Geometric and Topological Properties of Marginally Outer Trapped Surfaces

by

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Abstract

Black holes are one of the most important predictions of the general theory of relativity. Thus, mathematicians are always looking for ways to better study and understand them. An important step towards our goal is to define what exactly a black hole is. The causal mathematical definition is not very useful for many calculations and applications. As a result, we will define the concept of marginally outer trapped surfaces. Marginally outer trapped surfaces or MOTS are closed, spacelike, two-surfaces for which the outgoing null expansion $\theta_{(\ell)} = 0$ [6].

In this thesis, we will review both historic and recent developments regarding MOTS. To this end, we begin with the general ideas, concepts, and strategies. Then the necessary tools to construct and analyze the MOTS are recalled. The various specific constructions and their properties (both successes and defects) are discussed. Finally, some of the (actual and potential) applications of MOTS and specific constructions are briefly mentioned.

To those who tried, but failed.

Lay summary

The modern theory of gravity was introduced by Albert Einstein in 1915. In General Relativity there is a one-to-one relationship between geometry and gravity. Black holes are one of the most interesting predictions of general relativity. The Schwarzschild solution or Schwarzschild black hole is named in honor of Karl Schwarzschild, who found this exact solution in 1915 and published it in January 1916. It was the first exact solution of the Einstein field equations other than the trivial flat space solution. Since then black holes have become an important as well as interesting part of GR. In the early days of general relativity, nobody believed that black holes actually exist. However observational evidence of their existence is now overwhelming [13, 1].

One definition of black holes which is very common is that a black hole is a region of spacetime from which even light cannot escape. But this global definition is not very useful for understanding the dynamics of black holes. In this thesis, we want to answer these questions: how can we define a black hole locally? And how can that definition be used to better understand things like black hole mergers? We begin with the definition of a marginal outer trapped surface (MOTS) and then we will discuss what we know about them and what is our goal for the future.

Acknowledgements

I would like to acknowledge the territory in which the Memorial University of Newfoundland and Labrador, St. John's campus, is located as the ancestral home-lands of the Beothuk, and the island of Newfoundland as the ancestral homelands of the Mi'kmaq and Beothuk. I would also like to recognize the Inuit of Nunatsiavut and NunatuKavut and the Innu of Nitassinan, and their ancestors, as the original people of Labrador.

I had the great honor of working with Professor Booth and Professor Cox.

Statement of contribution

My contribution for this thesis is to collect ideas and theorems from the different papers listed in my references and write a review on MOTS and their related problems. Figures used in this thesis that were not produced by the author are cited to their original paper. Suggested edits are from my supervisors Dr. Ivan Booth and Dr. Graham Cox.

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List of symbols

M	Four dimensional spacetime manifold
Σ	Spacelike hypersurface in M
$\mathcal{S}$	Surface in Σ
g_{ab}	Lorentzian metric on four-dimensional space M
h_{ij}	Riemannian metric on three dimensional space Σ
q_{AB}	Riemannian metric on two dimensional space $\mathcal{S}$
$R_{abcd} := g_{ae}R^{e}{}_{bcd}$	Riemannian curvature tensor on M
$R_{ab} := g^{cd}R_{acbd}$	Ricci tensor
R	Ricci scalar
$R^{(2)}$	Ricci scalar on $\mathcal{S}$
$G_{ab} := R_{ab} - \frac{1}{2}Rg_{ab}$	Einstein tensor
$\mathcal{K}$	Gauss curvature of $\mathcal{S}$
$\mathcal{L}_X$	Lie derivative in the X direction
$\chi(\mathcal{S})$	Euler characteristic of $\mathcal{S}$
∇_a	Covariant derivative on M
D_i	Covariant derivative on Σ
d_A	Covariant derivative on $\mathcal{S}$

List of abbreviations

GR	General Relativity
MOTS	Marginally outer trapped surface
BH	Black Hole

Chapter 1

Introduction

This thesis is a review of marginally outer trapped surfaces, so our goal in this chapter is to explain why we are interested in these. To do this, we first need to introduce the theory of general relativity which was proposed by Albert Einstein in 1915.

Before general relativity, Newton's gravity was the best theory of gravity. This theory changed the history of science and our understanding of gravity forever. However, we all know his theory had some flaws: the most famous one is its failure to predict the perihelion shift of Mercury. General relativity (GR), as summarized by John Wheeler, says "matter tells space how to curve, and space tells matter how to move"[33]. From this quote we know that geometry plays an important role in general relativity. We will show what exactly is general relativity and how gravity is related to the geometry of spacetime. Einstein's famous equation for gravity is

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}. \tag{1.1}$$

The right-hand side of this equation has the energy-momentum tensor T_{ab} and the left-hand side contains the geometry of spacetime: R_{ab} is the Ricci curvature and R is the Ricci scalar. The right-hand side describes the energy-momentum of matter and contains information like the energy of fields and their flux. The left-hand side describes the geometry of spacetime. If we want to solve this equation we should put some restrictions on either the energy-momentum tensor or the geometry of spacetime. In Newtonian gravity, the background space is $\mathbb{R}^3$. However in Einstein's gravity our spacetime is a 4-dimensional Lorentzian manifold (M, g). We are using the usual convention $(-+++)$ for the metric g. The Einstein field equations (1.1) are second-order non-linear partial differential equations for the metric. Because both sides of the equation are symmetric two-index tensors, we have ten coupled equations. The Ricci tensor and Ricci scalar contain Christoffel symbols, their derivatives, and the inverse of the metric. Solving these equations is extremely hard. To make progress, we restrict our attention to particular classes of solutions. For example, as a trivial vacuum solution with energy-momentum tensor $T_{ab} = 0$, we have the flat Minkowski spacetime solution

$$\eta_{ab} dx^a dx^b = -dt^2 + dx^2 + dy^2 + dz^2. \tag{1.2}$$

Non-trivial exact vacuum solutions are much harder to find. To make this task easier we can put symmetry conditions on our metric. We will consider several symmetries, including spherical symmetry, axisymmetric symmetry and also stationary spacetime, which has a time symmetry.

Now we will examine some well-known solutions of GR. The most famous one is the Schwarzchild solution [30]. Karl Schwarzschild discovered this black hole solution in 1915 a very short period of time after GR was published. Schwarzschild considered a static and spherical spacetime. In static, spherical coordinates, the solution is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{1.3}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{1.4}$$

and M is the mass of the object. Also $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$ and $r > 0$

The next solution is the Kerr black hole, which was published in 1963 [26]. This solution is a rotating generalization of the Schwarzschild black hole:

$$\begin{aligned} ds^2 = &-\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right) dt^2 - \frac{2a\sin^2\theta\left(r^2 + a^2 - \Delta\right)}{\Sigma} dt d\phi \\ &+ \left[\frac{\left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right] \sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2, \end{aligned} \tag{1.5}$$

where $a = J/M$ is angular momentum per unit mass,

$$\Sigma = r^2 + a^2\cos^2\theta \tag{1.6}$$

and

$$\Delta = r^2 + a^2 - 2Mr. \tag{1.7}$$

Black holes are the most famous and definitely craziest prediction of General Relativity. In the beginning BHs (much like gravitational waves) were regarded as mathematical objects or even science-fiction, even Einstein didn't believe in them as physical objects [15]. But we now know they are physical objects for which clear evidence exists [1, 13]. So we are now even more curious about their properties. In studying their properties it's important to have a proper mathematical definition. The causal definition of black holes, as discussed in [23], is not local. Instead, it is a global property of the causal structure of an entire spacetime; one cannot properly identify a black hole region without that global knowledge [6]. So now our mission is to look for an alternative definition to describe the black holes.

We are doing so by introducing the concept of marginally outer trapped surfaces. The goal of this thesis is to write a review on marginally outer trapped surfaces or simply MOTS. The structure of this thesis is as follows. We introduce in Chapter 2 the background material we need for defining MOTS. In Chapter 3 we introduce the concept of MOTS and we will find MOTS in a spherical spacetime. In Chapter 4 we will study the stability operator for MOTS; we want to see whether they are stable or not. In Chapter 5 we will introduce a recently developed method [10] to find MOTS in axisymmetric spacetime. Finally, in Chapter 6 we will study MOTS in higher dimensions and then the relationship between the stability of a MOTS and its topology.

Chapter 2

Background

In this chapter, we introduce definitions which set the background for the general theory of relativity. Before general relativity, Newtonian gravity was the way to go. There the gravitational force was an instantaneous force which affects objects no matter how much distance there is between them. But when Einstein published his paper in 1915 [14], he changed our understanding of gravity. His earlier 1905 work on special relativity [16] also changed our view about space and time when he introduced the concept of spacetime. In this section, we will briefly go through Einstein's equations which describe gravity in a geometric way. As mentioned earlier, general relativity basically uses the geometry of spacetime to describe gravity and so we need to understand the geometry of our spacetime. In simple terms, gravity is just the curvature of spacetime. As such, in this chapter, we will heavily rely on geometric definitions. Marginally outer trapped surfaces, or just simply MOTS, are closely related to geodesics or minimal surfaces.

2.1 Geometric definitions

In this section, we give the geometric definitions that we need in future sections.

Definition 2.1.1 (Isometry [34]). If $\phi : M \to M$ is a diffeomorphism and we have metric tensor g_{ab} we say ϕ is an **isometry** if and only if $\phi^* g_{ab} = g_{ab}$.

Definition 2.1.2 (Conformal Isometry [34]). A **conformal isometry**, ϕ, on a manifold M with metric g_{ab}, is defined to be a diffeomorphism $\phi : M \to M$ for which there is a positive function Ω such that $\phi^* g_{ab} = \Omega^2 g_{ab}$.

Definition 2.1.3 (Orbit [34])**.** Let G be a Lie group, let B be a manifold, and consider a C^∞ map $\phi : G \times B \to B$. We write $\phi(g, p)$ as $\phi_g(p)$. The map ϕ is a left action of G on B if (i) for each fixed $g \in G$, the map $\phi_g : B \to B$ is a diffeomorphism and (ii) for all $g_1, g_2 \in G$, we have $\phi_{g_1} \circ \phi_{g_2} = \phi_{g_1 g_2}$. For each $p \in B$ the set $O = \{\phi_g(p) \mid g \in G\}$ of p is called the **orbit** of p under G.

Definition 2.1.4 (Local Frames [27])**.** A **local frame** is a choice of smooth vector fields $v_1,...,v_i...,v_N$ defined on an open set $U \subset M$ such that at each point $p \in U$, $v_1,...,v_i...,v_N$ forms a basis of the tangent space T_pM.

Definition 2.1.5 (Volume [27])**.** Given M and Riemannian metric g_{ab} we define

$$d\text{vol}M := (\sqrt{\det g})v^1 \wedge \cdots \wedge v^N, \tag{2.1}$$

where $v^1,...,v^N$ denote the local coframe dual to $v_1,...,v_N$ frame and $\det g = \det\left(g\left(v_i, v_j\right)\right)$.

2.2 Spacetime

Though in the realm of GR, we usually work in four dimensions, this is not always the case. So we define spacetime as a D-dimensional manifold (M, g) with a Lorentzian metric. Then we have three distinct groups of vectors in the tangent space of our manifold M. Suppose we have $p \in M$ and $X \in T_pM$ then we can have one of these cases.

1) If $g(X, X) < 0$ we say that X is **timelike**. Curves with timelike tangent vectors are traveling with speed slower than light.

2) If $g(X, X) > 0$ we say that X is **spacelike**. Curves with spacelike tangent vectors would be travelling faster than the speed of light.

3) If $g(X, X) = 0$ we say that X is **lightlike** or **null**. Curves with null tangent vectors are traveling at the speed of light.

Definition 2.2.1 (Causal curve)**.** A curve is a **casual curve** if the tangent vector is timelike or null at all points on the curve.

Definition 2.2.2 (Strongly Causal [34])**.** A spacetime (M, g_{ab}) is said to be **strongly causal** if for all $p \in M$ and every neighborhood O of p, there exists a neighborhood

V of p contained in O such that no causal curve intersects V more than once. The motivation behind this definition is to prevent the existence of closed, timelike curves, which are viewed as unphysical since they correspond to time travel.

2.2.1 Geometry of Spacetime

Definition 2.2.3 (Isometry Group [34])**.** If we have Riemannian manifold (M, g_{ab}) the **isometry group** is the group of Riemannian isometries $F : (M, g) \to (M, g)$ with the function composition as group operation.

Definition 2.2.4 (Spherical symmetry [34])**.** A spacetime is **spherically symmetric** if its isometry group contains a subgroup isomorphic to the group SO(3), and the orbits of this subgroup are two-dimensional spheres. From the physics point of view, these spherically symmetric spacetimes are invariant under all rotations around the same point.

Definition 2.2.5 (Stationary spacetime [34])**.** A spacetime is said to be **stationary** if there exists a one-parameter group of isometries, ϕ_t, whose orbits are timelike curves. This group of isometries expresses the time translation symmetry of the spacetime. Equivalently, a stationary space-time is one which possesses a timelike Killing vector field, ξ^a. A spacetime is said to be **static** if it is stationary and if, in addition, there exists a (spacelike) hypersurface Σ which is orthogonal to the orbits of the isometry.

Definition 2.2.6 (Axisymmetric spacetime [34])**.** A spacetime is said to be **axisymmetric** if there exists a one-parameter group of isometries χ_ϕ whose orbits are closed spacelike curves. This implies the existence of a spacelike Killing field ψ^a whose integral curves are closed.

2.2.2 Asymptotically flat spacetime

In general relativity sometimes we want to study an isolated system and its behavior. For example, if we're going to study a black hole, we want to ignore the effect of distant matter and study the black hole in an otherwise empty universe. Asymptotically flat spacetimes represent ideally isolated systems in general relativity [34] for which spacetime far from the centre approaches flat (Minkowski) space. Intuitively by null infinity we mean where all null geodesics end up and for spacelike infinity this is end of all outward traveling spacelike geodesics.

Definition 2.2.7 (Asymptotically Flat Spacetime [34]). A vacuum spacetime (M, g_{ab}) is said to be **asymptotically flat** at null and spatial infinity if there exists a spacetime $\left(\tilde{M}, \tilde{g}_{ab}\right)$-with $\tilde{g}_{ab}$ C^{∞} everywhere except possibly at a point i^0 where it is $C^{>0}$(i.e. meaning $\tilde{g}_{ab}$ will be continuous) and a conformal isometry $\psi : M \to \psi[M] \subset \tilde{M}$ with conformal factor Ω satisfying the following conditions :

(1) $\overline{J^+ (i^0)} \cup \overline{J^- (i^0)} = \tilde{M} - M$ where $\overline{J^+ (i^0)}$ is the closure of the causal future, $J^+ (i^0)$.

(2) There exists an open neighborhood,V, of $\dot{M} = i^0 \cup \mathscr{I}^+ \cup \mathscr{I}^-$ such that the spacetime $(V, \tilde{g}_{ab})$ is strongly causal. $\dot{M}$ is the boundary of M.

(3) Ω can be extended to a function on all of $\tilde{M}$ which is C^2 at i^0 and C^{∞} elsewhere.

2.3 General Relativity

In this section, we will briefly go through Einstein's equations. As we mentioned earlier, gravity is caused by the curvature of spacetime. Suppose (M, g) is our spacetime. In general relativity, continuous matter distributions and fields are again described by a stress-energy tensor T_{ab} [34] and we have Riemannian curvature tensor R_{abcd}. Then the Einstein field equations are

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}. \tag{2.2}$$

2.3.1 Riemann Tensor

Before defining the Riemann tensor we need to define the Lie bracket which is used in the definition of Riemann tensor.

Definition 2.3.1 (Lie Bracket [25]). Suppose we have a manifold M. For vector fields X, Y on M, the **Lie bracket** is

$$[X, Y] := X^b \frac{\partial Y^a}{\partial x^b} \frac{\partial}{\partial x^a} - Y^b \frac{\partial X^a}{\partial x^b} \frac{\partial}{\partial x^a} \tag{2.3}$$

As we know, curvature of spacetime plays a crucial role in GR, so we need to give the proper definitions of the curvatures we are using for spacetime.

Definition 2.3.2 (Riemann tensor [25])**.** Given a covariant derivative ∇, where is a Levi-Civita connection associated with the metric, if we have three vector fields X,Y and Z we can define the curvature tensor

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{2.4}$$

If we write in local coordinates we will get

$$R\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right)\frac{\partial}{\partial x^c} = R^d_{cab}\frac{\partial}{\partial x^d}. \tag{2.5}$$

Now we have

$$R_{abcd} = g_{ae} R^e{}_{bcd}. \tag{2.6}$$

We define the Ricci tensor

$$R_{ab} = g^{cd} R_{acbd} \tag{2.7}$$

and the Ricci scalar is

$$R = g^{ab} R_{ab}. \tag{2.8}$$

Next we define the Gauss curvature.

Definition 2.3.3 (Gauss Curvature [27])**.** If we have a two-dimensional manifold M, we will define the **Gauss curvature** based on the Riemann tensor,

$$R_{abcd} = \mathcal{K}\left(g_{ac} g_{bd} - g_{ab} g_{cd}\right), \tag{2.9}$$

where the function $\mathcal{K} = \mathcal{K}(x)$ is called Gauss curvature. Contracting both sides by $g_{ac} g_{bd}$ we can see $\mathcal{K} = R/2$.

Definition 2.3.4 (Extrinsic Curvature)**.** If we have Σ as a hypersurface in M we can define the **extrinsic curvature** K_{ij} of Σ as follow:

$$K_{ij} = e^a_i e^b_j \nabla_a u_b, \tag{2.10}$$

where $e^a_i = \frac{\partial x^a}{\partial y^i}$ and $y^i (i = 1, 2, 3)$ are coordinates intrinsic to the hypersurface Σ. in a local coordinate system x^a for M, Σ is described locally by the immerision $x^a = x^a\left(y^i\right)$.

Definition 2.3.5 (Gauss-Codazzi equations)**.** The **Gauss-Codazzi equations** relate the induced metric and extrinsic curvature of a hypersurface to the geometry of the ambient Lorentzian manifold surface:

$$R_{abcd}e_i^a e_j^b e_k^c e_l^d = R_{ijkl} - (K_{il}K_{jk} - K_{ik}K_{jl}) \tag{2.11}$$

$$R_{cabd}u^c e_i^a e_j^b e_k^d = D_k K_{ij} - D_j K_{ik} \tag{2.12}$$

Here u^c is the future-oriented unit normal to Σ.

2.4 Dominant energy condition

If T_{ab} is the energy-momentum tensor, we will say T_{ab} satisfies the **dominant energy condition** if for all future-directed causal vectors X and Y we have

$$T_{ab}X^a Y^b \geq 0. \tag{2.13}$$

In particular this implies the weak (null) energy condition which says that if X is timelike (null) then $T_{ab}X^a X^b \geq 0$ which, from the physics point of view, says that the energy density of matter, as measured by an observer whose four-velocity is X^a, is non-negative. Physicists believe that for all classical matter the energy density of matter for every timelike X^a is nonnegative. The dominant energy condition is interpreted as saying that the speed of energy flow of matter is always less than the speed of light [34] .

2.5 Gauss-Bonnet Theorem

In this part we will quickly review the Gauss-Bonnet theorem. The importance of this theorem is that it relates the topology of a manifold to its geometry. First we will introduce the Euler characteristic χ. This is a topological invariant which means it will not change if we continuously deform our manifold. For example χ for a sphere is two, while for a torus it is zero.

Theorem 1 (Gauss-Bonnet Theorem [32])**.** *Let M be a compact, oriented, two-dimensional Riemannian manifold with Gaussian curvature $\mathcal{K}$. Then we have the following relationship:*

$$\chi(M) = \frac{1}{2\pi} \int_M \mathcal{K} d\mathrm{Vol} M. \tag{2.14}$$

2.6 Geodesics

People including myself often use the term of "the shortest curve between two points" as a casual definition for geodesics. However, geodesics are not necessarily minimizing but they are critical points of the length functional.

Definition 2.6.1 (Geodesics)**.** Suppose we have a Riemannian manifold (M, g) with a connection ∇. A curve $\gamma : I \to M$ is a geodesic if

$$\nabla_{\gamma'} \gamma' = 0. \tag{2.15}$$

As an example, the most famous geodesics are straight lines in Euclidean space. If our curve is arc length parameterized then the geodesic equations will take the coordinate form

$$\frac{dT^a}{ds} + \Gamma^a_{bc} T^b T^c = 0, \tag{2.16}$$

where $T^a = d\gamma^a/ds$ is the unit tangent vector. For example, for a two-dimensional sphere we have

$$ds^2 = R^2 d\Theta^2 + R^2 \sin^2 \Theta d\Phi^2, \tag{2.17}$$

and these two equations will give us the following equation for the geodesics:

$$\ddot{\Theta} - (\sin\Theta \cos\Theta)\dot{\Phi}^2 = 0 \tag{2.18}$$

and

$$\ddot{\Phi} + 2(\cot\Theta)\dot{\Theta}\dot{\Phi} = 0. \tag{2.19}$$

Solutions to these two equations are great circles on the sphere.

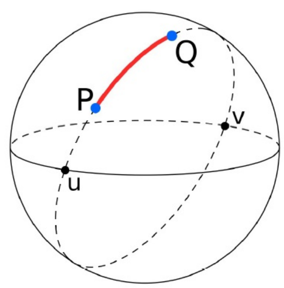

Figure 2.1: The red arc is the shortest curve between p and q, however the dotted path is not but both of them are geodesics [35]

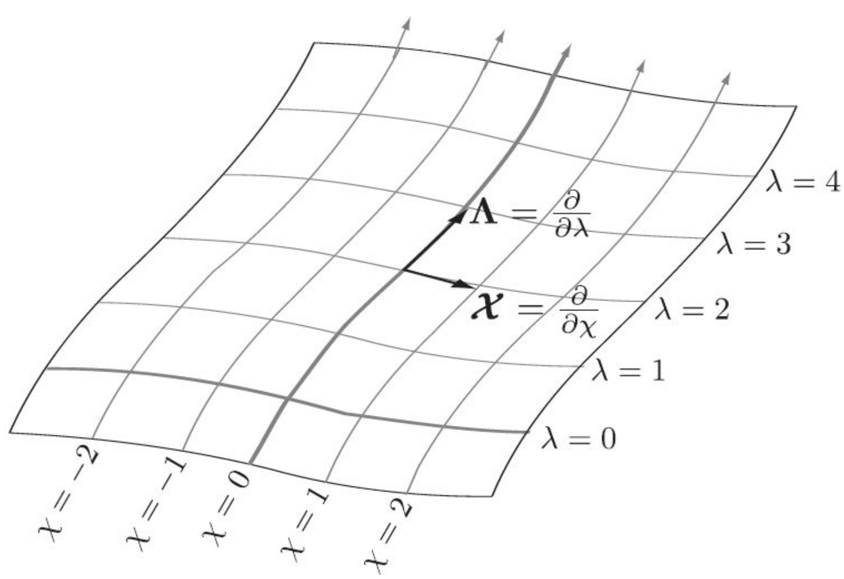

Figure 2.2: A congruence of geodesics [7]

2.6.1 First Variation Of Length

As we mentioned earlier and as is demonstrated in Figure (2.1), geodesics are not necessarily the shortest curve between two points. So we need some sort of a test for determining whether the geodesic is locally minimal or some kind of saddle point. Similarly to a multi-variable function, we will use the first and the second derivative test for the length functional. As you can see in Figure 2.2 we consider a congruence of geodesics. So we want to see what will happen if we deform a geodesic to a nearby curve which is not necessarily geodesic. χ labels which curve we are on and λ is an affine parameter on the geodesic. These two λ and χ form a coordinate system for the congruence of geodesics. The tangent vectors to these curves are

$$\Lambda = \frac{\partial}{\partial \lambda} \tag{2.20}$$

and

$$\mathcal{X} = \frac{\partial}{\partial \chi}. \tag{2.21}$$

Because they are coordinate vectors we have

$$[\mathcal{X}, \Lambda] = 0. \tag{2.22}$$

Let T be the unit tangent vector to curves of constant χ and N be the unit tangent vector to curves of constant λ. Then

$$\Lambda = \alpha T \text{ and } \mathcal{X} = \varphi N \tag{2.23}$$

for some functions α and φ. By construction $\mathcal{X}$ is perpendicular to Λ on Γ and we can choose $\alpha|_\Gamma = 1$. The overdot and δ are shorthand for derivative operators:

$$\dot{} = \Lambda^a \nabla_a = \frac{\partial}{\partial \lambda}, \quad \delta = \chi^a \nabla_a = \frac{\partial}{\partial \chi}. \tag{2.24}$$

So

$$\dot{T}^b = \Lambda^a \nabla_a T^b \tag{2.25}$$

but $\Lambda = \alpha T$ so we have

$$\dot{T}^b = \alpha T^a \nabla_a T^b. \tag{2.26}$$

Before starting our calculations we should mention a few important results. We will start with calculating $T^a\nabla_a T^b$ which we can write it with respect to the basis N and T:

$$T^a\nabla_a T^b = AN^b + BT^b \tag{2.27}$$

for some coefficents A and B. We know $T^b N_b = 0$ so

$$T_b T^a\nabla_a T^b = T_b(AN^b + BT^b). \tag{2.28}$$

The left-hand side $T_b T^a\nabla_a T^b = 0$ and we know $T^b N_b = 0$ so we will have

$$0 = AN^b T_b + BT^b T_b = B. \tag{2.29}$$

Hence

$$T^a\nabla_a T^b = \kappa N^b \tag{2.30}$$

for some function which we call curvature κ. Similarly for the $T^a\nabla_a N^b$ we can show

$$T^a\nabla_a N^b = -\kappa T^b. \tag{2.31}$$

For the $N^a\nabla_a T^b$ we can write the same way for $N^a\nabla_a T^b$

$$N^a\nabla_a T^b = \widetilde{A}T^b + \widetilde{B}N^b, \tag{2.32}$$

for some $\widetilde{A}$ and $\widetilde{B}$. Then by the same methods we have

$$N^a\nabla_a T^b = -\mu N^b \tag{2.33}$$

and

$$N^a\nabla_a N^b = \mu T^b, \tag{2.34}$$

for some μ. Therefore

$$\dot{T} = \alpha\kappa N \quad \text{and } \dot{N} = -\alpha\kappa T \tag{2.35}$$

and

$$\delta N = \varphi\mu T \quad \text{and } \delta T = -\varphi\mu N. \tag{2.36}$$

We calculate the Lie bracket as

$$0 = [\mathcal{X}, \Lambda]^a = \mathcal{X}^c \nabla_c \Lambda^a - \Lambda^c \nabla_c \mathcal{X}^a. \tag{2.37}$$

However $\mathcal{X}^c \nabla_c \Lambda^a = \delta \Lambda^a$ and $\Lambda = \alpha T$ so $\Lambda^b \nabla_b \mathcal{X}^a = \frac{d}{d\lambda}(\varphi N^a)$. Then by the product rule, we have

$$\begin{aligned} [\mathcal{X}, \Lambda]^a &= (\delta\alpha) T^a + \alpha \delta T^a - \dot{\varphi} N^a - \varphi \dot{N}^a \\ &= (\delta\alpha) T^a - \alpha\varphi\mu N^a - \dot{\varphi} N^a + \alpha\varphi\kappa T^a \\ &= (\delta\alpha + \alpha\varphi\kappa) T^a - (\dot{\varphi} + \alpha\mu\varphi) N^a = 0. \end{aligned}$$

However T and N are linearly independent so the coefficients must individually vanish:

$$\delta\alpha + \alpha\varphi\kappa = 0 \tag{2.38}$$

and

$$\dot{\varphi} + \alpha\mu\varphi = 0. \tag{2.39}$$

Therefore

$$\delta\alpha = -\alpha\varphi\kappa, \quad \dot{\varphi} = -\alpha\varphi\mu. \tag{2.40}$$

Next, let us define the length functional. The length of a curve in Riemannian geometry is:

$$\ell(\chi) = \int_{\lambda_1}^{\lambda_2} d\lambda \sqrt{g_{ab}\Lambda^a\Lambda^b} = \int_{\lambda_1}^{\lambda_2} d\lambda\alpha. \tag{2.41}$$

Letting $\delta = \partial/\partial\chi$ and using overdots for derivatives with respect to λ, we want to calculate the first variation. We have:

$$\delta\ell = \int_\Gamma d\lambda\delta\alpha = -\int_\Gamma d\lambda\alpha\varphi\kappa. \tag{2.42}$$

$ds = \alpha d\lambda$ so we have

$$\delta\ell = -\int_\Gamma ds\kappa\varphi. \tag{2.43}$$

We want an extremal length so should have

$$\delta\ell|_\Gamma = 0 \text{ for all } \varphi. \tag{2.44}$$

Hence we have for $\kappa = 0$ for $\mathcal{X} = 0$. That is, $\kappa = 0$ for a geodesic.

2.6.2 Second Variation Of Length

We assume the first variation of length is zero. Now we calculate the second variation,

$$\delta^2 \ell = -\int_\Gamma d\lambda \delta\left(\alpha\kappa\varphi\right). \tag{2.45}$$

Because of the product rule, we will have

$$\delta^2 \ell = \int_\Gamma d\lambda [\delta(\alpha\varphi)\kappa + \alpha^2\varphi\delta\kappa]. \tag{2.46}$$

Then

$$\dot{\varphi} = -\alpha\mu\varphi \tag{2.47}$$

and

$$\delta\alpha = -\alpha\kappa^2. \tag{2.48}$$

From earlier (2.35):

$$T^a\nabla_a\mathcal{X}^b = \frac{1}{\alpha}\Lambda^a\nabla_a\left(\varphi N^b\right) \tag{2.49}$$

so

$$T^a\nabla_a\mathcal{X}^b = \frac{1}{\alpha}\left(\dot{\varphi}N^b - \alpha\kappa\varphi T^a\right) \tag{2.50}$$

and therefore

$$T^a\nabla_a\mathcal{X}^b = \frac{\dot{\varphi}}{\alpha}N^b - \kappa\varphi T^a. \tag{2.51}$$

We first calculate

$$\delta\kappa = \delta\left(N^bT^a\nabla_aT^b\right). \tag{2.52}$$

With the product rule we will get

$$\delta\kappa = \delta N^b(T^a\nabla_aT_b) + \delta T^a(N^b\nabla_aT^b) + \delta(\nabla_aT^b)(N^bT^a) \tag{2.53}$$

so we can rewrite

$$\delta\kappa = \left(\delta N^bT^a + N^b\delta T^a\right)\nabla_aT_b + N^bT^a\chi^c\nabla_c\nabla_aT_b \tag{2.54}$$

but we have $\delta T^a = \frac{\varphi}{\alpha} N^a$ and $\delta N^a = -\frac{\dot{\varphi}}{\alpha} T^a$ and also by the definition of the Riemann tensor

$$R_{cabd} T^d + \nabla_a \nabla_c T_b = \nabla_c \nabla_a T_b. \tag{2.55}$$

Therefore

$$\delta\kappa = \frac{\dot{\varphi}}{\alpha} \left(-T^b T^a + N^b N^a \right) \nabla_a T_b + N^b T^a X^c \left[R_{cabd} T^d + \nabla_a \nabla_c T_b \right]. \tag{2.56}$$

We can change the order of χ and Λ (2.22). Therefore

$$N^b N^a \nabla_a T_b = N^b T^a \nabla_a N_b.$$

We also have $\mathcal{X} = \varphi N$, so if we replace them and use the previous result we have

$$\delta\kappa = -\frac{\dot{\varphi}}{\alpha} \left(T_b N^a \nabla_a N^b \right) + \varphi R_{cabd} N^c T^a N^b T^d + N^b T^a \chi^c \nabla_a \nabla_c T_b. \tag{2.57}$$

In our case we are interested in curves on 2D manifolds so we can rewrite our Riemannian curvature tensor in terms of Gauss curvature (2.9). Then

$$\varphi R_{cabd} N^c T^a N^b T^d = \varphi \mathcal{K} q_{cb} q_{ad} N^c T^a N^b T^d - \varphi \mathcal{K} q_{cd} q_{ab} N^c T^a N^b T^d \tag{2.58}$$

and we have

$$\varphi \mathcal{K} q_{cb} q_{ad} N^c T^a N^b T^d = \varphi \mathcal{K} N^c N_c T^a T_a. \tag{2.59}$$

But $N^c N_c = 1$ and $T^a T_a = 1$ so we obtain

$$\varphi \mathcal{K} q_{cb} q_{ad} N^c T^a N^b T^d = \varphi \mathcal{K}. \tag{2.60}$$

For the second part, we have

$$\varphi \mathcal{K} q_{ad} q_{bc} N^c T^a N^b T^d = 0 \tag{2.61}$$

and so

$$\delta\kappa = -\frac{\dot{\varphi}}{\alpha} \mu + \mathcal{K} \varphi + N^b T^a \nabla_a \left(\mathcal{X}^c \nabla_c T_b \right) - N^b \left(T^a \nabla_a \mathcal{X}^c \right) \nabla_c T_b \tag{2.62}$$

$$= \frac{\dot{\varphi}}{\alpha} \left(\frac{\dot{\varphi}}{\alpha \varphi} \right) + \mathcal{K} \varphi + N^b T^a \nabla_a \left(\frac{\dot{\varphi}}{\alpha} N_b \right) + N^b (\frac{\dot{\varphi}}{\alpha} N^c - \kappa \varphi T^c) \nabla_c T_b \tag{2.63}$$

$$= \frac{1}{\varphi}\left(\frac{\dot{\varphi}}{\alpha}\right)^2 + \mathcal{K}\varphi + \frac{1}{\alpha}\Lambda^a\nabla_a\left(\frac{\dot{\varphi}}{\alpha}\right) + \frac{\dot{\varphi}}{\alpha}T_bN^c\nabla_cN^b + \kappa\varphi N^bT^c\nabla_cT_b \tag{2.64}$$

$$= \frac{1}{\varphi}\left(\frac{\dot{\varphi}}{\alpha}\right)^2 + \mathcal{K}\varphi + \frac{1}{\alpha}\Lambda^a\nabla_a\left(\frac{\dot{\varphi}}{\alpha}\right) + \frac{\dot{\varphi}}{\alpha}\mu + \kappa^2\varphi \tag{2.65}$$

$$= \frac{1}{\varphi}\left(\frac{\dot{\varphi}}{\alpha}\right)^2 + \mathcal{K}\varphi + \frac{1}{\alpha}\Lambda^a\nabla_a\left(\frac{\dot{\varphi}}{\alpha}\right) + \frac{\dot{\varphi}}{\alpha}\left(-\frac{\dot{\varphi}}{\varphi\alpha}\right) + \kappa^2\varphi \tag{2.66}$$

$$= T^a\nabla_a\left(T^b\nabla_b\varphi\right) + \left(\mathcal{K} + \kappa^2\right)\varphi. \tag{2.67}$$

Then we have

$$\delta^2\ell = -\int_\Gamma d\lambda\left(\delta(\alpha\varphi)\kappa + \alpha\varphi\left[\varphi_{TT} + \left(\mathcal{K} + \kappa^2\right)\varphi\right]\right) \tag{2.68}$$

$$= -\int_\Gamma d\lambda\left(\alpha\mathcal{K}\delta\varphi + \kappa\varphi\delta\alpha + \alpha\varphi\varphi_{TT} + \alpha\varphi\left[\mathcal{K} + \kappa^2\right]\varphi\right) \tag{2.69}$$

$$= -\int_\Gamma d\lambda\left(\alpha\mathcal{K}\delta\varphi + \kappa\varphi\left[-\alpha\kappa\varphi\right] + \alpha\varphi\varphi_{TT} + \alpha\varphi\mathcal{K}\varphi + \alpha\kappa^2\varphi^2\right) \tag{2.70}$$

$$= -\int_\Gamma d\lambda\left(\alpha^2\varphi\left[\varphi_{TT} + \mathcal{K}\varphi\right] + \alpha\kappa\delta\varphi\right). \tag{2.71}$$

Therefore

$$\delta^2\ell = \int_\Gamma ds(\varphi J\varphi + \kappa\delta\varphi) \tag{2.72}$$

where $J\varphi := -\varphi_{TT} - \mathcal{K}\varphi$ is called the Jacobi operator. Note that for a geodesic $\kappa = 0$. We can rewrite the eigenvalue problem for the Jacobi operator as $\left(\frac{d^2}{ds^2} + \mathcal{K}\right)\varphi = -\xi\varphi$ where ξ is an eigenvalue and φ the corresponding eigenfunction. The Jacobi operator can be understood via Sturm-Liouville theory. Then it has real eigenvalues and we can set them in order $\xi_0 < \xi_1 < \xi_2 \cdots < \xi_n < \cdots \to \infty$. We call the smallest eigenvalue the principal eigenvalue. So if the principal eigenvalue $\xi_0 > 0$, then we have minimum length. If $\xi_0 < 0$, the curve is not a minimum. If $\xi_0 = 0$, the test is inconclusive, and we need to go to the third variation of length functional.

2.6.3 Third Variation

In this section [8], we assume the first and second variations are zero in the φ direction. For the third variation first we will calculate $\delta(\alpha\kappa)$ with the product rule, we have

$$\delta(\alpha\kappa) = \alpha\delta\kappa + \kappa\delta\alpha \tag{2.73}$$

$$= \alpha(\varphi_{TT} + \left[\mathcal{K} + \kappa^2\right]\varphi) - \alpha\kappa^2\varphi \tag{2.74}$$

$$= \alpha\left(\varphi_{TT} + \kappa\varphi\right). \tag{2.75}$$

For a general function H, we want to calculate δH_T. With our notation, H_T is equal to $T^a\nabla_a H$. We can replace T with Λ and with our notation and using the product rule we will get

$$\delta H_T = \delta\left(\frac{1}{\alpha}\dot{H}\right) = -\frac{\delta\alpha}{\alpha^2}\dot{H} + \frac{1}{\alpha}\delta\dot{H}. \tag{2.76}$$

But we have $\delta\alpha = -\alpha\varphi\kappa$ so we get

$$= \frac{1}{\alpha}\kappa^2\varphi\dot{H} + \frac{d}{ds}(\delta H) = \frac{d}{ds}(\delta H) + \kappa^2\varphi\frac{dH}{ds}. \tag{2.77}$$

If we replace H by φ_T we will have

$$\delta\varphi_{TT} = \frac{d}{ds}\left(\delta\varphi_T\right) + \kappa\varphi\varphi_{TT}. \tag{2.78}$$

So the final form of the third variation will be

$$\varphi\delta\varphi_{TT} = \varphi\left(\frac{d}{ds}\left(\delta\varphi_T\right) + \kappa\varphi\varphi_{TT}\right) \tag{2.79}$$

$$= \frac{d}{ds}\left(\varphi\delta\varphi_T\right) - \varphi_T\delta\varphi_T + \kappa\varphi^2\varphi_{TT} \tag{2.80}$$

$$= \frac{d}{ds}\left(\varphi\delta\varphi_T\right) - \varphi_T\left(\frac{d}{ds}\delta\varphi + \kappa\varphi\varphi_T\right) + \kappa\varphi^2\varphi_{TT} \tag{2.81}$$

$$= \frac{d}{ds}\left(\varphi\delta\varphi_T\right) - \varphi_T\left(\frac{d}{ds}\delta\varphi + \kappa\varphi\varphi_T\right) + \kappa\varphi^2\varphi_{TT} \tag{2.82}$$

$$= \frac{d}{ds}\left(\varphi\delta\varphi_T\right) - \frac{d}{ds}\left(\varphi_T\delta\varphi\right) + \varphi_{TT}\delta\varphi - \kappa\varphi\varphi_T^2 + \kappa\varphi\varphi_{TT} \tag{2.83}$$

$$= \frac{d}{ds}\left(\varphi\delta\varphi_T - \varphi_T\delta\varphi\right) + \varphi_{TT}\delta\varphi + \kappa\varphi\left(\varphi\varphi_{TT} - \varphi_T^2\right). \tag{2.84}$$

Therefore we will have

$$\varphi\delta(J\varphi) = \varphi\delta\varphi_{TT} + \varphi^2\delta\mathcal{K} + \mathcal{K}\varphi\delta\varphi \tag{2.85}$$

$$\frac{d}{ds}\left(\varphi\delta\varphi_T - \varphi_T\delta\varphi\right) + \left(\varphi_{TT} + \kappa\varphi\right)\delta\varphi + \varphi^3\mathcal{K}_N + \kappa\varphi\left(\varphi\varphi_{TT} - \varphi_T^2\right) \tag{2.86}$$

Therefore

$$\delta^3\ell = -\int_\Gamma d\lambda\delta(\alpha\varphi J\varphi + \alpha\kappa\delta^2\varphi) \tag{2.87}$$

$$\delta^3\ell = -\int_\Gamma d\lambda\left[(-\alpha\kappa\varphi)\left(\varphi J\varphi + \kappa\delta^2\varphi\right) + \alpha\delta\varphi J\varphi + \alpha\varphi\delta(J\varphi) + \delta(\alpha\kappa)\delta\varphi + \alpha\kappa\delta^2\varphi\right] \tag{2.88}$$

$$= -\int_\Gamma d\lambda\alpha(\kappa\varphi^2 J\varphi - \kappa^2\varphi\delta\varphi + (J\varphi)\,\delta\varphi + (J\varphi)\delta\varphi + \kappa\delta^2\varphi + (J\varphi)\delta\varphi \tag{2.89}$$

$$+ \varphi^3\mathcal{K}_N + \kappa\varphi\left(\varphi^2\varphi_{TT} - \varphi_T^2\right) \tag{2.90}$$

$$= -\int_\Gamma ds(\kappa\delta^2\varphi + \left(3J\varphi - \kappa^2\varphi\right)\delta\varphi + \varphi^2\left(\varphi\mathcal{K}_N - \kappa J\varphi\right) + \kappa\varphi(\varphi\varphi_{TT} - \varphi_T^2) \tag{2.91}$$

$$= \int_\Gamma ds(\kappa\delta^2\varphi + 3J\varphi - \kappa^2))\delta\varphi + \varphi^2\left(\varphi\mathcal{K}_N - \kappa J\varphi\right) + \kappa\varphi\left(\varphi\varphi_{TT}^2 - \varphi_T^2\right) \tag{2.92}$$

but most of the terms are zero because the first and second variation vanish so we get

$$\delta^3\ell = -\int_\Gamma ds\mathcal{K}_N\varphi^3. \tag{2.93}$$

2.7 Minimal surfaces

We now consider minimal surfaces: the surface generalization of a geodesic. For geodesics, we considered a one-parameter family of curves between two fixed points. Here we will consider the analogous case for the area functional. Suppose we have Σ as our D dimensional Riemannian manifold and we have a hypersurface $\mathcal{S}$ in Σ. We want to know how the surface area of $\mathcal{S}$ changes when we deform that surface in the direction of a normal vector. We start with the definition of area functional:

$$\text{Area}(\mathcal{S}) = \int_{\mathcal{S}} 1dA. \tag{2.94}$$

Let the normal vector to $\mathcal{S}$ in Σ be n and φ be a smooth function. We now calculate how the area of $\mathcal{S}$ changes if deform it in the direction of $n\varphi$. Suppose $q = \det\left(q_{AB}\right)$ where q_{AB} is the induced metric on surface $\mathcal{S}$. Then (2.94) becomes

$$A = \int_{\mathcal{S}} \sqrt{q}d^{D-1}x. \tag{2.95}$$

We now consider a congruence of curves which pass through $\mathcal{S}$ and which may be parameterized as $X\left(\lambda, x^A\right)$ such that: 1) $X\left(0, x^A\right)$ parametrizes $\mathcal{S}$ itself and 2) $\frac{\partial}{\partial\lambda}\big|_{\lambda=0} = n\varphi$. We can then use this to evolve $\mathcal{S}$ into a family of surfaces $\mathcal{S}_\lambda$, then the first variation of the area is:

$$\frac{\partial A}{\partial\lambda} = \int_{\mathcal{S}} \frac{1}{2\sqrt{q}} \mathcal{L}_{\frac{\partial}{\partial\lambda}} q d^{D-1}x. \tag{2.96}$$

So we will get

$$\frac{\partial A}{\partial\lambda} = \int_{\mathcal{S}} \frac{1}{\sqrt{q}} q q^{AB} \mathcal{L}_{\frac{\partial}{\partial\lambda}} q_{AB} d^{D-1}x \tag{2.97}$$

$$= \int_{\mathcal{S}} \frac{1}{2\sqrt{q}} q q^{AB} \left(2\varphi K_{AB}\right) d^{D-1}x. \tag{2.98}$$

The trace of extrinsic curvature is

$$K = q^{AB} K_{AB} \tag{2.99}$$

Therefore we will get

$$\frac{\partial A}{\partial\lambda} = \int_{S} \sqrt{q}\varphi K d^{D-1}x. \tag{2.100}$$

So minimal surfaces are the surfaces with $K = 0$.

Now we want to see how our minimal surface will change if we deform it in the direction normal to the surface. We want to derive the second variation of area based on the information which we have about our surface, not with the respect to the ambient space, so we will first derive some relationships and formulas which will be useful in our future calculations.

We know $\left(\frac{\partial}{\partial\lambda}\right)^i = \varphi n^i \Rightarrow (d\lambda)_i = \frac{1}{\varphi} n_i$. Then we have

$$n^i D_i n_j = n^i D_i \left(\varphi [d\lambda]_j\right). \tag{2.101}$$

Applying the product rule for the covariant derivative, we obtain

$$n^i D_i n_j = (n^i D_i \varphi)[d\lambda]_j + \varphi n^i D_i D_j \lambda \tag{2.102}$$

$$= \left[\frac{1}{\varphi} n^i D_i \varphi\right] n_j + \varphi n^i D_j D_i \lambda \tag{2.103}$$

$$= \left[\frac{1}{\varphi} n^i D_i \varphi\right] n_j + \varphi n^i D_j \left[\frac{1}{\varphi} n_i\right] \tag{2.104}$$

$$= \left[\frac{1}{\varphi} n^i D_i \varphi\right] n_j - \frac{1}{\varphi} D_j \varphi \tag{2.105}$$

so we will get

$$n^i D_i n_j = \frac{1}{\varphi}\left(n^i n_j - h^i_j\right) D_i \varphi = -\frac{1}{\varphi} D_j \varphi. \tag{2.106}$$

For the covariant derivative of extrinsic curvature, we have:

$$\frac{\partial}{\partial \lambda} K = D_\Lambda K \tag{2.107}$$

$$= \Lambda^k D_k K. \tag{2.108}$$

We have $\Lambda = \varphi n$ so we will have

$$\varphi n^k D_k K = \varphi n^k D_k \left[q^{ij} D_i n_j\right] \tag{2.109}$$

$$= \varphi n^k D_k \left[q^{ij} D_i n_j\right]. \tag{2.110}$$

We have $h^{ij} = q^{ij} + n^i n^j$ so we will have

$$\varphi n^k D_k K = \varphi n^k D_k \left[\left(h^{ij} - n^i n^j\right) D_i n_j\right]. \tag{2.111}$$

We have

$$n^i n_i = 1 \tag{2.112}$$

so we have

$$n^i n^j D_i n_j = 0. \tag{2.113}$$

Therefore we have

$$n^k D_k[-n^i n^j D_i n_j] = 0. \tag{2.114}$$

Again with the help of the product rule, we have

$$\varphi n^k \nabla_k [h^{ij} D_i n_j] = \varphi n^k D_k (h^{ij})(D_i n_j) + \varphi n^k D_k (D_i n_j)(h^{ij}). \tag{2.115}$$

But $D_k h^{ij} = 0$ so we will get

$$\varphi n^k D_k[h^{ij} D_i n_j] = \varphi h^{ij} n^k D_k D_i n_j. \tag{2.116}$$

With the definition of the Riemann tensor we have

$$\varphi n^k D_k[h^{ij} D_i n_j] = \varphi h^{ij} n^k \left(\mathcal{R}_{kij}{}^l n_l + D_i D_k n_j\right) \tag{2.117}$$
$$= \varphi \left(-h^{ij} \mathcal{R}_{ikjl} n^k n^l + h^{ij} n^k D_i D_k n_j\right) \tag{2.118}$$
$$= -\varphi \mathcal{R}_{kl} n^k n^l + \varphi h^{ij} n^k D_i D_k n_j. \tag{2.119}$$

We can rewrite the Ricci tensor by

$$\mathcal{R} = \left(h^{ik} + n^i n^k\right)\left(h^{jl} + n^j n^l\right) \mathcal{R}_{ijkl} \tag{2.120}$$
$$= q^{ik} q^{jl} \mathcal{R}_{ijkl} + q^{ik} n^j n^l \mathcal{R}_{ijkl} + q^{jl} n^i n^k \mathcal{R}_{ijkl} + n^i n^j n^k n^l \mathcal{R}_{ijkl}. \tag{2.121}$$

We can relabel b to a and d to c and use the following Riemannian tensor symmetries:

$$\mathcal{R}_{jikl} = -\mathcal{R}_{ijkl} \tag{2.122}$$

and also we have

$$\mathcal{R}_{ijkl} = -\mathcal{R}_{ijlk} \tag{2.123}$$

so

$$q^{jl} n^i n^k \mathcal{R}_{ijkl} = -q^{jl} n^i n^k \mathcal{R}_{jikl}. \tag{2.124}$$

Using the skew symmetry we will get

$$-q^{jl} n^i n^k \mathcal{R}_{jikl} = -(-q^{jl} n^i n^k \mathcal{R}_{jilk}) \tag{2.125}$$

so we can rewrite

$$q^{jl} n^i n^k \mathcal{R}_{ijkl} = q^{jl} n^i n^k \mathcal{R}_{jilk}$$

with the help of skew symmetries for the first two indices and two last indices. Relabeling

$$q^{jl} n^i n^k \mathcal{R}_{ijkl} = q^{ik} n^j n^l \mathcal{R}_{ijkl}. \tag{2.126}$$

We want to show the last term of (2.121) is zero,

$$n^i n^j n^k n^l \mathcal{R}_{ijkl} = 0.$$

We know

$$\mathcal{R}_{ijkl} = -\mathcal{R}_{jikl}. \tag{2.127}$$

Relabeling i to j, the term should be the same so

$$n^i n^j n^k n^l \mathcal{R}_{ijkl} = n^i n^j n^k n^l \mathcal{R}_{jikl}. \tag{2.128}$$

But with the use of skew symmetry property of Riemannian tensor

$$n^i n^j n^k n^l \mathcal{R}_{ijkl} = -n^i n^j n^k n^l \mathcal{R}_{ijkl} \tag{2.129}$$

so

$$n^i n^j n^k n^l \mathcal{R}_{ijkl} = 0. \tag{2.130}$$

Therefore we will have

$$\mathcal{R} = q^{ik} h^{jl} \mathcal{R}_{ijkl} + 2q^{ik} n^j n^l \mathcal{R}_{ijkl}. \tag{2.131}$$

So

$$\mathcal{R} = q^{AC} q^{BD} \left(R_{ABCD} + K_{AC} K_{BD} - K_{AD} K_{BC}\right) + 2h^{ik} n^j n^l R_{ijlk} \tag{2.132}$$

$$= q^{AC} q^{BD} \left(R_{ABCD} + K_{AC} K_{BD} - K_{AD} K_{BC}\right) + 2h^{ik} n^j n^l R_{ijkl}. \tag{2.133}$$

As a result, we obtain

$$R_{jl} n^j n^l = \frac{1}{2} \left(\mathcal{R} - R + K^2 - K_{AB} K^{AB}\right). \tag{2.134}$$

Now we expand the second term. With the use of the inverse of the product rule, we get

$$h^{ij} n^k D_i D_k n_j = h^{ij} [D_a \left(n^K D_k n_j\right) - \left(D_a n^k\right) \left(D_k n_j\right)]. \tag{2.135}$$

We want to calculate the term $h^{ij}(D_i n^k)(D_k n_j)]$. First we have

$$h^{ij} = q^{ij} + n^i n^j$$

and also can write

$$D_i n^k = D_i n_l h^{lk}. \tag{2.136}$$

The covariant derivative of the metric is zero and so we can write this as

$$D_i n^k = h^{lk} D_i n_l. \tag{2.137}$$

Then

$$h^{ij}(D_i n^c)(D_k n_j) = (q^{ij} + n^i n^j)(D_i n_k)(q^{kl} + n^k n^l)(D_l n_j) \tag{2.138}$$

but we have $n^i n^k D_i n_k$ and $n^j n^l D_l n_j$ are both zero so we will have

$$h^{ij}(D_i n^k)(D_j n_j) = q^{ij} q^{kl} D_i n_k D_l n_j \tag{2.139}$$

$$= h^{ij} D_a \left(\frac{-1}{\varphi} d_j \varphi \right) - (q^{ij})(D_i n_k)(q^{kl})(D_l n_j). \tag{2.140}$$

Applying the product rule for covariant derivative, we have

$$\frac{-1}{\varphi} d_j \varphi = -\frac{1}{\varphi} D_i d_j \varphi + \frac{1}{\varphi^2} [D_i \varphi][d_j \varphi] \tag{2.141}$$

so

$$= \left(q^{ij} + n^i n^j\right) \left(-\frac{1}{\varphi} D_i d_j \varphi + \frac{1}{\varphi^2} \left[D_i \varphi\right] \left[d_j \varphi\right] \right) - K_{ik} K^{ik}. \tag{2.142}$$

The induced covariant derivative on $\mathcal{S}$ is defined as follows

$$d_i A_j \equiv q_i{}^k q_j{}^l D_k A_l. \tag{2.143}$$

Then

$$q^{ij} \frac{1}{\varphi} D_i d_j \varphi = \frac{1}{\varphi} q^{ij} D_i d_j \varphi \tag{2.144}$$

$$= q^{ij} q_i{}^k q_j{}^l D_k A_l \tag{2.145}$$

$$= d^i d_j \varphi. \tag{2.146}$$

We have

$$n^i n^j \frac{1}{\varphi^2}[D_i\varphi][d_j\varphi] = 0 \tag{2.147}$$

and

$$n^i D_i n_j = -\frac{1}{\varphi} d_j\varphi \tag{2.148}$$

so we can rewrite

$$\begin{aligned}
&= -\frac{1}{\varphi} d^i d_i\varphi - \frac{1}{\varphi} n^i n^j D_i\left[d_j\varphi\right] + \frac{1}{\varphi^2}\left(d_i\varphi\right)\left(d^i\varphi\right) - K_{ik}K^{ik} && (2.149)\\
&= -\frac{1}{\varphi} d^2\varphi + \frac{1}{\varphi} d_j\varphi n^i D_i n^j + \frac{1}{\varphi^2}\left(d_i\varphi\right)\left(d^i\varphi\right) - K_{AB}K^{AB} && (2.150)\\
&= -\frac{1}{\varphi} d^2\varphi - \frac{1}{\varphi^2}\left(d_j\varphi\right)\left(d^j\varphi\right) + \frac{1}{\varphi^2}\left(d_i\varphi\right)\left(d^i\varphi\right) - K_{AB}k^{AB} && (2.151)\\
&= -\frac{1}{2} d^2\varphi - K_{AB}K^{AB}. && (2.152)
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\frac{\partial}{\partial\lambda}K &= -\frac{1}{2}\varphi\left(\mathcal{R} - R + K^2 - K_{AB}K^{AB}\right) - d^2\varphi - \varphi K_{AB}K^{AB} && (2.153)\\
&= -d^2\varphi - \frac{1}{2}\varphi\left(\mathcal{R} - R + K^2 + K_{AB}K^{AB}\right). && (2.154)
\end{aligned}$$

For a minimal surface $K = 0$ and so

$$\frac{\partial}{\partial\lambda}(\sqrt{q}\varphi K) = \sqrt{q}\varphi\frac{\partial K}{\partial\lambda} + K\frac{\partial}{\partial\lambda}(\sqrt{q}\varphi). \tag{2.155}$$

As a result, the second term is zero, so we will get

$$\frac{\partial}{\partial\lambda}(\sqrt{q}\varphi K) = \sqrt{q}\varphi\frac{\partial K}{\partial\lambda} + K\frac{\partial}{\partial\lambda}(\sqrt{q}\varphi) = -\sqrt{q}\varphi\left(d^2\varphi + \frac{1}{2}\left[\mathcal{R} - R + K_{AB}K^{AB}\right]2\varphi\right). \tag{2.156}$$

We define the trace-free part of K_{AB} as

$$\sigma_{AB} = K_{AB} - \frac{1}{n-1}Kq_{AB}. \tag{2.157}$$

Then the second variation for the area is:

$$\delta^2 A\big|_{K=0} = -\int_{\mathcal{S}} \sqrt{q}\varphi \left(d^2 + \frac{1}{2}\left[\mathcal{R} - R + K_{AB}K^{AB}\right]\right) \varphi d^{n-1}x \qquad (2.158)$$

Now $L = d^2 + \frac{1}{2}[\mathcal{R} - R + \sigma_{AB}\sigma^{AB}]$ is a self-adjoint elliptic operator on $\mathcal{S}$. It is called the stability operator.

$\mathcal{S}$ is compact and so the following properties are true [27]:

(1) The eigenvalues ξ are real, and there is a smallest one: ξ_0 the principal eigenvalue.

(2) The eigenfunctions form a basis for the deformations.

(3) If the principal eigenvalue $\xi_0 \geq 0$ we say that the surface is **stable** so in this case no smooth deformations can make the area smaller.

If the principal eigenvalue is $\xi_0 > 0$ we say that the surface is **strictly stable**.

If the principal eigenvalue is $\xi_0 < 0$ we say that the surface is **unstable** so there are deformations that make the area smaller.

We didn't calculate the third variation for minimal surface because it's very complicated.

Chapter 3

MOTS

In this chapter, we will begin with a brief history of trapped surfaces and their importance, and then give some definitions which are necessary for defining marginally outer trapped surfaces (MOTS). We explain the importance of understanding MOTS and our interest in them.

Our ultimate goal is to define what a black hole really is. How can we mathematically define them without any ambiguity? If we use the standard causal definition of a black hole, we need to have a global knowledge of spacetime: A causal black hole is a region of spacetime from which light can never escape and its boundary is the event horizon. Because of this definition, such a black hole is impossible to observe. Moreover, from a numerical relativity point of view, event horizon, the boundaries of causal finding black holes, is very costly [24]. So these are some of the reasons we are interested in a quasi-local definition of black holes. By introducing MOTS we can think about them as a generalization of minimal surfaces. As a result, we can use the same mathematical tools and methods to study them.

As noted above, the definition of an event horizon depends on the whole future evolution of the spacetime.

Definition 3.0.1 (Event Horizon [34])**.** Let (M, g_{ab}) be an asymptotically flat spacetime with associated unphysical spacetime $\left(\tilde{M}, \tilde{g}_{ab}\right)$. We say that (M, g_{ab}) is **strongly asymptotically predictable** if in the unphysical spacetime, there is an open region $\tilde{V} \subset \tilde{M}$ with $\overline{M \cap J^-(\mathscr{I}^+)} \subset \tilde{V}$ such that $\left(\tilde{V}, \tilde{g}_{ab}\right)$ is globally hyperbolic, where

$J^{-}(\mathscr{I}^{+})$ is the causal past of future null infinity. A strongly asymptotically predictable spacetime is said to contain a **black hole** if M is not contained in $J^{-}(\mathscr{I}^{+})$. The **black hole region** B of such a spacetime is defined to be $B = M - J^{-}(\mathscr{I}^{+})$ and the boundary of B in M, $H = \dot{J}^{-}(\mathscr{I}^{+}) \cap M$, is called the **event horizon**.

An alternative definition of a black hole was proposed by Penrose. This is based on the idea of a trapped surface [28]. To understand what a trapped surface is, first picture a two-sphere in Minkowski space, taken as a set of points some fixed radial distance from the origin, embedded in a constant-time slice. If we follow null rays emanating into spacetime from this spatial sphere, one set (pointed inward) will describe a shrinking set of spheres, while the other (pointed outward) will describe a growing set of spheres. But this would not be the case for a sphere of fixed radius $r < 2GM$ in the Schwarzschild geometry; inside the event horizon, both sets of null rays emanating from such a sphere would evolve to smaller values of r (since r is a timelike coordinate), and thus to smaller areas. This is what is meant by a trapped surface: a compact, spacelike, two-dimensional submanifold with the property that outgoing future directed light rays converge in both directions everywhere on the submanifold [12].

Definition 3.0.2 (Trapped Surface [34])**.** A compact, two-dimensional, spacelike submanifold $\mathcal{S}$ has both inward and outward directed null expansions that are negative is called a **trapped surface**.

Definition 3.0.3 (MOTS)**.** If we have a closed surface $\mathcal{S}$ with $\theta_{+} = 0$ and no restriction on θ_{-}, then $\mathcal{S}$ is called a **marginally outer trapped surface (MOTS)**.

First, we will set our notation. The notation we using here is based on the paper [11].

Suppose $(\mathcal{M}, g_{ab}, \nabla_a)$ is a smooth four-dimensional spacetime with signature $(-+++)$. In the case of three-dimensional spacelike surface in spacetime, we will use (Σ, q_{ij}, D_i). Let $(\mathcal{S}, q_{AB}, d_A)$ be a smooth two-dimensional spacelike surface in that spacetime.

The metric on $\mathcal{S}$ is the pullback of the full four-metric

$$q_{AB} = e^a_A e^b_B g_{ab}. \tag{3.1}$$

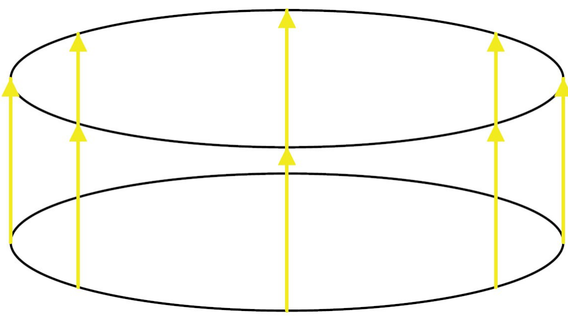

Figure 3.1: A surface with expansion $\theta_+ = 0$

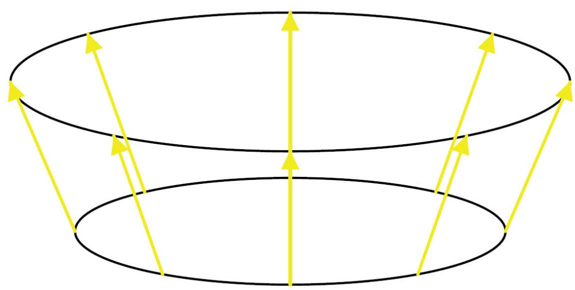

Figure 3.2: A surface with expansion $\theta_+ > 0$

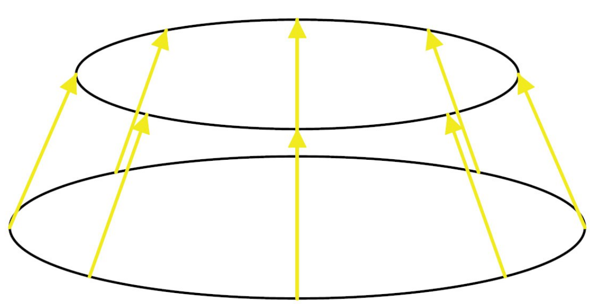

Figure 3.3: A surface with expansion $\theta_+ < 0$

If we consider coordinates on the $\mathcal{M}$ as $\{x^a\}$ and we parameterize $\mathcal{S}$ with $x^\alpha = x^a\left(y^A\right)$ then $e^a_A = \frac{\partial x^a}{\partial y^A}$. We will take two future pointing null normals to $\mathcal{S}$ such that $\ell^+ \cdot \ell^- = -1$. We can write extrinsic curvatures with respect to these two normals and divide them into trace and trace-free parts. We call the trace part the expansion $\theta_\pm$ and the trace-free part the shear $\sigma^\pm_{AB}$. This then is the form of the extrinsic curvature

$$k^\pm_{AB} := e^a_A e^b_B \nabla_a \ell^\pm_b = \frac{1}{2}\theta_\pm q_{AB} + \sigma^\pm_{AB} \tag{3.2}$$

where $\theta_\pm = q^{ab}\nabla_a \ell^\pm_b$, and

$$q^{ab} = e^a_A e^b_B q^{AB} = g^{ab} + \ell^a_+ \ell^b_- + \ell^a_- \ell^b_+. \tag{3.3}$$

If our surface $\mathcal{S}$ is closed and orientable we will call ℓ^+ and ℓ^- the outgoing and ingoing null normals, respectively and they will give us outgoing and ingoing expansions $\theta_\pm$ respectively.

As in [19], we shall call an open surface $\mathcal{S}$ with one of the expansions vanishing a marginally outer trapped open surface (MOTOS) and, by convention, call the vanishing expansion θ_+.

Chapter 4

MOTS stability operator

In this section, we will derive the stability operator for MOTS. We are interested in the MOTS stability operator because it plays an important role in understanding these surfaces. For close to four decades, versions of it have been used to characterize whether a particular MOTS (locally) separates outer trapped from untrapped regions and hence can be thought of as a black hole boundary [9]. Further, a strictly stable MOTS in a slice Σ forms a boundary between trapped and untrapped regions in that slice, in the sense that there is a two-sided neighbourhood that contains no complete outer trapped surfaces outside S and no complete outer untrapped surfaces inside. Thus, it can usefully be thought of as a generalization of the apparent horizon in Σ [9]. Minimal surfaces are a special case of MOTS so after deriving the MOTS stability operator, we recover the stability operator for minimal surfaces. After deriving the MOTS stability operator we will discuss some applications that relate the stability to the symmetry of the background space. The second part will be based on [9].

4.1 Stability operator

For a given initial MOTS $\mathcal{S}$, consider a smooth deformation to a one parameter family of surfaces $\mathcal{S}_\rho$ such that $\mathcal{S}_0 = \mathcal{S}$. Then the unit normal vector r^a to $\mathcal{S}$ naturally extends to a field r^a over the region covered by $\mathcal{S}_\rho$ and we can write the tangent vector to the curves that generate this family of deformations as

$$\frac{\partial}{\partial \rho} = \psi r.$$

The stability operator on $\mathcal{S}$ is defined as

$$L_{\mathcal{S}}\psi := \delta_{\psi r^a}\theta_{(\ell)} := \left.\frac{\partial}{\partial \rho}\right|_{\rho=0} \theta_{(\ell)}.$$

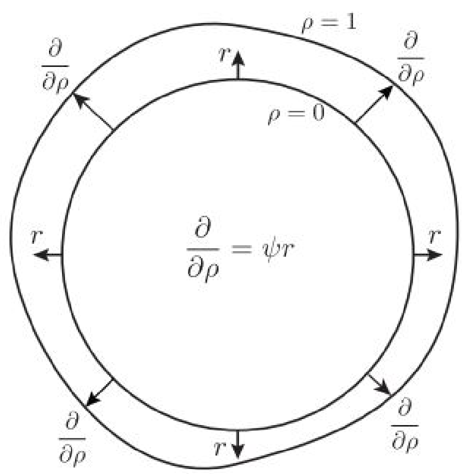

Figure 4.1: Deformation of a surface generated by a covariant vector field $\frac{\partial}{\partial \rho}$[2]

Definition 4.1.1 (Stable MOTS)**.** $\mathcal{S}$ is said to be **strictly stable** or **marginally stable**, respectively, if there exists a not everywhere vanishing, non-negative ψ such that $L_{\mathcal{S}}\psi > 0$ or $L_{\mathcal{S}}\psi = 0$ respectively. It is **unstable** if no such ψ exists.

4.2 Derivation of stability operator

The extrinsic geometry of $\mathcal{S}$ can be understood by considering how the null normals vary along the surface. We have null normals ℓ^+ and ℓ^-, which we defined by

$$\ell^+ = \left(u^\alpha + r^\beta\right) \tag{4.1}$$

and

$$\ell^{-} = \frac{1}{2}\left(u^{\alpha} - r^{\beta}\right). \tag{4.2}$$

If we take r^a as our unit spacelike normal to our surface $\mathcal{S}$ then the trace of our extrinsic curvature would be

$$k^{(r)} = q^{AB} k^{(r)}_{AB}.$$

Also if we write the trace of extrinsic curvature with respect of the unit timelike normal u^a we will get

$$k^{(u)} = q^{ij} K^{(u)}_{ij}.$$

We can choose r_j to be perpendicular for surfaces with ρ constant. As a result we will have

$$e^i_A \mathcal{L}_{\psi r} r_i = 0$$

where $\mathcal{L}$ is Lie derivative. For the variation of $k^{(r)}$ we have

$$\frac{\partial}{\partial \rho}\left(k^{(r)}\right) = \psi r^k D_k \left(q^{ij} D_i r_j\right)$$

$$\frac{\partial}{\partial \rho}\left(k^{(r)}\right) = \psi r^k D_k \left(\left[h^{ij} - r^i r^j\right] D_i r_j\right)$$

and from the rules for covariant derivative we have

$$\frac{\partial}{\partial \rho}\left(k^{(r)}\right) = \psi r^k D_k (h^{ij} D_i r_j) - \psi r^k D_k (r^i r^j D_i r_j).$$

Now with the product rule we will get

$$\begin{aligned}\frac{\partial}{\partial \rho}\left(k^{(r)}\right) = \psi r^k \big(& D_k(h^{ij}) D_i r_j + D_k(D_i r_j) h^{ij} - D_k(r^i)(r^j D_i r_j) - D_k(r^j)(r^i D_i r_j) \\ & - D_k D_i r_j (r^i r^j)\big)\end{aligned}$$

so we should simplify some of the terms. First we have $D_k(h^{ij}) = 0$. Also we have $r^j r_j = 1$, therefore

$$D_i(r^j r_j) = 0$$

$$D_k(r^i)(r^j D_i r_j) - D_k(r^j)(r^i D_i r_j) - D_k D_i r_j (r^i r^j)) = 0.$$

Further, because the covariant derivative of the metric is zero we have

$$\psi r^k D_k(h^{ij}) D_i r_j = 0 \tag{4.3}$$

so

$$\frac{\partial}{\partial\rho}\left(k^{(r)}\right) = \psi h^{ij} r^k D_k D_i r_j .$$

According to the definition of the Riemann tensor, we have

$$R_{kijl} r^l = D_k D_i r_j - D_i D_k r_j$$

which we can combine with the two previous lines to find

$$\frac{\partial}{\partial\rho}\left(k^{(r)}\right) = \psi h^{ij} r^k \left(R_{kijl} r^l + D_i D_k r_j\right),$$

so the variation of $k^{(r)}$ will be

$$\frac{\partial}{\partial\rho}\left(k^{(r)}\right) = \psi h^{ij} r^k \left(R_{kijl} r^d + D_a D_c r_b\right).$$

We have

$$\psi h^{ij} r^k R_{kijl} r^l = -\psi h^{ik} r^j r^l R_{ijkl}. \tag{4.4}$$

Also

$$R = h^{ik} h^{jl} R_{ijkl}.$$

So we will get

$$\begin{aligned} R &= \left(q^{ik} + r^i r^k\right)\left(q^{jk} + r^j r^l\right) R_{ijkl} \\ &= q^{ik} q^{jl} R_{ijkl} + 2 q^{ik} r^j r^l R_{ijkl} \\ &= q^{ik} q^{jl} R_{ijkl} + 2 h^{ik} r^j r^k R_{ijkl}. \end{aligned}$$

Therefore we have

$$h^{ik} r^j r^l R_{ijkl} = \frac{1}{2}\left(R - q^{ik} q^{jl} R_{ijkl}\right). \tag{4.5}$$

However with the Gauss–Codazzi equations (2.3.5) we can get

$$h^{ik} r^j r^l R_{ijkl} = \frac{1}{2}\left(R - {}^{(2)}R - k^{(r)}_{ij} k^{ij}_{(r)} + k^2\right), \tag{4.6}$$

so

$$\psi h^{ij} r^k R_{kijl} r^d = -\frac{1}{2} \psi \left(R - {}^{(2)}R - k_{ij}^{(r)} k_{(r)}^{ij} + k^2 \right) \tag{4.7}$$

Next,

$$\psi h^{ij} r^k D_i D_k r_j = \psi q^{ij} r^k D_i D_k r_j + \psi r^i r^j r^k D_i D_k r_j, \tag{4.8}$$

so we have

$$\begin{aligned} \psi h^{ij} r^k D_i D_k r_j &= \psi q^{ij} D_i \left(r^k D_k r_j \right) - \psi q^{ij} \left(D_i r^k \right) \left(D_k r_j \right) + \psi r^i r^k D_i \left(r^j D_k r_j \right) \\ &- \psi r^i r^k \left(D_i r^j \right) \left(D_k r_j \right) \\ &= \psi q^{ij} d_i \left(-\frac{1}{\psi} d_i \psi \right) - \psi k_{ij}^{(r)} k_{(r)}^{ij} - \psi \left(r^i D_i r^j \right) \left(r^k D_k r_j \right). \end{aligned}$$

Therefore we will get

$$\psi h^{ij} r^k D_i D_k r_j = -d^2 \psi - \psi k_{AB}^{(r)} k_{(r)}^{AB},$$

and so

$$\frac{\partial}{\partial \rho} \left(k^{(r)} \right) = -d^2 \psi - \frac{1}{2} \psi k_{AB}^{(r)} k_{(r)}^{AB} - \frac{1}{2} \psi \left(R - {}^{(2)}R + k_{(r)}^2 \right). \tag{4.9}$$

Next we will have the extrinsic curvature of $\mathcal{S}$ based on unit timelike normal u so

$$\frac{\partial}{\partial \rho} \left(k^{(u)} \right) = \psi r^k D_k \left(q^{ij} K_{ij} \right). \tag{4.10}$$

Then

$$\frac{\partial}{\partial \rho} \left(k^{(u)} \right) = \psi r^k D_k \left(K - K_{ij} r^i r^j \right) \tag{4.11}$$

$$= \psi \left(r^k D_k K - r^i r^j r^k D_k K_{ij} - K_{ij} r^i r^k D_k r^j - K_{ij} r^j r^k D_k r^i \right) \tag{4.12}$$

$$= \psi \left(r^k D_k K - r^i \left[h^{jk} - q^{jk} \right] D_k K_{ij} - 2 \psi K_{ij} r^i \left[r^k D_k r^j \right] \right). \tag{4.13}$$

We have

$$\frac{\partial}{\partial \rho} \left(k^{(u)} \right) = \psi r^k \left(D_k K - D_j K_c^j \right) + \psi r^i q^{jk} D_k K_{ij} - 2 \psi K_{ij} r^i \left(-\frac{1}{\psi} d^j \psi \right). \tag{4.14}$$

Therefore we obtain

$$\frac{\partial}{\partial \rho}\left(k^{(u)}\right) = -\psi G_{ab} u^a r^b + \psi q^{jk} D_k \left(K_{ij} r^i\right) - \psi q^{jk} K_{ij} D_k \left(r^i\right) + 2\left[q_i^j K_{jk} u^k\right] d^i \psi \tag{4.15}$$

$$= -\psi G_{ab} u^a r^b + \psi q^{jk} D_k \left(q_j^l K_{li} r^i + r_j \left[K_{li} r^l r^i\right]\right) \tag{4.16}$$

$$- \psi q^{jk} K_{ji} \left[q_l^i + r_l r^i\right] D_k r^l + 2\tilde{\omega}^A d_A \psi \tag{4.17}$$

which gives us

$$\frac{\partial}{\partial \rho}\left(k^{(u)}\right) = -\psi G_{ab} u^a r^b + \psi d_C \tilde{\omega}^C + \psi \left(K_{ab} r^a r^b\right) k^{(r)} - \psi k_{AB}^{(u)} k_{(r)}^{AB} + 2\tilde{\omega}^A d_A \psi \tag{4.18}$$

where

$$\tilde{\omega}_A = e_A^a \ell_b^- \nabla_a (\ell^{+b}) \tag{4.19}$$

$$= -\frac{1}{2} e_A^a \left(u_b - r_b\right) \nabla_a \left(u^b + r^b\right) \tag{4.20}$$

$$= e_A^a r^b \nabla_a u_b. \tag{4.21}$$

In terms of the extrinsic curvature, we can write

$$\tilde{\omega}_A = e_A^a K_{ab} r^b. \tag{4.22}$$

For Σ as a hypersurface we have

$$\frac{1}{2} R = G_{ab} u^a u^b - \frac{1}{2}\left(K^2 - K_{ij} K^{ij}\right). \tag{4.23}$$

And we have

$$K_{ij} = k_{ij}^{(u)} + \tilde{\omega}_i r_j + \tilde{\omega}_j r_i + K_{rr} r_i r_j \tag{4.24}$$

so

$$K = k^{(u)} + K_{rr} \tag{4.25}$$

and

$$K_{ij} K^{ij} = k_{ij}^{(u)} k_{(u)}^{ij} + 2\tilde{\omega}^A \tilde{\omega}_A + K_{rr}^2. \tag{4.26}$$

Therefore we can get

$$K^2 - K_{ij}K^{ij} = k_{(u)}^2 + 2K_{rr}k^{(u)} - k_{AB}^{(u)}k_{(u)}^{AB} - 2\|\tilde{\omega}\|^2. \tag{4.27}$$

With the help of the two previous equations we have

$$\frac{1}{2}R = G_{ab}u^a u^b - \frac{1}{2}\left(k_{(u)}^2 + 2K_{rr}k^{(u)} - k_{AB}^{(u)}k_{(u)}^{AB} - 2\|\tilde{\omega}\|^2\right) \tag{4.28}$$

and so

$$\frac{\partial}{\partial\rho}(k^{(r)}) = -d^2\psi + \frac{\psi}{2}\left({}^{(2)}R - k_{(r)}^2 - k_{AB}^{(r)}k_{(r)}^{AB}\right) - \psi(G_{ab}u^a u^b + \frac{1}{2}[k_{(u)}^{AB}k_{AB}^{(u)} - k_{(u)}^2] \\ -K_{rr}k^{(u)} + \|\tilde{\omega}\|^2).$$

We have $\ell = u + r$ and $\theta_{(\ell)} = k_{(u)} + k_{(r)}$ so the variation of $\theta_{(\ell)}$ will be

$$\frac{\partial}{\partial\rho}\left(\theta_{(\ell)}\right) = \frac{\partial}{\partial\rho}\left(k_{(u)}\right) + \frac{\partial}{\partial\rho}\left(k_{(r)}\right). \tag{4.29}$$

Combining our equations we obtain

$$\frac{\partial}{\partial\rho}\left(\theta_{(\ell)}\right) = -\psi G_{ab}u^a r^b + \psi d_C\tilde{\omega}^C + \psi\left(K_{ij}r^i r^j\right)k^{(r)} - \psi k_{AB}^{(u)}k_{(r)}^{AB} + 2\tilde{\omega}^A d_A\psi - d^2\psi \\ +\frac{\psi}{2}\left({}^{(2)}R - k_{(r)}^2 - k_{AB}^{(r)}k_{(r)}^{AB}\right) - \psi\left(G_{ab}u^a u^b + \frac{1}{2}\left[k_{(u)}^{AB}k_{AB}^{(u)} - k_{(u)}^2\right]\right. \\ \left. -K_{rr}k^{(u)} + \|\tilde{\omega}\|^2\right), \tag{4.30}$$

so

$$\frac{\partial}{\partial\rho}\left(\theta_{(\ell)}\right) = -d^2\psi + 2\tilde{\omega}^A d_A\psi + \psi\left(\tfrac{1}{2}{}^{(2)}R - \|\tilde{\omega}\|^2 + d_A\tilde{\omega}^A - G_{ab}u^a\ell^b\right) \\ +\psi K_{rr}\left(k_{(u)} + k_{(r)}\right) + \psi\left(-k_{AB}^{(u)}k_{(r)}^{AB} - \tfrac{1}{2}k_{(r)}^2 - \tfrac{1}{2}k_{AB}^{(r)}k_{(r)}^{AB}\right. \\ \left. +\tfrac{1}{2}k_u^2 - \tfrac{1}{2}k_{(u)}^{AB}k_{AB}^{(u)}\right). \tag{4.31}$$

Again because of $\theta_{(\ell)} = k_{(u)} + k_{(r)}$ we get

$$\frac{\partial}{\partial\rho}\left(\theta_{(\ell)}\right) = -d^2\psi + 2\tilde{\omega}^A d_A\psi + \psi\left(\frac{1}{2}{}^{(2)}R - \|\tilde{\omega}\|^2 + d_A\tilde{\omega}^A - G_{ab}u^a\ell^b\right) + \psi K_{rr}\theta_{(\ell)} \\ + \psi\left(\frac{1}{2}\left(k_{(u)}^2 - k_{(r)}^2\right) - \left[k_{AB}^{(u)} + k_{AB}^{(r)}\right]\left[k_{(u)}^{AB} + k_{(r)}^{AB}\right]\right), \tag{4.32}$$

and so the final expression

$$\frac{\partial}{\partial\rho}\left(\theta_{(\ell)}\right) = -d^2\psi + 2\tilde{\omega}^A d_A\psi + \psi\left(\frac{1}{2}{}^{(2)}R - \|\tilde{\omega}\|^2 + d_A\tilde{\omega}^A - G_{\alpha\beta}u^\alpha\ell^\beta\right) + \psi K_{rr}\theta_{(\ell)} + \psi\left(\theta_{(\ell)}\theta_{(r)} - \frac{1}{2}\theta_{(\ell)}^2 - \sigma_{(\ell)}^{AB}\sigma_{AB}^{(\ell)}\right). \tag{4.33}$$

Now for MOTS $\theta_{(\ell)} = 0$ we will have

$$\frac{\partial}{\partial\rho}\left(\theta_{(\ell)}\right) = -d^2\psi + 2\tilde{\omega}^A d_A\psi + \psi\left(\frac{1}{2}{}^{(2)}R - \|\tilde{\omega}\|^2 + d_A\tilde{\omega}^A - G_{\alpha\beta}u^\alpha\ell^\beta - \sigma_{(\ell)}^{AB}\sigma_{AB}^{(\ell)}\right). \tag{4.34}$$

We call (4.2) the MOTS stability operator.

4.3 Applications

This subsection is based on [9] and provides some examples of unstable MOTS. As we can see in Figure 4.2, MOTS can have very complicated self-intersecting geometries. Such exotic surfaces have also been shown to play a key role in black hole mergers, with a complicated series of MOTS pair creations and annihilations ultimately destroying the original pair of apparent horizons and resulting in a single final apparent horizon [10, 29, 9]. All exotic MOTSs so far observed in either exact or numerical solutions have been found to be unstable [9]. So our goal in this section is to provide some theorem to show why most MOTS are unstable in spacelike slices of highly symmetric spacetimes. First, we need to give some definitions.

Definition 4.3.1. A non-trivial vector field X on Σ is a **symmetry** of (Σ, h, K) if $\mathcal{L}_X g = \mathcal{L}_X K = 0$. It is a symmetry of a surface $\mathcal{S}$ if, in addition, it is everywhere tangent to $\mathcal{S}$.

Theorem 2. *Suppose $\mathcal{S}$ is a MOTS and X is a symmetry of (Σ, h, K) but not of $\mathcal{S}$. Then 0 is an eigenvalue of $L_{\mathcal{S}}$. Moreover,*

1) $\mathcal{S}$ is marginally stable if and only if X is nowhere tangent to $\mathcal{S}$,

2) $\mathcal{S}$ is unstable if and only if X is tangent to $\mathcal{S}$ at some point.

As a result of this theorem, we can say that any non-spherical MOTS in a spherically symmetric slice of Schwarzschild is unstable [9]. As a result the MOTS appearing in figure (4.2) are unstable.

Theorem 3. *Suppose $\mathcal{S}$ is an embedded MOTS and X is a* ***symmetry*** *of* (Σ, h, K).

If $\mathcal{S}$ bounds a compact region $A \subset \Sigma$ then it is unstable under any of the following conditions:

1) X is not a symmetry of $\mathcal{S}$,

2) X is a coordinate vector field, on A

3) $\chi(\mathcal{S}) \neq 0$ and X has no zeros in A.

So for any initial data set with translational symmetry, all MOTSs are unstable and therefore the boundary of the trapped region is not a MOTS [9]. A typical example of such initial data would be that for a cosmological spacetime in which all points lie in the trapped region and so there is no boundry.

Theorem 4. *Let X be a symmetry of a three-dimensional initial data set* (Σ, h, K). *If $\mathcal{S}$ is a stable MOTS that bounds a compact region in Σ and has $\chi(\mathcal{S}) \neq 0$, then it must intersect the zero set of X.*

Therefore, if (Σ, h, K) satisfies the dominant energy condition (2.4), then any MOTS that bounds a compact region and is strictly stable must intersect the zero set of X.

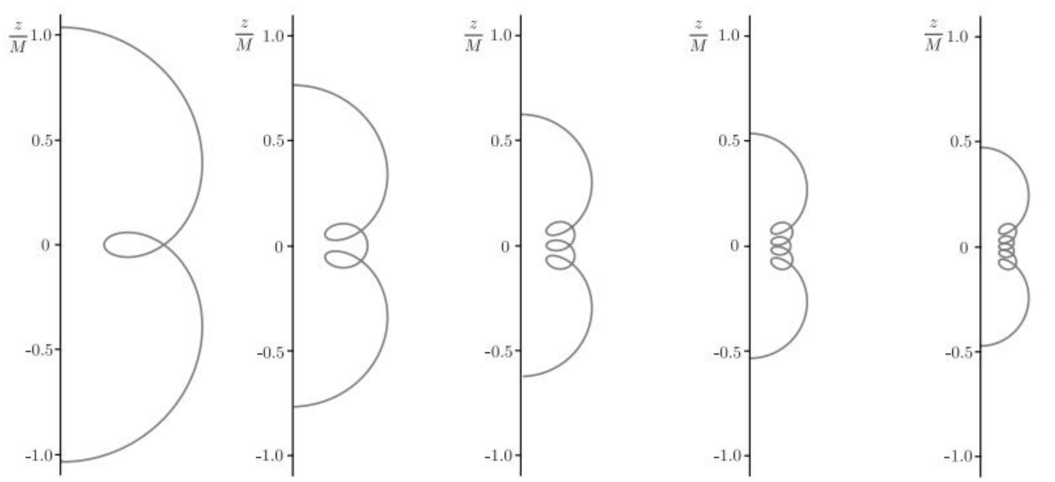

Figure 4.2: Some exotic MOTS in the Schwarzchild spacetime. These are found in constant time Painlevé-Gullstand slices and are all inside usual $r = 2M$ horizon [9]

Chapter 5

MOTS finder

In this chapter, we want to go through how we can find MOTS in axisymmetric spacetimes. The basic idea of the method is to rewrite the $\theta_{(\ell)} = 0$ condition for an axisymmetric surface in three-dimensional space into a pair of coupled ODEs for the generating curve. We will rotate our curve to make a surface like figure (5.1). Because of the close relationship between these equations and those of geodesic curves this method is called MOTSodesics.

Consider a spacelike half plane $\left\{\bar{\Sigma}, \bar{h}_{ab}, \bar{D}_a\right\}$ where a and b range over the Cartesian coordiante (ρ, z) where $\{\rho > 0, -\infty < z < \infty\}$. We will rotate $\bar{\Sigma}$ to get a three manifold which is axisymmetric $\{\Sigma, h_{ij}, D_i\}$. If we use the coordinate we will have the metric on Σ as

$$h_{ij}^{(\rho,\phi,z)} = \begin{bmatrix} \bar{h}_{\rho\rho} & 0 & \bar{h}_{\rho z} \\ 0 & R^2 & 0 \\ \bar{h}_{\rho z} & 0 & \bar{h}_{zz} \end{bmatrix}, \tag{5.1}$$

where $R(\rho, z)$ is the circumferential radius and where ϕ is the coordinate along the orbits of the Killing field ϕ which preserves the induced 2-metric q_{AB} on $\mathcal{S}$ and which vanishes precisely at the two poles.

Now we consider start with a curve

$$\gamma : (\rho, z) = (P(s), Z(s))$$

which is in $\bar{\Sigma}$ and arc length parametrized with parameter s. We will denote the derivative with respect to s with a dot. For the curve γ we have tangent vector

$$T = \dot{P}\frac{\partial}{\partial\rho} + \dot{Z}\frac{\partial}{\partial z} \tag{5.2}$$

with

$$h_{ab}T^aT^b = 1. \tag{5.3}$$

The unit normal one-form to the curve is

$$\tilde{N} = \sqrt{\bar{h}}(-\dot{Z}\ \mathrm{d}\rho + \dot{P}\ \mathrm{d}z) \tag{5.4}$$

or as a vector

$$N = \frac{1}{\sqrt{h}}\left(-\left(h_{21}\dot{P} + h_{22}\dot{Z}\right)\frac{\partial}{\partial\rho} + \left(h_{11}\dot{P} + h_{12}\dot{Z}\right)\frac{\partial}{\partial z}\right) \tag{5.5}$$

where $\bar{h} = \det\left(\bar{h}_{ab}\right)$ we have

$$T^a\bar{D}_aT^b = \kappa N^b. \tag{5.6}$$

Equivalently

$$N_bT^a\bar{D}_aT^b = \kappa N_bN^b. \tag{5.7}$$

We have $N_bN^b = 1$ and so

$$\kappa = N_bT^a\bar{D}_aT^b. \tag{5.8}$$

Now we will rotate the curve γ into a two surface

$$\{\mathcal{S}, q_{AB}, D_A\} \tag{5.9}$$

where indices A and B range over (s, ϕ). The induced metric will be

$$q_{AB} = \begin{bmatrix} 1 & 0 \\ 0 & R^2 \end{bmatrix}$$

where R is the circumferential radius of (ρ, z).

The trace of the extrinsic curvature of $\mathcal{S}$ relative to the timelike normal u to Σ is

$$k_u = q^{ij}K_{ij} = k^u_{ab}T^aT^b + k_{\phi\phi} \tag{5.10}$$

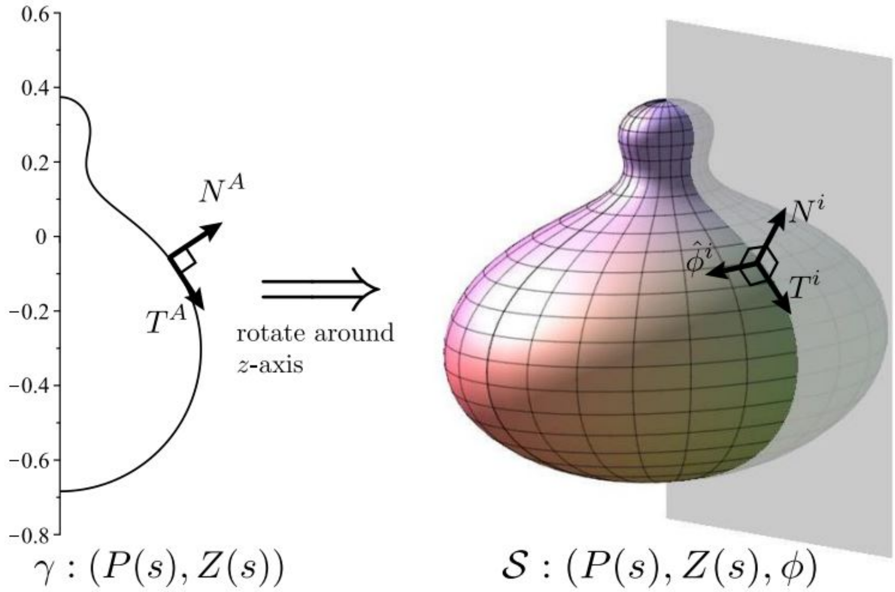

Figure 5.1: Making a surface by rotating a curve[10]

where

$$q^{ij} = T^i T^j + \hat{\phi}^i \hat{\phi}^j$$

and $\hat{\phi}^i = \frac{1}{R}\frac{\partial}{\partial \phi}$.

The expansions based on $\ell^{\pm}$ null normals are

$$2\theta_+ = k_u + \left(-N_b \left(T^a \bar{D}_a T^b\right) + N^a \bar{D}_a(\ln R)\right) \tag{5.11}$$

we want to have $\theta_+ = 0$ and so we should have

$$\kappa = N_b \left(T^a \bar{D}_a T^b\right) = N^a \bar{D}_a(\ln R) + k_u. \tag{5.12}$$

From this we will get two differential equation with $P(s)$ and $Z(s)$.

$$\begin{bmatrix} \ddot{P} \\ \ddot{Z} \end{bmatrix}^a = \dot{T}^a = -\bar{\Gamma}^a_{bc} T^b T^c + \kappa N^a \tag{5.13}$$

5.1 Example

In this section, we want to see one example of using this method in action.

Example 5.1.1. We want to find MOTSs for the Schwarzschild black hole in Painlevé-Gullstrand coordinates [24]. We have

$$ds^2 = -\left(1 - \frac{2M}{\sqrt{\rho^2+z^2}}\right) dT^2 + 2\sqrt{\frac{2M}{\sqrt{\rho^2+z^2}}}\frac{\rho d\rho + z dz}{\sqrt{\rho^2+z^2}} dT + \left(1 - \frac{2M}{\sqrt{\rho^2+z^2}}\right)^{-1} \frac{\rho^2 d\rho^2 + z^2 dz^2}{\rho^2+z^2} + \rho^2 d\phi^2$$

So for the induced metric on T= constant we will have

$$h_{ij}dx^i dx^j = d\rho^2 + dz^2 + \rho^2 d\phi^2. \tag{5.14}$$

And for the extrinsic curvature

$$K_{ij}dx^i dx^j = \sqrt{\frac{M}{2}}\left(\frac{\rho^2 - 2z^2}{r^{7/2}}d\rho^2 + \frac{6\rho z}{r^{7/2}}d\rho dz + \frac{z^2 - 2\rho^2}{r^{7/2}}dz^2 - \frac{2\rho^2}{r^{3/2}}d\phi^2\right) \tag{5.15}$$

where $r = \sqrt{\rho^2 + z^2}$. So now for $R = \rho$ we have

$$N = -\dot{Z}\frac{d}{d\rho} + \dot{P}\frac{d}{dz}, \tag{5.16}$$

as normal vector so we will get

$$k_u = -\sqrt{\frac{M}{2}}\left(\frac{3(Z\dot{P} - P\dot{Z})^2}{r^{7/2}} + \frac{1}{r^{3/2}}\right). \tag{5.17}$$

Therefore the MOTSodesics equations will be

$$\begin{aligned} \ddot{P} &= \frac{\dot{Z}^2}{P} \pm k_u \dot{Z} \\ \ddot{Z} &= -\frac{\dot{P}\dot{Z}}{P} \mp k_u \dot{P}. \end{aligned} \tag{5.18}$$

We can see $r = 2M$ is a solution for these equations but this system also straightforward to solve numerically. Curves in Figure (4.2) were found in this way. See [24] for more details.

Chapter 6

MOTS in higher dimensions

In the case of minimal surfaces and geodesics, we are using the Jacobi operator and stability operator to study their behavior. We use them to determine whether the critical points of the length functional and area functional, respectively, are minima or saddle points. A relatively recent interpretation of MOTS stability has proven very successful in proving higher dimensional results for the topology of black holes, which relies on showing that under certain assumptions on either the initial data set containing the MOTS or the spacetime, the MOTS is of positive Yamabe type, i.e., admits a metric of constant positive scalar curvature [21].

We are interested in the topology of black holes and how they will look, so the question is can black holes have different topology? For example is there any black hole in dimension $3+1$ with the topology of a torus? Is that even possible? Or do we have some restrictions? To answer these kinds of questions, the first important step is Hawking's Theorem [23, 22]. It states that the event horizon of a 4-dimensional asymptotically flat stationary black hole spacetime that satisfies the dominant energy condition will have the topology of a sphere. In proving this theorem Hawking used the Gauss-Bonnet theorem, so we cannot directly generalize it to higher dimensions. Gauss-Bonnet relates the geometry of a manifold to its topology. For more details, see Chapter 2, definition (2.5).

Nowadays there is more interest in higher dimensional black holes so we want to know which properties they share. Galloway, in his paper [20], studies the topology of black holes in higher dimensions. We start with a short review on Hawking's theorem and then go to the Galloway generalization. The counter-example to the notion that all black holes should be spherical is a great discovery by Emparan and Reall of a black ring with the $S^1 \times S^2$ horizon topology [17].

Now we will start with a review on Hawking's theorem [22].

6.1 Hawking's Theorem

Before starting to state Hawking's theorem we will introduce two definitions.

Definition 6.1.1 (locally outermost MOTS [3])**.** A marginally outer trapped surface $\mathcal{S}$ is called **locally outermost** in Σ, if and only if there exists a two-sided neighborhood of S such that its exterior part does not contain any weakly outer trapped surface.

Definition 6.1.2 (Outermost MOTS [5])**.** A MOTS $\mathcal{S}$ is **outermost** in Σ if there is no other MOTS in the complement of the region which bounds with a possibly empty inner boundary.

Remark: Being the outermost MOTS is stronger than being stable but weaker than being strictly stable [3].

Theorem 5. *(Hawking's Theorem [20]) Let M^4 be a four-dimensional asymptotically flat stationary black hole spacetime obeying the dominant energy condition. Then cross sections of the event horizon are topologically 2-spheres.*

By a cross-section, we mean a smooth compact (without boundary) 2-surface that is obtained by intersecting $H = \partial I^- (\mathscr{I}^+)$ with a spacelike hypersurface.

As we can see in the caption of Figure (6.1) cross-sections of a Schwarzchild event horizon are outermost MOTS. We know from paper [4] that outermost MOTS are stable. So instead of proving Hawking's theorem for outermost MOTS we will only prove it for strictly stable MOTS.

Hawking assumes that $\mathcal{S}$ is not spherical. His idea was to deform $\mathcal{S}$ to an outer trapped surface $\mathcal{S}'$. If $\mathcal{S}$ is not a 2-sphere, and hence has genus (i.e., number of handles) $g \geq 1$, the Gauss-Bonnet theorem and dominant energy condition are then used to show that $\frac{\partial \theta}{\partial t}\big|_{t=0} < 0$. It follows that for sufficiently small $t > 0, \theta(t) < 0$ which implies that $\mathcal{S}_t'$ is outer trapped. Hence, $\mathcal{S}$ must be a 2-sphere.

Proof. $\mathcal{S}$ is a strictly stable MOTS so we can use the stability operator (4.33). In our case for the event horizon we have $\frac{\partial}{\partial\rho}\left(\theta_{(\ell)}\right) > 0$ for every non-negative function ψ because our horizon is stable, so we can use $\psi = 1$ so $d^2\psi$, $2\,\tilde{\omega}^A d_A$ are zero. So we will be left with

$$0 < \frac{1}{2}R^{(2)} - \|\tilde{\omega}\|^2 + d_A\tilde{\omega}^A - G_{\alpha\beta}u^\alpha\ell^\beta. \tag{6.1}$$

Now if we rewrite it we have

$$\frac{1}{2}R^{(2)} > \|\tilde{\omega}\|^2 - d_A\tilde{\omega}^A + G_{\alpha\beta}u^\alpha\ell^\beta. \tag{6.2}$$

Now we will take integral of both sides

$$\int_{\mathcal{S}} \frac{1}{2}R^{(2)}d\mathrm{Vol}\mathcal{S} > \int_{\mathcal{S}}(\|\tilde{\omega}\|^2 - d_A\tilde{\omega}^A + G_{\alpha\beta}u^\alpha\ell^\beta)d\mathrm{Vol}\mathcal{S}. \tag{6.3}$$

In our case, $\mathcal{S}$ is a 2-dimensional surface, and the Ricci scalar is just Gaussian curvature up to a factor of 2. On the right hand sight of the integral we have $\|\tilde{\omega}\|^2 \geq 0$ and also because of dominant energy conditions $G_{\alpha\beta}u^\alpha\ell^\beta \geq 0$ and also $\int_{\mathcal{S}} d_A\tilde{\omega}^A = 0$. So now we have

$$\int_{\mathcal{S}} \frac{1}{2}R^{(2)}d\mathrm{Vol}\mathcal{S} > 0.$$

With the use of Gauss-Bonnet theorem we have $\chi > 0$ so $\mathcal{S}$ must have topology of a 2-sphere. □

Note that there is no restriction on the topology of unstable MOTS. For example of toroidal unstable MOTS in Schwarzschild spacetimes. In particular see [31].

6.2 Galloway's Theorem

First, we need to give some definitions.

Definition 6.2.1 (Positive Yamabe Type Manifold)**.** A smooth compact manifold is said to be of **positive Yamabe type** if it admits a Riemannian metric of positive scalar curvature.

Definition 6.2.2 (Connected sum)**.** Given connected n-manifolds M_1 and M_2 and regular coordinate balls $B_i \subseteq M_i$, the subspaces $M_i' = M_i \backslash B_i$ are n-manifolds with boundary whose boundaries are homeomorphic to $\mathbb{S}^{n-1}$ If $f : \partial M_2' \to \partial M_1'$ the adjunction space $M_1' \cup_f M_2'$, denoted by $M_1 \# M_2$ is the **connected sum** of M_1 and M_2.

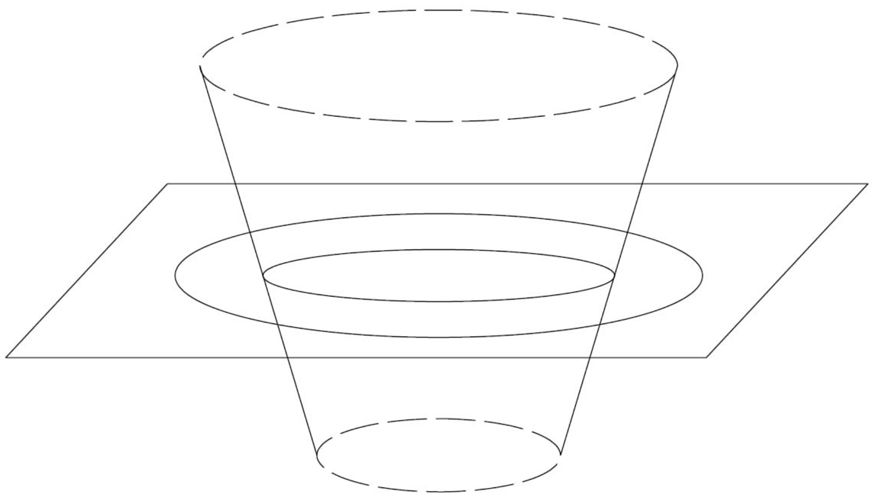

Figure 6.1: Cross sections of the event horizon in asymptotically flat stationary black hole spacetimes obeying the DEC are outermost MOTSs [20].

Galloway's theorem works for outermost MOTSs.

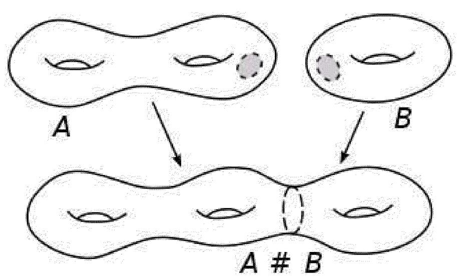

Figure 6.2: Connected sum of two manifolds [35].

Theorem 6 (Galloway-Schoen's Theorem [20])**.** *Let Σ^n, $n \geq 3$ be a spacelike hypersurface in a spacetime that satisfies the dominant energy condition. If $\mathcal{S}^{n-1}$ is an outermost MOTS in Σ^n then $\mathcal{S}^{n-1}$ is of positive Yamabe type, unless $\mathcal{S}$ is Ricci flat (flat if $n = 3, 4$).*

Now we will consider two special cases.

Case 1: If $\mathcal{S}$ is two dimensional, being a positive Yamabe manifold means having a metric with positive Gaussian curvature. As a result, with the help of the Gauss-Bonnet theorem, $\mathcal{S}$ must be a topological 2-sphere.

Case 2: If $\mathcal{S}$ is three-dimensional we have different possibilities which are stated in the theorem below.

Theorem 7. *If $\mathcal{S}$ is a compact orientable 3-manifold of positive Yamabe type then $\mathcal{S}$ must be diffeomorphic to (i) a spherical space (e.g. a 3-sphere S^3), (ii) $S^1 \times S^2$, or (iii) a connected sum of the previous two types.*

6.3 Black holes in higher dimensions

In this section, we want to give an example of black holes in more than four dimensions. We give an example showing that Hawking's theorem does not hold in five dimensions. Instead, we will show that the result is consistent with Galloway's theorem.

Example 6.3.1. Take a neutral black string in five dimensions, constructed as the direct product of the Schwarzschild solution and a line, so the geometry of the horizon is $\mathbf{R} \times S^2$. Imagine bending this string to form a circle, so the topology is now $S^1 \times S^2$. In principle this circular string tends to contract, decreasing the radius of the S^1 due to its tension and gravitational self-attraction. However, we can make the string rotate along the S^1 and balance these forces against the centrifugal repulsion. Then we end up with a neutral rotating black ring: a black hole with an event horizon of topology $S^1 \times S^2$ [18] . This solution is a rotating black ring in vacuum and was discovered by Roberto Emparan and Harvey S. Reall [19]

$$ds^2 = -\frac{F(y)}{F(x)}\left(dt - CR\frac{1+y}{F(y)}d\psi\right)^2 + \frac{R^2}{(x-y)^2}F(x)[-\frac{G(y)}{F(y)}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2]$$

where

$$F(\xi) = 1 + \lambda\xi, \quad G(\xi) = \left(1 - \xi^2\right)(1 + \nu\xi) \tag{6.4}$$

and

$$C = \sqrt{\lambda(\lambda - \nu)\frac{1+\lambda}{1-\lambda}}. \tag{6.5}$$

The dimensionless parameters λ and ν must lie in the range

$$0 < \nu \leq \lambda < 1. \tag{6.6}$$

λ and ν are parameters that characterize the shape and rotation velocity of the ring. The coordinates vary in the ranges $-\infty \leq y \leq -1$, $-1 \leq x \leq 1$ and $-\infty < t < +\infty$, and angles $0 < \phi, \psi < 2\pi$. Because this is a solution in the vacuum spacetime $T_{ab} = 0$, so our solution totally satisfies the dominant energy condition so we have some restrictions on the topology of black ring according to (7). From the example we can see the black rings will have their horizons at constant values of y, or r so the topology of black ring will be $S^1 \times S^2$.

Chapter 7

Discussion and Conclusion

In this chapter, we summarize our intention for this thesis and explain its structure. The goal of this thesis is to review the topology and geometry of MOTS. So, we started the thesis with an introduction to black hole and the problem of identifying them. We mentioned that using event horizon is not very practical for calculation. So that is the reason we will try to use the concept of marginally outer trapped surfaces or MOTS. But first, we needed to define some background concepts and definitions to achieve of our goal.

Therefore we introduced asymptotically flat spacetime. We used this to define the event horizon. After that, we defined different curvatures such as Gauss curvature. Gauss curvature plays and important role in the topology of MOTS. After that we defined geodesics and minimal surfaces and their stability operator. MOTS are closely related to minimal surfaces. Therefore it's essential to know about minimal surfaces and their stability operator.

In chapter three we discussed about event horizon and we defined MOTS. The problem with event horizon is we need to know the whole spacetime. So this will cause computational problems.

After defining MOTS in chapter three we discussed the MOTS stability operator. Similar to minimal surfaces we can discuss about stability operator for MOTS. However for MOTS its extremely difficult to calculate higher-order of stability operators. So one of tasks that can be done in a future thesis is to do those calculations. Also in chapter four, we saw that in highly symmetric spacetimes we have unstable MOTS. As a result we checked the relation between the symmetries of spacetimes and unstability of MOTS.

In chapter five our main goal is to find MOTS in axisymmetric spacetime using a method known as the MOTSodesic.

Finally, in chapter six we discussed the Hawking and Galloway theorem. These theorems restrict the allowed topologies of black holes in four dimensional spacetime and higher. In the case of four-dimensional asymptotically flat spacetime with a black hole that satisfies the dominant energy condition the topology of black hole should be a topological two-sphere.

Bibliography

[1] B. P. Abbott et al. Observation of Gravitational Waves from a Binary Black Hole Merger. *Physical Review Letters.*, 116(6):061102, 2016.

[2] Sharmin Akhter. Rigidity of marginally outer trapped surfaces in reissner-nordstrom spacetime. Master's thesis, Memorial University of Newfoundland, 2021.

[3] Lars Andersson, Marc Mars, and Walter Simon. Local existence of dynamical and trapping horizons. *Physical Review Letters*, 95(11), September 2005.

[4] Lars Andersson, Marc Mars, and Walter Simon. Stability of marginally outer trapped surfaces and existence of marginally outer trapped tubes. *Advances in Theoretical and Mathematical Physics*, pages 853–888, 2007.

[5] Lars Andersson and Jan Metzger. The area of horizons and the trapped region. *Communications in Mathematical Physics*, 290(3):941–972, 2009.

[6] Ivan Booth. Black-hole boundaries. *Canadian Journal of Physics*, 83(11):1073–1099, November 2005.

[7] Ivan Booth. A middle road to differential geometry (unpublished), March 2021.

[8] Ivan Booth. Unpublished note, April 2025.

[9] Ivan Booth, Graham Cox, and Juan Margalef-Bentabol. Symmetry and instability of marginally outer trapped surfaces. *Classical and Quantum Gravity*, 41(11):115003, May 2024.

[10] Ivan Booth, Robie A Hennigar, and Daniel Pook-Kolb. Ultimate fate of apparent horizons during a binary black hole merger. i. locating and understanding axisymmetric marginally outer trapped surfaces. *Physical Review D*, 104(8):084083, 2021.

[11] Ivan Booth, Robie A. Hennigar, and Daniel Pook-Kolb. Ultimate fate of apparent horizons during a binary black hole merger. i. locating and understanding axisymmetric marginally outer trapped surfaces. *Physical Review D*, 104(8), October 2021.

[12] Sean M Carroll. *Spacetime and geometry.* Cambridge University Press, 2019.

[13] Event Horizon Telescope Collaboration, K Akiyama, A Alberdi, W Alef, K Asada, R Azuly, et al. First M87 event horizon telescope results. i. the shadow of the supermassive black hole. *Astrophys. J. Lett*, 875(1):L1, 2019.

[14] Albert Einstein. Die feldgleichungen der gravitation. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften*, pages 844–847, 1915.

[15] Albert Einstein. On a stationary system with spherical symmetry consisting of many gravitating masses. *Annals of Mathematics*, pages 922–936, 1939.

[16] Albert Einstein et al. Zur elektrodynamik bewegter körper. *Annalen der physik*, 17(10):891–921, 1905.

[17] Roberto Emparan and Harvey S Reall. A rotating black ring solution in five dimensions. *Physical Review Letters*, 88(10):101101, 2002.

[18] Roberto Emparan and Harvey S Reall. Black rings. *Classical and Quantum Gravity*, 23(20):R169, 2006.

[19] Roberto Emparan and Harvey S Reall. Black holes in higher dimensions. *Living Reviews in Relativity*, 11(1):1–87, 2008.

[20] Gregory J Galloway. *Constraints on the topology of higher dimensional black holes.* Cambridge University Press Cambridge; New York, 2012.

[21] Gregory J. Galloway and Richard Schoen. A generalization of hawking's black hole topology theorem to higher dimensions. *Communications in Mathematical Physics*, 266(2):571–576, June 2006.

[22] Stephen W Hawking. Black holes in general relativity. *Communications in Mathematical Physics*, 25:152–166, 1972.

[23] Stephen W Hawking and George FR Ellis. *The large scale structure of space-time.* Cambridge university press, 2023.

[24] Robie A Hennigar, Kam To Billy Chan, Liam Newhook, and Ivan Booth. The interior MOTSs of spherically symmetric black holes. *Physical review letters D*, 2021.

[25] Jürgen Jost and Jeurgen Jost. *Riemannian geometry and geometric analysis*, volume 42005. Springer, 2008.

[26] Roy P Kerr. Gravitational field of a spinning mass as an example of algebraically special metrics. *Physical review letters*, 11(5):237, 1963.

[27] Dan A Lee. *Geometric relativity*, volume 201. American Mathematical Society, 2021.

[28] Roger Penrose. Gravitational collapse and space-time singularities. *Phys. Rev. Lett.*, 14:57–59, Jan 1965.

[29] Daniel Pook-Kolb, Robie A Hennigar, and Ivan Booth. What happens to apparent horizons in a binary black hole merger? *Physical Review Letters*, 127(18):181101, 2021.

[30] Karl Schwarzschild. Über das gravitationsfeld eines massenpunktes nach der einsteinschen theorie. *Sitzungsberichte der königlich preussischen Akademie der Wissenschaften*, pages 189–196, 1916.

[31] Kam To Billy Sievers, Liam Newhook, Sarah Muth, Ivan Booth, Robie A Hennigar, and Hari K Kunduri. Marginally outer trapped tori in black hole spacetimes. *Physical Review D*, 109(12):124023, 2024.

[32] Kristopher Tapp. *Differential geometry of curves and surfaces.* Springer, 2016.

[33] Kip S Thorne, Charles W Misner, and John Archibald Wheeler. *Gravitation.* Freeman San Francisco, 2000.

[34] Robert M Wald. *General relativity.* University of Chicago press, 2010.

[35] Wikipedia contributors. Geodesics. `https://en.wikipedia.org/wiki/Geodesic`, 2024. Accessed on: 2024-04-14.