Creation of entanglement by quantum channels

by

© **Kamanpreet Singh Manoor**

A thesis submitted to the School of Graduate Studies in partial fulfillment of the requirements for the degree of Master of Science.

Department of Mathematics and Statistics

Memorial University

August 2024

St. John's, Newfoundland and Labrador, Canada

Abstract

This thesis examines the creation of entanglement in open quantum systems through the application of quantum channels. The primary objective is to understand the transition from a product state, initially uncorrelated, to an entangled state upon the application of quantum channels. The study emphasizes the immediate effects of quantum channels on entanglement production, employing mathematical modeling and computational simulations to explore various system configurations and quantum channel parameters. Key results include the identification of conditions under which quantum channels produce only classical correlations and the determination of necessary and sufficient conditions for entanglement generation by specific subclasses of completely positive, trace-preserving (CPTP) maps.

To my parents

Lay summary

This thesis explores how entanglement, a fundamental phenomenon in quantum mechanics where particles become interconnected regardless of the distance between them, can be generated in open quantum systems through the use of quantum channels. Unlike closed systems, which are isolated from their surroundings, open systems continuously interact with their environment, exchanging energy and information in ways that profoundly affect their behavior.

The research focuses on the transformation of a product state - an initial state where the components are uncorrelated - into an entangled state through the application of quantum channels. Quantum channels are mathematical operations that can represent interactions or measurement processes in quantum systems. By applying these channels, the study aims to determine the types and quantities of correlations, both classical and quantum, that are produced.

In particular, the research identifies specific conditions under which quantum channels produce only classical correlations, as well as the necessary and sufficient conditions for these channels to generate entanglement. Additionally, the study examines a d-dimensional quantum system interacting with a field, providing insights into how factors such as the energy scales of the environment and the strength of coupling affect the production of entanglement.

The findings are strictly theoretical. This research contributes to the broader field of quantum mechanics by offering deeper insights into the fundamental processes governing entanglement in open quantum systems.

Acknowledgements

Foremost, I would like to thank my supervisor Professor Marco Merkli for the continuous support, advice and guidance throughout my master's degree. My questions were always answered with valuable and inspiring suggestions. I would like to thank him, especially for his patience and kindness during my time of research and writing.

Thank you to my parents and sister, for the loving childhood, motivation and support throughout my life.

I would like to express my gratitude to Professor Marco Merkli and my family for constant encouragement throughout the writing process and invaluable suggestions on the thesis.

I also want to thank the School of Graduate Studies for the financial support and the Department of Mathematics and Statistics for providing me teaching assistantships.

Statement of contribution

This thesis is a collaboration of work by Kamanpreet Singh Manoor and Dr. Marco Merkli. The research question investigated throughout the thesis was proposed by Dr. Marco Merkli. The thesis itself was written by Kamanpreet Singh Manoor with the input and guidance of Dr. Marco Merkli. Supervision and editing of the thesis was done by Dr. Marco Merkli.

Table of contents

Main results of the thesis

We consider open quantum complexes consisting of a system S and a reservoir R. Our main interest is to study the entanglement between S and R produced by quantum channels applied to SR. A quantum channel is a map on the SR states which may represent an interacting dynamics or a measurement process (generally, a quantum channel is a completely positive, trace preserving – CPTP – map on SR states). Our main question is: Given a product state of S and R (no correlation), what kind of correlation and how much of it is produced by applying a quantum channel? Correlations can be classical or quantum (=entanglement).
We consider different parameters in the quantum channel and various system configurations and we quantify the degree of entanglement using measures such as purity and concurrence. Our results highlight the intricate relationship between system parameters, bath properties and the production of entanglement. The theoretical findings are validated through numerical simulations.

• Our first main result is Theorem 2.1.1 of Chapter 2. In this theorem we exhibit a condition for a general CPTP map to produce only classical correlations. This means that any channel satisfying the condition is not able to entangle the system and the reservoir.
A second result is Theorem 2.2.1, in which we find a condition on a subclass of CPTP maps, which is *necessary and sufficient* to produce entanglement.

• In Chapter 3 we consider a more concrete family of quantum channels acting on a complex of a d-dimensional system interacting with a quantum field. Such models are used to describe matter-radiation effects in quantum theory. Our main results here are as follows. In Theorem 3.1.1 we find the reduced state of S after the application of the quantum channel. We analyze the case of the open qubit ($d = 2$) in detail, discussing its entanglement with the reservoir by studying the qubit purity (Lemma (3.1.1)). We conclude in particular that large energy scales of the reservoir diminish the qubit-field entanglement and so does a strong coupling to the ultra-violet (fast) modes of the reservoir. On the other hand, the infra-red (or, slow) modes of the reservoir enhance the production of the entanglement. Our analytical results are illustrated with numerics.

We further study the situation where the system is composed of N subsystems, in particular, an N qubit register. We find the reduced system state after application of the quantum channel and the further reduction to only two qubits. This allows us to study the intra-qubit entanglement generated by the quantum channel.

Chapter 1

Quantum Mechanics: Fundamentals

This chapter provides an overview of fundamental concepts in quantum mechanics, particularly focusing on the framework of closed and open systems, entanglement, and quantum field theory. The content in this chapter follows standard presentations found [2], [4], [3], [1], [6]. Specifically, Section 1.1 and 1.2 discusses the fundamentals of quantum mechanics and entanglement, drawing from [4], [2] and [1]. Section 1.3 and 1.4, which covers open quantum systems, follows the treatments in [3], [5], [6]. The derivation of the Born-Markov equation in Section 1.5 is guided by [5], [4], while Section 1.6 introduces elements of quantum field theory as discussed by [7]. These references provide a deeper exploration of the topics discussed and are essential for further understanding.

1.1 Quantum Mechanics of Closed Systems

In quantum mechanics, a closed system is defined as a physical system entirely isolated from external influences. This isolation ensures that the system's state and its temporal evolution are governed solely by its internal dynamics, aligning with the predictive capabilities of quantum mechanics. The mathematical framework of closed quantum systems can be elucidated through the following key concepts.

1.1.1 State

The state of a physical system in quantum mechanics encapsulates all the information about the system and is represented by a ray in a Hilbert space. These states, as mathematical entities, are crucial for predicting physical outcomes based on quantum theory.

Definition 1.1.1. *A Hilbert space (denoted by $\mathcal{H}$) is a vector space over the field of complex numbers $\mathbb{C}$ with the following properties:*

a. *Vectors in $\mathcal{H}$, denoted as $|\psi\rangle$, are combined with an inner product $\langle\cdot|\cdot\rangle : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$, which has the following characteristics:*

   i. *Positivity: $\langle\psi|\psi\rangle \geq 0$ for all $|\psi\rangle$, with equality if and only if $|\psi\rangle$ is the zero vector.*

   ii. *Linearity in the second argument: $\langle\phi| (a|\psi_1\rangle + b|\psi_2\rangle) = a\langle\phi|\psi_1\rangle + b\langle\phi|\psi_2\rangle$.*

   iii. *Conjugate symmetry: $\langle\phi|\psi\rangle = \overline{\langle\psi|\phi\rangle}$.*

b. *Completeness: $\mathcal{H}$ is complete, meaning every Cauchy sequence in $\mathcal{H}$ has a limit within $\mathcal{H}$, with respect to the norm defined by $\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$.*

This structure allows $\mathcal{H}$ to support comprehensive physical predictions. The completeness ensures stability under limits, critical for physical applications such as scattering theory and quantum field theory.

Definition 1.1.2. *Two vectors $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ are **orthogonal** if their inner product is zero, $\langle\psi|\phi\rangle = 0$.*

Definition 1.1.3. *A set $\{|\psi_i\rangle\}_i$ forms an basis for $\mathcal{H}$ if every vector $|\phi\rangle \in \mathcal{H}$ can be uniquely expressed as a linear combination of these basis vectors:*

$$|\phi\rangle = \sum_i c_i |\psi_i\rangle ,$$

*where c_i are complex coefficients, and the series converges in the norm of $\mathcal{H}$. The Hilbert space is said to be equal to the **span** of basis set (span$\{|\psi_i\rangle\}_i$). The dimension of $\mathcal{H}$ is defined by the cardinality of its basis set, which can be finite or infinite.*

The inner product in Hilbert space not only supports algebraic operations but also encodes physical properties such as probability amplitudes and expectation values.

A ray in $\mathcal{H}$ represents all vectors that differ only by an overall phase, forming an equivalence class denoted by $\{e^{i\alpha} |\psi\rangle : \alpha \in \mathbb{R}\}$. Conventionally, a vector $|\psi\rangle$ with unit norm, $\langle\psi|\psi\rangle = 1$, is chosen to represent the state, as the overall phase of a state has no observable physical consequences.

The notation $\langle\psi|$ denotes the dual vector corresponding to $|\psi\rangle$, crucial for defining measurements and transitions between states as per the dual space of $\mathcal{H}$.

Definition 1.1.4. *The dual space of $\mathcal{H}$, denoted $\mathcal{H}^*$, consists of all bounded linear functionals mapping $\mathcal{H}$ to $\mathbb{C}$. Each functional f in $\mathcal{H}^*$ satisfies:*

i. Linearity: $f(\alpha |\psi_1\rangle + \beta |\psi_2\rangle) = \alpha f(|\psi_1\rangle) + \beta f(|\psi_2\rangle)$.

ii. Boundedness: For every $|\psi\rangle \in \mathcal{H}$, there exists a constant $C \geq 0$ such that $|f(|\psi\rangle)| \leq C\||\psi\rangle\|$, with the smallest C being the norm of f, $\|f\|$.

The profound connection between $\mathcal{H}$ and its dual is articulated through the Riesz Representation Theorem, which asserts that every functional in $\mathcal{H}^*$ corresponds uniquely to a vector in $\mathcal{H}$, forming a foundational link between abstract mathematical states and their physical interpretations.

1.1.2 Observables

In quantum mechanics, observables are defined as self-adjoint operators on a Hilbert space, associated with measurable physical properties like position, momentum, and spin. These operators play a critical role in describing how measurements affect quantum systems.

Definition 1.1.5. *An operator A on a Hilbert space $\mathcal{H}$ is a linear map, meaning that for any vectors $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ and scalars $a, b \in \mathbb{C}$, the operation of A is defined by:*

$$A(a |\psi\rangle + b |\phi\rangle) = aA |\psi\rangle + bA |\phi\rangle .$$

This property ensures that A respects the linear structure of $\mathcal{H}$.

Notations such as $A|\psi\rangle$, $A(|\psi\rangle)$, and $|A\psi\rangle$ are commonly used interchangeably to indicate the vector in $\mathcal{H}$ resulting from the application of operator A to $|\psi\rangle$.

Definition 1.1.6. *The adjoint of an operator A, denoted A^*, is defined such that for all vectors $|\psi\rangle, |\phi\rangle \in \mathcal{H}$, it satisfies the equation:*

$$\langle\phi|A\psi\rangle = \langle A^*\phi|\psi\rangle .$$

An operator is termed self-adjoint if $A = A^*$, a fundamental requirement for all observables in quantum mechanics.

Definition 1.1.7. *A scalar $\lambda \in \mathbb{C}$ is an eigenvalue of A if there exists a non-zero vector $|\psi\rangle \in \mathcal{H}$ such that:*

$$A|\psi\rangle = \lambda|\psi\rangle .$$

The vector $|\psi\rangle$ is called an eigenvector corresponding to λ. When multiple linearly independent eigenvectors correspond to the same eigenvalue λ, it is said to be degenerate. The eigenspace $E(\lambda)$ associated with λ consists of all vectors, including the zero vector, satisfying $A|\phi\rangle = \lambda|\phi\rangle$.

Definition 1.1.8. *A projection operator P_λ associated with an eigenvalue λ exhibits the following properties:*

- $P_\lambda^2 = P_\lambda$ *(idempotence),*
- $P_\lambda^* = P_\lambda$ *(self-adjointness).*

This operator projects any vector in $\mathcal{H}$ onto the eigenspace associated with λ, crucial for the measurement process.

Eigenvalues of an observable are real and represent potential outcomes of measurements, directly correlating to observable physical quantities such as energy levels or momentum.

Moreover, the corresponding eigenvectors define the state of the system post-measurement, a phenomenon known as wavefunction collapse. This feature underpins the probabilistic nature of quantum mechanics and bridges mathematical formalism with observable phenomena.

The representation of an operator as a sum of products of eigenvalues and their corresponding projection operators is known as the spectral decomposition of an operator. This decomposition is particularly insightful in separable Hilbert spaces with a countable basis, where the spectral theorem provides a powerful tool for understanding the behavior of quantum systems.

Theorem 1.1.1 (Spectral Decomposition)**.** *In a separable Hilbert space $\mathcal{H}$ with a countable basis, the spectral decomposition of a self-adjoint operator A is given by:*

$$A = \sum_i \lambda_i P_i,$$

where λ_i are the eigenvalues and P_i are the projection operators onto the eigenspaces corresponding to these eigenvalues. Each P_i is defined such that $P_i^2 = P_i$ and $P_i^* = P_i$, denoting its idempotence and self-adjointness.

This decomposition implies that the eigenvectors associated with A form a complete orthonormal set within $\mathcal{H}$. 'Completeness' refers to the property that any vector in $\mathcal{H}$ can be uniquely represented as a linear combination of the eigenvectors of A. 'Orthonormality' ensures that these eigenvectors are mutually orthogonal and each has a norm of one.

The set of all bounded observables acting on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. A bounded operator A on $\mathcal{H}$ is a linear operator for which there exists a constant $C \geq 0$ such that $\|A\psi\| \leq C\|\psi\|$ for all $\psi \in \mathcal{H}$. This ensures that A maps bounded sets to bounded sets.

1.1.3 Measurement

Measurement in quantum mechanics is a fundamental process through which information about the state of a physical system is acquired. When an observable A is measured, the system's state collapses to an eigenstate of A, with the corresponding eigenvalue representing the measured value.

Consider an initial quantum state $|\psi\rangle$ just before the measurement. The probability of obtaining a particular eigenvalue λ_n of A is given by the square of the norm of the projection of $|\psi\rangle$ onto the eigenspace associated with λ_n:

$$\mathrm{Prob}(\lambda_n) = \|P_n |\psi\rangle\|^2 = \langle\psi| P_n |\psi\rangle , \tag{1.1}$$

where P_n is the projection operator corresponding to λ_n. Following the measurement, the quantum system's state becomes:

$$\frac{P_n |\psi\rangle}{\|P_n |\psi\rangle\|}. \tag{1.2}$$

This state reflects the post-measurement condition where the system has collapsed to the observed eigenstate. If the measurement of A is immediately repeated, the same result will be obtained, confirming that the system remains in the same eigenstate.

The expectation value of the outcome when measuring A, assuming a separable Hilbert space, can be computed as:

$$\langle A\rangle = \sum_n \lambda_n \mathrm{Prob}(\lambda_n) = \sum_n \lambda_n \langle\psi| P_n |\psi\rangle = \langle\psi| \sum_n \lambda_n P_n |\psi\rangle = \langle\psi| A |\psi\rangle . \tag{1.3}$$

1.1.4 Dynamics

The evolution of a quantum system is governed by the Schrödinger equation, analogous to Newton's laws of motion in classical mechanics. This fundamental equation is represented as:

$$\frac{d}{dt} |\psi(t)\rangle = -i\hbar H(t) |\psi(t)\rangle , \tag{1.4}$$

where $H(t)$ is the Hamiltonian of the system, a self-adjoint operator that dictates the energy and dynamics of the system at any given time.

Remark : In quantum mechanics, it is common practice to use units where the reduced Planck constant ($\hbar$) is set to 1. This simplification is often employed to streamline equations and calculations, making the mathematical expressions more concise. In these units, the Schrödinger equation takes the form:

$$\frac{d}{dt}\left|\psi(t)\right\rangle = -iH(t)\left|\psi(t)\right\rangle, \tag{1.5}$$

This choice of units does not affect the physical interpretations of the results but simplifies the formalism of quantum mechanics by removing $\hbar$ from the equations.

If the state of the system at an initial time t_0 is given by $|\psi(t_0)\rangle$, the state at any later time t can be described using the unitary propagator $U(t, t_0)$:

$$\left|\psi(t)\right\rangle = U(t, t_0)\left|\psi(t_0)\right\rangle, \tag{1.6}$$

where $U(t, t_0)$ is unitary, satisfying $U^*(t, t_0)U(t, t_0) = U(t, t_0)U^*(t, t_0) = \mathbb{1}$, and $\mathbb{1}$ is the identity operator. The evolution equation for $U(t, t_0)$ can then be derived from the Schrödinger equation:

$$\frac{d}{dt}U(t, t_0) = -iH(t)U(t, t_0), \qquad U(t_0, t_0) = \mathbb{1}. \tag{1.7}$$

We can formally write the evolution as a Dyson expansion using the time order operator $\mathcal{T}$,

$$U(t, t_0) = \mathcal{T}e^{-i\int_{t_0}^{t} H(s)\,ds}$$

In the special case where the Hamiltonian does not depend on time, i.e., $H(t) = H$, the propagator simplifies to:

$$U(t - t_0) = e^{-iH(t-t_0)}. \tag{1.8}$$

Such a system, where H is constant, is termed an **isolated system**.

The deterministic yet probabilistic nature of quantum mechanics is encapsulated by the

Schrödinger equation, which describes both the static properties through the Hamiltonian and the dynamic transitions through the unitary evolution of the system.

1.1.5 Composite Systems

In quantum mechanics, when considering two independent quantum systems, A and B, with respective state spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, the composite system AB is described in the tensor product space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. The state of the composite system can be expressed as:

$$|\psi_{AB}\rangle = \sum_{ij} a_{ij} \, |\psi_i\rangle_A \otimes |\psi_j\rangle_B \,, \tag{1.9}$$

where $|\psi_i\rangle_A \in \mathcal{H}_A$ and $|\psi_j\rangle_B \in \mathcal{H}_B$ are basis vectors of the respective Hilbert spaces, and a_{ij} are complex coefficients.

Observables in the composite system AB are represented by operators that act on $\mathcal{H}_{AB}$. These operators are typically constructed from the observables of systems A and B using tensor products, allowing for interactions between the systems that preserve their individual properties. The general form of such an operator is:

$$O_{AB} = \sum_{ij} A_i \otimes B_j, \tag{1.10}$$

where A_i are operators on $\mathcal{H}_A$ and B_j are operators on $\mathcal{H}_B$. This formulation ensures that measurements on one system do not affect the state of the other system unless the operators explicitly include interaction terms.

An operator M that acts only on system A can also be represented in the composite system as:

$$M_{AB} = M \otimes \mathbb{1}_B, \tag{1.11}$$

where $\mathbb{1}_B$ is the identity operator on $\mathcal{H}_B$. This representation allows M to operate on $\mathcal{H}_A$ while acting trivially on $\mathcal{H}_B$, illustrating the principle of locality in quantum mechanics.

This framework adequately captures the essence of quantum physics for closed systems. However, it raises questions, such as the Schrödinger equation's linearity in relation to nature's non-linear phenomena and the intrinsic randomness of the measurement process, known as the measurement problem in quantum theory. But, the main problem we are going to consider is the description of subsytem of a composite system.

1.1.6 Density Matrix Formalism

The density matrix formalism provides a comprehensive mathematical framework for describing the statistical states of quantum systems, encompassing both pure and mixed states. This formalism is essential for systems prepared under conditions that lead to statistical mixtures rather than well-defined pure states.

Definition and Properties of the Density Matrix

A density matrix ρ describes the state of a quantum system within a Hilbert space $\mathcal{H}$. It is defined as a linear operator with the following properties:

Definition 1.1.9. *A density matrix ρ associated with a Hilbert space $\mathcal{H}$ is a linear operator on $\mathcal{H}$ that satisfies:*

(i) $\mathrm{Tr}(\rho) = 1$ *(unit trace),*

(ii) $\rho^* = \rho$ *(self-adjointness),*

(iii) $\rho \geq 0$ *(positive semi-definiteness).*

These properties ensure that ρ adequately represents a physical state, with probabilities derived as expectation values that are real and non-negative.

The general form of a density matrix in an ensemble of states is:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{1.12}$$

where p_i are probabilities satisfying $\sum_i p_i = 1$, and $|\psi_i\rangle$ are the states in which the system can be found. The system is in a pure state if ρ can be expressed as $|\psi\rangle\langle\psi|$ for some state $|\psi\rangle$, and in a mixed state if it is a convex combination of such outer products.

Pure State Condition:

$$\rho^2 = \rho \quad \text{and} \quad \mathrm{Tr}(\rho^2) = 1. \tag{1.13}$$

Mixed State Condition:

$$\mathrm{Tr}(\rho^2) < 1. \tag{1.14}$$

Measurement in Density Matrix Formalism

Upon measurement of an observable $\mathcal{A}$ with eigenstates $|\lambda_i\rangle$, the density matrix ρ transforms according to the projection postulate. The probability of obtaining a specific outcome λ_i is given by:

$$p_i = \mathrm{Tr}(\rho E_i), \tag{1.15}$$

where E_i is the projection operator that maps state vectors to the eigenspace corresponding to λ_i. The state of the system after the measurement is:

$$\rho' = \frac{P_i \rho P_i}{\mathrm{Tr}(\rho P_i)}. \tag{1.16}$$

Dynamics of the Density Matrix

The time evolution of the density matrix in quantum mechanics is governed by the Liouville-von Neumann equation. For a time-dependent Hamiltonian $H(t)$, the equation is derived from the fundamental postulate that the evolution must be unitary, preserving the trace and hermiticity of ρ. Assuming the unitary evolution operator $U(t)$ that satisfies the

Schrödinger equation:

$$i\frac{d}{dt}U(t) = H(t)U(t), \tag{1.17}$$

and its initial condition $U(0) = I$, the density matrix at time t is given by:

$$\rho(t) = U(t)\rho(0)U^*(t). \tag{1.18}$$

Differentiating $\rho(t)$ with respect to time, we get:

$$\frac{d}{dt}\rho(t) = \frac{d}{dt}U(t)\rho(0)U^*(t) + U(t)\rho(0)\frac{d}{dt}U^*(t). \tag{1.19}$$

Using the Schrödinger equation for $U(t)$ and its adjoint, this simplifies to the Liouville-von Neumann equation:

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[H(t), \rho(t)], \tag{1.20}$$

where $[H(t), \rho(t)]$ denotes the commutator of $H(t)$ and $\rho(t)$.

Composite Systems

If systems A and B are described by density matrices ρ_A and ρ_B respectively, the joint state of the composite system (without correlations) is represented by:

$$\rho_{AB} = \rho_A \otimes \rho_B. \tag{1.21}$$

The general form, following (1.9) is given by

$$\rho_{AB} = \sum_{ijkl} c_{ijkl} \ \left|\psi_i\right\rangle_A \left\langle\psi_k\right| \otimes \left|\psi_j\right\rangle_B \left\langle\psi_l\right|. \tag{1.22}$$

where c_{ijkl} are complex coefficients.

1.2 Entanglement

Entanglement is a pivotal concept in quantum mechanics that illustrates profound non-classical correlations existing between the components of a composite quantum system. This key phenomenon fundamentally distinguishes quantum mechanics from classical physics by demonstrating the inseparability of state descriptions across subsystems, regardless of their spatial separation.

Definition 1.2.1. ***For Pure States:*** *In a composite system AB, described within the tensor product space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, a pure state $|\psi_{AB}\rangle$ is defined as entangled if it cannot be represented as a tensor product of states from each subsystem. Mathematically, this condition is expressed as:*

$$|\psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \tag{1.23}$$

for any $|\psi_A\rangle \in \mathcal{H}_A$ and $|\psi_B\rangle \in \mathcal{H}_B$. States that satisfy the product condition are termed separable or factorizable.

For Mixed States: *A mixed state, represented by a density matrix ρ_{AB}, is considered entangled if it cannot be decomposed into a convex combination of product states from the subsystems. This is formally stated as:*

$$\rho_{AB} \neq \sum_i p_i (\rho_A^i \otimes \rho_B^i) \tag{1.24}$$

where $p_i \geq 0$, $\sum_i p_i = 1$, and ρ_A^i, ρ_B^i are density matrices on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

1.2.1 Characterization of Entangled and Separable States

Schmidt Decomposition

The Schmidt Decomposition offers a robust framework for analyzing entanglement in pure states. Any pure state $|\psi_{AB}\rangle$ in $\mathcal{H}_{AB}$ can be decomposed as:

$$|\psi_{AB}\rangle = \sum_i \lambda_i \left|\phi_A^i\right\rangle \otimes \left|\phi_B^i\right\rangle \tag{1.25}$$

Here, λ_i are the non-negative real numbers known as Schmidt coefficients, and $|\phi_A^i\rangle$, $|\phi_B^i\rangle$ form complete orthonormal sets in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. The number of non-zero Schmidt coefficients is called Schmidt rank or Schmidt number. The state is identified as entangled if Schmidt rank is not equal to one, indicating the presence of non-trivial correlations between the subsystems.

Concurrence

Concurrence serves as a quantifiable measure of entanglement, particularly useful in two-qubit systems. For a state of two qubits described by the density matrix ρ, the concurrence $C(\rho)$ is given by

$$C(\rho) = \max\left(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\right), \tag{1.26}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq 0$ are the eigenvalues of the matrix

$$\xi(\rho) = \rho(\sigma^y \otimes \sigma^y)\,\overline{\rho}\,(\sigma^y \otimes \sigma^y), \tag{1.27}$$

with

$$\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and where $\overline{\rho}$ is the entry-wise complex conjugate of ρ, when written in the σ^z basis (Pauli matrices).

A concurrence value greater than zero signifies entanglement, with the maximum value of 1 indicating maximal entanglement.

The Peres-Horodecki criterion, also known as the Positive Partial Transpose (PPT) criterion, is particularly notable. According to this criterion, a state ρ_{AB} is entangled if the partial transpose of ρ_{AB} has negative eigenvalues. This criterion is both necessary and sufficient for systems of dimensions 2×2 and 2×3, but only necessary for higher dimensions.

Entanglement witnesses are another tool used to distinguish entangled states. These are Hermitian operators that exhibit non-negative expectation values for all separable states and negative expectation values for certain entangled states. This tool provides a practical method for experimentally detecting entanglement in quantum systems.

1.3 Pictures of Quantum Mechanics

In quantum mechanics, the choice of a mathematical framework, or "picture," for describing the dynamics of a physical system is dictated by the problem's specific requirements. The three main pictures—Schrödinger, Heisenberg, and Interaction—offer different perspectives on the time evolution of states and observables.

Schrödinger Picture

In the Schrödinger picture, the state vectors $|\psi(t)\rangle$ evolve over time according to the Schrödinger equation, while operators may also have explicit time dependence that is not a result of the unitary evolution. This explicit time dependence in operators can arise from external time-dependent parameters or conditions imposed on the system.

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle \tag{1.28}$$

where $U(t)$ is the time-evolution operator defined by the differential equation:

$$\frac{d}{dt}U(t) = -iH(t)U(t), \quad U(0) = \mathbb{1} \tag{1.29}$$

Here, $\mathbb{1}$ denotes the identity operator, and $H(t)$ is the system's Hamiltonian.

Heisenberg Picture

In the Heisenberg picture, state vectors are considered static, and all dynamical evolution is attributed to the operators. This perspective is especially useful for handling systems with time-dependent Hamiltonians.

$$|\psi(t)\rangle^H = U^*(t)\,|\psi(t)\rangle^S = |\psi(0)\rangle^S \tag{1.30}$$

where $U(t)$ is the unitary evolution operator that solves the time-dependent Schrödinger equation. The superscripts S, H denote Schrödinger and Heisenberg picture respectively. The observables in Heisenberg and Schrödinger picture are related as :

$$A^H(t) = U^*(t)A^S(t)U(t) \tag{1.31}$$

The evolution of operators in the Heisenberg picture is given by Heisenberg equation:

$$\frac{d}{dt}A^H(t) \;=\; i[H(t), A^H(t)] + U^*(t)\left(\frac{\partial}{\partial t}A^S(t)\right)U(t) \tag{1.32}$$

which is straightforward to derive from (1.31) and (1.29).

Interaction Picture

The Interaction picture is particularly valuable for systems where it is beneficial to separate the Hamiltonian into a free part and an interaction part, which is common in quantum field theory. This picture can be thought of as a hybrid of the Schrödinger and Heisenberg pictures.

In the Interaction picture, both the state vectors and operators are time-dependent. The time-dependency of observables is due to unitary evolution rather than time-dependent external parameters. The Hamiltonian is naturally separated into a free part, $H_0(t)$, and an interaction part, $H_{int}(t)$. This separation is crucial for tackling problems in quantum field theory and allows for a hybrid approach combining elements of both Schrödinger and Heisenberg pictures. The state vectors evolve with the free part of the Hamiltonian, $H_0(t)$:

$$|\psi(t)\rangle^I = U_0^*(t)\,|\psi(t)\rangle^S \tag{1.33}$$

where $U_0(t)$ is the evolution operator for the free Hamiltonian $H_0(t)$, $|\psi(t)\rangle^S$ is the state vector in the Schrödinger picture. We find the evolution of $|\psi(t)\rangle^I$ as follows:

$$\frac{d}{dt}\,|\psi(t)\rangle^I = -i\,H_{int}^I(t)\,|\psi(t)\rangle^I \tag{1.34}$$

where $H_{int}^I(t) = U_0^*(t)H_{int}^S(t)U_0(t)$ Thus, we define evolution of observables against the backdrop of H_0 :

$$A^I(t) = U_0^*(t)A^S(t)U_0(t) \tag{1.35}$$

The differential equation for density operators in interaction picture translates to :

$$\frac{d}{dt}\rho^I(t) = -i[H_{int}^S(t), \rho^I(t)] \tag{1.36}$$

The evolution equation for operators in the Interaction picture combines the free evolution and the interaction:

$$\frac{d}{dt}A^I(t) = i[H_0(t), A^I(t)] + U_0^*(t)\left(\frac{\partial}{\partial t}A^S(t)\right)U_0(t) \tag{1.37}$$

where $A^S(t)$ includes any explicit time dependence in the Schrödinger picture.

Note : Unless otherwise mentioned the state vectors and observables are represented in Scrödinger picture.

1.4 Open Systems

In quantum mechanics, an open quantum system refers to a system that interacts with its environment, leading to potential changes in its quantum state due to this interaction. This contrasts with a closed system, where the system evolves in isolation.

Consider a quantum system S and its environment E. Let $\mathcal{H}_S$ and $\mathcal{H}_E$ represent the Hilbert spaces associated with S and E, respectively. The basis for $\mathcal{H}_S$ is denoted by $\{|i\rangle_S\}_i$ and for $\mathcal{H}_E$ by $\{|k\rangle_E\}_k$. The composite system SE, comprising both S and E, is described by the Hilbert space $\mathcal{H}_{SE} = \mathcal{H}_S \otimes \mathcal{H}_E$, with a basis given by $\{|i\rangle_S \otimes |k\rangle_E\}_{i,k}$.

The state of the composite system is described by a density operator ρ_{SE}, which in general can be expressed as:

$$\rho_{SE} = \sum_{ijkl} c_{ijkl} \, |i\rangle_S \langle j| \otimes |k\rangle_E \langle l| \,, \tag{1.38}$$

where c_{ijkl} are complex coefficients.

1.4.1 Partial Trace

To analyze the state of just one subsystem, say S, while ignoring the environment (system E), we use the partial trace operation over the environment. The partial trace over E, denoted Tr_E, is defined as follows:

$$\rho_S = \mathrm{Tr}_E(\rho_{SE}) = \sum_k (\mathbb{1}_S \otimes \langle k|_E) \rho_{SE} (\mathbb{1}_S \otimes |k\rangle_E), \tag{1.39}$$

where $\mathbb{1}_S$ is the identity operator on $\mathcal{H}_S$. This operation effectively sums over all possible states of the environment, yielding a reduced density matrix ρ_S that describes the state of S independently of E.

1.4.2 Kraus Operators

Given $\rho_{SE}(t_0)$ and a unitary map evolving it in time, $U(t-t_0)$, we have

$$\rho_{SE}(t) = U(t-t_0)\rho_{SE}(t_0)U^*(t_0). \tag{1.40}$$

We can get the density matrix representing the state of S at time t by taking partial trace over E.

$$\rho_S(t) = \mathrm{Tr}_E\left(\rho_{SE}(t_0) = \mathrm{Tr}_E\left(U(t-t_0)\rho_{SE}(t_0)U^*(t-t_0)\right)\right). \tag{1.41}$$

For an open system where the environment E might be complex or not fully known, describing the state evolution of the system S can be challenging. Kraus operators[3] provide a method to encapsulate the effect of the environment in a set of operators $\{K_\alpha\}_\alpha$ that act only on the state space of S, simplifying the description of its dynamics.

Consider initial state, at time t_0 as a sum of two contributions:

$$\rho_{SE}(t_0) = \rho_S(t_0) \otimes \rho_E(t_0) + \rho_{corr}(t_0) \tag{1.42}$$

where $\mathrm{Tr}_S\ \rho_{corr}(t_0) = \mathrm{Tr}_E\ \rho_{corr}(t_0) = 0$. ρ_{corr} contains all correlations between system and environment. In general, $\rho_E(t_0) = \sum_i \lambda_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$ where $\left|\psi_i\right\rangle \in \mathcal{H}_E$. The state of S at time t is given by

$$\begin{aligned}
\rho_S(t) &= \mathrm{Tr}_E\big\{U(t-t_0)\big[\rho_S(t_0)\otimes\rho_E(t_0)+\rho_{corr}(t_0)\big]U^*(t-t_0)\big\} \\
&= \sum_i \lambda_i \,\mathrm{Tr}_E\left\{U(t-t_0)\big[\rho_A(t_0)\otimes\left|\psi_i\right\rangle\left\langle\psi_i\right|\big]U^*(t-t_0)\right\} \\
&+ \mathrm{Tr}_E\left\{\big[U(t-t_0)\rho_{corr}(t_0)U^*(t-t_0)\big]\right\} \\
&= \sum_\alpha K_\alpha(t,t_0)\rho_S(t_0)K_\alpha^*(t,t_0) + \delta\rho(t,t_0)
\end{aligned} \tag{1.43}$$

where $\alpha = (i,j)$ - an ordered pair, $\{\left|j\right\rangle\}_j$ form a basis set in $\mathcal{H}_E$, $\delta\rho(t,t_0) = \mathrm{Tr}_E\left\{\big[U(t-t_0)\rho_{corr}(t_0)U^*(t-t_0)\right.$ and $K_\alpha(t,t_0) = \sqrt{\lambda_i}\left\langle j\right| U(t-t_0)\left|\psi_i\right\rangle$. The operators K_α are called

Kraus operators. In case $\rho_{corr}(t_0) = 0$, We can simply write

$$\rho_S(t) = \sum_\alpha K_\alpha(t, t_0)\rho_S(t_0)K^*(t, t_0) \tag{1.44}$$

The equation (1.44) defining the evolution of the system in terms of Kraus operators is called Kraus Operator Sum Representation (OSR). We can think of OSR as a map, Φ, from initial state $\rho_S(t_0)$ at time t_0 to final state $\rho_S(t)$ at a later time t.

$$\rho_S(t) = \Phi(\rho(t_0)) = \sum_\alpha K_\alpha(t, t_0)\rho_S(t_0)K^*(t, t_0) \qquad \Phi : \rho_S(t_0) \mapsto \rho_S(t) \tag{1.45}$$

The resulting system state $\rho_S(t)$ should be a valid state. The map Φ satisfies the following properties (ensuring that the resulting density matrix $\rho_S(t)$ is a valid density matrix):

i. Trace preserving : From definition of Krauss operator, $K_\alpha \equiv \sqrt{\lambda_i}\,\langle j|\, U(t - t_0)\,|\psi_i\rangle$, we have $\sum_\alpha K^*(t, t_0)K(t, t_0) = \mathbb{1}$. This ensures that $\mathrm{Tr}\ \rho_S(t) = \mathrm{Tr}\ \Phi(\rho_S(t_0)) = \mathrm{Tr}\Big[\sum_\alpha K_\alpha(t, t_0)\rho_S(t_0)K^*_\alpha(t, t_0)\Big] = \mathrm{Tr}\ \rho_S(t_0)$.

ii. Linear : Using linearity of Trace operation, we deduce the linearity of Φ.

$$\begin{aligned}
\Phi(a\rho_1(t_0) + b\rho_2(t_0)) &= \sum_\alpha \mathrm{Tr}\Big(K_\alpha(t, t_0)a\rho_1(t_0)K^*(t, t_0)\Big) \\
&+ \sum_\alpha \mathrm{Tr}\Big(K_\alpha(t, t_0)b\rho_2(t_0)K^*(t, t_0)\Big) \\
&= a\sum_\alpha \mathrm{Tr}\Big(K_\alpha(t, t_0)\rho_1(t_0)K^*(t, t_0)\Big) \\
&+ b\sum_\alpha \mathrm{Tr}\Big(K_\alpha(t, t_0)\rho_2(t_0)K^*(t, t_0)\Big) \\
&= a\Phi(\rho_1) + b\Phi(\rho_2)
\end{aligned}$$

iii. Positivity : This property ensures that the map Φ transforms positive operators into positive operators. To demonstrate that $\Phi(\rho(t_0)) \geq 0$, it is sufficient to show that

$\langle\nu|\,\Phi(\rho(t_0))\,|\nu\rangle \geq 0$ for all $|\nu\rangle \in \mathcal{H}_S$, as this implies that all eigenvalues of $\Phi(\rho(t_0))$ are non-negative. Let $|w_\alpha\rangle = K_\alpha^\dagger\,|\nu\rangle$. Let $\rho(t_0) = \lambda_i\,|i\rangle\,\langle i|$ be the spectral decomposition of $\rho(t_0)$ at time t_0, where $\{|i\rangle\}_i$ forms basis set for $\mathcal{H}_S$. Then:

$$\begin{aligned}\langle\nu|\,\rho(t_0)\,|\nu\rangle &= \sum_\alpha \langle\nu|\,K_\alpha\rho(t_0)K_\alpha^\dagger\,|\nu\rangle \\ &= \sum_\alpha \langle w_\alpha|\left(\sum_i \lambda_i\,|i\rangle\,\langle i|\right)|w_\alpha\rangle \\ &= \sum_{\alpha,i} \lambda_i\,|\langle w_\alpha|\,i\rangle|^2 \geq 0.\end{aligned}$$

iv. Complete Positivity : The map should not only preserve positivity but also extend this property when the system is part of a larger system. This is critical for maintaining the physical validity of the state under entangling operations. This ensures that for any extension of the system state $\rho_{S'}(t_0) = \rho_S(t_0) \otimes \sigma$ (where σ is any state in an auxiliary Hilbert space), the map defined as $\Phi \otimes \mathbb{1}$ (where $\mathbb{1}$ is the identity on the auxiliary space) also produces a valid quantum state.

1.4.3 General Formalism

In quantum mechanics, the Hamiltonian plays a crucial role as the generator of unitary evolution within closed systems. Extending this concept to open quantum systems prompts the question of whether a similar generator exists that can describe their evolution comprehensively. This inquiry leads us to the formulation of an evolution equation for an open system S, which can be expressed in the form:

$$\dot{\rho}_S(t) = \mathcal{L}\rho_S(t_0). \tag{1.46}$$

Here, $\mathcal{L}$ represents the generator that drives the evolution of the system's density matrix $\rho_S(t)$. Equation (1.46) is known as the master equation for the state $\rho_S(t)$.

To understand the action of $\mathcal{L}$, consider the identity:

$$\mathcal{L}\rho_S(t) = \mathrm{Tr}_\mathrm{R}\left\{U(t-t_0)\rho_{SR}(t_0)U^*(t-t_0)\right\}, \tag{1.47}$$

where Tr_R denotes the trace over the degrees of freedom of reservoir, and $U(t-t_0)$ is the unitary evolution operator for the combined system-reservoir state $\rho_{SR}(t_0)$. At first glance, this may appear merely as a reformulation; however, the true utility of the master equation emerges when we impose certain assumptions about the system-environment interactions and their dynamics. These assumptions enable us to determine the appropriate time evolution of $\rho_S(t)$ even when direct computation of the exact global dynamics $\rho_{SR}(t_0)$ is infeasible.

In the subsequent section, we will derive the Born-Markov master equation [4] under a specific set of assumptions. For systems that adhere to these assumptions, the resulting master equation facilitates the computation of the time evolution of $\rho_S(t)$ for any given time t. It is a first-order differential equation that is local in time, expressed as:

$$\frac{d}{dt}\rho_S(t) = \mathcal{L}\rho_S(t) = -i[H'_S, \rho_S(t)] + \mathcal{D}\rho_S(t). \tag{1.48}$$

This formulation emphasizes that the rate of change of $\rho_S(t)$ at any time t depends solely on the state of $\rho_S(t)$ at that instant, without reference to its state at any other times $t' \neq t$.

- The term $\left[H'_S, \rho_S(t)\right]$ represents the unitary component of the evolution, dictated by the modified system Hamiltonian H'_S. This Hamiltonian H'_S differs from the system's free Hamiltonian, which would govern its evolution in the absence of environmental interactions.

- The non-unitary component $\mathcal{D}\rho_S(t)$ encapsulates the effects of decoherence and dissipation attributed to the environmental influence, significantly impacting the system's dynamics.

1.5 Born-Markov Equation

Consider a composite quantum system denoted as SR, where S represents the system of interest and R the reservoir (surrounding environment). The state of the composite system at time t is described by the density matrix $\rho_{SR}(t)$. The dynamics of this composite system are governed by the Hamiltonian

$$H = H_S \otimes \mathbb{1}_R + \mathbb{1}_S \otimes H_{\mathrm{R}} + H_{int}, \tag{1.49}$$

where H_S and H_{R} are free Hamiltonian of system and environment and H_{int} is the Hamiltonian describing interaction between between S and R. The identity operators $\mathbb{1}_S$, $\mathbb{1}_R$ ensure that each Hamiltonian component acts in the correct space.

For notational convenience, the time dependence of the system and environment Hamiltonians in the Schrödinger picture is dropped. This does not affect the generality of the derivation.

In the following section, we will use the interaction picture for the derivation. Therefore, we will omit the superscripts S, I (used to denote the Scroödinger picture and the Interaction picture) as it is understood that the density matrices and operators are in the interaction picture.

Dynamics in Interaction Picture

The time evolution of the density operator ρ_{SR} is governed by the von Neumann equation. In the interaction picture, this is expressed as:

$$\begin{aligned} \frac{d}{dt}\rho_{SR}(t) &= -i\big[H_{int}(t), \rho_{SR}(t)\big] \quad \text{(Differential form)} \\ \rho_{SR}(t) &= \rho_{SR}(0) - i\int_0^t dx\, \big[H_{int}(x), \rho_{SR}(x)\big] \quad \text{(Integral form)} \end{aligned}$$

Substituting integral form in differential equation, we get

$$\frac{d}{dt}\rho_{SR}(t) = -i\big[H_{int}(t), \rho_{SR}(0) - i\int_0^t dx\, \big[H_{int}(x), \rho_{SR}(x)\big]\big] \tag{1.50}$$

To find the dynamics of subsystem S, we can trace out reservoir R.

$$\frac{d}{dt}\rho_S(t) = -i\int_0^t dx\ \mathrm{Tr_R}\left\{\big[H_{int}(t), \rho_{SR}(0) - i\left[H_{int}(x), \rho_{SR}(x)\right]\big]\right\} \tag{1.51}$$

Assumption - Trace of Commutator with Initial State Vanishes : Assuming

$$\mathrm{Tr_R}[H_{int}(t), \rho_{SR}(0)] = 0$$

simplifies the dynamics significantly, eliminating the first term in the above equation. This assumption posits that the initial state of the composite system is such that the trace of its commutator with the interaction Hamiltonian $H_{int}(t)$ vanishes.

As a result, the evolution of the reduced density matrix $\rho_{\mathrm{S}}(t)$ simplifies to:

$$\frac{d}{dt}\rho_S(t) = -\int_0^t dx\ \mathrm{Tr_R}\left\{\big[H_{int}(t), \big[H_{int}(x), \rho_{\mathrm{SR}}(x)\big]\big]\right\} \tag{1.52}$$

Born Approximation : To obtain a closed equation of motion for ρ_S only, we assume that the interaction is weak such that the influence on the environment is small. This is also known as the Born approximation. Mathematically, this translates to

$$\rho_{SR}(t) \approx \rho_S(t) \otimes \rho_{\mathrm{R}} \tag{1.53}$$

where $\rho_R = \rho_R(0)$. In other words, the state of reservoir remains stationary. Using Born approximation, (1.52) is

$$\frac{d}{dt}\rho_S(t) = -\int_0^t dx\ \mathrm{Tr_R}\left\{\big[H_{int}(t), \big[H_{int}(x), \rho_S(x) \otimes \rho_R\big]\big]\right\} \tag{1.54}$$

Markov Approximation : To obtain a Markovian master equation, we first substitute x by $t - x$ in the integrand, which does not change the bounds of the integration. Then, we can understand the parameter x as indicating how far we go backwards in time to account for memory effects, which last for a characteristic timescale τ_R. Under the Markov approximation, we assume that these effects are short-lived. Hence, we can neglect the contribution of $\rho(x)$ with $x < t$ in (1.54). This leads to

$$\rho_S(x) \approx \rho_S(t).$$

In addition, the integrand decays very quickly for $x >> \tau_R$. Then, we may replace the upper bound of the integration by infinity, and obtain a Markovian master equation,

$$\frac{d}{dt}\rho_S(t) = -\int_0^\infty dx \ \mathrm{Tr}_R \left\{ \left[H_{int}(t), \left[H_{int}(t-x), \rho_S(t) \otimes \rho_R \right] \right] \right\} \tag{1.55}$$

The two approximation we have made so far are often grouped together as the **Born-Markov** approximation. However, the equation cannot be cast into a Lindblad form.

Going to Frequency Domain

Consider $H_{int}(t)$ in its general form,

$$H_{int}(t) = g \sum_\alpha A_\alpha(t) \otimes B_\alpha(t), \tag{1.56}$$

where g has units of energy, A_α, B_α are Hermitian operators that act only on system and reservoir respectively.

$$A_\alpha(t) = e^{iH_S t} A_\alpha(0) e^{-iH_S t}, \tag{1.57}$$

$$B_\alpha(t) = e^{iH_R t} B_\alpha(0) e^{-iH_R t} \tag{1.58}$$

(1.55) takes the form

$$\begin{aligned}\frac{d}{dt}\rho_S(t) &= -g^2\int_0^\infty dx\sum_{\alpha,\beta}\mathrm{Tr}_R\big\{\big[A_\alpha(t)\otimes B_\alpha(t),\\ &\qquad \big[A_\beta(t-x)\otimes B_\beta(t-x),\rho_S(t)\otimes\rho_R\big]\big]\big\},\end{aligned}\tag{1.59}$$

Expanding the double commutator, we get

$$\begin{aligned}&\mathrm{Tr}_R\big\{\big[A_\alpha(t)\otimes B_\alpha(t),\big[A_\beta(t-x)\otimes B_\beta(t-x),\rho_S(t)\otimes\rho_R\big]\big]\big\}\\ =\;& A_\alpha(t)A_\beta(t-x)\rho_S(t)\mathrm{Tr}\big[B_\alpha(t)B_\beta(t-x)\rho_R\big]\\ -\;& A_\beta(t-x)\rho_S(t)A_\alpha(t)\mathrm{Tr}\big[B_\beta(t-x)\rho_R B_\alpha(t)\big]\\ -\;& A_\alpha(t)\rho_S(t)A_\beta(t-x)\mathrm{Tr}\big[\rho_R B_\beta(t-x)B_\alpha(t)\big]\\ +\;& \rho_S(t)A_\beta(t-x)A_\alpha(t)\mathrm{Tr}\big[B_\alpha(t)\rho_R B_\beta(t-x)\big]\end{aligned}\tag{1.60}$$

We define the reservoir two-point correlation function,

$$\begin{aligned}\mathcal{C}_{\alpha\beta}(t,t-x) &\equiv \mathrm{Tr}\big(B_\alpha(t)B_\beta(t-x)\rho_R\big)\\ &= \mathrm{Tr}\big(e^{iH_Rt}B_\alpha(0)e^{-iH_Rt}e^{iH_R(t-x)}B_\beta(0)e^{-iH_R(t-x)}\rho_R\big)\\ &= \mathrm{Tr}\big(e^{iH_Rx}B_\alpha(0)e^{-iH_Rx}B_\beta(0)\rho_R\big)\\ &= \mathcal{C}_{\alpha\beta}(x,0)\equiv\mathcal{C}_{\alpha\beta}(x)\end{aligned}$$

where we have used the assumption that the reservoir state is stationary i.e. $[\rho_{\mathrm{R}},H_{\mathrm{R}}]=0$. Note that $\mathcal{C}^*_{\alpha\beta}(x)=\mathcal{C}_{\alpha\beta}(-x)$. Further, the fourth and third term in (1.60) are Hermitian conjugate of first and second term respectively,

$$\frac{d}{dt}\rho_S(t)=-g^2\sum_{\alpha\beta}\int_0^\infty dx\,\big\{\mathcal{C}_{\alpha\beta}(x)\big[A_\alpha(t),A_\beta(t-x)\rho_S(t)\big]+\mathrm{h.c.}\big\}.\tag{1.61}$$

Let the spectral decomposition of system Hamiltonian H_S be $H_S = \sum_a \lambda_a |\lambda_a\rangle \langle\lambda_a|$ where λ_a is eigenvalue corresponding to eigenvector $|\lambda_a\rangle$. We have

$$A_\alpha(t) = e^{iH_S t} A_\alpha(0) e^{-iH_S t} = \sum_{a,b} e^{-i(\lambda_b - \lambda_a)t} |\lambda_a\rangle \langle\lambda_a| A_\alpha(0) |\lambda_b\rangle \langle\lambda_b| = \sum_\omega A_\alpha(\omega) e^{-i\omega t}$$

where $\omega \equiv \lambda_b - \lambda_a$ is termed as Bohr frequency, and

$$A_\alpha(\omega) \equiv \sum_{\lambda_b - \lambda_b = \omega} \langle\lambda_a| A_\alpha(0) |\lambda_b\rangle |\lambda_a\rangle \langle\lambda_b| = A_\alpha^*(-\omega) \tag{1.62}$$

Consider two terms in the commutator $[A_\alpha(t), A_\beta(t-x)\rho_S(t)]$:

$$\begin{aligned} A_\alpha(t) A_\beta(t-x) \rho_S(t) &= \sum_{\omega,\omega'} e^{i\omega x} e^{i(\omega' - \omega)t} A_\alpha^*(\omega') A_\beta(\omega) \rho_S(t) \\ A_\beta(t-x) \rho_S(t) A_\alpha(t) &= \sum_{\omega,\omega'} e^{i\omega x} e^{i(\omega' - \omega)t} A_\beta(\omega) \rho_S(t) A_\alpha^*(\omega') \end{aligned}$$

Using above equations, (1.61) translates to

$$\frac{d}{dt}\rho_S(t) = -g^2 \sum_{\alpha\beta} \sum_{\omega,\omega'} \int_0^\infty dx\, e^{i\omega x} e^{i(\omega' - \omega)t} \{\mathcal{C}_{\alpha\beta}(x) \big[A_\alpha^*(\omega'), A_\beta(\omega)\rho_S(t)\big]\} + \text{h.c.} \tag{1.63}$$

The entire x dependence is in the factor $e^{i\omega x}$ and correlation function. We collect everything that is x-dependent into one function

$$\Gamma_{\alpha,\beta}(\omega) = \int_0^\infty dx\, e^{i\omega x}\, \mathcal{C}_{\alpha\beta}(x) \tag{1.64}$$

This allows us to write (1.63) as

$$\frac{d}{dt}\rho_S(t) = -g^2 \sum_{\alpha\beta} \sum_{\omega,\omega'} \big\{\Gamma_{\alpha,\beta}(\omega) e^{i(\omega' - \omega)t} \big[A_\alpha^*(\omega'), A_\beta(\omega)\rho_S(t)\big] + \text{h.c.}\big\} \tag{1.65}$$

Γ as defined has dimensions of time, and $g^2\Gamma$ has units of frequency.

Secular Approximation : The secular approximation, also called rotating wave approximation, is referred to as the omission of all terms with $\omega \neq \omega^{'}$, as these terms oscillate fast and average out.

Define

$$\gamma_{\alpha\beta}(\omega) \equiv \int_{-\infty}^{\infty} dx\, e^{i\omega x} \mathcal{C}_{\alpha\beta}(x) \tag{1.66}$$

$$\mathcal{C}_{\alpha\beta}(x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega^{'}\, e^{-i\omega^{'} x} \gamma_{\alpha\beta}(\omega^{'}) \tag{1.67}$$

Note that γ and correlation function $\mathcal{C}$ are full fourier transform and inverse fourier transform of other respectively. Also, $\gamma(\omega)$ is Hermitian matrix. Then,

$$\begin{aligned}
\Gamma_{\alpha\beta}(\omega) &= \int_{0}^{\infty} dx\, e^{i\omega x} \mathcal{C}_{\alpha\beta}(x) \\
&= \int_{0}^{\infty} dx\, e^{i\omega x} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega^{'}\, e^{-i\omega^{'} x} \gamma_{\alpha\beta}(\omega^{'}) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega^{'} \gamma_{\alpha\beta}(\omega^{'}) \int_{0}^{\infty} dx\, e^{i(\omega-\omega^{'})x}.
\end{aligned}$$

We can use the following established result regarding dirac δ function,

$$\int_{0}^{\infty} dx\, e^{ixy} = \pi\delta(y) + i\mathcal{P}\left(\frac{1}{y}\right), \qquad \mathcal{P}\left(\frac{1}{y}\right)[f] = \lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} dx\, \frac{f(y)}{y},$$

where $P(1/y)$ is the Cauchy principal value for smooth functions f with compact support on the real line $\mathbb{R}$. Substituting above in expression for Γ, we get

$$\Gamma_{\alpha\beta}(\omega) = \frac{1}{2}\gamma_{\alpha\beta}(\omega) + iS_{\alpha\beta}(\omega) \tag{1.68}$$

$$S_{\alpha\beta}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega^{'}\, \gamma_{\alpha\beta}(\omega^{'}) \mathcal{P}\left(\frac{1}{\omega - \omega^{'}}\right) = S^{*}_{\beta\alpha}(\omega) \tag{1.69}$$

Finally, using the secular approximation i.e. $\omega = \omega^{'}$ and (1.68), Born-Markov equation

takes the form

$$\frac{d}{dt}\rho_S(t) = -g^2 \sum_{\alpha\beta}\sum_{\omega}\left\{\left(\frac{1}{2}\gamma_{\alpha\beta}(\omega) + iS_{\alpha\beta}(\omega)\right)\left[A^*_\alpha(\omega), A_\beta(\omega)\rho_S(t)\right] + \text{h.c.}\right\}$$

Relabelling α, β indices for Hermitian conjugate term, and simplifying the commutator, we get

$$\begin{aligned}\frac{d}{dt}\rho_S(t) &= -i[H_{LS}(t), \rho_S(t)] \\ &+ g^2 \sum_{\alpha,\beta,\omega} \gamma_{\alpha\beta}(\omega)\big(A_\beta(\omega)\rho_S(t)A^*_\alpha(\omega) \\ &- \frac{1}{2}\big\{A^*_\alpha(\omega)A_\beta(\omega), \rho_S(t)\big\}\big). \end{aligned} \tag{1.70}$$

where $H_{LS}(t) = g^2 \sum_\omega \sum_{\alpha\beta} S_{\alpha\beta}(\omega)A^*_\alpha(\omega)A_\beta(\omega)$ is called the Lamb shift Hamiltonian. It is straightforward to verify that H_{LS} is Hermitian.

Transformation back to Schrödinger Equation

We, first, show that $[H_S, H_{LS}] = 0$. Using, (1.62), definition of $A_\alpha(\omega)$

$$\begin{aligned}\left[H_S, A^*_\alpha(\omega)\right] &= \left[H_S, A_\alpha(-\omega)\right] \\ &= \sum_{\lambda_b - \lambda_a = \omega} \left[H_S, \langle\lambda_a| A_\alpha(0) |\lambda_b\rangle |\lambda_a\rangle \langle\lambda_b|\right] \\ &= \sum_{\lambda_b - \lambda_a = \omega} (\lambda_a - \lambda_b) \langle\lambda_a| A_\alpha(0) |\lambda_b\rangle |\lambda_a\rangle \langle\lambda_b| \\ &= \omega A^*_\alpha(\omega)\end{aligned}$$

Similarly, $[H_S, A_\alpha(\omega)] = -\omega A_\alpha(\omega)$. Thus, we have $[H_S, A^\dagger_\alpha(\omega)A_\beta(\omega)] = 0$. Thus, it becomes evident that $[H_S, H_{LS}] = 0$.

We will revert to using superscripts since we are returning to the Schrödinger picture.

Recall that

$$\rho_S^I(t) = e^{iH_S t}\rho_S^S(t)e^{-iH_S t},$$

so

$$\frac{d}{dt}\rho_S^I(t) = i[H_S, \rho_S^S(t)] + e^{iH_S t}\frac{d\rho_S^S(t)}{dt}e^{-iH_S t},$$

hence

$$\frac{d\rho_S^S(t)}{dt} = -i[H_S, \rho_S^S(t)] + e^{-iH_S t}\frac{d\rho_S^I(t)}{dt}e^{iH_S t}. \tag{1.71}$$

Using, (1.62), again

$$\begin{aligned}
& e^{-iH_S t}A_\beta(\omega)\rho_S^I(t)A_\alpha^*(\omega)e^{iH_S t} \\
= & \sum_{\substack{\lambda_i-\lambda_j=\omega \\ \lambda_k-\lambda_l=\omega}} e^{-iH_S t}\left|\lambda_j\right\rangle\left\langle\lambda_j\right|A_\beta(0)\left|\lambda_i\right\rangle\left\langle\lambda_i\right|e^{iH_S t}\rho_S^S(t)e^{-iH_S t} \\
& \times \left|\lambda_k\right\rangle\left\langle\lambda_k\right|A_\alpha^*(0)\left|\lambda_l\right\rangle\left\langle\lambda_l\right|e^{iH_S t} \\
= & \sum_{\substack{\lambda_i-\lambda_j=\omega \\ \lambda_k-\lambda_l=\omega}} e^{i(-\lambda_j+\lambda_i-\lambda_k+\lambda_l)t}\left|\lambda_j\right\rangle\left\langle\lambda_j\right|A_\beta(0)\left|\lambda_i\right\rangle \\
& \times \left\langle\lambda_i\right|\rho_S^S(t)\left|\lambda_k\right\rangle\left\langle\lambda_k\right|A_\alpha^*(0)\left|\lambda_l\right\rangle\left\langle\lambda_l\right| \\
= & \sum_{\substack{\lambda_i-\lambda_j=\omega \\ \lambda_k-\lambda_l=\omega}} \left|\lambda_j\right\rangle\left\langle\lambda_j\right|A_\beta(0)\left|\lambda_i\right\rangle\left\langle\lambda_i\right|\rho_S^S(t)\left|\lambda_k\right\rangle\left\langle\lambda_k\right|A_\alpha^*(0)\left|\lambda_l\right\rangle\left\langle\lambda_l\right| \\
= & A_\beta(\omega)\rho_S^S(t)A_\alpha^*(\omega),
\end{aligned}$$

and

$$\begin{aligned}
& e^{-iH_S t} A^*_\alpha(\omega) A_\beta(\omega) \rho^I_S(t) e^{iH_S t} \\
= & \sum_{\substack{\lambda_i - \lambda_j = \omega \\ \lambda_k - \lambda_l = \omega}} e^{-iH_S t} \left|\lambda_j\right\rangle \left\langle\lambda_j\right| A^*_\alpha(0) \left|\lambda_i\right\rangle \left\langle\lambda_i\right| e^{iH_S t} e^{-iH_S t} \\
& \times \left|\lambda_k\right\rangle \left\langle\lambda_k\right| A_\beta(0) \left|\lambda_l\right\rangle \left\langle\lambda_l\right| e^{iH_S t} \rho^S_S(t) e^{-iH_S t} e^{iH_S t} \\
= & \sum_{\substack{\lambda_i - \lambda_j = \omega \\ \lambda_k - \lambda_l = \omega}} e^{i(-\lambda_j + \lambda_i - \lambda_k + \lambda_l)t} \left|\lambda_j\right\rangle \left\langle\lambda_j\right| A_\beta(0) \left|\lambda_i\right\rangle \\
& \times \left\langle\lambda_i|\lambda_k\right\rangle \left\langle\lambda_k\right| A_\beta(0) \left|\lambda_l\right\rangle \left\langle\lambda_l\right| \rho^S_S(t) \\
= & \sum_{\substack{\lambda_i - \lambda_j = \omega \\ \lambda_k - \lambda_l = \omega}} \left|\lambda_j\right\rangle \left\langle\lambda_j\right| A_\beta(0) \left|\lambda_i\right\rangle \left\langle\lambda_i|\lambda_k\right\rangle \left\langle\lambda_k\right| A_\beta(0) \left|\lambda_l\right\rangle \left\langle\lambda_l\right| \rho^S_S(t) \\
= & A^*_\alpha(\omega) A_\beta(\omega) \rho^S_S(t),
\end{aligned}$$

This shows that

$$\begin{aligned}
& e^{-iH_S t} \big(A_\beta(\omega) \rho^I_S(t) A^*_\alpha(\omega) - \frac{1}{2} \big\{ A^*_\alpha(\omega) A_\beta(\omega), \rho^I_S(t) \big\} \big) e^{iH_S t} \\
= & \big(A_\beta(\omega) \rho^S_S(t) A^*_\alpha(\omega) - \frac{1}{2} \big\{ A^*_\alpha(\omega) A_\beta(\omega), \rho^S_S(t) \big\} \big) \qquad (1.72)
\end{aligned}$$

Also, since we showed that H_S and H_{LS} commute,

$$\begin{aligned}
e^{-iH_S t} \Big[H_{LS}, \rho^I_S(t) \Big] e^{iH_S t} &= e^{-iH_S t} H_{LS} e^{iH_S t} e^{-iH_S t} \rho^I_S(t) e^{iH_S t} \\
&- e^{-iH_S t} \rho^I_S(t) e^{iH_S t} e^{-iH_S t} H_{LS} e^{iH_S t} \\
&= \Big[H_{LS}, \rho^S_S(t) \Big] \qquad (1.73)
\end{aligned}$$

Substituting (1.74) in (1.71) and using (1.73) and (1.72), we get the equation in Schrödinger

picture.

$$\begin{aligned}\frac{d}{dt}\rho_S(t) &= -i[H_S + H_{LS}, \rho_S(t)] \\ &+ g^2 \sum_{\alpha,\beta,\omega} \gamma_{\alpha\beta}(\omega)\big(A_\beta(\omega)\rho_S(t)A^*_\alpha(\omega) \\ &- \frac{1}{2}\big\{A^*_\alpha(\omega)A_\beta(\omega), \rho_S(t)\big\}\big). \end{aligned} \tag{1.74}$$

In above equation, we have removed the superscript S.

1.6 Elements of quantum field theory

In this section we will define Fock space, creation and annihilation opertors and Weyl operators.[7]

1.6.1 Bosons and Fermions

An ideal quantum gas consists of identical, non-interacting quantum particles. A single particle is described by a complex Hilbert space $\mathcal{H}$. For example, if $\mathcal{H} = L^2(\mathbb{R}^3, d^3x)$ (The space $L^2(\mathbb{R}^3, d^3x)$ is a specific type of Hilbert space consisting of square-integrable functions over three-dimensional Euclidean space.), a normalized vector $\psi \in \mathcal{H}$ is the particle's wave function, with $|\psi(x)|^2$ representing the probability density of finding the particle at $x \in \mathbb{R}^3$.

The Hilbert space of n distinguishable particles is given by the n-fold tensor product:

$$\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}.$$

To describe processes involving particle creation and annihilation, we build the direct sum Hilbert space:

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n\geq 0} \mathcal{H}^{\otimes n},$$

where $\mathcal{H}^0 = \mathbb{C}$ represents the vacuum sector. An element $f \in \mathcal{F}(\mathcal{H})$ is a sequence $f = \{f_n\}_{n\geq 0}$, with $f_n \in \mathcal{H}^{\otimes n}$.

For indistinguishable particles, symmetry under particle exchange is essential. Bosons are described by symmetric state vectors, while Fermions are described by anti-symmetric state vectors:

$$\frac{1}{n!}\sum_{\sigma\in S_n} f_{\sigma(1)}\otimes\cdots\otimes f_{\sigma(n)},$$

for Bosons, and

$$\frac{1}{n!}\sum_{\sigma\in S_n} \epsilon(\sigma) f_{\sigma(1)}\otimes\cdots\otimes f_{\sigma(n)},$$

for Fermions, where $\epsilon(\sigma)$ is the signature of permutation σ.

1.6.2 Creation and Annihilation Operators

For $f \in \mathcal{H}$, the annihilation operator $a(f)$ and creation operator $a^*(f)$ are defined as:

$$\begin{aligned} a(f)f_1\otimes\cdots\otimes f_n &= \sqrt{n}\langle f|f_1\rangle f_2\otimes\cdots\otimes f_n, \\ a^*(f)f_1\otimes\cdots\otimes f_n &= \sqrt{n+1}f\otimes f_1\otimes\cdots\otimes f_n. \end{aligned} \tag{1.75}$$

These operators satisfy the canonical commutation relations (CCR) for Bosons:

$$[a(f),a^*(g)] = \langle f|g\rangle\mathbb{1}, \quad [a(f),a(g)] = [a^*(f),a^*(g)] = 0. \tag{1.76}$$

For Fermions, the canonical anti-commutation relations (CAR) are:

$$\{a(f),a^*(g)\} = \langle f|g\rangle\mathbb{1}, \quad \{a(f),a(g)\} = \{a^*(f),a^*(g)\} = 0. \tag{1.77}$$

1.6.3 Weyl Operators

On a mathematical level, working with unbounded operators requires careful handling. Unbounded operators can grow without limit, which makes them challenging to handle

in mathematical analysis. Bosonic creation and annihilation operators are examples of unbounded operators because they can be applied repeatedly to create states with an increasing number of particles.

For example, applying the Bosonic creation operator $a^*(f)$ multiple times increases the number of particles without any restriction. This process can go on indefinitely, leading to an unbounded increase in the norm of the state.

In contrast, Fermionic creation and annihilation operators are bounded. This is due to the Pauli exclusion principle, which states that no two fermions can occupy the same quantum state. As a result, applying a Fermionic creation operator to an already occupied state yields zero. This limitation ensures that the Fermionic operators do not grow indefinitely, making them easier to handle mathematically. To address this, Weyl operators are introduced. They provide a set of bounded operators that are, in a sense, equivalent to the creation and annihilation operators.

For $f \in \mathcal{H}$, the Weyl operator $W(f)$ is defined as:

$$W(f) = e^{i\phi(f)},$$

where

$$\phi(f) = \frac{a(f) + a^*(f)}{\sqrt{2}}. \tag{1.78}$$

The operators $\phi(f)$ are known as field operators. They combine the creation and annihilation operators into a single self-adjoint operator. Note that Weyl operator is bounded. Since $\phi(f)$ is self-adjoint, the exponential of a self-adjoint operator, $e^{i\phi(f)}$, is unitary. Consequently, the Weyl operator is a unitary operator, and the boundedness of the Weyl operator follows from the boundedness of unitary operators.

Following are a few basic properties of Weyl operator :

• *Unitarity of* $W(f)$:

$$W(f)^* = W(-f) = (W(f))^{-1} \tag{1.79}$$

Given that $\phi(f)$ is self-adjoint, we have:

$$W(f)W(-f) = e^{i\phi(f)}e^{-i\phi(f)} = e^{i(\phi(f)-\phi(f))} = 1$$

and also,

$$W(f)^* = \left(e^{i\phi(f)}\right)^* = e^{-i\phi^*(f)} = e^{-i\phi(f)} = W(-f)$$

- *Canonical Commutation Relation (CCR) :*

$$W(f)W(g) = e^{-\frac{i}{2}\mathrm{Im}\langle f|g\rangle}W(f+g) \tag{1.80}$$

Considering the commutator:

$$\begin{aligned}
[\phi(f),\phi(g)] &= \frac{1}{2}[a^*(f)+a(f), a^*(g)+a(g)] \\
&= \frac{1}{2}([a^*(f),a(g)]+[a(f),a^*(g)]) \\
&= \frac{1}{2}(\langle f|g\rangle - \langle g|f\rangle) \\
&= i\ \mathrm{Im}\ \langle f|g\rangle
\end{aligned}$$

We can use above commutation relation and the Baker-Campbell-Hausdorff formula:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$$

$$\begin{aligned}
W(f)W(g) &= e^{i\phi(f)}e^{i\phi(g)} \\
&= e^{i\phi(f)+i\phi(g)+\frac{1}{2}[i\phi(f),i\phi(g)]} \\
&= e^{i(\phi(f)+\phi(g))-\frac{i}{2}\mathrm{Im}\langle f|g\rangle} \\
&= W(f+g)e^{-\frac{i}{2}\mathrm{Im}\langle f|g\rangle}
\end{aligned}$$

• *Time Evolution of* $\phi(f)$ *:*

$$e^{itN(f)}\phi(f)e^{-itN(f)} = \Psi(e^{itf}) \tag{1.81}$$

where $N(f) = a^*(f)a(f)$. (1.80) imply that:

$$a^*(f)N(f) = (N(f) - 1)a^*(f)$$

from which we obtain:

$$a^*(f)e^{-itN(f)} = e^{-it(N(f)-1)}a^*(f) = e^{-itN(f)}e^{-it}a^*(f)$$

and thus:

$$e^{itN(f)}a^*(f)e^{-itN(f)} = e^{-it}a^*(f)$$

Taking the adjoint gives:

$$e^{itN(f)}a(f)e^{-itN(f)} = e^{it}a(f)$$

• *Time Evolution of* $W(f)$ *:*

$$e^{itN(f)}W(f)e^{-itN(f)} = W(e^{it}f) \tag{1.82}$$

This follows directly from the Time Evolution of $\phi(f)$.

Chapter 2

Entanglement produced by quantum channels

While the previous material was general, well known information about quantum theory, we start now to present our own, new results.

In this chapter, we investigate the conditions under which quantum channels produce either classical correlations or entanglement between a system and its reservoir. The primary motivation for this exploration lies in understanding the transition from a product state to an entangled state when subjected to a quantum channel. Specifically, the chapter aims to identify the necessary and sufficient conditions for completely positive, trace-preserving (CPTP) maps to generate entanglement.

2.1 Quantum channels producing classical correlation

Consider a bipartite ('system-reservoir') quantum system described by the Hilbert space

$$\mathcal{H} = \mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{R}}. \tag{2.1}$$

A density matrix ρ on the bipartite Hilbert space $\mathcal{H}$ is called *separable* if it can be approximated in trace norm by a convex combination of product states [13, 8].[1] In other words, ρ is separable if for $n \in \mathbb{N}$, there are density matrices ρ_n^{S}, ρ_n^{R} on $\mathcal{H}_{\mathrm{S}}$, $\mathcal{H}_{\mathrm{R}}$, respectively, and probabilities $0 \leq p_n \leq 1$, $\sum_{n\geq 1} p_n = 1$, such that

$$\rho = \sum_{n\geq 1} p_n \, \rho_n^{\mathrm{S}} \otimes \rho_n^{\mathrm{R}}, \tag{2.2}$$

where the series converges in the trace norm of $\mathcal{H}$. If ρ is not separable, then it is called *entangled*. Equivalently, the term inseparable is used for entangled [9]. If the sum in (2.2) consists of a single term, $\rho = \rho^{\mathrm{S}} \otimes \rho^{\mathrm{R}}$, then the state is called *factorized*. Otherwise it is called *classically correlated*.

Let K_α, $\alpha \in I$ be bounded (Kraus) operators on $\mathcal{H}$ satisfying

$$\sum_{\alpha} K_\alpha^* K_\alpha = \mathbb{1} \tag{2.3}$$

and define the completely positive, trace preserving (CPTP) map on $\mathcal{B}(\mathcal{H})$,

$$\Lambda[X] = \sum_{\alpha} K_\alpha X K_\alpha^*. \tag{2.4}$$

An operator X on $\mathcal{H}_{\mathrm{S}}$ is called *digonalizable* if there is an orthonormal family of vectors ψ_l and numbers $E_l \in \mathbb{C}$ such that

$$X = \sum_{l=1}^{N} E_l |\psi_l\rangle\langle\psi_l|. \tag{2.5}$$

Note that we allow that $E_k = E_l$ for some $k \neq l$ (degenerate eigenvalues of X).

[1] The word 'separable' is used for states and, in a different context, for Hilbert spaces – a separable Hilbert space is one which has a countable orthonormal basis. In the original paper [13], separable states are called *classically correlated* states.

Theorem 2.1.1 (Kraus operators producing classical correlation)**.** *Suppose there is a diagonalizable operator L_{S} on $\mathcal{H}_{\mathrm{S}}$ and a density matrix ρ_{S} on $\mathcal{H}_{\mathrm{S}}$ satisfying*

$$[L_{\mathrm{S}}, \rho_{\mathrm{S}}] = 0 \quad \textit{and} \quad [L_{\mathrm{S}} \otimes \mathbb{1}, K_\alpha] = 0 \quad \forall \alpha. \tag{2.6}$$

Let ρ_{R} be any density matrix of the reservoir and Λ any CPTP map (2.4)*. Then we have the following.*

(1) *The state $\Lambda[\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}]$ is separable and, denoting by Tr_{R} the partial trace over $\mathcal{H}_{\mathrm{R}}$, we have*

$$\mathrm{Tr}_{\mathrm{R}}\big(\Lambda[\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}]\big) = \rho_{\mathrm{S}}. \tag{2.7}$$

Furthermore,

(i) *If ρ_{S} is a pure state, then $\Lambda[\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}]$ is* factorized*.*

(ii) *If ρ_{S} is a mixed state, then $\Lambda[\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}]$ is* classically correlated*.*

(2) *If $\Lambda[\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}]$ is pure then ρ_{S} is pure.*

A special case of Theorem 1 is when $\rho_{\mathrm{S}} = L_{\mathrm{S}}$, leading directly to the

Corollary 2.1.1. *Suppose that $\rho_{\mathrm{S}} \otimes \mathbb{1}$ and K_α commute for all α. Then for any density matrix ρ_{R} on $\mathcal{H}_{\mathrm{R}}$, the state $\Lambda[\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}]$ is separable and* (2.7) *holds. Moreover, $\Lambda[\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}]$ is factorized if and only if ρ_{S} is pure.*

Discussion. Generally, the density matrix $\Lambda[\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}]$ can be entangled or classically correlated, according to the choice of the Kraus operators K_α. Theorem 2.1.1 says that even though there may be correlations in $\Lambda[\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}]$, those correlations are *classical* and moreover, those correlations *do not influence at all* the reduced state of S, as per (2.7).

Proof. We start by proving (1). Since L_{S} is diagonalizable, by (2.5) we have $L_{\mathrm{S}} = \sum_l E_l P_l$, where

$$P_l = |\psi_l\rangle\langle\psi_l|.$$

Since $L_{\rm S}\otimes\mathbb{1}$ and K_α commute, we have

$$(P_l\otimes\mathbb{1})K_\alpha=(P_l\otimes\mathbb{1})K_\alpha(P_l\otimes\mathbb{1}). \tag{2.8}$$

Next we note that

$$(P_l\otimes\mathbb{1})K_\alpha(P_l\otimes\mathbb{1})=P_l\otimes M_{l,\alpha}, \tag{2.9}$$

for some bounded operator $M_{l,\alpha}$ on $\mathcal{H}_{\rm R}$. Indeed, using that

$$K_\alpha=\sum_m A_{m,\alpha}\otimes B_{m,\alpha}, \tag{2.10}$$

for some operators $A_{m,\alpha}$ and $B_{m,\alpha}$ on $\mathcal{H}_{\rm S}$ and $\mathcal{H}_{\rm R}$, respectively, we have

$$(|\psi_l\rangle\langle\psi_l|\otimes\mathbb{1})K_\alpha(|\psi_l\rangle\langle\psi_l|\otimes\mathbb{1})=|\psi_l\rangle\langle\psi_l|\otimes\sum_m\langle\psi_l,A_{m,\alpha}\psi_l\rangle B_{m,\alpha}, \tag{2.11}$$

so that

$$M_{l,\alpha}=\sum_m\langle\psi_l,A_{m,\alpha}\psi_l\rangle B_{m,\alpha}. \tag{2.12}$$

It follows from (2.8), (2.9) and $\sum_l P_l=\mathbb{1}$ that

$$K_\alpha=\sum_l(P_l\otimes\mathbb{1})K_\alpha=\sum_l P_l\otimes M_{l,\alpha}. \tag{2.13}$$

Due to the property $\sum_\alpha K_\alpha^*K_\alpha=\mathbb{1}$ we obtain

$$\mathbb{1}=\sum_{\alpha,l,l'}(P_l\otimes M_{l,\alpha}^*)(P_{l'}\otimes M_{l',\alpha})=\sum_l P_l\otimes Q_l, \tag{2.14}$$

where

$$Q_l=\sum_\alpha M_{l,\alpha}^*M_{l,\alpha}\geq 0. \tag{2.15}$$

Applying (2.14) to the vector $\psi_k\otimes\chi_{\rm R}$, for any fixed k and any vector $\chi_{\rm R}$ in $\mathcal{H}_{\rm R}$ gives

$\psi_k \otimes \chi_{\rm R} = \psi_k \otimes Q_k \chi_{\rm R}$, which implies that $Q_k \chi_{\rm R} = \chi_{\rm R}$, so that

$$Q_l = \sum_\alpha M_{l,\alpha}^* M_{l,\alpha} = \mathbb{1}, \quad \forall l. \tag{2.16}$$

Using (2.13) and the fact that $\rho_{\rm S}$ commutes with P_l, we obtain

$$\begin{aligned} K_\alpha(\rho_{\rm S} \otimes \rho_{\rm R}) K_\alpha^* &= \sum_{l,l'} (P_l \otimes M_{l,\alpha})(\rho_{\rm S} \otimes \rho_{\rm R})(P_{l'} \otimes M_{l',\alpha}^*) \\ &= \sum_l \rho_{\rm S} P_l \otimes M_{l,\alpha} \rho_{\rm R} M_{l,\alpha}^*. \end{aligned} \tag{2.17}$$

Next, $\rho_{\rm S} P_l = P_l \rho_{\rm S} P_l = p_l P_l$, where $p_l = \langle \psi_l, \rho_{\rm S} \psi_l \rangle = {\rm Tr} \rho_{\rm S} P_l$. It follows from this and (2.17) that

$$\Lambda[\rho_{\rm S} \otimes \rho_{\rm R}] = \sum_l p_l P_l \otimes \rho_{{\rm R},l}, \tag{2.18}$$

where

$$\rho_{{\rm R},l} = \sum_\alpha M_{l,\alpha} \rho_{\rm R} M_{l,\alpha}^*. \tag{2.19}$$

It is clear that $\rho_{{\rm R},l} \geq 0$ and furthermore, due to (2.16),

$${\rm Tr}_{\rm R}\, \rho_{{\rm R},l} = {\rm Tr}_{\rm R}\big(\rho_{\rm R} \sum_\alpha M_{l,\alpha}^* M_{l,\alpha}\big) = {\rm Tr}_{\rm R}\, \rho_{\rm R} = 1.$$

Thus $\rho_{{\rm R},l}$ is a density matrix on $\mathcal{H}_{\rm R}$. If $\rho_{\rm S} = |\psi\rangle\langle\psi|$ is a pure state, then $|\psi\rangle$ is an eigenvector of $L_{\rm S}$ (with eigenvalue $\langle \psi, L_{\rm S} \psi \rangle$). In this case, we can choose one of the ψ_l to be equal to ψ. Then (2.18) is of factorized form. Otherwise several p_l are nonzero and $\Lambda[\rho_{\rm S} \otimes \rho_{\rm R}]$, (2.18), is a classically correlated state. Finally we take the partial trace over $\mathcal{H}_{\rm R}$,

$${\rm Tr}_{\rm R} \Lambda[\rho_{\rm S} \otimes \rho_{\rm R}] = \sum_l p_l P_l = \rho_{\rm S}. \tag{2.20}$$

This completes the proof of points (1) in Theorem 2.1.1.

Now we show (2). Suppose that $\Lambda[\rho_{\rm S} \otimes \rho_{\rm R}]$ is pure. Then the sum over l in (2.18) must reduce to a single term, for otherwise the rank of the operator given by this sum would

exceed 1. This means that all but one of the p_l must vanish, so $\rho_{\rm S}$ is pure. □

2.2 Channels originating from a single Kraus operator

We consider the setup (2.1), (2.4) where Λ is given by a single Kraus operator K with $K^*K = \mathbb{1}$. Let $\{\psi_l\}$ be an orthonormal basis of $\mathcal{H}_{\rm S}$ and set $P_l = |\psi_l\rangle\langle\psi_l|$. We consider Kraus operators of the form

$$K = \sum_l P_l \otimes M_l, \tag{2.21}$$

where M_l is an operator on $\mathcal{H}_{\rm R}$. An example of such a K is

$$K = \exp\big[-i\tau \sum_l P_l \otimes H_{{\rm R},l}\big],$$

where $\tau \in \mathbb{R}$ and the $H_{{\rm R},l}$ are self-adjoint operators (Hamiltonians) on $\mathcal{H}_{\rm R}$. In this example, $M_l = e^{-i\tau H_{{\rm R},l}}$. Any SR Hamiltonian $H_{\rm SR}$ which commutes with all the $P_l \otimes \mathbb{1}$ is of the form $H_{\rm SR} = \sum_l P_l \otimes H_{{\rm R},l}$ for some $H_{{\rm R},l}$.

Theorem 2.2.1 (Necessary and sufficient condition for $\Lambda[\rho_{\rm S} \otimes \rho_{\rm R}]$ to be separable)**.**
Let $\rho_{\rm S} = |\Omega_{\rm S}\rangle\langle\Omega_{\rm S}|$ and $\rho_{\rm R} = |\Omega_{\rm R}\rangle\langle\Omega_{\rm R}|$ be pure states on $\mathcal{H}_{\rm S}$ and $\mathcal{H}_{\rm R}$, respectively. Let $\Lambda[\cdot] = K[\cdot]K^*$ with K satisfying (2.21) *and assume that*

$$|\langle M_l\Omega_{\rm R}, M_r\Omega_{\rm R}\rangle| < 1 \quad \textit{for all } l, r \textit{ such that } l \neq r. \tag{2.22}$$

Then we have

$$\Lambda[\rho_{\rm S} \otimes \rho_{\rm R}] \textit{ is separable} \quad \Longleftrightarrow \quad \rho_{\rm S} \otimes \mathbb{1} \textit{ commutes with } K.$$

Remarks: (1) As $K^*K = \mathbb{1}$ we have $M_l^*M_l = \mathbb{1}$ for every l, so that

$$\|M_l\Omega_{\rm R}\|^2 = \langle M_l\Omega_{\rm R}, M_l\Omega_{\rm R}\rangle = 1$$

for all l. It follows that $|\langle M_l\Omega_{\rm R}, M_r\Omega_{\rm R}\rangle| \leq 1$ (with equality if $l = r$). The assumption (2.22) is equivalent to saying that the vectors $M_l\Omega_{\rm R}$ and $M_r\Omega_{\rm R}$ are not parallel. In other words, (2.22) is equivalent to: $\forall l \neq r \ \nexists \alpha \in \mathbb{R}$ such that $\Omega_{\rm R} = e^{i\alpha}M_l^*M_r\Omega_{\rm R}$, or simply, $\Omega_{\rm R}$ is not an eigenvector of $M_l^*M_r$ with an eigenvalue on the unit circle in $\mathbb{C}$.

(2) In standard, explicit models where the reservoir is given by a quantum field in thermal equilibrium, the condition (2.22) is satisfied. We show this in section 2.3.

Proof of Theorem 2.2.1. Since $\rho_{\rm S} \otimes \rho_{\rm R}$ is pure and Λ is obtained by a single Kraus operator, the state $\Lambda[\rho_{\rm S} \otimes \rho_{\rm R}]$ is also pure. Therefore we have

$$\Lambda[\rho_{\rm S} \otimes \rho_{\rm R}] \text{ is separable} \quad \Longleftrightarrow \quad \rho'_{\rm S} := {\rm Tr}_{\rm R}\big(\Lambda[\rho_{\rm S} \otimes \rho_{\rm R}]\big) \text{ is pure.}$$

Writing $\langle X\rangle_{\rm R} = \langle \Omega_{\rm R}, X\Omega_{\rm R}\rangle$ for reservoir operators X, we obtain $\rho'_{\rm S} = \sum_{l,r} P_l\rho_{\rm S}P_r\langle M_r^*M_l\rangle_{\rm R}$ and

$$ {\rm Tr}_{\rm S}(\rho'_{\rm S})^2 = \sum_{l,r} \big|\langle \psi_l, \rho_{\rm S}\psi_r\rangle\big|^2 \big|\langle M_l^*M_r\rangle_{\rm R}\big|^2. \tag{2.23}$$

Similarly, as $\rho_{\rm S} = \sum_{l,r} P_l\rho_{\rm S}P_r$ we have

$$ {\rm Tr}_{\rm S}(\rho_{\rm S})^2 = \sum_{l,r} \big|\langle \psi_l, \rho_{\rm S}\psi_r\rangle\big|^2 = 1. \tag{2.24}$$

The last equality is due to the fact that $\rho_{\rm S}$ is pure. Combining (2.23) and (2.24) yields

$$\begin{aligned} 1 - {\rm Tr}_{\rm S}(\rho'_{\rm S})^2 &= \sum_{l,r} \big|\langle \psi_l, \rho_{\rm S}\psi_r\rangle\big|^2 \big(1 - \big|\langle M_l^*M_r\rangle_{\rm R}\big|^2\big) \\ &= \sum_{l\neq r} \big|\langle \psi_l, \rho_{\rm S}\psi_r\rangle\big|^2 \big(1 - \big|\langle M_l^*M_r\rangle_{\rm R}\big|^2\big). \end{aligned} \tag{2.25}$$

To arrive at the last equality, we note that the 'diagonal' terms ($l = r$) in the double sum vanish, as $\langle M_l^*M_l\rangle_{\rm R} = 1$ for all l. Next, we observe that

$$|\langle M_l^*M_r\rangle_{\rm R}| = |\langle M_l\Omega_{\rm R}, M_r\Omega_{\rm R}\rangle| \leq \|M_l\Omega_{\rm R}\| \, \|M_r\Omega_{\rm R}\| = 1,$$

because $\|M_l\Omega_{\rm R}\|^2 = \langle M_l^* M_l\rangle_{\rm R} = 1$ for all l. We conclude that each summand in (2.25) is non-negative. Therefore,

$$\begin{aligned}\rho'_{\rm S} \text{ is pure} \quad &\Leftrightarrow \quad 1 - {\rm Tr}_{\rm S}(\rho'_{\rm S})^2 = 0 \\ &\Leftrightarrow \quad \Big\{\big|\langle\psi_l, \rho_{\rm S}\psi_r\rangle\big|^2 \big(1 - \big|\langle M_l^* M_r\rangle_{\rm R}\big|^2\big) = 0 \quad \forall l, r \text{ such that } l \neq r\Big\}.\end{aligned}$$

Due to the assumption in the theorem, $|\langle M_l^* M_r\rangle_{\rm R}| < 1$ for $l \neq r$ and so $\rho'_{\rm S}$ is pure if and only if $\langle\psi_l, \rho_{\rm S}\psi_r\rangle = 0$ for all l, r with $l \neq r$. The latter condition is equivalent with $\rho_{\rm S}$ being diagonal in the basis $\{\psi_l\}$, which is equivalent to $\rho_{\rm S}$ commuting with every P_l, which is also equivalent to $\rho_{\rm S} \otimes \mathbb{1}$ commuting with K. This completes the proof of Theorem 2.2.1. □

2.3 Illustrative example: A qubit-reservoir complex

We consider the qubit-reservoir state obtained from a CPTP map (2.4) with a single Kraus operator,

$$\Lambda[\rho_{\rm S} \otimes \rho_{\rm R}] = K\big(\rho_{\rm S} \otimes \rho_{\rm R}\big)K^*, \quad K = e^{-i\tau\sigma_z\otimes\phi(f)}, \tag{2.26}$$

where σ_z is the Pauli-z matrix and $\phi(f)$ was defined in (1.78).

The physical meaning is that K evolves the system plus reservoir state $\rho_{\rm S} \otimes \rho_{\rm R}$ for a duration τ according to the interacting Hamiltonian $H = \sigma_z \otimes \phi(f)$. For $g \in \mathcal{H}_R$, we introduce the *Weyl operator*

$$W(g) = e^{i\phi(g)}, \tag{2.27}$$

which is a unitary operator acting on $\mathcal{F}(\mathcal{H}_R)$.

Lemma 2.3.1. *Let $P_\pm$ be the eigenprojections of σ_z, $P_\pm\sigma_z = \pm P_\pm$. The state* (2.26) *is*

given by

$$\begin{aligned}\Lambda[\rho_{\mathrm{S}}\otimes\rho_{\mathrm{R}}] &= P_+\rho_{\mathrm{S}}P_+\otimes W(\tau f)^*\rho_{\mathrm{R}}W(\tau f)\\ &+ P_+\rho_{\mathrm{S}}P_-\otimes W(\tau f)^*\rho_{\mathrm{R}}W(\tau f)^*\\ &+ P_-\rho_{\mathrm{S}}P_+\otimes W(\tau f)\rho_{\mathrm{R}}W(\tau f)\\ &+ P_-\rho_{\mathrm{S}}P_-\otimes W(\tau f)\rho_{\mathrm{R}}W(\tau f)^*. \qquad (2.28)\end{aligned}$$

The following discussion is an illustration of the result of Corollary 2.1.1.

Discussion.

(1) Suppose that $\rho_{\mathrm{S}}\otimes\mathbf{1}_{\mathrm{R}}$ commutes with K, which is equivalent with ρ_{S} commuting with $P_\pm$. Then $\rho_{\mathrm{S}}P_+ = p\rho_{\mathrm{S}}$ for some $0\leq p\leq 1$ and $\rho_{\mathrm{S}}P_- = (1-p)P_-$ (where p, $1-p$ are the populations of ρ_{S}) and (2.28) reduces to

$$\begin{aligned}\Lambda[\rho_{\mathrm{S}}\otimes\rho_{\mathrm{R}}] &= pP_+\otimes W(\tau f)^*\rho_{\mathrm{R}}W(\tau f)\\ &+ (1-p)P_-\otimes W(\tau f)\rho_{\mathrm{R}}W(\tau f)^*. \qquad (2.29)\end{aligned}$$

This shows that $\Lambda[\rho_{\mathrm{S}}\otimes\rho_{\mathrm{R}}]$ is separable. Since ρ_{S} is pure if and only if $p=0$ or $p=1$, (2.29) shows that

$$\rho_{\mathrm{S}}\text{ is pure}\quad\Leftrightarrow\quad\Lambda[\rho_{\mathrm{S}}\otimes\rho_{\mathrm{R}}]\text{ is factorized.}$$

Furthermore, by taking the partial trace over the reservoir in (2.29) and using the cyclicity of the trace, the unitarity of the Weyl operators and that ρ_{R} has trace one, we readily obtain $\mathrm{Tr}_{\mathrm{R}}\big(\Lambda[\rho_{\mathrm{S}}\otimes\rho_{\mathrm{R}}]\big) = P_+\rho_{\mathrm{S}}+P_-\rho_{\mathrm{S}} = \rho_{\mathrm{S}}$, as predicted by (2.7).

(2) Suppose that ρ_{S} does not commute with K. Then $P_+\rho_{\mathrm{S}}P_-\neq 0$. This shows that generally, if $\rho_{\mathrm{S}}\otimes\mathbb{1}_{\mathrm{R}}$ and K do not commute, then Λ can create entanglement when applied to $\rho_{\mathrm{S}}\otimes\rho_{\mathrm{R}}$. In fact, the previous discussion shows the following result.

Proof of Lemma 2.3.1. By using the spectral decomposition $\sigma_z = (+1)P_+ + (-1)P_-$, we

obtain

$$K = e^{-i\tau P_+ \otimes \phi(f) + i\tau P_- \otimes \phi(f)} = e^{-i\tau P_+ \otimes \phi(f)}\, e^{i\tau P_- \otimes \phi(f)}, \tag{2.30}$$

where the last equality holds since the two operators in the exponent commute. Next we note that

$$\begin{aligned} e^{-i\tau P_+ \otimes \phi(f)} &= \mathbb{1} + \sum_{n \geq 1} \frac{(-i)^n}{n!} P_+ \otimes \big(\phi(\tau f)\big)^n \\ &= \big(\mathbb{1} - P_+\big) \otimes \mathbb{1} + P_+ \otimes W(-\tau f). \end{aligned} \tag{2.31}$$

We use (2.31) and the analogous expression for $e^{i\tau P_- \otimes \phi(f)}$ in (2.30), and that $P_+ P_- = 0$, to see that

$$K = P_+ \otimes W(-\tau f) + P_- \otimes W(\tau f). \tag{2.32}$$

Using (2.32) and $W(g)^* = W(-g)$ for $g \in \mathfrak{h}$ in (2.26), we get

$$\begin{aligned} \Lambda[\rho_{\rm S} \otimes \rho_{\rm R}] &= \big(P_+ \otimes W(-\tau f) + P_- \otimes W(\tau f)\big) \\ &\times \big(\rho_{\rm S} \otimes \rho_{\rm R}\big)\big(P_+ \otimes W(\tau f)^* + P_- \otimes W(-\tau f)^*\big). \end{aligned} \tag{2.33}$$

Multiplying out the terms in (2.33) gives (2.28). □

Corollary 2.3.1. *Suppose that $\rho_{\rm S}$ and $\rho_{\rm R}$ are pure. Then the state* (2.26) *is separable if and only if $\rho_{\rm S} \otimes \mathbf{1}_{\rm R}$ commutes with K.*

Examples of a pure reservoir state are the vacuum (or, zero temperature) state $\Omega_{\rm R}$, or coherent states $W(f)\Omega_{\rm R}$, or superpositions of states with finitely many excitations.

Proof. We have already shown in (2.29) that if $\rho_{\rm S} \otimes \mathbb{1}$ commutes with K, then (2.26) is separable (in fact factorized). Let us now show the converse. Assume that $\Lambda[\rho_{\rm S} \otimes \rho_{\rm R}]$ is separable. As this is a pure state we know that $\mathrm{Tr}_{\rm R}\big(\Lambda[\rho_{\rm S} \otimes \rho_{\rm R}]\big)$ is pure as well. From (2.28) we obtain

$$\mathrm{Tr}_{\rm R}\big(\Lambda[\rho_{\rm S} \otimes \rho_{\rm R}]\big) = P_+ \rho_{\rm S} P_+ + \eta\, P_+ \rho_{\rm S} P_- + \overline{\eta}\, P_- \rho_{\rm S} P_+ + P_- \rho_{\rm S} P_-, \tag{2.34}$$

where

$$\eta = \mathrm{Tr}_{\mathrm{R}}\big(W(\tau f)^* \rho_{\mathrm{R}} W(\tau f)^*\big) = \mathrm{Tr}_{\mathrm{R}}\big(\rho_{\mathrm{R}} W(-2\tau f)\big). \tag{2.35}$$

Writing

$$\rho_{\mathrm{S}} = \begin{pmatrix} p & z \\ \overline{z} & 1-p \end{pmatrix}$$

in the $P_\pm$ basis, we get from (2.34)

$$\mathrm{Tr}_{\mathrm{R}}\big(\Lambda[\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}]\big) = \begin{pmatrix} p & \eta z \\ \overline{\eta z} & 1-p \end{pmatrix}. \tag{2.36}$$

The latter matrix represents a pure state, so we have $p(1-p) = |\eta|^2|z|^2$. But ρ_{S} is also pure and thus $p(1-p) = |z|^2$. It follows that $|z|^2 = |\eta|^2|z|^2$. If $z = 0$ then $p \in \{0, 1\}$ and ρ_{S} commutes with σ_z, so $\rho_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{R}}$ commutes with K. If $z \neq 0$ then $|\eta| = 1$. As $\rho_{\mathrm{R}} = |\Omega_{\mathrm{R}}\rangle\langle\Omega_{\mathrm{R}}|$ is pure, with $\Omega_{\mathrm{R}} \in \mathcal{F}(\mathfrak{h})$ a normalized vector, we have from (2.35), $1 = |\langle\Omega_{\mathrm{R}}, W(-2\tau f)\Omega_{\mathrm{R}}\rangle|$. Now both Ω_{R} and $W(-2\tau f)\Omega_{\mathrm{R}}$ are vectors of norm one, so the latter constraint implies that $W(-2\tau f)\Omega_{\mathrm{R}} = e^{i\alpha}\Omega_{\mathrm{R}}$ for some $\alpha \in [0, 2\pi)$. This means that Ω_{R} is an eigenvector of the field operator $\phi(-2\tau f)$ (see (2.27)). However, unless $\tau = 0$, $\phi(-2\tau f)$ has purely absolutely continuous spectrum covering $\mathbb{R}$ (the field operator corresponds to a position operator). Therefore, $|\eta| = 1$ implies that $\tau = 0$, so $K = \mathbb{1}$ and then again $\rho_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{R}}$ commutes with K. □

2.3.1 An inverse problem

Suppose a quantum channel determined by a single Kraus operator K is given, and so is a reservoir density matrix ρ_{R}. Suppose further that a given 'target' system density matrix ρ is given. We define the set

$$\mathcal{S}_\rho = \big\{\rho_{\mathrm{S}} : \rho = \mathrm{Tr}_{\mathrm{R}}\big(\Lambda[\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}]\big)\big\}.$$

Note that $\mathcal{S}_\rho$ is a convex set: for all $0 \leq p_j \leq 1$, $\sum_j p_j = 1$, $j = 1, \ldots, n$

$$\rho_{\mathrm{S}}^j \in \mathcal{S}_\rho,\ j = 1, \ldots, n \quad \Rightarrow \quad \sum_{j=1}^{n} p_j \rho_{\mathrm{S}}^j \in \mathcal{S}_\rho.$$

Lemma 2.3.2. *Consider the qubit in a bosonic environment with Λ as in* (2.26). *Given any ρ_{R} with associated η,* (2.35) *and any system target density matrix $\rho = \begin{pmatrix} p & z \\ \overline{z} & 1-p \end{pmatrix}$, the set $\mathcal{S}_\rho = \{\rho_{\mathrm{S}}\}$ consists of the single point*

$$\rho_{\mathrm{S}} = \begin{pmatrix} p & z/\eta \\ \overline{z}/\overline{\eta} & 1-p \end{pmatrix}.$$

Proof. This is a simple consequence of the formula (2.36). □

Chapter 3

‘Matter-radiation’ channels

In the following text, we extend the analysis to specific models of quantum channels, particularly those describing matter-radiation interactions. In particular, we examine the behavior of a d-level quantum system interacting with a quantum field, focusing on how the energy scales of the reservoir and the strength of coupling affect the entanglement between the system and its environment, with specific attention to qubit systems (system) in Bosonic fields (environment).

3.1 A d-level system interacting with a field

Consider a d-level system coupled to the field, so that $\mathcal{H}_{\mathrm{S}} = \mathbb{C}^d$. We take a family of initial SR states of the form

$$\rho_{\mathrm{SR}}^{\mathrm{in}} = \Lambda[\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}] = K(\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}})K^*, \tag{3.1}$$

where

$$K = K(\lambda, \mu) = e^{-i(\lambda G \otimes \phi(f) + \mu H_{\mathrm{R}})}, \tag{3.2}$$

where G is a hermitian operator on the system Hilbert space, $f \in L^2(\mathbb{R}^3, d^3k)$ is a coupling function and λ, μ are real parameters. K implements an interaction of the system with the

field for a time which is shorter than the characteristic time of the system Hamiltonian, so that the latter can be dropped and does not show in the exponential in (3.2). We now also have the possibility to study the strong interaction regime, when λ is large.

In a first step, we find a useful expression for K and the reduced system density matrix after the application of Λ. We write the hermitian operator G as

$$G = \sum_k \gamma_k P_k, \qquad P_k = |\psi_k\rangle\langle\psi_k| \tag{3.3}$$

where the γ_k are the eigenvalues (possibly repeated) and the P_k are rank-one orthogonal projections (with ψ_k forming an orthonormal basis of $\mathcal{H}_{\rm S}$) satisfying

$$\sum_k P_k = \mathbb{1}_{\rm S}. \tag{3.4}$$

We also recall the definition of the field Weyl operator, (2.27).

Theorem 3.1.1. *The state of the system obtained by tracing out the reservoir in* (3.1) *is given by*

$$\begin{aligned} \rho_{\rm S}^{\rm in} &\equiv \mathrm{Tr}_{\rm R}\rho_{\rm SR}^{\rm in} \\ &= \sum_{k,\ell} \exp\Big[-\frac{i}{2}\lambda^2(\gamma_k^2-\gamma_\ell^2)\big\langle f\big|\frac{\sin(\mu\omega)-\mu\omega}{\mu^2\omega^2}f\big\rangle\Big]\,\eta_{k,\ell}\,P_k\rho_{\rm S}P_\ell, \end{aligned} \tag{3.5}$$

where

$$\eta_{k,\ell} = \mathrm{Tr}_{\rm R}\Big[\rho_{\rm R}W\Big(i(\gamma_k-\gamma_\ell)\lambda\frac{e^{i\mu\omega}-1}{\mu\omega}f\Big)\Big]. \tag{3.6}$$

For a reservoir in a thermal equilibrium state $\rho_{{\rm R},\beta}$ at inverse temperature $\beta<\infty$, the $\eta_{k,\ell}$ are

$$\eta_{k,\ell}(\beta) = \exp\Big[-\frac{\lambda^2}{4}(\gamma_k-\gamma_\ell)^2\big\langle f,\coth(\beta\omega/2)\big[\frac{\sin(\mu\omega/2)}{\mu\omega/2}\big]^2 f\big\rangle\Big]. \tag{3.7}$$

Formula (3.7) follows from the fact that the thermal characteristic function is given by

$$\mathrm{Tr}_{\rm R}(\rho_{{\rm R},\beta}W(f)) = \exp\Big[-\tfrac{1}{4}\langle f,\coth(\beta\omega/2)f\rangle\Big], \tag{3.8}$$

and $1-\cos(x) = 2\sin^2(x/2)$. In terms of the orthonormal basis $\{|\psi_k\rangle\}_k$, (3.3), the purity of $\rho_{\rm S}^{\rm in}$ is calculated from (3.5),

$$\begin{aligned}\mathcal{P}(\lambda,\mu) &\equiv \mathrm{Tr}_{\rm S}\big(\rho_{\rm S}^{\rm in}\big)^2 \\ &= \sum_k \big|\langle\psi_k, \rho_{\rm S}\psi_k\rangle\big|^2 + \sum_{k,\ell;\, k\neq\ell} |\eta_{k,\ell}|^2 \big|\langle\psi_k, \rho_{\rm S}\psi_\ell\rangle\big|^2. \end{aligned} \tag{3.9}$$

Proof of Theorem 3.1.1. We fist show that the operator K, (3.2), can be written as

$$K = \exp\Big[-\frac{i}{2}\lambda^2 G^2\big\langle f\big|\frac{\sin(\mu\omega)-\mu\omega}{\mu^2\omega^2}f\big\rangle\Big]\sum_k P_k\otimes W\Big(i\gamma_k\lambda\frac{1-e^{-i\mu\omega}}{\mu\omega}f\Big)e^{-i\mu H_{\rm R}}. \tag{3.10}$$

From the Trotter product formula, we have

$$K = \lim_{N\to\infty}\big(e^{-\frac{i\lambda}{N}G\otimes\phi(f)}e^{-\frac{i\mu}{N}H_{\rm R}}\big)^N.$$

We examine Nth power of the operator using (3.4),

$$\begin{aligned}\big(e^{-\frac{i\lambda}{N}G\otimes\phi(f)}e^{-\frac{i\mu}{N}H_{\rm R}}\big)^N &= \sum_k P_k\otimes\big(W(-\tfrac{\lambda}{N}\gamma_k f)e^{-\frac{i\mu}{N}H_{\rm R}}\big)^N \\ &= \sum_k P_k\otimes W(-\tfrac{\lambda}{N}\gamma_k f)W(-\tfrac{\lambda}{N}\gamma_k e^{-i\mu\omega/N}f)\cdots W(-\tfrac{\lambda}{N}\gamma_k e^{-i\mu\omega(N-1)/N}f)e^{-i\mu H_{\rm R}} \\ &= \sum_k P_k\otimes e^{-\frac{i}{2}\Phi_N}W\Big(-\tfrac{\lambda}{N}\gamma_k\sum_{\ell=0}^{N-1}e^{-i\mu\omega\ell/N}f\Big)e^{-i\mu H_{\rm R}}. \end{aligned} \tag{3.11}$$

In the second equality, we have used that the free field dynamics satisfies

$$e^{-i\alpha H_{\rm R}}W(h)e^{i\alpha H_{\rm R}} = W(e^{-i\alpha\omega}h). \tag{3.12}$$

The phase Φ_N in (3.11) is accumulated according to the CCR

$$W(f)W(g) = e^{-\frac{i}{2}\mathrm{Im}\langle f|g\rangle}W(f+g). \tag{3.13}$$

It is given by

$$\Phi_N = \frac{\gamma_k^2}{N^2}\,\mathrm{Im}\sum_{\ell=0}^{N-2}\Big\langle \sum_{r=0}^{\ell} e^{-i\mu\omega r/N} f \Big| e^{-i\mu\omega(\ell+1)/N} f\Big\rangle. \tag{3.14}$$

Taking $N \to \infty$ in (3.11) gives (3.10).

The formula (3.5) follows from the definition (3.1) and a simple calculation. □

3.1.1 The case $d = 2$: Qubit interacting with field

Consider the setup of Section 3.1 above with $d = 2$, so the system is a qubit (or spin $1/2$) and choose $G = \sigma_x$, that is,

$$K = K(\lambda, \mu) = e^{-i(\lambda\sigma_x\otimes\phi(f)+\mu H_{\mathrm{R}})}. \tag{3.15}$$

We have (see (3.3)) $\gamma_1 = 1$, $\gamma_2 = -1$. The reduced density matrix (3.5), written as a 2×2 matrix in the eigenbasis $\{|+\rangle, |-\rangle\}$ of σ_z, takes the form

$$\rho_{\mathrm{S}}^{\mathrm{in}} = \begin{pmatrix} x & y \\ \bar{y} & 1-x \end{pmatrix}, \tag{3.16}$$

where

$$\begin{aligned} x &= \tfrac{1}{2} - (\,\tfrac{1}{2} - [\rho_{\mathrm{S}}]_{++})\,\mathrm{Re}\,\eta + (\mathrm{Im}\,[\rho_{\mathrm{S}}]_{+-})\,\mathrm{Im}\,\eta & (3.17)\\ y &= \mathrm{Re}\,[\rho_{\mathrm{S}}]_{+-} + i(\tfrac{1}{2} - [\rho_{\mathrm{S}}]_{++})\mathrm{Im}\,\eta + i\,(\mathrm{Im}\,[\rho_{\mathrm{S}}]_{+-})\mathrm{Re}\,\eta & (3.18)\\ \eta &= \mathrm{Tr}_{\mathrm{R}}\Big[\rho_{\mathrm{R}} W\Big(2i\lambda\frac{e^{i\mu\omega}-1}{\mu\omega} f\Big)\Big]. \end{aligned}$$

Here, $[\rho_{\mathrm{S}}]_{+-} = \langle +|\rho_{\mathrm{S}}|-\rangle$ and so on are the matrix elements of ρ_{S} in the σ_z eigenbasis. The purity of $\rho_{\mathrm{S}}^{\mathrm{in}}$, (3.9), is given in terms of x, y by

$$\mathcal{P} = x^2 + (1-x)^2 + 2|y|^2. \tag{3.19}$$

Substituting (3.17) and (3.18) in (3.19) gives the following result.

Lemma 3.1.1. *The purity of* (3.16) *is given by*

$$\begin{aligned}\mathcal{P}(\lambda,\mu) &= \tfrac{1}{2}+2\Big((\tfrac{1}{2}-[\rho_{\rm S}]_{++})^2+(\mathrm{Im}\,[\rho_{\rm S}]_{+-})^2\Big)|\eta|^2+2\big(\mathrm{Re}\,[\rho_{\rm S}]_{+-}\big)^2\\ &= \mathcal{P}_0\,|\eta|^2+\big(\tfrac{1}{2}+2(\mathrm{Re}[\rho_{\rm S}]_{+-})^2\big)(1-|\eta|^2),\end{aligned}\tag{3.20}$$

where $\mathcal{P}_0$ is the purity of the state $\rho_{\rm S}$ in (3.1) *before the action of the channel Λ. (Equivalently, $\mathcal{P}_0$ is the purity of $\rho_{\rm S}^{\rm in}$ for $\lambda=0$.)*

Lemma 3.1.1 shows that the purity is a strictly increasing function of $|\eta|$ (use the first expression on the right side), unless $[\rho_{\rm S}]_{++}=\frac{1}{2}$ and $[\rho_{\rm S}]_{+-}\in\mathbb{R}$. In the latter case, $\mathcal{P}=\mathcal{P}_0=\frac{1}{2}+2([\rho_{\rm S}]_{+-})^2$ is independent of η.

We use the qubit purity to measure entanglement between $\rm S$ and $\rm R$ in the state $\rho_{\rm SR}^{\rm in}$, (3.1). For this purpose, let us take $\rho_{\rm S}$ in (3.1) to be a pure qubit state and let $\rho_{\rm R}$ be the thermal equilibrium state, $\rho_{\rm R}=\omega_{{\rm R},\beta}$. For low temperatures, that is $\beta\to\infty$, the state $\rho_{\rm S}\otimes\rho_{\rm R}$ is approximately a pure state and hence so is the initial state $\rho_{\rm SR}^{\rm in}$, (3.1). Consequently, the purity $\mathcal{P}$ of reduced system state is a good measure for the entanglement in $\rho_{\rm SR}^{\rm in}$. $\mathcal{P}=1$ means that there is no entanglement between $\rm S$ and $\rm R$ in the state $\rho_{\rm SR}^{\rm in}$ while $\mathcal{P}=1/2$ corresponds to maximal entanglement.

In the thermal case η is given by (3.7), which becomes for small temperatures,

$$|\eta|\to|\eta_\infty|=\exp\Big[-\lambda^2\big\langle f,\big[\frac{\sin(\mu\omega/2)}{\mu\omega/2}\big]^2 f\big\rangle\Big],\qquad \beta\to\infty.\tag{3.21}$$

As $\rho_{\rm S}$ is pure we have $\mathcal{P}_0=1$ using (3.20) and (3.21) we obtain,

$$\mathcal{P}(\lambda,\mu)\to\tfrac{1}{2}\big(1+|\eta_\infty|^2\big)+2(\mathrm{Re}[\rho_{\rm S}]_{+-})^2\big(1-|\eta_\infty|^2\big),\quad \beta\to\infty.\tag{3.22}$$

For $\lambda=0$ the operator K, (3.2), acts trivially on the system density matrix in (3.1). We then have $|\eta_\infty|=1$ and (3.22) gives the correct value 1 for the purity of a pure state. For strong coupling, that is λ large, we have $|\eta_\infty|\ll 1$ and the purity approaches the value

$$\lim_{\lambda\to\infty}\mathcal{P}(\lambda,\mu)=\tfrac{1}{2}+2(\mathrm{Re}[\rho_{\rm S}]_{+-})^2.$$

It is minimal ($=\frac{1}{2}$) for $\mathrm{Re}[\rho_{\mathrm{S}}]_{+-}=0$, in which case the qubit is maximally entangled with the field. In particular, if ρ_{S} is diagonal in the σ_z basis, say if ρ_{S} is an equilibrium state, then at strong coupling, S and R are maximally entangled.

Next we consider $\lambda=\mu$ and call the common value τ (see (3.15)),

$$K=K(\tau)=e^{-i\tau(\sigma_x\otimes\phi(f)+H_{\mathrm{R}})}.$$

We look at large values of τ. Consider a radially symmetric f of the form $f(\omega/\omega_c)$, where $\omega=|k|\geq 0$, and where $\omega_c>0$ is an 'ultraviolet cutoff' parameter, so that $f(k)$ is small for $\omega\gg\omega_c$. As

$$\frac{\sin^2(\omega t/2)}{\omega^2}\approx\frac{\pi}{4}t\delta(0)$$

for large t (where $\delta(0)$ is the Dirac delta function), we obtain for $\omega_c\tau\gg 1$

$$\begin{aligned}\Big\langle f,\big[\frac{\sin(\tau\omega/2)}{\omega/2}\big]^2 f\Big\rangle &\propto \int_0^\infty |f(\omega/\omega_c)|^2\sin^2(\omega\tau/2)d\omega\\ &= \omega_c\int_0^\infty |f(\omega)|^2\sin^2(\omega\,\omega_c\tau/2)d\omega\\ &\approx \tfrac{\pi}{4}\omega_c^2\tau\,\big(\omega^2|f(\omega)|^2\big)|_{\omega=0}.\end{aligned}\tag{3.23}$$

For $f(\omega)\sim\omega^{-1/2}$ as $\omega\ll 1$, (3.23) is proportional to $\omega_c^2\tau$ and thus $|\eta_\infty|$ (3.21) decays exponentially quickly in τ. If the infra-red behaviour of f is more regular,

$$f(\omega)\sim\omega^p\quad\text{as }\omega\ll 1\tag{3.24}$$

for $-\frac{1}{2}<p\leq 1$, then the decay is slower than exponential, and for $p>1$ the limit as $\omega_c\tau\to\infty$ of the term on the leftmost side of (3.23) tends to a nonzero value, see [12]. In the case $p\leq 1$ we thus have $|\eta_\infty|\to 0$ as $\omega_c\tau\to\infty$, and we obtain

$$\lim_{\omega_c\tau\to\infty}\mathcal{P}(\tau,\tau)=\tfrac{1}{2}+2(\mathrm{Re}[\rho_{\mathrm{S}}]_{+-})^2\qquad(\text{for } p\leq 1).$$

We approach maximal entanglement (minimal purity) between the qubit and the reservoir in the state $\rho_{\mathrm{SR}}^{\mathrm{in}}$ for $\omega_c\tau$ large.

For our numerical simulation, we choose

$$f(k) = \tfrac{1}{\mathcal{N}}\,\omega^p e^{-\omega/\omega_c}, \quad \omega = |k|, \tag{3.25}$$

where $p \geq -\frac{1}{2}$, $\omega_c > 0$ and $\mathcal{N}$ is chosen so that $\|f\|_{L^2(\mathbb{R}^3, d^3k)} = 1$. We take

$$\rho_{\mathrm{S}} = |-\rangle\langle -|,$$

which is the equilibrium state at zero temperature (ground state) relative to the qubit Hamiltonian σ_z. (Here, $\sigma_z|-\rangle = -|-\rangle$.)

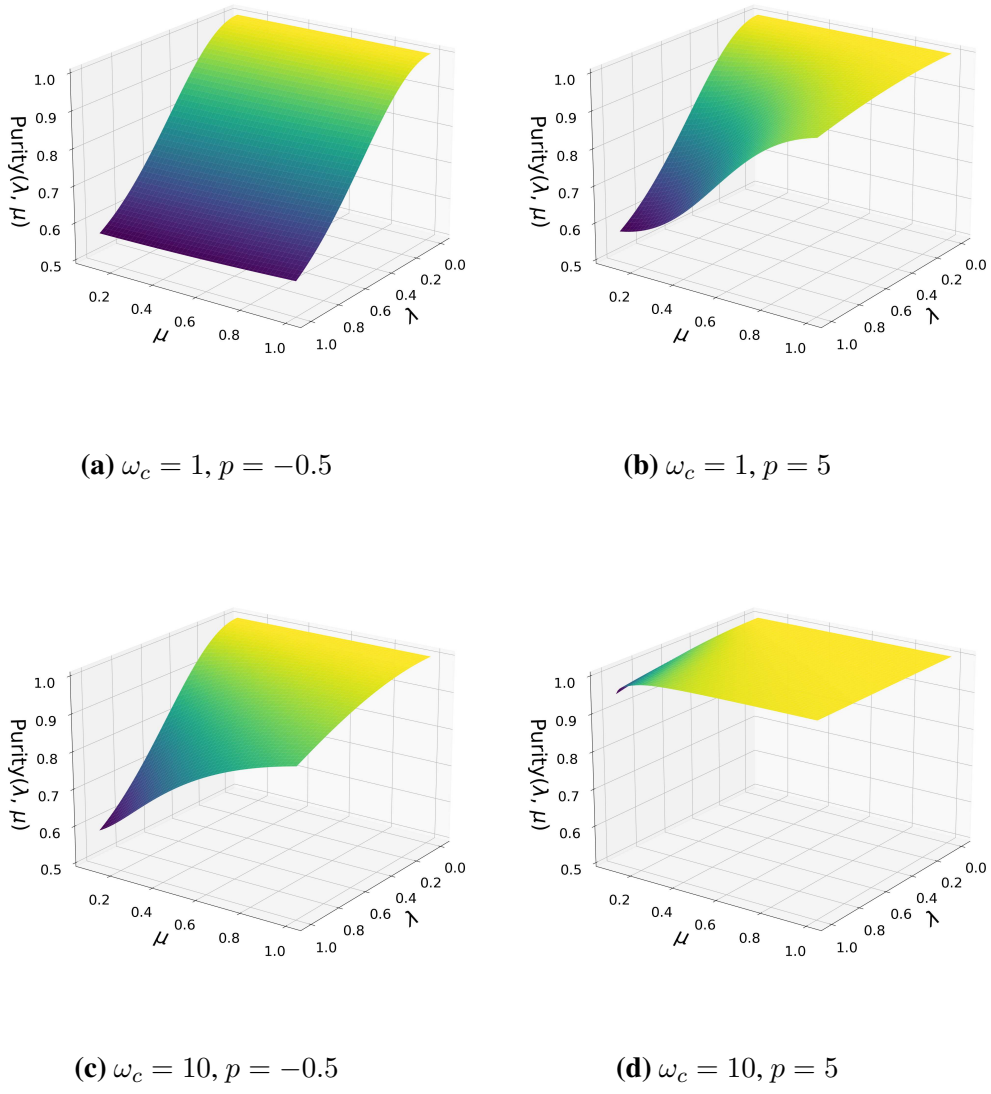
Observing Figure 1 provides us with the following insights:

• *Strong interaction favours qubit-reservoir entanglement.* As the interaction constant λ increases, the purity $\mathcal{P}(\lambda, \mu)$ decreases, so the qubit-reservoir entanglement increases. This is expected: bigger qubit-reservoir interaction implies increased entanglement.

• *Large reservoir energy decreases qubit-reservoir entanglement.* As μ increases, the purity increases as well, which means that the qubit-reservoir entanglement decreases. The parameter μ can be viewed as a scaling of the energy of the reservoir particles. As this energy increases the reservoir becomes less entangled with the qubit.

• *Infra-red reservoir modes favour qubit-reservoir entanglement.* If the infra-red reservoir modes are strongly coupled to the qubit (p small) then the purity is small which means that the qubit-reservoir entanglement is large (panels (a) and (c)). Conversely, if the infra-red modes are suppressed in the interaction (p large) then purity increases and so entanglement decreases (panels (b) and (d)).

• *Ultra-violet reservoir modes lower qubit-reservoir entanglement.* Comparing panels (a) and (b) to (c) and (d) shows that as the ultra-violet cutoff ω_c increases the purity increases as well, so the qubit-reservoir entanglement diminishes.



(a) $\omega_c = 1, p = -0.5$

(b) $\omega_c = 1, p = 5$

(c) $\omega_c = 10, p = -0.5$

(d) $\omega_c = 10, p = 5$

Figure 1: The purity $\mathcal{P}$ of single qubit interacting with the field according to (3.15) with f in (3.25), where p determines the infra-red behaviour, ω_c is the ultra-violet cutoff. The expression for the qubit purity $\mathcal{P}(\lambda, \mu)$ is given by (3.20).

3.2 N d-level systems interacting with a field

The Hilbert space of N d-level systems is given by the tensor product

$$\mathcal{H}_{\mathrm{S}} = \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d. \tag{3.26}$$

Analogously to Section 3.1 we take a family of initial SR states of the form

$$\rho_{\mathrm{SR}}^{\mathrm{in}} = \Lambda[\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}}] = K(\rho_{\mathrm{S}} \otimes \rho_{\mathrm{R}})K^*, \tag{3.27}$$

where

$$K = K(\lambda, \mu) = e^{-i(\lambda G \otimes \phi(f) + \mu H_{\mathrm{R}})}. \tag{3.28}$$

Here, the operator G acting on $\mathcal{H}_{\mathrm{S}}$ is symmetric in the subsystems, given by

$$G = \sum_{k=1}^{N} A^{(k)}, \qquad A^{(k)} = \mathbb{1} \otimes \cdots \otimes A \otimes \cdots \otimes \mathbb{1} \quad (A \text{ on the } k\text{th factor}) \tag{3.29}$$

where A is a self-adjoint operator on $\mathbb{C}^d$ with spectral decomposition

$$A = \sum_k \gamma_k P_k,$$

and where the P_k are a family of disjoint orthogonal rank-one projections with associated eigenvalues γ_k. We define

$$F = \big\langle f \big| \frac{\sin(\mu\omega) - \mu\omega}{\mu^2\omega^2} f \big\rangle. \tag{3.30}$$

Theorem 3.2.1. *Let $\rho_{\mathrm{S}} = \rho_{\mathrm{S}_1} \otimes \cdots \otimes \rho_{\mathrm{S}_N}$. The state of the system obtained by tracing out*

the reservoir in (3.27) *is given by*

$$\begin{aligned}\rho_{\rm S}^{\rm in} &= {\rm Tr}_{\rm R}(\rho_{\rm SR}^{\rm in}) \\ &= \sum_{k_1,\cdots,k_N}\sum_{\ell_1,\cdots,\ell_N}\exp\left[-\frac{i\lambda^2 F}{2}\Big(\big(\sum_{i=1}^N\gamma_{k_i}\big)^2-\big(\sum_{i=1}^N\gamma_{\ell_i}\big)^2\Big)\right] \\ &\quad\times\eta(\vec{k},\vec{\ell})\,(P_{k_1}\rho_{{\rm S}_1}P_{\ell_1})\otimes\cdots\otimes(P_{k_N}\rho_{{\rm S}_N}P_{\ell_N}),\end{aligned} \tag{3.31}$$

where

$$\eta(\vec{k},\vec{\ell})={\rm Tr}_{\rm R}\Big[\rho_{\rm R}W\Big(i\lambda\sum_{i=1}^N\big(\gamma_{k_i}-\gamma_{\ell_i}\big)\frac{e^{i\mu\omega}-1}{\mu\omega}f\Big)\Big]. \tag{3.32}$$

Proof of Theorem 3.2.1. We start by calculating the operator K, (3.28) using the Trotter formula,

$$K=\lim_{M\to\infty}\big(e^{-\frac{i\lambda}{M}G\otimes\phi(f)}e^{-\frac{i\mu}{M}H_{\rm R}}\big)^M.$$

Denoting as usual the Weyl operators by $W(h)=e^{i\phi(h)}$, we get

$$e^{-\frac{i\lambda}{M}G\otimes\phi(f)}e^{-\frac{i\mu}{M}H_{\rm R}}=\sum_{k_1,\ldots,k_N}P_{k_1}\otimes\ldots\otimes P_{k_N}\otimes W\Big(\tfrac{\lambda}{M}f_{\vec{k}}\Big)e^{-\frac{i\mu}{M}H_{\rm R}} \tag{3.33}$$

where

$$f_{\vec{k}}=-\sum_{j=1}^N\gamma_{k_j}f.$$

Taking the Mth power of (3.33) gives

$$\begin{aligned}&\Big(e^{-\frac{i\lambda}{M}G\otimes\phi(f)}e^{-\frac{i\mu}{M}H_{\rm R}}\Big)^M \\ &= \sum_{k_1,\ldots,k_N}P_{k_1}\otimes\ldots\otimes P_{k_N}\otimes W\big(\tfrac{\lambda}{M}f_{\vec{k}}\big)W\big(\tfrac{\lambda}{M}e^{-i\mu\omega/M}f_{\vec{k}}\big)\cdots \\ &\qquad\cdots W\big(\tfrac{\lambda}{M}e^{-i\mu\omega(M-1)/M}f_{\vec{k}}\big)e^{-i\mu H_{\rm R}} \\ &= \sum_{k_1,\ldots,k_N}P_{k_1}\otimes\ldots\otimes P_{k_N}\otimes e^{-\frac{i}{2}\Phi_M}W\Big(\tfrac{\lambda}{M}\sum_{\ell=0}^{M-1}e^{-i\mu\omega\ell/M}f_{\vec{k}}\Big)e^{-i\mu H_{\rm R}},\end{aligned} \tag{3.34}$$

where the phase is

$$\Phi_M = \frac{\lambda^2}{M^2}\big(\sum_{i=1}^{N}\gamma_{k_i}\big)^2\,\mathrm{Im}\sum_{\ell=0}^{M-2}\Big\langle\sum_{r=0}^{\ell}e^{-i\mu\omega r/M}f\Big|e^{-i\mu\omega(\ell+1)/M}f\Big\rangle.$$

The second equality in (3.34) is obtained as in (3.11) using (3.12) and the phase Φ_M is calculated taking into account (3.13). We now take $M\to\infty$,

$$\begin{aligned}K &= \sum_{k_1,\ldots,k_N}\exp\Big[-\frac{i\lambda^2F}{2}\big(\sum_{i=1}^{N}\gamma_{k_i}\big)^2\Big]\;P_{k_1}\otimes\cdots\otimes P_{k_N}\\ &\quad\otimes W\Big(i\lambda\big(\sum_{i=1}^{N}\gamma_{k_i}\big)\frac{1-e^{-i\mu\omega}}{\mu\omega}f\Big)e^{-i\mu H_{\mathrm{R}}},\end{aligned}\tag{3.35}$$

where we recall that F is given in (3.30). The system-reservoir density matrix (3.27) for $\rho_{\mathrm{S}}=\rho_{\mathrm{S}_1}\otimes\cdots\otimes\rho_{\mathrm{S}_N}$ is thus

$$\begin{aligned}\rho_{\mathrm{SR}}^{\mathrm{in}} &= \sum_{k_1,\cdots,k_N}\sum_{\ell_1,\cdots,\ell_N}\exp\Big[-\frac{i\lambda^2F}{2}\Big(\big(\sum_{i=1}^{N}\gamma_{k_i}\big)^2-\big(\sum_{i=1}^{N}\gamma_{\ell_i}\big)^2\Big)\Big]\\ &\quad\times(P_{k_1}\rho_{\mathrm{S}_1}P_{\ell_1})\otimes\cdots\otimes(P_{k_N}\rho_{\mathrm{S}_N}P_{\ell_N})\\ &\quad\otimes W\Big(i\lambda\big(\sum_{i=1}^{N}\gamma_{k_i}\big)\frac{1-e^{-i\mu\omega}}{\mu\omega}f\Big)e^{-i\mu H_{\mathrm{R}}}\,\rho_{\mathrm{R}}\,e^{i\mu H_{\mathrm{R}}}\,W\Big(-i\lambda\big(\sum_{i=1}^{N}\gamma_{\ell_i}\big)\frac{1-e^{-i\mu\omega}}{\mu\omega}f\Big).\end{aligned}$$

We obtain the reduced register density matrix by taking the partial trace over reservoir. Now

$$\begin{aligned}&\mathrm{Tr}_{\mathrm{R}}\Big[W\Big(i\lambda\big(\sum_{i=1}^{N}\gamma_{k_i}\big)\frac{1-e^{-i\mu\omega}}{\mu\omega}f\Big)e^{-i\mu H_{\mathrm{R}}}\,\rho_{\mathrm{R}}\,e^{i\mu H_{\mathrm{R}}}\,W\Big(-i\lambda\big(\sum_{i=1}^{N}\gamma_{\ell_i}\big)\frac{1-e^{-i\mu\omega}}{\mu\omega}f\Big)\Big]\\ &= \mathrm{Tr}_{\mathrm{R}}\Big[\rho_{\mathrm{R}}e^{i\mu H_{\mathrm{R}}}W\Big(i\lambda\sum_{i=1}^{N}(\gamma_{k_i}-\gamma_{\ell_i})\frac{1-e^{-i\mu\omega}}{\mu\omega}f\Big)e^{-i\mu H_{\mathrm{R}}}\Big]\end{aligned}\tag{3.36}$$

and (3.31) follows from (3.12). This completes the proof of Theorem 3.2.1. □

Our next goal is to further reduce $\rho_{\mathrm{S}}^{\mathrm{in}}$ to the first two of the N subsystems. We take

the partial trace of (3.31) over the Hilbert spaces $\mathcal{H}_{\mathrm{S}_3}, \ldots, \mathcal{H}_{\mathrm{S}_N}$. As $\mathrm{Tr}_{\mathrm{S}}(P_k \rho_{\mathrm{S}_j} P_\ell) = \delta_{k,\ell}[\rho_{\mathrm{S}_j}]_{k,k}$, we obtain

$$\begin{aligned} \rho^{\mathrm{in}}_{\mathrm{S},2} &:= \mathrm{Tr}_{\mathrm{S}_3,\ldots,\mathrm{S}_N}\big(\rho^{\mathrm{in}}_{\mathrm{S}}\big) \qquad (3.37) \\ &= \sum_{k_1,k_2,\ell_1,\ell_2} \sum_{k_3,\ldots,k_N} \exp\Big[-\frac{i\lambda^2 F}{2}\Big(\big(\sum_{j=1}^{2}\gamma_{k_j}+\gamma\big)^2 - \big(\sum_{j=1}^{2}\gamma_{\ell_j}+\gamma\big)^2\Big)\Big] \\ &\quad \times \eta\left(\prod_{j\geq 3}[\rho_{\mathrm{S}_j}]_{k_j,k_j}\right)(P_{k_1}\rho_{\mathrm{S}_1}P_{\ell_1})\otimes(P_{k_2}\rho_{\mathrm{S}_2}P_{\ell_2}). \end{aligned}$$

where

$$\begin{aligned} \gamma &= \sum_{j=3}^{N}\gamma_{k_j} \\ \eta &= \mathrm{Tr}_{\mathrm{R}}\Big[W\big(i\lambda\sum_{j=1,2}(\gamma_{k_j}-\gamma_{\ell_j})\frac{e^{i\mu\omega}-1}{\mu\omega}f\big)\Big]. \end{aligned}$$

If the system state ρ_{S} in (3.27) is taken to be symmetric in the N subsystems, say $\rho_{\mathrm{S}} = \rho_1 \otimes \cdots \otimes \rho_1$ for a d-level density matrix ρ_1, then $\rho^{\mathrm{in}}_{\mathrm{S}}$ is also symmetric, that is invariant under permutations of the N subsystems. In this case $\rho^{\mathrm{in}}_{\mathrm{S},2}$, (3.37) is the the same if the reduction is taken with respect any of the two subsystems. We will use $\rho^{\mathrm{in}}_{\mathrm{S},2}$ to study the entanglement between two of the subsystems.

3.2.1 N-qubit register

The Hilbert space for N qubits is given by $\mathcal{H}_{\mathrm{S}}$, (3.26) with $d = 2$. Let us consider the interaction ($A = \sigma_x$ in (3.29))

$$G = \sum_{k=1}^{N}\sigma_x^{(k)}, \qquad \sigma_x^{(k)} = \mathbb{1}\otimes\cdots\otimes\sigma_x\otimes\cdots\otimes\mathbb{1} \qquad (\sigma_x \text{ on the } k\text{th factor}). \qquad (3.38)$$

The operator $G \otimes \phi(f)$ in (3.2) represents an interaction of the N qubits with a common ('collective') environment [11]. The eigenvalues of σ_x are $\gamma_1 = 1$ and $\gamma_2 = -1$ with

associated eigenprojections P_1, P_2. The two-qubit density matrix was derived in (3.37). We now take each qubit initial state to be the ground state (relative to σ_z),

$$\rho_{\mathrm{S}_1} = \rho_{\mathrm{S}_2} = \cdots = \rho_{\mathrm{S}_N} = |-\rangle\langle -|, \qquad \text{where} \qquad |-\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.39}$$

is the eigenvector of σ_z with eigenvalue -1. In this notation, we have

$$P_1 = \tfrac{1}{2} \left| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right|, \quad P_2 = \tfrac{1}{2} \left| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right|.$$

Proposition 3.2.1. *The two-qubit density matrix* (3.37) *has the following expression,*

$$\begin{aligned} \rho^{\mathrm{in}}_{\mathrm{S},2} \;=\; & \tfrac{1}{4}\mathbb{1} \otimes \mathbb{1} \\ & -\tfrac{1}{4}\eta_1 \big[\cos\big(2F\lambda^2\big)\big]^{N-2} \left(e^{-2iF\lambda^2} P_1 + e^{2iF\lambda^2} P_2\right) \otimes \sigma_+^x + \mathrm{h.c.} \\ & -\tfrac{1}{4}\eta_1 \big[\cos\big(2F\lambda^2\big)\big]^{N-2} \, \sigma_+^x \otimes \left(e^{-2iF\lambda^2} P_1 + e^{2iF\lambda^2} P_2\right) + \mathrm{h.c.} \\ & +\tfrac{1}{4}\left(\eta_2 \big[\cos\big(4F\lambda^2\big)\big]^{N-2} \, \sigma_+^x \otimes \sigma_+^x \;+\; \sigma_+^x \otimes (\sigma_+^x)^* \right) + \mathrm{h.c.}, \end{aligned} \tag{3.40}$$

where we recall that F is given in (3.30) *and where*

$$\eta_r = \mathrm{Tr}_{\mathrm{R}}\Big[\rho_{\mathrm{R}} W\Big(2^r i\lambda \frac{e^{i\mu\omega} - 1}{\mu\omega} f\Big)\Big], \qquad r = 1, 2. \tag{3.41}$$

Proof of Proposition 3.2.1. With the ρ_{S_ℓ} given in (3.39) we obtain

$$P_j \rho_{\mathrm{S}_\ell} P_j = \tfrac{1}{2} P_j, \; j = 1, 2 \quad \text{and} \quad P_1 \rho_{\mathrm{S}_\ell} P_2 = -\tfrac{1}{4} \left| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right| = -\tfrac{1}{2}\sigma_+^x,$$

where σ_+^x is the raising operator in the σ_x basis, acting as $\sigma_+^x \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$, $\sigma_+^x \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We have $[\rho_{\rm S}]_{k,k} = \frac{1}{2}$ and the sum over $k_3, \ldots, k_N$ in (3.37) becomes

$$\begin{aligned}
&\sum_{k_3,\ldots,k_N} \Big(\prod_{j\geq 3} [\rho_{\rm S_j}]_{k_j,k_j}\Big) \exp\Bigg[-i\lambda^2 F \sum_{j=1,2} (\gamma_{k_j} - \gamma_{\ell_j}) \sum_{j\geq 3} \gamma_{k_j}\Bigg] \\
&= (\tfrac{1}{2})^{N-2} \sum_{k_3,\ldots,k_N} \prod_{r\geq 3} \exp\Bigg[-i\gamma_{k_r}\lambda^2 F \sum_{j=1,2} (\gamma_{k_j} - \gamma_{\ell_j})\Bigg] \\
&= (\tfrac{1}{2})^{N-2} \Big(\sum_{k=1,2} \exp\Bigg[-i\gamma_k \lambda^2 F \sum_{j=1,2} (\gamma_{k_j} - \gamma_{\ell_j})\Bigg]\Big)^{N-2} \\
&= \Big[\cos\big(\lambda^2 F \sum_{j=1,2} (\gamma_{k_j} - \gamma_{\ell_j})\big)\Big]^{N-2}. \qquad (3.42)
\end{aligned}$$

The first line in (3.40) is the term $k_1 = \ell_1$ and $k_2 = \ell_2$, the second line is $k_1 = \ell_1$ and $k_2 \neq \ell_2$, the third line is $k_1 \neq \ell_1$ and $k_2 = \ell_2$, and the last line is $k_1 \neq \ell_1$ and $k_2 \neq \ell_2$. This completes the proof of Propostion 3.2.1. □

Next we analyze the entaglement between two qubits using the entanglement measure of *concurrence* which was defined in (1.26). We concentrate on *Gaussian centered* states $\rho_{\rm R}$, which means that

$$\mathrm{Tr}_{\rm R}\big(\rho_{\rm R} W(h)\big) = e^{-\langle h, Ch\rangle},$$

where C is a non-negative linear operator. With the definition (3.41), this implies that

$$\eta_2 = \eta_1^4, \qquad \eta_1 \geq 0. \qquad (3.43)$$

Physically relevant specific examples where the relation (3.43) is satisfied are thermal states, for which we have

$$\eta_1 = \exp\Bigg[-\frac{4\lambda^2}{\mu^2} \int_{\mathbb{R}^3} \frac{\sin^2(\mu\omega/2)}{\omega^2} |f(\omega)|^2 \coth\big(\frac{\beta\omega}{2}\big)\, d^3k\Bigg]. \qquad (3.44)$$

Note: It is always true that $|\eta_r| \leq 1$ because the absolute value of the average of a Weyl

operator in any state ρ_{R} cannot exceed 1. (To prove this, take for instance a purification of ρ_{R}, then $|\mathrm{Tr}_{\mathrm{R}}(\rho_{\mathrm{R}}W(h))| = |\langle\Omega_{\mathrm{R}}, \pi(W(h))\Omega_{\mathrm{R}}\rangle| \leq 1$ by Cauchy-Schwarz since Ω_{R} is normalized and $\|\pi(W(h))\| = 1$).

Proposition 3.2.2. *Suppose that* (3.43) *is satisfied and set* $\eta \equiv \eta_1$. *Let* $N \geq 2$. *The concurrence (see* (1.26)*) of* $\rho^{\mathrm{in}}_{\mathrm{S},2}$, (3.40) *is given by is given by :*

$$\begin{aligned} & C\big(\xi(\rho^{\mathrm{in}}_{\mathrm{S},2})\big) \\ &= \tfrac{1}{4}\max\left(0, \sqrt{\Big(4\eta\big(\cos 2F\lambda^2\big)^{N-2}\big(\sin 2F\lambda^2\big)\Big)^2 + \Big(1-\eta^4\big(\cos 4F\lambda^2\big)^{N-2}\Big)^2}\right. \\ &\quad \left. -\Big(1-\eta^4\big(\cos 4F\lambda^2\big)^{N-2}\Big)\right). \end{aligned} \tag{3.45}$$

Note. We see from (3.45) that the concurrence of $\xi^{\mathrm{in}}_{\mathrm{S},2}$ vanishes as $N \to \infty$ (unless $|\cos(2F\lambda^2)| = 1$). This observation aligns with the quantum DeFinetti theorem, suggesting that since $\xi(\rho^{\mathrm{in}}_{\mathrm{S},2})$ constitutes the two-qubit marginal of an exchangeable (symmetric) N-qubit state, it inevitably becomes separable as $N \to \infty$.

Proof of Proposition 3.2.2. In Appendix A we calculate (3.40) as a matrix in the ordered orthonormal basis

$$\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$$

of two qubits, where $|0\rangle = |{+}{+}\rangle\,,\ |1\rangle = |{+}{-}\rangle\,,\ |2\rangle = |{-}{+}\rangle\,,\ |3\rangle = |{-}{-}\rangle$. Here, $|\pm\rangle$ are the eigenvectors of σ_z, $\sigma_z|\pm\rangle = \pm|\pm\rangle$. The result is

$$\rho^{\mathrm{in}}_{\mathrm{S},2} \;=\; \begin{pmatrix} \frac{1}{2}-a-c & 0 & 0 & -ib-a \\ 0 & a & a & 0 \\ 0 & a & a & 0 \\ ib-a & 0 & 0 & \frac{1}{2}-a+c \end{pmatrix}, \tag{3.46}$$

where

$$\begin{aligned} a &= \tfrac{1}{8} - \tfrac{1}{8}\eta^4\big[\cos\big(4F\lambda^2\big)\big]^{N-2} \\ b &= \tfrac{1}{2}\eta\big[\cos\big(2F\lambda^2\big)\big]^{N-2}\sin\big(2F\lambda^2\big) \\ c &= \tfrac{1}{2}\eta\big[\cos\big(2F\lambda^2\big)\big]^{N-2}\cos\big(2F\lambda^2\big). \end{aligned} \tag{3.47}$$

We then calculate, in Appendix A, the quantity (1.27),

$$\xi(\rho_{\mathrm{S},2}^{\mathrm{in}}) = \begin{pmatrix} \big(\frac{1}{2}-a\big)^2 - c^2 + b^2 + a^2 & 0 & 0 & 2(\frac{1}{2}-a-c)(-ib-a) \\ 0 & 2a^2 & 2a^2 & 0 \\ 0 & 2a^2 & 2a^2 & 0 \\ 2(\frac{1}{2}-a+c)(ib-a) & 0 & 0 & \big(\frac{1}{2}-a\big)^2 - c^2 + b^2 + a^2 \end{pmatrix}.$$

The eigenvalues of $\xi(\rho_{\mathrm{S},2}^{\mathrm{in}})$ are given by

$$\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 4a^2 \\ \lambda_3 &= \big(\sqrt{\big(\tfrac{1}{2}-a\big)^2 - c^2} + \sqrt{b^2+a^2}\big)^2 \\ \lambda_4 &= \big(\sqrt{\big(\tfrac{1}{2}-a\big)^2 - c^2} - \sqrt{b^2+a^2}\big)^2. \end{aligned} \tag{3.48}$$

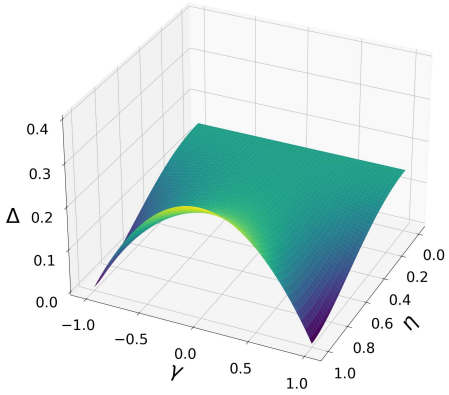
The eigenvalues of $\xi(\rho)$ always non-negative for any 2-qubit density matrix ρ [15]. It is immediately clear that $\lambda_1, \lambda_2 \geq 0$. To check that $\lambda_3, \lambda_4 \geq 0$ we need to verify that

$$\Delta := (\tfrac{1}{2}-a)^2 - c^2 \geq 0. \tag{3.49}$$

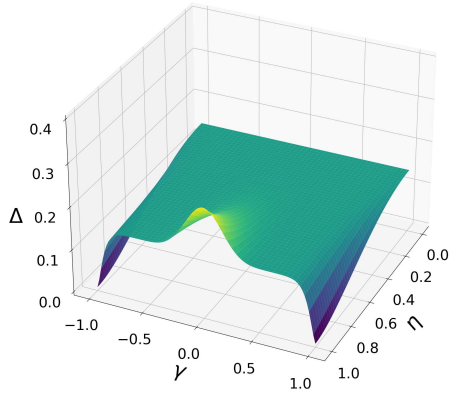
It is surprisingly difficult to verify this analytically, except for specific parameter values. We thus give a numerical verification of (3.49). To do this, we use the parameters

$$\gamma := \cos\big(2F\lambda^2\big) \in [-1,1], \qquad \eta \in [0,1]. \tag{3.50}$$

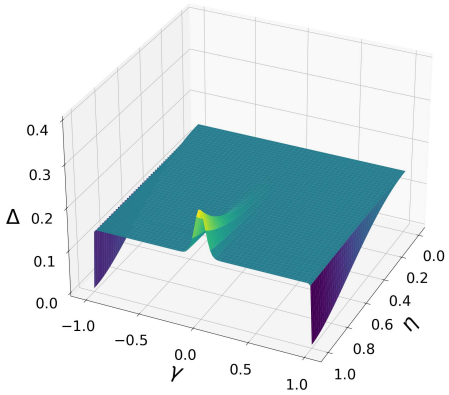
All of a, b, c in (3.47) can be written in terms of γ, η. Figure 2 shows that condition (3.49) is always satisfied.



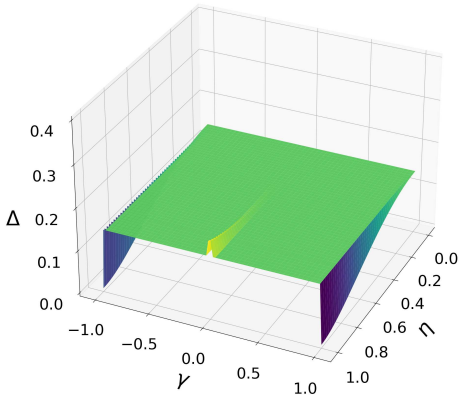
(a) $N = 2$



(b) $N = 10$



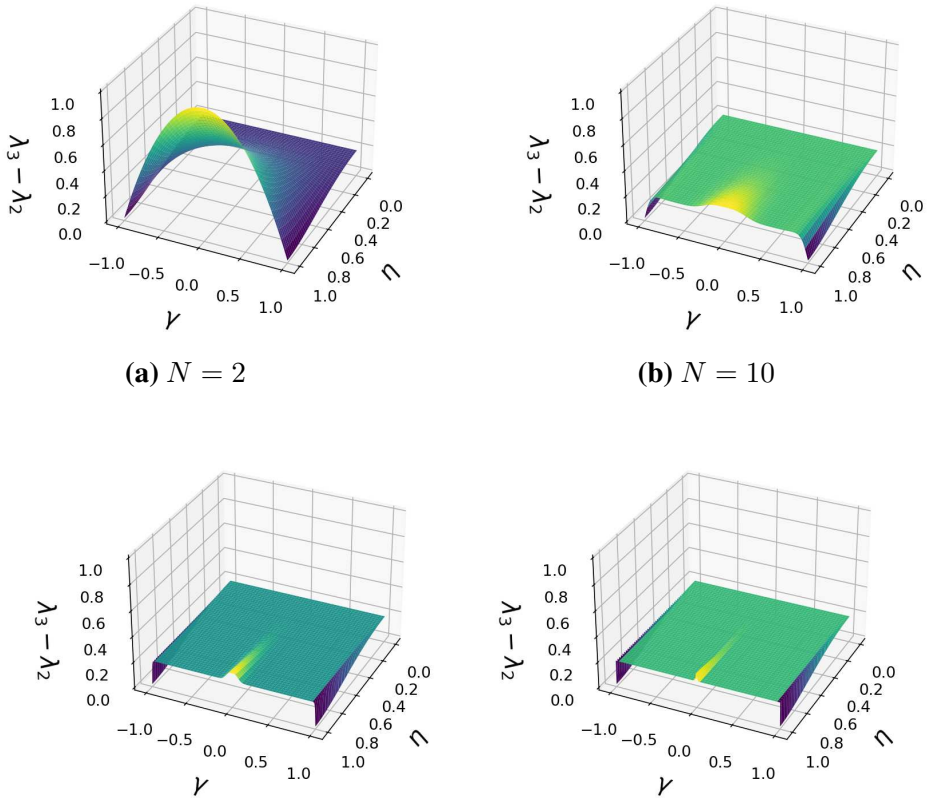
(c) $N = 100$



(d) $N = 5000$

Figure 2: The quantity Δ defined in (3.49) as a function of γ and η, (3.50), for $N = 2, 10, 100, 5000$. The graphs show that condition (3.49) is satisfied: $\Delta \geq 0$.

In order to obtain the concurrence (1.26), we need to order the eigenvalues (3.48). Clearly λ_1 is the smallest and $\lambda_3 \geq \lambda_4$. We check that $\lambda_3 \geq \lambda_2$ numerically in Figure 3. (Again, it turns out not to be simple to do this analytically, except for specific values of the parameters.)



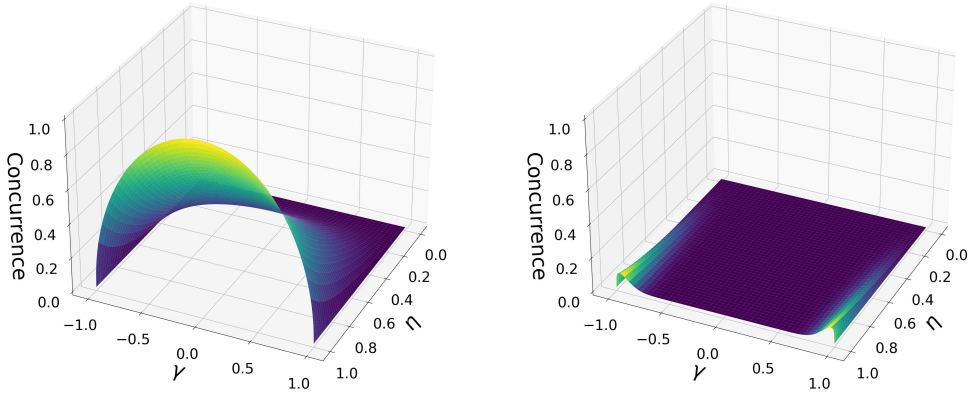
(a) $N = 2$ **(b)** $N = 10$

(c) $N = 100$ **(d)** $N = 5000$

Figure 3: The difference of the eiganvalues $\lambda_3 - \lambda_2$ as a function of η, γ, (3.50), for different values of N. The graph shows that $\lambda_3 \geq \lambda_2$.

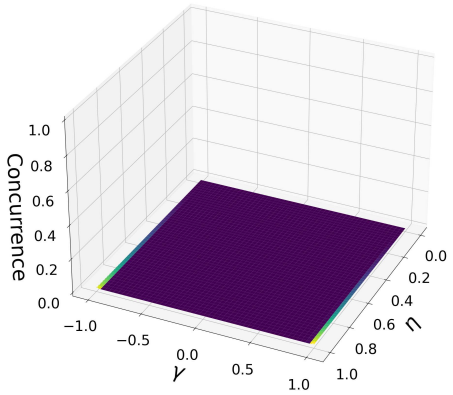
Thus λ_3 is the maximum eigenvalue. The concurrence is thus $\max(0, \sqrt{\lambda_3} - \sqrt{\lambda_2} - \sqrt{\lambda_4})$, which is the formula (3.45). This concludes the proof of Proposition 3.2.2. □

In Figure 4 we plot the two-qubit concurrence $C(\xi(\rho^{\text{in}}_{\text{S},2}))$ for varying numbers of total qubits in the system, $N = 2, 10, 100$ and 5000. It is clear that concurrence tends to

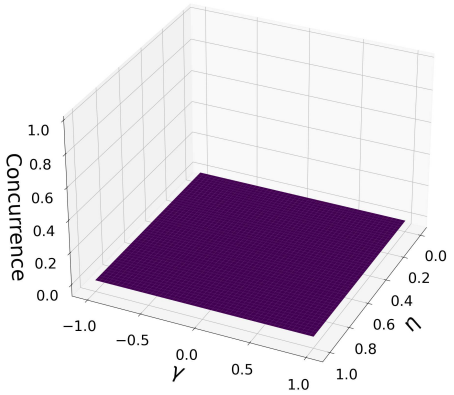


(a) $N = 2$

(b) $N = 10$



(c) $N = 100$



(d) $N = 5000$

Figure 4: Concurrence $C(\xi(\rho^{\text{in}}_{\text{S},2}))$, (3.45) as a function of η, $\gamma = \cos(2F\lambda^2)$ for different values of N.

diminish as N increases (DeFinetti theorem). We observe that for fixed N and a constant value of γ, entanglement increases with an increase in η. For even N, the concurrence is symmetric in γ.

Bibliography

[1] M. A. Nielsen, I. L. Chuang: *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge (2010).

[2] J. Preskill: *Lecture Notes on Quantum Computation*. Available at: http://www.theory.caltech.edu/ preskill/ph219/.

[3] A. Rivas, S.F. Huelga: *Open quantum systems: an introduction* In: Springer Briefs in Physics (2012) https://doi.org/10.48550/arXiv.1104.5242

[4] Daniel A. Lidar: *Lecture Notes on the Theory of Open Quantum Systems*, (2020). https://doi.org/10.48550/arXiv.1902.00967

[5] H.-P. Breuer, F. Petruccione: *The Theory of Open Quantum Systems*. Oxford University Press, Oxford (2002).

[6] M. Schlosshauer: *Decoherence and the Quantum-to-Classical Transition*. Springer, Berlin (2007).

[7] Merkli, M. *The Ideal Quantum Gas.* In: Attal, S., Joye, A., Pillet, C.A. (eds) *Open Quantum Systems I*. Lecture Notes in Mathematics, vol **1880**. Springer, Berlin, Heidelberg. (2006). https://doi.org/10.1007/3-540-33922-1_5

[8] R. Clifton, H. Halvorson: *Bipartite-mixed-states of infinite-dimensional systems are generically nonseparable*, Phys. Rev. A **61**, 012108 (1999)

[9] M. Horodecki, P. Horodecki, and R. Horodecki: *Mixed-State Entanglement and Distillation: Is there a "Bound" Entanglement in Nature?*, Phys. Rev. Lett. **80** no. 24 (1998)

[10] M. Merkli: *Correlation Decay and Markovianity in Open Systems*, Ann. Henri Poincaré **24**, 751–782 (2023)

[11] M. Merkli, G.P. Berman, R.T. Sayre, S. Gnanakaran, M. Könenberg, A.I. Nesterov, H. Song: *Dynamics of a Chlorophyll Dimer in Collective and Local Thermal Environments*, J. Math. Chem. **54**(4), 866-917 (2016)

[12] A. Trushechkin: *Long-Term Behaviour in an Exactly Solvable Model of Pure Decoherence and the Problem of Markovian Embedding*, Mathematics 2024, **12**(1), 1, https://doi.org/10.3390/math12010001

[13] R.F. Werner: *Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model*, Phys. Rev. A **40**, 4277 (1989)

[14] K. Yuasa, S. Tasaki, P. Facchi, G. Kimura, H. Nakazato, I. Ohba, S. Pascazio: *On the assumption of initial factorization in the master equation for weakly coupled systems II: solvable models.* Ann. Phys. **322**, 657-676 (2007)

[15] Sam A. Hill and William K. Wootters Phys. Rev. Lett. **78**, 5022

Appendix A

Supplementary Calculations for Proposition 3.2.2

The matrix form of the operators is calculated in the ordered orthonormal basis

$$\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$$

of two qubits where $|0\rangle = |{+}{+}\rangle$, $|1\rangle = |{+}{-}\rangle$, $|2\rangle = |{-}{+}\rangle$, $|3\rangle = |{-}{-}\rangle$. $|{+}{-}\rangle$ are the eigenvectors of σ_z, $\sigma_z |\pm\rangle = \pm |\pm\rangle$ represent the states of each qubit. This specific order is maintained throughout the calculations. The matrices P_1, P_2, σ_+^x, and $\left(\sigma_+^x\right)^*$ (in the basis $|\pm\rangle$ of single qubit) are defined as follows:

$$P_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad P_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{A.1}$$

$$\sigma_+^x = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad \left(\sigma_+^x\right)^* = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \tag{A.2}$$

The expression for $\rho_{\mathrm{S},2}^{\mathrm{in}}$ is given by:

$$
\begin{aligned}
\rho_{\mathrm{S},2}^{\mathrm{in}} &= \frac{1}{4}\mathbb{1}\otimes\mathbb{1} \\
&- \frac{1}{4}\eta_1\left[\cos\left(2F\lambda^2\right)\right]^{N-2}\left(e^{-2iF\lambda^2}P_1 + e^{2iF\lambda^2}P_2\right)\otimes\sigma_+^x + \text{h.c.} \\
&- \frac{1}{4}\eta_1\left[\cos\left(2F\lambda^2\right)\right]^{N-2}\sigma_+^x\otimes\left(e^{-2iF\lambda^2}P_1 + e^{2iF\lambda^2}P_2\right) + \text{h.c.} \\
&+ \frac{1}{4}\left(\eta_2\left[\cos\left(4F\lambda^2\right)\right]^{N-2}\sigma_+^x\otimes\sigma_+^x + \sigma_+^x\otimes(\sigma_+^x)^*\right) + \text{h.c.}
\end{aligned}
\tag{A.3}
$$

We calculate the intermediate terms as follows:

$$
T_1 = \frac{1}{4}\mathbb{1}\otimes\mathbb{1} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\tag{A.4}
$$

$$
e^{-2iF\lambda^2}P_1 + e^{2iF\lambda^2}P_2 = \begin{pmatrix} \cos(2F\lambda^2) & -i\sin(2F\lambda^2) \\ -i\sin(2F\lambda^2) & \cos(2F\lambda^2) \end{pmatrix}
$$

$$
\begin{aligned}
T_2 &= -\frac{1}{4}\eta_1\left[\cos\left(2F\lambda^2\right)\right]^{N-2}\left(e^{-2iF\lambda^2}P_1 + e^{2iF\lambda^2}P_2\right)\otimes\sigma_+^x \\
&= -\frac{1}{4}\eta_1\left[\cos\left(2F\lambda^2\right)\right]^{N-2}\begin{pmatrix} \cos(2F\lambda^2) & -i\sin(2F\lambda^2) \\ -i\sin(2F\lambda^2) & \cos(2F\lambda^2) \end{pmatrix}\otimes\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \\
&= -\frac{1}{8}\eta_1\left[\cos\left(2F\lambda^2\right)\right]^{N-2} \\
&\times \begin{pmatrix} \cos(2F\lambda^2) & -\cos(2F\lambda^2) & -i\sin(2F\lambda^2) & i\sin(2F\lambda^2) \\ \cos(2F\lambda^2) & -\cos(2F\lambda^2) & -i\sin(2F\lambda^2) & i\sin(2F\lambda^2) \\ -i\sin(2F\lambda^2) & i\sin(2F\lambda^2) & \cos(2F\lambda^2) & -\cos(2F\lambda^2) \\ -i\sin(2F\lambda^2) & i\sin(2F\lambda^2) & \cos(2F\lambda^2) & -\cos(2F\lambda^2) \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
T_3 &= T_2^* = -\frac{1}{8}\eta_1 \left[\cos\left(2F\lambda^2\right)\right]^{N-2} \\
&\times \begin{pmatrix} \cos(2F\lambda^2) & \cos(2F\lambda^2) & i\sin(2F\lambda^2) & i\sin(2F\lambda^2) \\ -\cos(2F\lambda^2) & -\cos(2F\lambda^2) & -i\sin(2F\lambda^2) & -i\sin(2F\lambda^2) \\ i\sin(2F\lambda^2) & i\sin(2F\lambda^2) & \cos(2F\lambda^2) & \cos(2F\lambda^2) \\ -i\sin(2F\lambda^2) & -i\sin(2F\lambda^2) & -\cos(2F\lambda^2) & -\cos(2F\lambda^2) \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
T_4 &= -\frac{1}{4}\eta_1 \left[\cos\left(2F\lambda^2\right)\right]^{N-2} \sigma_+^x \otimes \left(e^{-2iF\lambda^2} P_1 + e^{2iF\lambda^2} P_2\right) \\
& e^{-2iF\lambda^2} P_1 + e^{2iF\lambda^2} P_2 = \begin{pmatrix} \cos(2F\lambda^2) & -i\sin(2F\lambda^2) \\ -i\sin(2F\lambda^2) & \cos(2F\lambda^2) \end{pmatrix} \\
T_4 &= -\frac{1}{4}\eta_1 \left[\cos\left(2F\lambda^2\right)\right]^{N-2} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} \cos(2F\lambda^2) & -i\sin(2F\lambda^2) \\ -i\sin(2F\lambda^2) & \cos(2F\lambda^2) \end{pmatrix} \\
T_4 &= -\frac{1}{8}\eta_1 \left[\cos\left(2F\lambda^2\right)\right]^{N-2} \\
&\times \begin{pmatrix} \cos(2F\lambda^2) & -i\sin(2F\lambda^2) & -\cos(2F\lambda^2) & i\sin(2F\lambda^2) \\ -i\sin(2F\lambda^2) & \cos(2F\lambda^2) & i\sin(2F\lambda^2) & -\cos(2F\lambda^2) \\ \cos(2F\lambda^2) & -i\sin(2F\lambda^2) & -\cos(2F\lambda^2) & i\sin(2F\lambda^2) \\ -i\sin(2F\lambda^2) & \cos(2F\lambda^2) & i\sin(2F\lambda^2) & -\cos(2F\lambda^2) \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
T_5 &= T_4^* = -\frac{1}{8}\eta_1 \left[\cos\left(2F\lambda^2\right)\right]^{N-2} \\
&\times \begin{pmatrix} \cos(2F\lambda^2) & i\sin(2F\lambda^2) & \cos(2F\lambda^2) & i\sin(2F\lambda^2) \\ i\sin(2F\lambda^2) & \cos(2F\lambda^2) & i\sin(2F\lambda^2) & \cos(2F\lambda^2) \\ -\cos(2F\lambda^2) & -i\sin(2F\lambda^2) & -\cos(2F\lambda^2) & -i\sin(2F\lambda^2) \\ -i\sin(2F\lambda^2) & -\cos(2F\lambda^2) & -i\sin(2F\lambda^2) & -\cos(2F\lambda^2) \end{pmatrix}
\end{aligned}
$$

$$T_2+T_3+T_4+T_5 = -\frac{1}{2}\eta_1\left[\cos\left(2F\lambda^2\right)\right]^{N-2} \times \begin{pmatrix} \cos(2F\lambda^2) & 0 & 0 & i\sin(2F\lambda^2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i\sin(2F\lambda^2) & 0 & 0 & -\cos(2F\lambda^2) \end{pmatrix} \tag{A.5}$$

$$T_6 = \frac{1}{4}\eta_2\left[\cos\left(4F\lambda^2\right)\right]^{N-2}\sigma_+^x\otimes\sigma_+^x$$

$$\sigma_+^x\otimes\sigma_+^x = \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\otimes\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1 & -1 & -1 & 1\\ 1 & -1 & -1 & 1\\ 1 & -1 & -1 & 1\\ 1 & -1 & -1 & 1\end{pmatrix}$$

$$T_6 = \frac{1}{16}\eta_2\left[\cos\left(4F\lambda^2\right)\right]^{N-2}\begin{pmatrix}1 & -1 & -1 & 1\\ 1 & -1 & -1 & 1\\ 1 & -1 & -1 & 1\\ 1 & -1 & -1 & 1\end{pmatrix}$$

$$T_7 = T_6^* = \frac{1}{16}\eta_2\left[\cos\left(4F\lambda^2\right)\right]^{N-2}\begin{pmatrix}1 & 1 & 1 & 1\\ -1 & -1 & -1 & -1\\ -1 & -1 & -1 & -1\\ 1 & 1 & 1 & 1\end{pmatrix}$$

$$T_6 + T_7 = \frac{1}{8}\eta_2 \left[\cos\left(4F\lambda^2\right)\right]^{N-2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{A.6}$$

$$T_8 = \frac{1}{4}\sigma_+^x \otimes (\sigma_+^x)^*$$

$$\sigma_+^x \otimes (\sigma_+^x)^* = \frac{1}{4} \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

$$T_8 = \frac{1}{16} \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

$$T_9 = T_8^* = \frac{1}{16} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

$$T_8 + T_9 = \frac{1}{8} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{A.7}$$

By substituting (A.4), (A.5), (A.6), and (A.7) into (A.3), we obtain:

$$\rho_{\mathrm{S},2}^{\mathrm{in}} = \begin{pmatrix} \frac{1}{2}-a-c & 0 & 0 & -ib-a \\ 0 & a & a & 0 \\ 0 & a & a & 0 \\ ib-a & 0 & 0 & \frac{1}{2}-a+c \end{pmatrix} \tag{A.8}$$

where

$$\begin{aligned} a &= \frac{1}{8} - \frac{1}{8}\eta_2 \left[\cos\left(4F\lambda^2\right)\right]^{N-2} \\ b &= \frac{1}{2}\eta_1 \left[\cos\left(2F\lambda^2\right)\right]^{N-2} \sin\left(2F\lambda^2\right) \\ c &= \frac{1}{2}\eta_1 \left[\cos\left(2F\lambda^2\right)\right]^{N-2} \cos\left(2F\lambda^2\right) \end{aligned} \tag{A.9}$$

Now, we proceed to calculate the concurrence. First, consider the following

$$\left(\sigma_y \otimes \sigma_y\right) \overline{\rho}_{\mathrm{S},2}^{\mathrm{in}} \left(\sigma_y \otimes \sigma_y\right),$$

where $\overline{\rho}_{\mathrm{S},2}^{\mathrm{in}}$ in the element-wise conjugate of $\rho_{\mathrm{S},2}^{\mathrm{in}}$. Note that

$$\begin{aligned} \left(\sigma_y \otimes \sigma_y\right) &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\ \overline{\rho}_{\mathrm{S},2}^{\mathrm{in}} &= \begin{pmatrix} \frac{1}{2}-a-c & 0 & 0 & ib-a \\ 0 & a & a & 0 \\ 0 & a & a & 0 \\ -ib-a & 0 & 0 & \frac{1}{2}-a+c \end{pmatrix} \end{aligned}$$

Using above definitions,

$$
\begin{aligned}
(\sigma_y \otimes \sigma_y)\,\overline{\rho}^{\text{in}}_{\text{S},2}\,(\sigma_y \otimes \sigma_y) &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2}-a-c & 0 & 0 & ib-a \\ 0 & a & a & 0 \\ 0 & a & a & 0 \\ -ib-a & 0 & 0 & \frac{1}{2}-a+c \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} ib+a & 0 & 0 & -\frac{1}{2}+a-c \\ 0 & a & a & 0 \\ 0 & a & a & 0 \\ -\frac{1}{2}+a+c & 0 & 0 & -ib+a \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}-a+c & 0 & 0 & -ib-a \\ 0 & a & a & 0 \\ 0 & a & a & 0 \\ ib-a & 0 & 0 & \frac{1}{2}-a-c \end{pmatrix}
\end{aligned}
$$

Consequently, the quantity $\xi(\rho^{\text{in}}_{\text{S},2})$ is

$$
\begin{aligned}
&\xi(\rho^{\text{in}}_{\text{S},2}) = \rho^{\text{in}}_{\text{S},2}\,(\sigma_y \otimes \sigma_y)\,\overline{\rho}^{\text{in}}_{\text{S},2}\,(\sigma_y \otimes \sigma_y) \\
&= \begin{pmatrix} \frac{1}{2}-a-c & 0 & 0 & -ib-a \\ 0 & a & a & 0 \\ 0 & a & a & 0 \\ ib-a & 0 & 0 & \frac{1}{2}-a+c \end{pmatrix} \begin{pmatrix} \frac{1}{2}-a+c & 0 & 0 & -ib-a \\ 0 & a & a & 0 \\ 0 & a & a & 0 \\ ib-a & 0 & 0 & \frac{1}{2}-a-c \end{pmatrix} \\
&= \begin{pmatrix} \left(\frac{1}{2}-a\right)^2 - c^2 + b^2 + a^2 & 0 & 0 & 2(-ib-a)\left(\frac{1}{2}-a-c\right) \\ 0 & 2a^2 & 2a^2 & 0 \\ 0 & 2a^2 & 2a^2 & 0 \\ 2(ib-a)\left(\frac{1}{2}-a+c\right) & 0 & 0 & \left(\frac{1}{2}-a\right)^2 - c^2 + b^2 + a^2 \end{pmatrix}
\end{aligned}
$$

We set for shortness

$$\begin{aligned}
\alpha &= \left(\frac{1}{2}-a\right)^2 - c^2 + b^2 + a^2 \\
\beta &= 2(-ib-a)\left(\tfrac{1}{2}-a-c\right) \\
\gamma &= 2(ib-a)\left(\tfrac{1}{2}-a+c\right) \\
\delta &= 2a^2
\end{aligned}$$

Then, $\xi(\rho^{\text{in}}_{\text{S},2})$ takes the form

$$\xi(\rho^{\text{in}}_{\text{S},2}) = \begin{pmatrix} \alpha & 0 & 0 & \beta \\ 0 & \delta & \delta & 0 \\ 0 & \delta & \delta & 0 \\ \gamma & 0 & 0 & \alpha \end{pmatrix}$$

The eigenvalues of $\xi(\rho^{\text{in}}_{\text{S},2})$ are calculated to be

$$\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 &= 2\delta = 4a^2 \\
\lambda_3 &= \alpha + \sqrt{\beta\gamma} \\
&= \left(\frac{1}{2}-a\right)^2 - c^2 + b^2 + a^2 + 2\sqrt{\left(\left(\frac{1}{2}-a\right)^2 - c^2\right)(b^2+a^2)} \\
&= \left(\sqrt{\left(\tfrac{1}{2}-a\right)^2 - c^2} + \sqrt{b^2+c^2}\right)^2 \\
\lambda_4 &= \alpha - \sqrt{\beta\gamma} \\
&= \left(\frac{1}{2}-a\right)^2 - c^2 + b^2 + a^2 - 2\sqrt{\left(\left(\frac{1}{2}-a\right)^2 - c^2\right)(b^2+a^2)} \\
&= \left(\sqrt{\left(\tfrac{1}{2}-a\right)^2 - c^2} - \sqrt{b^2+c^2}\right)^2
\end{aligned}$$

where a, b, c were defined in (A.9). We have numerically verified (2, 3) that for the considered value ranges the quantity $\left(\frac{1}{2} - a\right)^2 - c^2$ is indeed a positive. Thus, validating the expressions for λ_3 and λ_4.