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Research Article

Analysis of a Heterogeneous Trader Model for Asset Price Dynamics

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We examine an asset pricing model of Westerhoff (2005). The model incorporates heterogeneous beliefs among traders, specifically fundamentalists and trend-chasing chartists. The form of the model is shown here to be a nonlinear planar map. Since it contains a single parameter, the model may be considered the simplest effective model yet derived for financial asset pricing with heterogeneous trading. Analysis of the map yields results for stability and bifurcations of fixed points and periodic orbits. The model has intricate attractor basin behavior and global bifurcations to chaos: symmetric homoclinic bifurcation and boundary crisis.

1. Introduction

The notion that the interaction of investor classes can be expressed as discrete dynamical systems is not new. Following the seminal models of Brock and Hommes [1, 2], a number of influential models, including [3–17], have been formulated and analyzed using a dynamical system approach. See [18–20] for informative recent surveys on this flourishing line of research. The basis of all such models is the empirical evidence that traders are heterogeneous, tending to form groups relying on different but simple and fundamental trading rules. Because the models are naturally formulated to consider the interactions of various identifiable trading groups and because the observable variables such as asset price and trading volume are essentially discrete, the models fall into the branch of mathematics known as discrete dynamical systems, maps, or difference equation systems. The study of nonlinear maps has been a very active area of mathematics for more than 30 years due to its wide application and astonishing range of behavior.

Westerhoff [17] developed a simple asset pricing model taking into account fundamentalists and trend-chasing chartists. Fundamentalists, or “smart money” traders, base their decisions on the belief that prices eventually tend to return to their fundamental value. Chartists use technical trading rules, past trends, and extrapolation of data to predict future prices.

In Section 2, the basic assumptions of Westerhoff’s asset pricing model are discussed, and we reformulate the model in the standard form of a nonlinear planar map. In Section 3, we prove results for stability and bifurcations of fixed points and period-2 cycles in this map and use graphical techniques with illustrations generated by iDMC [21] to investigate its global bifurcations involving chaos and local attractor basin structures. This contrasts with [17], where purely numerical simulations are performed. Because of its functional form containing a single parameter, we consider the Westerhoff model [17] to be the simplest effective model yet derived for financial asset pricing based on heterogeneous trading. As such, its thorough analysis in this paper will help to guide the formulation and analysis of more detailed models of markets with interacting heterogeneous agents.

2. The Model

Following [17], we assume the price of an asset at time $t + 1$ to depend upon the demand of the speculators in the previous period. If there is excess demand, then the price increases. Otherwise, the price remains the same or decreases.

Let P be the logarithm of the asset price. The change in P at time $t + 1$ is proportional to the sum of the orders generated by fundamentalists and chartists, resulting in a map

$$P_{t+1} = P_t + N\left(D_t^F + D_t^C\right). \tag{2.1}$$

Here $N > 0$ is a measure of the strength of the demand, the aggressiveness of speculators toward the particular asset. The quantity $N(D_t^F + D_t^C)$ is the total excess demand for the asset at time t.

Expressions can be given for the orders generated by each trader type. Since fundamentalists trust that prices converge to their perceived fundamental value over time, D_t^F can be expressed as

$$D_t^F = F - P_t, \tag{2.2}$$

where F is the logarithm of the fundamental value of the asset. This value is considered to be constant and known. If the current price of the asset is larger than the perceived fundamental value then fundamentalists assume that the asset is overpriced, and hence the excess demand for the asset decreases. Likewise, if the current price is smaller than the fundamental price, then fundamentalists assume that the asset is underpriced, and the demand increases. If this were the only group of traders present, the asset price in the next period would coincide with this increase or decrease in demand for the asset. However, there exists a different group of traders called trend-chasing chartists who must also be considered.

The orders generated for the asset by trend chasers at time t, denoted by D_t^C, are given as

$$D_t^C = (P_t - F)V_{t-1}, \tag{2.3}$$

where

$$V_{t-1} = N\left|D_{t-1}^F\right| + N\left|D_{t-1}^C\right|. \tag{2.4}$$

Here, V_{t-1} represents the trading volume at time $t-1$. Trend chasers buy when the price is high and sell when the price is low, assuming that prices will continue the upward or downward trend. Chartists consider V_{t-1} to provide clues about how reliable their extrapolations may be. More specifically, a high trading volume when current prices exceed the fundamental price causes trend chasers to purchase more of the asset, whereas a low trading volume under the same condition would cause chartists to purchase less of the asset.

The total excess demand for the asset at time t can be written as

$$\begin{aligned} E_t &= N\left(D_t^F + D_t^C\right) \\ &= N(F - P_t + V_{t-1}(P_t - F)), \end{aligned} \tag{2.5}$$

and hence the asset price at time $t+1$ can be written as

$$P_{t+1} = P_t + N(F - P_t + V_{t-1}(P_t - F)). \tag{2.6}$$

Since the deviation from the fundamental value is important and the actual fundamental value of the asset is not, we can assume the fundamental value is unity (i.e., $F \equiv 0$) without loss of generality. Thus, from the above equations, we obtain the recurrence relation

$$\begin{aligned} P_{t+1} &= P_t(1 - N + NV_{t-1}), \\ V_t &= N|P_t|(1 + |V_{t-1}|) \end{aligned} \tag{2.7}$$

which is somewhat novel in that it allows us to predict asset price from the current log asset price (P_t) and the trading volume from the *previous* trading period (V_{t-1}). Writing this system in the standard form of a planar map, we arrive at our final model:

$$\begin{aligned} P &\longmapsto P(1 - N + NV), \\ V &\longmapsto N|P|(1 + V). \end{aligned} \tag{2.8}$$

Since the model contains only a single parameter N, we believe it may be the simplest effective model yet derived for financial asset pricing with heterogeneous trading.

The model as presented in [17] does not contain system (2.7), and the stated final model in that publication is in error. However, we wish to emphasize that we have used the same assumptions as in [17] for formulating the model, and these assumptions are fundamentally sound. Also, the numerical simulations in [17] were calculated from the defining conditions (2.1)–(2.4), not the stated final model in that paper, and are correct. Our goal here is to show that by properly reformulating the model as a standard planar map we can bring a considerable theory to be applied and can gain a better understanding of the pricing behavior.

3. Analysis

We refer the reader to [22, 23] for general theory of fixed points, stability, and bifurcations of discrete and continuous dynamical systems. We determine the fixed points of the map (2.8) by solving the following algebraic system:

$$\begin{aligned} \overline{P} &= \overline{P}\left(1 - N + N\overline{V}\right), \\ \overline{V} &= N\left|\overline{P}\right|\left(1 + \overline{V}\right). \end{aligned} \tag{3.1}$$

This map has three fixed points:

$$\left(\overline{P}, \overline{V}\right) = (0, 0), \tag{3.2}$$

$$\left(\overline{P}, \overline{V}\right) = \left(\frac{1}{2N}, 1\right), \tag{3.3}$$

$$\left(\overline{P}, \overline{V}\right) = \left(\frac{-1}{2N}, 1\right). \tag{3.4}$$

The local stability of the fixed points can be determined from eigenvalue analysis. Since the derivatives of (2.8) involve absolute values, we consider the two cases:

$$J_{(P>0)} = \begin{bmatrix} 1 - N + NV & NP \\ N + NV & NP \end{bmatrix}, \qquad J_{(P<0)} = \begin{bmatrix} 1 - N + NV & NP \\ -N - NV & -NP \end{bmatrix}. \tag{3.5}$$

In the limit as $(P, V) \to (0, 0)$, the Jacobian is

$$J(0, 0) = \begin{bmatrix} 1 - N & 0 \\ \pm N & 0 \end{bmatrix}. \tag{3.6}$$

Solving the characteristic equation $\det(J - \lambda I) = 0$ yields the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1 - N$. Fixed points of planar maps are asymptotically stable for $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Hence, (3.2) is asymptotically stable for $0 < N < 2$ and is unstable for $N > 2$.

Recall that a fixed point of a map is *nonhyperbolic* if at least one eigenvalue is on the unit circle. In parametrized systems, nonhyperbolic states are associated with possible changes in invariant subspaces and yield the possible local bifurcation points of the system. At $N = 0$, the fixed point (3.2) is nonhyperbolic with $\lambda_2 = +1$, and at $N = 2$, it is nonhyperbolic with $\lambda_2 = -1$. We can identify the bifurcations occurring at these parameter values. Since $|\lambda_1| < 1$

and λ_2 changes smoothly from $|\lambda_2| < 1$ to $\lambda_2 < -1$ as N increases through $N = 2$, a period-doubling bifurcation occurs at $\lambda_2 = -1$. It can be shown that the bifurcation at $N = 0$ is transcritical; however, this is not pertinent due to the practical restriction $N > 0$ in this model.

A similar process is carried out for fixed point (3.3). Substituting this fixed point into J, noting that $\overline{P} = 1/2N$ is positive, gives

$$J_1\left(\frac{1}{2N}, 1\right) = \begin{bmatrix} 1 & \frac{1}{2} \\ 2N & \frac{1}{2} \end{bmatrix}. \tag{3.7}$$

The eigenvalues are found to be $\lambda_{1,2} = 3/4 \pm \sqrt{1/16 + N}$. Here $\lambda_1 > 1$ for all $N > 0$, so (3.3) is unstable. As N increases through $N = 3$, λ_2 decreases through $\lambda_2 = -1$, resulting in the fixed point to change type from unstable saddle to unstable node.

Finally, fixed point (3.4) is examined. Substituting this value into J, noting that $P = -1/2N$ is negative, yields

$$J_2\left(\frac{-1}{2N}, 1\right) = \begin{bmatrix} 1 & -\frac{1}{2} \\ -2N & \frac{1}{2} \end{bmatrix}. \tag{3.8}$$

The eigenvalues are identical to the case of (3.3), and the same stability results are obtained.

Figure 1 is the orbit diagram corresponding to (2.7). From this figure it is evident that prices converge to their fundamental value if $0 < N < 2$. If $N > 2$, then prices alternate between two values, one that is lower and another that is higher than the fundamental value. At $N \approx 2.8$, a generic Neimark-Sacker bifurcation occurs for each of the points of the period-2 cycle. Instead of prices alternating between two points, they now alternate between values on the two limit cycles (Figure 1). Each of the two limit cycles is locally stable in period-2. The patchwork basin of attraction of each limit cycle is shown in Figure 2. The basin boundaries can be found analytically. They are the preiterates of the *critical curves* [24], the locus of points mapping to the fixed point $(\overline{P}, \overline{V}) = (0, 0)$. See [25] for another example of the Neimark-Sacker bifurcation in a nonlinear financial system.

The existence of the period-2 Neimark-Sacker bifurcation can be inferred using the second iterate of the map:

$$\begin{aligned} P &\longmapsto P(1 - N + NV)\left(1 - N + N^2|P|(1 + V)\right), \\ V &\longmapsto N|P(1 - N + NV)|(1 + N|P|(1 + V)). \end{aligned} \tag{3.9}$$

Fixed points of the second iterate correspond to fixed points or to components of period-2 cycles in the original map. The period-2 fixed points are the solutions of the algebraic system:

$$\begin{aligned} \overline{P} &= \overline{P}\left(1 - N + N\overline{V}\right)\left(1 - N + N^2\left|\overline{P}\right|\left(1 + \overline{V}\right)\right), \\ \overline{V} &= N\left|\overline{P}\left(1 - N + N\overline{V}\right)\right|\left(1 + N\left|\overline{P}\right|\left(1 + \overline{V}\right)\right). \end{aligned} \tag{3.10}$$

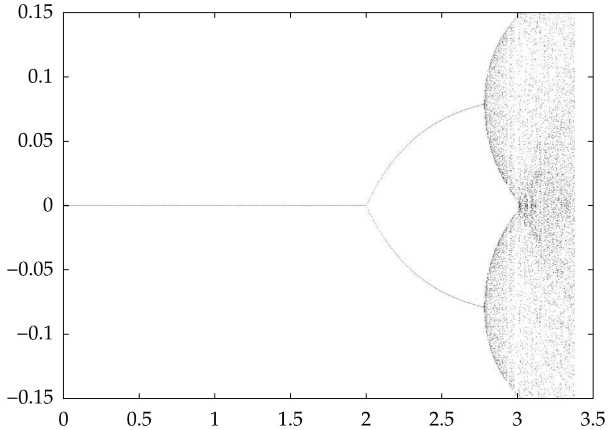

Figure 1: Orbit diagram: Log of asset price P for varying trading aggressiveness N.

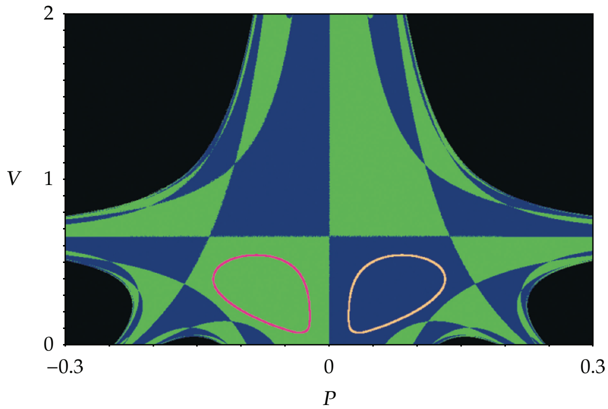

Figure 2: Phase diagram illustrating attractor basins for symmetric limit cycles ($N = 2.9$).

Since system (3.10) involves absolute values, we solve it for the four cases resulting from the possible signs of $\overline{P}$ and $(1 - N + N\overline{V})$. We recover the fixed points (3.2), (3.3), and (3.4) as solutions and obtain three period-2 cycles

$$\left(\overline{P}, \overline{V}\right) : \left(\frac{N-2}{2N(N-1)}, \frac{N-2}{N}\right) \longleftrightarrow \left(\frac{-(N-2)}{2N(N-1)}, \frac{N-2}{N}\right), \tag{3.11}$$

$$\left(\overline{P}, \overline{V}\right) : \left(\overline{P}\left(\overline{V}_1\right), \overline{V}_1\right) \longleftrightarrow \left(\overline{P}\left(\overline{V}_2\right), \overline{V}_2\right), \tag{3.12}$$

$$\left(\overline{P}, \overline{V}\right) : \left(-\overline{P}\left(\overline{V}_1\right), \overline{V}_1\right) \longleftrightarrow \left(-\overline{P}\left(\overline{V}_2\right), \overline{V}_2\right), \tag{3.13}$$

where

$$\overline{V}_{1,2} = (N-2) \pm \sqrt{\frac{(N-1)(N-2)(N-3)}{N}}, \qquad \overline{P}\left(\overline{V}_i\right) = \frac{2 - N + N\overline{V}_i - \overline{V}_i}{N\left(1 + \overline{V}_i\right)\left(1 - N + N\overline{V}_i\right)}. \tag{3.14}$$

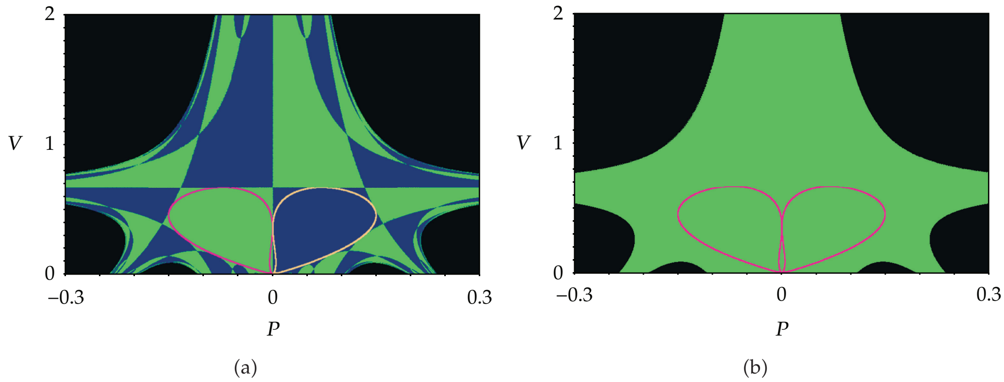

Figure 3: Phase diagrams illustrating symmetric homoclinic bifurcation. Diagram on right ($N = 3.010$) includes the period-2 attractor basins for the two limit cycles prior to the bifurcation. Diagram at left ($N = 3.011$) shows the attractor formed from the merging limit cycles and resulting loss of local attractor basin structure.

Period-2 orbit (3.11) exists for $N \geq 2$ and is created by supercritical period doubling [26] of fixed point (3.2), as shown in Figure 1. The orbits (3.12) and (3.13) exist for $N \geq 3$ and originate from saddle period doublings [26] of the fixed points (3.3) and (3.4), respectively.

Local stability analysis (see the appendix) proves generic Neimark-Sacker bifurcations occur at $N^* \approx 2.78$, creating stable limit cycles surrounding the unstable spiral points (3.11) in period-2.

Period-2 cycles (3.12) and (3.13) are found to possess one eigenvalue greater than unity for all $N \geq 3$. Thus, these period-2 cycles are saddles.

The limit cycle behavior exists until $N = 3.01$, where a symmetric homoclinic bifurcation occurs due to the collision of the two limit cycles formed by the Neimark-Sacker bifurcations with the unstable saddle point at the origin. This is illustrated in Figures 2 and 3.

As N increases through $N = 3.010$, the system becomes chaotic from the break-up of the limit cycles. The mechanism for transition to chaos via homoclinic connection tangency is similar to that of the delayed logistic map. See Aronson et al. [27] for a thorough exposition of this global bifurcation in the delayed logistic map. A complicating factor here is that there is a symmetric configuration. The two homoclinic orbits simultaneously approach the saddle point at the origin along the same side in "butterfly" configuration, rather than on opposite sides (i.e., "figure-eight" configuration), of the origin's stable manifold. It is well known [28] that such configurations lead to interesting and chaotic dynamics. Since price dynamics are chaotic for $3.0 < N < 3.3$, the asset price is not predictable for N in this range (Figure 4). As confirmed by numerical calculation of the maximal Lyapunov exponent, prices are purely chaotic for $N > 3$.

At $N \approx 3.4$, the system undergoes a boundary crisis [29]. A boundary crisis occurs when the boundary of a chaotic set collides with an unstable fixed point or unstable periodic trajectory. This causes annihilation of the chaotic set and its basin of attraction. The boundary crisis is illustrated in Figure 5, as the boundary of the chaotic attractor collides with unstable fixed points $(P, V) = (1/2N, 1)$ and $(P, V) = (-1/2N, 1)$. The former attractor becomes "leaky", and a typical trajectory will follow the attractor region as a transient but eventually escape to the attractor at infinity.

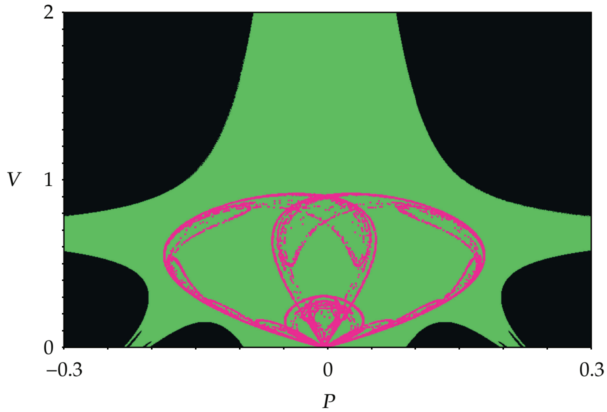

Figure 4: Phase diagram showing chaotic attractor ($N = 3.2$).

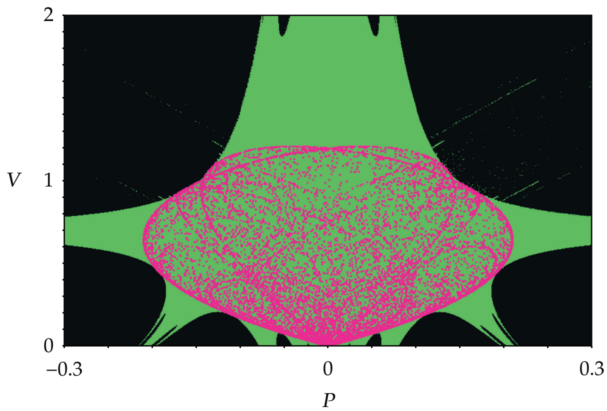

Figure 5: Boundary crisis ($N = 3.378$).

4. Conclusion

Heterogeneous trader models are a viable alternative to the usual stochastic calculus-based models of market behavior. By viewing markets from a different perspective, these models provide new insights for prediction and underlying mechanisms. The model studied here is the simplest known of heterogeneous trading type, in that it is restricted to the behavior of only two trader groups and incorporates just a single parameter. As such, the behavior described in this paper is expected to be generic behaviour common in all heterogeneous trader models. These behaviors are shown to be an approach to fundamental price, period-2 price oscillation, orbits on limit cycles or orbits alternating between limit cycles, and chaos. With increasing strength of demand, the markets become more volatile and less predictable. However, chartists and fundamentalists tend to take opposing actions. For example, chartist orders will generally counteract a strong reaction of fundamentalists to perceived mispricing of an asset. This interplay of the two groups can lead to reduced strength of demand and consequent calming of the markets.

Appendix

The Neimark-Sacker Bifurcation

The Neimark-Sacker bifurcation is the discrete equivalent of the Hopf bifurcation in continuous dynamical systems. A theorem for the Neimark-Sacker bifurcation can be given as follows (see [22]).

Theorem A.1. *Let* $\mathbf{F} : \mathbf{R}\times\mathbf{R}^2 \to \mathbf{R}^2$*; let* $(\mu, x) \mapsto \mathbf{F}(\mu, x)$ *be a* C^4 *map depending on* μ *and satisfying the following conditions:*

(i) $\mathbf{F}(\mu, 0) = 0$ *for* μ *near some fixed* μ^*,

(ii) $J(\mathbf{F})(\mu, 0)$ *has two nonreal eigenvalues* $\lambda(\mu)$ *and* $\overline{\lambda}(\mu)$ *for* μ *near* μ^* *with* $|\lambda(\mu^*)| = 1$,

(iii) $(d/d\mu)|\lambda(\mu)| > 0$ *at* $\mu = \mu^*$,

(iv) $\lambda^k(\mu^*) \neq 1$ *for* $k = 1, 2, 3, 4$.

Then, there is a smooth μ*-dependent change of coordinates bringing* $\mathbf{F}$ *into the form* $\mathbf{F}(\mu, x) = \mathbf{G}(\mu, x) + O(\|x\|^5)$*, as well as smooth functions* $a(\mu)$*,* $b(\mu)$*, and* $\omega(\mu)$ *so that the function* $\mathbf{G}(\mu, x)$ *is given in polar coordinates by*

$$\begin{pmatrix} r \\ \theta \end{pmatrix} \longmapsto \begin{pmatrix} |\lambda(\mu)|r - a(\mu)r^3 \\ \theta + \omega(\mu) + b(\mu)r^2 \end{pmatrix}. \tag{A.1}$$

If $a(\mu^*) > 0$*, then there is a neighborhood* U *of the origin and* $\delta > 0$ *such that, for* $|\mu - \mu^*| < \delta$ *and* $x^0 \in U$*, the* ω*-limit set of* x^0 *is the origin if* $\mu < \mu^*$ *and belongs to a closed invariant* C^1 *curve* $\Gamma(\mu)$ *encircling the origin if* $\mu > \mu^*$*. If* $a(\mu^*) < 0$*, then there is a neighborhood* U *of the origin and a* $\delta > 0$ *such that, for* $|\mu - \mu^*| < \delta$ *and* $x^0 \in U$*, the* α*-limit set of* x^0 *is the origin if* $\mu > \mu^*$ *and belongs to a closed invariant* C^1 *curve* $\Gamma(\mu)$ *encircling the origin if* $\mu < \mu^*$.

The sign of the coefficient $a(\mu^*)$ characterizes the type of the Neimark-Sacker bifurcation. If $a(\mu^*) > 0$, the bifurcation is supercritical and yields a stable limit cycle for $\mu > \mu^*$. If $a(\mu^*) < 0$, it is subcritical, yielding an unstable limit cycle for values $\mu < \mu^*$.

With the map at $N = N^*$ in the Jordan normal form:

$$\mathbf{x} \longmapsto \mathbf{f}(\mathbf{x}) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{pmatrix} \tag{A.2}$$

$a(N^*)$ can be calculated explicitly from the following [22]:

$$a = \operatorname{Re}\left[\frac{(1-2\lambda)\overline{\lambda}^2}{1-\lambda}\xi_{11}\xi_{20}\right] + \frac{1}{2}|\xi_{11}|^2 + |\xi_{02}|^2 - \operatorname{Re}\left(\overline{\lambda}\xi_{21}\right),$$

$$\xi_{20} = \frac{1}{8}\left[(g_1)_{x_1x_1} - (g_1)_{x_2x_2} + 2(g_2)_{x_1x_2} + i\left((g_2)_{x_1x_1} - (g_2)_{x_2x_2} - 2(g_1)_{x_1x_2}\right)\right],$$

$$\xi_{11} = \frac{1}{4}\left[(g_1)_{x_1x_1} + (g_1)_{x_2x_2} + i\left((g_2)_{x_1x_1} + (g_2)_{x_2x_2}\right)\right],$$

$$\xi_{02} = \frac{1}{8}\Big[(g_1)_{x_1x_1} - (g_1)_{x_2x_2} - 2(g_2)_{x_1x_2} + i\Big((g_2)_{x_1x_1} - (g_2)_{x_2x_2} + 2(g_1)_{x_1x_2}\Big)\Big],$$
$$\xi_{21} = \frac{1}{16}\Big[(g_1)_{x_1x_1x_1} + (g_1)_{x_1x_2x_2} + (g_2)_{x_1x_1x_2} + (g_2)_{x_2x_2x_2}$$
$$+i\Big((g_2)_{x_1x_1x_1} + (g_2)_{x_1x_2x_2} - (g_1)_{x_1x_1x_2} - (g_1)_{x_2x_2x_2}\Big)\Big]. \tag{A.3}$$

For example, the second iterate of the Westerhoff map (with $p > 0$) yields

$$\begin{aligned} P &\longmapsto P(1-N+NV)\Big(1-N+N^2P(1+V)\Big),\\ V &\longmapsto -NP(1-N+NV)\ (1+NP(1+V)), \end{aligned} \tag{A.4}$$

where absolute values are not necessary because the local attractor basins are nonfractal, and a fixed point $(\overline{\overline{P}},\overline{\overline{V}}) = ((N-2)/2N(N-1),(N-2)/N)$.

(i) We shift this fixed point to the origin, using $X = P - \overline{\overline{P}},\ Y = V - \overline{\overline{V}}$:

$$\begin{aligned} X &\longmapsto (3-N)X - \frac{(N-2)(3N-4)}{4(N-1)^2}Y + O(X,Y),\\ Y &\longmapsto (3N-4)X - \frac{(N-2)\left(4N^2-9N+6\right)}{4(N-1)^2}Y + O(X,Y). \end{aligned} \tag{A.5}$$

The eigenvalues are

$$\begin{aligned} \lambda_{1,2}(N) = &-\frac{8N^3-37N^2+52N-24}{8(N-1)^2}\\ &\pm\frac{\sqrt{512-2048N+3248N^2-2536N^3+969N^4-144N^5}}{8(N-1)^2}. \end{aligned} \tag{A.6}$$

(ii) The eigenvalues are nonreal for $N > 2.2087$, and $|\lambda(N)| = 1$ uniquely at $N^* = (7+\sqrt{17})/4$.

(iii) $(d/dN)|\lambda(N)| = \sqrt{21097-4607\sqrt{17}}/34 > 0$ at $N = N^*$.

(iv) Testing the nonresonance conditions, we find that $\lambda(N^*)$ is not a small root of unity:

$$\begin{gathered} \lambda(N^*) = \frac{33-9\sqrt{17}}{16} \pm \frac{\sqrt{-2210+594\sqrt{17}}}{16}i \approx -0.257 \pm 0.966i,\\ \lambda^2(N^*) \approx -0.878 \pm 0.50i, \quad \lambda^3(N^*) \approx -0.70 \pm 0.71i, \quad \lambda^4(N^*) \approx -0.51 \pm 0.86i. \end{gathered} \tag{A.7}$$

Using the eigenvectors at $N = N^*$ to bring the linear part of the map into the Jordan normal form and using the nonlinear part of the resulting map to determine an explicit value for $a(N^*)$, we calculate the following:

$$\begin{aligned}
\xi_{20} &= -\left(\frac{6+\sqrt{17}}{64}\right)A + \left(\frac{123+61\sqrt{17}}{64}\right)i,\\
\xi_{11} &= \left(\frac{5+\sqrt{17}}{128}\right)A - \left(\frac{81+23\sqrt{17}}{32}\right)i,\\
\xi_{02} &= \left(\frac{7+\sqrt{17}}{8}\right)i,\\
\xi_{21} &= \left(\frac{69+3\sqrt{17}}{128}\right)A - \left(\frac{697+175\sqrt{17}}{256}\right)i + \left(\frac{1895+457\sqrt{17}}{512\sqrt{17}}\right)Ai,
\end{aligned} \tag{A.8}$$

where $A := \sqrt{-2210+594\sqrt{17}}$.

This yields $a = (466653 - 116693\sqrt{17} + (697+175\sqrt{17})A)/4096 \approx 1.82$.

Since $a > 0$, the Neimark-Sacker bifurcation is supercritical and there exists a stable invariant closed orbit around the unstable fixed point $(\overline{P}, \overline{V})$ for parameter values $N > N^*$.

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