentially closed designs (9 CPU months). The only one belonging to the category of hard calculations was the determination of the chromatic indices, which consumed just under 8 CPU years. All remaining calculations were "easy".

Appendix

We use the same method for compressing STSs as in the supplement to [6]. That is, for the points we use the symbols `a-s` and represent an STS by a string of 57 symbols $x_1x_2\cdots x_{57}$. The symbol x_i is the largest element in the ith block. The other two symbols in the ith block are the smallest pair of symbols not occurring in earlier blocks under the *colexicographic* ordering of pairs: a pair y, z with $y < z$ is smaller than a pair y', z' with

$y' < z'$ iff $z < z'$, or $z = z'$ and $y < y'$. The order of the automorphism group is given after each design.

S1: `edgfhghijkllmnljompqporqsnsloqprmrsnnopsrqqprosqsrpsqrrss` (1)
S2: `cefggfhijijklmnokppqmrsolrsqnqpsnrmornsoqpsqporpqrsrsqsrs` (8)
S3: `cefghngjljrikoqplrnqmskmsnonsmrlpmoprqpqosopqsrrpsqqsrsrs` (3)
S4: `cefghigpojlijqmplrqokomsnnqpslrommnsrqprnsoprqsrspqqsrsrs` (1)
S5: `cefghfgjoiksmrlpnksqkmpsnlrnoqmmnqposrprqoorpqsrspqqrssrs` (6)
S6: `cefghigomjsinksllsjqkmropnlqrpomnrpqpqornsopqrsrpqsqsrsrs` (9)
S7: `cefihkgsojosmiqmnrlpjqklospnqlpormprnsprqonsoprqsrqqrssrs` (1)

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Table 9: Number of grids

Grid	#	Grid	#	Grid	#	Grid	#
21	1	58	421 406 261	95	5 466 378	132	19 595
22	1	59	455 538 873	96	4 452 414	133	17 568
23	1	60	483 962 320	97	3 625 512	134	17 390
24	6	61	505 587 977	98	2 964 501	135	15 125
25	27	62	519 737 441	99	2 419 681	136	14 765
26	44	63	525 975 481	100	1 984 363	137	12 845
27	156	64	524 399 635	101	1 625 523	138	12 707
28	403	65	515 397 821	102	1 340 634	139	10 911
29	1 012	66	499 528 245	103	1 103 378	140	10 689
30	2 577	67	477 877 986	104	915 322	141	9 228
31	6 067	68	451 447 963	105	756 727	142	9 097
32	13 721	69	421 183 378	106	629 794	143	7 629
33	29 607	70	388 549 216	107	522 121	144	7 495
34	62 549	71	354 553 810	108	439 478	145	6 593
35	125 648	72	320 163 173	109	365 162	146	6 407
36	246 636	73	286 220 933	110	310 349	147	5 325
37	461 547	74	253 571 165	111	256 766	148	5 266
38	840 481	75	222 621 207	112	219 625	149	4 318
39	1 484 562	76	193 840 439	113	183 979	150	4 386
40	2 534 581	77	167 454 239	114	157 625	151	3 507
41	4 196 398	78	143 611 784	115	133 530	152	3 515
42	6 739 474	79	122 366 578	116	115 251	153	2 820
43	10 522 877	80	103 592 757	117	97 139	154	2 838
44	15 960 510	81	87 177 751	118	85 923	155	2 265
45	23 562 586	82	72 978 536	119	72 545	156	2 455
46	33 871 296	83	60 813 771	120	65 014	157	1 830
47	47 412 716	84	50 428 258	121	55 582	158	1 905
48	64 736 436	85	41 665 785	122	50 393	159	1 433
49	86 205 567	86	34 306 651	123	43 478	160	1 552
50	112 103 389	87	28 141 430	124	40 275	161	1 124
51	142 489 811	88	23 037 710	125	34 759	162	1 284
52	177 059 163	89	18 809 436	126	32 578	163	913
53	215 192 146	90	15 344 880	127	28 746	164	1 010
54	256 144 342	91	12 489 931	128	27 080	165	766
55	298 709 622	92	10 159 180	129	23 884	166	843
56	341 446 147	93	8 261 382	130	23 163	167	557
57	382 864 465	94	6 721 096	131	20 281	168	664

Table 9: Number of grids (cont.)

Grid	#	Grid	#	Grid	#	Grid	#
169	490	194	80	219	2	249	3
170	527	195	19	220	23	250	2
171	324	196	90	221	2	252	10
172	429	197	21	222	14	254	1
173	267	198	70	223	5	255	2
174	383	199	8	224	33	256	7
175	206	200	97	225	5	258	1
176	328	201	16	226	8	260	7
177	153	202	39	227	5	262	1
178	232	203	6	228	31	264	8
179	126	204	79	229	2	267	2
180	223	205	5	230	4	272	7
181	128	206	25	231	3	276	4
182	207	207	13	232	21	280	4
183	109	208	59	234	10	284	3
184	155	209	4	235	1	288	5
185	75	210	51	236	26	294	1
186	149	211	2	238	5	300	1
187	57	212	46	239	1	303	1
188	159	213	10	240	26	308	1
189	45	214	14	242	1	312	3
190	91	215	2	243	1	320	2
191	44	216	38	244	7	336	2
192	123	217	3	245	1	384	1
193	36	218	15	248	11		

Table 10: Number of prisms

Prism	#	Prism	#	Prism	#	Prism	#
171	1	250	75 976	287	42 388 161	324	198 341 505
189	1	251	98 127	288	46 639 711	325	196 983 412
200	1	252	125 286	289	51 169 522	326	195 225 803
207	1	253	158 108	290	55 931 715	327	193 085 136
211	1	254	200 729	291	60 918 787	328	190 605 951
216	2	255	253 967	292	66 151 873	329	187 795 686
217	1	256	318 185	293	71 586 084	330	184 649 280
219	6	257	397 908	294	77 237 835	331	181 212 592
221	1	258	492 617	295	83 032 700	332	177 549 753
222	6	259	610 716	296	88 988 957	333	173 586 201
223	17	260	753 345	297	95 089 060	334	169 440 136
224	22	261	921 675	298	101 293 200	335	165 109 202
225	27	262	1 126 793	299	107 579 627	336	160 640 418
226	25	263	1 368 838	300	113 892 453	337	155 982 892
227	41	264	1 655 279	301	120 225 453	338	151 293 063
228	73	265	1 993 377	302	126 496 164	339	146 440 917
229	130	266	2 390 574	303	132 753 692	340	141 569 668
230	166	267	2 851 791	304	138 902 842	341	136 664 720
231	245	268	3 389 099	305	144 926 038	342	131 727 398
232	321	269	4 010 807	306	150 790 370	343	126 770 273
233	448	270	4 727 106	307	156 429 753	344	121 858 346
234	667	271	5 547 565	308	161 884 623	345	116 981 409
235	932	272	6 485 240	309	167 038 214	346	112 190 976
236	1 291	273	7 552 715	310	171 888 128	347	107 410 238
237	1 750	274	8 757 871	311	176 448 741	348	102 737 476
238	2 462	275	10 118 769	312	180 620 616	349	98 136 704
239	3 344	276	11 640 128	313	184 476 735	350	93 657 722
240	4 558	277	13 335 175	314	187 911 346	351	89 292 744
241	6 221	278	15 233 835	315	190 927 860	352	85 046 857
242	8 341	279	17 317 913	316	193 530 670	353	80 920 249
243	11 120	280	19 617 190	317	195 702 979	354	76 911 822
244	14 888	281	22 137 761	318	197 395 867	355	73 054 525
245	20 119	282	24 884 491	319	198 675 356	356	69 332 115
246	26 400	283	27 887 561	320	199 497 261	357	65 735 409
247	34 577	284	31 140 015	321	199 874 535	358	62 291 346
248	44 753	285	34 623 522	322	199 760 946	359	58 986 226
249	58 845	286	38 376 738	323	199 286 571	360	55 805 608

Table 10: Number of prisms (cont.)

Prism	#	Prism	#	Prism	#	Prism	#
361	52 776 788	398	5 210 998	435	424 445	472	95 566
362	49 877 144	399	4 870 806	436	400 992	473	92 467
363	47 109 094	400	4 555 184	437	375 930	474	89 604
364	44 477 939	401	4 255 687	438	356 584	475	86 116
365	41 956 665	402	3 975 185	439	335 932	476	83 388
366	39 596 950	403	3 710 635	440	318 533	477	80 516
367	37 316 718	404	3 468 155	441	300 617	478	78 206
368	35 158 337	405	3 235 022	442	286 646	479	74 644
369	33 131 446	406	3 021 856	443	271 545	480	72 289
370	31 199 621	407	2 817 205	444	258 555	481	68 924
371	29 360 909	408	2 632 611	445	245 429	482	67 293
372	27 626 089	409	2 454 635	446	235 409	483	63 891
373	25 997 783	410	2 292 545	447	224 067	484	62 065
374	24 455 068	411	2 137 919	448	214 575	485	58 964
375	22 993 528	412	1 995 564	449	205 399	486	56 790
376	21 604 049	413	1 861 521	450	197 610	487	54 505
377	20 310 057	414	1 737 449	451	188 729	488	52 492
378	19 075 074	415	1 616 932	452	182 542	489	49 354
379	17 916 453	416	1 509 591	453	176 060	490	47 536
380	16 819 109	417	1 404 929	454	168 815	491	45 253
381	15 795 662	418	1 314 772	455	162 976	492	43 832
382	14 826 839	419	1 225 935	456	158 019	493	40 816
383	13 907 432	420	1 144 721	457	152 147	494	39 536
384	13 050 725	421	1 067 065	458	148 600	495	37 181
385	12 241 906	422	995 655	459	142 312	496	35 949
386	11 482 906	423	927 859	460	138 498	497	33 708
387	10 762 834	424	868 000	461	134 174	498	32 268
388	10 084 561	425	811 642	462	130 272	499	30 063
389	9 453 238	426	758 276	463	125 969	500	28 901
390	8 853 538	427	709 328	464	122 632	501	27 030
391	8 294 860	428	663 317	465	117 860	502	25 906
392	7 771 024	429	619 097	466	115 901	503	24 000
393	7 269 785	430	582 159	467	111 021	504	23 162
394	6 806 485	431	544 981	468	108 594	505	21 754
395	6 363 581	432	513 193	469	104 985	506	20 937
396	5 960 984	433	479 631	470	101 572	507	19 322
397	5 569 324	434	452 765	471	98 344	508	18 497

Table 10: Number of prisms (cont.)

Prism	#	Prism	#	Prism	#	Prism	#
509	17 095	546	1 731	583	334	620	2 116
510	16 519	547	1 499	584	421	621	2 254
511	15 154	548	1 394	585	413	622	2 301
512	14 143	549	1 222	586	434	623	2 357
513	13 411	550	1 291	587	392	624	2 510
514	12 808	551	1 103	588	480	625	2 523
515	11 849	552	1 094	589	420	626	2 527
516	11 530	553	926	590	465	627	2 581
517	10 468	554	1 000	591	474	628	2 719
518	10 064	555	826	592	572	629	2 826
519	9 280	556	885	593	521	630	2 966
520	8 869	557	719	594	593	631	3 099
521	8 064	558	757	595	599	632	3 144
522	7 774	559	648	596	662	633	3 059
523	7 153	560	728	597	647	634	3 157
524	6 714	561	532	598	710	635	3 236
525	6 300	562	629	599	729	636	3 362
526	6 014	563	517	600	830	637	3 384
527	5 362	564	511	601	872	638	3 465
528	5 209	565	436	602	972	639	3 487
529	4 847	566	505	603	959	640	3 393
530	4 551	567	416	604	1 011	641	3 423
531	4 184	568	497	605	1 050	642	3 599
532	4 108	569	374	606	1 149	643	3 580
533	3 736	570	452	607	1 188	644	3 753
534	3 743	571	358	608	1 375	645	3 622
535	3 116	572	387	609	1 308	646	3 827
536	3 141	573	349	610	1 358	647	3 643
537	2 792	574	345	611	1 471	648	3 812
538	2 744	575	330	612	1 495	649	3 744
539	2 548	576	381	613	1 553	650	3 902
540	2 452	577	336	614	1 701	651	3 579
541	2 100	578	351	615	1 703	652	3 790
542	2 155	579	326	616	1 875	653	3 752
543	1 864	580	382	617	1 868	654	3 713
544	1 844	581	315	618	1 980	655	3 683
545	1 613	582	399	619	2 027	656	3 662

Table 10: Number of prisms (cont.)

Prism	#	Prism	#	Prism	#	Prism	#
657	3 649	694	1 495	731	194	768	27
658	3 597	695	1 380	732	211	769	22
659	3 637	696	1 400	733	175	770	21
660	3 667	697	1 250	734	177	771	14
661	3 567	698	1 324	735	164	772	21
662	3 416	699	1 141	736	152	773	12
663	3 464	700	1 136	737	154	774	24
664	3 326	701	1 010	738	147	775	10
665	3 370	702	1 024	739	116	776	16
666	3 370	703	931	740	116	777	5
667	3 294	704	935	741	88	778	13
668	3 155	705	833	742	123	779	3
669	3 170	706	844	743	89	780	5
670	3 123	707	729	744	97	781	8
671	3 023	708	759	745	75	782	10
672	3 036	709	669	746	103	783	6
673	2 903	710	666	747	68	784	9
674	2 895	711	636	748	90	785	5
675	2 735	712	624	749	79	786	10
676	2 797	713	597	750	91	787	5
677	2 606	714	564	751	56	788	9
678	2 600	715	511	752	60	789	2
679	2 416	716	531	753	44	790	8
680	2 493	717	433	754	65	791	8
681	2 302	718	455	755	43	792	12
682	2 238	719	439	756	45	793	1
683	2 215	720	394	757	46	795	2
684	2 072	721	359	758	39	796	2
685	2 115	722	366	759	42	797	2
686	2 023	723	334	760	35	798	3
687	1 880	724	326	761	28	799	1
688	1 868	725	262	762	30	800	2
689	1 724	726	306	763	23	801	1
690	1 645	727	229	764	40	805	1
691	1 620	728	253	765	15	806	5
692	1 595	729	253	766	16	807	1
693	1 497	730	218	767	19	808	4

Table 10: Number of prisms (cont.)

Prism	#	Prism	#	Prism	#	Prism	#
809	1	838	1	856	1	912	2
814	1	840	2	864	2	918	6
816	3	844	1	868	1	1152	3
818	1	846	2	870	4		
822	14	850	1	878	1		
832	1	852	1	888	2		

Table 11: Number of mitres

Mitre	#	Mitre	#	Mitre	#	Mitre	#
0	4	29	666 856 068	56	699 975	83	39
3	11	30	726 726 670	57	427 224	84	83
4	27	31	765 630 873	58	261 965	85	16
5	94	32	780 912 655	59	162 576	86	47
6	463	33	771 673 239	60	105 125	87	20
7	1 587	34	739 625 001	61	68 560	88	34
8	5 196	35	688 305 207	62	47 177	89	7
9	16 130	36	622 481 814	63	32 413	90	54
10	45 051	37	547 576 707	64	23 643	91	1
11	119 156	38	468 917 351	65	16 778	92	19
12	292 925	39	391 303 591	66	12 393	93	9
13	685 985	40	318 424 938	67	8 661	94	7
14	1 502 196	41	252 876 637	68	6 489	96	27
15	3 122 990	42	196 124 480	69	4 295	98	2
16	6 160 011	43	148 685 094	70	3 264	99	2
17	11 527 121	44	110 224 646	71	2 181	100	6
18	20 542 885	45	79 959 174	72	1 700	102	7
19	34 903 297	46	56 803 086	73	990	104	2
20	56 577 514	47	39 545 210	74	909	105	2
21	87 700 390	48	26 981 662	75	469	108	5
22	130 128 895	49	18 067 853	76	465	112	3
23	185 013 010	50	11 873 632	77	270	114	1
24	252 364 501	51	7 665 089	78	263	116	2
25	330 721 805	52	4 870 654	79	122	120	2
26	416 700 734	53	3 046 823	80	191	144	3
27	505 540 524	54	1 883 004	81	72		
28	591 121 831	55	1 150 672	82	96		

Table 12: Number of hexagons

Hexa	#	Hexa	#	Hexa	#	Hexa	#
0	1	34	724 247 745	66	436 234	98	110
2	1	35	714 131 642	67	326 333	99	62
4	8	36	685 867 252	68	239 208	100	77
5	2	37	642 422 184	69	179 527	101	33
6	18	38	587 540 455	70	134 495	102	74
7	42	39	525 307 321	71	100 405	103	17
8	275	40	459 726 499	72	75 980	104	35
9	1 060	41	394 271 746	73	57 056	105	28
10	3 888	42	331 862 444	74	43 803	106	28
11	13 543	43	274 475 233	75	31 922	107	10
12	42 046	44	223 366 811	76	26 629	108	136
13	119 420	45	179 088 397	77	17 366	109	10
14	315 586	46	141 683 536	78	13 996	110	17
15	769 997	47	110 703 052	79	9 867	111	10
16	1 750 488	48	85 587 484	80	8 815	112	14
17	3 711 050	49	65 546 910	81	5 888	113	1
18	7 390 282	50	49 813 749	82	5 139	114	17
19	13 851 974	51	37 586 617	83	3 120	115	1
20	24 536 316	52	28 199 864	84	2 880	116	18
21	41 147 211	53	21 046 347	85	1 883	117	4
22	65 593 940	54	15 677 184	86	2 264	118	1
23	99 604 643	55	11 622 883	87	1 127	120	10
24	144 448 598	56	8 623 668	88	1 016	121	1
25	200 532 422	57	6 370 044	89	615	122	4
26	266 967 992	58	4 713 086	90	1 645	124	8
27	341 559 277	59	3 483 045	91	436	126	16
28	420 712 045	60	2 580 662	92	408	128	1
29	499 765 074	61	1 909 874	93	249	132	3
30	573 401 076	62	1 419 396	94	234	144	12
31	636 579 383	63	1 050 752	95	150	171	1
32	684 620 989	64	786 486	96	248		
33	714 416 762	65	577 280	97	75		

Table 13: Number of crowns

Crown	#	Crown	#	Crown	#	Crown	#
0	4	75	84	112	39 276	149	11 026 967
24	8	76	225	113	46 162	150	12 418 598
28	1	77	98	114	55 376	151	13 951 975
32	7	78	230	115	65 172	152	15 649 589
34	1	79	146	116	78 169	153	17 504 989
36	17	80	303	117	92 109	154	19 550 198
40	4	81	184	118	110 533	155	21 784 052
42	2	82	352	119	130 711	156	24 217 202
44	3	83	271	120	155 188	157	26 857 968
45	1	84	507	121	183 383	158	29 735 229
46	3	85	409	122	218 318	159	32 838 784
48	17	86	625	123	256 913	160	36 187 030
49	2	87	538	124	304 546	161	39 768 756
50	4	88	788	125	357 058	162	43 644 429
51	3	89	745	126	420 855	163	47 762 633
52	19	90	1 103	127	493 066	164	52 146 277
54	23	91	997	128	580 012	165	56 809 902
55	5	92	1 448	129	678 149	166	61 761 576
56	37	93	1 460	130	794 787	167	66 960 296
57	9	94	1 941	131	925 609	168	72 455 628
58	16	95	1 999	132	1 080 365	169	78 205 211
59	6	96	2 809	133	1 256 516	170	84 194 952
60	72	97	2 861	134	1 462 493	171	90 422 800
61	5	98	3 569	135	1 691 178	172	96 907 778
62	22	99	3 832	136	1 960 531	173	103 579 676
63	16	100	5 157	137	2 262 445	174	110 428 354
64	65	101	5 644	138	2 612 802	175	117 487 564
65	16	102	7 012	139	3 008 486	176	124 638 538
66	77	103	7 868	140	3 455 009	177	131 927 624
67	19	104	9 735	141	3 958 995	178	139 275 613
68	71	105	11 111	142	4 536 189	179	146 638 317
69	32	106	13 806	143	5 178 047	180	154 028 623
70	81	107	15 906	144	5 903 381	181	161 359 146
71	55	108	19 655	145	6 715 687	182	168 619 294
72	173	109	22 619	146	7 629 172	183	175 716 385
73	65	110	27 800	147	8 645 817	184	182 665 320
74	144	111	32 269	148	9 772 477	185	189 374 242

Table 13: Number of crowns (cont.)

Crown	#	Crown	#	Crown	#	Crown	#
186	195 806 871	217	134 563 689	248	4 411 819	279	3 866
187	201 907 700	218	126 661 383	249	3 734 688	280	3 026
188	207 659 159	219	118 861 873	250	3 149 311	281	2 220
189	212 988 838	220	111 206 681	251	2 648 386	282	1 621
190	217 852 023	221	103 671 809	252	2 219 528	283	1 190
191	222 201 411	222	96 346 526	253	1 850 527	284	880
192	226 058 774	223	89 252 032	254	1 538 216	285	617
193	229 322 865	224	82 406 974	255	1 272 656	286	458
194	231 961 935	225	75 860 206	256	1 051 337	287	337
195	233 966 564	226	69 578 295	257	862 379	288	237
196	235 362 932	227	63 614 491	258	707 331	289	186
197	236 055 372	228	57 977 229	259	576 064	290	135
198	236 115 675	229	52 664 490	260	468 744	291	88
199	235 469 719	230	47 671 940	261	378 298	292	63
200	234 145 518	231	43 011 588	262	304 621	293	35
201	232 142 509	232	38 676 935	263	244 241	294	36
202	229 517 435	233	34 668 107	264	194 690	295	12
203	226 209 636	234	30 961 644	265	155 113	296	19
204	222 338 699	235	27 551 781	266	123 781	297	14
205	217 827 123	236	24 435 171	267	96 942	298	5
206	212 820 389	237	21 602 222	268	76 095	299	7
207	207 301 265	238	19 035 194	269	59 785	300	4
208	201 303 814	239	16 706 493	270	46 762	301	1
209	194 883 375	240	14 612 461	271	36 086	302	1
210	188 122 519	241	12 739 285	272	27 879	303	4
211	180 992 703	242	11 064 520	273	21 426	306	1
212	173 617 922	243	9 577 688	274	16 461	309	2
213	166 018 485	244	8 260 815	275	12 570	314	1
214	158 267 357	245	7 095 407	276	9 619		
215	150 399 412	246	6 078 552	277	7 117		
216	142 486 139	247	5 188 692	278	5 272		