UNIQUENESS OF EXTREMAL KERR-NEWMAN ISOLATED HORIZONS: QUASILOCAL INTRINSIC STRUCTURE, NEAR-HORIZON GEOMETRY AND DISTORTION OF EXTREMAL HORIZONS

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Uniqueness of Extremal Kerr-Newman Isolated Horizons: Quasilocal Intrinsic Structure, Near-Horizon Geometry and Distortion of Extremal Horizons

by

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Abstract

Our work begins with ref.[14] in which Lewandowski and Pawlowski obtained the unique solutions $\{P, U, B, \phi_1\}$ to the field equations restricting axisymmetric and electrovacuum extremal isolated horizons (IHs). After reviewing the boundary conditions and generic geometry of IHs, we construct the on-horizon data $\{K, \Phi_{11},\ \mathrm{Re}(\Psi_2), \pi^{(0)}, \omega_a, \text{etc}\}$ using the local uniqueness solutions $\{P, U, B, \phi_1\}$. Subsequently, we extend the adapted tetrad on an IH to cover the external regions and develop the method to reconstruct the near-horizon geometry of extremal IHs embedded in electrovacuum. This quasilocal method is applied to rebuild the near-horizon metrics of extremal Reissner-Nordström and Kerr horizons, which prove to be equivalent with those derived from the near-horizon limit of the corresponding global metrics. These results confirm that the local solutions $\{P, U, B, \phi_1\}$ describe the intrinsic structure of extremal Kerr-Newman-family horizons. The solutions $\{P, U, B, \phi_1\}$ lead to the first uniqueness theorem from quasilocal definitions of black holes, and this theorem implies that the intrinsic structure of extremal Kerr-Newman horizons cannot be distorted by external energy-matter distribution. This conjecture is examined and verified in conformastatically distorted extremal Reissner-Nordström spacetime.

Key Words Isolated Horizons, Adapted Tetrad, Near-Horizon Metric, Conformastatic Distortion, Newman-Penrose Formalism

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List of Symbols

$\hat{=}$ denotes equality on an NEH/WIH/IH;

$\simeq$ denotes equality in the near-horizon limit;

$\lrcorner$ denotes the hook operator, $l^a F_{ab} = \boldsymbol{l} \lrcorner \boldsymbol{F}$;

$\hat{\star}$ denotes the intrinsic Hodge dual operator with respect to the induced metric $\hat{h}_{ab}$;

ω_a denotes the rotation 1-form potential of an NEH/WIH/IH;

$\mathcal{D}$ denotes the induced connection on an NEH/WIH/IH;

$(\Delta, [l])$ denotes an WIH/IH with the equivalence class $[l]$;

$\hat{Q}$ hat over quantities denotes intrinsic quantities on a foliation leaf of an NEH/WIH/IH;

$O(r^m)$ denotes rank-m higher-order infinitesimals for Taylor expansion in near-horizon limit;

$Q_{,a}$ denotes partial derivative with respect to the variable x^a, $Q_{,a} := \partial_a Q$;

$Q_{;a}$ denotes covariant derivative w.r.t. the vector field $x^a \partial_a$, $Q_{;a} := x^a \nabla_a Q$;

Ψ_i $(i \in \{0, 1, 2, 3, 4\})$ denotes Weyl scalars in NP formalism;

Φ_{ij} $(i \in \{0, 1, 2\})$ denotes Ricci scalars in NP formalism;

ϕ_i $(i \in \{0, 1, 2\})$ denotes Maxwell scalars in NP formalism;

$\mathbb{R}$ denotes the set of real numbers.

Chapter 1

Introductory Remarks

In the realm of classical black hole physics, one of the most important results developed in the last four decades is a group of *global* uniqueness theorems. These theorems tell us that, the only *stationary*, *asymptotically flat*, *electrovacuum* and *nondegenerate* black-hole solutions to the Einstein-Maxwell equations are the Kerr-Newman family (for a comprehensive review, c.f. refs.[1][2] and the references therein as well as ref.[3]). Here the assumptions (asymptotic flatness, etc) needed for the proofs are imposed on the entire spacetime, which leads to the scenario of isolated black holes embedded in electrovacuum that extends to null infinity. The set of global uniqueness theorems was recently enriched with the proof regarding the uniqueness of *degenerate* (*extremal*) Kerr-Newman solutions in ref.[4].

While black holes have been extensively studied via the global approach, investigations from the quasilocal perspective have also achieved great progress in the last two decades. Some typical quasilocal definitions include *trapping horizons*[5], (generic) *isolated horizons*[6][7], *dynamical horizons*[8][9], *slowly evolving horizons*[10][11] etc., and they play important roles in numerical relativity, quantum gravity and other fields. One can refer to ref.[12] for a detailed review of quasilocal characterization of black holes.

Can we build uniqueness theorems from quasilocal approaches? This is quite a challenging problem and let's take isolated horizons (IHs) as an example to show where the difficulties usually arise. As will be shown shortly afterwards in Chapter 2, an IH is respectively said to be a vacuum or electrovacuum IH if it satisfies[7][13][14]

$$R_{AB} = \underleftarrow{R_{ab}} \mathrel{\hat{=}} 0\,, \quad \text{or} \quad R_{AB} - 8\pi T^{(EM)}_{AB} = \underleftarrow{R_{ab}} - \underleftarrow{8\pi T^{(EM)}_{ab}} \mathrel{\hat{=}} 0\,. \tag{1.1}$$

That is to say, we only require that the vacuum or electrovacuum Einstein(-Maxwell) equations hold *on the horizon*, regardless of the behaviors at external regions. Restrictions on IHs and other quasilocal definitions of black-hole horizons cannot be extended to the exteriors, and in general, it is impossible to fix the structure of the black-hole horizon without referring to external energy-matter distribution.

However, based on the geometry of generic IHs developed in ref.[13], equations restricting *axisymmetric* and *electrovacuum extremal* IHs were solved by Lewandowski and Pawlowski in ref.[14], and the solutions interpreted (though without further calculations) to represent the uniqueness of extremal Kerr-Newman horizons. As will be proved in Chapter 3 and Chapter 5, implications of these local solutions do agree with the intrinsic and near-horizon structures of the event horizon of extremal Kerr-Newman black holes.

The local uniqueness theorem in ref.[14] is really an amazing result. It is the first successful attempt to build black-hole uniqueness structures from quasilocal definitions. Moreover, it implies that, the intrinsic structure of extremal Kerr-Newman horizons is independent of matter and fields outside the horizon. Based on the local uniqueness as well as some other results[15], my supervisor, Dr Ivan Booth, put forward the conjecture that

The intrinsic structure of an extremal Kerr-Newman (i.e. axisymmetric and electrovacuum) horizon cannot be distorted by external energy-matter distribution.

This conjecture is partly examined in Chapter 6 of this thesis.

Hence, the two topics of this thesis have been introduced. Firstly, we will verify that the local uniqueness solutions correspond to extremal Kerr-Newman horizons. Secondly, we will study the intrinsic structure of extremal Kerr-Newman-family horizons in external distortion fields. To achieve these goals, the thesis is arranged as follows. In Chapter 2, we review the definition of IHs, rederive the boundary conditions and restriction equations on IHs and introduce the local uniqueness solutions $\{P^2,\ U,\ B,\ \phi_1\}$. In Chapter 3, we explicitly calculate the on-horizon data $\{K, \Phi_{11},\ \mathrm{Re}(\Psi_2), \pi^{(0)}, \omega_a, \mathrm{etc}\}$ using the local uniqueness solutions $\{P^2,\ U,\ B,\ \phi_1\}$, and interpret the meanings of three parameters $\{\alpha, A, \theta_0\}$. In Chapter 4, we extend the adapted tetrad on IHs to cover the exteriors, and develop the method to construct the near-horizon geometry of an IH embedded in electrovacuum using on-horizon data. In Chapter 5, we explicitly compute the near-horizon metrics of extremal Reissner-Nordström and Kerr horizons using the local method in Chapter 4, and prove their equivalence with those derived from near-horizon limit of the corresponding global metrics. In Chapter 6, we take the extremal Reissner-Nordström black hole as an example, treat it as a conformastatic metric, and investigate its superposition with external conformastatic fields.

Since we take an isolated-horizon approach to study black hole horizons (c.f. ref.[16] or ref.[17] for a complete review of IHs), the best mathematical language should be the null tetrad formalism developed by Newman and Penrose[18]. Unlike the traditional signature $\{(+,-,-,-),\ l^a n_a = 1, m^a \bar{m}_a = -1\}$ used in Newman-Penrose (NP) formalism[18][19], we will switch to $\{(-, +, +, +),\ l^a n_a = -1, m^a \bar{m}_a = 1\}$ throughout this thesis in accordance with the signature used for trapping surfaces. The consequences of this change and the principal NP equations are discussed in Appendix B. Also, we will employ the *tensorial* rather than *spinorial* version of NP formalism; for a unified formulation of these two versions, one can refer to Chapters 2 and 3 of ref.[20].

Chapter 2

Isolated Horizons: Boundary Conditions and Extremal Structures

The geometry and mechanics of *generic* (rotating and distorted) isolated horizons (IHs) were developed in refs.[6][7][13], and based on these we will rederive the boundary conditions of IHs and analyze the intrinsic structures of extremal IHs in this chapter. In fact, IHs date back to refs.[21]-[24], but these earliest works were imperfect as they only dealt with nonrotating and undistorted IHs.

Following the standard set up in refs.[6][7][13], we will introduce nonexpanding horizons (NEHs), weakly isolated horizons (WIHs) and IHs in sequence. NEHs are geometric prototypes of WIHs and IHs, on which we can establish all boundary conditions, and WIHs and IHs would naturally inherit all these conditions. Strengthening the concept of NEHs to WIHs, we will be able to define a valid surface gravity and generalize the black hole mechanics[7]. WIHs are sufficient in studying the physics on the horizon, but for geometric purposes, stronger restrictions can be imposed to WIHs so as to introduce IHs, where the equivalence class of null normals $[l]$ fully preserves the induced connection $\mathcal{D}$ on the horizon[13].

2.1 Boundary Conditions of Einstein-Maxwell IHs

2.1.1 Generic NEHs: Definition and Implications

Isolated horizons provide a local description for black holes in equilibrium with their exteriors. An IH is an NEH whose *extrinsic* structure is preserved, while an NEH is an enclosed null surface whose *intrinsic* structure is preserved. NEHs are geometric prototypes of IHs, so we will begin with NEHs to investigate the geometric characteristics.

A three-dimensional submanifold Δ is defined as a generic NEH if it respects the following conditions[7][13],

(i) Δ is null and topologically $S^2 \times \mathbb{R}$;

(ii) Along any null normal field l tangent to Δ, the outgoing expansion rate $\theta_{(l)} := \hat{h}^{ab}\hat{\nabla}_a l_b$ vanishes;

(iii) All field equations hold on Δ, and the stress-energy tensor T_{ab} on Δ is such that $V^a := -T^a_b l^b$ is a future-directed causal vector for any future-directed null normal l^a, $V^a V_a \leq 0$.

Condition (i) is fairly trivial and just states the general fact that from a $3+1$ perspective[26] an NEH[1] Δ is foliated by spacelike 2-spheres $\hat{\Delta} = S^2$, where S^2 emphasizes that $\hat{\Delta}$ is topologically compact with genus zero ($g = 0$). The signature of Δ is $(0, +, +)$ with a degenerate temporal coordinate, and the intrinsic geometry of a foliation leaf $\hat{\Delta} = S^2$ is nonevolutional. The property $\theta_{(l)} = 0$ in condition (ii) plays a pivotal role in defining NEHs and the rich implications encoded therein will be extensively discussed below. Condition (iii) makes one

[1]Following the conventions in refs.[6][7][13], the symbol Δ is adopted to denote an NEH; in the following context, Δ also refers to the standard symbol for the directional derivative $\Delta := n^a\nabla_a$ in NP formalism. We believe this won't cause an ambiguity.

feel free to apply the NP formalism of Einstein-Maxwell field equations to the horizon and its near-horizon vicinity; furthermore, the very energy inequality is motivated from the dominant energy condition[27][28] and is a sufficient condition for deriving many boundary conditions of NEHs.

Now let's work out the implications of the definition of NEHs. Being a null normal to Δ, l^a is automatically geodesic, $\kappa := -m^a l^b \nabla_b l_a \hat{=} 0$, and twist free, $\mathrm{Im}(\rho) = \mathrm{Im}(-m^a \bar{m}^b \nabla_b l_a) \hat{=} 0$. For an NEH , the outgoing expansion rate $\theta_{(l)}$ along l^a is vanishing, $\theta_{(l)} \hat{=} 0$, and consequently $\mathrm{Re}(\rho)$=$\mathrm{Re}(-m^a \bar{m}^b \nabla_b l_a) = -\frac{1}{2}\theta_{(l)} \hat{=} 0$. Moreover, according to the Raychaudhuri-NP *expansion-twist* equation (also for the *shear* equation Eq(2.5) below, c.f. page 56, Section 9(a) of ref.[19])[2],

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \frac{1}{2} R_{ab} l^a l^b \hat{=} 0 \,, \tag{2.1}$$

it follows that on Δ

$$\sigma\bar{\sigma} + \frac{1}{2} R_{ab} l^a l^b \hat{=} 0 \,, \tag{2.2}$$

where $\sigma := -m^b m^a \nabla_a l_b$ is the NP-shear coefficient. Due to the assumed energy condition (iii), we have $R_{ab} l^a l^b = R_{ab} l^a l^b - \frac{1}{2} R g_{ab} l^a l^b = 8\pi T_{ab} l^a l^b$ $(c = G = 1)$, and therefore $R_{ab} l^a l^b$ is nonnegative on Δ. The product $\sigma\bar{\sigma}$ is of course nonnegative, too. Consequently, $\sigma\bar{\sigma}$ and $R_{ab} l^a l^b$ must be simultaneously zero on Δ, i.e. $\sigma \hat{=} 0$ and $R_{ab} l^a l^b \hat{=} 0$. As a summary,

$$\kappa \hat{=} 0\,, \quad \mathrm{Im}(\rho) \hat{=} 0\,, \quad \mathrm{Re}(\rho) \hat{=} 0\,, \quad \sigma \hat{=} 0\,, \quad R_{ab} l^a l^b \hat{=} 0. \tag{2.3}$$

Thus, the isolated horizon Δ is nonevolutional and all foliation leaves $\hat{\Delta} = S^2$ look identical with one another. The relation $R_{ab} l^a l^b = 8\pi \cdot T_{ab} l^a l^b = 8\pi \cdot T^a_b l^b \cdot l_a \hat{=} 0$ implies that the causal vector $-T^a_b l^b$ in condition (iii) is proportional to l^a and $R_{ab} l^b$ is proportional to l_a on

[2] Following the conventions in refs.[6][7][13], the symbol $\hat{=}$ means equality on NEHs and their spacelike cross-sections, while hat over quantities ($\hat{\Delta}$, $\hat{\Gamma}^a_{bc}$, etc) would denote intrinsic quantities on a foliation leaf.

the horizon Δ; that is, $-T^a_b l^b \hat{=} c l^a$ and $R_{ab} l^b \hat{=} c l_a$, $c \in \mathbb{R}$. Applying this result to the related Ricci-NP scalars, we get $\Phi_{00} := \frac{1}{2} R_{ab} l^a l^b \hat{=} \frac{c}{2}\, l_b l^b \hat{=} 0$, $\Phi_{01} = \overline{\Phi_{10}} := \frac{1}{2} R_{ab} l^a m^b \hat{=} \frac{c}{2}\, l_b m^b \hat{=} 0$, thus

$$R_{ab} l^b \hat{=} c l_a\,, \quad \Phi_{00} \hat{=} 0\,, \quad \Phi_{10} = \overline{\Phi_{01}} \hat{=} 0\,. \tag{2.4}$$

The vanishing of Ricci-NP quantities $\{\Phi_{00}\,, \Phi_{01}\,, \Phi_{10}\}$ signifies that, there is no energy-momentum flux of *any* kind of charge *across* the horizon, such as electromagnetic waves, Yang-Mills flux or dilaton flux. Also, there should be no gravitational waves crossing the horizon; however, gravitational waves are propagation of perturbations of the spacetime continuum rather than flows of charges, and therefore depicted by four Weyl quantities Ψ_i $(i = 0, 1, 3, 4)$ (excluding Ψ_2) rather than Ricci quantities Φ_{ij}. According to the Raychaudhuri-NP *shear* equation

$$D\sigma = \sigma(\rho + \bar{\rho}) + \Psi_0 = -2\sigma\theta_{(l)} + \Psi_0\,, \tag{2.5}$$

or the NP field equation on the horizon

$$D\sigma - \delta\kappa = (\rho + \bar{\rho})\sigma + (3\varepsilon - \bar{\varepsilon})\sigma - (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)\kappa + \Psi_0 \hat{=} 0\,, \tag{2.6}$$

it follows that $\Psi_0 := C_{abcd} l^a m^b l^c m^d \hat{=} 0$. Moreover, the NP equation

$$\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha} + \beta) - \sigma(3\alpha - \bar{\beta}) + (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa - \Psi_1 + \Phi_{01} \hat{=} 0 \tag{2.7}$$

implies that $\Psi_1 := C_{abcd} l^a n^b l^c m^d \hat{=} 0$. To sum up, we have

$$\Psi_0 \hat{=} 0\,, \quad \Psi_1 \hat{=} 0\,, \tag{2.8}$$

which means that (c.f. Sections 2.1.1 and 2.1.2 in ref.[29]), geometrically, a principal null direction of Weyl's tensor is repeated twice and l^a is aligned with the principal direction; physically, no gravitational waves (transverse component Ψ_0 and longitudinal component Ψ_1) enter the black hole[29][30]. This result is consistent with the physical scenario defining NEHs.

Remarks The tensor form of Raychaudhuri's equation for a null congruence reads (c.f. ref.[31] for a relatively comprehensive review)

$$\pounds_l \theta_{(l)} = -\frac{1}{2}\theta_{(l)}^2 + \tilde{\kappa}_{(l)}\theta_{(l)} - \sigma_{ab}\sigma^{ab} + \tilde{\omega}_{ab}\tilde{\omega}^{ab} - R_{ab}l^a l^b\,, \tag{2.9}$$

where $\tilde{\kappa}_{(l)}$ is defined such that $\tilde{\kappa}_{(l)}l^b := l^a\nabla_a l^b$. The quantities in Raychaudhuri's equation are related with NP spin coefficients via (c.f. Chapter 2 of ref.[29], Chapter 9 of ref.[30], Section 2.3 of ref.[32], and Chapters 2 and 3 of ref.[28], where null congruences and the meanings of spin coefficients employed in this section are extensively discussed)

$$\theta_{(l)} = -(\rho + \bar{\rho}) = -2\text{Re}(\rho)\,, \quad \theta_{(n)} = \mu + \bar{\mu} = 2\text{Re}(\mu)\,, \tag{2.10}$$

$$\sigma_{ab} = -\sigma\bar{m}_a\bar{m}_b - \bar{\sigma}m_a m_b\,, \tag{2.11}$$

$$\tilde{\omega}_{ab} = \frac{1}{2}\big(\rho - \bar{\rho}\big)\big(m_a\bar{m}_b - \bar{m}_a m_b\big) = \text{Im}(\rho)\cdot\big(m_a\bar{m}_b - \bar{m}_a m_b\big)\,, \tag{2.12}$$

where Eq(2.10) follows directly from $\hat{h}^{ab} = \hat{h}^{ba} = m^b\bar{m}^a + \bar{m}^b m^a$ and

$$\begin{aligned}\theta_{(l)} &= \hat{h}^{ba}\nabla_a l_b = m^b\bar{m}^a\nabla_a l_b + \bar{m}^b m^a\nabla_a l_b \\ &= m^b\bar{\delta}l_b + \bar{m}^b\delta l_b = -(\rho + \bar{\rho})\,,\end{aligned} \tag{2.13}$$

$$\begin{aligned}\theta_{(n)} &= \hat{h}^{ba}\nabla_a n_b = \bar{m}^b m^a\nabla_a n_b + m^b\bar{m}^a\nabla_a n_b \\ &= \bar{m}^b\delta n_b + m^b\bar{\delta}n_b = \mu + \bar{\mu}\,.\end{aligned} \tag{2.14}$$

Moreover, a null congruence is *hypersurface orthogonal* if $\text{Im}(\rho) = 0$ (c.f. Section 2.1.3 of ref.[29]).

2.1.2 Constraints from Electromagnetic Fields

Vacuum NEHs on which $\{\Phi_{ij} \hat{=} 0\,,\Lambda \hat{=} 0\}$ are the simplest types of NEHs, but in general there can be various physically meaningful fields surrounding an NEH, among which we are mostly interested in electrovacuum fields with $\Lambda \hat{=} 0$. This is the simplest extension of vacuum NEHs, being a special kind of Einstein-Maxwell NEHs with no electromagnetic media overspreading the external regions. The nonvanishing energy-stress tensor for electromagnetic fields reads[27]

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right), \tag{2.15}$$

where F_{ab} refers to the antisymmetric ($F_{ab} = -F_{ba}$, $F^a_a = 0$) electromagnetic field strength, and T_{ab} is trace-free ($T^a_a = 0$) by definition and respects the dominant energy condition. (One should be careful with the antisymmetry of F_{ab} in defining Maxwell-NP scalars ϕ_i).

The boundary conditions derived in the previous section are applicable to generic NEHs. In the electromagnetic case, Φ_{ij} can be specified in a more particular way. By the NP formalism of Einstein-Maxwell equations, one has (c.f. Section 2.2.1 of ref.[29])

$$\Phi_{ij} = 2\,\phi_i\,\overline{\phi_j}\,, \quad i,j \in \{0,1,2\}\,, \tag{2.16}$$

where ϕ_i denote the three Maxwell-NP scalars. As an alternative to Eq(2.4), we can see that the condition $\Phi_{00} = 0$ also results from the NP equation

$$D\rho - \bar{\delta}\kappa = (\rho^2 + \sigma\bar{\sigma}) + (\varepsilon + \bar{\varepsilon})\rho - \bar{\kappa}\tau - (3\alpha + \bar{\beta} - \pi)\,\kappa + \Phi_{00} \hat{=} 0 \tag{2.17}$$

as $\kappa \hat{=} \rho \hat{=} \sigma = 0$, so

$$\Phi_{00} \hat{=} 0 \quad \Leftrightarrow \quad 2\,\phi_0\,\overline{\phi_0} \hat{=} 0 \quad \Rightarrow \quad \phi_0 = \overline{\phi_0} \hat{=} 0\,. \tag{2.18}$$

It follows straightforwardly that

$$\Phi_{01} = \overline{\Phi_{10}} = 2\,\phi_0\,\overline{\phi_1} \hat{=} 0\,, \quad \Phi_{02} = \overline{\Phi_{20}} = 2\,\phi_0\,\overline{\phi_2} \hat{=} 0\,. \tag{2.19}$$

These results demonstrate that, there are no electromagnetic waves across (Φ_{00}, Φ_{01}) or along (Φ_{02}) the NEH except the null geodesics generating the horizon. It is also worthwhile to point out that, the supplementary equation $\Phi_{ij} = 2\,\phi_i\,\overline{\phi_j}$ in Eq(2.16) is only valid for electromagnetic fields; for example, in the case of Yang-Mills fields there will be $\Phi_{ij} = \mathrm{Tr}\,(F_i \bar{F}_j)$ where F_i ($i \in \{0, 1, 2\}$) are Yang-Mills-NP scalars (c.f. page 27, Appendix A.2 of ref.[34]).

2.1.3 Adapted Frames and Newman-Unti-type Tetrad

Null Tetrad Adapted to NEHs

Usually, null tetrads adapted to spacetime properties are employed to achieve the most succinct NP descriptions. For example, a null tetrad can be adapted to principal null directions once the Petrov type is known[29][30]; also, at some typical boundary regions such as null infinity, timelike infinity, spacelike infinity, black hole horizons and cosmological horizons, tetrads can be adapted to boundary structures. Similarly, a *preferred* tetrad adapted to on-horizon geometric behaviors is employed in the literature to further investigate NEHs (c.f. refs.[7][13][23][24][25][36], etc.).

As indicated from the 3 + 1 perspective from condition (i) in the definition, an NEH Δ is foliated by spacelike hypersurfaces $\hat{\Delta}_v = S^2_v$ transverse to its null normal along an ingoing null coordinate v, where we follow the standard notation of ingoing Eddington-Finkelstein null coordinates and use v to label the 2-dimensional leaves S^2_v at v = constant; that is, $\Delta = \hat{\Delta} \times [v_0, v_1] = S^2 \times [v_0, v_1]$. v is set to be future-directed and choose the first tetrad covector n_a as $n_a = -dv$ ($\pounds_v n^a = -1$)[7][13], and then there will be a unique vector field l^a as null normals to S^2_v satisfying the cross-normalization $l^a n_a = -1$ and affine parametrization $Dv = 1$; such choice of $\{l^a, n^a\}$ would actually yields a preferred foliation of Δ. While $\{l^a, n^a\}$ are related to the extrinsic properties and null generators (i.e. null flows/geodesic congruence on Δ), the remaining two complex null vectors $\{m^a, \bar{m}^a\}$ are to span the intrinsic geometry of a foliation leave S^2_v, tangent to Δ and transverse to $\{l^a, n^a\}$; that is, $\pounds_l \boldsymbol{m} \hat{=} \pounds_l \bar{\boldsymbol{m}} \hat{=} 0$.

Now let's check the consequences of this kind of adapted tetrad. Since

$$\pounds_l \boldsymbol{m} \hat{=} 0 = [\boldsymbol{l}, \boldsymbol{m}] \Rightarrow \delta D - D\delta = (\bar{\alpha} + \beta - \bar{\pi})D + \kappa\Delta - (\bar{\rho} + \varepsilon - \bar{\varepsilon})\delta - \sigma\bar{\delta} \hat{=} 0\,, \tag{2.20}$$

with $\kappa \hat{=} \rho \hat{=} \sigma \hat{=} 0$, we have

$$\pi \hat{=} \alpha + \bar{\beta}, \quad \varepsilon \hat{=} \bar{\varepsilon}\,. \tag{2.21}$$

Also, in such an adapted frame, the derivative $\pounds_{\bar{m}} \boldsymbol{m}$ on $\hat{\Delta}_v = S^2_v$ should be purely intrinsic; thus in the commutator

$$\pounds_{\bar{m}} \boldsymbol{m} = [\bar{\boldsymbol{m}}, \boldsymbol{m}] = \bar{\delta}\delta - \delta\bar{\delta} = (\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta - (\bar{\beta} - \alpha)\delta - (\bar{\alpha} - \beta)\bar{\delta}\,, \tag{2.22}$$

the coefficients for the directional derivatives D and Δ must be zero, that is

$$\bar{\mu} \hat{=} \mu\,, \quad \pounds_{\bar{m}} \boldsymbol{m} \hat{=} (\alpha - \bar{\beta})\delta - (\bar{\alpha} - \beta)\bar{\delta}\,, \tag{2.23}$$

so the ingoing null normal field n^a is twist-free by Im(μ)=Im($\bar{m}^a m^b \nabla_b n_a$)=0, and 2μ =

2Re(μ) equals the ingoing expansion rate $\theta_{(n)}$.

The concrete form of such an adapted tetrad will be constructed in Section 4.1, which works for both the horizon and its external vicinity. The construction is inspired by the classic Newman-Unti tetrad[33] used to study asymptotic behaviors at *null infinity*.

Newman-Unti Tetrad for Null Infinity

In this subsection, we will briefly review how Newman-Unti (NU) tetrad works, which also prepares us for the discussion in Section 4.1. The NU tetrad reads (c.f. [33] and Section IV of ref.[18]; or page 29, Appendix B of ref.[34])

$$\begin{aligned} l^a\partial_a &= \partial_r\,, \\ n^a\partial_a &= \partial_u + U\partial_r + X\partial_\varsigma + \bar{X}\partial_{\bar{\varsigma}}\,, \\ m^a\partial_a &= \omega\partial_r + \xi^3\partial_\varsigma + \xi^4\partial_{\bar{\varsigma}}\,, \\ \bar{m}^a\partial_a &= \bar{\omega}\partial_r + \bar{\xi}^3\partial_{\bar{\varsigma}} + \bar{\xi}^4\partial_\varsigma\,. \end{aligned} \tag{2.24}$$

For the NU tetrad, the foliation leaves are parameterized by the outgoing (advanced) null coordinate u with $l_a = du$, and r is the normalized affine coordinate along l^a ($Dr = l^a\partial_a r = 1$); the ingoing null vector n^a acts as the null generator at null infinity with $\Delta u = n^a\partial_a u = 1$. The coordinates $\{u, r, \varsigma, \bar{\varsigma}\}$ comprise two real affine coordinates $\{u, r\}$ and two complex stereographic coordinates $\{\varsigma := e^{i\phi}\cot\frac{\theta}{2}, \bar{\varsigma} = e^{-i\phi}\cot\frac{\theta}{2}\}$, where $\{\theta, \phi\}$ are the usual spherical coordinates on the cross-section $\hat{\Delta}_u = S^2_u$ (as shown in Appendix B of [34], *complex stereographic* rather than *real isothermal* coordinates are used just for the convenience of completely solving NP equations). For the NU tetrad, the null frame $\{l_a, n_a, m_a, \bar{m}_a\}$ is parallelly transported along the tangent vector field $l^a\partial_a$, thus

$$Dl_a = (\varepsilon + \bar{\varepsilon})l_a - \bar{\kappa}m_a - \kappa\bar{m}_a = 0\,, \tag{2.25}$$

$$Dn_a = -(\varepsilon + \bar{\varepsilon})n_a + \pi m_a + \bar{\pi}\bar{m}_a = 0\,, \tag{2.26}$$

$$Dm_a = \bar{\pi}l_a - \kappa n_a + (\varepsilon - \bar{\varepsilon})\bar{m}_a = 0\,, \tag{2.27}$$

$$D\bar{m}_a = \pi l_a - \bar{\kappa}n_a + (\bar{\varepsilon} - \varepsilon)m_a = 0\,, \tag{2.28}$$

which implies that

$$\kappa = \pi = \varepsilon = 0\,. \tag{2.29}$$

Furthermore, applying the commutators to u and taking Eq(2.24) and the relations $\{Du = 0\,, \Delta u = 1\,, \delta u = 0\,, \bar{\delta}u = 0\}$ into account, one obtains

$$(\Delta D - D\Delta)\,u = (\gamma + \bar{\gamma})(Du) - \bar{\tau}(\delta u) - \tau(\bar{\delta}u) \;=\; 0\,, \tag{2.30}$$

$$(\delta D - D\delta)\,u = (\bar{\alpha} + \beta)(Du) - \bar{\rho}(\delta u) - \sigma(\bar{\delta}u) \;=\; 0\,, \tag{2.31}$$

$$(\delta\Delta - \Delta\delta)\,u = -\bar{\nu}(Du) + (\tau - \bar{\alpha} - \beta)(\Delta u) + (\mu - \gamma + \bar{\gamma})(\delta u) + \bar{\lambda}(\bar{\delta}u) = 0\,, \tag{2.32}$$

$$(\bar{\delta}\delta - \delta\bar{\delta})\,u = (\bar{\mu} - \mu)(Du) + (\bar{\rho} - \rho)(\Delta u) + (\alpha - \bar{\beta})(\delta u) - (\bar{\alpha} - \beta)(\bar{\delta}u) = 0\,. \tag{2.33}$$

While the first two commutator equations are trivial, the last two commutators yield that

$$\rho = \bar{\rho}\,, \qquad \tau = \bar{\alpha} + \beta\,. \tag{2.34}$$

Eq(2.29) and Eq(2.34) constitute the basic gauge conditions for the NU tetrad. This tetrad naturally annihilates several spin coefficients and exerts restrictions between other spin coefficients. This is just what we are looking for to express the aforementioned boundary condition for NEHs. However, the NU tetrad is designed for null or spatial infinity and is inappropriate for near-horizon regions; for example, ε in Eq(2.29) is related to the surface gravity (i.e. acceleration along null geodesics) and should be nonzero close to NEHs except in extremal situations. In a word, the NU tetrad cannot serve as the adapted tetrad for IHs after transition to the horizons.

However, inspired by the method of NU tetrad Eq(2.24), one can construct a similar tetrad for the horizon and its vicinity which fully respects the properties discussed in the previous section by switching the roles of l^a and n^a, resetting the outgoing null vector field l^a as generators, and taking the ingoing (retarded) null coordinate v as the foliation parameter, as will be discussed in Section 4.1 (c.f. Section I of ref.[35] for the differences of asymptotic behaviors between null infinity and near-horizon vicinity). In advance, we introduce the new gauge conditions in modified NU tetrad for a preview,

$$\nu = \tau = \gamma = 0\,, \quad \mu = \bar{\mu}\,, \quad \pi = \alpha + \bar{\beta}\,, \tag{2.35}$$

which agree with the boundary conditions of NEHs.

2.2 Geometry and Connections of NEHs

Connections and Rotation 1-Form Potential

While the connections on the full spacetime outside an NEH are depicted by Levi-Civita connections whose components in a local coordinate system manifest themselves as Christoffel symbols, the connections on an NEH are given by (c.f. Appendix B.2 of ref.[13])

$$\mathcal{D}_a l^b = \underleftarrow{\nabla_a} l^b \mathrel{\hat{=}} \omega_a l^b\,, \tag{2.36}$$

$$\bar{m}^b \nabla_a n_b \mathrel{\hat{=}} -\pi n_a + \lambda m_a + \mu \bar{m}_a\,, \tag{2.37}$$

$$m^b \nabla_a \bar{m}_b \mathrel{\hat{=}} -\bar{m}^b \mathcal{D}_a m_b \mathrel{\hat{=}} (\varepsilon - \bar{\varepsilon}) n_a - (\alpha - \bar{\beta}) m_a + (\bar{\alpha} - \beta) \bar{m}_a\,. \tag{2.38}$$

With the properties Eq(2.21), Eq(2.23) and Eq(2.35) in the adapted tetrad, the commutator $\pounds_{n}\boldsymbol{l} = [\Delta, D]$ yields

$$\begin{aligned} \Delta l_a - D n_a &= (\gamma + \bar{\gamma}) l_a + (\varepsilon + \bar{\varepsilon}) n_a - (\bar{\tau} + \pi) m_a - (\tau + \bar{\pi}) \bar{m}_a \\ &= (\varepsilon + \bar{\varepsilon}) n_a - \pi m_a - \bar{\pi} \bar{m}_a \\ &= (\varepsilon + \bar{\varepsilon}) n_a - (\alpha + \bar{\beta}) m_a - (\bar{\alpha} + \beta) \bar{m}_a \,. \end{aligned} \tag{2.39}$$

Contracting Eq2.36 with n_b, one obtains the *rotation 1-form potential* ω_a,

$$\begin{aligned} \omega_a &\triangleq -n_b \mathcal{D}_a l^b \triangleq -\pounds_{n}\boldsymbol{l} \\ &\triangleq -(\varepsilon + \bar{\varepsilon}) n_a + \pi m_a + \bar{\pi} \bar{m}_a \\ &\triangleq -(\varepsilon + \bar{\varepsilon})\, n_a + (\alpha + \bar{\beta})\, m_a + (\bar{\alpha} + \beta)\, \bar{m}_a \,. \end{aligned} \tag{2.40}$$

So far, the main boundary conditions and connections on NEHs have been derived. An NEH is a geometric object, or a proto-IH; to study the mechanics, we need to strengthen the concept of NEHs and introduce weekly isolated horizons (WIHs) which have a well-defined valid surface gravity.

WIHs, Surface Gravity and IHs

A WIH $(\Delta, [l])$ is an NEH (Δ) equipped with an equivalence class $[l]$ of null normals satisfying[7][13]

$$\pounds_l \omega_a \triangleq 0 \;\text{ or }\; [\pounds_l, \mathcal{D}_a]\, l^a \triangleq 0\,, \quad \forall l \in [l]\,, \quad [l] = \left\{ l' | \, l' = c l\,,\; c \in \mathbb{R} \right\}. \tag{2.41}$$

Being a subset of NEHs, WIHs naturally inherit all the boundary conditions and geometry of NEHs. Similar to other Killing horizons, the surface gravity $\kappa_{(l)}$ for WIHs can be defined as the acceleration along null geodesics,

$$Dl^a = l^b \nabla_b l^a := \kappa_{(l)}\, l^a\,. \tag{2.42}$$

Compare this definition with the transportation equation

$$Dl^a = (\varepsilon + \bar{\varepsilon})l^a - \bar{\kappa} m^a - \kappa \bar{m}^a \mathrel{\hat{=}} (\varepsilon + \bar{\varepsilon})l^a\,, \tag{2.43}$$

and with the aid of Eq(2.21) it naturally follows that

$$\kappa_{(l)} \mathrel{\hat{=}} \varepsilon + \bar{\varepsilon} \mathrel{\hat{=}} 2\varepsilon. \tag{2.44}$$

After defining the surface gravity, the rotation 1-form potential ω_a and the geometric connections on a WIH $(\Delta = \hat{\Delta} \times \mathbb{R}, [l])$ become

$$\mathcal{D}_a l^b \mathrel{\hat{=}} \omega_a l^b \mathrel{\hat{=}} (-\kappa_{(l)} n_a + \pi m_a + \bar{\pi} \bar{m}_a)\, l^b\,, \tag{2.45}$$

$$\bar{m}^b \nabla_a n_b \mathrel{\hat{=}} -\pi n_a + \lambda m_a + \mu \bar{m}_a\,, \tag{2.46}$$

$$\bar{m}^b \mathcal{D}_a m_b \mathrel{\hat{=}} \zeta m_a - \bar{\zeta} \bar{m}_a\,, \tag{2.47}$$

where the commutator coefficient ζ is defined via $\zeta := \alpha - \bar{\beta}$. In these connection equations, the terms involving n_a arise from the foliation process, so the rotation 1-form potential $\hat{\omega}_a$ and the intrinsic connections on a foliation leaf $\hat{\Delta}_v = S_v^2$ are given by

$$\hat{\omega}_a \mathrel{\hat{=}} \pi m_a + \bar{\pi} \bar{m}_a\,, \tag{2.48}$$

$$\bar{m}^b \hat{\nabla}_a n_b \mathrel{\hat{=}} \lambda m_a + \mu \bar{m}_a\,, \tag{2.49}$$

$$m^b \hat{\mathcal{D}}_a \bar{m}_b \mathrel{\hat{=}} -\zeta m_a + \bar{\zeta} \bar{m}_a\,. \tag{2.50}$$

By the way, it follows from Eq(2.48) that

$$\hat{\omega}^a = \pi m^a + \bar{\pi}\bar{m}^a\,, \quad \hat{\omega}_a\,\hat{\omega}^a = 2\pi\bar{\pi}\,, \tag{2.51}$$

where the index of $\hat{\omega}^a$ is raised from $\hat{\omega}_a$ by the intrinsic inverse metric $\hat{h}^{ab} = m^a\bar{m}^b + \bar{m}^a m^b$.

As proved in ref.[13], an NEH can *always* be strengthened to produce a corresponding WIH, and within the framework of WIHs, we are able to carry out all necessary calculations for the geometry and mechanics of the horizon. However, we will still take this opportunity to introduce the definition of IHs[7][13] for completeness; that is, *an IH is a WIH with* $[\pounds_l, \mathcal{D}_a] \,\hat{=}\, 0$.

Weyl-NP Scalar Ψ_2

Besides the connection coefficients discussed above, another important quantity reflecting both geometrical and mechanical characters of WIHs is the Weyl-NP scalar Ψ_2 (the other four Weyl scalars being related to gravitational waves). The NP equation

$$\delta\alpha - \bar{\delta}\beta = (\mu\rho - \lambda\sigma) + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \varepsilon(\mu - \bar{\mu}) - \Psi_2 + \Phi_{11} + \Lambda \tag{2.52}$$

yields that (as Φ_{ii} are real by definition and $\Lambda = 0$)

$$\delta\alpha - \bar{\delta}\beta \,\hat{=}\, \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta - \Psi_2 + \Phi_{11}\,, \quad \bar{\delta}\bar{\alpha} - \delta\bar{\beta} \,\hat{=}\, \alpha\bar{\alpha} + \beta\bar{\beta} - 2\bar{\alpha}\bar{\beta} - \overline{\Psi}_2 + \Phi_{11}\,, \tag{2.53}$$

addition and subtraction of which give rise to the real and imaginary parts of Ψ_2 that

$$-\,2\mathrm{Re}(\Psi_2) \,\hat{=}\, \delta\zeta + \bar{\delta}\bar{\zeta} - 2\zeta\bar{\zeta} - 2\Phi_{11} \;=\; K - 2\Phi_{11}\,, \tag{2.54}$$

$$-\,2\mathrm{Im}(\Psi_2) \,\hat{=}\, \delta\zeta - \bar{\delta}\bar{\zeta} + 2\alpha\beta - 2\bar{\alpha}\bar{\beta}\,, \tag{2.55}$$

where

$$K := \delta\zeta + \bar{\delta}\bar{\zeta} - 2\zeta\bar{\zeta} \tag{2.56}$$

refers to Gaussian curvature of the cross-section $\hat{\Delta}_v = S^2_v$. Thus for electrovacuum

$$\Psi_2 = \frac{1}{2}\Big(- K + 2\Phi_{11} + i\,(\bar{\delta}\bar{\zeta} - \delta\zeta + 2\bar{\alpha}\bar{\beta} - 2\alpha\beta)\Big), \tag{2.57}$$

which reduces to

$$\Psi_2 = \frac{1}{2}\Big(- K + i\,(\bar{\delta}\bar{\zeta} - \delta\zeta + 2\bar{\alpha}\bar{\beta} - 2\alpha\beta)\Big), \tag{2.58}$$

for vacuum WIHs with $\Phi_{11} = 0$. Moreover, Eq(2.21) and Eq(2.48) yield that[7][13]

$$d\hat{\omega}_a = (\bar{\delta}\bar{\zeta} - \delta\zeta + 2\bar{\alpha}\bar{\beta} - 2\alpha\beta)\,\epsilon^{(2)} = 2\,\mathrm{Im}(\Psi_2)\,\epsilon^{(2)}\,, \tag{2.59}$$

where $\epsilon^{(2)}$ is the area 2-form of the cross-section $\hat{\Delta} = S^2$. Thus, the intrinsic *Gaussian curvature* and *rotational property* of a WIH are encoded into the *real* and *imaginary* part of Ψ_2 respectively. Furthermore, since the horizon cannot be flat ($K \neq 0$), Ψ_2 is always nonzero ($\Psi_2 \neq 0$); now recalling the boundary conditions Eq(2.8) that $\Psi_0 \hat{=} \Psi_1 \hat{=} 0$, thus geometrically a large class of WIHs will be of *Petrov-type D*[29][30], including WIHs induced from the renowned Kerr-Newman-family black-hole horizons.

2.3 Extremal Electrovacuum WIHs with Axisymmetry

2.3.1 Field Equations for Vacuum and Electrovacuum WIHs

The intrinsic field equations on a foliation leaf $\hat{\Delta} = S^2$ should be related to the pullbacks $\underleftarrow{R_{ab}m^a m^b}$, $\underleftarrow{R_{ab}\bar{m}^a\bar{m}^b}$ and $\underleftarrow{R_{ab}m^a\bar{m}^b}$ (c.f. ref.[13] and its Appendix B). Employing the boundary conditions in Eqs(2.3), (2.21), (2.23) and (2.48) and projecting the NP field equation

$$\delta\tau - \Delta\sigma = (\mu\sigma + \bar{\lambda}\rho) + (\tau + \beta - \bar{\alpha})\tau - (3\gamma - \bar{\gamma})\sigma - \kappa\bar{\nu} + \Phi_{02} \tag{2.60}$$

onto $\hat{\Delta} = S^2$, one obtains

$$\frac{1}{2} \underleftarrow{R_{ab} m^a m^b} \; : \quad \Phi_{02} \hat{=} 0\,; \tag{2.61}$$

while projection of

$$D\lambda - \bar{\delta}\pi = (\rho\lambda + \bar{\sigma}\mu) + \pi^2 + (\alpha - \bar{\beta})\pi - \nu\bar{\kappa} - (3\varepsilon - \bar{\varepsilon})\lambda + \Phi_{20}\,, \tag{2.62}$$

onto a foliation leaf yields

$$\frac{1}{2} \underleftarrow{R_{ab} \bar{m}^a \bar{m}^b} \; : \quad \Phi_{20} \hat{=} \kappa_{(l)}\lambda - \bar{\delta}\pi - \pi^2 - \zeta\pi \; = \; 0\,. \tag{2.63}$$

Now recall that for *electrovacuum*,

$$24\Lambda \; = 0 = \; R_{ab} g^{ab} \; = \; R_{ab}\Big(- 2l^a n^b + 2m^a \bar{m}^b \Big) \;\; \Rightarrow \;\; R_{ab} l^a n^b \; = \; R_{ab} m^a \bar{m}^b\,, \tag{2.64}$$

thus

$$\Phi_{11} := \frac{1}{4} R_{ab}(\, l^a n^b + m^a \bar{m}^b) = \frac{1}{2} R_{ab} l^a n^b = \frac{1}{2} R_{ab} m^a \bar{m}^a \tag{2.65}$$

Add together the following equations

$$D\mu - \delta\pi = (\bar{\rho}\mu + \sigma\lambda) + \pi\bar{\pi} - (\varepsilon + \bar{\varepsilon})\mu - (\bar{\alpha} - \beta)\pi - \nu\kappa + \Psi_2 + 2\Lambda\,, \tag{2.66}$$

$$\delta\alpha - \bar{\delta}\beta = (\mu\rho - \lambda\sigma) + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \varepsilon(\mu - \bar{\mu}) - \Psi_2 + \Phi_{11} + \Lambda\,, \tag{2.67}$$

and one obtains

$$\frac{1}{2}\underleftarrow{R_{ab}m^a\bar{m}^b} : \quad \Phi_{11} \hat{=} \kappa_{(l)}\mu - \delta\pi - \pi\bar{\pi} + \bar{\zeta}\pi + \delta\alpha - \bar{\delta}\beta - \alpha\bar{\alpha} - \beta\bar{\beta} + 2\alpha\beta$$
$$\hat{=} \kappa_{(l)}\mu - \frac{1}{2}\Big(\hat{\mathrm{div}}\,\hat{\omega} + 2\pi\bar{\pi} - K\Big)\,. \tag{2.68}$$

where $\hat{\mathrm{div}}\,\hat{\omega}$ refers to the divergence of the intrinsic rotation 1-form potential $\hat{\omega}$,

$$\hat{\mathrm{div}}\,\hat{\omega} \hat{=} \delta\pi + \bar{\delta}\bar{\pi} - \zeta\bar{\pi} - \bar{\zeta}\pi\,, \tag{2.69}$$

and K is the Gaussian curvature of $\hat{\Delta} = S^2$ as defined previously by Eq(2.56). Eq(2.61) appears trivial due to Eq(2.19) while Eq(2.63) and Eq(2.68) turn out to be the valid field equations projected onto foliation leaves.

Hence, for a WIH embedded in *vacuum* ($\Phi_{ij} = \Lambda = 0$), Einstein's equation yields from Eq(2.63) and Eq(2.68) that

$$R_{\bar{m}\bar{m}} \hat{=} 2\kappa_{(l)}\lambda - 2\bar{\delta}\pi - 2\zeta\pi - 2\pi^2 = 0\,, \tag{2.70}$$

$$R_{m\bar{m}} \hat{=} 2\kappa_{(l)}\mu - \hat{\mathrm{div}}\,\hat{\omega} - 2\pi\bar{\pi} + K = 0\,, \tag{2.71}$$

with the abbreviations therein defined as $R_{\bar{m}\bar{m}} := R_{ab}\bar{m}^a\bar{m}^b = \overline{R_{mm}} := \overline{R_{ab}m^am^b}$, $R_{m\bar{m}} := R_{ab}m^a\bar{m}^b$.

For a foliation leaf of an *electrovacuum* WIH ($\Phi_{ij} = 2\phi_i\overline{\phi_j}$, $\Lambda = 0$) which reduces to a vacuum WIH with $\Phi_{ij} = 0$, Einstein-Maxwell equations imply from Eq(2.63) and Eq(2.68) that

$$R_{\bar{m}\bar{m}} \hat{=} 2\kappa_{(l)}\lambda - 2\bar{\delta}\pi - 2\zeta\pi - 2\pi^2 = 0\,, \tag{2.72}$$

$$R_{m\bar{m}} \hat{=} 2\Phi_{11} \hat{=} 2\kappa_{(l)}\mu - \hat{\text{div}}\,\hat{\omega}_a - 2\pi\bar{\pi} + K = 4\phi_1\bar{\phi}_1\,, \tag{2.73}$$

where ϕ_1 respects the reduced Maxwell-NP equations that (c.f. Appendix B)

$$D\phi_1 \hat{=} 0\,, \tag{2.74}$$

$$\delta\phi_1 \hat{=} 0\,, \tag{2.75}$$

$$\Delta\phi_1 \hat{=} \delta\phi_2 - 2\mu\phi_1 + 2\beta\,\phi_2\,, \tag{2.76}$$

$$\bar{\delta}\phi_1 \hat{=} D\phi_2 - 2\pi\phi_1 + \kappa_{(l)}\,\phi_2\,. \tag{2.77}$$

2.3.2 Extremal Vacuum WIHs

For extremal WIHs, we have $\kappa_{(l)} = 0$ (c.f. [37] for *quasilocal* characterizations of extremality), and the vacuum equations Eq(2.70) and Eq(2.71) become

$$\bar{\delta}\pi + \zeta\pi + \pi^2 = 0\,, \tag{2.78}$$

$$\hat{\text{div}}\,\hat{\omega} + 2\pi\bar{\pi} - K = 0\,. \tag{2.79}$$

To solve these equations, Lewandowski and Pawlowski[14] decomposed the rotation 1-form potential $\hat{\omega}$ on $\hat{\Delta} = S^2$ into an exact part $\hat{\star}dU$ and a coexact part $d\ln B$ (this decomposition and its existence firstly appeared in Section III.C of ref.[13]),

$$\hat{\omega} = \hat{\star}dU + d\,\ln B\,. \tag{2.80}$$

Here U represents the *rotational scalar potential* which accounts for the gravitational contribution to the angular momentum of a WIH and is defined by[13]

$$\hat{\nabla}_L^2 U \mathrel{\hat{=}} 2\,\mathrm{Im}(\Psi_2)\,, \quad \hat{\nabla}_L^2 := \delta\bar{\delta} + \bar{\delta}\delta - \zeta\delta - \bar{\zeta}\bar{\delta}\,, \tag{2.81}$$

where $\hat{\nabla}_L^2$ is the 2-dimensional intrinsic Laplacian, while B in Eq(2.80) just stands for gauge freedom[13]. As a consequence, the coefficient π of $\hat{\omega}_a$ in Eq(2.48) can be rewritten into[14]

$$\pi = -i\bar{\delta}U + \bar{\delta}\ln B\,. \tag{2.82}$$

Since $\hat{\omega}_a$ in Eq(2.48) reflects the rotational properties, a WIH is non-rotating ($\hat{\omega}_a = 0$) if and only if $\pi = 0$, which implies by Eq(2.80) that $U = 0$ and $B = 1$. Substitute $\pi = \pi(U, B)$ and its complex conjugate $\bar{\pi} = i\delta U + \delta \ln B$ into the extremal vacuum equation Eq(2.78) and it follows that

$$\frac{1}{B}\Big(\bar{\delta}\delta B + \zeta\bar{\delta}B - 2i\bar{\delta}U \cdot \bar{\delta}B\Big) = (\bar{\delta}U)^2 + i\delta\bar{\delta}U + i\zeta\bar{\delta}U\,. \tag{2.83}$$

The divergence $\hat{\mathrm{div}}\,\hat{\omega}$ introduced in Eq(2.69) becomes

$$\hat{\mathrm{div}}\,\hat{\omega} = -\frac{2}{B^2}\delta B\bar{\delta}B + \frac{1}{B}\hat{\nabla}_L^2 B + i(\bar{\delta}\delta U - \delta\bar{\delta}U) + i(\bar{\zeta}\bar{\delta}U - \zeta\delta U)\,. \tag{2.84}$$

Recall the NP commutator

$$\bar{\delta}\delta - \delta\bar{\delta} = (\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta + (\alpha - \bar{\beta})\delta - (\bar{\alpha} - \beta)\bar{\delta} \mathrel{\hat{=}} \zeta\delta - \bar{\zeta}\bar{\delta}\,, \tag{2.85}$$

hence

$$\hat{\mathrm{div}}\,\hat{\omega} \mathrel{\hat{=}} -\frac{2}{B^2}\delta B\bar{\delta}B + \frac{1}{B}\hat{\nabla}_L^2 B\,, \tag{2.86}$$

and Eq(2.79) becomes

$$\frac{1}{B}\Big\{\hat{\nabla}_L^2 B + 2i\,\delta U\bar{\delta}B - 2i\,\bar{\delta}U\delta B\Big\} = -2\delta U\bar{\delta}U + K\,. \tag{2.87}$$

For vacuum WIHs, according to Eqs(2.58)(2.59)(2.81), the Weyl scalar Ψ_2 which combines the Gaussian curvature K and rotational scalar potential U is given by

$$\Psi_2 = \frac{1}{2}\Big(- K + i\,\hat{\nabla}_L^2 U\Big). \tag{2.88}$$

2.3.3 Extremal Electrovacuum WIHs

In the extremal electrovacuum case, Eq(2.72) and Eq(2.73) become

$$\bar{\delta}\pi + \zeta\pi + \pi^2 = 0\,, \tag{2.89}$$

$$\hat{\operatorname{div}}\,\hat{\omega} + 2\pi\bar{\pi} - K = -4\phi_1\bar{\phi}_1\,. \tag{2.90}$$

Substitute the decompositions Eq(2.80) and Eq(2.82) into Eq(2.89) and Eq(2.90) and one obtains the restriction equations[14] ,

$$\frac{1}{B}\Big\{\bar{\delta}\bar{\delta}B + \zeta\bar{\delta}B - 2i\bar{\delta}U\cdot\bar{\delta}B\Big\} = (\bar{\delta}U)^2 + i\bar{\delta}\bar{\delta}U + i\zeta\bar{\delta}U\,, \tag{2.91}$$

$$\frac{1}{B}\Big\{\hat{\nabla}_L^2 B - 2i\bar{\delta}U\delta B\Big\} = -2\delta U\bar{\delta}U + K - 4\phi_1\bar{\phi}_1\,; \tag{2.92}$$

also, for electrovacuum WIHs, according to Eqs(2.57)(2.59)(2.81), we have

$$\Psi_2 = \frac{1}{2}\Big(- K + 4\phi_1\bar{\phi}_1 + i\,\hat{\nabla}_L^2 U\Big). \tag{2.93}$$

It is notable that ϕ_1 can be reexpressed using the two auxiliary function $\{U,\ B\}$ as well[14]

,

$$\phi_1 = E_0 \frac{e^{2iU}}{B^2}\,, \quad \Rightarrow \Phi_{11} = \frac{2E_0^2}{B^4}\,, \tag{2.94}$$

where E_0 is a constant.

2.3.4 Solutions to Extremal Axisymmetric Electrovacuum

Assuming *axisymmetry* to the leaf $\hat{\Delta} = S^2$, labeling $\hat{\Delta} = S^2$ with real isothermal coordinates (x, φ) and introducing the following complex tetrad with tetrad function $P(x)$,

$$m^a = \frac{1}{2}\Big(\frac{1}{P(x)}\partial_x + iP(x)\,\partial_\varphi\Big), \quad m_a = P(x)\,dx + i\frac{1}{P(x)}\,d\varphi\,, \tag{2.95}$$

it is found in ref.[14] that, *rotating* electrovacuum solutions to Eq(2.91) and Eq(2.92) are

$$P^2(x) = \frac{4\pi(1+\alpha^2)}{A}\,\frac{A^2 + \frac{1-\alpha^2}{1+\alpha^2}(8\pi x)^2}{A^2 - (8\pi x)^2}\,, \tag{2.96}$$

$$U = \pm\arctan\Big(\sqrt{\frac{1-\alpha^2}{1+\alpha^2}}\frac{8\pi x}{A}\Big), \quad B = \Big(1 + \frac{1-\alpha^2}{1+\alpha^2}\frac{(8\pi x)^2}{A^2}\Big)^{\frac{1}{2}}, \tag{2.97}$$

$$\phi_1 = e^{i\theta_0}\,\frac{2\alpha\,\sqrt{\pi}A^{\frac{3}{2}}}{1+\alpha^2}\,\frac{\Big(A^2 - \frac{1-\alpha^2}{1+\alpha^2}(8\pi x)^2\Big) \pm 2i\sqrt{\frac{1-\alpha^2}{1+\alpha^2}}\,(8\pi x A)}{\Big(A^2 + \frac{1-\alpha^2}{1+\alpha^2}(8\pi x)^2\Big)^2}\,, \tag{2.98}$$

$$\alpha \in [0, 1)\,, \quad A \in (0, \infty)\,, \quad \theta_0 \in [0, 2\pi)\,, \tag{2.99}$$

while *nonrotating* solutions are given by

$$P^2(x) = \frac{8\pi A}{A^2 - (8\pi x)^2}\,, \quad \phi_1 = e^{i\theta_0}\Big(\frac{\pi}{A}\Big)^{\frac{1}{2}}. \tag{2.100}$$

Henceforth we will call Eqs(2.97-2.99) and Eq(2.100) the *local uniqueness solutions*.

2.4 Summary

In this chapter, we rederived the boundary conditions of NEHs/WIHs/IHs, studied the adapted tetrad and extensively analyzed the restriction equations for a foliation leaf. With these groundworks, we were finally able to introduce the local uniqueness solutions obtained in ref.[14]. Although the three concepts NEHs, WIHs and IHs are introduced in apparent hierarchy for a clear picture, there is no need to distinguish them in the remaining part of the thesis and we will simply refer to them as IHs henceforth.

It is argued in ref.[14] that, these solutions represent the uniqueness of intrinsic structures of extremal Kerr-Newman IHs. Is this argument correct? Before answering this question, we will first investigate the implications of these solutions.

Chapter 3

On-Horizon Data from Local Uniqueness Solutions

In this chapter, we will show how to properly read the local uniqueness solutions through construction of on-horizon data. The formulae employed in the subsequent calculations are provided in either Chapter 2 or ref.[14]. For reference convenience, we write the solutions again,

$$P^2(x) = \frac{4\pi(1+\alpha^2)}{A} \frac{A^2 + \frac{1-\alpha^2}{1+\alpha^2}(8\pi x)^2}{A^2 - (8\pi x)^2}, \tag{3.1}$$

$$U = \pm \arctan\Big(\sqrt{\frac{1-\alpha^2}{1+\alpha^2}} \frac{8\pi x}{A}\Big), \quad B = \Big(1 + \frac{1-\alpha^2}{1+\alpha^2}\frac{(8\pi x)^2}{A^2}\Big)^{\frac{1}{2}}, \tag{3.2}$$

$$\phi_1 = e^{i\theta_0} \frac{2\alpha\sqrt{\pi}A^{\frac{3}{2}}}{1+\alpha^2} \frac{\Big(A^2 - \frac{1-\alpha^2}{1+\alpha^2}(8\pi x)^2\Big) \pm 2i\sqrt{\frac{1-\alpha^2}{1+\alpha^2}}(8\pi x A)}{\Big(A^2 + \frac{1-\alpha^2}{1+\alpha^2}(8\pi x)^2\Big)^2}, \tag{3.3}$$

$$\alpha \in [0,1], \quad A \in (0,\infty), \quad \theta_0 \in [0, 2\pi), \tag{3.4}$$

where Eqs(2.97-2.99) and Eq(2.100) are unified by setting $\alpha \in [0, 1]$.

3.1 Implications of Local Uniqueness Solutions

Intrinsic Metric

The complex tetrad $\{m^a, \bar{m}^a\}$ spanning the leaf $\hat{\Delta}(x,\varphi) = S^2(x,\varphi)$ is set as[14]

$$m^a = \frac{1}{2}\left(\frac{1}{P}\partial_x + iP\partial_\varphi\right) = \frac{1}{2}\left(0, 0, \frac{1}{P}, iP\right), \quad \bar{m}^a = \frac{1}{2}\left(\frac{1}{P}\partial_x - iP\partial_\varphi\right) = \frac{1}{2}\left(0, 0, \frac{1}{P}, -iP\right), \tag{3.5}$$

and the dual bases satisfying the normalization $m^a \bar{m}_a = \bar{m}^a m_a = 1$ are given by

$$m_a = P\,dx + i\frac{1}{P}d\varphi = \left(0, 0, P, i\frac{1}{P}\right), \quad \bar{m}_a = P\,dx - i\frac{1}{P}d\varphi = \left(0, 0, P, -i\frac{1}{P}\right). \tag{3.6}$$

Thus, the intrinsic metric and its inverse are respectively

$$h_{ab}\,dx^a \otimes dx^b \triangleq m_a \bar{m}_b + \bar{m}_a m_b \triangleq 2\left(P^2 dx^2 + \frac{1}{P^2}d\varphi^2\right), \tag{3.7}$$

$$h^{ab}\,\partial x^a \otimes \partial x^b \triangleq m^a \bar{m}^b + \bar{m}^a m^b \triangleq \frac{1}{2}\left(\frac{1}{P^2}\partial_x^2 + P^2\partial_\varphi^2\right). \tag{3.8}$$

According to the local uniqueness solutions, we have

$$h_{ab} = \frac{8\pi\left[A^2(1+\alpha^2) + 64\pi^2 x^2(1-\alpha^2)\right]}{A(A^2 - 64\pi^2 x^2)}dx^2 + \frac{A(A^2 - 64\pi^2 x^2)}{2\pi\left[A^2(1+\alpha^2) + 64\pi^2 x^2(1-\alpha^2)\right]}d\varphi^2, \tag{3.9}$$

$$h^{ab} = \frac{A(A^2 - 64\pi^2 x^2)}{8\pi\left[A^2(1+\alpha^2) + 64\pi^2 x^2(1-\alpha^2)\right]}\partial_x^2 + \frac{2\pi\left[A^2(1+\alpha^2) + 64\pi^2 x^2(1-\alpha^2)\right]}{A(A^2 - 64\pi^2 x^2)}\partial_\varphi^2. \tag{3.10}$$

Intrinsic Connection Coefficients

Given the metric h_{ab}, it follows immediately with the intrinsic Levi-Civita connections

$$\begin{aligned}\Gamma^{x}_{xx} &= \frac{128\pi^2 A^2 x}{(A^2-64\pi^2x^2)\left[A^2(1+\alpha^2)+64\pi^2x^2(1-\alpha^2)\right]},\\ \Gamma^{x}_{\phi\phi} &= \frac{8A^4x(A^2-64\pi^2x^2)}{\left[A^2(1+\alpha^2)+64\pi^2x^2(1-\alpha^2)\right]^3},\\ \Gamma^{\phi}_{x\phi} = \Gamma^{\phi}_{\phi x} &= -\frac{128\pi^2A^2x}{(A^2-64\pi^2x^2)\left[A^2(1+\alpha^2)+64\pi^2x^2(1-\alpha^2)\right]}.\end{aligned} \tag{3.11}$$

Directional Derivative & Commutator Coefficient

The directional derivatives $\{\delta, \bar{\delta}\}$ and the commutator coefficients ζ become[14]

$$\delta = \bar{\delta} = m^x\partial_x = \frac{1}{2P}\partial_x, \quad \zeta = \alpha - \bar{\beta} \hat{=} \frac{1}{2}\bar{m}^a\bar{\delta}m_a - \frac{1}{2}\overline{\bar{m}^a\delta m_a} \hat{=} \frac{P_{,x}}{2P^2} = \bar{\zeta}. \tag{3.12}$$

Thus

$$\delta = \bar{\delta} = \frac{1}{4}\left(\frac{A}{\pi(1+\alpha^2)}\frac{A^2-(8\pi x)^2}{A^2+\frac{1-\alpha^2}{1+\alpha^2}(8\pi x)^2}\right)^{1/2}\partial_x, \tag{3.13}$$

$$\zeta = \bar{\zeta} = \frac{32\,x\,\pi^{3/2}A^{5/2}}{\left(A^2-64\pi^2x^2\right)^{1/2}\left[A^2(1+\alpha^2)+64\pi^2x^2(1-\alpha^2)\right]^{3/2}}. \tag{3.14}$$

Gaussian Curvature

Gaussian curvature of the horizon is given by

$$K = -\frac{1}{h_{xx}}\left(\partial_x\hat{\Gamma}^{\varphi}_{x\varphi} - \partial_\varphi\hat{\Gamma}^{\varphi}_{xx} + \hat{\Gamma}^{x}_{x\varphi}\hat{\Gamma}^{\varphi}_{xx} - \hat{\Gamma}^{x}_{xx}\hat{\Gamma}^{\varphi}_{x\varphi} + \hat{\Gamma}^{\varphi}_{x\varphi}\hat{\Gamma}^{\varphi}_{\varphi x} - \hat{\Gamma}^{\varphi}_{xx}\hat{\Gamma}^{\varphi}_{\varphi\varphi}\right), \tag{3.15}$$

or by Eq(2.56)

$$K := \delta\zeta + \bar{\delta}\bar{\zeta} - 2\zeta\bar{\zeta} \hat{=} 2\delta\zeta - 2\zeta^2, \tag{3.16}$$

which yield the same result that

$$K = \frac{16A^3\pi\left[A^2(1+\alpha^2) - 192\,\pi^2x^2(1-\alpha^2)\right]}{\left[A^2(1+\alpha^2) + 64\pi^2x^2(1-\alpha^2)\right]^3}. \tag{3.17}$$

Spin Coefficient $\pi^{(0)}$

The spin coefficient $\pi^{(0)}$ which reflects rotational character of the horizon, and can be reexpressed by the rotational scalar potential U via Eq(2.82)

$$\pi^{(0)} \hat{=} -i\bar{\delta}U + \bar{\delta}\,\ln B. \tag{3.18}$$

Hence,

$$\pi^{(0)} \hat{=} \frac{16\pi\sqrt{\pi}\,x\left(1-\alpha^2\right) \mp 2i\,A\sqrt{\pi}\,\sqrt{(1-\alpha^2)(1+\alpha^2)}}{\sqrt{\dfrac{A^2\left(1+\alpha^2\right)+64\,\pi^2x^2\left(1-\alpha^2\right)}{A\,(A^2-64\,\pi^2x^2)}}\left[A^2(1+\alpha^2)+64\pi^2x^2(1-\alpha^2)\right]}. \tag{3.19}$$

Rotation 1-Form Potential

The rotation 1-form potential in Eqs(2.48)(2.51)

$$\begin{aligned} \hat{\omega}_a &\hat{=} \pi m_a + \bar{\pi}\bar{m}_a \\ \hat{\omega}^a \hat{=} h^{ab}\hat{\omega}_b &\hat{=} \pi m^a + \bar{\pi}\bar{m}^a, \quad \hat{\omega}_a\hat{\omega}^a = 2\pi\bar{\pi} \end{aligned} \tag{3.20}$$

can be rewritten using $\{U, B\}$ into exact and coexact components as in Eq(2.80),

$$\hat{\omega}_a \hat{=} \hat{\star}dU + d\,\ln B, \tag{3.21}$$

where $\hat{\star}$ is the Hodge dual operator with respect to h_{ab}[14]. Thus,

$$\tilde{\omega}_a = (\pm)\frac{2A^2\left(A^2 - 64\pi^2x^2\right)\sqrt{(1-\alpha^2)(1+\alpha^2)}}{\left[A^2(1+\alpha^2) + 64\pi^2x^2(1-\alpha^2)\right]^2}d\varphi + \frac{64\pi^2x\left(1-\alpha^2\right)}{A^2(1+\alpha^2) + 64\pi^2x^2(1-\alpha^2)}dx\,. \tag{3.22}$$

Electrovacuum Field (Ricci-NP Scalar Φ_{11})

The electromagnetic field on the horizon can be described by the Ricci-NP scalar Φ_{11}, which is related with ϕ_1 via $\Phi_{11} = 2\phi_1\bar{\phi}_1$, thus

$$\begin{aligned}\Phi_{11} = & \frac{8\pi\alpha^2A^3}{\left(A^2(1+\alpha^2) + 64\pi^2x^2(1-\alpha^2)\right)^4} \\ & \left\{A^2\left(1+\alpha^2\right) - 64\pi^2x^2\left(1-\alpha^2\right) - 16i\pi xA\sqrt{(1-\alpha^2)(1+\alpha^2)}\right\} \\ & \left\{A^2\left(1+\alpha^2\right) - 64\pi^2x^2\left(1-\alpha^2\right) + 16i\pi xA\sqrt{(1-\alpha^2)(1+\alpha^2)}\right\}.\end{aligned} \tag{3.23}$$

Interestingly, the flexible parameter $\theta_0 \in [0, 2\pi)$ from ϕ_1 is removed in Φ_{11}.

With the elementary values above, more quantities can be computed via relevant equations in Chapter 2, such as Ψ_2 via Eq(2.93). Since the generic forms of those quantities are very complicated, they won't be listed here. Almost all quantities required in subsequent chapters have been calculated in this section.

3.2 Meanings of Flexible Parameters: α, A and θ_0

There are three flexible parameters in the local uniqueness solutions: $\alpha \in [0, 1]$, $A \in (0, \infty)$ and $\theta_0 \in [0, 2\pi)$; every set of exact values $\{\breve{\alpha}, \breve{A}, \breve{\theta}_0\}$ specifies an exact intrinsic horizon $\breve{h}_{ab}$ and generates a particular group of on-horizon data $\{\breve{K}, \breve{\phi}_1, \breve{\Phi}_{11}, \text{Re}(\breve{\Psi}_2), \breve{\pi}^{(0)}, \breve{\omega}_a, \text{etc}\}$.

Parameter α

The two local uniqueness solutions Eqs(2.97-2.99) and Eq(2.100) were derived individually in ref.[14], but it is easy to see that the nonrotating solution Eq(2.100) is just the rotating solution Eqs(2.97-2.99) with α fixed by $\alpha = 1$. Thus, α signifies the contribution of angular effects (actually electric effects) in balancing the extremal horizon; $\alpha = 1$ represents absence of rotation so that extremality attributes entirely to electric repulsion, while $\alpha = 0$ means rotation alone producing extremality with no help from electricity. For the interval $0 < \alpha < 1$, extremality results from joint function of rotation and electricity. Obviously, $\alpha = 1$ corresponds to extremal Reissner-Nordström IHs, $\alpha = 0$ to Kerr, while $0 < \alpha < 1$ to generic Kerr-Newman.

We can also express the function of α as

$$\alpha^2\,[\text{rotation}] + \alpha^2\,[\text{electricity}] = 1, \tag{3.24}$$

where we utilize α^2 rather than α because it is always the former that works in the local uniqueness solutions and the implied on-horizon data. This relation is in clear analogy with the restriction equation $M^2 = Q^2 + a^2$ (c.f. [38] and references therein for extension of $M^2 \geq Q^2 + a^2$ under distortion) for extremal Kerr-Newman-family black holes,

$$\frac{a^2}{M^2} + \frac{Q^2}{M^2} = 1\,, \;\Rightarrow\; \tilde{a}^2 + \tilde{Q}^2 = 1 \quad (\text{where}\,\tilde{a} := \frac{a}{M}\,,\; \tilde{Q} := \frac{Q}{M})\,. \tag{3.25}$$

comparison of which with the practical effects of α leads to

$$\alpha^2 = \left(\frac{Q}{M}\right)^2 = 1 - \left(\frac{a}{M}\right)^2. \tag{3.26}$$

Parameter A

According to ref.[14], the second parameter A refers to the horizon area. However, does the horizon area A, which sounds global, act like a good local parameter? Moreover, given an arbitrary value A (and an α), there will be a particular intrinsic metric h_{ab}; in theory, h_{ab} should *officially* yield another horizon area $\hat{A}(A, \alpha)$. Which one stands for the true horizon area? After concrete calculation using Eq(3.9), it turns out that

$$\hat{A}(A, \alpha) = \int_{-\frac{A}{8\pi}}^{\frac{A}{8\pi}} \sqrt{\det(\hat{h}_{ab})}\, dx \int_0^{2\pi} d\varphi = A\,, \tag{3.27}$$

where $x \in [-\frac{A}{8\pi}, \frac{A}{8\pi}]$, $\varphi \in [0, 2\pi)$. Thus, the coordinates $\{x, \varphi\}$ and the tetrad $m_a dx^a = P\,dx + i\frac{1}{P}d\varphi$ are set up such that the parameter A is more than some arbitrary real value; in fact, *A as an input would fix the true area of the horizon.*

The argument $A \in (0, \infty)$ indicates that, A is the *absolute* horizon area, and therefore the variable x is absolute as well by $x \in [-\frac{A}{8\pi}, \frac{A}{8\pi}]$. However, the other input parameter $\alpha \in [0, 1]$ takes *relative* values as $1 - \alpha^2 = \left(\frac{a}{M}\right)^2$. *Without loss of generality, we will relativize the horizon area A by setting $M = 1$, as suggested by Eq(3.25).* The convention $M = 1$ will bring us great convenience in subsequent calculations. Yet, what is the relativized domain for A?

The horizon area A is an *invariant* in the sense that it is independent of the coordinate systems chosen on the horizon. Hence, instead of relying solely on the arbitrary input A, we can determine the surface areas of extremal Kerr-Newman horizons using the induced metrics in the usual $\{\theta, \phi\}$ coordinates. As will be shown in Chapter 5, with the convention $M = 1$, we have $A = 4\pi$ for extremal Reissner-Nordström horizons, $A = 8\pi$ for Kerr and $A = 4\pi(2 - \alpha^2)$ for generic Kerr-Newman. Thus, the argument for the relativized horizon area A (and the unified α) are

$$4\pi \leq A \leq 8\pi\,, \quad \text{and} \quad 0 \leq \alpha \leq 1\,. \tag{3.28}$$

Henceforth in this thesis, the local uniqueness solution will be *updated* with these new arguments, and for $\{P^2,\ U,\ B,\ \phi_1\}$ in the solutions Eq(3.1), A and x should take relativized values.

Parameter θ_0

The third one θ_0 denotes the relative weight of electric and magnetic charges in the electromagnetic contribution to extremality[14]. θ_0 is introduced simply to enrich (or complicate) the local uniqueness solution and doesn't interfere our calculations. θ_0 only takes effect in ϕ_1, and is covered up in the practically effective quantity Φ_{11}; or mathematically equivalently, we can just set $\theta_0 = 0$ which means absence of magnetic charge.

3.3 Summary

In this chapter, the local uniqueness solutions were fully investigated. We worked out implications of the four functions $\{P^2,\ U,\ B,\ \phi_1\}$ by construction of on-horizon data $\{K, \Phi_{11},$ $\mathrm{Re}(\Psi_2), \pi^{(0)}, \omega_a,$ etc$\}$. We also figured out how to understand the three parameters $\{\alpha,\ A,\ \theta_0\}$ and relativized the solutions. With these preparations, we will compare the local uniqueness solutions with the Kerr-Newman horizons via their near-horizon structures in next two chapters.

Chapter 4

Near-Horizon Geometry of Extremal IHs Embedded in Electrovacuum

In this chapter, the near-horizon geometry (NHG) of isolated horizons which are embedded in *electrovacuum* will be investigated. By electrovacuum we mean that, it is electrovacuum at least in the near-horizon limit, while regions more distant from the horizon are unrestricted by this assumption and can be electrovacuum or not. We emphasize electrovacuum for the following two reasons:

(I) A generic isolated horizon is a system in equilibrium with the external environment, and there can be energy flows arbitrarily close to the horizon. It is impossible to reconstruct the near-horizon metric of such a generic IH using structural data on that horizon, because this task requires such information (like $\Psi_3^{(0)}$, $\Psi_4^{(0)}$, Λ) which cannot be obtained from the local uniqueness solution. Thus, based on the local uniqueness solution we are only able to investigate IHs embedded in electrovacuum where all required information to determine NHG can be retrieved from on-horizon data (e.g. $\mathrm{Re}(\Psi_2^{(0)})$, $\Phi_{11}^{(0)}$, K, $\hat{\omega}_a$).

(II) The usual method to derive the extremal Kerr-Newman NHG is employing the global metrics, which requires the entire spacetime to be stationary electrovacuum which is both axisymmetric and asymptotically flat. In the alternative to such derivation from a local approach, it would be natural and reasonable to inherit the assumption of electrovacuum (and axisymmetry) at least in the near-horizon limit.

4.1 Newman-Unti-type Tetrad and Tetrad Equations

To proceed, we need to extend the adapted tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$ *on* an IH (c.f. Section(2.1.3)) and set up the Gaussian null coordinates $\{v, r, y, z\}$ to cover all electrovacuum exteriors, including the near-horizon regions and possibly off-horizon regions (Note: by *near-horizon* we always mean in the near-horizon limit, while *off-horizon* refers to regions beyond the near-horizon limit). Description of this kind of tetrad can be traced back to ref.[36], the Bondi-type coordinates in which is essentially the same as the Gaussian null coordinates in use in this chapter. Moreover, in refs.[39]-[42], the authors employed similar (but not identical) tetrads to generalize various near-horizon physical effects from quasilocal perspectives.

Here we want to emphasize two references. Firstly, ref.[42] in which the authors investigated the extremal Kerr IHs and CFT correspondence, has been most helpful for us to overcome many mathematical difficulties in this chapter. Secondly, after the majority of Chapter 2 and this chapter were finished, ref.[43] which has a lot in common with our Chapters 2 and has an identical tetrad with our chapter 4 came into being. Although our work was done separately, ref.[43] did help us confirm and partly improve our results. The near-horizon structure of IHs is also studied very recently in more abstract mathematical language in ref.[44].

Tetrad and Coordinates Setup

Choose the first real null covector n_a as the gradient of foliation leaves (c.f. [33] and Section IV of [18] for analogous setup of the Newman-Unti tetrad),

$$n_a = -dv\,, \tag{4.1}$$

where v is the *ingoing* (retarded) Eddington-Finkelstein-type null coordinate, which labels the foliation cross-sections and acts as an affine parameter with regard to the outgoing null vector field $l^a\partial_a$,

$$Dv = 1\,, \quad \Delta v = \delta v = \bar{\delta} v = 0\,. \tag{4.2}$$

Introduce the second coordinate r as an affine parameter along the ingoing null vector field n^a, which obeys the normalization

$$n^a\partial_a r = -1 \Leftrightarrow n^a\partial_a = -\partial_r\,. \tag{4.3}$$

Unlike Schwarzschild-type coordinates, here $r = 0$ represents the horizon, while $r > 0$ ($r < 0$) corresponds to the exterior (interior) of an IH. Hereafter, we will often Taylor expand a scalar Q with respect to the horizon $r = 0$,

$$Q = \sum_{i=0} Q^{(i)} r^i = Q^{(0)} + Q^{(1)} r + \cdots + Q^{(n)} r^n + \ldots \tag{4.4}$$

where $Q^{(0)}$ refers to its on-horizon value. To avoid ambiguity, on-horizon quantities will be emphasized by a superscript (0) like $Q^{(0)}$ henceforth.

Now, the first real null tetrad vector n^a is fixed. To determine the remaining tetrad vectors $\{l^a, m^a, \bar{m}^a\}$ and their covectors, besides the basic cross-normalization conditions, it is also required that: (i) the outgoing null normal field l^a acts as the null generators; (ii) the null frame (covectors) $\{l_a, n_a, m_a, \bar{m}_a\}$ are parallelly propagated along $n^a\partial_a$; (iii) $\{m^a, \bar{m}^a\}$ spans the $\{t = \text{constant}, r = \text{constant}\}$ cross-sections which are labeled by *real* isothermal coordinates $\{y, z\}$.

Tetrads satisfying the above restrictions can be expressed in the general form that

$$\begin{aligned}
l^a\partial_a &= \partial_v + U\partial_r + X^3\partial_y + X^4\partial_z := D\,,\\
n^a\partial_a &= -\partial_r := \Delta\,,\\
m^a\partial_a &= \Omega\partial_r + \xi^3\partial_y + \xi^4\partial_z := \delta\,,\\
\bar{m}^a\partial_a &= \bar{\Omega}\partial_r + \bar{\xi}^3\partial_y + \bar{\xi}^4\partial_z := \bar{\delta}\,;
\end{aligned} \tag{4.5}$$

where it would be sufficient to define the four NP derivatives $\{D, \Delta, \delta, \bar{\delta}\}$ to be partial operators as they will only act on scalar fields (tetrad component functions, spin coefficients, Weyl-NP scalars, Ricci-NP scalars, etc) in the following contexts.

Eq(4.5) leads to the inverse metric via $g^{ab} = -l^a n^b - n^a l^b + m^a \bar{m}^b + \bar{m}^a m^b$,

$$g^{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2(U + \Omega\bar{\Omega}) & X^3 + \Omega\bar{\xi}^3 + \bar{\Omega}\xi^3 & X^4 + \Omega\bar{\xi}^4 + \bar{\Omega}\xi^4 \\ 0 & X^3 + \Omega\bar{\xi}^3 + \bar{\Omega}\xi^3 & 2\xi^3\bar{\xi}^3 & \xi^3\bar{\xi}^4 + \xi^4\bar{\xi}^3 \\ 0 & X^4 + \Omega\bar{\xi}^4 + \bar{\Omega}\xi^4 & \xi^3\bar{\xi}^4 + \xi^4\bar{\xi}^3 & 2\bar{\xi}^4\xi^4 \end{pmatrix}. \tag{4.6}$$

When one goes smoothly to the IH ($r = 0$), g^{ab} naturally reduces to the degenerate inverse metric $\hat{h}^{ab}$. This implies that the tetrad functions $\{U\,, X^3\,, X^4\,, \Omega\}$ are all $O(r)$ functions in the near-horizon limit and

$$U \hat{=} 0\,, \quad X^3 \hat{=} 0\,, \quad X^4 \hat{=} 0\,, \quad \Omega \hat{=} 0\,. \tag{4.7}$$

Hence, on the horizon, the four intrinsic derivatives become

$$\hat{D} \hat{=} \partial_v\,, \quad \hat{\Delta} \hat{=} -\partial_r\,, \quad \hat{\delta} \hat{=} \xi^{3(0)}\partial_y + \xi^{4(0)}\partial_z\,, \quad \hat{\bar{\delta}} \hat{=} \bar{\xi}^{3(0)}\partial_y + \bar{\xi}^{4(0)}\partial_z\,. \tag{4.8}$$

Direct Consequences of Tetrad Setup

Now let's check direct consequences of the tetrad setup. Because of parallel transportation along $n^a\partial_a$, the covector fields $\{l_a, n_a, m_a, \bar{m}_a\}$ respect the transportation equations that

$$\Delta l_a = (\gamma + \bar{\gamma})l_a - \bar{\tau}m_a - \tau\bar{m}_a \;=\; 0\,, \tag{4.9}$$

$$\Delta n_a = \nu m_a + \bar{\nu}\bar{m}_a - (\gamma + \bar{\gamma})n_a \;=\; 0\,, \tag{4.10}$$

$$\Delta m_a = (\gamma - \bar{\gamma})m_a + \bar{\nu}l_a - \tau n_a \;=\; 0\,, \tag{4.11}$$

$$\Delta m_a = (\gamma - \bar{\gamma})m_a + \nu l_a - \bar{\tau}n_a \;=\; 0\,, \tag{4.12}$$

which immediately yield

$$\nu = \tau = \gamma = 0\,. \tag{4.13}$$

Moreover, apply the commutators which are reduced by Eq(4.13) to the ingoing null coordinate v, and it follows from Eq(4.2) that

$$(\Delta D - D\Delta)\,v = (\varepsilon + \bar{\varepsilon})(\Delta v) - \pi(\delta v) - \bar{\pi}(\bar{\delta}v) \;=\; 0\,, \tag{4.14}$$

$$(\delta\Delta - \Delta\delta)\,v = -(\bar{\alpha} + \beta)(\Delta v) + \mu(\delta v) + \bar{\lambda}(\bar{\delta}v) \;=\; 0\,, \tag{4.15}$$

$$(\delta D - D\delta)\,v = (\bar{\alpha} + \beta - \bar{\pi})(Dv) + \kappa(\Delta v) - (\bar{\rho} + \varepsilon - \bar{\varepsilon})(\delta v) - \sigma(\bar{\delta}v) \;=\; 0\,, \tag{4.16}$$

$$(\bar{\delta}\delta - \delta\bar{\delta})\,\nu = (\bar{\mu} - \mu)(D\nu) + (\bar{\rho} - \rho)(\Delta\nu) + (\alpha - \bar{\beta})(\delta\nu) - (\bar{\alpha} - \beta)(\bar{\delta}\nu) = 0 . \tag{4.17}$$

and the last two relations imply

$$\pi = \alpha + \bar{\beta}, \quad \mu = \bar{\mu} . \tag{4.18}$$

Eq(4.13) and Eq(4.18) comprise the basic gauge conditions for the Newman-Unti-type tetrad Eq(4.5),

$$\nu = \tau = \gamma = 0, \quad \pi = \alpha + \bar{\beta}, \quad \mu = \bar{\mu}, \tag{4.19}$$

which holds for both on-horizon and near-horizon structures. Thus, the commutators for the tetrad Eq(4.5) with the gauge conditions Eq(4.19) are finally

$$\Delta D - D\Delta = (\varepsilon + \bar{\varepsilon})\Delta - \pi\delta - \bar{\pi}\bar{\delta}, \tag{4.20}$$

$$\delta D - D\delta = \kappa\Delta - (\rho + \varepsilon - \bar{\varepsilon})\delta - \sigma\bar{\delta}, \tag{4.21}$$

$$\delta\Delta - \Delta\delta = -\bar{\pi}\Delta + \mu\delta + \bar{\lambda}\bar{\delta}, \tag{4.22}$$

$$\bar{\delta}\delta - \delta\bar{\delta} = (\alpha - \bar{\beta})\delta - (\bar{\alpha} - \beta)\bar{\delta} . \tag{4.23}$$

Tetrad Functions from Commutators

The gauge conditions derived above would in turn set restrictions to the six tetrad functions in Eq(4.5). Compared with the parallel transportation of null frame in Eqs(4.10-4.12), it is more convenient to utilize the reduced commutators, where it is the tetrad vectors rather than null basis covectors that are involved and from which it follows that

$$\Delta l^a - Dn^a = (\varepsilon + \bar{\varepsilon})n^a - \pi m^a - \bar{\pi}\bar{m}^a , \tag{4.24}$$

$$\delta l^a - Dm^a = \kappa n^a - (\rho + \varepsilon - \bar{\varepsilon})m^a - \sigma\bar{m}^a . \tag{4.25}$$

Now expand $\Delta l^a - Dn^a$ in Eq(4.24) using Eq(4.5), and one obtains

$$\partial_r U = (\varepsilon + \bar{\varepsilon}) + \pi\Omega + \bar{\pi}\bar{\Omega}, \tag{4.26}$$

$$\partial_r X^3 = \pi\xi^3 + \bar{\pi}\bar{\xi}^3, \tag{4.27}$$

$$\partial_r X^4 = \pi\xi^4 + \bar{\pi}\bar{\xi}^4, \tag{4.28}$$

Expand $\delta n^a - \Delta m^a$ in Eq(4.25), thus

$$\partial_r\Omega = \bar{\pi} + \mu\Omega + \bar{\lambda}\bar{\Omega}, \tag{4.29}$$

$$\partial_r\xi^3 = \mu\xi^3 + \bar{\lambda}\bar{\xi}^3, \tag{4.30}$$

$$\partial_r\xi^4 = \mu\xi^4 + \bar{\lambda}\bar{\xi}^4. \tag{4.31}$$

With these r-derivatives, we hope to fix the six tetrad functions. It is notable that, a tetrad function Q can be dependent on all four coordinates $\{v, r, y, z\}$, but only the r-dependence is separated here while dependence on other coordinates are still encoded in $\partial_r Q$.

Remarks: The remaining two commutators

$$\delta n^a - \Delta m^a = -\bar{\pi}n^a + \mu m^a + \bar{\lambda}\bar{m}^a \tag{4.32}$$

$$\bar{\delta}m^a - \delta\bar{m}^a = (\alpha - \bar{\beta})m^a - (\bar{\alpha} - \beta)\bar{m}^a \tag{4.33}$$

contain another two set of supplementary equations, which, however, are unpractical in solving the tetrad functions. Expand $\delta l^a - Dm^a$,

$$\Omega\partial_r U - (\partial_u\Omega + U\partial_r\Omega) = -\kappa - (\rho + \varepsilon - \bar{\varepsilon})\Omega - \sigma\bar{\Omega}, \tag{4.34}$$

$$\Omega\,\partial_r X^3 - (\partial_u\xi^3 + U\partial_r\xi^3 + X^3\partial_y\xi^3 + X^4\partial_z\xi^3) = -(\rho + \varepsilon - \bar{\varepsilon})\xi^3 - \sigma\bar{\xi}^3, \tag{4.35}$$

$$\Omega\,\partial_r X^4 - (\partial_u \xi^4 + U\partial_r \xi^4 + X^3\partial_y \xi^4 + X^4\partial_z \xi^4) = -(\rho + \varepsilon - \bar{\varepsilon})\xi^4 - \sigma\bar{\xi}^4\,, \tag{4.36}$$

which yields the boundary conditions $\partial_u \Omega \hat{=} 0$, $\partial_u \xi^3 \hat{=} 0$, $\partial_u \xi^3 \hat{=} 0$ on the horizon. Expand $\bar{\delta} m^a - \delta \bar{m}^a$,

$$\bar{\Omega}\,\partial_r \Omega - \Omega\,\partial_r \bar{\Omega} = (\alpha - \bar{\beta})\Omega - (\bar{\alpha} - \beta)\bar{\Omega}\,, \tag{4.37}$$

$$\bar{\Omega}\,\partial_r \xi^3 + \hat{\bar{\delta}}\xi^3 - (\Omega\,\partial_r \bar{\xi}^3 + \hat{\delta}\bar{\xi}^3) = (\alpha - \bar{\beta})\,\xi^3 - (\bar{\alpha} - \beta)\,\bar{\xi}^3\,, \tag{4.38}$$

$$\bar{\Omega}\,\partial_r \xi^4 + \hat{\bar{\delta}}\xi^4 - (\Omega\,\partial_r \bar{\xi}^4 + \hat{\delta}\bar{\xi}^4) = (\alpha - \bar{\beta})\,\xi^4 - (\bar{\alpha} - \beta)\,\bar{\xi}^4\,. \tag{4.39}$$

which become the intrinsic relation $\hat{\bar{\delta}}\xi^i - \hat{\bar{\delta}}\xi^i \hat{=} (\alpha^{(0)} - \bar{\beta}^{(0)})\,\xi^{i(0)} - (\bar{\alpha}^{(0)} - \beta^{(0)})\,\bar{\xi}^{i(0)}$ on the horizon.

4.2 Exact Solutions of Tetrad Functions

In fact, the tetrad Eq(4.5) and the r-derivative equations of tetrad functions in Eqs(4.26-4.31) apply to both near-horizon and off-horizon regions and can be pulled back to the IH through smooth transition. Our goal is to solve the tetrad functions in the near-horizon limit by recasting them using on-horizon data so as to rebuild the NHG.

In the near-horizon limit ($r \to 0$), Eqs(4.26-4.31) act as

$$\partial_r U \;= (\varepsilon + \bar{\varepsilon}) + \pi\Omega + \bar{\pi}\bar{\Omega} \;\simeq\; 2\varepsilon^{(0)} + O(r)\;, \tag{4.40}$$

$$\partial_r \Omega \;= \bar{\pi} + \mu\Omega + \bar{\lambda}\bar{\Omega} \;\simeq\; \bar{\pi}^{(0)} + O(r)\,, \tag{4.41}$$

$$\partial_r X^3 = \pi\xi^3 + \bar{\pi}\bar{\xi}^3 \;\simeq\; \pi^{(0)}\xi^{3(0)} + \bar{\pi}^{(0)}\bar{\xi}^{3(0)} + O(r)\,, \tag{4.42}$$

$$\partial_r X^4 = \pi\xi^4 + \bar{\pi}\bar{\xi}^4 \;\simeq\; \pi^{(0)}\xi^{4(0)} + \bar{\pi}^{(0)}\bar{\xi}^{4(0)} + O(r)\,, \tag{4.43}$$

$$\partial_r \xi^3 \;= \mu\xi^3 + \bar{\lambda}\bar{\xi}^3 \;\simeq\; \mu^{(0)}\xi^{3(0)} + \bar{\lambda}^{(0)}\bar{\xi}^{3(0)} + O(r)\,, \tag{4.44}$$

$$\partial_r \xi^4 \;= \mu\xi^4 + \bar{\lambda}\bar{\xi}^4 \;\simeq\; \mu^{(0)}\xi^{4(0)} + \bar{\lambda}^{(0)}\bar{\xi}^{4(0)} + O(r)\,, \tag{4.45}$$

where the symbol $\simeq$ is employed to denote equality in the near-horizon limit (recall that $\hat{=}$ denotes equality on an IH). Since $\pi^{(0)}$ (or more extensively, π) reflects the rotational property of an IH, information regarding rotation of the horizon (and the spacetime dragging) is contained in the functions $\{U, \Omega, X^3, X^4\}$.

There are two ways to handle these equations: direct integration or further differentiation. In Eqs(4.40-4.2), only $\{\pi^{(0)}, \mu^{(0)}, \lambda^{(0)}\}$ can be determined by on-horizon data, so only Ω in Eq(4.42) can be obtained via straightforward quadrature,

$$\Omega \simeq \bar{\pi}^{(0)} r + O(r^2). \tag{4.46}$$

On the other hand, with the tetrad Eq(4.5) and the gauge conditions Eq(4.19), there will be

$$\partial_r \varepsilon = \bar{\pi}\alpha + \pi\beta + \Psi_2 + \Phi_{11} - \Lambda\,. \tag{4.47}$$

$$\partial_r \pi = \pi\mu + \bar{\pi}\lambda + \Psi_3 + \Phi_{21}\,, \tag{4.48}$$

$$\partial_r \mu = \mu^2 + \lambda\bar{\lambda} + \Phi_{22}\,, \tag{4.49}$$

$$\partial_r \lambda = 2\mu\lambda + \Psi_4\,, \tag{4.50}$$

so it is only practical to further differentiate $\partial_r U$, while $\{\partial_{rr} X^3,\, \partial_{rr} X^4,\, \partial_{rr}\xi^3,\, \partial_{rr}\xi^4\}$ requires input of $\{\Psi_3, \Psi_4, \Phi_{21}, \Phi_{22}\}$ which are not provided by local uniqueness solution and need to be solved separately. Another important reason that prevents us from solving r-derivatives of $\{X^3, X^4, \xi^3, \xi^4\}$ is that the analytical forms of $\{\xi^3, \xi^4\}$ rely on the choice of isothermal coordinates $\{y, z\}$ which are unchosen so far.

Now, let's calculate the second-order r-derivatives of U. It follows from Eq(4.47) and

Eq(4.40) that

$$\partial_r(\varepsilon+\bar{\varepsilon}) = 2\pi\bar{\pi} + 2\mathrm{Re}(\Psi_2) + 2\Phi_{11} - 2\Lambda\,, \tag{4.51}$$

thus

$$\begin{aligned}\partial_{rr}U &= 2\pi\bar{\pi} + 2\mathrm{Re}(\Psi_2) + 2\Phi_{11} - 2\Lambda + \pi\partial_r\Omega - \Omega\partial_r\pi + \bar{\pi}\partial_r\bar{\Omega} + \bar{\Omega}\partial_r\bar{\pi}\\ &\simeq 2\pi^{(0)}\bar{\pi}^{(0)} + 2\mathrm{Re}(\Psi_2^{(0)}) + 2\Phi_{11}^{(0)} + \pi^{(0)}\partial_r\Omega + \bar{\pi}^{(0)}\partial_r\bar{\Omega}\\ &\simeq 2\,(2\pi^{(0)}\bar{\pi}^{(0)} + \mathrm{Re}(\Psi_2^{(0)}) + \Phi_{11}^{(0)})\,.\end{aligned} \tag{4.52}$$

where $\{\Phi_{ii}, \Lambda\}$ are real by definition and $\Lambda \simeq 0$ for electrovacuum in the near-horizon limit. Integration of $\partial_{rr}U$ yields

$$U \simeq 2\,(2\pi^{(0)}\bar{\pi}^{(0)} + \mathrm{Re}(\Psi_2^{(0)}) + \Phi_{11}^{(0)})\,r^2 + 2\varepsilon^{(0)}\,r + O(r^3) \hat{=} 0 \quad (\text{nonextremal})\,, \tag{4.53}$$

$$U \simeq 2\,(2\pi^{(0)}\bar{\pi}^{(0)} + \mathrm{Re}(\Psi_2^{(0)}) + \Phi_{11}^{(0)})\,r^2 + O(r^3) \hat{=} 0 \quad (\text{extremal}, \varepsilon^{(0)} = 0)\,, \tag{4.54}$$

Eq(4.46) and Eq(4.54) constitute the exact near-horizon solutions for the tetrad functions Ω and U for extremal IHs.

Remarks To derive $\partial_{rr}U$, we can't just take second r-derivative of the near-horizon limit of $\partial_r U$; that is,

$$\partial_{rr}U \simeq \partial_r(\varepsilon+\bar{\varepsilon}) \simeq 2\,(\pi^{(0)}\bar{\pi}^{(0)} + \mathrm{Re}(\Psi_2^{(0)}) + \Phi_{11}^{(0)})\,, \tag{4.55}$$

$$\Rightarrow\ U \simeq 2\,(\pi^{(0)}\bar{\pi}^{(0)} + \mathrm{Re}(\Psi_2^{(0)}) + \Phi_{11}^{(0)})\,r^2 + 2\varepsilon^{(0)}\,r + O(r^3) \hat{=} 0 \quad (\text{nonextremal})\,, \tag{4.56}$$

$$\Rightarrow\ U \simeq 2\,(\pi^{(0)}\bar{\pi}^{(0)} + \mathrm{Re}(\Psi_2^{(0)}) + \Phi_{11}^{(0)})\,r^2 + O(r^3) \hat{=} 0 \quad (\text{extremal}, \varepsilon^{(0)} = 0)\,. \tag{4.57}$$

Compare Eq(4.57) with Eq(4.54) and it is easy to see that, although for *nonrotating* IHs ($\pi^0 = 0$) these two approximations are equal, Eq(4.57) cannot fully reflect the angular effects of generic (rotating) IHs.

So far, two tetrad functions $\{U, \Omega\}$ have been exactly solved. As with the remaining four functions $\{X^3, X^4, \xi^3, \xi^4\}$, we will just integrate Eqs(4.43-4.2) and obtain *formally* that

$$X^3 \simeq (\pi^{(0)}\xi^{3(0)} + \bar{\pi}^{(0)}\bar{\xi}^{3(0)})\, r + O(r^2) \hat{=} 0 \,, \tag{4.58}$$

$$X^4 \simeq (\pi^{(0)}\xi^{4(0)} + \bar{\pi}^{(0)}\bar{\xi}^{4(0)})\, r + O(r^2) \hat{=} 0 \,, \tag{4.59}$$

$$\xi^3 \simeq \xi^{3(0)} + (\mu^{(0)}\xi^{3(0)} + \bar{\lambda}^{(0)}\bar{\xi}^{3(0)})\, r + O(r^2) \hat{=} \xi^{3(0)} \,, \tag{4.60}$$

$$\xi^4 \simeq \xi^{4(0)} + (\mu^{(0)}\xi^{4(0)} + \bar{\lambda}^{(0)}\bar{\xi}^{4(0)})\, r + O(r^2) \hat{=} \xi^{4(0)} \,. \tag{4.61}$$

These are not exact solutions because $\xi^{3(0)}$ and $\xi^{4(0)}$ are undetermined yet. (Note for a quantity Q that $Q^{(0)} = Q(r = 0)$ and there isn't implicit dependence on r any more for $Q^{(0)}$.)

The tetrad Eq(4.5) cannot be completely determined unless all six tetrad functions get resolved. Fortunately, as shown in the next section, the roles of $\{X^3, X^4, \xi^3, \xi^4\}$ can be replaced by another on-horizon data, the rotation 1-form potential. Thus, we will continue using the $\{X^3, X^4, \xi^3, \xi^4\}$ obtained above in subsequent calculations until they are replaced by equivalent quantities.

4.3 Reconstruction of Near-Horizon Structure

For an *extremal* IH embedded in electrovacuum, with the tetrad functions given by Eq(4.46), Eq(4.54) and Eqs(4.58-4.61), the tetrad Eq(4.5) becomes

$$l^a\partial_a \simeq \partial_v + 2(2\pi^{(0)}\bar{\pi}^{(0)} + \mathrm{Re}(\Psi_2^{(0)}) + \Phi_{11}^{(0)})r^2\partial_r + (\pi^{(0)}\xi^{3(0)} + \bar{\pi}^{(0)}\bar{\xi}^{3(0)})r\partial_y + (\pi^{(0)}\xi^{4(0)} + \bar{\pi}^{(0)}\bar{\xi}^{4(0)})r\partial_z,$$

$$\begin{aligned} n^a\partial_a &\simeq -\partial_r\,, \\ m^a\partial_a &\simeq \bar{\pi}^{(0)}r\partial_r + [\xi^{3(0)} + (\mu^{(0)}\xi^{3(0)} + \bar{\lambda}^{(0)}\bar{\xi}^{3(0)})\,r]\partial_y + [\xi^{4(0)} + (\mu^{(0)}\xi^{4(0)} + \bar{\lambda}^{(0)}\bar{\xi}^{4(0)})\,r]\partial_z\,, \\ \bar{m}^a\partial_a &\simeq \pi^{(0)}r\partial_r + [\bar{\xi}^{3(0)} + (\mu^{(0)}\bar{\xi}^{3(0)} + \lambda^{(0)}\xi^{3(0)})\,r]\partial_y + [\bar{\xi}^{4(0)} + (\mu^{(0)}\bar{\xi}^{4(0)} + \lambda^{(0)}\xi^{4(0)})\,r]\partial_z\,. \end{aligned} \tag{4.62}$$

Thus, by Eq(4.6) we have the following components of the inverse metric,

$$g^{rr} \simeq 2(U + \Omega\bar{\Omega}) \simeq 2\,(3\pi^{(0)}\bar{\pi}^{(0)} + \mathrm{Re}(\Psi_2^{(0)}) + \Phi_{11}^{(0)})\,r^2 + O(r^3) = F\,r^2 + O(r^3)\,, \tag{4.63}$$

$$g^{ry} \simeq X^3 + \Omega\bar{\xi}^3 + \bar{\Omega}\xi^3 \simeq 2(\pi^{(0)}\xi^{3(0)} + \bar{\pi}^{(0)}\bar{\xi}^{3(0)})\,r + O(r^2) = G^3\,r + O(r^2)\,, \tag{4.64}$$

$$g^{rz} \simeq X^4 + \Omega\bar{\xi}^4 + \bar{\Omega}\xi^4 \simeq 2(\pi^{(0)}\xi^{4(0)} + \bar{\pi}^{(0)}\bar{\xi}^{4(0)})\,r + O(r^2) = G^4\,r + O(r^2)\,, \tag{4.65}$$

where F, G^3 and G^4 are two auxiliary functions defined as

$$\begin{aligned} F &:= 2\left(3\pi^{(0)}\bar{\pi}^{(0)} + \mathrm{Re}(\Psi_2^{(0)}) + \Phi_{11}^{(0)}\right), \\ G^3 &:= 2\left(\pi^{(0)}\xi^{3(0)} + \bar{\pi}^{(0)}\bar{\xi}^{3(0)}\right), \quad G^4 := 2\left(\pi^{(0)}\xi^{4(0)} + \bar{\pi}^{(0)}\bar{\xi}^{4(0)}\right). \end{aligned} \tag{4.66}$$

It is obvious that, F is purely comprised of on-horizon data as

$$\mathrm{Re}(\Psi_2^{(0)}) = \frac{1}{2}\Big(-K + 2\Phi_{11}^{(0)}\Big)\ \text{(Electrovacuum)} \quad \text{OR} \quad \mathrm{Re}(\Psi_2^{(0)}) = -\frac{1}{2}K\ \text{(Vacuum)}\,, \tag{4.67}$$

with K being Gaussian curvature of the horizon. Now, the inverse metric Eq(4.6) can be rewritten in the near-horizon limit into

$$g^{ab} \simeq \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & F\,r^2 + O(r^3) & G^3\,r + O(r^2) & G^4\,r + O(r^2) \\ 0 & G^3\,r + O(r^2) & 2\,\xi^{3(0)}\bar{\xi}^{3(0)} & \xi^{3(0)}\bar{\xi}^{4(0)} + \xi^{4(0)}\bar{\xi}^{3(0)} \\ 0 & G^4\,r + O(r^2) & \xi^{3(0)}\bar{\xi}^{4(0)} + \xi^{4(0)}\bar{\xi}^{3(0)} & 2\,\bar{\xi}^{4(0)}\xi^{4(0)} \end{pmatrix}, \tag{4.68}$$

and therefore the near-horizon metric is $g_{ab} \simeq$

$$\begin{pmatrix} \left(\frac{2(\xi^3\bar{\xi}^4+\xi^4\bar{\xi}^3)G^3G^4-2\xi^3\bar{\xi}^3(G^4)^2-2\xi^4\bar{\xi}^4(G^3)^2}{(\xi^3\bar{\xi}^4-\xi^4\bar{\xi}^3)^2}-F\right)r^2 & 1 & \frac{2\xi^4\bar{\xi}^4G^3-(\xi^3\bar{\xi}^4+\xi^4\bar{\xi}^3)G^4}{(\xi^3\bar{\xi}^4-\xi^4\bar{\xi}^3)^2}r & \frac{2\xi^3\bar{\xi}^3G^4-(\xi^3\bar{\xi}^4+\xi^4\bar{\xi}^3)G^3}{(\xi^3\bar{\xi}^4-\xi^4\bar{\xi}^3)^2}r \\ 1 & 0 & 0 & 0 \\ -\frac{-2\xi^4\bar{\xi}^4G^3+(\xi^3\bar{\xi}^4+\xi^4\bar{\xi}^3)G^4}{(\xi^3\bar{\xi}^4-\xi^4\bar{\xi}^3)^2}r & 0 & \frac{-2\xi^4\bar{\xi}^4}{(\xi^3\bar{\xi}^4-\xi^4\bar{\xi}^3)^2} & \frac{\xi^3\bar{\xi}^4+\xi^4\bar{\xi}^3}{(\xi^3\bar{\xi}^4-\xi^4\bar{\xi}^3)^2} \\ -\frac{-2\xi^3\bar{\xi}^3G^4+(\xi^3\bar{\xi}^4+\xi^4\bar{\xi}^3)G^3}{(\xi^3\bar{\xi}^4-\xi^4\bar{\xi}^3)^2}r & 0 & \frac{\xi^3\bar{\xi}^4+\xi^4\bar{\xi}^3}{(\xi^3\bar{\xi}^4-\xi^4\bar{\xi}^3)^2} & \frac{-2\xi^3\bar{\xi}^3}{(\xi^3\bar{\xi}^4-\xi^4\bar{\xi}^3)^2} \end{pmatrix} \tag{4.69}$$

where $\{\xi^3, \bar{\xi}^3, \xi^4, \bar{\xi}^4\}$ in g_{ab} should be read as $\{\xi^{3(0)}, \bar{\xi}^{3(0)}, \xi^{4(0)}, \bar{\xi}^{4(0)}\}$. and for a metric component $g_{ab} = \tilde{g}_{ab}r^m$, the asymptotic terms $O(r^{m+1})$ are omitted temporarily for editing convenience and will be resumed henceforth. It is useful to introduce the induced intrinsic metric $\hat{h}_{AB}$ for the IH,

$$\hat{h}_{AB} = \begin{pmatrix} \frac{-2\bar{\xi}^{4(0)}\xi^{4(0)}}{(\xi^{3(0)}\bar{\xi}^{4(0)}-\xi^{4(0)}\bar{\xi}^{3(0)})^2} & \frac{\xi^{3(0)}\bar{\xi}^{4(0)}+\xi^{4(0)}\bar{\xi}^{3(0)}}{(\xi^{3(0)}\bar{\xi}^{4(0)}-\xi^{4(0)}\bar{\xi}^{3(0)})^2} \\ \frac{\xi^{3(0)}\bar{\xi}^{4(0)}+\xi^{4(0)}\bar{\xi}^{3(0)}}{(\xi^{3(0)}\bar{\xi}^{4(0)}-\xi^{4(0)}\bar{\xi}^{4(0)})^2} & \frac{-2\xi^{3(0)}\bar{\xi}^{3(0)}}{(\xi^{3(0)}\bar{\xi}^{4(0)}-\xi^{4(0)}\bar{\xi}^{3(0)})^2} \end{pmatrix} \tag{4.70}$$

whose inverse is

$$\hat{h}^{AB} = \begin{pmatrix} 2\,\xi^{3(0)}\bar{\xi}^{3(0)} & \xi^{3(0)}\bar{\xi}^{4(0)}+\xi^{4(0)}\bar{\xi}^{3(0)} \\ \xi^{3(0)}\bar{\xi}^{4(0)}+\xi^{4(0)}\bar{\xi}^{3(0)} & 2\,\bar{\xi}^{4(0)}\xi^{4(0)} \end{pmatrix}, \tag{4.71}$$

and the metric tensor Eq(4.69) could be rewritten into a most compact form

$$g_{ab} \simeq \begin{pmatrix} (-F+\hat{h}_{AB}G^AG^B)\,r^2+O(r^3) & 1 & -\hat{h}_{AB}G^B\,r+O(r^2) \\ 1 & 0 & 0 \\ -\hat{h}_{AB}G^A\,r+O(r^2) & 0 & \hat{h}_{AB} \end{pmatrix}, \tag{4.72}$$

or by line element that[42]

$$
\begin{aligned}
ds^2 &\simeq (\hat{h}_{AB}G^A G^B - F)r^2 dv^2 + 2dvdr - \hat{h}_{AB}G^B r dv dy^A - \hat{h}_{AB}G^A r dv dy^B + \hat{h}_{AB}dy^A dy^B \\
&\simeq -F\, r^2 dv^2 + 2dvdr + \hat{h}_{AB}(dy^A - G^A\, rdv)(dy^B - G^B\, rdv)\,,
\end{aligned}
\tag{4.73}
$$

where the indices $\{A, B\}$ run over $\{3, 4\}$, and $\{y^3 = y, y^4 = z\}$. Specifically for static isolated horizons, $\pi^{(0)} = 0$, $G^3 = G^4 = 0$, $F = 2\,(\mathrm{Re}(\Psi_2^{(0)}) + \Phi_{11}^{(0)})$, and

$$
ds^2 \simeq -F\, r^2 dv^2 + 2dvdr + \hat{h}_{AB}dy^A dy^B\,. \tag{4.74}
$$

Eq(4.73) and Eq(4.74) tells us that, once $\{F, G^3, G^4\}$ are determined, the near-horizon metric of an IH would be absolutely fixed. For nonrotating case Eq(4.74), the last term $\hat{h}_{AB}dy^A dy^B$ is just the intrinsic metric of the horizon in chosen isothermal coordinates, but for generic rotating case Eq(4.73), $\{G^3,G^4\}$ are still unknown quantities and the NHG cannot be calculated directly.

It is now timely to set up the intrinsic coordinates, and we will identify $\{y^A,\ y^B\}$ to be $\{x,\ \varphi\}$ in accordance with the local uniqueness solutions. Immediately, Eq(4.74) becomes

$$
ds^2 \simeq -F\, r^2 dv^2 + 2dvdr + \hat{h}_{xx}dx^2 + \hat{h}_{\varphi\varphi}d\varphi^2\,. \tag{4.75}
$$

For generic IHs, after diagonalization of $\hat{h}_{AB}$ due to the choice of $\{x,\ \varphi\}$, $\{G^3,G^4\}$ which encodes information about rotation could be replaced by the rotation 1-form potential[42],

$$
\begin{aligned}
\hat{h}_{AB} &= m_A\bar{m}_B + \bar{m}_A m_B\,, \quad \hat{\omega}_A = \pi^{(0)}m_A + \bar{\pi}^{(0)}\bar{m}_A\,, \\
\Rightarrow\ \hat{h}_{AB}G^B &= 2\hat{\omega}_A\,, \quad G^B = \frac{2\hat{\omega}_A}{\hat{h}_{AB}}\,,
\end{aligned}
\tag{4.76}
$$

and therefore in Eq(4.73) become the calculable metric

$$ds^2 \simeq -F r^2 dv^2 + 2dvdr + \hat{h}_{xx}\left(dx - \frac{2\hat{\omega}_x}{\hat{h}_{xx}} rdv\right)^2 + \hat{h}_{\varphi\varphi}\left(d\varphi - \frac{2\hat{\omega}_\varphi}{\hat{h}_{\varphi\varphi}} rdv\right)^2, \qquad (4.77)$$

where $\hat{\omega}_x$ and $\hat{\omega}_\varphi$ are the x and φ components of the intrinsic rotation 1-form potential $\hat{\omega}_A$.

4.4 Addendum: More Insights into On-Horizon Properties

In Chapter 2, we have introduced the first set of boundary conditions *on* an extremal IH,

$$\varepsilon \triangleq \kappa \triangleq \rho \triangleq \sigma \triangleq 0; \quad \Psi_0 \triangleq \Psi_1 \triangleq 0; \quad \Phi_{00} \triangleq 0, \quad \Phi_{01} = \overline{\Phi_{10}} \triangleq 0, \quad \Phi_{02} = \overline{\Phi_{20}} \triangleq 0. \qquad (4.78)$$

Also, not only the NP quantities, but the null tetrad is coordinates-dependent, and for $r = 0$, the tetrad Eq(4.5) reduces to

$$\hat{D} \triangleq \partial_v, \quad \hat{\Delta} \triangleq -\partial_r, \quad \hat{\delta} \triangleq \xi^{3(0)}\partial_y + \xi^{4(0)}\partial_z, \quad \hat{\bar{\delta}} \triangleq \bar{\xi}^{3(0)}\partial_y + \bar{\xi}^{4(0)}\partial_z. \qquad (4.79)$$

In this section, we will investigate how spin coefficients, Ψ_i and Φ_{ij} which are defined for near-horizon and off-horizon regions behave on the horizon. This serves as a supplement to the on-horizon data by the local uniqueness solution and could yield the second set of boundary conditions.

Substitute the gauge conditions Eq(4.19), the on-horizon properties Eq(4.78) and the reduced tetrad Eq(4.79) into Einstein-Maxwell-NP equations and Bianchi-NP equations. Note that we are dealing with boundary values of the directional derivatives of NP quantities, e.g. $(\delta Q)\big|_{r=0} = (\delta Q)^{(0)} = \hat{\delta} Q$, rather than the derivatives of boundary values, e.g. $\hat{\delta}(Q^{(0)})$.

Boundary Equations

Firstly, we have the r-derivatives of spin coefficients and Weyl scalars,

$$\partial_r \kappa \hat{=} 0 \,, \tag{4.80}$$

$$\partial_r \sigma \hat{=} 0 \,, \tag{4.81}$$

$$\partial_r \rho \hat{=} \Psi_2^{(0)} \,, \tag{4.82}$$

$$\partial_r \lambda \hat{=} 2\mu^{(0)}\lambda^{(0)} + \Psi_4^{(0)} \,, \tag{4.83}$$

$$\partial_r \mu \hat{=} (\mu^{(0)})^2 + \lambda^{(0)}\bar{\lambda}^{(0)} + \Phi_{22}^{(0)} \,, \tag{4.84}$$

$$\partial_r \alpha \hat{=} \lambda^{(0)}\beta^{(0)} + \mu^{(0)}\alpha^{(0)} + \Psi_3^{(0)} \,, \tag{4.85}$$

$$\partial_r \beta \hat{=} \mu^{(0)}\beta^{(0)} + \alpha^{(0)}\bar{\lambda}^{(0)} + \Phi_{12}^{(0)} \,, \tag{4.86}$$

$$\partial_r \pi \hat{=} \pi^{(0)}\mu^{(0)} + \bar{\pi}^{(0)}\lambda^{(0)} + \Psi_3^{(0)} + \Phi_{21}^{(0)} \,, \tag{4.87}$$

$$\partial_r \varepsilon \hat{=} \bar{\pi}^{(0)}\alpha^{(0)} + \pi^{(0)}\beta^{(0)} + \Psi_2^{(0)} + \Phi_{11}^{(0)} \,. \tag{4.88}$$

$$\partial_r \Psi_0 + \hat{\delta}\Psi_1 - \partial_v \Phi_{02} + \hat{\delta}\Phi_{01} \hat{=} 0 \,, \tag{4.89}$$

$$\partial_r \Psi_1 + \hat{\delta}\Psi_2 - \partial_r \Phi_{01} - \hat{\bar{\delta}}\Phi_{02} \hat{=} 0 \,, \tag{4.90}$$

$$\partial_r \Psi_2 + \hat{\delta}\Psi_3 - \partial_u \Phi_{22} + \hat{\delta}\Phi_{21} \hat{=} 3\mu^{(0)}\Psi_2^{(0)} - 2\beta^{(0)}\Psi_3^{(0)} + 2\mu^{(0)}\Phi_{11}^{(0)} - 2\pi^{(0)}\Phi_{12}^{(0)} - 2(\bar{\pi}^{(0)} + \beta^{(0)})\Phi_{21}^{(0)} , \tag{4.91}$$

$$\partial_r \Psi_3 + \hat{\delta}\Psi_4 - \partial_r \Phi_{21} - \hat{\bar{\delta}}\Phi_{22} \hat{=} 4\mu^{(0)}\Psi_3^{(0)} - 4\beta^{(0)}\Psi_4^{(0)} - 2\lambda^{(0)}\Phi_{12}^{(0)} - 2\mu^{(0)}\Phi_{21}^{(0)} + 2\pi^{(0)}\Phi_{22}^{(0)} , \tag{4.92}$$

where, however, $\left(\partial_r \Psi_4\right)\Big|_{r=0}$ is still undetermined.

Secondly, there will be the (partial) evolution equations containing v-derivatives of spin coefficients and Weyl scalars,

$$\partial_v \rho \hat{=} \hat{\bar{\delta}} \kappa \,, \tag{4.93}$$

$$\partial_v \sigma \hat{=} \hat{\delta} \kappa \,, \tag{4.94}$$

$$\partial_v \alpha \hat{=} \hat{\bar{\delta}} \varepsilon \,, \tag{4.95}$$

$$\partial_v \beta \hat{=} \hat{\delta} \varepsilon \,, \tag{4.96}$$

$$\partial_v \lambda \hat{=} 2\alpha^{(0)}\pi^{(0)} + \hat{\bar{\delta}} \pi \,, \tag{4.97}$$

$$\partial_v \mu \hat{=} 2\beta^{(0)}\pi^{(0)} + \Psi_2^{(0)} + \hat{\delta} \pi \,, \tag{4.98}$$

$$\partial_v \Psi_1 - \hat{\bar{\delta}} \Psi_0 - \partial_v \Phi_{01} + \hat{\delta} \Phi_{00} \hat{=} 0 \,, \tag{4.99}$$

$$\partial_v \Psi_2 - \hat{\bar{\delta}} \Psi_1 - \partial_r \Phi_{00} - \hat{\bar{\delta}} \Phi_{01} \hat{=} 0 \,, \tag{4.100}$$

$$\partial_v \Psi_3 - \hat{\bar{\delta}} \Psi_2 - \partial_v \Phi_{21} + \hat{\delta} \Phi_{20} \hat{=} 3\pi^{(0)}\Psi_2^{(0)} - 2\pi^{(0)}\Phi_{11}^{(0)} \,, \tag{4.101}$$

$$\partial_v \Psi_4 - \hat{\bar{\delta}} \Psi_3 - \partial_r \Phi_{20} - \hat{\bar{\delta}} \Phi_{21} \hat{=} -3\lambda^{(0)}\Psi_2^{(0)} + 2(\alpha^{(0)} + 2\pi^{(0)})\Psi_3^{(0)} - 2\lambda^{(0)}\Phi_{11}^{(0)} + 2\alpha^{(0)}\Phi_{21}^{(0)} \,. \tag{4.102}$$

Since v is the ingoing null coordinate (a mixture of spatial and temporal scales) rather than a pure temporal coordinate, these v-derivatives are only *partial* evolution equations.

The last group containing intrinsic restrictions ($\hat{\delta}$ derivatives), and the remaining Bianchi equations describing the Ricci scalars,

$$\hat{\delta} \alpha - \hat{\bar{\delta}} \beta \hat{=} \alpha^{(0)}\bar{\alpha}^{(0)} + \beta^{(0)}\bar{\beta}^{(0)} - 2\alpha^{(0)}\beta^{(0)} - \Psi_2^{(0)} + \Phi_{11}^{(0)} \,, \tag{4.103}$$

$$\hat{\delta} \lambda - \hat{\bar{\delta}} \mu \hat{=} \pi^{(0)}\mu^{(0)} + (\bar{\alpha}^{(0)} - 3\beta^{(0)})\lambda^{(0)} - \Psi_3^{(0)} + \Phi_{21}^{(0)} \,. \tag{4.104}$$

$$\partial_v \Phi_{11} - \partial_r \Phi_{00} - \hat{\delta}\Phi_{10} - \hat{\bar{\delta}}\Phi_{01} \hat{=} 0\,, \tag{4.105}$$

$$\partial_v \Phi_{12} - \partial_r \Phi_{01} - \hat{\delta}\Phi_{11} - \hat{\bar{\delta}}\Phi_{02} \hat{=} 2\bar{\pi}^{(0)}\Phi_{11}^{(0)}\,, \tag{4.106}$$

$$\partial_v \Phi_{22} - \partial_r \Phi_{11} - \hat{\delta}\Phi_{21} - \hat{\bar{\delta}}\Phi_{12} \hat{=} 4\mu^{(0)}\Phi_{11}^{(0)} + 2(\pi^{(0)} + \bar{\beta}^{(0)})\Phi_{12}^{(0)} + 2(\bar{\pi}^{(0)} + \beta^{(0)})\Phi_{21}^{(0)}\,. \tag{4.107}$$

Spin Coefficients in Near-Horizon Limit

In the near-horizon limit, we can use the leading terms to approximate the spin coefficients. The two sets of boundary conditions Eq(4.78) and Eqs(4.80-4.89) give rise to

$$\kappa \simeq O(r) \simeq \kappa^{(2)} r^2 + O(r)\,, \tag{4.108}$$

$$\sigma \simeq O(r) \simeq \sigma^{(2)} r^2 + O(r)\,, \tag{4.109}$$

$$\rho \simeq O(0) \simeq \Psi_2^{(0)} r + O(r)\,, \tag{4.110}$$

$$\lambda \simeq O(0) \simeq \left(2\mu^{(0)}\lambda^{(0)} + \Psi_4^{(0)}\right) r + O(r)\,, \tag{4.111}$$

$$\mu \simeq O(0) \simeq \left((\mu^{(0)})^2 + \lambda^{(0)}\bar{\lambda}^{(0)} + \Phi_{22}^{(0)}\right) r + O(r)\,, \tag{4.112}$$

$$\alpha \simeq O(0) \simeq \left(\lambda^{(0)}\beta^{(0)} + \mu^{(0)}\alpha^{(0)} + \Psi_3^{(0)}\right) r + O(r)\,, \tag{4.113}$$

$$\beta \simeq O(0) \simeq \left(\mu^{(0)}\beta^{(0)} + \alpha^{(0)}\bar{\lambda}^{(0)} + \Phi_{12}^{(0)}\right) r + O(r)\,, \tag{4.114}$$

$$\pi \simeq O(0) \simeq \left(\pi^{(0)}\mu^{(0)} + \bar{\pi}^{(0)}\lambda^{(0)} + \Psi_3^{(0)} + \Phi_{21}^{(0)}\right) r + O(r)\,, \tag{4.115}$$

$$\varepsilon \simeq O(0) \simeq \left(\bar{\pi}^{(0)}\alpha^{(0)} + \pi^{(0)}\beta^{(0)} + \Psi_2^{(0)} + \Phi_{11}^{(0)}\right) r + O(r)\,. \tag{4.116}$$

First of all, as expected from the gauge conditions, it is obvious that $\mu = \bar{\mu}$. Moreover, here we want to stress the behavior of ρ. Recall that

$$\Psi_2^{(0)} = \frac{1}{2}\Big(- K + 4\phi_1^{(0)}\bar{\phi}_1^{(0)} + i\,\hat{\nabla}_L^2 U\Big)\,, \tag{4.117}$$

so for a generic rotating IH, $\rho \simeq \Psi_2^{(0)} r$ is imaginary, Im(ρ)$\neq$0; only for nonrotating IHs, $U = 0 \Rightarrow \mathrm{Im}(\Psi_2^{(0)}) = 0 \Rightarrow \mathrm{Im}(\rho)=0$. Consequently, the commutator

$$\pounds_{\bar{m}} \boldsymbol{m} = [\bar{\boldsymbol{m}}, \boldsymbol{m}] = \bar{\delta}\delta - \delta\bar{\delta} = (\bar{\rho} - \rho)\Delta - (\bar{\beta} - \alpha)\delta - (\bar{\alpha} - \beta)\bar{\delta} \tag{4.118}$$

becomes purely intrinsic (the coefficients for the directional derivatives Δ being zero) and the null congruence becomes hypersurface orthogonal only for nonrotating IHs. This can also be understood this way, that near-horizon null geodesics would be twisted by rotation of the IH.

How Does Extremality Work ?

Now it is clear that the extremality condition $\kappa_{(l)} = 2\varepsilon^{(0)} = 0$ serves as a boundary condition in integrating the field equations in the near-horizon limit. Taking spin coefficients as an example and rederiving the foregoing three groups of equations with $\varepsilon^{(0)} \neq 0$, one could find that the extremality condition has nothing to do with the r-derivatives, but would influence the evolutions; that is to say, Eqs(4.80-4.89) remain the same no matter $\varepsilon^{(0)} = 0$ or $\varepsilon^{(0)} \neq 0$, but two of the evolution equations Eq(4.98) and Eq(4.99) would become

$$\partial_u \lambda \mathrel{\hat{=}} 2\alpha^{(0)}\pi^{(0)} - 2\varepsilon^{(0)}\lambda^{(0)} + \hat{\bar{\delta}}\pi\,, \tag{4.119}$$

$$\partial_u \mu \mathrel{\hat{=}} 2\beta^{(0)}\pi^{(0)} - 2\varepsilon^{(0)}\mu^{(0)} + \Psi_2^{(0)} + \hat{\delta}\pi\,. \tag{4.120}$$

Moreover, as expected, an extra term containing $\varepsilon^{(0)}$ would appear in every Bianchi boundary equations above as there is v-derivative in each of these equations.

Chapter 5

NHMs of Extremal Kerr-Newman IHs

With the groundworks in Chapters 2 and 3 as well as the local method developed in Chapter 4, we will (re)construct the near-horizon metrics (NHMs) of extremal Reissner-Nordström and Kerr isolated horizons embedded in electrovacuum via Eq(4.75) and Eq(4.77) respectively in this chapter.

Contrary to such local constructions, existing methods would follow global approaches by taking the near-horizon limit of global Kerr-Newman metrics (e.g.[4]). If the NHMs derived locally prove to be equivalent with those derived globally after coordinate transformations, they will correspond to identical near-horizon (and also intrinsic) structures.

The parameters are set up as follows: for extremal Reissner-Nordström horizons, $\alpha = 1$ and $A = 4\pi$; for extremal Kerr horizons, $\alpha = 0$ and $A = 8\pi$; for extremal Kerr-Newman horizons, α flexible and $A = 4\pi(2 - \alpha^2)$. The choices of α was explained in Chapter 3, and the choices of A will be interpreted later in the addendum to this chapter.

5.1 Reconstruction of Extremal Reissner-Nordström NHM

Extremal Reissner-Nordström NHM from Global Metric

The metric of extremal Reissner-Nordström black hole is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)^2 dt^2 + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{5.1}$$

Taking the transformation

$$t \mapsto \frac{\tilde{t}}{\epsilon}, \quad r \mapsto M + \epsilon \tilde{r}, \quad \epsilon \to 0, \tag{5.2}$$

and then omitting the tildes, one obtains the near-horizon metric

$$\begin{aligned} ds^2 &\simeq -\frac{r^2}{M^2} dt^2 + \frac{M^2}{r^2} dr^2 + M^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \\ &\simeq -r^2 dt^2 + \frac{1}{r^2} dr^2 + \left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \ (M^2 = 1), \\ &\simeq -\rho^2 dt^2 + \frac{1}{\rho^2} d\rho^2 + \left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \ (r \mapsto \rho). \end{aligned} \tag{5.3}$$

This metric results from a global approach using the metric describing the entire extremal RN spacetime.

Extremal Reissner-Nordström NHM via on-Horizon Data

Now we will take a local approach to reconstruct the NHM of extremal RN isolated horizons embedded in electrovacuum. Setting the surface area to be $A = 4\pi$ (and $\alpha = 1$), we immediately have the local uniqueness solution:

$$P^2 = \frac{2}{1 - 4x^2}, \quad \phi_1^{(0)} = \frac{e^{i\theta_0}}{2}, \quad U = 0, \quad B = 0; \tag{5.4}$$

intrinsic metric:

$$h_{AB}: \quad ds^2 \triangleq \frac{4}{1-4x^2}dx^2 + (1-4x^2)d\varphi^2\,; \tag{5.5}$$

electromagnetic field:

$$\Phi_{11}^{(0)} = 2\phi_1^{(0)}\bar{\phi}_1^{(0)} = \frac{1}{2}\,; \tag{5.6}$$

nonzero connection coefficients:

$$\hat{\Gamma}^x_{xx} \triangleq \frac{4x}{1-4x^2}\,, \quad \hat{\Gamma}^x_{\varphi\varphi} \triangleq x(1-4x^2)\,, \quad \hat{\Gamma}^\varphi_{x\varphi} = \hat{\Gamma}^\varphi_{\varphi x} \triangleq -\frac{4x}{1-4x^2}\,; \tag{5.7}$$

Gaussian curvature of the horizon:

$$K = -\frac{1}{h_{xx}}\left(\partial_x\hat{\Gamma}^\varphi_{x\varphi} - \partial_\varphi\hat{\Gamma}^\varphi_{xx} + \hat{\Gamma}^x_{x\varphi}\hat{\Gamma}^\varphi_{xx} - \hat{\Gamma}^x_{xx}\hat{\Gamma}^\varphi_{x\varphi} + \hat{\Gamma}^\varphi_{x\varphi}\hat{\Gamma}^\varphi_{\varphi x} - \hat{\Gamma}^\varphi_{xx}\hat{\Gamma}^\varphi_{\varphi\varphi}\right) \triangleq 1\,, \tag{5.8}$$

or by commutator coefficient

$$\zeta^2 \triangleq \frac{2x^2}{1-4x^2}\,, \qquad K \triangleq 2\delta\zeta - 2\zeta^2 = 1\,. \tag{5.9}$$

Thus the metric function F is

$$F = 2\mathrm{Re}(\Psi_2^{(0)}) + 2\Phi_{11}^{(0)} = -K + 4\Phi_{11}^{(0)} = 1\,, \tag{5.10}$$

which leads to the near-horizon metric

$$g_{ab}: \quad ds^2 \simeq -r^2\,dv^2 + 2dvdr + \frac{4}{1-4x^2}dx^2 + (1-4x^2)d\varphi^2\,. \tag{5.11}$$

Equivalence of Two NHMs

With the coordinate transformation

$$v = t - \frac{1}{\rho}\,, \quad r = \rho\,, \quad x = \frac{\cos\theta}{2}\,, \quad \varphi = \phi\,, \tag{5.12}$$

the NHM Eq(5.11) becomes

$$g_{ab}: \quad ds^2 \simeq -\rho^2\,dt^2 + \frac{1}{\rho^2}\,d\rho^2 + (d\theta^2 + \sin^2\theta\,d\phi^2)\,, \tag{5.13}$$

which is identical with the NHM Eq(5.3) obtained from the global metric.

5.2 Reconstruction of Extremal Kerr NHM

Extremal Kerr NHM from Global Metric

The metric of extremal Kerr black hole ($M = a = J/M$) in Boyer-Lindquist coordinates can be written in the following two enlightening forms[27][45],

$$ds^2 = -\frac{\rho_K^2\Delta_K}{\Sigma^2}\,dt^2 + \frac{\rho_K^2}{\Delta_K}\,dr^2 + \rho_K^2 d\theta^2 + \frac{\Sigma^2\sin^2\theta}{\rho_K^2}(d\phi - \omega_K\,dt)^2\,, \tag{5.14}$$

$$= -\frac{\Delta_K}{\rho_K^2}\,(dt - M\sin^2\theta d\phi)^2 + \frac{\rho_K^2}{\Delta_K}\,dr^2 + \rho_K^2 d\theta^2 + \frac{\sin^2\theta}{\rho_K^2}\Big(Mdt - (r^2 + M^2)d\phi\Big)^2\,, \tag{5.15}$$

where

$$\rho_K^2 = r^2 + M^2\cos^2\theta\,,\ \Delta_K = (r-M)^2\,,\ \Sigma^2 = (r+M^2)^2 - M^2\Delta_K\sin^2\theta\,,\ \omega_K = \frac{2M^2 r}{\Sigma^2}\,. \tag{5.16}$$

Taking the transformation

$$t \mapsto \frac{\tilde{t}}{\epsilon}\,, \quad r \mapsto M + \epsilon\,\tilde{r}\,, \quad \phi \mapsto \tilde{\phi} + \frac{1}{2M\epsilon}\tilde{t}\,, \quad \epsilon \to 0\,, \tag{5.17}$$

and omitting the tildes, one obtains the near-horizon metric (this is also called extremal Kerr throat, c.f.[46])

$$
\begin{aligned}
ds^2 &\simeq \frac{1+\cos^2\theta}{2}\Big(-\frac{r^2}{2M^2}dt^2+\frac{2M^2}{r^2}dr^2+2M^2d\theta^2\Big)+\frac{4M^2\sin^2\theta}{1+\cos^2\theta}\Big(d\phi+\frac{rdt}{2M^2}\Big)^2 \\
&\simeq (1+\cos^2\theta)\Big(-\frac{r^2}{4}dt^2+\frac{1}{r^2}dr^2+d\theta^2\Big)+\frac{4\sin^2\theta}{1+\cos^2\theta}\Big(d\phi+\frac{r}{2}dt\Big)^2,\ (M^2:=1) \\
&\simeq (1+\cos^2\theta)\Big(-\frac{\rho^2}{4}dt^2+\frac{1}{\rho^2}d\rho^2+d\theta^2\Big)+\frac{4\sin^2\theta}{1+\cos^2\theta}\Big(d\phi+\frac{\rho}{2}dt\Big)^2,\ (r\mapsto\rho). \quad (5.18)
\end{aligned}
$$

Extremal Kerr NHM via on-Horizon Data

Now, from the local perspective, set $A = 8\pi$ and $\alpha = 0$ to Eq(2.97), and we have the local uniqueness solution

$$P^2 = \frac{1+x^2}{2(1-x^2)}, \quad U = \pm\arctan x, \quad B = \sqrt{1+x^2}, \quad \phi_1^{(0)} = 0; \tag{5.19}$$

intrinsic metric:

$$h_{AB}: \quad ds^2 \triangleq \frac{1+x^2}{1-x^2}dx^2 + \frac{4(1-x^2)}{1+x^2}d\varphi^2; \tag{5.20}$$

spin coefficient $\pi^{(0)}$ for rotation:

$$\pi^{(0)} = \sqrt{\frac{1-x^2}{2(1+x^2)}}\,\frac{x\mp i}{1+x^2}; \tag{5.21}$$

rotation 1-form potential:

$$\hat{\omega}_A = \frac{x}{1+x^2}dx \pm \frac{2(1-x^2)}{(1+x^2)^2}d\phi; \tag{5.22}$$

nonzero connection coefficients:

$$\hat{\Gamma}^x_{xx} \triangleq \frac{2x}{1-x^4}, \quad \hat{\Gamma}^x_{\varphi\varphi} \triangleq \frac{8x(1-x^2)}{(1+x^2)^3}, \quad \hat{\Gamma}^\varphi_{x\varphi} = \hat{\Gamma}^\varphi_{\varphi x} \triangleq -\frac{2x}{1-x^4}; \tag{5.23}$$

Gaussian curvature of the horizon:

$$K \hat{=} -\frac{1}{h_{xx}}\left(\partial_x \hat{\Gamma}^{\varphi}_{x\varphi} - \partial_\varphi \hat{\Gamma}^{\varphi}_{xx} + \hat{\Gamma}^{x}_{x\varphi}\hat{\Gamma}^{\varphi}_{xx} - \hat{\Gamma}^{x}_{xx}\hat{\Gamma}^{\varphi}_{x\varphi} + \hat{\Gamma}^{\varphi}_{x\varphi}\hat{\Gamma}^{\varphi}_{\varphi x} - \hat{\Gamma}^{\varphi}_{xx}\hat{\Gamma}^{\varphi}_{\varphi\varphi}\right) \hat{=} \frac{2-6x^2}{(1+x^2)^3}, \tag{5.24}$$

or

$$\zeta^2 \hat{=} \frac{2x^2}{(1-x^2)(1+x^2)^3}, \quad K \hat{=} 2\delta\zeta - 2\zeta^2 = \frac{2-6x^2}{(1+x^2)^3}. \tag{5.25}$$

As with the metric functions Eq4.66 for Kerr black holes, we have $\Phi^{(0)}_{11} = 0$, as well as $K = -2\mathrm{Re}(\Psi^{(0)}_2)$, thus the metric function is

$$F = 6\pi^{(0)}\bar{\pi}^{(0)} + 2\mathrm{Re}(\Psi^{(0)}_2) = \frac{1+6x^2-3x^4}{(1+x^2)^3}, \tag{5.26}$$

then one obtains the near-horizon metric

$$\begin{aligned} ds^2 \hat{=} & -\frac{1+6x^2-3x^4}{(1+x^2)^3} r^2 dv^2 + 2dvdr \\ & + \frac{1+x^2}{1-x^2}\left(dx - \frac{2x(1-x^2)}{(1+x^2)^2} r dv\right)^2 + \frac{4(1-x^2)}{1+x^2}\left(d\varphi \mp \frac{1}{1+x^2} r dv\right)^2. \end{aligned} \tag{5.27}$$

Equivalence of Two NHMs

Eq(5.27) is equivalent to Eq(5.18) via the coordinate transformation (c.f. [42] for a similar result), which also shows how '±' for the rotational scalar potential U works,

$$U = +\arctan(x): \quad v \mapsto t - \frac{1}{\rho}, \quad r \mapsto \frac{\rho}{2}(1+\cos^2\theta), \quad x \mapsto \cos\theta, \quad \varphi \mapsto \phi + \frac{1}{2}\ln\rho; \tag{5.28}$$

$$U = -\arctan(x): \quad v \mapsto t - \frac{1}{\rho}, \quad r \mapsto \frac{\rho}{2}(1+\cos^2\theta), \quad x \mapsto \cos\theta, \quad \varphi \mapsto \phi - \frac{1}{2}\ln\rho. \tag{5.29}$$

Thus, we have the transformation $x = \dfrac{\cos\theta}{2}$ for extremal Reissner-Nordström horizons and $x = \cos\theta$ for extremal Kerr horizons, which are consistent with the argument of x, $x \in [-\dfrac{A}{8\pi}, \dfrac{A}{8\pi}]$.

5.3 Summary

In this chapter, we verified that the near-horizon metrics of extremal Reissner-Nordström and Kerr horizons derived from the quasilocal approach do agree with those obtained from global metrics. Thus, the first goal of the thesis is now accomplished, and in the next chapter, we will study distortion of extremal Kerr-Newman horizons.

5.4 Addendum: Horizon Areas of Kerr-Newman-Family Black Holes

The intrinsic metric of the outer horizon of a generic Kerr-Newman black hole is[27][45]

$$\hat{h}_{AB}: \quad ds^2 \hat{=} \rho_{KN}^2 d\theta^2 + \frac{\sin^2\theta}{\rho_{KN}^2}\left\{\left(r_+^2 + a^2\right)^2 - \Delta_{KN} a^2 \sin^2\theta\right\} d\phi^2 , \tag{5.30}$$

where

$$a := \frac{J}{M}, \quad \rho_{KN} := r_+^2 + a^2 , \quad \Delta_{KN} := r_+^2 - 2Mr_+ + a^2 + Q^2 , \quad r_+ = M + \sqrt{M^2 - Q^2 - a^2} . \tag{5.31}$$

so its horizon area is

$$A = \int_0^\pi d\theta \int_0^{2\pi} d\phi \sqrt{\det(\hat{h}_{AB})} = 4\pi(r_+^2 + a^2) = 8\pi\left(M^2 - \frac{Q^2}{2} + M\sqrt{M^2 - Q^2 - a^2}\right). \tag{5.32}$$

It is interesting that only a (rotation) appears explicitly in the surface area formula while Q (electric charge) is concealed.

Now, let's compute the surface area of every Kerr-Newman family member.

(i) For extremal Kerr-Newman black holes, $r_+ = M = \sqrt{Q^2 + a^2}$,

$$A = 4\pi(Q^2 + 2a^2) = 4\pi(M^2 + a^2) = 4\pi(2M^2 - Q^2). \qquad (5.33)$$

(ii) For Kerr black holes, $Q = 0$, $r_+ = M + \sqrt{M^2 - a^2}$,

$$A = 4\pi(r_+^2 + a^2) = 8\pi(M^2 + M\sqrt{M^2 - a^2}); \qquad (5.34)$$

in the extremal case, $r_+ = a = M$,

$$A = 8\pi a^2 = 8\pi J = 8\pi M^2. \qquad (5.35)$$

(iii) For Reissner-Nordström black holes, $a = 0$ ($J = 0$), $r_+ = M + \sqrt{M^2 - Q^2}$,

$$A = 4\pi r_+^2 = 4\pi(2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2}); \qquad (5.36)$$

in the extremal case, $r_+ = Q = M$,

$$A = 4\pi Q^2 = 4\pi M^2. \qquad (5.37)$$

(iv) For Schwarzschild black holes, $a = 0$, $Q = 0$, $r_+ = 2M$,

$$A = 16\pi M^2. \qquad (5.38)$$

The local uniqueness solutions is actually consistent with the convention of $M = 1$. Thus, for extremal Kerr-Newman, $A = 4\pi(2 - \alpha^2)$ with $Q = \alpha M$, and $\alpha \in [0, 1]$ being the very input parameter in the local uniqueness solutions; for extremal Kerr, $A = 8\pi$; for extremal Reissner-Nordström, $A = 4\pi$. This explains our choices of horizon areas at the beginning of this chapter.

Also, we utilized the fact that, the horizon area A is an *invariant* in the sense that it is independent of the coordinate system chosen on the horizon ($\{x, \varphi\}$ or $\{\theta, \phi\}$).

Chapter 6

Conformastatic Distortion of Extremal Reissner-Nordström Horizons

As discussed in previous chapters, the *local* uniqueness theorem established in ref.[14] implies *Booth's conjecture* that the intrinsic structure of extremal Kerr-Newman horizons cannot be distorted by external energy-matter distribution. In this chapter, as the second goal in this thesis, we will partly examine this conjecture using distorted extremal Reissner-Nordström (ERN) spacetime.

Above all, we need to know the exact expression of the metric for an ERN black hole in appropriate distortion fields. Both nonextremal and extremal RN solutions belong to Weyl's family, and static, axisymmetric distortion of nonextremal RN black holes can be well described within the framework of Weyl solutions[47][48]. For the ERN metric, however, the easiest way to realize the distortion is by linear superposition of characteristic *conformastatic* potentials (as will be shown in Section 6.2).

This chapter proceeds as follows. In Section 6.1 and 6.2, Weyl metrics and conformastatic metrics are compared, and the distorted ERN metric is derived from the conformastatic perspective. In Section 6.3, the outgoing and ingoing expansion rates of null radial congruences are calculated, and it is easy to see that for distortion sources satisfying $\partial_r U < 0$ (such as circular disks), there exists one and only one *outer marginally trapped surface* (i.e. $\theta_{(l)} = 0$, $\theta_{(n)} < 0$, c.f.[12]) at $r = M$ which can be identified as the black-hole horizon. For more generic distortion where the signature of $\partial_r U$ is unspecified, the surface $r = M$ is still a marginally trapped surface (i.e. $\theta_{(l)} = 0$) and serves as a *candidate* for the black-hole horizon. The induced metric for the cross-section $\{v = \text{constant},\ r = M\}$ is unaffected by distortion sources, and furthermore, we compute all NP quantities in the distorted spacetime and compare their boundary values on the hypersurface $r = M$ with the isolated ERN horizon. Full comparison shows that, distorted and isolated ERN black holes have identical *intrinsic* (l^a-relevant) behaviors at $r = M$, and only n^a-relevant quantities $\{\mu, \Psi_3, \Phi_{11}, \Phi_{12}, \Phi_{22}\}$ depicting *extrinsic* behaviors are influenced by the distortion potential. These results indicate that, even under generic conformastatic distortion, $r = M$ still refers to the ERN black-hole horizon and the intrinsic structures of the horizon are undistorted.

6.1 ERN Solution: From Weyl to Conformastatic Metrics

ERN Solution as a Weyl Metric

All *static axisymmetric* solutions of Einstein-Maxwell equations can be written in the form of Weyl's metric[29][47],

$$ds^2 = -e^{2\psi(\rho,z)}dt^2 + e^{2\gamma(\rho,z)-2\psi(\rho,z)}(d\rho^2 + dz^2) + e^{-2\psi(\rho,z)}\rho^2 d\phi^2\,, \tag{6.1}$$

where $\psi(\rho, z)$ and $\gamma(\rho, z)$ are two metric potentials dependent on Weyl's canonical coordinates $\{\rho\,, z\}$. The coordinate system $\{t, \rho, z, \phi\}$ serves best for symmetries of Weyl's spacetime (c.f. Section 10.1 of ref.[29]) and usually acts like cylindrical coordinates, but is *incomplete* when describing a black hole as $\{\rho, z\}$ only cover the horizon and its exteriors. Three members in the Kerr-Newman family can be identified as Weyl-type metrics[29][47]:

(i) SCHWARZSCHILD METRIC

$$\psi_{SS} = \frac{1}{2}\ln\frac{L-M}{L+M}, \quad \gamma_{SS} = \frac{1}{2}\ln\frac{L^2-M^2}{l_+l_-}, \tag{6.2}$$

where

$$L = \frac{1}{2}(l_+ + l_-), \quad l_+ = \sqrt{\rho^2 + (z+M)^2}, \quad l_- = \sqrt{\rho^2 + (z-M)^2}, \tag{6.3}$$

thus Eq(6.1) becomes

$$ds^2 = -\frac{L-M}{L+M}dt^2 + \frac{(L+M)^2}{l_+l_-}(d\rho^2 + dz^2) + \frac{L+M}{L-M}\rho^2 d\phi^2, \tag{6.4}$$

and the transformation

$$\begin{aligned} &L + M = r, \quad l_+ + l_- = 2M\cos\theta, \quad z = (r-M)\cos\theta, \\ &\rho = \sqrt{r^2 - 2Mr}\,\sin\theta, \quad l_+l_- = (r-M)^2 - M^2\cos^2\theta, \end{aligned} \tag{6.5}$$

yields the common form of Schwarzschild metric in $\{t, r, \theta, \phi\}$ coordinates,

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2. \tag{6.6}$$

(ii) NONEXTREMAL REISSNER-NORDSTRÖM METRIC

$$\psi_{RN} = \frac{1}{2}\ln\frac{L^2 - (M^2 - Q^2)}{(L+M)^2}, \qquad \gamma_{RN} = \frac{1}{2}\ln\frac{L^2 - (M^2 - Q^2)}{l_+l_-}, \tag{6.7}$$

where

$$L = \frac{1}{2}(l_+ + l_-)\,, \quad l_+ = \sqrt{\rho^2 + (z + \sqrt{M^2 - Q^2})^2}\,, \quad l_- = \sqrt{\rho^2 + (z - \sqrt{M^2 - Q^2})^2}\,, \tag{6.8}$$

thus

$$ds^2 = -\frac{L^2 - (M^2 - Q^2)}{(L + M)^2}dt^2 + \frac{(L + M)^2}{l_+ l_-}(d\rho^2 + dz^2) + \frac{(L + M)^2}{L^2 - (M^2 - Q^2)}\rho^2 d\phi^2\,, \tag{6.9}$$

and the transformation

$$\begin{aligned} &L + M = r\,, \quad l_+ + l_- = 2\sqrt{M^2 - Q^2}\cos\theta\,, \quad z = (r - M)\cos\theta\,, \\ &\rho = \sqrt{r^2 - 2Mr + Q^2}\sin\theta\,, \quad l_+ l_- = (r - M)^2 - (M^2 - Q^2)\cos^2\theta\,, \end{aligned} \tag{6.10}$$

yields the nonextremal Reissner-Nordström metric in Schwarzschild-type coordinates,

$$ds^2 = -\Big(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\Big)dt^2 + \Big(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\Big)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\,. \tag{6.11}$$

(iii) Extremal Reissner-Nordström Metric

$$\psi_{ERN} = \ln\frac{L}{L + M}\,, \quad \gamma_{ERN} = 0\,, \tag{6.12}$$

where

$$L = \sqrt{\rho^2 + z^2}\,, \tag{6.13}$$

thus

$$ds^2 = -\frac{L^2}{(L + M)^2}dt^2 + \frac{(L + M)^2}{L^2}(d\rho^2 + dz^2 + \rho^2 d\phi^2)\,, \tag{6.14}$$

and the transformation

$$L + M = r\,, \quad z = L\cos\theta\,, \quad \rho = L\sin\theta\,, \tag{6.15}$$

yields the ERN metric in Schwarzschild-type coordinates,

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\,. \tag{6.16}$$

Among these solutions, the Schwarzschild and nonextremal RN metrics can be statically and axisymmetrically distorted (or superposed) by external sources within the framework of Weyl's solutions[47][49]. However, the complete procedure of distorting a nonextremal RN metric is highly nonlinear and is invalid for ERN metrics. As will be shown in Section 6.5, distortion Ψ_D is imposed on the *linearized (uncharged)* RN potential Ψ_{RN}, and the true RN potential $\psi_{RN} + \psi_D$ can be derived via

$$e^{-\psi_{RN}-\psi_D} = \frac{1}{2}\left(1 + \frac{C}{\sqrt{C^2-1}}\right) e^{-\sqrt{C^2-1}\,(\Psi_{RN}+\Psi_D)} + \frac{1}{2}\left(1 - \frac{C}{\sqrt{C^2-1}}\right) e^{\sqrt{C^2-1}\,(\Psi_{RN}+\Psi_D)}\,, \tag{6.17}$$

with $C = M/|Q|$. When applying this transformation to ERN solutions ($C = 1$), both mathematical and physical problems arise, which encourages us to search for alternative methods of ERN distortion.

ERN Solution as a Conformastatic Metric

Weyl's metrics Eq(6.1) with the vanishing potential $\gamma(\rho, z)$ (like the ERN metric) constitute a special subclass which have only one potential $\psi(\rho, z)$ to be identified. Extending this subclass by canceling the restriction of axisymmetry, one obtains the family of *conformastatic* solutions (yet still in Weyl's coordinates)

$$ds^2 = -e^{2\lambda(\rho,z,\phi)}dt^2 + e^{-2\lambda(\rho,z,\phi)}\left(d\rho^2 + dz^2 + \rho^2 d\phi^2\right), \tag{6.18}$$

where we use λ as the single metric function in place of ψ in Eq(6.1) to emphasize that they are different by axial symmetry (ϕ-dependence). Substitute the metric Eq(6.18) into Einstein-Maxwell equations,

$$R_{ab} = 8\pi T_{ab}, \tag{6.19}$$

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right), \tag{6.20}$$

$$F_{ab} = A_{b,a} - A_{a,b}, \tag{6.21}$$

$$(\sqrt{g}\,F^{ab})_{,b} = 0, \tag{6.22}$$

and we obtain that $\lambda(\rho, z, \phi)$ is solution to the reduced field equations[50]

$$\nabla_L^2\lambda = e^{-2\lambda}\,\nabla\Phi\,\nabla\Phi, \tag{6.23}$$

$$\nabla_L^2\Phi = 2\nabla\lambda\,\nabla\Phi, \tag{6.24}$$

$$\lambda_{,i}\lambda_{,j} = e^{-2\lambda}\,\Phi_{,i}\Phi_{,j}, \tag{6.25}$$

where $\nabla_L^2 := \partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \partial_{zz} + \frac{1}{\rho^2}\partial_{\phi\phi}$ and ∇ are respectively the Laplacian and gradient operators, while Φ refers to the electromagnetic scalar potential $A_a = (\Phi, 0, 0, 0)$. We will introduce a special class of solutions to this family of equations given by ref.[50]. Assume the functional relation $\Phi = \Phi(\lambda)$, and then Eq(6.25) yields

$$\left(\Phi_{,\lambda}\right)^2 = e^{2\lambda} \Rightarrow \Phi = \pm e^{\lambda} + k_1, \tag{6.26}$$

where k_1 is an integration constant. With this result, Eq(6.23) and Eq(6.24) would combine into a single equation

$$\nabla_L^2 \lambda = \nabla\lambda\nabla\lambda\,. \tag{6.27}$$

Suppose that λ is dependent on an auxiliary function U which is a solution to the Laplace equation ($\nabla_L^2 U = 0$), $\lambda = \lambda(U)$, and thus Eq(6.27) yields

$$\lambda_{,UU} - \left(\lambda_{,U}\right)^2 = 0 \Rightarrow e^{\lambda} = \frac{k_2}{U + k_3}\,, \tag{6.28}$$

where $\{k_2,\ k_3\}$ are arbitrary integration constants, too.

With the general solution given by Eq(6.26) and Eq(6.28), we will now apply physically meaningful boundary conditions to fix the integration constants $\{k_1,\ k_2,\ k_3\}$[50]. For sources of finite extension, asymptotic flatness should be resumed at null or spatial infinity; thus, there should be $e^{\lambda} \to 1$ and $\Phi \to 0$ at infinity, which in turn requires that $U \to 0$ at infinity and $k_2 = k_3$, $k_1 = \mp 1$. Hence, Eq(6.26) and Eq(6.28) become

$$e^{\lambda} = \frac{k}{U + k}\,, \qquad \Phi = \pm\Big(\frac{k}{U + k} - 1\Big)\,. \tag{6.29}$$

Actually, $U(\rho, z, \phi)$ represents the gravitational potential of a compact Newtonian source; U is negative definite and goes to zero $U \to 0^-$ at null/spatial infinity. Astrophysically meaningful solutions also requires $k < 0$. For example, the energy density of a conformastatic disk is[50][51]

$$\epsilon = -\frac{k\Sigma}{(U + k)^2}\,, \qquad \Sigma = \frac{U_{,z}}{2\pi}\,, \tag{6.30}$$

where $\Sigma > 0$ is the Newtonian mass density. To guarantee $\epsilon > 0$, we must have $k < 0$.

In the sense of Eq(6.18) and Eq(6.29), the ERN metric is identified by

$$U_{ERN} = \frac{Mk}{\sqrt{\rho^2 + z^2}}, \quad e^{\lambda} = \Big(\frac{M}{\sqrt{\rho^2 + z^2}} + 1\Big)^{-1}, \tag{6.31}$$

which shows again that, as expected, when reaching infinity $\sqrt{\rho^2 + z^2} \to \infty$, one has $U \to 0^-$; when approaching the horizon $\sqrt{\rho^2 + z^2} \to 0^+$, one encounters an infinitely deep potential well $U \to -\infty$.

Now it is possible to distort an ERN black hole based on Eq(6.29) and Eq(6.31) from the conformastatic perspective.

6.2 Conformastatic Distortion and Tetrad Setup

As we know, Laplace's equation is linear and allows superposition of solutions (for a detailed discussion, c.f.[52]). Hence, we can add another harmonic function $U(\rho, z, \phi)$ to U_{ERN}, and according to Eq(6.29), the resultant potential $U_{ERN} + U(\rho, z, \phi)$ yields

$$e^{\lambda} = \frac{k}{U_{ERN} + U(\rho, z, \phi) + k}, \quad \Phi = \pm\Big(\frac{k}{U_{ERN} + U(\rho, z, \phi) + k} - 1\Big). \tag{6.32}$$

Substitute Eq(6.32) into Eq(6.18) and we obtain the distorted ERN metric

$$ds^2 = -\Big(\frac{M}{\sqrt{\rho^2 + z^2}} + \frac{U}{k} + 1\Big)^{-2} dt^2 + \Big(\frac{M}{\sqrt{\rho^2 + z^2}} + \frac{U}{k} + 1\Big)^2 \big(d\rho^2 + dz^2 + \rho^2 d\phi^2\big). \tag{6.33}$$

Transform Eq(6.33) from Weyl coordinates to Schwarzschild-type coordinates via Eq(6.13) and (6.15),

$$\sqrt{\rho^2 + z^2} + M = r, \quad z = (r - M)\cos\theta, \quad \rho = (r - M)\sin\theta, \tag{6.34}$$

and it follows that

$$U_{ERN} + U(\rho, z, \phi) = \frac{Mk}{r-M} + U(r,\theta,\phi), \quad e^{\lambda} = \Big(\frac{r}{r-M} + \frac{U(r,\theta,\phi)}{k}\Big)^{-1}, \tag{6.35}$$

and

$$ds^2 = -\tilde{F}dt^2 + \tilde{F}^{-1}dr^2 + \Big[r + \frac{U}{k}(r-M)\Big]^2\Big(d\theta^2 + \sin^2\theta\, d\phi^2\Big). \tag{6.36}$$

where

$$\tilde{F} := \Big(\frac{r}{r-M} + \frac{U(r,\theta,\phi)}{k}\Big)^{-2}. \tag{6.37}$$

Introduce the ingoing Eddington-Finkelstein coordinate v in place of t via $dv = dt + \dfrac{dr}{\tilde{F}}$, and Eq(6.36) becomes

$$ds^2 = -\tilde{F}dv^2 + 2dvdr + \Big[r + \frac{U}{k}(r-M)\Big]^2\Big(d\theta^2 + \sin^2\theta\, d\phi^2\Big), \tag{6.38}$$

so the Lagrangian for null radial geodesics ($\mathscr{L} = 0$, $\dot{\theta} = 0$ and $\dot{\phi} = 0$) reads

$$2\mathscr{L} = -\tilde{F}\dot{v}^2 + 2\dot{v}\dot{r} = \dot{v}\,(2\dot{r} - \tilde{F}\dot{v}) = 0, \tag{6.39}$$

with an *ingoing* solution

$$\dot{v} = 0, \tag{6.40}$$

and an *outgoing* solution

$$\dot{r} = \frac{\tilde{F}}{2}\dot{v}. \tag{6.41}$$

Thus, for an ingoing observer ($\dot{r} = -1$ for $n^a\partial_a$), the real tetrad can be set as

$$\begin{aligned} l^a\partial_a &= \Big(1, \frac{\tilde{F}}{2}, 0, 0\Big), \quad n^a\partial_a = \Big(0, -1, 0, 0\Big), \\ l_a dx^a &= \Big(-\frac{\tilde{F}}{2}, 1, 0, 0\Big), \quad n_a dx^a = \Big(-1, 0, 0, 0\Big), \end{aligned} \tag{6.42}$$

where we also utilized the cross-normalization condition $l^a n_a = n^a l_a = -1$, $l^a l_a = n^a n_a = 0$ as well as the requirement that $g_{ab} + l_a n_b + n_a l_b$ span the induced metric h_{AB} for cross-sections of $\{v = \text{constant}, r = \text{constant}\}$. Thus, h_{AB} and its inverse are given by

$$h_{AB} dx^A \otimes dx^B = \left[r + \frac{U}{k}(r - M)\right]^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right), \tag{6.43}$$

$$h^{AB} \partial_A \otimes \partial_B = \left[r + \frac{U}{k}(r - M)\right]^{-2} \left(\partial_\theta^2 + (\sin\theta)^{-2} \partial_\phi^2\right), \tag{6.44}$$

and the remaining two complex tetrad (co)vectors can be constructed via the nonholonomic method (c.f. Chapter 9 of ref.[30]),

$$m^a \partial_a = \frac{1}{\sqrt{2}} \frac{k}{\Delta_D} \left(0, 0, 1, i(\sin\theta)^{-1}\right), \quad m_a dx^a = \frac{1}{\sqrt{2}} \frac{\Delta_D}{k} \left(0, 0, 1, i \sin\theta\right), \tag{6.45}$$

where

$$\Delta_D := (r - M)U + kr, \tag{6.46}$$

and Δ_D is negative definite if $r \geq M$ due to $\{U < 0, k < 0\}$.

6.3 Properties of Distorted ERN Spacetime

6.3.1 Outer Marginally Trapped Surfaces for $\partial_r U > 0$

In the tetrad Eqs(6.42)(6.45), the outgoing and ingoing expansion rates are respectively

$$\theta_{(l)} = -2(\rho + \bar{\rho}) = \frac{k^2}{\Delta_D^3}(r - M)^2 \left[(r - M)\partial_r U + U + k\right], \tag{6.47}$$

$$\theta_{(n)} = \mu + \bar{\mu} = -\frac{2}{\Delta_D}\left[(r - M)\partial_r U + U + k\right], \tag{6.48}$$

Recall that for $r \geq M$, $\{U < 0,\ k < 0,\ \Delta_D < 0\}$. Thus, for distortion sources satisfying $\partial_r U < 0$, such as disks of finite extension surrounding the ERN black hole, we always have $\theta_{(l)} > 0$ and $\theta_{(n)} < 0$ (untrapped surfaces) for $r > M$, while $\theta_{(l)} = 0$ and $\theta_{(n)} < 0$ (outer marginally trapped surface) for $r = M$.

Thus, for the superposed metric Eq(6.38) with $\partial_r U < 0$ and in the domain $r \geq M$, there exists one and only one outer marginally trapped surface which is the hypersurface $r = M$. We also know that $r = M$ corresponds to the horizon of undistorted ERN black holes. With $r = M$, Eq(6.43) and Eq(6.44) exactly reduce to the induced metric of the ERN horizon,

$$h_{AB} dx^A \otimes dx^B = M^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \quad h^{AB}\partial_A \otimes \partial_B = M^{-2}\left(\partial_\theta^2 + (\sin\theta)^{-2}\,\partial_\phi^2\right). \tag{6.49}$$

6.3.2 NP Quantities under Generic Distortion

For generic conformastatic distortion where the signature of $\partial_r U$ is unspecified, we can only conclude from $\{\theta_{(l)}, \theta_{(n)}\}$ that, there exists a marginally trapped surface ($\theta_{(l)}$=0) at $r = M$, although Eq(6.43) and Eq(6.44) would still reduce to Eq(6.49). To examine whether $r = M$ still corresponds to the black hole horizon or not, we will compare all NP quantities of the distorted and isolated cases for their behaviors at $r = M$.

Spin Coefficients for Ingoing Observers

In the tetrad Eqs(6.42)(6.45), the spin coefficients for an ingoing observer are given by

$$\kappa = \frac{\sqrt{2}\,k^3}{2\Delta_D^4}(r-M)^3\left(\partial_\theta U + i\,\csc\theta\,\partial_\phi U\right), \tag{6.50}$$

$$\rho = -\frac{k^2}{2\,\Delta_D^3}(r-M)^2\left[(r-M)\partial_r U + U + k\right], \tag{6.51}$$

$$\sigma = 0, \quad \tau = 0; \tag{6.52}$$

$$\mu = -\frac{1}{\Delta_D}\Big[(r-M)\partial_r U + U + k\Big], \tag{6.53}$$

$$\nu = 0, \quad \lambda = 0, \quad \pi = 0; \tag{6.54}$$

$$\alpha = -\frac{\sqrt{2}\,k}{4\,\Delta_D^2}\Big[(r-M)(\partial_\theta U - i\csc\theta\,\partial_\phi U) + \Delta_D \cot\theta\Big], \tag{6.55}$$

$$\beta = \frac{\sqrt{2}\,k}{4\Delta_D^2}\Big[(r-M)(\partial_\theta U + i\csc\theta\,\partial_\phi U) + \Delta_D \cot\theta\Big], \tag{6.56}$$

$$\gamma = 0, \quad \pi = \alpha + \bar{\beta}, \tag{6.57}$$

$$\varepsilon = -\frac{k^2}{2\,\Delta_D^3}(r-M)\Big[(r-M)^2\,\partial_r U - Mk\Big]. \tag{6.58}$$

Weyl-NP Scalars

The Weyl-NP scalars are given by

$$\Psi_4 = 0, \tag{6.59}$$

$$\begin{aligned}\Psi_0 = &\frac{k^4(r-M)^3}{2\,\Delta_D^5}\Big\{ \cot\theta\,\partial_\theta U - \partial_{\theta\theta} U + \csc\theta\,\partial_{\phi\phi} U + 2i\,\csc\theta\,(\cot\theta\,\partial_\phi U - \partial_{\theta\phi} U)\Big\} + \\ &\frac{5k^4(r-M)^4}{2\,\Delta_D^6}\Big\{ (\partial_\theta U)^2 - \csc^2\theta\,(\partial_\phi U)^2 + 2i\,\csc\theta\,\partial_\theta U\,\partial_\phi U \Big\},\end{aligned} \tag{6.60}$$

$$\begin{aligned}\Psi_1 = &\frac{\sqrt{2}k^3(r-M)}{8\Delta_D^5}\Big\{ -11(r-M)^3(\partial_r U)(\partial_\theta + i\,\csc\theta\,\partial_\phi)U \\ &+ 3\Big[r^3 - M^3 + 3rM^2U - 3r^2MU + rk(r-M)^2\Big](\partial_\theta + i\,\csc\theta\,\partial_\phi)(\partial_r U) \\ &+ \Big(15rMk + 8rMU - 11M^2k - 4r^2k - 4r^2U - 4M^2U\Big)(\partial_\theta + i\,\csc\theta\,\partial_\phi)U \Big\},\end{aligned} \tag{6.61}$$

$$\begin{aligned}\Psi_2 = &\frac{k^2(r-M)^2}{6\Delta_D^4}\Bigg\{ \frac{\Delta_D}{r-M}\Big[(\partial_{\theta\theta} + \csc^2\theta\,\partial_{\phi\phi} - 2(r-M)\Delta_D\partial_{rr}\Big]U \\ &+ 2(r-M)\Big[3(r-M)\partial_r U + U\Big]\partial_r U - \Big[(\partial_\theta U)^2 + \csc^2\theta\,(\partial_\phi U)^2\Big] \\ &+ \frac{\Delta_D}{r-M}\cot\theta\,\partial_\theta U + \frac{2k(r^3 + 13M^2r - 8r^2M - 6M^3)\partial_r U}{(r-M)^2} - \frac{6Mk(k+U)}{r-M}\Bigg\},\end{aligned} \tag{6.62}$$

$$\Psi_3 = \frac{\sqrt{2}\,k\,(r-M)}{4\Delta_D^3}\left\{\left[(r-M)U\partial_r U - \frac{Mk}{r-M} - \Delta_D\partial_r\right](\partial_\theta - i\,\csc\theta\,\partial_\phi)U\right\}. \qquad (6.63)$$

Ricci-NP Scalars

The Ricci-NP scalars are given by

$$\Phi_{20} = 0, \qquad (6.64)$$

$$\Phi_{22} = -\frac{1}{\Delta_D}\Big[(r-M)\partial_{rr}U + 2\partial_r U\Big], \qquad (6.65)$$

$$\begin{aligned}\Phi_{00} = \frac{k^4(r-M)^3}{\Delta_D^6}\Big\{ &-\Big[(r^3-M^3)U + r^3k\Big]\partial_{rr}U \\ &- 2\Delta_D\Big[(\partial_{\theta\theta} + \csc^2\theta\,\partial_{\phi\phi}) + (r-M)\partial_r + \cot\theta\,\partial_\theta\Big]U \\ &+(r-M)\Big[rM(k+3U)\partial_{rr}U + 6(\partial_\theta U)^2 + 6\csc^2\theta\,(\partial_\phi U)^2\Big]\Big\},\end{aligned} \qquad (6.66)$$

$$\begin{aligned}\Phi_{01} = \frac{\sqrt{2}k^3(r-M)^2}{8\Delta_D^5}\Big\{ &-5\Big[(r-M)^2\partial_r U - Mk\Big](\partial_\theta - i\,\csc\theta\,\partial_\phi)U \\ &+(r-M)\Delta_D(\partial_\theta - i\,\csc\theta\,\partial_\phi)(\partial_r U)\Big\},\end{aligned} \qquad (6.67)$$

$$\begin{aligned}\Phi_{11} = \frac{k^2(r-M)^2}{4\Delta_D^4}\Big\{ &-\frac{\Delta_D}{r-M}(\partial_\theta + \partial_{\theta\theta} + \csc^2\theta\,\partial_{\phi\phi})U - (r-M)\Delta_D\partial_{rr}U \\ &+2(r-M)(\partial_r U)\Big[(r-M)\partial_r U - U\Big] + (\partial_\theta U)^2 + \csc^2\theta\,(\partial_\phi U)^2\Big\} \\ -\frac{k^2}{4\Delta_D^4}\Big\{&U[(r-M)^4 + 4r^3M]\partial_{rr}U + 2k(r^3 + 2M^3 - 3M^2r)\partial_r U - 2M^2k^2\Big\},\end{aligned} \qquad (6.68)$$

$$\Phi_{12} = -\frac{\sqrt{2}\,k\,(r-M)}{4\Delta_D^3}\Big[(r-M)\partial_r U - \Delta_D\partial_r - \frac{Mk}{r-M}\Big]\Big(\partial_\theta U + i\csc\theta\,\partial_\phi\,U\Big), \qquad (6.69)$$

$$\begin{aligned}\Lambda = -\frac{k^2(r-M)}{12\Delta_D^4}\Big\{ &\Delta_D\Big[(r-M)^2\partial_{rr} + \partial_{\theta\theta} + \csc^2\theta\,\partial_{\phi\phi} + 2(r-M)\partial_r + \cot\theta\,\partial_\theta\Big]U \\ &-(r-M)[(\partial_\theta U)^2 + \csc^2\theta\,(\partial_\phi U)^2]\Big\}.\end{aligned} \qquad (6.70)$$

6.3.3 NP Quantities for Undistorted ERN Spacetime

The metric for an undistorted ERN black hole is

$$ds^2 = -\tilde{G}dv^2 + \tilde{G}^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 , \quad \text{where} \quad \tilde{G} = \left(1 - \frac{M}{r}\right)^2 . \tag{6.71}$$

Replace t with the ingoing null coordinate v such that $dt = dv + \dfrac{dr}{\tilde{G}}$, and we have

$$ds^2 = -\tilde{G}dv^2 + 2dvdr + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 . \tag{6.72}$$

Following the same procedure as Section 6.2 for distorted ERN spacetime, the null tetrad adapted to ingoing null radial geodesics is

$$\begin{aligned} l^a\partial_a &= \left(1, \frac{\tilde{G}}{2}, 0, 0\right), \quad n^a\partial_a = \left(0, -1, 0, 0\right), \\ l_a dx^a &= \left(-\frac{\tilde{G}}{2}, 1, 0, 0\right), \quad n_a dx^a = \left(-1, 0, 0, 0\right), \\ m^a\partial_a &= \frac{1}{\sqrt{2}}\left(0, 0, \frac{1}{r}, \frac{i}{r\sin\theta}\right), \quad m_a dx^a = \frac{1}{\sqrt{2}}\left(0, 0, r, i\sin\theta\right). \end{aligned} \tag{6.73}$$

In this tetrad, the NP quantities are given by

$$\kappa = \tau = \sigma = 0 , \quad \pi = \nu = \lambda = 0 , \quad \gamma = 0 ; \tag{6.74}$$

$$\rho = -\frac{(r-M)^2}{2r^3} , \tag{6.75}$$

$$\mu = -\frac{1}{r} , \tag{6.76}$$

$$\alpha = -\frac{\sqrt{2}\cot\theta}{4r} , \tag{6.77}$$

$$\beta = \frac{\sqrt{2}\cot\theta}{4r} , \tag{6.78}$$

$$\varepsilon = \frac{(r-M)M}{2r^3}; \tag{6.79}$$

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \tag{6.80}$$

$$\Psi_2 = -\frac{(r-M)M}{r^4}; \tag{6.81}$$

$$\Phi_{00} = \Phi_{10} = \Phi_{20} = \Phi_{12} = \Phi_{22} = \Lambda = 0, \tag{6.82}$$

$$\Phi_{11} = \frac{M^2}{2r^4}. \tag{6.83}$$

It is obvious that the distorted NP quantities in Section 6.3.2 would reduce to the corresponding undistorted quantities in the absence of distortion potential($U = 0$).

6.3.4 Distorted and Isolated ERN: Full Comparison

Compare the NP quantities in Section 6.3.3 and Section 6.3.4 on the hypersurface $r = M$, and we find that they match each other for the following quantities (where $\hat{=}$ *denotes equality on the surface* $r = M$),

$$\sigma = \tau = 0, \quad \nu = \lambda = \pi = 0, \quad \gamma = 0; \ \Psi_4 = 0, \quad \Phi_{20} = 0, \quad \pi = \alpha + \bar{\beta}, \tag{6.84}$$

$$\kappa \hat{=} 0, \quad \rho \hat{=} 0, \quad \varepsilon \hat{=} 0, \tag{6.85}$$

$$\alpha \hat{=} -\frac{\sqrt{2}\cot\theta}{4M}, \quad \beta \hat{=} \frac{\sqrt{2}\cot\theta}{4M}, \tag{6.86}$$

$$\Psi_0 \hat{=} 0, \quad \Psi_1 \hat{=} 0, \quad \Psi_2 \hat{=} 0, \quad \Phi_{00} \hat{=} 0, \quad \Phi_{01} \hat{=} 0, \quad \Phi_{20} = 0, \quad \Lambda \hat{=} 0, \tag{6.87}$$

while the only differences occur in n^a-relevant quantities $\{\mu, \Psi_3, \Phi_{11}, \Phi_{12}, \Phi_{22}\}$:

Quantity	Definition	Distorted ERN	Undistorted ERN
μ	$\mu := \bar{m}^a \delta n_a$	$-\dfrac{1}{M} - \dfrac{U}{kM}$	$-\dfrac{1}{M}$
Ψ_3	$\Psi_3 := C_{abcd} l^a n^b \bar{m}^c n^d$	$\dfrac{1}{2M^2} - \dfrac{U}{k^2}(\partial_{rr} - \partial_{\phi\phi})U$	0
Φ_{11}	$\Phi_{11} \hat{=} \dfrac{1}{2} R_{ab} l^a n^b$	$\dfrac{1}{2M^2} - \dfrac{U}{k^2}(\partial_{rr} - \partial_{\phi\phi})U$	$\dfrac{1}{2M^2}$
Φ_{12}	$\Phi_{12} := \dfrac{1}{2} R_{ab} \bar{m}^a n^b$	$\dfrac{\sqrt{2}(\partial_\theta + i \csc\theta\, \partial_\phi)U}{4kM^3}$	0
Φ_{22}	$\Phi_{22} := \dfrac{1}{2} R_{ab} n^a n^b$	$-\dfrac{2\partial_r U}{kM}$	0

where $\Phi_{11} := \frac{1}{4} R_{ab}(l^a n^b + m^a \bar{m}^b)$ is reduced to $\Phi_{11} \hat{=} \frac{1}{2} R_{ab} l^a n^b$ for electrovacuum ($\Lambda \hat{=} 0$ for $r = M$), c.f. Eq(2.64).

6.4 Summary

In this chapter, we proved that, for an ERN black hole under generic conformastatic distortion, the hypersurface $r = M$ is a marginally trapped surface ($\theta_{(l)} \hat{=} 0$), whose *intrinsic* structures, including the induced matric h_{AB}, spacetime connections (ℓ-relevant spin coefficients), electromagnetic field (ℓ-relevant Φ_{ij}) and curvature scalars (ℓ-relevant Ψ_i), coincide with those of the horizon of isolated ERN black holes. This is sufficient to conclude that, the intrinsic structures of ERN horizons cannot be distorted by conformastatic energy-matter distribution outside the horizon. Yet, as illustrated by the presence of U in $\{\mu, \Psi_3, \Phi_{11}, \Phi_{12}, \Phi_{22}\}$ at $r = M$, the distortion fields do influence the *extrinsic* structures of the horizon. These conclusions agree with the results expected from the local uniqueness theorem.

6.5 Addendum: Distortion of Nonextremal RN Spacetime

Weyl-Distorted Nonextremal RN Metric

The field equations governing the Weyl potentials $\{\psi(\rho, z), \gamma(\rho, z)\}$ in Eq(6.1) are[29][47]

$$\psi_{,\rho\rho} + \frac{1}{\rho}\psi_{,\rho} + \psi_{,zz} = -e^{-2\psi}(\Phi_{,\rho}^2 + \Phi_{,z}^2), \tag{6.88}$$

$$(\rho\, e^{-2\psi}\,\Phi_{,\rho})_{,\rho} + (\rho\, e^{-2\psi}\Phi_{,z})_{,z} = 0, \tag{6.89}$$

$$\gamma_{,\rho} = \rho\left(\psi_{,\rho}^2 - \psi_{,z}^2 + 2e^{-2\psi}(\Phi_{,\rho}^2 - \Phi_{,z}^2)\right), \tag{6.90}$$

$$\gamma_{,z} = 2\rho\left(\psi_{,\rho}\psi_{,z} - e^{-2\psi}\Phi_{,\rho}\Phi_{,z}\right), \tag{6.91}$$

$$\gamma_{,\rho\rho} + \gamma_{,zz} = \nabla_L^2\psi - (\psi_{,\rho}^2 + \psi_{,z}^2). \tag{6.92}$$

For vacuum spacetimes, $\Phi = 0$ and Eqs(6.88)-(6.92) are reduced into

$$\psi_{,\rho\rho} + \frac{1}{\rho}\psi_{,\rho} + \psi_{,zz} = 0, \tag{6.93}$$

$$\gamma_{,\rho} = \rho\left(\psi_{,\rho}^2 - \psi_{,z}^2\right), \tag{6.94}$$

$$\gamma_{,z} = 2\rho\,\psi_{,\rho}\psi_{,z}, \tag{6.95}$$

$$\gamma_{,\rho\rho} + \gamma_{,zz} = \nabla_L^2\psi - (\psi_{,\rho}^2 + \psi_{,z}^2). \tag{6.96}$$

The Weyl potentials $\{\psi(\rho, z), \gamma(\rho, z)\}$ for the RN metric are solutions to Eqs(6.88)-(6.92) rather than Eqs(6.93)-(6.96). Since Eq(6.88) is the nonlinear Poisson equation with nonvanishing electromagnetic field sources, superposition of given solutions no longer yields new solutions. Thus, in order to distort the RN metric, we need to *linearize* Poisson's equation Eq(6.88) by transforming it into Laplace's equation.

Suppose there exists a functional relationship $e^{2\psi} = f(\Phi)$ and it follows from Eq(6.89) that

$$\frac{f}{\rho}\Phi_{,\rho} - f_{,\Phi}\left[\left(\Phi^2_{,\rho}\right)^2 + \left(\Phi^2_{,z}\right)^2\right] + f\left(\Phi_{,\rho\rho} + \Phi_{,zz}\right) = 0\,. \tag{6.97}$$

Differentiate $e^{2\psi} = f(\Phi)$ twice with regard to ρ and z respectively and add them together, one obtains

$$2f\left(\psi_{,\rho\rho} + \psi_{,zz}\right) + 4f^2\left[\left(\psi^2_{,\rho}\right)^2 + \left(\psi^2_{,z}\right)^2\right] = f_{,\Phi\Phi}\left[\left(\Phi^2_{,\rho}\right)^2 + \left(\Phi^2_{,z}\right)^2\right] + f_{,\Phi}\left(\Phi_{,\rho\rho} + \Phi_{,zz}\right). \tag{6.98}$$

Insert Eq(6.88) and Eq(6.97) into Eq(6.98) and it implies that

$$\frac{d^2 f}{d\Phi^2} = 2\,. \tag{6.99}$$

Direct integration of this equation yields that $e^{2\psi} = \Phi^2 + \tilde{C}\Phi + B$. To resume asymptotic flatness at spatial infinity, we need that $\lim\limits_{r\to\infty} e^{2\psi} = -\lim\limits_{r\to\infty} g_{tt} = \lim\limits_{r\to\infty}(\Phi^2 + \tilde{C}\Phi + B) = 1$, and thus there should be $\lim\limits_{r\to\infty}\Phi = 0$ and $B = 1$. Also, replace the integral constant $\tilde{C}$ by $-2C$ for mathematical convenience in subsequent calculations (as in ref.[47]), thus,

$$e^{2\psi} = \Phi^2 - 2C\Phi + 1\,. \tag{6.100}$$

Now, introduce the linearized (uncharged) metric potential[47]

$$\Psi(\rho, z) = \int (\Phi^2 - 2C\Phi + 1)^{-1} d\Phi = \int e^{-2\psi} d\Phi, \tag{6.101}$$

and Eqs(6.88)-(6.92) are reduced to

$$\Psi_{,\rho\rho} + \frac{1}{\rho}\Psi_{,\rho} + \Psi_{,zz} = 0, \tag{6.102}$$

$$\gamma_{,\rho} = (C^2 - 1)\rho\left(\Psi^2_{,\rho} + \Psi^2_{,z}\right), \tag{6.103}$$

$$\gamma_{,z} = (C^2 - 1)\, 2\rho\,\Psi_{,\rho}\Psi_{,z}, \tag{6.104}$$

$$\gamma_{,\rho\rho} + \gamma_{,zz} = (C^2 - 1)\left\{\nabla^2_L\Psi - (\Psi^2_{,\rho} - \Psi^2_{,z})\right\}. \tag{6.105}$$

One could seek for $\Psi(\rho, z)$ instead of $\psi(\rho, z)$ by solving Laplace's equation Eq(6.102), and integrate for $\gamma(\rho, z)$ via Eq(6.103). The potentials $\psi(\rho, z)$ can be retrieved from $\Psi(\rho, z)$ by[47]

$$e^{-\psi} = \frac{1}{2}\Big(1 + \frac{C}{\sqrt{C^2-1}}\Big)e^{-\sqrt{C^2-1}\Psi} + \frac{1}{2}\Big(1 - \frac{C}{\sqrt{C^2-1}}\Big)e^{\sqrt{C^2-1}\Psi}, \quad \text{where} C = \frac{M}{|Q|}. \tag{6.106}$$

For the nonextremal RN metric, we have

$$\Psi_{RN} = \frac{1}{2}\Big(\frac{M^2}{Q^2} - 1\Big)^{-\frac{1}{2}} \ln\frac{L - \sqrt{M^2 - Q^2}}{L + \sqrt{M^2 - Q^2}}, \tag{6.107}$$

$$\psi_{RN} = \frac{1}{2}\ln\frac{L^2 - (M^2 - Q^2)}{(L + M)^2}, \qquad \gamma_{RN} = \frac{1}{2}\ln\frac{L^2 - (M^2 - Q^2)}{l_+ l_-}, \tag{6.108}$$

Using the method of superposition by adding another solution Ψ_D to Eq(6.102) which is *regular at the outer RN radius*, then the potential $\Psi_{RN}+\Psi_D$ will yield a distorted RN solution $\psi_{RN} + \psi_D$, where ψ_D is related to Ψ_D via

$$e^{-\psi_{RN}-\psi_D} = \frac{1}{2}\Big(1 + \frac{C}{\sqrt{C^2-1}}\Big)e^{-\sqrt{C^2-1}(\Psi_{RN}+\Psi_D)} + \frac{1}{2}\Big(1 - \frac{C}{\sqrt{C^2-1}}\Big)e^{\sqrt{C^2-1}(\Psi_{RN}+\Psi_D)}. \tag{6.109}$$

Thus, one *formally* obtains the distorted metric (omitting the subscript D)

$$ds^2 = -e^{2\psi}\frac{L^2 - (M^2 - Q^2)}{(L+M)^2}dt^2 + e^{2\gamma-2\psi}\frac{(L+M)^2}{l_+ l_-}(d\rho^2 + dz^2) + e^{-2\psi}\frac{(L+M)^2}{L^2 - (M^2 - Q^2)}\rho^2 d\phi^2 , \tag{6.110}$$

which can be transformed into the usual $\{t, r, \theta, \phi\}$ coordinates via Eq(6.10),

$$ds^2 = -e^{2\psi}\Big(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\Big)dt^2 + e^{2\gamma-2\psi}\Big\{\Big(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\Big)^{-1}dr^2 + r^2 d\theta^2\Big\} + e^{-2\psi}r^2\sin^2\theta\, d\phi^2 . \tag{6.111}$$

As we can see, the *whole* distortion process is actually highly nonlinear, and Eq(6.106) or Eq(6.109) is not applicable to the ERN solution ($|C| = 1$).

Spin Coefficients for Weyl-Distorted Nonextremal RN

The Lagrangian for Weyl-distorted nonextremal RN spacetime is

$$\mathscr{L} = \frac{1}{2}\Big\{-H(r)\,e^{2\psi}\dot{t}^2 + e^{-2\psi+2\gamma}\Big(H(r)^{-1}\dot{r}^2 + r^2\dot{\theta}^2\Big) + e^{-2\psi}r^2\sin^2\theta\,\dot{\phi}^2\Big\} . \tag{6.112}$$

where $H(r) := 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$. There are two constants of motion,

$$p_t = He^{2\psi}\dot{t} := E_0 , \quad p_\phi = r^2\sin^2\theta\, e^{-2\psi}\dot{\phi} := J_0 , \tag{6.113}$$

and the tangent vector field for null radial geodesics ($\mathscr{L} = 0$, $\dot{\theta} = 0$ and $\dot{\phi} = 0$) is given by (the method is similar to the calculations of Kerr-Newman-family black holes using NP formalism in ref.[19])

$$\dot{t} = \frac{1}{He^{2\psi}} , \quad \dot{r} = \pm e^{-\gamma} , \quad \dot{\theta} = 0 , \quad \dot{\phi} = 0 . \tag{6.114}$$

Set up the tetrad for null an *ingoing* observer,

$$
\begin{aligned}
l^a &= \frac{1}{\sqrt{2}}(1, He^{2\psi-\gamma}, 0, 0)\,, \quad n^a = \frac{1}{\sqrt{2}}(H^{-1}e^{-2\psi}, -e^{-\gamma}, 0, 0) \\
m^a &= \frac{1}{\sqrt{2}}(0, 0, r^{-1}e^{\psi-\gamma}, i(r\sin\theta)^{-1}e^{\psi})\,, \quad \bar{m}^a = \frac{1}{\sqrt{2}}(0, 0, r^{-1}e^{\psi-\gamma}, -i(r\sin\theta)^{-1}e^{\psi})\,,
\end{aligned}
\tag{6.115}
$$

$$
\begin{aligned}
l_a &= \frac{1}{\sqrt{2}}(-He^{2\psi}, e^{\gamma}, 0, 0)\,, \quad n_a = \frac{1}{\sqrt{2}}(-1, -H^{-1}e^{-2\psi+\gamma}, 0, 0) \\
m_a &= \frac{1}{\sqrt{2}}(0, 0, re^{-\psi+\gamma}, ir\sin\theta\, e^{-\psi})\,, \quad \bar{m}_a = \frac{1}{\sqrt{2}}(0, 0, re^{-\psi+\gamma}, -ir\sin\theta\, e^{-\psi})\,.
\end{aligned}
\tag{6.116}
$$

and the spin coefficients are given by

$$\kappa = -\frac{1}{2\sqrt{2}}r^{-1}He^{3\psi-\gamma}\,(2\partial_\theta\psi - \partial_\theta\gamma) \tag{6.117}$$

$$\rho = -\frac{1}{2\sqrt{2}}He^{2\psi-\gamma}\,(2r^{-1} - 2\partial_r\psi + \partial_r\gamma) \tag{6.118}$$

$$\sigma = -\frac{1}{2\sqrt{2}}He^{2\psi-\gamma}\partial_r\gamma \tag{6.119}$$

$$\tau = -\frac{1}{2\sqrt{2}}r^{-1}e^{\psi-\gamma}\partial_\theta\gamma \tag{6.120}$$

$$\nu = \frac{1}{2\sqrt{2}}r^{-1}H^{-1}e^{-\psi-\gamma}\,(2\partial_\theta\psi - \partial_\theta\gamma) \tag{6.121}$$

$$\mu = -\frac{1}{2\sqrt{2}}e^{-\gamma}(2r^{-1} - 2\partial_r\psi + \partial_r\gamma) \tag{6.122}$$

$$\lambda = -\frac{1}{2\sqrt{2}}e^{-\gamma}\partial_r\gamma \tag{6.123}$$

$$\pi = \frac{1}{2\sqrt{2}}r^{-1}e^{\psi-\gamma}\partial_\theta\gamma \tag{6.124}$$

$$\alpha = \frac{1}{2\sqrt{2}}r^{-1}e^{\psi-\gamma}\,\partial_\theta\psi \tag{6.125}$$

$$\beta = \frac{1}{2\sqrt{2}}r^{-1}e^{\psi-\gamma}\,\partial_\theta\psi \tag{6.126}$$

$$\gamma = \frac{1}{4\sqrt{2}}e^{-\gamma}\Big\{\Big(\frac{M}{r^2} - \frac{Q^2}{r^3}\Big)H^{-1} + \partial_r\psi\Big\} \tag{6.127}$$

$$\varepsilon = \frac{1}{2\sqrt{2}}He^{2\psi-\gamma}\Big\{2\Big(\frac{M}{r^2} - \frac{Q^2}{r^3}\Big)H^{-1} + 2\partial_r\psi - \partial_r\gamma\Big\}\,. \tag{6.128}$$

Chapter 7

Conclusions and Prospective Study

In this thesis, two principal goals have been achieved based on the local uniqueness solutions $\{P, U, B, \phi_1\}$ in ref.[14]:

(I) A *local* method is developed to reconstruct the near-horizon metric of an axisymmetric extremal IH embedded in electrovacuum, and we verified that $\{P, U, B, \phi_1\}$ do represent the structures of extremal Kerr-Newman horizons;

(II) *Booth's conjecture* which is implied from the local uniqueness theorem is partly examined, and we found that the intrinsic structure of the horizon of an extremal Reissner-Nordström black hole cannot be distorted by *conformastatic* energy-matter distribution in the exterior.

In prospective investigations following this thesis, we will try to figure out the answers of the following three problems:

I. Equivalence of Generic Extremal Kerr-Newman NHMs

In Chapter 5, we explicitly calculated the NHMs of two special types of axisymmetric extremal IHs embedded in electrovacuum: (i) static and electrically charged with $\{\alpha = 1, M = 4\pi\}$; (ii) rotating and electrically neutral with $\{\alpha = 0, M = 8\pi\}$. We proved that, the former matches the NHM of extremal RN horizons, while the later is equivalent to the NHM of extremal Kerr horizons. To confirm that the local uniqueness solutions do describe extremal Kerr-Newman horizons, we still need to verify the generic case with $\{0 < \alpha^2 < 1, M = 4\pi(2 - \alpha^2)\}$.

From the global, extremal Kerr-Newman black holes ($r_+^2 = M^2 + Q^2$) are described by[27]

$$ds^2 = -\Big(1 - \frac{2Mr - Q^2}{\rho_{KN}}\Big)dt^2 - \frac{2a\sin^2\theta\,(2Mr - Q^2)}{\rho_{KN}}dtd\phi + \rho_{KN}\Big(\frac{dr^2}{\Delta_{KN}} + d\theta^2\Big) + \frac{\Sigma^2}{\rho_{KN}}d\phi^2, \quad (7.1)$$

where

$$\Delta_{KN} := r^2 - 2Mr + a^2 + Q^2\,,\quad \rho_{KN} := r^2 + a^2\cos^2\theta\,,\quad \Sigma^2 := (r^2 + a^2)^2 - \Delta_{KN}a^2\sin^2\theta\,. \quad (7.2)$$

Taking the near-horizon transformation

$$t \mapsto \frac{\tilde{t}}{\epsilon}\,,\quad r \mapsto M + \epsilon\tilde{r}\,,\quad \phi \mapsto \tilde{\phi} + \frac{a}{r_0^2\epsilon}\tilde{t}\,,\quad \epsilon \to 0\,,\quad \big(r_0^2 := M^2 + a^2\big) \quad (7.3)$$

and omitting the tildes, one obtains the NHM (we follow the denotations used in ref.[4])

$$ds^2 \simeq \Big(1 - \frac{a^2}{r_0^2}\sin^2\theta\Big)\Big(-\frac{r^2}{r_0^2}dt^2 + \frac{r_0^2}{r^2}dr^2 + r_0^2 d\theta^2\Big) + r_0^2\sin^2\theta\Big(1 - \frac{a^2}{r_0^2}\sin^2\theta\Big)^{-1}\Big(d\phi + \frac{2arM}{r_0^4}dt\Big)^{-1}. \quad (7.4)$$

From the local perspective, however, the NHM

$$ds^2 \simeq -F\, r^2 dv^2 + 2dvdr + \hat{h}_{xx}\Big(dx - \frac{2\hat{\omega}_x}{\hat{h}_{xx}}\, r dv\Big)^2 + \hat{h}_{\varphi\varphi}\Big(d\varphi - \frac{2\hat{\omega}_\varphi}{\hat{h}_{\varphi\varphi}}\, r dv\Big)^2 \tag{7.5}$$

for the generic case $\{0 < \alpha^2 < 1,\ M = 4\pi(2 - \alpha^2)\}$ will explicitly manifest itself in an extremely complicated and unreadable form, which makes it difficult to find out the equivalence transformation.

Our proposal is, we should at least verify the equivalence between Eq(7.4) and Eq(7.5) for some particular yet random values $\breve{\alpha}^2$ of $0 < \alpha^2 < 1$. For example, with $\alpha^2 = \frac{1}{2}$, we have the local constructions for Eq(7.5) that

$$h_{AB}:\quad ds^2 = \frac{2(27 + 16x^2)}{3(9 - 16x^2)}\, dx^2 + \frac{6(9 - 16x^2)}{27 + 16x^2}\, d\varphi^2\,, \tag{7.6}$$

$$\pi^{(0)}\overline{\pi^{(0)}} = \frac{12(9 - 16x^2)}{(27 + 16x^2)^2}\,,\quad K = \frac{2592(9 - 16x^2)}{(27 + 16x^2)^3}\,,\quad \Phi_{11} = \frac{108}{(27 + 16x^2)^2}\,, \tag{7.7}$$

$$\hat{\omega}_A = \frac{16x}{27 + 16x^2} dx \pm \frac{36\sqrt{3}(9 - 16x^2)}{(27 + 16x^2)^3}\, d\varphi\,, \tag{7.8}$$

$$\begin{aligned} \Rightarrow\quad ds^2 \simeq\ & -\frac{72(27 - 16x^2)(3 + 16x^2)}{(27 + 16x^2)^3} r^2 dv^2 + 2dvdr \\ & + \frac{2(27 + 16x^2)}{3(9 - 16x^2)}\left[dx - \frac{48x(9 - 16x^2)}{(27 + 16x^2)^2} r dv\right]^2 + \frac{6(9 - 16x^2)}{27 + 16x^2}\left[d\varphi - \frac{12\sqrt{3}}{(27 + 16x^2)^2}\right]^2, \end{aligned} \tag{7.9}$$

while Eq(7.4) becomes(with $M = 1$)

$$ds^2 \simeq \Big(1 - \frac{1}{3}\sin^2\theta\Big)\Big(-\frac{2}{3}dt^2 + \frac{3}{2r^2}dr^2 + \frac{3}{2}d\theta^2\Big) + \frac{3}{2}\sin^2\theta\Big(1 - \frac{1}{3}\sin^2\theta\Big)^{-1}\Big(d\phi + \frac{4\sqrt{2}}{9}dt\Big)^{-1}, \tag{7.10}$$

and the two NHMs Eq(7.9) and Eq(7.10) should be equivalent via appropriate coordinate transformations.

Besides near-horizon metrics, we are also interested in a quasilocal characterization of generic near-horizon geometry and topology of extremal IHs. We believe that the advanced discussion in refs.[54]-[56] from *global* perspectives would shed light on this goal.

II. Unified Distortion of Extremal and Nonextremal RN Horizons

For a Weyl-distorted nonextremal RN black hole, the spacetime metric is (c.f. Section 6.5)

$$ds^2 = -H(r)\, e^{2\psi} dt^2 + e^{-2\psi}\Big\{ e^{2\gamma}\Big(H(r)^{-1} dr^2 + r^2 d\theta^2\Big) + r^2 \sin^2\theta\, d\phi^2 \Big\}\,, \tag{7.11}$$

where $H(r) := 1 - \dfrac{2M}{r} + \dfrac{Q^2}{r^2}$, and therefore the induced metric for the outer horizon $r = r_+ := r + \sqrt{M^2 - Q^2}$ is

$$h_{AB} = \underleftarrow{g_{ab}}: \quad ds^2 \hat{=} e^{2\psi - 4\bar{u}}\, r_+^2 d\theta^2 \ + e^{-2\psi}\, r_+^2 \sin^2\theta\, d\phi^2\,, \tag{7.12}$$

where $\gamma(r_+, \theta) \hat{=} 2\psi(r_+, \theta) - 2\bar{u}$ on the horizon[48]. h_{AB} yields the Gaussian curvature that

$$K \hat{=} \frac{e^{4\bar{u} - 2\psi}}{r_+^2}\Big(\partial_{\theta\theta}\psi + 3\cot\theta\, \partial_\theta\psi - 2(\partial_\theta\psi)^2 + 1\Big)\,. \tag{7.13}$$

as opposed to $K = \dfrac{1}{r_+^2}$ for isolated RN. Hence, contrary to ERN horizons, intrinsic structure of the nonextremal outer horizon is distorted by Weyl-type external sources. This result is expected because the Schwarzschild horizon can also be considerably distorted by Weyl sources, as shown in ref.[57].

Thus, distortion happens on nonextremal RN horizons and ceases for extremal RN horizons. There might exist a smooth transition connecting these two situations, and we will look for a unified treatment distortion of the extremal and nonextremal RN solutions. Considering that Eq(6.106) has an effective near-extremal ($C \to 1$) limit,

$$\lim_{C\to 1}\left\{\frac{1}{2}\Big(1+\frac{C}{\sqrt{C^2-1}}\Big)e^{-\sqrt{C^2-1}\,\Psi}+\frac{1}{2}\Big(1-\frac{C}{\sqrt{C^2-1}}\Big)e^{\sqrt{C^2-1}\,\Psi}\right\}=\Psi-1\,, \tag{7.14}$$

we suspect that a unified description might still be realized within the framework of Weyl distortion.

III. Further Investigation of Distortion of Extremal IHs

The *distortibility* of extremal Kerr-Newman horizons is quite an important problem for our understanding of the geometry and mechanics of extremal black holes, so we will explicitly calculate more examples of Kerr-Newman horizons exposed in different kinds of distortion fields. Hopefully, we could finally find a rigorous mathematical proof from either the quasilocal or global approach for the distortibility problem.

Moreover, the local uniqueness theorem requires the horizon to be axisymmetric. Thus, we also hope to find out whether the intrinsic structure of a *generic* (not necessarily axisymmetric) extremal horizon can be distorted or not.

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Appendices

Appendix A.

Sign Convention

There are two kinds of sign conventions for spacetime metrics in general relativity, i.e. $(-, +, +, +)$ and $(+, -, -, -)$. When one switches from one signature to the other, the signs of some typical quantities have the behaviors listed below.

Metric tensor and its inverse	g_{ab} , g^{ab}	Changing sign
Christoffel symbols of first kind	$\Gamma_{abc} = \frac{1}{2}\left(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc}\right)$	Changing sign
Christoffel symbols of second kind	$\Gamma^a_{bc} = \frac{1}{2} g^{ad}\left(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc}\right)$	No change
Riemann tensor	$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed}$	No change
Ricci tensor	$R_{ab} = R^c{}_{acb}$	No change
Scalar curvature	$R = g^{ab} R_{ab}$	Changing sign
Einstein tensor	$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$	No change
Energy-momentum tensor	$8\pi G T_{ab} = G_{ab}$	No change
Cosmological constant	$R_{ab} - \frac{1}{2} g_{ab} R + g_{ab}\Lambda = G_{ab}$	Changing sign

Throughout this thesis, the convention $(-, +, +, +)$ has been employed.

Appendix B.

Newman-Penrose Formalism

B..1 Null Tetrad, NP Quantities and Tetrad Equations

The light cone is one of the most fundamental *local* structures at a spacetime point; in this spirit, a special kind of tetrad (moving frame) called *complex null tetrad* can be constructed. Such a tetrad contains two real null (co)vectors $\{ l, n\}$ and two complex null (co)vectors $\{ m, \bar{m}\}$. Being null (co)vectors, the self-normalizations are vanishing, $l_a l^a = n_a n^a = m_a m^a = \bar{m}_a \bar{m}^a = 0$, so the following two pairs of *cross*-normalizations are adopted

$$l_a n^a = -1 = l^a n_a , \quad m_a \bar{m}^a = 1 = m^a \bar{m}_a, \tag{B..1}$$

while contractions between the two pairs also vanish, $l_a m^a = l_a \bar{m}^a = n_a m^a = n_a \bar{m}^a = 0$. Here the indices can be raised and lowered by the global metric g_{ab} which in turn can be obtained via

$$g_{ab} = -l_a n_b - n_a l_b + m_a \bar{m}_b + \bar{m}_a m_b , \quad g^{ab} = -l^a n^b - n^a l^b + m^a \bar{m}^b + \bar{m}^a m^b , \tag{B..2}$$

There are four directional covariant derivatives along with each null tetrad vector,

$$D := \nabla_{l} = l^a \nabla_a \,, \; \Delta := \nabla_{\mathbf{n}} = n^a \nabla_a \,, \; \delta := \nabla_{\mathbf{m}} = m^a \nabla_a \,, \; \bar{\delta} := \nabla_{\bar{\mathbf{m}}} = \bar{m}^a \nabla_a \,, \tag{B..3}$$

which are reduced to $\{D = l^a \partial_a \,, \Delta = n^a \partial_a \,, \delta = m^a \partial_a \,, \bar{\delta} = \bar{m}^a \partial_a\}$ when acting on scalar functions. Instead of using index notations as in orthogonal tetrads, each Ricci rotation coefficient in the null tetrad is assigned a lower-case Greek letter, which constitute the 12 complex spin coefficients,

$$\begin{aligned} \kappa &:= -m^a D l_a = -m^a l^b \nabla_b l_a \,, \quad \tau := -m^a \Delta l_a = -m^a n^b \nabla_b l_a \,, \\ \sigma &:= -m^a \delta l_a = -m^a m^b \nabla_b l_a \,, \quad \rho := -m^a \bar{\delta} l_a = -m^a \bar{m}^b \nabla_b l_a \,; \end{aligned} \tag{B..4a}$$

$$\begin{aligned} \pi &:= \bar{m}^a D n_a = \bar{m}^a l^b \nabla_b n_a \,, \quad \nu := \bar{m}^a \Delta n_a = \bar{m}^a n^b \nabla_b n_a \,, \\ \mu &:= \bar{m}^a \delta n_a = \bar{m}^a m^b \nabla_b n_a \,, \quad \lambda := \bar{m}^a \bar{\delta} n_a = \bar{m}^a \bar{m}^b \nabla_b n_a \,; \end{aligned} \tag{B..4b}$$

$$\begin{aligned} -\varepsilon &:= \frac{1}{2}(n^a D l_a - \bar{m}^a D m_a) = \frac{1}{2}(n^a l^b \nabla_b l_a - \bar{m}^a l^b \nabla_b m_a) \,, \\ -\gamma &:= \frac{1}{2}(n^a \Delta l_a - \bar{m}^a \Delta m_a) = \frac{1}{2}(n^a n^b \nabla_b l_a - \bar{m}^a n^b \nabla_b m_a) \,, \\ -\beta &:= \frac{1}{2}(n^a \delta l_a - \bar{m}^a \delta m_a) = \frac{1}{2}(n^a m^b \nabla_b l_a - \bar{m}^a m^b \nabla_b m_a) \,, \\ -\alpha &:= \frac{1}{2}(n^a \bar{\delta} l_a - \bar{m}^a \bar{\delta} m_a) = \frac{1}{2}(n^a \bar{m}^b \nabla_b l_a - \bar{m}^a \bar{m}^b \nabla_b m_a) \,. \end{aligned} \tag{B..4c}$$

Applying the directional derivative operators to tetrad vectors yields the transportation equations

$$\begin{aligned} D l^a &= (\varepsilon + \bar{\varepsilon}) l^a - \bar{\kappa} m^a - \kappa \bar{m}^a \,, \quad \Delta l^a = (\gamma + \bar{\gamma}) l^a - \bar{\tau} m^a - \tau \bar{m}^a \,, \\ \delta l^a &= (\bar{\alpha} + \beta) l^a - \bar{\rho} m^a - \sigma \bar{m}^a \,, \quad \bar{\delta} l^a = (\alpha + \bar{\beta}) l^a - \bar{\sigma} m^a - \rho \bar{m}^a \,; \end{aligned} \tag{B..5a}$$

$$Dn^a = \pi m^a + \bar{\pi}\bar{m}^a - (\varepsilon + \bar{\varepsilon})n^a\,, \quad \Delta n^a = \nu m^a + \bar{\nu}\bar{m}^a - (\gamma + \bar{\gamma})n^a\,,$$
$$\delta n^a = \mu m^a + \bar{\lambda}\bar{m}^a - (\bar{\alpha} + \beta)n^a\,, \quad \bar{\delta} n^a = \lambda m^a + \bar{\mu}\bar{m}^a - (\alpha + \bar{\beta})n^a\,; \tag{B..5b}$$

$$Dm^a = (\varepsilon - \bar{\varepsilon})m^a + \bar{\pi}l^a - \kappa n^a\,, \quad \Delta m^a = (\gamma - \bar{\gamma})m^a + \bar{\nu}l^a - \tau n^a\,,$$
$$\delta m^a = (\beta - \bar{\alpha})m^a + \bar{\lambda}l^a - \sigma n^a\,, \quad \bar{\delta} m^a = (\alpha - \bar{\beta})m^a + \bar{\mu}l^a - \rho n^a\,; \tag{B..5c}$$

$$D\bar{m}^a = (\bar{\varepsilon} - \varepsilon)\bar{m}^a + \pi l^a - \bar{\kappa} n^a\,, \quad \Delta \bar{m}^a = (\gamma - \bar{\gamma})\bar{m}^a + \nu l^a - \bar{\tau} n^a\,,$$
$$\delta \bar{m}^a = (\beta - \bar{\alpha})\bar{m}^a + \mu l^a - \bar{\rho} n^a\,, \quad \bar{\delta} \bar{m}^a = (\alpha - \bar{\beta})\bar{m}^a + \lambda l^a - \bar{\sigma} n^a\,. \tag{B..5d}$$

The metric-compatibility or twist-freeness of the covariant derivative is recast into the commutators of the directional derivatives,

$$\begin{aligned}
\Delta D - D\Delta &= (\gamma + \bar{\gamma})D + (\varepsilon + \bar{\varepsilon})\Delta - (\bar{\tau} + \pi)\delta - (\tau + \bar{\pi})\bar{\delta}\,,\\
\delta D - D\delta &= (\bar{\alpha} + \beta - \bar{\pi})D + \kappa\Delta - (\bar{\rho} + \varepsilon - \bar{\varepsilon})\delta - \sigma\bar{\delta}\,,\\
\delta\Delta - \Delta\delta &= -\bar{\nu}D + (\tau - \bar{\alpha} - \beta)\Delta + (\mu - \gamma + \bar{\gamma})\delta + \bar{\lambda}\bar{\delta}\,,\\
\bar{\delta}\delta - \delta\bar{\delta} &= (\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta + (\alpha - \bar{\beta})\delta - (\bar{\alpha} - \beta)\bar{\delta}\,.
\end{aligned} \tag{B..6}$$

The 10 independent components of Weyl's tensor can be encoded into 5 complex Weyl-NP scalars,

$$\Psi_0 := C_{abcd}l^a m^b l^c m^d\,, \quad \Psi_1 := C_{abcd}l^a n^b l^c m^d\,, \quad \Psi_2 := C_{abcd}l^a m^b \bar{m}^c n^d\,,$$
$$\Psi_3 := C_{abcd}l^a n^b \bar{m}^c n^d\,, \quad \Psi_4 := C_{abcd}n^a \bar{m}^b n^c \bar{m}^d\,. \tag{B..7}$$

The 10 independent components of the Ricci tensor are encoded into 4 *real* scalars $\{\Phi_{00}, \Phi_{11}, \Phi_{22}, \Lambda\}$ and 3 *complex* scalars $\{\Phi_{10}, \Phi_{20}, \Phi_{21}\}$ (with their complex conjugates),

$$\Phi_{00} := \frac{1}{2}R_{ab}l^a l^b\,, \quad \Phi_{11} := \frac{1}{4}R_{ab}(\,l^a n^b + m^a \bar{m}^b)\,, \quad \Phi_{22} := \frac{1}{2}R_{ab}n^a n^b\,, \quad \Lambda := \frac{R}{24}\,; \tag{B..8a}$$

$$\Phi_{01} := \frac{1}{2}R_{ab}l^a m^b\,, \qquad \Phi_{10} := \frac{1}{2}R_{ab}l^a \bar{m}^b = \overline{\Phi_{01}}\,,$$
$$\Phi_{02} := \frac{1}{2}R_{ab}m^a m^b\,, \qquad \Phi_{20} := \frac{1}{2}R_{ab}\bar{m}^a \bar{m}^b = \overline{\Phi_{02}}\,, \qquad \text{(B..8b)}$$
$$\Phi_{12} := \frac{1}{2}R_{ab}\bar{m}^a n^b\,, \qquad \Phi_{21} := \frac{1}{2}R_{ab}m^a n^b = \overline{\Phi_{12}}\,;$$

in these definitions, R_{ab} could be replaced by its trace-free part $Q_{ab} = R_{ab} - \frac{1}{4}g_{ab}R$ (as in Appendix E of ref.[27]) or by the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$ because of the normalization relations around Eq(B..1). Also, Φ_{11} is reduced to $\Phi_{11} = \frac{1}{2}R_{ab}l^a n^b = \frac{1}{2}R_{ab}m^a \bar{m}^a$ for electrovacuum ($\Lambda = 0$) (c.f. Eq(2.64) in Chapter 2).

The first systematic formulation of the complex null tetrad method attributes to Newman and Penrose (NP)[18]. Compared with ref.[18], in this appendix we adopt the signature $(-,+,+,+)$ rather than $(+,-,-,-)$, and follow Eq(B..1) rather than $\{l^a n_a = 1\,, m^a \bar{m}_a = -1\}$, as used in Chapter 2 of ref.[29] and Appendix E of ref.[27], because this is the usual choice nowadays in studying trapping null surfaces. The standard NP formalism in the original sign convention can be found in refs.[18][19].

B..2 Einstein-Maxwell-NP Equations

In an orthonormal tetrad, Einstein-Maxwell equations are locally reexpressed by Cartan's first and second structure equations, while in a complex null tetrad, the dynamical equations are the Newman-Penrose equations,

$$D\rho - \bar{\delta}\kappa = (\rho^2 + \sigma\bar{\sigma}) + (\varepsilon + \bar{\varepsilon})\rho - \bar{\kappa}\tau - \kappa(3\alpha + \bar{\beta} - \pi) + \Phi_{00}\,, \tag{B..9}$$

$$D\sigma - \delta\kappa = (\rho + \bar{\rho})\sigma + (3\varepsilon - \bar{\varepsilon})\sigma - (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)\kappa + \Psi_0\,, \tag{B..10}$$

$$D\tau - \Delta\kappa = (\tau + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma + (\varepsilon - \bar{\varepsilon})\tau - (3\gamma + \bar{\gamma})\kappa + \Psi_1 + \Phi_{01}\,, \tag{B..11}$$

$$D\alpha - \bar{\delta}\varepsilon = (\rho + \bar{\varepsilon} - 2\varepsilon)\alpha + \beta\bar{\sigma} - \bar{\beta}\varepsilon - \kappa\lambda - \bar{\kappa}\gamma + (\varepsilon + \rho)\pi + \Phi_{10}\,, \tag{B..12}$$

$$D\beta - \delta\varepsilon = (\alpha + \pi)\sigma + (\bar{\rho} - \bar{\varepsilon})\beta - (\mu + \gamma)\kappa - (\bar{\alpha} - \bar{\pi})\varepsilon + \Psi_1\,, \tag{B..13}$$

$$D\gamma - \Delta\varepsilon = (\tau + \bar{\pi})\alpha + (\bar{\tau} + \pi)\beta - (\varepsilon + \bar{\varepsilon})\gamma - (\gamma + \bar{\gamma})\varepsilon + \tau\pi - \nu\kappa + \Psi_2 + \Phi_{11} - \Lambda\,, \tag{B..14}$$

$$D\lambda - \bar{\delta}\pi = (\rho\lambda + \bar{\sigma}\mu) + \pi^2 + (\alpha - \bar{\beta})\pi - \nu\bar{\kappa} - (3\varepsilon - \bar{\varepsilon})\lambda + \Phi_{20}\,, \tag{B..15}$$

$$D\mu - \delta\pi = (\bar{\rho}\mu + \sigma\lambda) + \pi\bar{\pi} - (\varepsilon + \bar{\varepsilon})\mu - (\bar{\alpha} - \beta)\pi - \nu\kappa + \Psi_2 + 2\Lambda\,, \tag{B..16}$$

$$D\nu - \Delta\pi = (\pi + \bar{\tau})\mu + (\bar{\pi} + \tau)\lambda + (\gamma - \bar{\gamma})\pi - (3\varepsilon + \bar{\varepsilon})\nu + \Psi_3 + \Phi_{21}\,, \tag{B..17}$$

$$\Delta\lambda - \delta\nu = -(\mu + \bar{\mu})\lambda - (3\gamma - \bar{\gamma})\lambda + (3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu - \Psi_4\,, \tag{B..18}$$

$$\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha} + \beta) - \sigma(3\alpha - \bar{\beta}) + (\rho - \bar{\rho})\tau + (\mu - \bar{\mu})\kappa - \Psi_1 + \Phi_{01}\,, \tag{B..19}$$

$$\delta\alpha - \bar{\delta}\beta = (\mu\rho - \lambda\sigma) + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \gamma(\rho - \bar{\rho}) + \varepsilon(\mu - \bar{\mu}) - \Psi_2 + \Phi_{11} + \Lambda\,, \tag{B..20}$$

$$\delta\lambda - \bar{\delta}\mu = (\rho - \bar{\rho})\nu + (\mu - \bar{\mu})\pi + (\alpha + \bar{\beta})\mu + (\bar{\alpha} - 3\beta) - \Psi_3 + \Phi_{21}\,, \tag{B..21}$$

$$\delta\nu - \Delta\mu = (\mu^2 + \lambda\bar{\lambda}) + (\gamma + \bar{\gamma})\mu - \bar{\nu}\pi + (\tau - 3\beta - \bar{\alpha})\nu + \Phi_{22}\,, \tag{B..22}$$

$$\delta\gamma - \Delta\beta = (\tau - \bar{\alpha} - \beta)\gamma + \mu\tau - \sigma\nu - \varepsilon\bar{\nu} - (\gamma - \bar{\gamma} - \mu)\beta + \alpha\bar{\lambda} + \Phi_{12}\,, \tag{B..23}$$

$$\delta\tau - \Delta\sigma = (\mu\sigma + \bar{\lambda}\rho) + (\tau + \beta - \bar{\alpha})\tau - (3\gamma - \bar{\gamma})\sigma - \kappa\bar{\nu} + \Phi_{02}\,, \tag{B..24}$$

$$\Delta\rho - \bar{\delta}\tau = -(\rho\bar{\mu} + \sigma\lambda) + (\bar{\beta} - \alpha - \bar{\tau})\tau + (\gamma + \bar{\gamma})\rho + \nu\kappa - \Psi_2 - 2\Lambda\,, \tag{B..25}$$

$$\Delta\alpha - \bar{\delta}\gamma = (\rho + \varepsilon)\nu - (\tau + \beta)\lambda + (\bar{\gamma} - \bar{\mu})\alpha + (\bar{\beta} - \bar{\tau})\gamma - \Psi_3\,. \tag{B..26}$$

Also, the Weyl-NP scalars Ψ_i and the Ricci-NP scalars Φ_{ij} defined in Eq(B..7) and Eq(B..8) respectively can be calculated indirectly from the above NP equations after obtaining the spin coefficients rather than directly using their definitions.

The six independent components of the Faraday-Maxwell 2-form (i.e. the electromagnetic field strength tensor) F_{ab} can be encoded into three complex Maxwell scalars

$$\phi_0 := -F_{ab}l^a m^b\,,\quad \phi_1 := -\frac{1}{2}F_{ab}(l^a n^a - m^a\bar{m}^b)\,,\quad \phi_2 := F_{ab}n^a\bar{m}^b\,, \tag{B..27}$$

and therefore the eight real Maxwell equations $d\mathbf{F} = 0$ and $d^{\star}\mathbf{F} = 0$ (as $\mathbf{F} = dA$) can be transformed into four complex equations,

$$D\phi_1 - \bar{\delta}\phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2\,, \tag{B..28}$$

$$D\phi_2 - \bar{\delta}\phi_1 = -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\varepsilon)\phi_2\,, \tag{B..29}$$

$$\Delta\phi_0 - \delta\phi_1 = (2\gamma - \mu)\phi_0 - 2\tau\phi_1 + \sigma\phi_2\,, \tag{B..30}$$

$$\Delta\phi_1 - \delta\phi_2 = \nu\phi_0 - 2\mu\phi_1 + (2\beta - \tau)\phi_2\,, \tag{B..31}$$

with the Ricci-NP scalars Φ_{ij} related to Maxwell scalars by

$$\Phi_{ij} = 2\,\phi_i\,\overline{\phi_j}\,,\quad (i, j \in \{0, 1, 2\})\,. \tag{B..32}$$

To sum up, Eqs(B..9)-(B..26), Eqs(B..28)-(B..31) and Eq(B..32) constitute the Einstein-Maxwell equations in Newman-Penrose formalism.

When switching from $\{(+, -, -, -)\,, l^a n_a = 1\,, m^a\bar{m}_a = -1\}$ to $\{(-, +, +, +)\,, l^a n_a = -1\,, m^a\bar{m}_a = 1\}$, definitions of the spin coefficients, Weyl-NP scalars Ψ_i and Ricci-NP scalars Φ_{ij} need to change their signs; this way, the Einstein-Maxwell equations can be left unchanged.