Research Article

Bipartite Toughness and $k$-Factors in Bipartite Graphs

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1. Introduction

Toughness, like connectivity, is an important invariant in graphs. There has been extensive work on toughness (see the survey in [1]) since Chvátal introduced the concept in 1973 [2]. The toughness $t(G)$ of a graph $G$ is the minimum value of $|S|/w(G - S)$, where $S \subset V(G)$ is a proper subset of the vertices of $G$ and $w(G - S) > 1$ is the number of connected components after removing $S$ from $G$. (If $G$ is a complete graph so that $w(G - S)$ is always equal to 1, then $t(G)$ is set to be $\infty$.) That is, for any integer $k > 1$, $G$ cannot be split into $k$ connected components by removing less than $k \cdot t(G)$ vertices. We also say that $G$ is $t(G)$-tough. Chvátal made a number of conjectures in [2], including the famous 2-tough conjecture saying that every 2-tough graph has a Hamiltonian cycle. Having inspired many interesting results, the 2-tough conjecture itself was showed to be false by Bauer et al. in 2000 [3].

A subgraph $H$ of $G$ is called a factor of $G$ if $H$ is a spanning subgraph of $G$. An important class of factors is $k$-factors, also called regular degree factors, where every vertex of $G$ has degree $k$ in $H$. (Note that a perfect matching is a 1-factor, and a Hamiltonian cycle is a connected 2-factor.) There has been extensive work on the conditions of existence of various factors in graphs. Many results can be found in the latest survey by Plummer [4].
Figure 1: The bound of bipartite toughness in Theorems 1.2 and 1.3, illustrated with $n = 1000$. The $x$-axis is $k$ and $y$-axis is $\log(t^B(G))$. The bound is given by $f_1$ on the left and $f_2$ on the right of $k = (n + 4)/4$.

It is natural to expect that toughness, yet another measure of the connectivity of a graph, ought to relate to the existence of $k$-factors in graphs. Enomoto et al. [5–7] proved that every $k$-tough graph contains a $k$-factor if it satisfies trivial necessary conditions, and there are $(k - \varepsilon)$-tough graphs for any $\varepsilon > 0$ that do not contain a $k$-factor. Consider a bipartite graph $G = (X, Y; E)$, where $X \cup Y = V(G)$ is a partition of $V(G)$ and $E$ is the edge set of $G$ with each edge having one end in $X$ and the other in $Y$. Katerinis [8] proved that every 1-tough bipartite graph has a 2-factor. Recall that the toughness of a bipartite graph $G = (X, Y; E)$ is at most 1 because the removal of $X$ from $G$ (assuming $|X| \geq |Y|$) results in an independent set $Y$. Therefore, it is not possible to use toughness to predict the existence of $k$-factors in balanced bipartite graphs for any $k \geq 3$.

1.1. Bipartite toughness

In this paper, we introduce bipartite toughness, which is analogous to the concept of toughness but reflects the bipartition of $V(G)$. The bipartite toughness $t^B(G)$ of a bipartite graph $G = (X, Y; E)$ is the minimum value of $|S|/w(G - S)$, where $S$ is a proper subset of $X$ or $Y$ and $w(G - S) > 1$ is the number of connected components after removing $S$ from $G$. We set $t^B(G) = \infty$ for complete bipartite graphs, just like $t(G) = \infty$ for complete graphs.

A bipartite graph can have a regular degree factor only if $|X| = |Y|$. Therefore, in the rest of the paper, we consider only a balanced bipartite graph with $|X| = |Y| = n$. For a subset $S$ of $V(G)$, we use $N(S)$ to denote the set of vertices adjacent to at least one vertex in $S$. For two disjoint subsets $S$ and $T$ of $V(G)$, we use $e_G(S,T)$ to stand for the number of edges having one end in $S$ and the other in $T$. Other terminologies and notations used in this paper follow [9] and other references.

Bipartite toughness $t^B(G)$ measures the connectivity of a bipartite graph better than toughness $t(G)$ does. In contrast to toughness $t(G)$ that is at most 1 in a bipartite graph, $t^B(G)$ can be arbitrarily big. For example, in a complete bipartite graph with one edge deleted,
Let $t(G) = O(n)$, which approaches to $\infty$, is just like $t(G) = O(n)$ in a complete graph with one edge deleted. Interestingly, $t^B(G)$ a better invariant to predict the existence of $k$-factors in balanced bipartite graphs, for any $k$. Furthermore, by their definitions, calculating $t^B(G)$ in a bipartite graph is easier than calculating $t(G)$ since one is a subtask of the other.

1.2. Our results

Let $G = (X, Y; E)$ be a balanced bipartite graph with $|X| = |Y| = n$ and $1 \leq k \leq n$ be an integer. In this paper, we prove the following three theorems.

Theorem 1.1. Let $m = \lfloor(n-1)/2 \rfloor$. If $t^B(G) > m/(m+2)$, then $G$ has a 1-factor.

Theorem 1.2. For $k \geq 2$ and $n \geq 4k-4$, if $t^B(G) > f_1 = (2k-1)(n-1)/(kn+1)$, then $G$ has a $k$-factor.

Theorem 1.3. For $n \leq 4k-4$, if $t^B(G) > f_2 = (n-1)/(2\sqrt{kn+1}-2k+1)$, then $G$ has a $k$-factor.

These theorems together give a sharp bound of $t^B(G)$ for $G$ to have a $k$-factor, for $k = 1, \ldots, n$. (See Figure 1. Note that $m/(m-2) = f_1$ when $k = 1$ and $n$ is odd; and $f_1 = f_2$ when $n = 4k-4$.)

The bound of $t^B(G)$ is sharp in the following senses.

(a) For Theorem 1.1, let $m = \lfloor(n-1)/2 \rfloor$ and construct a balanced bipartite graph $G = (X, Y; E)$ as follows. Let $X = S \cup P$ and $Y = T \cup Q$, where $|P| = |T| = n - m$, $|S| = |Q| = m$, and $|X| = |Y| = n$. Let $E$ be comprised of all possible edges between $X$ and $Q$ and all possible edges between $S$ and $Y$. If $n$ is even, then we add into $E$ an edge between $P$ and $T$. Here, $|S| + e_G(X-S, T) - |T| = -1$ so that by Lemma 2.1 below, $G$ has no 1-factor. On the other hand, it is not hard to verify that $t^B(G) = m/(m+2)$ in this construction of $G$. Therefore, $m/(m+2)$ is a sharp bound.

(b) For Theorem 1.2, for integers $k \geq 2$ and $r \geq 2$, construct a balanced bipartite graph $G_r = (X, Y; E)$ as follows. Let $X = S \cup P$ and $Y = T \cup Q$, where $|P| = |T| = kr - 1$, $|S| = |Q| = (k-1)r - 1$, and $|X| = |Y| = n = (2k-1)r - 2 \geq 4k - 5$. Let $E$ be comprised of all possible edges between $X$ and $Q$, all possible edges between $S$ and $Y$, and a 1-factor between $P$ and $T$. Here, $k|S| + e_G(X-S, T) - k|T| = -1$ so that by Lemma 2.1 below, $G_r$ has no $k$-factor. On the other hand, it is not hard to verify that $t^B(G_r) = (n-1)/(n-|S|) = (2k-1)(n-1)/(kn+1) = f_1$ in $G_r$. Therefore, $f_1$ is a sharp bound.

(c) For Theorem 1.3. Let $n/4 < k < n$ and $\sqrt{kn+1} = t$ be an integer. Obviously, $n/2 < t < n$. Construct a balanced bipartite graph $G = (X, Y; E)$ as follows. Let $X = S \cup P$ and $Y = T \cup Q$, where $|P| = |T| = t$, $|S| = |Q| = n-t$, and $|X| = |Y| = n$. Let $E$ be comprised of all possible edges between $X$ and $Q$, all possible edges between $S$ and $Y$, and a $(2k-t)$-factor between $P$ and $T$. Then $k|S| + e_G(X-S, T) - k|T| = kn-t = k(n-t) + (2k-t) - kt = kn - t^2 = -1$. Again, by Lemma 2.1 below, $G$ has no $k$-factor. Moreover, it is not hard to verify that $t^B(G) = (n-1)/(2\sqrt{kn+1}-2k+1)$. Therefore, $f_2$ is also a sharp bound.

It is also worth to mention that, unlike Enomoto et al.’s well-known result that $k$-tough graphs have $k$-factors, in our results the bound of $t^B(G)$ is much smaller than $k$, in fact less than 2 for most $k$ (see Figure 1). This looks counterintuitive but it is due to
2. Proofs of the theorems

The following lemma will be needed in the proofs of theorems.

**Lemma 2.1.** Let $G = (X, Y; E)$ be a balanced bipartite graph, where $|X| = |Y| = n$, and let $k \geq 1$ be an integer. Then the following three statements are equivalent:

(i) $G$ has a $k$-factor;

(ii) $G$ has $k$ edge-disjoint 1-factors;

(iii) for any $S \subseteq X$ and $T \subseteq Y$, $k|S| + e_G(X - S, T) - k|T| \geq 0$.

**Proof.** (i) and (ii) follow from the K"onig-Hall theorem [9, Theorem 5.2 and Lemma 5.2], a regular degree bipartite graph has a perfect matching. Therefore, a $k$-factor of a bipartite graph $G$ can be partitioned into a collection of $k$ edge-disjoint perfect matchings (1-factors). (ii) to (i) is trivial.

(i) and (iii): the equivalence of (i) and (iii) can be deduced from the max-flow min-cut theorem [10, 11]. Convert $G = (X, Y; E)$ into a network by (a) adding a source vertex $s$ with $k$ multiedges between $s$ and each vertex $x \in X$; (b) adding a sink vertex $t$ with $k$ multiedges between $t$ and each vertex $y \in Y$; and (c) orienting each edge into a directed arc going from $s$ to $X$, from $X$ to $Y$, or from $Y$ to $t$ (see Figure 2). Clearly, $G$ has a $k$-factor if and only if the network has a $kn$-flow from $s$ to $t$. Any cut in the network that separates $s$ and $t$ contains at least $kn$ forward edges. For any $S \subseteq X$ and $T \subseteq Y$, consider the cut shown in dashed line in Figure 2, we have

$$k|S| + e_G(X - S, T) + k|Y - T| \geq kn = k|T| + k|Y - T|,$$  

so that

$$k|S| + e_G(X - S, T) - k|T| \geq 0. \quad (2.2)$$
Proof of Theorem 1.1 (By contradiction). Suppose $G$ has no $k$-factor and $n \geq 4k - 4$, we will infer that $t^B(G) \leq f_1$. According to Lemma 2.1, there exist $S \subseteq X$ and $T \subseteq Y$ such that $k|S| + e_G(X - S, T) - k|T| < 0$. Let $s = |S|$ and $t = |T|$. Then

$$e_G(X - S, T) \leq kt - ks - 1. \quad (2.3)$$

Obviously, $t > s$. We can further assume that

$$s + t \leq n. \quad (2.4)$$

Because, if $s + t > n$, then we can let $S' = X - S$ and $T' = Y - T$ and have $|S'| + |T'| < n$, $|S'| > |T'|$, and $k|T'| + e_G(S', Y - T') - k|S'| = k|S| + e_G(X - S, T) - k|T|$. By symmetry, this converts to the case of $s + t \leq n$.

We then have two cases to consider.

Case 1.

$$k(t - s) \leq t. \quad (2.5)$$

If $k = 1$, then $w(G - S) \geq t + 1 - (t - s - 1) = s + 2$ by (2.3). By $t > s$ and (2.4), we have $s \leq m$, where $m = [(n - 1)/2]$. Thus

$$t^B(G) \leq \frac{|S|}{w(G - S)} \leq \frac{s}{s + 2} \leq \frac{m}{m + 2}. \quad (2.6)$$

This completes the proof of Theorem 1.1. (Note that when $k = 1$, we have only Case 1 to consider.)

Proof of Theorem 1.2 (Continue the proof of Theorem 1.1). Now suppose $k \geq 2$, by (2.5), we have $t \leq ks/(k - 1)$. Let $T' = T \cap N(X - S)$. Then by (2.3), $|T'| \leq kt - ks - 1$. Let $T'' = (Y - T) \cup T'$. Then $|T''| \leq n - t + (kt - ks - 1) < n$ and $w(G - T'') \geq n - s + 1$. Therefore,

$$t^B(G) \leq \frac{|T''|}{w(G - T'')} \leq \frac{n + (k - 1)t - ks - 1}{n - s + 1}. \quad (2.7)$$

Case 1.1. If $n - s \leq ks/(k - 1)$, then we have $s \geq (k - 1)n/(2k - 1)$. By (2.4) and (2.7),

$$t^B(G) \leq \frac{n + (k - 1)(n - s) - ks - 1}{n - s + 1} = 2k - 1 - \frac{(k - 1)n + 2k}{n - s + 1} \leq 2k - 1 - \frac{(k - 1)n + 2k}{n - (k - 1)n/(2k - 1) + 1} = \frac{(2k - 1)(n - 1)}{kn + 2k - 1} \leq \frac{(2k - 1)(n - 1)}{kn + 1} = f_1. \quad (2.8)$$
Case 1.2. If $n - s > ks / (k - 1)$, then we have $s < (k - 1)n / (2k - 1)$. By (2.5) and (2.7),

$$t^B(G) \leq \frac{n - 1}{n - s + 1} \leq \frac{n - 1}{n - (k - 1)n / (2k - 1) + 1} = \frac{(2k - 1)(n - 1)}{kn + 2k - 1} \leq \frac{(2k - 1)(n - 1)}{kn + 1} = f_1.$$  

(2.9)

Case 2.

$$k(t - s) > t. \tag{2.10}$$

Let $d$ be the unique integer satisfying

$$t \cdot d < k(t - s) \leq (d + 1)t. \tag{2.11}$$

By (2.10), $1 \leq d \leq k - 1$. By (2.3) and (2.11), there is a vertex $y_0 \in T$ that is adjacent to at most $d$ vertices in $X - S$. Let $T' = Y - \{y_0\}$ so $|T'| = n - 1$ and $w(G - T') \geq n - s - d + 1$. By (2.4) and (2.11), we have $s \leq [(k - d)n - 1] / (2k - d)$. Therefore,

$$t^B(G) \leq \frac{n - 1}{n - s - d + 1} \leq \frac{n - 1}{n - ((k - d)n - 1) / (2k - d) - d + 1}. \tag{2.12}$$

Define a function $g(d) = n - [(k - d)n - 1] / (2k - d) - d + 1$. It is easy to verify that, by the assumption of $n \geq 4k - 4$, $g(1) \leq g(2)$. Since $g(d)$ is a convex function, it follows that $f(1) \leq f(d)$ for $d > 1$. By (2.12),

$$t^B(G) \leq \frac{n - 1}{f(d)} \leq \frac{n - 1}{f(1)} = \frac{(2k - 1)(n - 1)}{(kn + 1)} = f_1. \tag{2.13}$$

This completes the proof of Theorem 1.2. \qed

Proof of Theorem 1.3 (By contradiction). Indeed, we will prove that the result in Theorem 1.3 holds for all $1 \leq k \leq n$. The condition of $n \geq 4k - 4$ in Theorem 1.3 is only because that $f_2$ is not as tight a bound as $f_1$ when $n < 4k - 4$.

Suppose $G$ has no $k$-factor, we will infer that $t^B(G) \leq f_2$. According to Lemma 2.1, there exist $S \subseteq X$ and $T \subseteq Y$ such that

$$e_C(X - S, T) \leq kt - ks - 1, \tag{2.14}$$

where $s = |S|$ and $t = |T|$. Like in the proof of Theorems 1.1 and 1.2, we can still assume (2.4).
Suppose \( y_0 \) is vertex in \( T \) that is adjacent to the least number (denoted by \( d \)) of vertices in \( X - S \). By (2.14), we have \( t d \leq kt - ks - 1 \). Then with (2.4), we further have \( s \leq [(k - d)n - 1]/(2k - d) \). Let \( T' = Y - \{ y_0 \} \), then \( |T'| = n - 1 \) and \( w(G - T') \geq n - s - d + 1 \). Therefore,

\[
t^b(G) \leq \frac{|T'|}{w(G - T')} \leq \frac{n - 1}{n - s - d + 1} \leq \frac{n - 1}{n - ((k - d)n - 1)/(2k - d) - d + 1}
\]

\[
= \frac{n - 1}{(2k - d) + (kn + 1)/(2k - d) - 2k + 1} \leq \frac{n - 1}{2\sqrt{kn + 1} - 2k + 1} = f_2 .
\]

This completes the proof of Theorem 1.3. \( \square \)

3. Conclusion and future work

We have defined a new invariant in bipartite graphs called bipartite toughness and provided a sharp bound of it for a balanced bipartite graph to have a \( k \)-factor, for \( k \) from 1 through \( n \). We view this as a big improvement from using toughness to predict \( k \)-factors in bipartite graphs, as toughness of a bipartite graph is at most 1 and it cannot predict \( k \)-factors for any \( k \geq 3 \).

There is also research on computational complexity of toughness. In general, recognizing toughness of a graph is NP-hard [12]. Furthermore, 1-tough of graphs is also NP-hard [13], and even 1-tough of bipartite graphs is NP-hard [14] too. Toughness in claw-free \((K_1,3\)-free\) graphs [15], 1-tough in split graphs [14], and toughness in split graphs [16] have been shown in \( P \). In the future, it would be very interesting to determine the complexity of bipartite toughness.

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References


