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THE BEST APPROXIMATION AND AN EXTENSION OF A FIXED POINT THEOREM OF F.E. BROWDER

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ABSTRACT In this paper, the KKM principle has been used to obtain a theorem on the best approximation of a continuous function with respect to an affine map. The main result provides extensions of some well-known fixed point theorems.

KEY WORDS AND PHRASES. Best approximation, fixed point theorem, KKM-map, p-affine map, inward set.

MATHEMATICS SUBJECT CLASSIFICATIONS: Primary 47H10, Secondary 54H25.

Let E be a locally convex topological vector space and C a non-empty subset of E. A mapping $p : C \times E \to [0, \infty)$ is a convex map iff for each fixed $x \in C, p(x, \cdot) : E \to (0, \infty)$ is a convex function. For $x \in C$, the inward set $I_C(x) = \{x + r(y - x) : y \in C, r > 0\}$. Browder [1] proved the following extension of the Schauder's fixed point theorem.

THEOREM 1. ***(Browder).*** *Let C be a compact, convex subset of E and $f : C \to E$ a continuous map. If $p : C \times E \to [0, \infty)$ is a continuous convex map satisfying*

(1) for each $x \neq f(x)$, there exists a $y \in I_C(x)$ with $p(x, f(x) - y) < p(x, f(x) - x)$, then f has a fixed point.

It may be stated that the importance of Theorem 1 stems from p being a continuous convex map instead of a continuous seminorm on E. In this paper, we use the KKM principle to obtain a result on the 'best approximation' that yields Theorem 1 with relaxed hypothesis on compactness.

Let X be a non-empty subset of E. Recall that a mapping $F : X \to 2^E$ is a KKM map if $F(x) \neq \emptyset$ for each $x \in X$, and for any finite subset $A = \{x_1, x_2, \ldots, x_n\} \subseteq X, C_0(A) \subseteq \cup\{F(x_i) : i = 1, 2, \ldots, n\}$, where $C_0(A)$ denotes the convex hull of A. Observe that if F is a KKM map, then $x \in F(x)$ for each $x \in X$.

It is shown by Fan [2] that if $F : X \to 2^E$ is a closed valued KKM map, then the family $\{F(x) : x \in X\}$ has the finite intersection property.

As an immediate consequence of the above result, we have:

LEMMA 2. *If X is a non-empty compact, convex subset of E and $F : X \to 2^E$ is a closed valued KKM map, then $\cap\{F(x) : x \in X\} \neq \emptyset$.*

PROOF. Define a map $G : X \to 2^X$ by

$$G(x) = F(x) \cap X.$$

Then $G(x)$ is a nonempty compact subset of X and G is a KKM map. Consequently, by [2], $\{G(x) : x \in X\}$ has the finite intersection property. Since X is compact, it follows that $\cap\{G(x) : x \in X\} \neq \emptyset$, and hence, $\cap\{F(x) : x \in X\} \neq \emptyset$. $\square$

The following lemma is essentially due to Kim [3]. We give a proof for completeness.

Note: In the following, $C_0(A)$: stands for the closed convex hull of A.

LEMMA 3. *If A and B are compact, convex subsets of E, then $C_0(A \cup B)$ is a compact, convex subset of E.*

PROOF. Since A and B are convex, it follows $C_0(A \cup B) = \{\lambda x + \mu y : x \in A, y \in B, \lambda, \mu \in [0,1] \text{ and } \lambda + \mu = 1\}$. Clearly, $C_0(A \cup B)$ is a closed and convex subset of E. To show that $C_0(A \cup B)$ is compact,let $C = [0,1] \times [0,1] \times A \times B$ and $D = \{\lambda x + \mu y : x \in A, y \in B, \lambda, \mu \in [0,1]\}$. Then C is a compact subset of $Y = [0,1] \times [0,1] \times E \times E$ in the product topology on Y. Further, the mapping $f : Y \to E$ defined by $f(\lambda, \mu, x, y) = \lambda x + \mu y$ being continuous, it follows that $D = f(C)$ is a compact subset of E and, hence, $C_0(A \cup B) \subseteq D$ is compact. $\square$

LEMMA 4. *Let X be a non-empty convex subset of E and $F : X \to 2^E$ a closed valued KKM map. If there exists a compact, convex set $S \subseteq X$ such that $\cap\{F(x) : x \in S\}$ is non-empty and compact, then $\cap\{F(x) : x \in X\} \neq \emptyset$.*

PROOF. Let $C = \cap\{F(x) : x \in S\}$. Then C is non-empty and a compact subset of E. To prove the lemma, it suffices to show that the family $\{F(x) \cap C : x \in X\}$ has the finite intersection property. To prove this, let A be a finite subset of X. Then $C_0(A)$ is compact and by Lemma 3, $D = C_0(S \cup C_0(A))$ is a compact and convex subset of X. Consequently, by Lemma 2, $\cap\{F(x) : x \in D\} \neq \emptyset$. This implies that $\cap\{F(x) \cap C : x \in A\} \neq \emptyset$. Thus, $\{F(x) \cap C : x \in X\}$ has the finite intersection property. Since C is compact and $F(x)$ is closed for each $x \in X$, it follows that $\cap\{F(x) \cap C : x \in X\} \neq \emptyset$. This implies that $\cap\{F(x) : x \in X\} \neq \emptyset$.

Let X be a non-empty convex subset of E and $p : X \times E \to [0, \infty)$ a convex map. A mapping $g : X \to X$ is a p-affine map iff for each triple $\{x, x_1, x_2\} \subseteq X, y \in E$, and $\lambda, \mu \in [0,1]$ with $\lambda + \mu = 1$,

$$p(x, y - g(\lambda x_1 + \mu x_2)) \leq \max\{p(x, y - g(x_i)) : i = 1, 2\}.$$

Note: If g is linear or affine in the sense of Prolla [4], then p being convex, it follows that g is p-affine in the above sense. It is immediate that if g is p-affine, then for any finite set $A = \{x_1, x_2, \ldots, x_n\} \subseteq X$ and $\lambda_i \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$,

$$p(x, y - g(\sum_{i=1}^{n} \lambda_i x_i)) \leq \max\{p(x, y - g(x_i)) : i = 1, 2, \ldots, n\}$$

for each $x \in X, y \in E$. $\square$

The following is the main result of this paper.

THEOREM 5. *Let X be a nonempty convex subset of E and $p : X \times E \to [0, \infty)$ a continuous convex map. Let $f : X \to E$ and $g : X \to X$ be continuous mappings with g p-affine. Suppose there exist a compact, convex set $S \subseteq X$ and a compact set $K \subseteq X$ such that*

(2) for each $y \in X \backslash K$ there exists an $x \in S$ such that $p(y, f(y) - g(y)) > p(y, f(y) - g(x))$. Then there exists a $u \in X$ that satisfies

(3) $p(u, f(u) - g(u)) = \inf\{p(u, f(u) - g(x)) : x \in X\} = \inf\{p(u, f(y) - z) : z \in cl\ I_X(g(u))\}$.

PROOF. We first prove the left equality. For this, we define a mapping $G : X \to 2^X$ by

$$G(x) = \{y \in X : p(y, f(y) - g(y)) \leq p(y, f(y) - g(x))\}.$$

Clearly, $x \in G(x)$ and it follows that $G(x)$ is closed for each $x \in X$. We show that G is a KKM map. Let $y = \sum_{i=1}^{n} \lambda_i x_i, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, x_i \in X$ for each i. Suppose $y \notin \cup\{G(x_i), i = 1,2,\ldots,n\}$. Then for each $i = 1,2,\ldots,n$,

$$p(y, f(y) - g(y)) > p(y, f(y) - g(x_i)).$$

This implies that $p(y, f(y) - g(y)) = p(y, f(y) - g(\sum_{i=1}^{n} \lambda_i x_i)) \leq \max\{p(y, f(y) - g(x_i)), i = 1,2,\ldots,n\} < p(y, f(y) - g(y))$. This inequality is impossible and, consequently, $y \in \cup\{G(x_i) : i = 1,2,\ldots,n\}$, that is, G is a closed valued map. Now, since S is a compact convex subset of X, it follows by Lemma 2 that $C = \cap\{G(x) : x \in S\}$ is a nonempty closed subset of X. We show that $C \subseteq K$. Suppose $y \in C$ and assume that $y \in X \backslash K$. Then by hypothesis there exists an $x \in S$ such that $p(y, f(y) - g(y)) > p(y, f(y) - g(x))$. This implies that $y \notin G(x)$ for an $x \in S$ and, hence, $y \notin C$, contradicting the initial supposition. Thus, $C \subseteq K$ and, hence, $\cap\{G(x) : x \in S\}$ is a nonempty compact subset of K. Hence by Lemma 4, $\cap\{G(x) : x \in X\} \neq \phi$. If $u \in \cap\{G(x) : x \in X\}$, then for each $x \in X$, $p(u, f(u) - g(u)) \leq p(u, f(u) - g(x))$. Further, since $u \in X$, it follows that $p(u, f(u) - g(u)) = \inf\{p(u, f(u) - g(x)) : x \in X\}$. This proves the first equality in (3). To prove right side of the equality in (3) we first show that for each $z \in I_X(g(u)) \backslash X, p(u, f(u) - g(u)) \leq p(u, f(u) - z)$. Now $z \in I_X(g(u)) \backslash X$ implies that there is a $y \in X$ and $r > 1$ such that $y = \frac{1}{r}z + (1 - \frac{1}{r})g(u)$. Hence, by the first equality and p being convex, it follows that $p(u, f(u) - g(u)) \leq p(u, f(u) - y) \leq \frac{1}{r}p(u, f(u) - z) + (1 - \frac{1}{r})p(u, f(u) - g(u))$, that is, $p(u, f(u) - g(u)) \leq p(u, f(u) - z)$ for each $z \in I_X(g(u)) \backslash X$. Since the last inequality is also true for any $z \in X$, it follows that $p(u, f(u) - g(u)) \leq p(u, f(u) - z)$ for each $z \in I_X(g(u))$. Further, since the functions f, g, and p are continuous and $g(u) \in I_X(g(u))$, it follows that $p(u, f(u) - g(u)) = \inf\{p(u, f(u) - z) : z \in \text{cl}\,(I_X(g(u)))\}$. This proves the second equality in (3). □

As a simple consequence of Theorem 2, we have

COROLLARY 6. *Suppose X is a compact, convex subset of $E, p : X \times E \to [0,\infty)$ a continuous convex function and $f : X \to E$ a continuous function. Then for any continuous p-affine map $g : X \to X$, there exists a $u \in X$ that satisfies (3). Further,*

(i) if $f(x) \in \text{cl}\,(I_X(g(x)))$ for each $x \in X$ then $p(u, f(u) - g(u)) = 0$,

(ii) if for each $x \in X$, with $f(x) \neq g(x)$ there exists a $y \in \text{cl}\,(I_X(g(x)))$ such that $p(x, f(x) - y) < p(x, f(x) - g(x))$, then $f(u) = g(u)$.

PROOF. Set $S = K = X$ in Theorem 5. Since $X \backslash K = \phi$, condition (2) in Theorem 5 is satisfied. Hence, there is a $u \in X$, that satisfies (3). Clearly, (i) implies $p(u, f(u) - g(u)) = 0$. To prove (ii), suppose $f(u) \neq g(u)$. Then by hypothesis $p(u, f(u) - z) < p(u, f(u) - g(u))$ for some $z \in \text{cl}\,(I_X(g(u)))$. The last inequality contradicts (3). Hence, $f(u) = g(u)$. □

It may be remarked that if g is the identity mapping of X, then Corollary 6 yields Browder's Theorem 1 and also extends a recent result of Sehgal, Singh, and Gastl [5] if f therein is a single valued map.

For the next result, let $\boldsymbol{P}$ denote the family of nonnegative continuous convex functions on $X \times E$. Note if p_1 and $p_2 \in \boldsymbol{P}$, then so is $p_1 + p_2$. Also, if p is a continuous seminorm on E, then p generates a nonnegative continuous convex function on $X \times E$ defined by $\hat{p}(x, y) = p(y)$. A mapping $g : X \to X$ is $\boldsymbol{P}$ affine if it is p-affine for each $p \in \boldsymbol{P}$.

The result below is an extension of an earlier result of Fan.

THEOREM 7. *Let X be a compact, convex subset of E and $f : X \to E$ a continuous function. Then for any continuous $\boldsymbol{P}$ affine map $g : X \to X$,*

(4) either $f(u) = g(u)$ for some $u \in X$,

(5) or there exists a $p \in \boldsymbol{P}$ *and a* $u \in X$ *with* $0 < p(u, f(u) - g(u)) = \inf\{p(u, f(u) - z) : z \in cl\,(I_X(g(u)))$.

In particular, if $f(x) \in cl\,(I_X(g(x)))$ *for each* x, *then (5) holds.*

PROOF. It follows by Theorem 5 that for each $p \in \boldsymbol{P}$ there is a $u = u_p \in X$ such that $p(u, fu - gu) = \inf\{p(u, f(u) - z) : z \in \mathrm{cl}(I_X(g(u)))\}$. If for some $p, p(u, f(u) - g(u)) > 0$, then (5) is true. Suppose then, $p(u_p, f(u_p) - g(u_p)) = 0$ for each $p \in \boldsymbol{P}$. Set $A_p = \{u \in X : p(u, f(u) - g(u)) = 0\}$. Then A_p is a nonempty compact subset of X. Furthermore, the family $\{A_p : p \in \boldsymbol{P}\}$ has the finite intersection property. Consequently, there is a $u \in X$ that satisfies

(6) $p(u, f(u) - g(u)) = 0$ for each $p \in \boldsymbol{P}$.

If $f(u) \neq g(u)$, then since E is separated, there exists a continuous seminorm p on E such that $p(f(u) - g(u)) \neq 0$ and, hence, $\hat{p}(u, f(u) - g(u)) > 0$, contradicting (6). Thus, $f(u) = g(u)$. Hence, (5) holds in the alternate case. □

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