Brushing Directed Graphs

by

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Abstract

Brushing of graphs is an edge searching strategy where the searching agents are called brushes. We focus on brushing directed graphs based on a new model that we have developed. The fundamentals of this model have been influenced by previous studies about brushing undirected graphs. We discuss strategies to brush directed graphs as well as values and bounds for the brushing number of directed graphs. As the main results we give an upper bound for the brushing number of directed acyclic graphs. We also establish exact values for the brushing number of transitive tournaments, complete directed graphs and rotational tournaments. The behaviour of the brushing number of several other types of directed graphs are also taken into consideration.

To my loving husband

Lay summary

A graph is a collection of a set of points, called vertices, and a set of lines called edges connecting pairs of vertices. As an analogy consider a network of pipes. A point where two or more pipes join can be taken as a vertex and pipes are the edges. The direction in which the water flows in the pipe indicates the direction of the edge in the graph. An edge with a direction is called an arc. A graph with directed edges is called a directed graph. In this thesis we consider directed graphs. Brushing graphs is a process in which a graph is initially considered dirty and it needs cleaning, including all the vertices and arcs using cleaning agents called brushes. For an example of a situation that can be modelled by brushing, consider a network of pipes that have to be cleaned because of algae contamination. We assign brushes to each vertex. To reduce the recontamination, when a vertex is cleaned, a brush must travel down each connected pipe (in the direction of the corresponding arc). When a vertex disperses its brushes to the set of arcs connected to it, we say that the vertex has fired. Once a brush traverses an arc, that arc is cleaned. We introduce a set of rules for the behaviour of brushes in a directed graph. In this thesis we focus on how to clean a graph using the minimum number of brushes, which is known as the brushing number of a graph. In some situations we give an upper bound for the brushing number of a certain type of graphs. In order to find the brushing number of a graph we need to establish the number of brushes that should be placed at each vertex of a graph and the order in which the vertices fire. We call this procedure a strategy for brushing the graph. When we have a strategy to brush the graph then we can find the brushing number or the upper bound for the brushing number of the graph. To get an idea about the brushing number of some types of graphs considered in this thesis we use the brushing number of another type of graphs. In the final chapter we present some questions about generalising the upper bounds that we have established for certain types of directed graphs.

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Statement of contribution

Dr. David Pike and Dr. Jared Howell suggested the idea of the research and supervised the entirety of the research. All the included results were proven by the author independently or with the help of the supervisors (Dr. David Pike and Dr. Jared Howell). The manuscript was done with the supervision of Dr. David Pike and Dr. Jared Howell.

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Chapter 1

Introduction

In this thesis the process of cleaning directed graphs will be introduced. Graph cleaning is a process inspired by the concepts of edge searching and chip firing. We will give a brief overview of these two concepts in Section 1.1.

In order to understand the contents of this thesis we first review some basic graph theory. A *graph* G is a triple containing a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates each edge with one or two vertices called its endpoints. A vertex u is *adjacent* to a vertex v if they are joined by an edge. If vertex v is an endpoint of edge e, then v is said to be *incident* to e, and e is incident to v. The *degree* of a vertex $v \in V(G)$ denoted by $\deg(v)$ is the number of edges with that vertex as an end-point. The *minimum degree* of graph G is denoted by $\delta(G)$ where $\delta(G) = \min\{deg(v) \mid v \in V(G)\}$. A *loop* is an edge that joins a vertex to itself. A *multi-edge* is a collection of two or more edges having identical endpoints. A *simple graph* is a graph with no loops or multi-edges.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *path* is a simple graph or a subgraph whose vertices can be sequentially listed (without repetition) so that two vertices are adjacent if and only if one vertex succeeds the other with respect to the ordering in the list. A graph G is *connected* if each pair of vertices in G belongs to a path. Figure 1.1 gives an example of a connected simple graph. A *directed graph* is a graph that consists of a set of vertices connected by directed edges called *arcs*. In a directed graph a vertex with only outgoing incident arcs is called a *source* and a vertex with only incoming incident arcs is called a *sink*. In Figure 1.2 v is a source and u is a sink. For additional information on graph theory we recommend the textbooks [4, 8, 15].

The process of cleaning graphs, which was introduced for undirected graphs, will be adapted for directed graphs in this thesis. In the remainder of this chapter we will discuss cleaning undirected graphs and then we will introduce some terminology for directed graphs and cleaning directed graphs. Finally an overview of Chapters 2, 3 and 4 will be given.

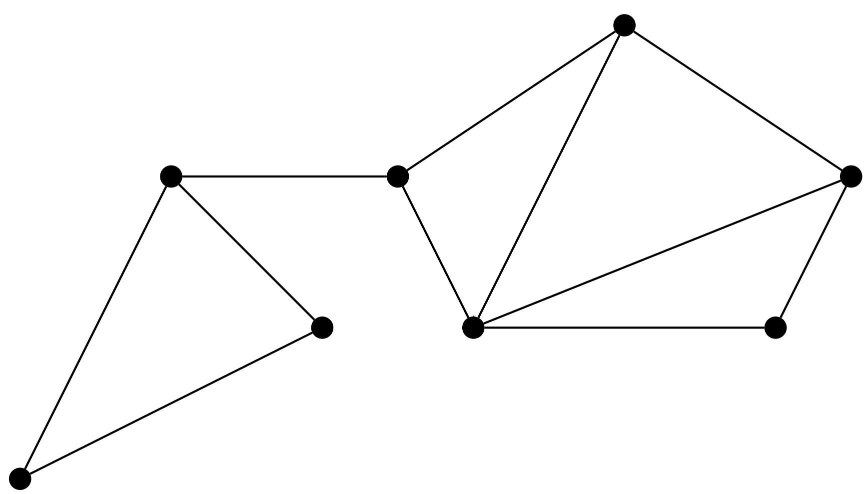

Figure 1.1: Graph G which is a connected graph.

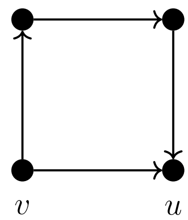

Figure 1.2: A graph with a source vertex v and a sink u.

1.1 Brushing Undirected Graphs

In this thesis we will discuss a cleaning model which was inspired by the chip firing game and searching. Through this section we will briefly discuss the history of graph searching and related problems. As a motivating example consider the problem of capturing an intruder in a network that is represented by a graph. As mentioned in [1] there are two parts to this problem. Searchers and intruders move from vertex to vertex, and along the edges of the graph in a discrete or continuous way. A searcher captures an intruder when they both occupy the same position at the same time. If an intruder may be located only at vertices, then the process of seeking to capture an intruder is called 'searching'. If the intruder is located at vertices or along edges, then the process of seeking to capture an intruder is called 'sweeping'. The problem initially was inspired by spelunkers. The discussion about searching for a person lost in a cave system dates back far as 1967 [5]. A group of spelunkers in the 1970s consulted Tory Parsons, who was a mathematician at Pennsylvania State University, to seek advice regarding their search techniques. Parsons composed the problem in terms of graph theory. He published two papers [12, 13], about sweeping graphs. His work laid the foundation to the study of searching and sweeping graphs. Graph searching provides mathematical models for real-world problems such as eliminating a computer virus in a network, computer games and counterterrorism. In all searching problems, there are searchers (or cops) trying to capture some intruder (robber, or fugitive). A fundamental optimization question here is: What is the minimum number of searchers required to capture the intruder?

The cleaning process of a graph is related to the chip firing game and searching. In chip firing [3] there is an initial configuration of chips on vertices and a vertex is said to be ‘primed’ if the vertex has at least as many chips as its degree. When a primed vertex ‘fires’ it sends one chip along each incident edge. In the study of chip firing games commonly raised questions have been variants of “is this process finite or infinite?”, “how many chips are needed to produce a cycle?” and “how long will it take for an infinite game to become a repeated cycle of chip configurations?”.

A cleaning model which is a combination of chip firing and searching was introduced by McKeil [10]. As an analogy consider a network of pipes that have to be periodically cleaned of a contaminant such as algae. This is accomplished by having cleaning agents, called *brushes*, assigned to some vertices. To reduce the recontamination, when a vertex is *cleaned*, a brush must travel down each incident contaminated edge. Once a brush traverses an edge, that edge is *cleaned*. A graph G has been cleaned once every edge of G has been cleaned. In [10] McKeil defined a pairing of each vertex v of G with the number of brushes located on v at time t, and its set of clean incident edges as a *brush configuration* on a graph G at time t. McKeil also defined the *brush number* of a graph as the number of brushes required by a brush configuration. McKeil focused on two distinct objectives namely, minimum brush number and minimum cleaning time. In [10] *dispersal mode* was defined as the rule that chooses the way that vertices can fire. Two types of dispersal modes, namely parallel and sequential, are given. McKeil has developed two models, regular cleaning model and open cleaning model. For both models McKeil presented strategies to minimise the number of brushes used and number of time steps taken to clean a graph. Also in the two models more than one brush can travel through an edge and brushes can travel through cleaned edges.

The regular cleaning model adapts regular brush dispersal rules from chip firing. When a vertex has at least the number of brushes equal to its degree, the vertex can fire, whether the incident edges are clean or not. Under this particular model, for complete graphs, results for the brush number, cleaning time, cycle time, cleaning frequency and cleaning redundancy are given in [10]. Further McKeil in [10] considers regular cleaning with time constraints. She has obtained results regarding the one step cleaning of a bipartite graph and a complete graph. During one step cleaning of a complete graph with n vertices, the brush number is obtained as $(n-1)^2$.

In the open cleaning model if the number of brushes at a vertex is at least the number of dirty incident edges of the vertex, then the vertex fires. Therefore, the number of brushes a vertex needs to fire changes through the cleaning process. Under the open cleaning model bounds on the brush number of graphs in general and specifically bounds for the brush number of trees are given in [10]. Further the basic cyclic properties of cleaning sequences are discussed. In [10] the minimum number of brushes that can be used to clean G, over all possible configurations in the open cleaning model, is defined as the *open cleaning minimum brush number* of a graph G, denoted by $b(G)$. The following theorem in [10] gives lower and upper bounds for the brush number of a tree.

Theorem 1.1.1. *If T is a tree and $\ell(T)$ is the leaf set of T then*

$$\left\lceil \frac{\Delta(T)}{2} \right\rceil \leq b(T) \leq \left\lceil \frac{|\ell(T)|}{2} \right\rceil.$$

As given in [10] under the bounds for brush number of graphs in general, for any graph G, $\delta(G) \leq b(G) \leq |E(G)|$.

A cleaning model in which the number of brushes that can traverse an edge is restricted is known as a cleaning model with edge capacity restrictions. The first model for brushing with edge capacity restrictions is given in [11]. In that model every edge in a graph G is initially considered dirty and a fixed number of brushes begin on a set of vertices. At each step of the process, a vertex v may be cleaned (instead of fired) if the number of brushes on v is greater than or equal to the number of dirty incident edges. When G is cleaned, every dirty edge must be traversed by only one brush; moreover, brushes cannot traverse a clean edge. A graph cleaning model which differs from [11] is presented by Bryant, Francetić, Gordinowicz, Pike and Prałat in [6] where edges are allowed to be traversed by multiple brushes. They have developed general bounds for the brushing number of an undirected graph, including bounds that are given in terms of parameters such as cutwidth and bisection width. One of the main results in [6] is the 'Reversibility Theorem', which is given below.

Theorem 1.1.2. *[6] Given an initial configuration ω_0, suppose G can be cleaned yielding final configuration ω_n, $n = |V(G)|$. Then, given initial configuration $\tau_0 = \omega_n$, G can be cleaned yielding the final configuration $\tau_n = \omega_0$.*

As mentioned in [6] the notion of reversing the cleaning process is adapted from [11]. Further [6] proves that the brushing number of any graph G is greater than or equal to the cutwidth of the graph G and the brushing number is greater than or equal to the bisection width of the graph G. Two specific classes of graphs, namely Cartesian products and trees have been studied in [6]. For Cartesian products of graphs an upper bound on the brushing number is proved and exact values for the brushing number of m by n grids and hypercubes are also given. In [6] it is proved that for a tree T with $\ell(T)$ leaves, $\frac{(\ell(T)+1)}{2}$ is the brushing number when $\ell(T)$ is odd.

When $\ell(T)$ is even the brushing number is either $\frac{\ell(T)}{2}$ or $\frac{\ell(T)}{2} + 1$. This implies that trees with an even number of leaves are divided into two groups depending on their brushing number. Moreover Penso, Rautenbach and Ribeiro de Almeida proved that trees with $\ell(T)$ leaves and brushing number $\frac{\ell(T)}{2}$ can be recognized efficiently in [14].

1.2 Directed Graphs

1.2.1 Definitions and Terminology

As the contents of this thesis will be about directed graphs, now we will review some terminology about directed graphs. A *directed graph* or a *digraph* G consists of a vertex set $V(G)$ and an arc set $A(G)$ such that each arc of $A(G)$ is an ordered pair of vertices. We say that an *arc* is directed from its tail to its head. A *simple directed graph* is a directed graph with no self-loops and no multi arcs. In a simple directed graph G, an arc from u to v where $u, v \in V(G)$ is commonly denoted (u, v) (or sometimes uv). Throughout this thesis we consider simple directed graphs only.

For an undirected graph G with vertex set $V(G)$ and the edge set $E(G)$, the *neighbourhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all neighbours of v, i.e., $N_G(v) = \{u \mid (u, v) \in E(G)\}$. For a directed graph G, consider an arc $(a, b) \in A(G)$. We say that vertex b is an *out-neighbour* of a and vertex a is an *in-neighbour* of b. We use $N_G^+(v)$ to indicate the *set of out-neighbours* and $N_G^-(v)$ to indicate the *set of in-neighbours* of $v \in V(G)$. Further when G is a directed graph, for a vertex $v \in V(G)$, $\deg^+(v)$ denotes the *out-degree* of v where $\deg^+(v) = |N_G^+(v)|$. Similarly $\deg^-(v)$ denotes the *in-degree* of v where $\deg^-(v) = |N_G^-(v)|$. A source of G is a $v \in V(G)$ with $\deg^-(v) = 0$. A sink of G is a $v \in V(G)$ with $\deg^+(v) = 0$. For further details refer to [16]. Given a directed graph, its *underlying graph* is the undirected graph obtained by replacing every arc with an undirected edge.

A *tournament* is a directed graph in which there is exactly one arc between each pair of vertices. A tournament $\mathcal{T}$ is *transitive* if and only if for all vertices $u, v, w \in V(\mathcal{T})$ if (u, v) and (v, w) are arcs of $\mathcal{T}$ then (u, w) is an arc of $\mathcal{T}$. Note that if (u, v) and (v, w) are arcs in $\mathcal{T}$, we must have $w \neq u$, and then (w, u) is an arc of $\mathcal{T}$ if and only if (u, w) is not an arc of $\mathcal{T}$. It follows that a tournament $\mathcal{T}$ is not transitive if and only if there exist vertices u, v, w, such that $(u, v), (v, w)$ and (w, u) are all arcs of $\mathcal{T}$. Transitive tournaments are acyclic directed graphs. Figure 1.3 and Figure 1.4 provide further clarification on this definition. Each vertex in a transitive tournament has a different out-degree. A transitive tournament on n vertices has the degree sequence $\{n-1, n-2, \ldots, 0\}$.

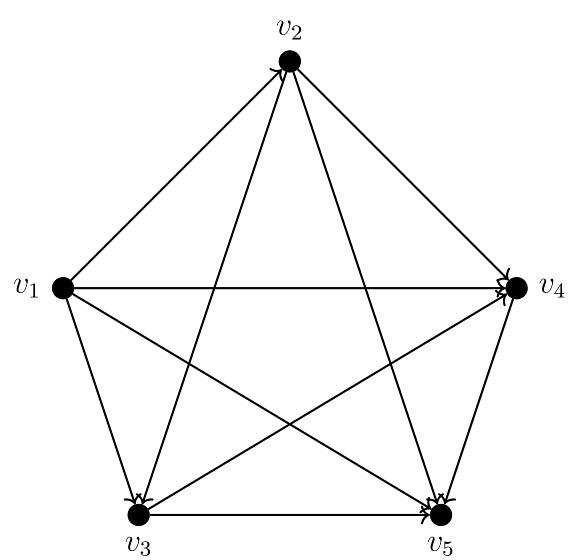

Figure 1.3: A transitive tournament $\mathcal{T}$ on five vertices.

A *complete directed graph* is a simple directed graph such that between each pair of its vertices, both (oppositely directed) arcs exist. A *walk* in an undirected graph

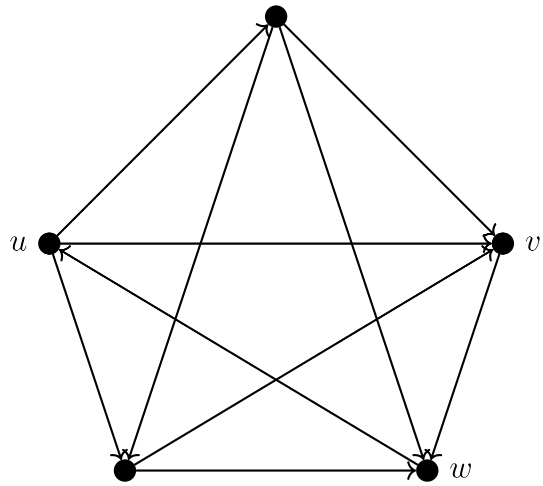

Figure 1.4: A tournament $\mathcal{T}^{'}$ on five vertices which is not transitive. Note that (u, v) and (v, w) are arcs in $\mathcal{T}^{'}$ and (u, w) is not an arc.

G is an alternating sequence of vertices and edges,

$$W = v_0, e_1, v_1, e_1, \dots, e_n, v_n$$

such that for $j = 1, 2, \dots, n$, the vertices v_{j-1} and v_j are the end points of the edge e_j. If G is a directed graph and each arc e_j is directed from v_{j-1} to v_j, then W is a *directed walk*. A *directed trail* in a directed graph is a directed walk such that no arc occurs more than once. A *directed cycle* in a directed graph is a non-empty directed trail in which only the first and last vertices are equal. When a directed graph has no directed cycles, it is called a *directed acyclic graph*. Every directed acyclic graph has at least one source and at least one sink.[8]. A *component* of an undirected graph G is a connected subgraph H such that no subgraph of G that properly contains H is connected. Basically a component is a maximal connected subgraph [8]. In the present thesis when we consider the components of a directed graph D, we refer to the components of the underlying graph G. See Figure 1.5 for an example.

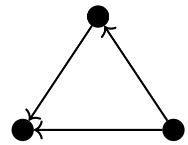

Figure 1.5: A directed graph G with one component.

1.2.2 Model for Brushing Directed Graphs

In the present thesis, we have adapted the model in [6] by making some modifications relevant for directed graphs. In the process of cleaning a directed graph G, an isolated vertex can fire provided that it has a brush, whereas a vertex $v \in V(G)$ that is not isolated is able to fire only if the present number of brushes at v is at least the out-degree of v. When a vertex v fires, the brushes on v clean v, and each outgoing arc incident with v is traversed by at least one brush that is fired from v, thereby cleaning the dirty outgoing arcs that were incident with v. At the end of the step each brush from firing vertex v moves to the vertex adjacent to v at the endpoint of the arc it traversed and the excess brushes are allowed to remain at v. Arcs are allowed to be traversed by more than one brush, and multiple brushes can simultaneously traverse an arc; however each arc can be traversed during at most one step of the cleaning process. We allow vertices to fire one at a time in a sequence. In this model we do not fire multiple vertices that are ready to fire at the same time step (although a way of simultaneously firing vertices when brushing undirected graphs is described as *parallel dispersal mode* in [10]). Before beginning the process of cleaning we decide the sequence in which the vertices fire. Since we allow multiple brushes to traverse an arc, it is possible that a vertex which fires later in the sequence might accumulate brushes from vertices which have been fired earlier in the sequence.

As an example refer the graph G in Figure 1.6. At the beginning of the process of cleaning graph G in Figure 1.6, put one brush each at the vertices v_1, v_2 and v_3.

Then v_1, v_2 and v_3 fire sequentially and v_4 accumulates three brushes. Then v_4 fires and these three brushes traverse the arc (v_4, v_5). Then v_5 receives three brushes and v_5 fires. Then vertices v_6, v_7 and v_8 get one brush each. Then v_6, v_7 and v_8 fire sequentially and brushes stay at these vertices as these are sink vertices. Now we have cleaned graph G with three brushes.

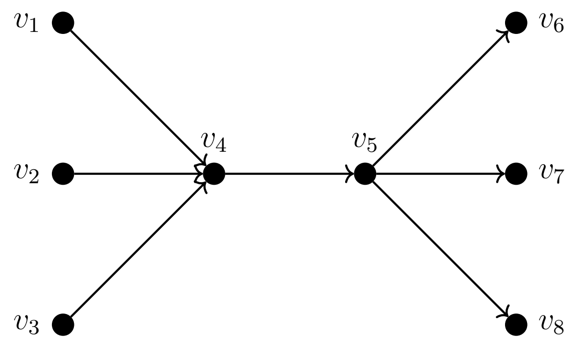

Figure 1.6: Graph G; an example of a graph where multiple brushes simultaneously traverse an arc during brushing.

The process of cleaning terminates when a clean graph is obtained, or else the process stops in a situation in which there are dirty vertices but none are capable of firing. Let $B_G^t(v)$ denote the number of brushes at vertex v of G, at time t ($t = 0, 1, 2, \ldots, n$) and $B_G^0(v_i)$ denote the number of brushes at the vertex v_i in the initial configuration. A vertex v_i fires in between time $i-1$ and i. The *brushing number* of a graph is the minimum number of brushes needed for some initial configuration to clean the graph. For a graph G, the brushing number is denoted by $B(G)$. As an example refer to Figure 1.7. The process of brushing the underlying graph G' of the directed graph G in Figure 1.7 with the cleaning model in [11] is given in Figure 1.8. In Figures 1.7 and 1.8 black nodes represent unfired (or dirty) vertices. The edges and arcs in thick lines represent dirty edges and dirty arcs. The white nodes represent fired (or clean) vertices. The edges and arcs in dashed lines represent clean edges and clean arcs. The number of brushes at a vertex in a time step is given inside the node.

If there is no number inside a vertex then that vertex does not have any brushes at the particular time step.

Based on the cleaning model given for directed graphs, the following Theorem 1.2.1 can be presented.

Theorem 1.2.1. *If G is a directed graph, then* $\max\{\deg^+(v) \mid v \in V(G)\} \le B(G) \le |A(G)|$.

Proof. In a directed graph G if v_M is the vertex with maximum out-degree, to fire v_M we need $\deg^+(v_M)$ brushes. Therefore $\deg^+(v_M) \le B(G)$. Also, in graph G when all the vertices have been fired, then at least one brush should have traversed each arc, which implies $|A(G)|$ is an upper bound for the brushing number. □

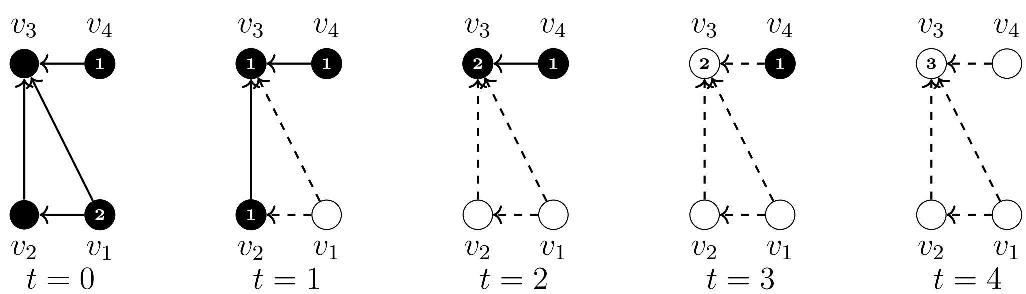

Figure 1.7: A directed graph G with $B(G) = 3$.

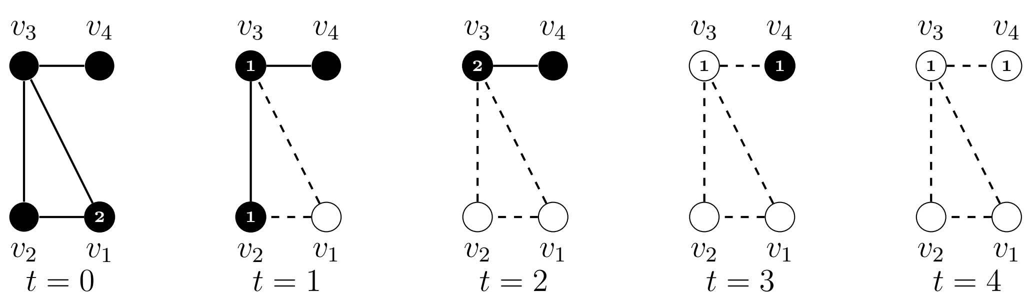

Figure 1.8: Graph G' with brush number 2.

1.3 Outline of the Thesis

In Chapter 2 of this thesis the brushing number of directed acyclic graphs will be discussed in detail. We specifically discuss the brushing number of transitive tournaments and use it to arrive at conclusions about the brushing number of directed acyclic graphs in general. In Chapter 3 we take several other types of directed graphs into consideration. We also show that other concepts like path decomposition and width of the edge poset of a graph are associated with the brushing number of a graph. In the final chapter we present some questions about generalising the upper bounds that we have established for certain types of directed graphs. For an example, in Theorem 3.0.25 we give an upper bound for rotational tournaments. In the final chapter we ask the question, what is the upper bound for the brushing number of regular tournaments?

Chapter 2

Brushing Directed Acyclic Graphs

In this chapter we will mainly focus on brushing transitive tournaments and directed acyclic graphs. Denote $\mathfrak{G}_k$ as the set of all directed acyclic graphs on k vertices. For $k = 2$, there are two directed acyclic graphs as shown in Figure 2.1 with brushing number 2 and 1 respectively. For $k \geq 2$ we will find upper bounds for the brushing number of all elements of $\mathfrak{G}_k$ in Section 2.2. To do so, first we will discuss a brushing strategy for transitive tournaments and we will determine the brushing number of transitive tournaments in Section 2.1. If one edge of a transitive tournament is taken away, it is still a directed acyclic graph. We will discuss the brushing number of such graphs as well.

Figure 2.1: The set of directed acyclic graphs with two vertices.

2.1 Transitive tournaments

In this section first we introduce a strategy that can be used to brush a transitive tournament as follows.

Strategy 2.1.1. In this strategy, a series of steps for brushing a transitive tournament with n vertices will be presented. Consider the set of vertices $\{v_1, v_2, v_3, \ldots, v_n\}$ with $\deg^+(v_i) = n - i$, for all $1 \leq i \leq n$. Let $B_G^t(v)$ be the number of brushes at vertex v of G, at time t ($t = 0, 1, 2, \ldots, n$). For each $i \in 1, 2, \ldots, n$ the vertex v_i fires in between time $i - 1$ and i. Let $\ell = \lfloor \frac{n}{2} \rfloor$. Let $B_G^0(v_k)$ denote the number of brushes at the k^{th} vertex in the initial configuration such that

$$B_G^0(v_k) = \begin{cases} n - (2k - 1) & \text{if } k = 1, 2, \ldots, \ell, \\ 0 & \text{otherwise.} \end{cases}$$

So no vertex with index greater than ℓ receives a brush in the initial configuration. As vertex v_1 with out-degree $n - 1$ has $n - 1$ brushes, v_1 fires. Now

$$B_G^1(v_k) = \begin{cases} 0 & \text{if } k = 1, \\ n - (2k - 2) & \text{if } k = 2, 3, \ldots, \ell, \\ 1 & \text{otherwise.} \end{cases}$$

Observe that the number of brushes at v_2 is $n - 2$. So v_2 fires. Then

$$B_G^2(v_k) = \begin{cases} 0 & \text{if } k = 1, 2, \\ n - (2k - 3) & \text{if } k = 3, 4, \ldots, \ell, \\ 2 & \text{otherwise.} \end{cases}$$

After the i^{th} vertex fires ($i = 1, 2, \ldots, \ell$)

$$B_G^i(v_k) = \begin{cases} 0 & \text{if } k = 1, 2, \ldots, i, \\ n - 2k + i + 1 & \text{if } k = i + 1, i + 2, \ldots, \ell, \\ i & \text{otherwise.} \end{cases}$$

When $v_1, v_2, \ldots, v_{i-1}$ have all fired v_i is able to fire, as it will have $n - (2i - 1) + i - 1 = n - i = \deg^+(v_i)$ brushes, where $i = 1, 2, \ldots, \ell$. After the vertices $v_1, v_2, \ldots, v_{\ell-1}$ fire, v_ℓ obtains $\ell - 1$ brushes. But initially, there were $n - (2\ell - 1)$ brushes in v_ℓ. Therefore, in total v_ℓ has $n - \ell$ brushes, which is equal to its out-degree. Then v_ℓ fires. Initially there were no brushes in $v_{\ell+1}$. After v_ℓ fires, $v_{\ell+1}$ has accumulated ℓ brushes. As $\deg^+(v_{\ell+1}) = n - (\ell + 1)$, $\deg^+(v_{\ell+1}) = \lfloor \frac{n-1}{2} \rfloor$ and then $n - (\ell + 1) = \deg^+(v_{\ell+1}) \leq \ell$. Therefore $v_{\ell+1}$ fires. Similarly, we can also show that $v_{\ell+2}, v_{\ell+3}, \ldots, v_n$ are able to fire in sequence.

As an example we will walk through the process of brushing the transitive tournament $\mathcal{T}$ in Figure 1.3. According to the Strategy 2.1.1 set an initial configuration of six brushes in $\mathcal{T}$ such that

$$B_G^0(v_k) = \begin{cases} 5-(2k-1) & \text{if } k = 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

So no vertex with index greater than 2 receives brushes in the initial configuration. As vertex v_1 with out-degree four has four brushes, v_1 fires. Now

$$B_G^1(v_k) = \begin{cases} 0 & \text{if } k = 1, \\ 3 & \text{if } k = 2, \\ 1 & \text{otherwise.} \end{cases}$$

Observe that the number of brushes at v_2 is three. So v_2 fires. Then

$$B_G^2(v_k) = \begin{cases} 0 & \text{if } k = 1, 2, \\ 2 & \text{otherwise.} \end{cases}$$

Observe that the number of brushes at v_3 is two. So v_3 fires. Then

$$B_G^3(v_k) = \begin{cases} 0 & \text{if } k = 1, 2, 3, \\ 3 & \text{otherwise.} \end{cases}$$

The out-degree of v_4 is one and the number of brushes at v_4 is three. So v_4 fires. The excess two brushes remain at v_4. Then

$$B_G^4(v_k) = \begin{cases} 0 & \text{if } k = 1, 2, 3, \\ 2 & \text{if } k = 4, \\ 4 & \text{otherwise.} \end{cases}$$

The sink vertex v_5 of $\mathcal{T}$ has four brushes now. Then v_5 fires, but the brushes remain at v_5. Then

$$B_G^5(v_k) = \begin{cases} 0 & \text{if } k = 1, 2, 3 \\ 2 & \text{if } k = 4, \\ 4 & \text{otherwise.} \end{cases}$$

Now the transitive tournament $\mathcal{T}$ is clean. Therefore $B(\mathcal{T}) \leq 6$.

We will shortly present a lemma that will assist with finding the brushing number of a transitive tournament. Apart from the brushing number it is interesting to notice the paths that brushes travel in directed graphs. After establishing the brushing number of a transitive tournament we will have a corollary related to the paths that brushes traverse in a transitive tournament. Further, the brushing number of a transitive tournament when one edge is taken away will be discussed at the end of this subsection.

The upcoming Theorem 2.1.3 in this thesis will be about establishing the brushing number of a transitive tournament. In order to prove the equality in Theorem 2.1.3 we will need Lemma 2.1.2, for which we will introduce some notations. Let G be a directed graph. The set of edges with one end in S and the other end in T, where $S, T \subseteq V(G)$ is denoted by $[S, T]$. An *edge-cut* is a set of edges of the $[S, \bar{S}]$ where $\emptyset \subset S \subset V(G)$ and $\bar{S} = \{v \in V(G) | v \notin S\}$. The directed subgraph of G whose vertex set is S and whose arc set is the set of those arcs of G that have both ends in S is called the *subgraph of G induced by S* and is denoted by $G[S]$.

Lemma 2.1.2. *If G is a directed graph and $S \subseteq V(G)$ such that no arcs go from $\bar{S}$ to S, then $B(G) \geq |[S, \bar{S}]|$. Moreover, when G is a transitive tournament $B(G) \geq |S||\bar{S}|$.*

Proof. Let G be a directed graph and $S \subseteq V(G)$ such that no arcs go from $\bar{S}$ to S. As a result of the edge-cut $[S, \bar{S}]$ the two subgraphs of G, $G[S]$ and $G[\bar{S}]$, are obtained. The subgraphs $G[S]$ and $G[\bar{S}]$ can be brushed as two separate graphs. But consider G where $A(G) = A(G[S]) \cup A(G[\bar{S}]) \cup [S, \bar{S}]$ and the three sets of arcs $A(G[S])$, $A(G[\bar{S}])$ and $[S, \bar{S}]$ are pairwise disjoint. When brushing G, we have to use one brush for each arc in $[S, \bar{S}]$. Therefore $B(G) \geq |[S, \bar{S}]|$.

When G is a transitive tournament, consider a set $S \subseteq V(G)$ such that no arcs go from $\bar{S}$ to S. Then $|[S, \bar{S}]| = |S||\bar{S}|$ as every vertex in S is adjacent to every vertex in $\bar{S}$. Therefore $B(G) \geq |S||\bar{S}|$. □

Now we have all the tools necessary to prove Theorem 2.1.3.

Theorem 2.1.3. *If G is a transitive tournament with n vertices, then*

$$B(G) = \begin{cases} \frac{n^2-1}{4} & \text{if } n \text{ is odd,} \\ \frac{n^2}{4} & \text{if } n \text{ is even.} \end{cases}$$

Proof. When Strategy 2.1.1 is used to brush G, $B(G) \leq \sum_{k=1}^{\ell} B_G^0(v_k)$. When n is even,

$$B(G) \leq \underbrace{n-1+n-3+\cdots+1}_{\text{sum of odd numbers between 0 and } n} = \left(\frac{n}{2}\right)^2 = \frac{n^2}{4}.$$

When n is odd,

$$B(G) \leq \underbrace{n-1+n-3+\cdots+2}_{\text{sum of even numbers between 0 and } n} = \left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+1\right) = \frac{(n^2-1)}{4}.$$

By using Lemma 2.1.2 for a transitive tournament G with $S = \{v_1, v_2, v_3, \ldots, v_\ell\}$, we get the following lower bound on $B(G)$.

$$B(G) \geq \begin{cases} \frac{n^2-1}{4} & \text{if } n \text{ is odd,} \\ \frac{n^2}{4} & \text{if } n \text{ is even.} \end{cases}$$

Therefore

$$B(G) = \begin{cases} \frac{n^2-1}{4} & \text{if } n \text{ is odd,} \\ \frac{n^2}{4} & \text{if } n \text{ is even.} \end{cases}$$

□

The obtained brushing number for a transitive tournament implies that Strategy 2.1.1 is an optimal strategy. As the movement of brushes in a transitive tournament is observed in Theorem 2.1.3, we can obtain the following corollary.

Corollary 2.1.4. *If G is a transitive tournament that is cleaned using Strategy 2.1.1, then there is at least one brush that travels a Hamiltonian path during brushing G.*

Proof. When using Strategy 2.1.1 to brush transitive tournament G, every vertex obtains one brush from the preceding vertex that just fired. □

So far in this section we have discussed brushing transitive tournaments and facts related to this process. Then the following question arises: if one of the arcs in a transitive tournament is removed, what will be the brushing number of the resulting directed graph? Studying these types of directed graphs will be useful when building an upper bound for directed acyclic graphs, later in Section 2.2. Note that when an arc e is removed from a directed graph G we denote it by $G - e$.

Theorem 2.1.5. *Let G be a transitive tournament on $n \geq 3$ vertices. If $e \in A(G)$, then $B(G-e) \leq B(G)$.*

Proof. Label the vertices of G in the descending order of their out-degree, so that

$$\deg^+_G(v_1) > \deg^+_G(v_2) > \cdots > \deg^+_G(v_n).$$

Let $\ell = \lfloor \frac{n}{2} \rfloor$. Strategy 2.1.1 will be used for cleaning G and based on that a strategy to clean $G-e$ using no more than $B(G)$ brushes will be given in this proof. Let $L = \{v_1, v_2, v_3, v_4, \ldots, v_\ell\}$ and $R = V(G) \setminus L$. Recall that $B^t_G(v)$ denotes the number of brushes at vertex v of G, at time t ($t = 0, 1, 2, \ldots, n$) and $B^0_G(v_i)$ denotes the number of brushes at the vertex v_i in the initial configuration. A vertex v_i fires in between time $i-1$ and i. Observe that for each $v_i \in R \setminus \{v_{\ell+1}\}$, $B^{i-1}_G(v_i) > \deg^+_G(v_i)$. We will describe an initial configuration B^0_{G-e} to clean $G-e$, whereby the vertices will fire in the sequence $v_1, v_2, \ldots, v_n$.

Case I: $e = (v_a, v_b)$, where $v_a \in L \setminus \{v_\ell\}$ and $v_b \in R \setminus \{v_{\ell+1}\}$

As the arc e is not present in $G-e$, let $B^0_{G-e}(v_a) = B^0_G(v_a) - 1$ and $B^0_{G-e}(v) = B^0_G(v)$ for each $v \in V(G) \setminus \{v_a\}$. Then for $0 \leq t \leq a-1$,

$$B^t_{G-e}(v) = \begin{cases} B^t_G(v) - 1 & \text{if } v = v_a, \\ B^t_G(v) & \text{otherwise.} \end{cases}$$

Now for $t \in \{a, a+1, \dots, b-1\}$,

$$B^t_{G-e}(v) = \begin{cases} B^t_G(v) - 1 & \text{if } v = v_b, \\ B^t_G(v) & \text{otherwise.} \end{cases}$$

Observe that $B^{b-1}_{G-e}(v_b) = B^{b-1}_G(v_b) - 1 \geq \deg^+_G(v_b) = \deg^+_{G-e}(v_b)$ and so v_b can fire between time b and $b-1$. Consequently for $t \in \{b, b+1, \dots, n\}$,

$$B^t_{G-e}(v) = \begin{cases} B^t_G(v) - 1 & \text{if } v = v_b, \\ B^t_G(v) & \text{otherwise.} \end{cases}$$

Thus $B(G-e) \leq B(G) - 1$.

Case II: (a) $e = (v_a, v_b)$, where $v_a \in L \setminus \{v_\ell\}$, $v_b = v_{\ell+1}$ and n is even.
As the arc e is not present in $G-e$, let $B^0_{G-e}(v_a) = B^0_G(v_a) - 1$ and $B^0_{G-e}(v) = B^0_G(v)$ for each $v \in V(G) \setminus \{v_a\}$.

Then for $t \in \{0, 1, \dots, a-1\}$,

$$B^t_{G-e}(v) = \begin{cases} B^t_G(v) - 1 & \text{if } v = v_a, \\ B^t_G(v) & \text{otherwise.} \end{cases}$$

Now for $t \in \{a, a+1, \dots, \ell\}$,

$$B^t_{G-e}(v) = \begin{cases} B^t_G(v) - 1 & \text{if } v = v_{\ell+1}, \\ B^t_G(v) & \text{otherwise.} \end{cases}$$

Observe that $B^{\ell}_{G-e}(v_{\ell+1}) = B^{\ell}_{G}(v_{\ell+1}) - 1 = \deg^{+}_{G-e}(v_{\ell+1})$ and so $v_{\ell+1}$ can fire. Consequently for $t \in \{\ell+1, \ell+2, \ldots, n\}$,

$$B^{t}_{G-e}(v) = \begin{cases} B^{t}_{G}(v) - 1 & \text{if } v = v_{\ell+1}, \\ B^{t}_{G}(v) & \text{otherwise.} \end{cases}$$

Thus $B(G-e) = B(G) - 1$.

Case II: (b) $e = (v_a, v_b)$, where $v_a \in L \setminus \{v_\ell\}$, $v_b = v_{\ell+1}$ and n is odd. As the arc e is not present in $G-e$, and $B^{\ell}_{G}(v_{\ell+1}) = \deg^{+}_{G}(v_{\ell+1})$ let

$$B^{0}_{G-e}(v) = \begin{cases} B^{0}_{G}(v) - 1 & \text{if } v = v_a, \\ B^{0}_{G}(v) + 1 & \text{if } v = v_{\ell+1}, \\ B^{0}_{G}(v) & \text{otherwise.} \end{cases}$$

Then for $t \in \{0, 1, \ldots, a-1\}$,

$$B^{t}_{G-e}(v) = \begin{cases} B^{t}_{G}(v) - 1 & \text{if } v = v_a, \\ B^{0}_{G}(v) + 1 & \text{if } v = v_{\ell+1}, \\ B^{t}_{G}(v) & \text{otherwise.} \end{cases}$$

Now for $t \in \{a, a+1, \ldots, \ell\}$,

$$B^{t}_{G-e}(v) = B^{t}_{G}(v) \text{ for all } v \in V(G).$$

Observe that $B^{\ell}_{G-e}(v_{\ell+1}) = B^{\ell}_{G}(v_{\ell+1}) = \deg^{+}_{G-e}(v_{\ell+1})$ and so $v_{\ell+1}$ can fire. Consequently for $t \in \{\ell+1, \ell+2, \ldots, n\}$,

$$B^{t}_{G-e}(v) = B^{t}_{G}(v) \text{ for all } v \in V(G).$$

Thus $B(G-e) \leq B(G)$.

Case III: $e = (v_a, v_b)$, where $v_a \in L \setminus \{v_\ell\}$ and $v_b \in L$

As the arc e is not present in $G-e$, and $B^{\ell}_{G}(v_b) = \deg^{+}_{G}(v_b)$ let

$$B^{0}_{G-e}(v) = \begin{cases} B^{0}_{G}(v) - 1 & \text{if } v = v_a, \\ B^{0}_{G}(v) + 1 & \text{if } v = v_b, \\ B^{0}_{G}(v) & \text{otherwise.} \end{cases}$$

Then for $t \in \{0, 1, \ldots, a-1\}$,

$$B^{t}_{G-e}(v) = \begin{cases} B^{t}_{G}(v) - 1 & \text{if } v = v_a, \\ B^{0}_{G}(v) + 1 & \text{if } v = v_b, \\ B^{t}_{G}(v) & \text{otherwise.} \end{cases}$$

Now for $t \in \{a, a+1, \ldots, b\}$

$$B^{t}_{G-e}(v) = B^{t}_{G}(v) \text{ for all } v \in V(G).$$

Observe that $B^{b-1}_{G-e}(v_b) = \deg^{+}_{G-e}(v_b)$ and so v_b can fire. Consequently for $t \in \{b, b+1, \ldots, n\}$,

$$B^{t}_{G-e}(v) = B^{t}_{G}(v) \text{ for all } v \in V(G).$$

Thus $B(G-e) \leq B(G)$.

Case IV: $e=(v_a,v_b)$, where $v_a \in R$ and $v_b \in R \setminus \{v_{\ell+1}\}$

Let $B^0_{G-e}(v) = B^0_G(v)$ for each $v \in V(G)$. Then for $t \in \{0,1,\ldots,a-1\}$, $B^t_{G-e}(v) = B^t_G(v)$. Now for $t \in \{a,a+1,\ldots,b-1\}$,

$$B^t_{G-e}(v) = \begin{cases} B^t_G(v)-1 & \text{if } v=v_b, \\ B^t_G(v) & \text{otherwise.} \end{cases}$$

Observe that $B^{b-1}_{G-e}(v_b) = B^{b-1}_G(v_b)-1 \geq \deg^+_{G-e}(v_b)$ and so v_b can fire. Consequently for $t \in \{b,b+1,\ldots,n\}$,

$$B^t_{G-e}(v) = \begin{cases} B^t_G(v)-1 & \text{if } v=v_b, \\ B^t_G(v) & \text{otherwise.} \end{cases}$$

Thus $B(G-e) \leq B(G)$.

Case V: (a) $e=(v_\ell,v_{\ell+1})$ and n is odd

As the arc e is not present in $G-e$, and $B^\ell_G(v_{\ell+1}) = \deg^+_G(v_{\ell+1})$ let

$$B^0_{G-e}(v) = \begin{cases} B^0_G(v)-1 & \text{if } v=v_\ell, \\ B^0_G(v)+1 & \text{if } v=v_{\ell+1}, \\ B^0_G(v) & \text{otherwise.} \end{cases}$$

Then for $t \in \{0, 1, \ldots, \ell - 1\}$,

$$B^t_{G-e}(v) = \begin{cases} B^t_G(v) - 1 & \text{if } v = v_\ell, \\ B^0_G(v) + 1 & \text{if } v = v_{\ell+1}, \\ B^t_G(v) & \text{otherwise.} \end{cases}$$

Now for $t \in \{a, a+1, \ldots, \ell\}$,

$$B^t_{G-e}(v) = B^t_G(v) \text{ for all } v \in V(G).$$

Observe that $B^\ell_{G-e}(v_{\ell+1}) = \deg^+_{G-e}(v_{\ell+1})$ and so $v_{\ell+1}$ can fire. Consequently for $t \in \{\ell + 1, \ell + 2, \ldots, n\}$,

$$B^t_{G-e}(v) = B^t_G(v) \text{ for all } v \in V(G).$$

Thus $B(G - e) \leq B(G)$.

Case V: (b) $e = (v_\ell, v_{\ell+1})$ and n is even

As the arc e is not present in $G - e$, let $B^0_{G-e}(v_\ell) = B^0_G(v_\ell) - 1$ and $B^0_{G-e}(v) = B^0_G(v)$ for each $v \in V(G) \setminus \{v_\ell\}$.

Then for $t \in \{0, 1, \ldots, \ell - 1\}$,

$$B^t_{G-e}(v) = \begin{cases} B^t_G(v) - 1 & \text{if } v = v_\ell, \\ B^t_G(v) & \text{otherwise.} \end{cases}$$

Now for $t = \ell$,

$$B_{G-e}^{t}(v) = \begin{cases} B_G^t(v) - 1 & \text{if } v = v_{\ell+1}, \\ B_G^t(v) & \text{otherwise.} \end{cases}$$

Observe that $B_{G-e}^{\ell}(v_{\ell+1}) = B_G^{\ell}(v_{\ell+1}) - 1 = \deg_{G-e}^{+}(v_{\ell+1})$ and so $v_{\ell+1}$ can fire. Consequently for $t \in \{\ell+1, \ell+2, \ldots, n\}$,

$$B_{G-e}^{t}(v) = \begin{cases} B_G^t(v) - 1 & \text{if } v = v_{\ell+1}, \\ B_G^t(v) & \text{otherwise.} \end{cases}$$

Thus $B(G-e) \leq B(G) - 1$.

Case VI: $e = (v_\ell, v_b)$ where $v_b \in R \setminus \{v_{\ell+1}\}$

As the arc e is not present in $G-e$, let $B_{G-e}^{0}(v_\ell) = B_G^0(v_\ell) - 1$ and $B_{G-e}^{0}(v) = B_G^0(v)$ for each $v \in V(G) \setminus \{v_\ell\}$. Then for $t \in \{0, 1, \ldots, \ell-1\}$,

$$B_{G-e}^{t}(v) = \begin{cases} B_G^t(v) - 1 & \text{if } v = v_\ell, \\ B_G^t(v) & \text{otherwise.} \end{cases}$$

Now for $t \in \{\ell, \ell+1, \ldots, b-1\}$,

$$B_{G-e}^{t}(v) = \begin{cases} B_G^t(v) - 1 & \text{if } v = v_b, \\ B_G^t(v) & \text{otherwise.} \end{cases}$$

Observe that $B_{G-e}^{b-1}(v_b) = B_G^{b-1}(v_b) - 1 \geq \deg^{+}(v_b)$ and so v_b can fire. Consequently for $t \in \{b, b+1, \ldots, n\}$,

$$B^t_{G-e}(v) = \begin{cases} B^t_G(v) - 1 & \text{if } v = v_b, \\ B^t_G(v) & \text{otherwise.} \end{cases}$$

Thus $B(G - e) \leq B(G) - 1$. □

In contrast to this result for transitive tournaments, in general for directed acyclic graphs it is possible for the brushing number to increase when an arc is removed. For an example consider the graph G in Figure 2.2. The brushing number of G is two. When the arc e is removed, the brushing number of $G - e$ is four.

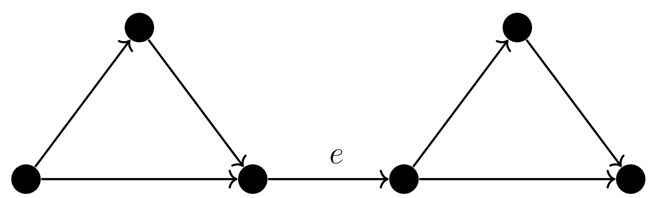

Figure 2.2: Graph G with brushing number two.

2.2 Directed Acyclic Graphs

Recall that $\mathfrak{G}_k$ denotes the set of all directed acyclic graphs on k vertices. By this point we have established the results necessary to find the upper bounds for the brushing number of all elements of $\mathfrak{G}_k$. As previously illustrated in Figure 2.1 the upper bound for the brushing number of graphs in $\mathfrak{G}_2$ is two. We will continue with the case where $k = 3$.

Lemma 2.2.1. *If $G \in \mathfrak{G}_3$ then $B(G) \leq 3$.*

Proof. All graphs of $\mathfrak{G}_3$ are shown in Figure 2.3 and each can be brushed with at most three brushes. □

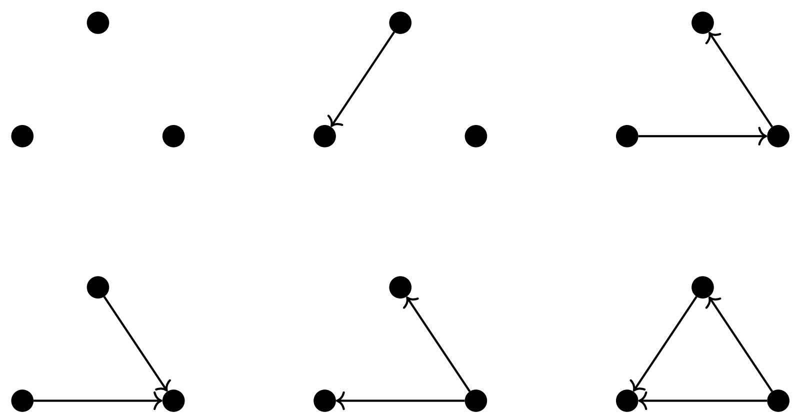

Figure 2.3: All graphs of $\mathfrak{G}_3$.

The upper bound for the brushing number of graphs in $\mathfrak{G}_2$ and $\mathfrak{G}_3$ deviates from the general result obtained for the upper bound that we will establish for $k \geq 6$ (Theorem 2.2.4). Mathematical induction is going to be used in the proof of Theorem 2.2.4, by considering the two cases where k is odd and even respectively. As the base cases of this proof we will give the upper bound for the brushing number of the two sets $\mathfrak{G}_4$ and $\mathfrak{G}_5$ as separate lemmata before proving Theorem 2.2.4. In the following proofs, $S_{n,m} \subseteq \mathfrak{G}_n$ will denote the set of directed acyclic graphs with n vertices and m components.

Lemma 2.2.2. *If* $G \in \mathfrak{G}_4$ *then* $B(G) \leq 4$.

Proof. There is one graph $G_1 \in S_{4,4} \subset \mathfrak{G}_4$ with four components which is shown in Figure 2.4. The graph G_1 can brushed with four brushes. Also there is one graph

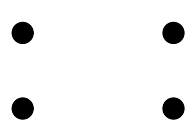

Figure 2.4: $G_1 \in \mathfrak{G}_4$ with four components.

$G_2 \in S_{4,3} \subset \mathfrak{G}_4$ with three components which is shown in Figure 2.5. The graph G_2 can brushed with three brushes. The graphs of $S_{4,2} \subset \mathfrak{G}_4$ are shown in Figure 2.6 and

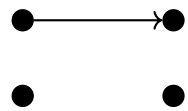

Figure 2.5: $G_2 \in \mathfrak{G}_4$ with three components.

each can be brushed with at most three brushes.

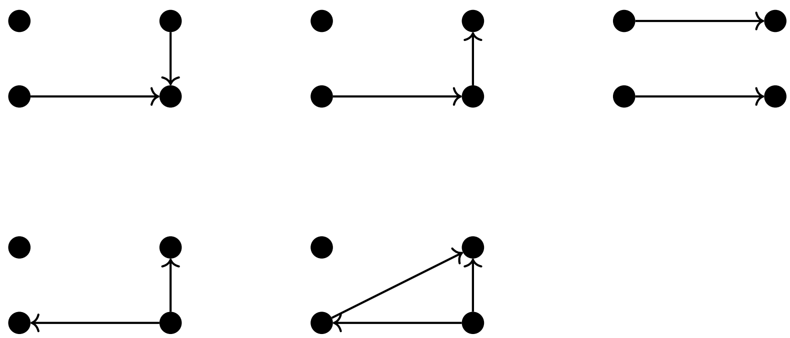

Figure 2.6: The graphs $S_{4,2} \subset \mathfrak{G}_4$.

Now we will consider the set of graphs $S_{4,1} \subset \mathfrak{G}_4$. For the graphs in $S_{4,1}$ we will calculate the brushing number of graphs with $6, 5, 3$ and 4 edges respectively. In $S_{4,1}$ the graphs with six edges are transitive tournaments. The brushing number of a transitive tournament with four vertices is 4 as proved in Theorem 2.1.3. Consequently from Theorem 2.1.5, it follows that when G is a directed acyclic graph with four vertices, five edges and one component $B(G) \leq 4$. The set of graphs from $S_{4,1}$ with three edges is illustrated in Figure 2.7.

All graphs in Figure 2.7 can be brushed with three brushes. Now consider the set of graphs from $S_{4,1}$ with four edges. Let $G \in S_{4,1}$ with four edges. Each graph G has a spanning sub-tree H with three edges. If $e \in A(G)$ and $e \notin A(H)$, then $G - e$ is connected. As $G - e = H$ will be one of the graphs in $S_{4,1}$ with three edges, which can be brushed using at most three brushes, the brushing number of $G \in S_4$ is at

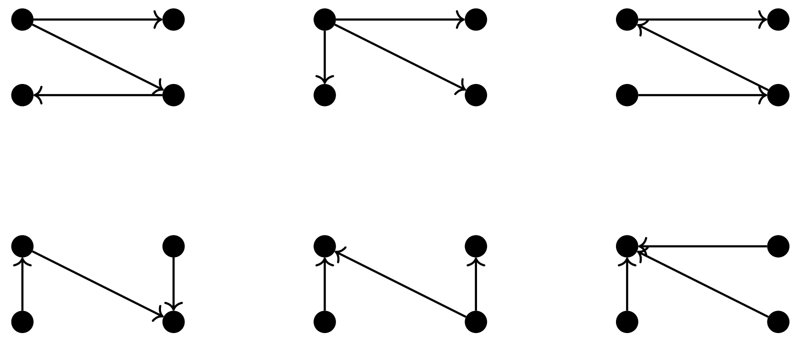

Figure 2.7: The set of graphs from $S_{4,1}$ with three edges.

most 4. □

Lemma 2.2.3. *If* $G \in \mathfrak{G}_5$ *then* $B(G) \leq 6$.

Proof. Consider a graph $G \in \mathfrak{G}_5$. Pick a source u and a sink v from G and consider the remaining set of vertices, $S = V(G) \setminus \{u, v\}$. Note that $|S| = 3$. Let X, Y and Z be subsets of S such that $X = N^+(u) \cap N^-(v)$, $Y = \{N^+(u) \cup N^-(v)\} \setminus X$, and $Z = S \setminus \{X \cup Y\}$. Note $|X| + |Y| + |Z| = 3$. Also $0 \leq |X| \leq 3$, $0 \leq |Y| \leq 3$, and $0 \leq |Z| \leq 3$. The graph $G - \{u, v\}$ is a directed acyclic graph on 3 vertices.

First consider the case where $G - \{u, v\}$ has less than three components. In G place $|X|$ brushes at u to brush paths of the form (u, x, v) where $x \in X$. Let $Y_1 \subset Y$ be the set of vertices with incident arcs of the form (u, y) where $y \in Y_1$. Let $Y_2 \subset Y$ be the set of vertices with incident arcs of the form (y, v) where $y \in Y_2$. Also $|Y| = |Y_1| + |Y_2|$. Then place $|Y_1|$ brushes at u to brush arcs of the form (u, y) and place $|Y_2|$ brushes, one brush at each vertex of Y_2, to brush arcs of the form (y, v) where $y \in Y_2$. Also if there is one arc from u to v, place one more brush at u. From Lemma 2.2.1 we know that when $G - \{u, v\}$ has less than three components the brushing number of $G - \{u, v\}$ is at most 2. Then place up to 2 brushes on set S in a way such that it will be consistent with a strategy that successfully brushes $G - \{u, v\}$. Now first fire vertex u and then all the vertices in S in accordance with the strategy used to brush

$G - \{u, v\}$. After all vertices in S fire, v accumulates brushes as v is the sink. Finally vertex v fires. Therefore,

$$B(G) \leq |X| + |Y| + 1 + 2 \leq 3 + 1 + 2 = 6.$$

When $G - \{u, v\}$ has three components, place $|X| + |Y_1|$ brushes at u and also one brush at each vertex of Y_2. As $G - \{u, v\}$ has no arcs now, place $|Z|$ brushes, one brush each at every vertex in Z. Then fire the vertices in a sequence similar to the previous case. Therefore,

$$B(G) \leq |X| + |Y| + |Z| + 1 \leq 3 + 1 = 4.$$

□

Having established the upper bounds for the brushing number of all $G \in \mathfrak{G}_k$, for $k = 2, 3, 4, 5$, now we will consider all $k \geq 6$ in the following theorem.

Theorem 2.2.4. *If $k \geq 6$ and $G \in \mathfrak{G}_k$, then*

$$B(G) \leq \begin{cases} \frac{k^2-1}{4} & \text{if } k \text{ is odd,} \\ \frac{k^2}{4} & \text{if } k \text{ is even.} \end{cases}$$

Proof. The method of mathematical induction is used in this proof.

Case I : k is odd.

All directed acyclic graphs on five vertices have brushing number 6 or less as established in Lemma 2.2.3.

For $n = k$, we will assume as our inductive hypothesis that for all $G \in \mathfrak{G}_n$, $B(G) \leq \frac{n^2-1}{4}$.

Now consider a graph $G' \in \mathfrak{G}_{n+2}$. Pick a source u and a sink v from G' and consider the remaining set of vertices, $S = V(G') \setminus \{u, v\}$. Note that $|S| = n$. Let X, Y, and Z be subsets of S such that $X = N^+(u) \cap N^-(v)$, $Y = \{N^+(u) \cup N^-(v)\} \setminus X$, and $Z = S \setminus \{X \cup Y\}$.

Note $|X| + |Y| + |Z| = n$. Also $0 \leq |X| \leq n$, $0 \leq |Y| \leq n$, and $0 \leq |Z| \leq n$. The graph $G' - \{u, v\}$ is a directed acyclic graph on n vertices. Thus, the brushing number of this remaining graph will be less than or equal to $\frac{n^2-1}{4}$, by the inductive hypothesis. In G (with u and v), place $|X|$ brushes at u to brush paths of the form (u, x, v) where $x \in X$. Let $Y_1 \subset Y$ be the set of vertices with incident arcs of the form (u, y) where $y \in Y_1$. Let $Y_2 \subset Y$ be the set of vertices with incident arcs of the form (y, v) where $y \in Y_2$. Also $|Y| = |Y_1| + |Y_2|$. Then place $|Y_1|$ brushes at u to brush arcs of the form (u, y) and place $|Y_2|$ brushes, one brush at each vertex of Y_2, to brush arcs of the form (y, v) where $y \in Y_2$. Also if there is one arc from u to v, place one more brush at u. Therefore,

$$B(G') \leq |X| + |Y| + 1 + \frac{n^2-1}{4} \leq n + 1 + \frac{n^2-1}{4} = \frac{n^2+4n+3}{4} = \frac{(n+2)^2-1}{4}.$$

Case II : k is even.

Consider $k = 4$ as the base case when k is even. All directed acyclic graphs with four vertices can be brushed with 4 or less brushes as established in Lemma 2.2.2.

Then we can proceed in a manner similar to the odd case. When $G \in \mathfrak{G}_n$, where n is even,

$$B(G) \leq |X| + |Y| + 1 + \frac{n^2}{4} \leq n + 1 + \frac{n^2}{4} = \frac{n^2+4n+4}{4} = \frac{(n+2)^2}{4}.$$

□

Chapter 3

Other Types of Directed Graphs

In this chapter we will find the brushing number of several types of graphs such as complete graphs, trees and rotational tournaments. For some graphs we will discuss the relationship between the brushing number and a specific parameter in the graph. Recall that a *complete directed graph* is a simple directed graph such that between each pair of its vertices, both (oppositely directed) arcs exist.

Theorem 3.0.1. *If G is a complete directed graph, then*

$$B(G) = \frac{|A(G)|}{2}.$$

Proof. Every vertex of a complete directed graph G with n vertices has out-degree $n-1$. Therefore, the total number of arcs $|A(G)| = n(n-1)$ (because every arc of G has one tail and one head.) Consider a complete directed graph G with n vertices, $v_1, v_2, \ldots, v_n$. The vertices of G are labelled arbitrarily $v_1, v_2, \ldots, v_n$. Recall that $B_G^t(v)$ is the number of brushes at vertex v of G, at time t ($t = 0, 1, 2, \ldots, n$). A vertex v_i fires in between time $i-1$ and i. Consider an initial configuration of brushes for G where

$$B_G^0(v_i) = n - i \text{ for } i = 1, 2, \ldots, n.$$

As vertex v_1 with out-degree $n-1$ has $n-1$ brushes, v_1 fires. Now

$$B_G^1(v_i) = \begin{cases} 0 & \text{if } i = 1, \\ n - i + 1 & \text{otherwise.} \end{cases}$$

Observe that the number of brushes at v_2 is $n-1$. So v_2 fires. Then

$$B_G^2(v_i) = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{if } i = 2, \\ n - i + 2 & \text{otherwise.} \end{cases}$$

Thus, after the k^{th} vertex fires,

$$B_G^k(v_i) = \begin{cases} k - i & \text{if } i = 1, 2, \ldots, k-1, \\ 0 & \text{if } i = k \\ n - i + k & \text{otherwise.} \end{cases}$$

When $\{v_1, v_2, \ldots, v_{k-1}\}$ have all fired v_k is able to fire, as it will have $n - k + k - 1 = n - 1 = \deg^+(v_k)$ brushes, where $2 \leq k \leq n$. Similarly, we can also show that $v_{k+1}, v_{k+2}, \ldots, v_n$ are able to fire in sequence. The total number of brushes used is equal to $(n-1) + (n-2) + (n-3) + \cdots + 2 + 1 + 0 = n(n-1)/2 = \frac{|A(G)|}{2}$ and hence $B(G) \leq \frac{|A(G)|}{2}$.

For each $i = 1, 2, \ldots, n$ let m_i be the number of in-neighbour vertices fired before firing the i^{th} vertex. For vertex v_i to fire

$$B_G^0(v_i) + m_i \geq (n-1). \tag{3.1}$$

Summing the inequality 3.1 for all $i \in \{1, 2, \ldots, n\}$ and observing that $m_i = i - 1$ for $i \in \{1, 2, \ldots, n\}$ yields

$$\sum_{i=1}^{n} B_G^0(v_i) + \sum_{i=1}^{n} m_i \geq n(n-1),$$

where $B_G^0(v_i)$ is any valid initial brushing configuration that successfully cleans the graph. If $B_G^0(v_i)$ is an optimal brushing configuration then $\sum_{i=1}^{n} B_G^0(v_i) = B(G)$. Hence

$$B(G) \geq n(n-1) - \sum_{i=1}^{n}(i-1)$$

and

$$B(G) \geq n(n-1) - \frac{(n^2 - 3n)}{2}.$$

Therefore

$$B(G) \geq \frac{n(n-1)}{2} = \frac{|A(G)|}{2}.$$

□

Definition 3.0.2. *A directed tree is a directed graph whose underlying graph is a tree. A rooted tree is a directed tree with a distinguished vertex r, called the* root *such that for every other vertex v, the unique path from r to v is a directed path from r to v.*

Theorem 3.0.3. *If G is a rooted tree with k leaves, then $B(G) = k$.*

Proof. Let $L \subseteq V(G)$ be the set of leaves of G, where $|L| = k$. Since there exist k sinks, $B(G) \geq k$ (by Lemma 2.1.2).

There is a unique directed path from a root r to a leaf $\ell \in L$. The union of the arcs of these paths is the arc set of G. Now place k brushes at the root r of the graph G. To clean the graph G one brush can travel in each path from r to each $\ell \in L$. Therefore, $B(G) \leq k$. □

Definition 3.0.4. *A path decomposition of a directed graph G is a set of arc-disjoint directed paths such that the union of the arcs of these paths is the arc set of G. If M is a path decomposition of a directed graph G, and there does not exist a path decomposition of G with less than $|M|$ paths, then M is a minimum path decomposition of G.*

Theorem 3.0.5. *If M is a minimum path decomposition of a directed acyclic graph G then $B(G) \leq |M|$.*

Proof. Obtain a minimum path decomposition M of a directed acyclic graph G. The paths of G that we consider in this proof are elements of M. Let a_i be the number of paths of M that begin at $v_i \in V(G)$ where $V(G) = \{v_1, v_2, \ldots, v_{|V(G)|}\}$. For all $v_i \in V(G)$ let $B_G^0(v_i) = a_i$. Let ω be the length of the longest path in G. For each $k \in \{0, 1, \ldots, \omega\}$, let S_k be the set of vertices with distance k from the furthest source vertex in G and let $S_{<k} = S_0 \cup S_1 \cup \cdots \cup S_{k-1}$. As S_0 is the set of source vertices in G then for $v_i \in S_0$, $\deg^+(v_i) = a_i$. So vertices in S_0 fire between time steps 0 and 1.

Consider $v_i \in S_k$ where $k \in \{1, 2, \ldots, \omega\}$. Note that vertex v_i receives brushes from its in-neighbours in $S_{<k}$ before the k^{th} time step. Observe that $\deg^+(v_i)$ is equal to the sum of the number of paths in M that start at v_i and the number of paths in M that include v_i as a middle vertex. Hence $\deg^+(v_i)$ is at most the sum of a_i and the number of brushes that v_i receives from its in-neighbours in $S_{<k}$, which is equal to $B_G^k(v_i)$. Therefore $\deg^+(v_i) \leq B_G^k(v_i)$. Then each vertex of S_k fires between the time steps k and $k+1$ where $k \in \{1, 2, \ldots, \omega\}$. Hence $B(G) \leq \sum_{v \in V(G)} B_G^0(v) = |M|$. □

Example 3.0.6. In the graph G illustrated in Figure 3.1, a minimum path decomposition M has five paths. The graph G can be brushed using a minimum of three brushes.

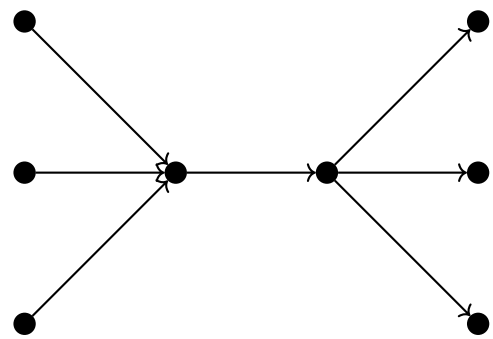

Figure 3.1: Graph G that can be brushed using 3 brushes.

Definition 3.0.7. *The transpose of a directed graph G is another directed graph G^T on the same set of vertices with all the arcs reversed compared to the orientation of the corresponding arcs in G.*

Lemma 3.0.8. *If G is a directed acyclic graph, then $B(G^T) \leq B(G)$.*

Proof. Let G be a directed acyclic graph with the vertex set $\{v_1, v_2, \ldots, v_n\}$ and optimal initial brushing configuration $B_G^0(v_i) = a_i$ for all $v_i \in V(G)$ that successfully cleans the graph G. Observe that at some time t' the graph G is clean and the brushing configuration is $B_G^{t'}(v_i) = b_i$, for all $v_i \in V(G)$. Throughout the cleaning process of G each brush in the initial brushing $B_G^0(v_i)$ traverses a directed path. If the transpose of these directed paths is considered then we obtain a path decomposition M' of G^T. Define the initial brushing configuration of G^T as $B_{G^T}^0(v_i) = B_G^{t'}(v_i) = b_i$. Then b_i is the number of paths of M' that begin at v_i in G^T.

Now we use a similar approach as in Theorem 3.0.5. Let ω be the length of the longest path in M'. For each $k \in \{0, 1, \ldots, \omega\}$ let S_k be the set of vertices with distance k from the furthest source vertex in G^T and $S_{<k} = S_0 \cup S_1 \cup \cdots \cup S_{k-1}$. As

S_0 is the set of source vertices in G^T for $v_i \in S_0$, $\deg^+_{G^T}(v_i) = b_i$. So vertices in S_0 fire between time steps 0 and 1.

Consider $v_i \in S_k$ where $k \in \{1, 2, \ldots, \omega\}$. Note that vertex v_i receives brushes from its in-neighbours in $S_{<k}$ before the k^{th} time step. Observe that $\deg^+_{G^T}(v_i)$ is equal to the sum of the number of paths in M' that start at v_i and the number of paths in M' that include v_i as a middle vertex. Hence $\deg^+_{G^T}(v_i)$ is at most the sum of b_i and the number of brushes that v_i receives from its in-neighbours in $S_{<k}$, which is equal to $B^k_G(v_i)$. Therefore $\deg^+_{G^T}(v_i) \leq B^k_G(v_i)$. Then each vertex of S_k fires between the time steps k and $k+1$ where $k \in \{1, 2, \ldots, \omega\}$. Hence $B(G^T) \leq \sum\limits_{v \in V(G^T)} B^0_{G^T}(v) = \sum\limits_{v \in V(G)} B^{t_1}_G(v) = \sum\limits_{v \in V(G)} B^0_G(v) = B(G)$. □

Theorem 3.0.9. *If G is a directed acyclic graph, then $B(G^T) = B(G)$.*

Proof. From Lemma 3.0.8 we have $B(G^T) \leq B(G)$. We also have $B(G) = B((G^T)^T) \leq B(G^T)$. □

For the upcoming theorems in this thesis we will need the following definitions.

Definition 3.0.10. *A partial order is a binary relation $\preceq$ on a set X that is*

1. *reflexive: for all $x \in X, x \preceq x$;*

2. *anti symmetric: for all $x, y \in X$, if $x \preceq y$ and $y \preceq x$, then $x = y$;*

3. *transitive: for all $x, y, z \in X$, if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.*

Definition 3.0.11. *A poset, or partially ordered set $P = (X, \preceq)$ is a pair consisting of a set X, called the domain, and a partial order $\preceq$ on X.*

Definition 3.0.12. *Elements x, y of a poset P are comparable if either $x \preceq y$ or $y \preceq x$.*

Definition 3.0.13. *For elements x and y if $x \preceq y$ and $x \neq y$ then we say x is less than element y, written $x \prec y$.*

Definition 3.0.14. *The comparability directed graph of the poset $P = (X, \preceq)$ is the directed graph with vertex set X such that there is an arc from x to y if and only if $x \preceq y$.*

Definition 3.0.15. *The element y covers the element x in a poset if $x \prec y$ and there is no element z such that $x \prec z \prec y$.*

Definition 3.0.16. *The cover graph of a poset $P = (X, \preceq)$ is the graph with vertex set X such that x, y are adjacent if and only if one of them covers the other.*

Definition 3.0.17. *A Hasse diagram of poset P is a straight-line drawing of the cover graph such that the lesser element of each adjacent pair is lower in the drawing.*

Example 3.0.18. Let X = {1,2,3,4,6} and let $\preceq$ be the divisibility relation on X. That is, $x \preceq y$ if and only if y/x is an integer. The comparability directed graph and the Hasse diagram for $P = (X, \preceq)$ are as shown in Figure 3.2.

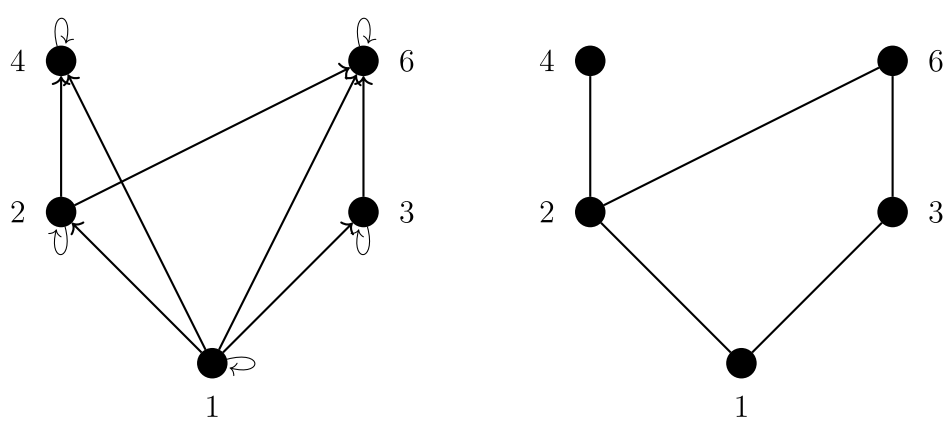

Figure 3.2: Comparability directed graph and the Hasse diagram for divisibility relation of a set X.

Consider a directed graph G. Let $\preceq$ be a binary relation on $A(G)$ such that when $a, b \in A(G)$, $a \preceq b$ if and only if the head of arc a meets the tail of arc b at the same vertex. We define the comparability directed graph of the poset $P(G) = (A(G), \preceq)$ as the *edge poset graph* of G. We observed that when G is a directed graph, the number of paths in the edge poset graph of G appears to be an upper bound for the brushing number of graph G. But we were unable to obtain the desired proof. Hence Conjecture 3.0.19 is presented.

Conjecture 3.0.19. *Let G be a directed graph with the poset $P(G) = (A(G), \preceq)$ and every maximal directed path in G is directed from a source u towards a sink v. If M is a minimum path decomposition of the edge poset graph of G, then $B(G) \leq |M|$.*

When G is a directed graph with the poset $P(G) = (A(G), \preceq)$ and $A' \subset A(G)$, we say that A' is an *antichain* if every distinct pair of points from A' is incomparable in $P(G)$. The *width* of a poset $P(G) = (A(G), \preceq)$, denoted by $width(P(G))$, is the largest w for which there exists an antichain of w points in $P(G)$.

Theorem 3.0.20. *Given a directed acyclic graph G with the poset $P(G) = (A(G), \preceq)$, $B(G) \geq width(P(G))$.*

Proof. For a directed acyclic graph G, each arc in any largest antichain should be traversed by a different brush. □

The following principle is described in [4].

Definition 3.0.21. ***Principle of Directional Duality***

Any statement about a directed graph has an accompanying dual statement, obtained by applying the statement to the transpose of the directed graph and reinterpreting it in terms of the original directed graph.

Theorem 3.0.22. *If T is a directed tree on n vertices, the set of source vertices of T*

is S_1 and the set of sink vertices is S_2, then

$$B(T) \geq \max\left\{\sum_{v\in S_1}|N^+(v)|, \sum_{v\in S_2}|N^-(v)|\right\}.$$

Proof. The arcs incident with each $v \in S_1$ require a unique brush (no brush can clean two of the incident arcs). Therefore $B(T) \geq \sum_{v\in S_1}|N^+(v)|$. By the principle of directional duality $B(T) \geq \sum_{v\in S_2}|N^-(v)|$. □

Example 3.0.23. The equality of the above theorem does not always hold. For an example consider the graph G illustrated in Figure 3.3.

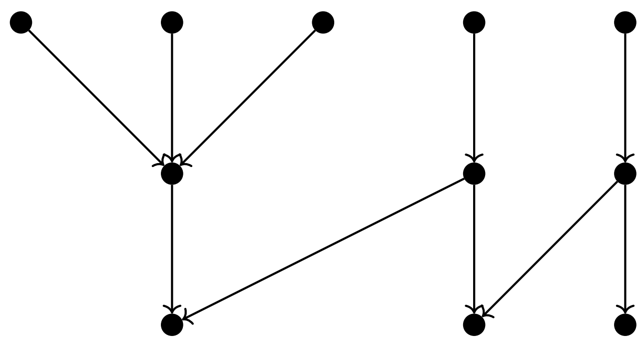

Figure 3.3: Graph T is a tree with $B(T) = 7$, $\sum_{v\in S_1}|N^+(v)| = \sum_{v\in S_2}|N^-(v)| = 5$.

Definition 3.0.24. *[8] Let Γ be an abelian group of odd order $n = 2m+1$ with identity 0. Let S be an m-element subset of $\Gamma\setminus\{0\}$ such that for every $x, y \in S$, $x+y \neq 0$. That is, choose exactly one element from each of the m 2-sets of the form $\{x, -x\}$, where x ranges over all $x \in \Gamma \setminus \{0\}$. Form the directed graph D with vertex set $V(D) = \Gamma$ and arc set $A(D)$ defined by: arc $(x, y) \in A(D)$ if and only if $y - x \in S$. Then D is called a rotational tournament with symbol set S and is denoted $R_\Gamma(S)$, or simply $R(S)$ if the group Γ is understood.*

A *regular tournament* is a tournament T in which there is an integer s so that $\deg^+(v) = s$ for all vertices $v \in T$. The rotational tournament $R_\Gamma(S)$, where $|\Gamma| = n$,

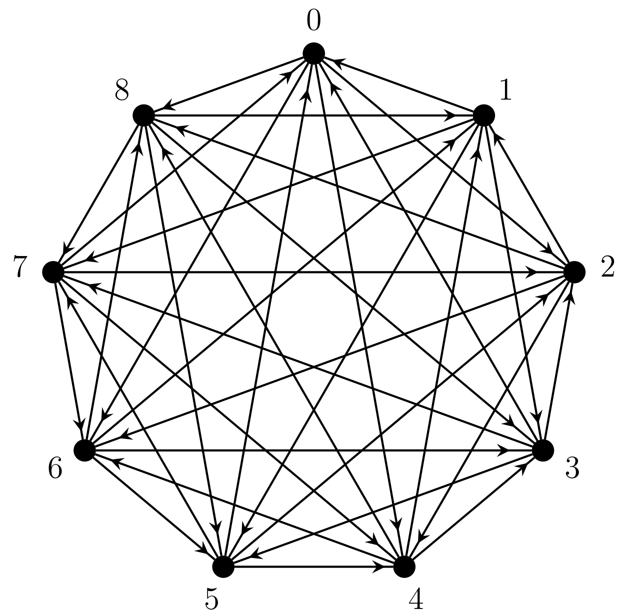

Figure 3.4: The regular, rotational tournament $R(\{2,4,6,8\})$.

is a regular tournament on n vertices. Each vertex in a rotational tournament with n vertices has out-degree $\frac{n-1}{2}$. Figure 3.4 is an example for a regular, rotational tournament with 9 vertices and $S = \{2,4,6,8\}$. (This example is borrowed from [8]). For further information about rotational tournaments refer to [8] and [2].

Theorem 3.0.25. *If G is a rotational tournament with n vertices then $B(G) = \frac{n^2-1}{8}$.*

Proof. Let $B^t_G(v)$ be the number of brushes at vertex v of G, at time t ($t = 0, 1, 2, \ldots, n$). A vertex v_i fires in between time $i-1$ and i. Let $B^0_G(v_k)$ denote the number of brushes at the k^{th} vertex in the initial configuration.

$$B^0_G(v_k) = \begin{cases} \frac{n-1}{2} - (k-1) & \text{if } k = 1, 2, \ldots, \frac{n-1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

So no vertex with index greater than $\frac{n-1}{2}$ receives a brush in the initial configuration. Recall that $\deg^+(v_i) = \frac{n-1}{2}$, for all $0 \le i \le n$. As vertex v_1 with out-degree $\frac{n-1}{2}$

has $\frac{n-1}{2}$ brushes, v_1 fires. Now

$$B_G^1(v_k) = \begin{cases} \frac{n-1}{2} - (k-1) + 1 & \text{if } k = 2, 3, \ldots, \frac{n-1}{2} \\ 1 & \text{if } k = \frac{n-1}{2} + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the number of brushes at v_2 is $\frac{n-1}{2}$. So v_2 fires. Then

$$B_G^2(v_k) = \begin{cases} \frac{n-1}{2} - (k-1) + 2 & \text{if } k = 3, 4, \ldots, \frac{n-1}{2} \\ 2 & \text{if } k = \frac{n-1}{2} + 1 \\ 1 & \text{if } k = \frac{n-1}{2} + 2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus after v_j vertex fires, where $1 \leq j \leq \frac{n-1}{2}$,

$$B_G^j(v_k) = \begin{cases} \frac{n-1}{2} - (k-1) + j & \text{if } k = j+1, j+2, \ldots, \frac{n-1}{2} \\ j - (\ell - 1) & \text{if } k = \frac{n-1}{2} + \ell, (\ell = 1, 2, \ldots, j) \\ 0 & \text{otherwise.} \end{cases}$$

When $\{v_1, v_2, \ldots, v_j\}$ have all fired v_{j+1} is able to fire, as it will have $\frac{n-1}{2} - (j + 1 - 1) + j - 1 = \frac{n-1}{2} = \deg^+(v_{j+1})$ brushes, where $1 < j+1 \leq \frac{n-1}{2}$. After $v_{\frac{n-1}{2}+1}$ fires,

$$B_G^{\frac{n-1}{2}+1}(v_k) = \begin{cases} \frac{n-1}{2} - (\ell - 2) & \text{if } k = \frac{n-1}{2} + \ell, (\ell = 2, 3, \ldots, \frac{n+1}{2}) \\ 0 & \text{otherwise.} \end{cases}$$

After $v_{\frac{n-1}{2}+r}$ fires, where $2 \leq r \leq \frac{n+1}{2}$,

$$B_G^{\frac{n-1}{2}+r}(v_k) = \begin{cases} \frac{n-1}{2} - [\ell - (r+1)] & \text{if } k = \frac{n-1}{2} + \ell, (\ell = r+1, r+2, \ldots, \frac{n+1}{2}) \\ r - k & \text{if } k = 1, 2, \ldots, r-1 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we can also show that all $v_{\frac{n-1}{2}+r}$, where $2 \leq r \leq \frac{n+1}{2}$, are able to fire in sequence.

When the above brushing strategy is used for G we obtain,

$$B(G) \leq \sum_{k=1}^{\frac{n-1}{2}} B_G^0(v_k) = \frac{n-1}{2} + \frac{n-1}{2} - 1 + \frac{n-1}{2} - 2 + \cdots + 1 = \frac{n^2-1}{8}.$$

Let m_i be the number of in-neighbour vertices fired before firing the i^{th} vertex. For vertex v_k to fire

$$B_G^0(v_k) + m_k \geq \frac{n-1}{2} \tag{3.2}$$

$$m_k = \begin{cases} k-1 & \text{if } k = 1, 2, \ldots, \frac{n-1}{2} \\ \frac{n-1}{2} & \text{otherwise.} \end{cases}$$

Summing the inequality 3.2 for all $k \in \{1, 2, \ldots, n\}$ we get

$$\sum_{k=1}^{n} B_G^0(v_k) + \sum_{k=1}^{n} m_k \geq \frac{n(n-1)}{2},$$

$$B(G) \geq \frac{n(n-1)}{2} - \left(\sum_{k=1}^{\frac{n(n-1)}{2}} (k-1) + \sum_{k=\frac{(n+1)}{2}}^{n} \frac{(n-1)}{2} \right)$$

and

$$B(G) \geq \frac{n(n-1)}{2} - \left(\frac{(n-3)}{2}\frac{(n-1)}{2}\frac{1}{2} + \frac{(n-1)}{2}\frac{(n+1)}{2}\right).$$

Therefore

$$B(G) \geq \frac{n^2-1}{8}.$$

□

Corollary 3.0.26. *If G is a rotational tournament that is cleaned using the strategy given in Theorem 3.0.25 then there is at least one brush that travels a Hamiltonian path during brushing G.*

Proof. When using the strategy given in Theorem 3.0.25 to brush a rotational tournament G, every vertex obtains at least one brush from the preceding vertex that just fired. □

Recall the definition of minimum path decomposition from Definition 3.0.4. If M is a minimum path decomposition of directed graph G, $|M|$ is called the *path number* of G and following the notation of [9] it is denoted by $pn(G)$. A lower bound on $pn(G)$ can be given by considering the degree sequence of G. For each $v \in V(G)$ let $d_v = \deg^+_G(v) - \deg^-_G(v)$. Observe that in any path decomposition of G at least d_v paths should start at v. Hence $pn(G) \geq \frac{1}{2}\sum_{v\in V(G)} |d_v|$.

Definition 3.0.27. *[7] A perfect decomposition of a directed graph G is a set $P = \{P_1, \ldots, P_r\}$ of arc disjoint paths of G that together cover $E(G)$ where $r = \frac{1}{2}\sum_{v\in V(G)} |d_v|$.*

Theorem 3.0.28. *[9] If G is a directed acyclic graph, then it has a perfect decomposition.*

Theorem 3.0.29. *If G is a directed acyclic graph, with a perfect decomposition $P = \{P_1, \ldots, P_r\}$, then $B(G) \leq r$.*

Proof. The strategy used in Theorem 3.0.5 can be generalised for any path decomposition. Therefore by following an approach similar to that of Theorem 3.0.5 we obtain $B(G) \leq r$. □

Comparing Theorem 3.0.5 and Theorem 3.0.29 we obtain a better upper bound for the $B(G)$ of a directed acyclic graph which is $pn(G) \geq r \geq B(G)$.

Chapter 4

Summary and Discussion

In this thesis we have focused on brushing a directed graph using the minimum number of brushes possible. For this purpose we have developed a new cleaning model for brushing directed graphs and defined the brushing number of directed graphs. We have developed theorems regarding the brushing number and upper bounds for the brushing number of different types of directed graphs.

In Chapter 2 we have developed a strategy to brush transitive tournaments. Using this strategy we have established the brushing number of transitive tournaments and hence we have obtained an upper bound for the brushing number of directed acyclic graphs. We developed another upper bound for the brushing number of directed acyclic graphs using path decompositions. Comparing Theorem 3.0.5 and Theorem 3.0.29 we have obtained a better upper bound for the brushing number of a directed acyclic graph G which is $pn(G) \geq r \geq B(G)$. It remains as an open problem to find an example of a directed acyclic graph with the property $pn(G) > r$.

In Chapter 3 we have given exact values for the brushing number of complete directed graphs and rooted trees. Theorem 3.0.25 gives the brushing number of a rotational tournament. The set of rotational tournaments is a subset of regular tournaments. Based on the observations made on regular tournaments and results obtained about rotational tournaments we present the following conjecture.

Conjecture 4.0.1. *If G is regular tournament, then* $B(G) \leq \frac{n^2-4n+7}{4}$.

In Chapter 3 we also have discussed the brushing number of directed trees. The brushing number of a rooted tree is given in Theorem 3.0.3 and a lower bound for the brushing number of a directed tree is given Theorem 3.0.22. An upper bound for the brushing number of any directed tree T with n vertices can be obtained by considering the minimum path decomposition of T via Theorem 3.0.5. We suggest that building an algorithm which calculates the brushing number of a directed tree using path decompositions will be an interesting application of Theorem 3.0.5. Another open problem that can be considered is finding the brushing number of directed graph products. For an example the research can be continued to find the relationships between the brushing number of a directed graph G and the Cartesian product of G with itself, which is $G \square G$.

Another aspect to consider is the comparison of the brushing number of a directed graph and the brushing number of its underlying undirected graph. For an example graph G_1 in Figure 4.1 has brushing number four and graph G_2 has brushing number one. But G_1 and G_2 have the same underlying graph (with brushing number one).

(a) G_1 (b) G_2

Figure 4.1: Graphs G_1 and G_2 with the same underlying undirected graph.

The cleaning model presented in this thesis allows any number of brushes to traverse an arc of a directed graph. If we implement edge capacity restrictions it would create a new cleaning model for brushing directed graphs. The cleaning model that we have introduced in this thesis is based on minimising the number of brushes that can be used to clean a given directed graph. It remains as an open problem to explore what is the most efficient cleaning sequence for a directed graph, that is the one that minimises the number of brushes used as well as minimises the time taken to clean the graph. One way to achieve this goal is to fire multiple vertices that are ready to fire at the same time step. A way of firing brushes similar to this for undirected graphs is described as *parallel dispersal mode* in [10].

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