Spatio-Temporal Dynamics of Some Reaction-Diffusion Population Models in Heterogeneous Environments

by

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Abstract

Spatial and temporal evolutions are very important topics in epidemiology and ecology. This thesis is devoted to the study of global dynamics of some reaction-diffusion models incorporating environmental heterogeneities.

As biological invasions significantly impact ecology and human society, how invasive species' growth and spatial spread interact with the environment becomes a significant challenging problem. We start with an impulsive time-space periodic model to describe a single species with a birth pulse in the reproductive stage in Chapter 2. In-host viral infections commonly involve hepatitis B virus (HBV), hepatitis C virus (HCV), and human immunodeficiency virus (HIV). To explore the effects of the spread heterogeneity on the spread of within-host virus, we propose a time-delayed nonlocal reaction-diffusion model and obtain the threshold-type results in terms of the basic reproduction ratio in Chapter 3. In Chapter 4, we then explore the existence and nonexistence of traveling wave solutions for such a non-monotone system on an unbounded domain, and show that there is a minimum wave speed for traveling waves connecting the infection-free equilibrium and the endemic equilibrium. Mosquito-borne diseases are transmitted by the bite of infected mosquitoes, including Zika, West Nile, Chikungunya, dengue, and malaria. To investigate the effects of spatial and temporal heterogeneity on the spread of the Chikungunya virus, we develop a nonlocal periodic reaction-diffusion model of Chikungunya disease with periodic time delays in Chapter 5. We further establish two threshold-type results regarding the global dynamics of mosquito growth and disease transmission, respectively. At the end of this thesis, a brief summary and some future works are presented.

To my dearest family

Lay summary

The mathematical modeling and analysis of species growth, within-host virus infections and transmissions, and mosquito-borne diseases, are essential topics in epidemiology and ecology. This thesis is devoted to the study of the global dynamics of some reaction-diffusion population models with environmental heterogeneities.

We start with a time-space periodic population growth model with impulsive birth. We explore the dynamics of the model in both unbounded and bounded spatial domains, respectively. We conduct numerical simulations to verify our analytical results and illustrate some interesting findings. Moreover, we present a time-delayed nonlocal reaction-diffusion model of within-host viral infections. We derive the basic reproduction ratio of the system and use it to present threshold-type results. Numerically, we illustrate the analytical results. Furthermore, we investigate the dynamics of such a non-monotone system on an unbounded domain and implement numerical simulations to illustrate the long-term behavior of solutions. Finally, we develop a nonlocal periodic reaction-diffusion model of Chikungunya disease with periodic time delays, and obtain two threshold-type results on the global dynamics for the growth of mosquitoes and the disease transmission, respectively. We also carry out numerical simulations for the Chikungunya transmission in Ceará, Brazil, to investigate the effects of spatial heterogeneity on the disease transmission.

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Statement of contribution

Chapters 2–5 of this thesis consist of the following papers:

Chapter 2: Zhimin Li, Xiao-Qiang Zhao, A time-space periodic population growth model with impulsive birth, Zeitschrift für angewandte Mathematik und Physik, 75, (2024), 83.

Chapter 3: Zhimin Li, Xiao-Qiang Zhao, Global dynamics of a time-delayed nonlocal reaction-diffusion model of within-host viral infections, Journal of Mathematical Biology, 88 (2024), 38.

Chapter 4: Zhimin Li, Xiao-Qiang Zhao, Traveling waves for a time-delayed nonlocal reaction-diffusion model of within-host viral infections, Journal of Differential Equations, Revised.

Chapter 5: Zhimin Li, Xiao-Qiang Zhao, Global dynamics of a nonlocal periodic reaction-diffusion model of Chikungunya disease, Journal of Dynamics and Differential Equations, 2023. https://doi.org/10.1007/s10884-023-10267-1

The work of the above papers was performed by the author under the supervision of Professor Xiaoqiang Zhao.

List of symbols

Symbols	Definitions
α	the Kuratowski measure of noncompactness
$\mathcal{C}$	the set of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}$
Ω	the bounded spatial habitat $\Omega \subset \mathbb{R}^m$ with smooth boundary $\partial\Omega$
$\bar{\Omega}$	the closure of the set Ω
$C(\bar{\Omega}, \mathbb{R}^m)$	the set of continuous functions from $\bar{\Omega}$ to the m-dimensional real number space $\mathbb{R}^m$
Δ	the Laplacian operator
∇	the gradient operator
$Df(a)$	the Fréchet derivative of f at a
Γ	the Green function
ν	the outward normal vector to $\partial\Omega$
$\mathscr{R}_0$	the basic reproduction number (ratio)
X^+	the positive cone of the ordered Banach space X
$\omega(x)$	the omega limit set of x
$\omega(\Phi)$	the exponential growth bound of the evolution family $\Phi(t, s)$

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Chapter 1

Preliminaries

In this chapter, we present some terminologies and known results which will be employed throughout this thesis. They are involved in the study of global attractor, chain transitivity, uniform persistence, monotone dynamics, and the theory of the basic reproduction ratio.

1.1 Global attractor and chain transitivity

Let X be a metric space with metric d and $f : X \to X$ a continuous map. For a nonempty invariant set M (i.e., $f(M) = M$), the set $W^s(M) := \{x \in X : \lim_{n\to\infty} d(f^n(x), M) = 0\}$ is called the stable set of M. The omega limit set of x is defined as

$$\omega(x) = \{y \in X : f^{n_k}(x) \to y, \text{ for some } n_k \to +\infty\}.$$

A negative orbit through $x = x_0$ is a sequence $\gamma^-(x) = \{x_k\}_{-\infty}^{0}$ such that $f(x_{k-1}) = x_k$ for integers $k \leq 0$. If $\gamma^+(x) = \{f^n(x) : n \geq 0\}$ is precompact (i.e., it is contained in a compact set), then $\omega(x)$ is nonempty, compact, and invariant [151, Section 1.1].

Recall that for any subsets $A, B \subseteq X$, we define $d(x, A) := \inf_{y\in A} d(x, y)$ and $\delta(B, A) := \sup_{x\in B} d(x, A)$. The Kuratowski measure of noncompactness, α, is defined as

$$\alpha(B) = \inf\{r : B \text{ has a finite cover of diameter } \leq r\},$$

for any bounded set B of X. A continuous map $f : X \to X$ is said to be compact (completely continuous) if f maps any bounded set to a precompact set in X.

Lemma 1.1.1. [151, Lemma 1.1.2] *The following statements are valid:*

(a) Let $I \subseteq [0, \infty)$ be unbounded, and $\{A_t\}_{t\in I}$ be a nonincreasing family of nonempty closed subset (i.e., $t \leq s$ implies $A_s \subseteq A_t$). Assume that $\alpha(A_t) \to 0$ as $t \to \infty$. Then $A_\infty = \cap_{t\geq 0} A_t$ is nonempty and compact, and $\delta(A_t, A_\infty) \to 0$ as $t \to \infty$.

(b) For each $A \subseteq X$ and $B \subseteq X$, we have $\alpha(B) \leq \alpha(A) + \delta(B, A)$.

For a subset $B \subseteq X$, let $\gamma^+(B) := \cup_{m\geq 0} f^m(B)$ be the positive orbit of B for f, and

$$\omega(B) := \cap_{n\geq 0}\overline{\cup_{m\geq n} f^m(B)}$$

the omega limit set of B. We say that a subset $A \subseteq X$ attracts a subset $B \subseteq X$ for f if $\lim_{n\to\infty} \delta(f^n(B), A) = 0$.

Lemma 1.1.2. [151, Lemma 1.1.3] *Let B be a subset of X and assume that there exists a compact subset C of X which attracts B for f. Then $\omega(B)$ is nonempty, compact, invariant for f and attracts B.*

Definition 1.1.1. *A continuous mapping $f : X \to X$ is said to be point dissipative if there is a bounded set B_0 in X such that B_0 attracts each point in X; α-condensing (α-contraction of order k, $0 \leq k < 1$) if f takes bounded sets to bounded sets and $\alpha(f(B)) < \alpha(B)$ ($\alpha(f(B)) \leq k\alpha(B)$) for any nonempty closed bounded set $B \subseteq X$ with $\alpha(B) > 0$; α-contracting if $\lim_{n\to\infty} \alpha(f^n(B)) = 0$ for any bounded subset $B \subseteq X$; asymptotically smooth if for any nonempty closed bounded set $B \subseteq X$ for which $f(B) \subseteq B$, there is a compact set $J \subseteq B$ such that J attracts B.*

Definition 1.1.2. *A subset $A \subseteq X$ is said to be an attractor for f if A is nonempty, compact, and invariant, and A attracts some open neighborhood U of itself; a global attractor for $f : X \to X$ is an attractor that attracts every point in X; and a strong global attractor for f if A attracts every bounded subset of X.*

Definition 1.1.3. *Let $A \subseteq X$ be a nonempty invariant set. A is said to be internally chain transitive if for any $a, b \in A$ and any $\epsilon > 0$, there is a finite sequence $x_1, ..., x_m$ in A with $x_1 = a, x_m = b$ such that $d(f(x_i), x_{i+1}) < \epsilon, 1 \leq i \leq m - 1$. The sequence $\{x_1, ..., x_m\}$ is called an ϵ-chain in A connecting a and b.*

Lemma 1.1.3. [151, Lemma 1.2.1] *Let $f : X \to X$ be a continuous map. Then the omega (alpha) limit set of any precompact positive (negative) orbit is internally chain transitive.*

Theorem 1.1.1. [151, Theorem 1.2.1] *Let A be an attractor and C a compact internally chain transitive set for $f : X \to X$. If $C \cap W^s(A) \neq \emptyset$, then $C \subseteq A$.*

Theorem 1.1.2. [151, Theorem 1.1.3] *Let $f : X \to X$ be a continuous map. Assume that f is point dissipative on X, and one of the following condition holds:*

- *(i) f^{n_0} is compact for some integer $n_0 \geq 1$, or*
- *(ii) f is asymptotically smooth, and for each bounded set $B \subseteq X$, there exists $k = k(B) \geq 0$ such that the positive orbit $\gamma^+(f^k(B))$ is bounded.*

Then there is a strong global attractor A for f.

1.2 Uniform persistence and coexistence states

Let $f : X \to X$ be a continuous map and $X_0 \subseteq X$ an open set. Define $\partial X_0 := X \setminus X_0$, and $M_\partial := \{x \in \partial X_0 : f^n(x) \in \partial X_0, n \geq 0\}$, which may be empty.

Theorem 1.2.1. [151, Theorem 1.3.1 and Remark 1.3.1] *Assume that*

(C1) $f(X_0) \subseteq X_0$ *and* f *has a global attractor* A*;*

(C2) There exists a finite sequence $\mathcal{M} = \{M_1, ..., M_k\}$ *of disjoint, compact, and isolated invariant sets in* ∂X_0 *such that*

- *(a)* $\Omega(M_\partial) := \cup_{x \in M_\partial} \omega(x) \subseteq \cup_{i=1}^{k} M_i$*;*
- *(b) No subset of* $\mathcal{M}$ *forms a cycle in* ∂X_0*;*
- *(c) Each* M_i *is isolated in* X*;*
- *(d)* $W^s(M_i) \cap X_0 = \emptyset$ *for each* $1 \leq i \leq k$*.*

Then there exists $\delta > 0$ *such that for any compact internally chain transitive set* L *with* $L \not\subseteq M_i$ *for all* $1 \leq i \leq k$*, we have* $\inf_{x \in L} d(x, \partial X_0) > \delta$*.*

Definition 1.2.1. *A function* $f : X \to X$ *is said to be uniformly persistent with respect to* $(X_0, \partial X_0)$ *if there exists* $\eta > 0$ *such that* $\liminf_{n \to \infty} d(f^n(x), \partial X_0) \geq \eta$ *for all* $x \in X_0$*. If " inf" in this inequality is replaced with " sup", then* f *is said to be weakly uniformly persistent with respect to* $(X_0, \partial X_0)$*.*

Theorem 1.2.2. [151, Theorem 1.3.6] *Assume that* f *is asymptotically smooth and uniformly persistent with respect to* $(X_0, \partial X_0)$*, and that* f *has a global attractor* A*. Then* $f : (X_0, d) \to (X_0, d)$ *has a global attractor* A_0*. Moreover, if a subset* B *of* X_0 *has the property that* $\gamma^+(f^k(B))$ *is strongly bounded for some* $k \geq 0$*, then* A_0 *attracts* B *for* f*.*

In order to establish the existence of coexistence steady state (i.e., the fixed point in X_0) for uniformly persistent dynamical systems, we always assume that X is a closed subset of a Banach space E, and that X_0 is a convex and relatively open subset of X. Then $\partial X_0 := X \setminus X_0$ is relatively closed in X. Given a set $A \subset E$, let $co(A)$ be the convex hull of A and $\overline{co}(A)$ the closed convex hull of A, respectively.

Theorem 1.2.3. [151, Theorem 1.3.8] *Assume that* f *is* α*-condensing. If* $f : X_0 \to X_0$ *has a global attractor* A_0*, then* f *has a fixed point* $x_0 \in A_0$*.*

Definition 1.2.2. [151, Definition 1.3.4] *Let* X *be a closed and convex subset of a Banach space* E*, and* $f : X \to X$ *a continuous map. Define* $\widehat{f}(B) = \overline{co}(f(B))$ *for each* $B \subset X$*.* f *is said to be convex* α*-contracting if* $\lim_{n \to \infty} \alpha(\widehat{f^n}(B)) = 0$ *for any bounded subset* $B \subset X$*.*

Theorem 1.2.4. [151, Theorem 1.3.9] *Assume that f is convex α-contracting. If $f : X_0 \to X_0$ has a global attractor A_0, then f has a fixed point $x_0 \in A_0$.*

We have the following result on the existence of coexistence steady states for uniformly persistent systems.

Theorem 1.2.5. [151, Theorem 1.3.10] *Assume that*

(i) f is point dissipative and uniformly persistent with respect to $(X_0, \partial X_0)$;

(ii) One of the following two conditions holds:

(a) f^{n_0} is compact for some integer $n_0 \geq 1$, or

(b) Positive orbits of compact subsets of X are bounded;

(iii) Either f is α-condensing or f is convex α-contracting.

Then $f : X_0 \to X_0$ admits a global attractor A_0, and f has a fixed point in A_0.

Remark 1.2.1. *Theorems 1.2.3 and 1.2.5 are still valid if we assume that X is an open subset of a Banach space E and $f : X \to X$ is α-condensing or convex α-contracting.*

For an autonomous semiflow $\Phi(t) : X \to X, t \geq 0$, we have the following result.

Theorem 1.2.6. [151, Theorem 1.3.11] *Let $\Phi(t)$ be a continuous-time semiflow on X with $\Phi(t)(X_0) \subset X_0$ for all $t \geq 0$. Assume that either $\Phi(t)$ is α-condensing for each $t > 0$, or $\Phi(t)$ is convex α-contracting for each $t > 0$, and that $\Phi(t) : X_0 \to X_0$ has a global attractor A_0. Then $\Phi(t)$ has an equilibrium $x_0 \in A_0$, i.e., $\Phi(t)x_0 = x_0, \forall t \geq 0$.*

Let X be a complete metric space with metric d, and let $\omega > 0$. A family of mappings $\Phi(t) : X \to X, t \geq 0$, is called an ω-periodic semiflow on X if it admits the following properties:

(i) $\Phi(0) = I$, where I is the identity map on X;

(ii) $\Phi(t + \omega) = \Phi(t) \circ \Phi(\omega), \forall t \geq 0$;

(iii) $\Phi(t)x$ is continuous in $(t, x) \in [0, \infty) \times X$.

The mapping $\Phi(\omega)$ is called the Poincaré map associated with this periodic semiflow. In particular, if above (ii) holds for any $\omega > 0$, $\Phi(t)$ is called an autonomous semiflow.

Definition 1.2.3. [151, Definition 3.1.1] *A periodic semiflow $\Phi(t)$ is said to be uniformly persistent with respect to $(X_0, \partial X_0)$ if there exists $\eta > 0$ such that for any $x \in X_0$,*

$$\liminf_{t\to\infty} d(\Phi(t)x, \partial X_0) \geq \eta.$$

Theorem 1.2.7. [151, Theorem 3.1.1] *Let $\Phi(t)$ be an ω-periodic semiflow on X with $\Phi(t)X_0 \subseteq X_0, \forall t \geq 0$, and $\Phi = \Phi(\omega)$. Assume that $\Phi : X \to X$ is asymptotically smooth and has a global attractor. Then uniform persistence of Φ with respect to $(X_0, \partial X_0)$ implies that of $\Phi : X \to X$. More precisely, $\Phi : X_0 \to X_0$ admits a global attractor $A_0 \subseteq X_0$, and the compact set $A_0^* = \cup_{0\leq t\leq\omega}\Phi(t)A_0 \subseteq X_0$ attracts every point in X_0 for $\Phi(t)$ in the sense that $\liminf_{t\to\infty} d(\Phi(t)x, A_0^*) = 0$ for any $x \in X_0$.*

1.3 Monotone and subhomogeneous systems

Definition 1.3.1. (Irreducibility). *A nonnegative matrix $L \geq 0$ is said to be reducible if there exists a permutation matrix P such that:*

$$P^T LP = \begin{pmatrix} X & Y \\ \mathbf{0} & Z \end{pmatrix},$$

where X and Z are square matrices and $\mathbf{0}$ is a rectangular zero block. If L is not reducible, then it is called irreducible.

Theorem 1.3.1. (Perron-Frobenius Theorem) [34, Chapter XIII.3, Theorem 3] *Let $L \geq 0$ be an irreducible matrix. Then there exists a simple positive eigenvalue $r(L) > 0$ of L which has an associated positive eigenvector and which has the highest modulus of any other eigenvalue of L.*

Let E be an ordered Banach space with positive cone P such that $int(P) \neq \emptyset$. For any $x, y \in E$, we write $x \geq y$ if $x - y \in P$; $x > y$ if $x - y \in P \setminus \{0\}$; and $x \gg y$ if $x - y \in int(P)$. If $a < b$, we define $[a, b]_E := \{x \in E : a \leq x \leq b\}$. The cone P is said to be normal if there exists a constant M such that $0 \leq x \leq y$ implies that $\|x\| \leq M\|y\|$.

Definition 1.3.2. *A linear operator $L : E \to E$ is said to be positive if $L(P) \subseteq P$; strongly positive if $L(P \setminus \{0\}) \subseteq int(P)$.*

Definition 1.3.3. *Let E be a Banach space, $K \subseteq E$ be a cone with $K \neq \{0\}$, and L be a positive and bounded linear operator. $r(L)$ is called the principal eigenvalue if there exists some $x \in K \setminus \{0\}$ such that $Lx = r(L)x$.*

Theorem 1.3.2. (Krein-Rutman Theorem) [44, Theorems 7.1 and 7.2] *Assume that a compact operator $L : E \to E$ is positive and $r(L)$ is the spectral radius of L. If*

$r(L) > 0$, then $r(L)$ is an eigenvalue of L with an eigenfunction $x > 0$. Moreover, if L is strongly positive, then $r(L) > 0$ and it is an algebraically simple eigenvalue with an eigenfunction $x \gg 0$; there is no other eigenvalue with the associated eigenfunction $x \gg 0$; $|\lambda| < r(L)$ for all eigenvalues $\lambda \neq r(L)$.

Let L be a positive and bounded linear operator on E. We use $\mathcal{N}(L)$ and $\mathcal{R}(L)$ to denote the null space and range, respectively, of L. Recall that the spectrum set of L is defined as

$$\sigma(L) = \{\lambda \in \mathbb{C} : \lambda I - L \text{ has no bounded inverse}\}.$$

Moreover, $\sigma(L)$ can be written as follows:

$$\begin{aligned}\sigma(L) = &\{\lambda \in \mathbb{C} : \mathcal{N}(\lambda I - L) \neq \{0\}\} \\ &\cup \{\lambda \in \mathbb{C} : \mathcal{N}(\lambda I - L) = \{0\} \text{ and } \overline{\mathcal{R}(\lambda I - L)} \neq X\} \\ &\cup \{\lambda \in \mathbb{C} : \mathcal{N}(\lambda I - L) = \{0\} \text{ and } \mathcal{R}(\lambda I - L) \text{ is not closed}\}.\end{aligned}$$

L is said to be a Fredholm operator if $\mathcal{R}(L)$ is closed and both of *dim* $\mathcal{N}(L)$ and *codim* $\mathcal{R}(L)$ are finite. The Fredholm index is $ind(L) = dim\ \mathcal{N}(L) - codim\ \mathcal{R}(L)$. The definition of the essential spectrum of L is given as follows (see, e.g., [105, Section 7.5]):

$$\sigma_e(L) = \{\lambda \in \sigma(L) : \lambda I - L \text{ is not a Fredholm operator with } ind(\lambda I - L) = 0\}.$$

The spectral radius and essential spectral radius of L are denoted by $r(L)$ and $r_e(L)$, respectively.

Theorem 1.3.3. (Weak version of the Generalized Krein-Rutman Theorem) [93, Corollary 2.2] *Let E be a Banach space with a total cone $K \subseteq E$ (i.e., $E = \overline{K - K}$), and L be a positive and bounded linear operator on E. If the essential spectral radius $r_e(L)$ of L is less than the spectral radius $r(L)$ of L, then there exists some $x \in K \setminus \{0\}$ such that $Lx = r(L)x$.*

Theorem 1.3.4. (Generalized Krein-Rutman Theorem) [93] and [151, Lemma 2.5.2] *Let E be a Banach space, $K \subseteq E$ be a cone with nonempty interior, and $L : E \to E$ be a strongly positive and bounded linear operator. If $r_e(L) < r(L)$, then $r(L)$ is an algebraically simple eigenvalue of L with an eigenvector $v \in int(K)$, and all other eigenvalues of L have their absolute values less than $r(L)$.*

Definition 1.3.4. [151, Definition 2.1.1] *Let U be a subset of E, and $f : U \to U$ a continuous map. The map f is said to be monotone if $x \geq y$ implies that $f(x) \geq f(y)$; strictly monotone if $x > y$ implies that $f(x) > f(y)$; strongly monotone if $x > y$ implies that $f(x) \gg f(y)$.*

Lemma 1.3.1. [137, Lemma 2.1] *Let P be normal, and $S : E \to E$ a continuous and monotone map. If $x^* \in E$ is a locally attractive fixed point of S, then x^* is Lyapunov stable for S.*

Recall that a subset K of E is said to be order convex if $[u,v]_E \subseteq K$ whenever $u,v \in K$ satisfy $u < v$.

Definition 1.3.5. [151, Definition 2.3.1] *Let $U \subseteq P$ be a nonempty, closed and order convex set. A continuous map $f : U \to U$ is said to be subhomogeneous if $f(\lambda x) \geq \lambda f(x)$ for any $x \in U$ and $\lambda \in [0,1]$; strictly subhomogeneous if $f(\lambda x) > \lambda f(x)$ for any $x \in U$ with $x \gg 0$ and $\lambda \in (0,1)$; strongly subhomogeneous if $f(\lambda x) \gg \lambda f(x)$ for any $x \in U$ with $x \gg 0$ and $\lambda \in (0,1)$.*

Theorem 1.3.5. [151, Theorem 2.3.2] *Assume that $f : U \to U$ satisfies either*

(1) f is monotone and strongly subhomogeneous, or

(2) f is strongly monotone and strictly subhomogeneous.

If $f : U \to U$ admits a nonempty compact invariant set $K \subset int(P)$, then f has a fixed point $e \gg 0$ such that every nonempty compact invariant set of f in $int(P)$ consists of e.

Denote the Fréchet derivative of f at $u = a$ by $Df(a)$ if it exists, and let $r(Df(a))$ be the spectral radius of the linear operator $Df(a) : E \to E$.

Theorem 1.3.6. (Threshold dynamics) [151, Theorem 2.3.4] *Let either $V = [0,b]_E$ with $b \gg 0$ or $V = P$. Assume that*

(i) $f : V \to V$ satisfies either

  (1) f is monotone and strongly subhomogeneous, or

  (2) f is strongly monotone and strictly subhomogeneous;

(ii) $f : V \to V$ is asymptotically smooth, and every positive orbit of f in V is bounded;

(iii) $f(0) = 0$, and $Df(0)$ is compact and strongly positive.

Then exists threshold dynamics:

(a) If $r(Df(0)) \leq 1$, then every positive orbit in V converges to 0.

(b) If $r(Df(0)) > 1$, then there exists a unique fixed point $u^* \gg 0$ in V such that every positive orbit in $V \setminus \{0\}$ converges to u^*.

1.4 Abstract functional differential equations

Consider a reaction-diffusion system with delays on a bounded domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega$ smooth:

$$\begin{aligned} &\frac{\partial u^i(t,x)}{\partial t} = d_i \Delta u^i(t,x) + g_i(t,x,u_t^1(\cdot,x),...,u_t^n(\cdot,x)),\ t > a, x \in \Omega, \\ &\alpha_i(x)u^i(t,x) + k_i \frac{\partial u^i(t,x)}{\partial \nu} = \beta_i(t,x),\ t > a, x \in \partial\Omega, \\ &u^i(a+\theta,x) = \phi^i(\theta,x), \ -\tau \leq \theta \leq 0, x \in \Omega, \end{aligned} \tag{1.1}$$

where Δ is the Laplacian operator on Ω, and $\frac{\partial}{\partial \nu}$ is the outward normal derivative on $\partial\Omega$, $a \geq 0$ and $i = 1,...,n$. It is assumed that the coefficients in system (1.1) satisfy the following:

(A1) There is a subset Σ_0 of $\{1,...,n\}$ such that $d_i = 0$ for all $i \in \Sigma_0$ and $d_i > 0$ for all $i \in \Sigma_0^c$;

(A2) $\alpha_i : \bar{\Omega} \to [0,\infty)$ is C^1 and $\beta_i : [0,\infty) \times \bar{\Omega} \to \mathbb{R}$ is C^2 for $i = 1,...,n$;

(A3) If $i \in \Sigma_0^c$ then $k_i = 1$ and if $i \in \Sigma_0$ then $\alpha_i = 0, d_i \equiv 0$, and $\beta_i \equiv 0$.

The underlying assumptions on g are as follows:

(B1) g is continuous from $[0,\infty) \times \bar{\Omega} \times C_\Lambda^n$ into $\mathbb{R}^n$, where

$$C_\Lambda^n = \{\phi \in C([-\tau,0],\mathbb{R}^n) : \phi(\theta) \in \Lambda, \forall -\tau \leq \theta \leq 0\},$$

and note that C_Λ^n is a closed convex subset of $\mathbb{R}^n$;

(B2) For each $R > 0$, there exist $v = v(R) \in (0,1]$ and $L = L(R) \in (0,\infty)$ such that

$$|g_i(t,x,\phi) - g_i(s,x,\psi)| \leq L(|t-s|^v + \Sigma_{j=1}^n |\phi_j - \psi_j|),$$

for all $t,s \in [0,R]$, $x \in \bar{\Omega}$, $\phi,\psi \in C([-\tau,0],\mathbb{R}^n)$ with $\|\phi\|,\|\psi\| \leq R$, and $i = 1,...,n$;

(B3) $\lim_{k\to 0^+} \frac{1}{k}\text{dist}(\phi(0) + kg(t,x,\phi);\Lambda) = 0,\ \forall (t,x,\phi) \in [0,\infty) \times \bar{\Omega} \times C_\Lambda^n$. (The subtangential condition on g relative to the set Λ).

Suppose $v^\pm = (v_i^\pm)_1^n$ are continuously differentiable functions from $[a-\tau,c) \times \bar{\Omega}$ into Λ, where $a < c \leq \infty$, that they are C^2 in $x \in \Omega$, $i \in \Sigma_0^c$, and that

$$v^-(t,x) \leq v^+(t,x),\ [v^-(t,x),v^+(t,x)] \subseteq \Lambda, \forall (t,x) \in [a-\tau,c) \times \bar{\Omega}.$$

Let $g^{\pm} = (g_i^{\pm})_1^n$ be continuous functions from $[0,\infty) \times \bar{\Omega} \times C([-\tau,0],\mathbb{R}^n)$ into $\mathbb{R}^n$ and assume the following differential inequalities are satisfied:

$$\begin{aligned}
&\frac{\partial v_i^+(t,x)}{\partial t} \geq d_i \Delta v_i^+(t,x) + g_i^+(t,x,v_t^+(\cdot,x)),\ a<t<c, x\in\Omega,\\
&\alpha_i(x)v_i^+(t,x) + \frac{\partial v_i^+(t,x)}{\partial \nu} = \beta_i^+(t,x) \geq \beta_i(t,x),\ a<t<c, x\in\partial\Omega,\\
&v_i^+(a+\theta,x) = \phi_i^+(\theta,x) \geq \phi^i(\theta,x),\ \ -\tau\leq\theta\leq 0, x\in\Omega,
\end{aligned} \tag{1.2}$$

and

$$\begin{aligned}
&\frac{\partial v_i^-(t,x)}{\partial t} \leq d_i \Delta v_i^-(t,x) + g_i^-(t,x,v_t^-(\cdot,x)),\ a<t<c, x\in\Omega,\\
&\alpha_i(x)v_i^-(t,x) + \frac{\partial v_i^-(t,x)}{\partial \nu} = \beta_i^-(t,x) \leq \beta_i(t,x),\ a<t<c, x\in\partial\Omega,\\
&v_i^-(a+\theta,x) = \phi_i^-(\theta,x) \leq \phi^i(\theta,x),\ \ -\tau\leq\theta\leq 0, x\in\Omega.
\end{aligned} \tag{1.3}$$

Proposition 1.4.1. [79, Proposition 1] *Suppose that $v^{\pm}$ and $f^{\pm}$ are as in (1.2), (1.3), and those (A1)-(A3) and (B1)-(B3) are satisfied with (B3) replaced by the following:*

(B4) if $k \in \{1,...,n\}$ and $(t,x,\phi) \in [a,c) \times \bar{\Omega} \times C([-\tau,0],\mathbb{R}^n)$ with $v^-(t+\theta,x) \leq \phi(\theta) \leq v^+(t+\theta,x)$ for all $-\tau \leq \theta \leq 0$.

Then

(i) $\phi_k(0) = v_k^+(t,x)$ implies that $g_k(t,x,\phi) \leq g_k^+(t,x,v_t^+(\cdot,x))$, and

(ii) $\phi_k(0) = v_k^-(t,x)$ implies that $g_k(t,x,\phi) \geq g_k^-(t,x,v_t^-(\cdot,x))$.

Then system (1.1) has a unique noncontinuable mild solution u on $[a,b)$, where $b \geq c$, and this solution satisfies

$$v^-(t,x) \leq u(x,t) \leq v^+(t,x),\ \forall (t,x) \in [a,c) \times \bar{\Omega}.$$

Let $\mathbb{X}$ be a Banach space with the norm denoted $\|\cdot\|_{\mathbb{X}}$, let $\tau > 0$, and denote by $X := C([-\tau,0],\mathbb{X})$ the space of all continuous functions $\phi : [-\tau,0] \to \mathbb{X}$ with $\|\phi\|_X = \max_{\theta\in[-\tau,0]} \|\phi(\theta)\|_{\mathbb{X}}$, $\forall \phi \in X$.

Suppose that $T = \{T(t,s) : t \geq s \geq 0\}$ is a family of bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$ that satisfy

(T1) $T(t,t)x \equiv x$ and $T(t,s)T(s,r)x = T(t,r)x$ for all $t \geq s \geq r \geq 0$;

(T2) For each $x \in \mathbb{X}$ the map $(t,s) \to T(t,s)x$ is continuous for $t \geq s \geq 0$;

(T3) There exist numbers $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|T(t,s)\| \equiv \sup\{\|T(t,s)x\|_{\mathbb{X}} : \|x\|_{\mathbb{X}} \leq 1\} \leq Me^{\omega(t-s)},\ \forall t \geq s \geq 0.$$

Such a family T is a C_0 linear evolution system, and if $T(t,s) \equiv T(t-s)$ for $t \geq s \geq 0$, then T is a C_0 linear semigroup.

It is assumed throughout this section that the following assumptions are satisfied:

(H1) D is a closed convex subset of $[-\tau, \infty) \times \mathbb{X}$ and $D(t) \equiv \{x \in \mathbb{X} : (t,x) \in D\}$ is nonempty for each $t \geq -\tau$;

(H2) $\mathcal{D}$ is the closed subset of $[0,\infty) \times X$ defined by $\mathcal{D} \equiv \{(t,\phi) : \phi(\theta) \in D(t+\theta),\ \forall \theta \in [-\tau, 0]\}$. Also, $\mathcal{D}(t) \equiv \{\phi \in X : (t,\phi) \in \mathcal{D}\}, \forall t \geq 0$, and we assume that $\mathcal{D}(t)$ is nonempty for each set $t \geq 0$;

(H3) F is continuous from $D(F)$ into $\mathbb{X}$ where $\mathcal{D} \subset D(F) \subset [0,\infty) \times X$.

If $b > 0$ and u is a continuous function from $[-\tau, b]$ into $\mathbb{X}$ and $t \in [0,b]$, then u_t denotes the member of X defined by $u_t(\theta) = u(t+\theta)$ for $-\tau \leq \theta \leq 0$. Assuming (H1)-(H3), we consider the abstract integral equation

$$\begin{cases} u(t,\phi) = T(t,0)\phi + \displaystyle\int_0^t T(t,s)F(s,u_s)ds, \ 0 \leq t < b, \\ u_0 = \phi, \end{cases} \tag{1.4}$$

where $\phi \in \mathcal{D}(0)$ is given. A function $u : [-\tau, b) \to \mathbb{X}$ is a solution to (1.4) if u is continuous, $u_0 = \phi$, $(t, u_t) \in \mathcal{D}$ for all $t \in [0,b)$, and u satisfies the first equation in (1.4). Define

$$\operatorname{dist}(x, D(t)) \equiv \inf\{\|x-y\|_{\mathbb{X}} : y \in D(t)\} \text{ for } x \in \mathbb{X}, t \geq 0.$$

Corollary 1.4.1. [79, Corollary 4] *Suppose that K is a closed, convex subset of $\mathbb{X}$, and (T1)-(T3) and (H1)-(H3) are satisfied with $D(t) \equiv K$ for all $t \geq 0$. Suppose further that F is locally Lipschitz continuous with respect to t and ϕ and that*

(a) $T(t,s) : K \to K$ *for* $t \geq s \geq 0$, *and*

(b) $\lim_{k\to 0^+} \frac{1}{k}\operatorname{dist}(\phi(0) + kF(t,\phi), K) = 0$ *for* $(t,\phi) \in \mathcal{D}$.

Then (1.4) has a unique noncontinuable solution u on $[0,b)$ from some $b > 0$ and $u(t) \in K$ for all $-\tau \leq t < b$.

Theorem 1.4.1. (Generalized Arzela-Ascoli Theorem) [151] *Let $a < b$ be two real numbers and X be a complete metric space. Assume that a sequence of functions $\{f_n\}$ in $C([a,b], X)$ satisfies the following conditions:*

(i) The family $\{f_n(s)\}_{n\geq 1}$ is uniformly bounded on $[a,b]$;

(ii) For each $s \in [a,b]$, the set $\{f_n(s) : n \geq 1\}$ is precompact in X;

(iii) The family $\{f_n(s)\}_{n\geq 1}$ is equi-continuous on $[a,b]$.

Then $\{f_n\}$ has a convergent subsequence in $C([a,b],X)$, that is, there exists a subsequence of functions $\{f_{n_k}(s)\}$ which converges in X uniformly for $s \in [a,b]$.

1.5 Basic reproduction ratio

The basic reproduction number (ratio) $\mathscr{R}_0$ is one of the most critical concepts in population biology. In epidemiology, $\mathscr{R}_0$ represents the expected number of secondary cases produced by a typical infected individual during its entire period of infectiousness in a completely susceptible population, as stated in [18]. The parameter $\mathscr{R}_0$ serves as a threshold value for measuring the effort required to control the infectious disease. The threshold criterion asserts that the disease can invade if $\mathscr{R}_0 > 1$, whereas it cannot if $\mathscr{R}_0 < 1$. Since the seminal contributions of Diekmann, Heesterbeek and Metz [18], as well as van den Driessche and Watmough [19], there have been numerous studies on the analysis of $\mathscr{R}_0$ for various autonomous infectious disease models. Several works have been conducted on the theory and applications of $\mathscr{R}_0$ for model systems in order to investigate population models in the periodic environment (see, e.g., [5–7,49,52,112,126] and the references therein). Later, the theory of $\mathscr{R}_0$ has been developed for periodic and time-delayed compartmental population models in [150]. Recently, Liang, Zhang and Zhao [67] extended such a theory to abstract functional differential equations whose solution maps may be noncompact. More recently, Zhao [152] established the theory of $\mathscr{R}_0$ for autonomous FDEs in which both the infection (production) and the internal transition have time delays. In this section, we present the theory of $\mathscr{R}_0$ for periodic abstract FDEs developed by [67], the theory of $\mathscr{R}_0$ for autonomous FDEs developed by [152], and the numerical algorithm for the computation of $\mathscr{R}_0$.

1.5.1 Periodic abstract functional differential equations

Let X be a Banach space with a normal and reproducing cone X^+, and $\hat{X}$ be a Banach space with $\hat{X} \hookrightarrow X$. Let $\tau \geq 0$ be a given number, and $C = C([-\tau,0],X)$ equipped with the maximum norm $\|\cdot\|_C$ and a positive cone $C^+ = C([-\tau,0],X^+)$.

Let $(V(t))_{0\leq t\leq w}$ be a family of ω-periodic closed linear operators with the following properties:

(i) $D(V(t)) = \hat{X},\ \forall t \in [0,\omega]$.

(ii) There is some $\lambda_0 \in \mathbb{R}$ such that $\{\lambda \in \mathbb{C} : \Re\lambda \geq \lambda_0\} \subseteq \rho(-V(t)),\ \forall t \in [0,\omega]$ and $\|(\lambda + V(t))^{-1}\|_X \leq \frac{c}{1+|\lambda|},\ \forall \lambda \in \mathbb{C}$ with $\Re\lambda \geq \lambda_0,\ \forall t \in [0,\omega]$.

(iii) $V(\cdot) : [0, \omega] \to \mathcal{L}(\hat{X}, X)$ is Hölder continuous.

Assume that $F(\cdot) : \mathbb{R} \to \mathcal{L}(C, X)$ is ω-periodic, $F(t)\phi$ is continuous jointly in $(t, \phi) \in \mathbb{R} \times C$ and the operator norm of $F(t)$ is uniformly bounded for all $t \in [0, \omega]$.

We consider a linear and ω-periodic functional differential system:

$$\frac{du(t)}{dt} = F(t)u_t - V(t)u(t), \ t \geq 0. \tag{1.5}$$

Let $\Phi(t, s), t \leq s$, be the evolution operators associated with the following system

$$\frac{du(t)}{dt} = -V(t)u(t). \tag{1.6}$$

Let $C_\omega(\mathbb{R}, X)$ be the ordered Banach space of all continuous and ω-periodic functions from $\mathbb{R}$ to X, with the maximum norm. Then we define two linear operators on $C_\omega(\mathbb{R}, X)$ by

$$[Lv](t) = \int_0^\infty \Phi(t, t-s)F(t-s)v(t-s+\cdot)ds, \ \forall t \in \mathbb{R}, v \in C_\omega(\mathbb{R}, X),$$

and

$$[\hat{L}v](t) = F(t) \int_0^\infty \Phi(t+\cdot, t-s+\cdot)v(t-s+\cdot)ds, \ \forall t \in \mathbb{R}, v \in C_\omega(\mathbb{R}, X).$$

Let A and B be two bounded linear operators on C_ω defined by

$$[Av](t) = \int_0^\infty \Phi(t, t-s)v(t-s)ds, \ [Bv](t) = F(t)v_t, \quad \forall t \in \mathbb{R}, \ v \in C_\omega.$$

It then follows that $L = A \circ B$ and $\hat{L} = B \circ A$, and hence L and $\hat{L}$ have the same spectral radius. Furthermore, motivated by the concept of next generation operators (see, e.g., [7, 18, 112, 126, 151]), we define the spectral radius of L and $\hat{L}$ as the basic reproduction number $\mathscr{R}_0 := r(L) = r(\hat{L})$ for periodic system (1.5).

Let $\{U(t, s, \lambda) : t \geq s\}$ be the evolution operators on C of the following linear periodic system with $\lambda \in [0, +\infty)$:

$$\frac{du(t)}{dt} = \lambda F(t)u_t - V(t)u(t), \ t \geq 0. \tag{1.7}$$

Then $U(\omega, 0, 1) = U(\omega, 0)$ be the Poincaré map of system (1.5) on C. We present the following assumptions:

(H1) Each operator $F(t) : C \to X$ is positive in the sense that $F(t)C^+ \subseteq X^+$.

(H2) For any $t \geq s$, $\Phi(t, s)$ is a positive operator on X, and $\omega(\Phi) < 0$.

(H3) The positive linear operator L possesses the principal eigenvalue.

(H4) The positive linear operators $U(\omega, 0, \lambda)$ possesses the isolated principal eigenvalue with finite multiplicity for any $\lambda \in [0, +\infty)$ whenever $r(U(\omega, 0, \lambda)) \geq 1$.

(H5) Either the principal eigenvalue of L is isolated, or there exists an integer $n_0 > 0$ such that L^{n_0} is strongly positive.

(H6) Each operator $\Phi(t, s)$ is compact on X for $t > s$.

Note that under the assumptions (H1) and (H2), (H6) is sufficient for (H3)-(H5) to hold.

Theorem 1.5.1. [67, Theorem 3.7] *Let (H1)-(H5) hold. The following statements are valid:*

(i) $\mathscr{R}_0 = 1$ *if and only if* $r(U(\omega, 0)) = 1$.

(ii) $\mathscr{R}_0 > 1$ *if and only if* $r(U(\omega, 0)) > 1$.

(iii) $\mathscr{R}_0 < 1$ *if and only if* $r(U(\omega, 0)) < 1$.

Thus, $\mathscr{R}_0 - 1$ *has the same sign as* $r(U(\omega, 0)) - 1$.

Theorem 1.5.2. [150, Theorem 2.2] and [67, Theorem 3.8] *If* $\mathscr{R}_0 > 0$, *then* $\lambda = {\mathscr{R}_0}^{-1}$ *is the unique solution of* $r(U(\omega, 0, \lambda)) = 1$.

1.5.2 Autonomous functional differential equations

Let (X, X^+) be an ordered Banach space with the positive cone X^+ being normal and solid. Let $\tau \in \mathbb{R}_+$ be given, and $E = C([-\tau, 0], X)$ equipped with the maximum norm $\| \cdot \|_E$ and the positive cone $E^+ = C([-\tau, 0], X^+)$. Then (E, E^+) is an ordered Banach space. Let $\mathcal{L}(E, X)$ be the space of all bounded and linear operators from E to X. For any $L \in \mathcal{L}(E, X)$, we define $\hat{L} \in \mathcal{L}(X, X)$ by

$$\hat{L}x = L(\hat{x}), \forall x \in X,$$

where $\hat{x}(\theta) = x, \forall \theta \in [-\tau, 0]$. We consider the following abstract autonomous FDE:

$$\frac{du(t)}{dt} = Au(t) + B(u_t), \tag{1.8}$$

where A is a closed linear operator in X with a dense domain $D(A)$ and $B \in \mathcal{L}(E, X)$. Assume that

(H) $A : D(A) \to X$ generates a strongly continuous positive semigroup $T_A(t)$ on X, and B is positive in the sense that $B(E^+) \subseteq X^+$.

By the standard semigroup theory (see, e.g., [79]), it follows that for any $\phi \in E$, system (1.8) has a unique mild solution $u(t, \phi)$ on $[0, \infty)$ with $u_0 = \phi$, and its solution maps generate a positive semigroup $\mathcal{T}(t)$ on E.

Recall that the exponential growth bound of the semigroup $\mathcal{T}(t)$ is defined as

$$\omega(\mathcal{T}) = \inf\{\bar{\omega} : \exists M \geq 1 \text{ such that } \|\mathcal{T}(t)\| \leq M e^{\bar{\omega} t}, \forall t \geq 0\},$$

and the spectral bound of the closed linear operator $A + \hat{B}$ in X is defined as

$$s(A + \hat{B}) = \sup\{\Re\lambda : \lambda \in \sigma(A + \hat{B})\},$$

where $\sigma(A + \hat{B})$ is the spectrum of $A + \hat{B}$.

Theorem 1.5.3. [152, Theorem 3.2] *Assume that A and B satisfy (H), and $T_A(t)$ is compact on X for each $t > 0$. Then $\omega(\mathcal{T})$ and $s(A + \hat{B})$ have the same sign.*

Let $T_N(t)$ be the strongly continuous semigroup generated by a closed linear operator N in X with a dense domain $D(N)$, and $L, \mathcal{F} \in \mathcal{L}(E, X)$. Next, we consider the following autonomous linear FDE:

$$\frac{du(t)}{dt} = Nu(t) + L(u_t) + \mathcal{F}(u_t) = \mathcal{F}(u_t) - \mathcal{V}(u_t), \tag{1.9}$$

where $-V\phi := N\phi(0) + L\phi$. It then follows that (1.9) generates a semigroup $U(t)$ on E, and the autonomous linear FDE

$$\frac{du(t)}{dt} = -\mathcal{V}(u_t) \tag{1.10}$$

generates a semigroup $\Psi(t)$ on E, respectively.

To introduce the basic reproduction number for system (1.9), throughout this section we assume that

(H1) $\mathcal{F}$ is positive in the sense that $\mathcal{F}(E^+) \subseteq X^+$.

(H2) $T_N(t)$ is a positive semigroup, L is positive, and $\omega(\Psi) < 0$.

With the help of [102, Theorem 1], as applied to (1.10), we may use the essentially same arguments as those for [152, Section 2] to derive the next generation operator as $\hat{F} \circ \hat{V}^{-1}$, where $-\hat{V} := N + \hat{L}$. Thus, we define the basic reproduction number for system (1.9) to be $\mathscr{R}_0 = r(\hat{F} \circ \hat{V}^{-1})$.

Theorem 1.5.4. [152, Theorem 3.3] *Assume that $T_N(t)$ is compact on X for each $t > 0$. Then $\mathscr{R}_0 - 1$ and $\omega(U)$ have the same sign.*

Let $U_\lambda(t)$ be the solution semigroup of the following linear system with $\lambda \in [0, +\infty)$:

$$\frac{du(t)}{dt} = \lambda \mathcal{F}(u_t) - \mathcal{V}(u_t). \tag{1.11}$$

Theorem 1.5.5. [152, Theorem 3.4] *Assume that $T_N(t)$ is compact on X for each $t > 0$. If $\mathscr{R}_0 > 0$, then $\lambda = \mathscr{R}_0^{-1}$ is the unique positive solution of $r(U_\lambda(t_0)) = 1$ for any given $t_0 > 0$, and also the unique positive solution of $\omega(U_\lambda) = 0$.*

1.5.3 Numerical computation of $\mathscr{R}_0$

Lemma 1.5.1. [67, Lemma 2.5] *Assume that (C, C^+) is an ordered Banach space with C^+ being normal and $int(C^+) \neq \emptyset$, which is equipped with the norm $\|\cdot\|_C$. Let L be a positive bounded linear operator. Choose $v_0 \in int(C^+)$ and define $a_n = \|Lv_{n-1}\|_C, v_n = \frac{Lv_{n-1}}{a_n}, \forall n \geq 1$. If $\lim_{n\to\infty} a_n$ exists, then $r(L) = \lim_{n\to\infty} a_n$.*

For a periodic system, for any given $\lambda \in [0, +\infty)$, we choose $v_0 \in int(C^+)$ and define

$$a_n = \|U(\omega, 0, \lambda)v_{n-1}\|_C, \ v_n = \frac{U(\omega, 0, \lambda)v_{n-1}}{a_n}, \ \forall n \geq 1.$$

By Lemma 1.5.1, it then follows that if $\lim_{n\to+\infty} a_n$ exists, then $r(U(\omega, 0, \lambda)) = \lim_{n\to+\infty} a_n$. Thus, we can solve $r(U(\omega, 0, \lambda)) = 1$ for λ numerically via the bisection method, which is an approximation of $\frac{1}{\mathscr{R}_0}$ due to Theorem 1.5.2.

Similarly, for an autonomous system, for any given $\lambda \in [0, +\infty)$, we choose $v_0 \in int(C^+)$ and define

$$a_n = \|U_\lambda(t_0)v_{n-1}\|_C, \ v_n = \frac{U_\lambda(t_0)v_{n-1}}{a_n}, \ \forall n \geq 1,$$

for any given $t_0 > 0$. By Lemma 1.5.1, it then follows that if $\lim_{n\to+\infty} a_n$ exists, then $r(U_\lambda(t_0)) = \lim_{n\to+\infty} a_n$. Also, we can solve $r(U_\lambda(t_0)) = 1$ for λ numerically via the bisection method, which is an approximation of $\frac{1}{\mathscr{R}_0}$ due to Theorem 1.5.5.

Chapter 2

A time-space periodic population growth model with impulsive birth

This chapter is devoted to the study of spatio-temporal dynamics for a time-space periodic population growth model with impulsive birth. We first formulate a discrete-time semiflow by the one-year solution map, and obtain a threshold-type result for the semiflow with spatially periodic initial data. Then we establish the existence and the computational formulas of the spreading speeds and prove the coincidence of the spreading speeds with the minimal speeds of spatially periodic traveling waves in the monotone case. Further, we investigate the global dynamics of this model in a bounded spatial domain. Finally, we conduct numerical simulations to verify our analytical results and illustrate some interesting findings.

2.1 Introduction

Reproductive synchrony refers to the temporal clustering of reproductive events among individuals within a restricted time, such as mating, spawning, and births, which is widely recorded in plant and animal populations (see [84, 100]). It may arise simply as an adaptive mechanism of environmental seasonality in climate or resources and the seasonal growth of plant and animal populations. Some modeling studies have been performed to understand the mechanisms of reproductive synchrony and its effects on population growth. To catch the every-other-day egg laying synchrony pattern observed in colonies of gulls, two difference equation models with juvenile-adult structure were proposed and analyzed in [116, 117]. Such disturbances typically occur for a short time; nevertheless, they significantly impact the population density and number. Classical differential equations may not well describe the phenomena where the critical drivers are non-continuous processes. A well-accepted modeling approach that includes the reproductive synchrony is to divide the cycle into two seasons, a nonreproduction season and a short reproduction season. Recently, impulsive differential equations have been introduced to models and characterize these hybrid discrete-continuous processes (see, e.g., [8, 62, 128, 136, 137]), and the impulsive harvesting was also considered in [81, 82].

Suppose the duration of the cycle is fixed to be one year. In that case, the differential system with the impulsive reproduction rate may define an abstract difference equation or a discrete-time semiflow, which describes the evolution of offspring density from this year to the following year [53, 136]. In this vein, a reaction-diffusion system model with impulsive seasonal reproduction and individual dispersal was proposed in [27], and an impulsive integro-differential model was analyzed in [136] to capture the dynamics of an invading species with an impulsive reproduction stage and a nonlocal dispersal stage. Further, the problem of persistence and extinction was addressed in [128] for moving animal species with birth pulse and habitat shift. More recently, the propagation dynamics of species growth with annually synchronous emergence of adults was studied via an impulsive reaction-diffusion model in [8], where the critical domain size was also determined for the persistence of species in a bounded spatial domain with a lethal exterior.

To take into account of climate changes and seasonality, it is more reasonable to incorporate the temporal and spatial variations into these impulsive models. Periodicity is one of the simplest environmental heterogeneities and is a good candidate for approximating complex heterogeneity (see, e.g., [24, 25, 48, 123]). There are also similar situations in the ecological environment. This corresponds to a river with a series of pools and riffles, for example (see, e.g., [75, Section 2.5]). Another example is that in continuous mountains, the living environment of species is closely related to the altitude and light, which make some parameters change periodically between mountains. The purpose of the current paper is to incorporate the spatio-temporal periodic variability into the model proposed in [8] and study the global dynamics of the resulting model by appealing to the theory of monotone evolution systems and the comparison arguments. In the case of an unbounded spatial domain, we obtain sufficient and necessary conditions for the monostable structure of the associated time-space periodic system with spatially periodic initial data, which is different from the spatially homogeneous system discussed in [8]. Moreover, we need to overcome certain difficulties induced by the spatio-temporal periodicity to derive the computational formulas of the spreading speeds. In the case of a bounded spatial domain, we choose to use the fractional power space to deal with the Neumann, Robin type, and Dirichlet boundary conditions in a unified way. Our numerical simulations also give rise to some interesting findings in the presence of spatio-temporal periodicity.

The rest of the chapter is organized as follows. In section 2.2, we formulate a time-space periodic reaction-diffusion model with an annual impulsive maturation emergence term to describe the spatial evolution of adult population density, and show that solutions of this model generate a discrete-time semiflow. In section 2.3, we obtain a threshold-type result for the semiflow with spatially periodic initial data, establish the existence and the computational formulas of the spreading speeds in both monotone and non-monotone cases, and further show that in the monotone case, the spreading speeds coincide with the minimal speeds of spatially periodic traveling waves. In section 2.4, we study the global dynamics of the model in a bounded spatial domain. Numerical simulations are conducted to interpret the obtained analytical results, and a brief discussion concludes the chapter.

2.2 Model formulation

Let $u_n(t,x)$ and $v_n(t,x)$ be the density of adult population and immature population, respectively, at time $t \in [0,1]$ and location $x \in \mathbb{R}$ within year $n \in \mathbb{N}$. Then the adult population densities $N_n(x)$ and $N_{n+1}(x)$ at the beginning of year n and year $n+1$ are

$$N_n(x) = u_n(0,x) \text{ and } N_{n+1}(x) = u_n(1,x) = u_{n+1}(0,x).$$

Assume that the immature individuals are reproduced at density $g(x, N_n(x))$ at the beginning of n-th year and location x, and hence, $v_n(0,x) = g(x, N_n(x))$.

The birth function $g(x,N)$ can be chosen as $g_1(x,N) = \frac{p(x)N}{q(x)+N}$ or $g_2(x,N) = Ne^{r(x)-k(x)N}$, where $p(x)$ and $q(x)$ are positive spatial periodic functions with the same period, and $r(x)$ and $k(x)$ are also positive spatial periodic functions with the same period. As mentioned in Section 2.1, this corresponds to a river with a series of pools and riffles (see, e.g., [75, Section 2.5]). When p, q, r and k are positive constants, $\frac{pN}{q+N}$ and Ne^{r-kN} are the Beverton-Holt function and the Ricker function, respectively. Note that functions g_1 and g_2 are monotone and non-monotone with respect to N, respectively.

Assume that these immature individuals develop into the adult stage after time τ for $\tau \in (0,1]$. The synchronized maturation in the n-th year, the adult population density $u_n(t,x)$ has an abrupt increase at time $t = \tau$ and location x, that is,

$$u_n(\tau^+, x) = u_n(\tau^-, x) + R(x; N_n),$$

where $R(x; N_n)$ describes the density of newly emerging matured individuals at time t in the n-th year. Assume that the dispersal is symmetric and follows Fick's diffusion law. Then the evolution of the adult $u_n(t,x)$ is governed by

$$\frac{\partial u_n}{\partial t} = \frac{\partial}{\partial x}(D_M(t,x)\frac{\partial u_n}{\partial x}) + f(t,x,u_n),\ 0 < t \leq 1,$$

on other time instances of one year, where $D_M(t,x)$ is the random diffusion rate and $f(t,x,u)$ is the death rate function. The evolution of the immature $v_n(t,x)$ is governed by

$$\frac{\partial v_n}{\partial t} = \frac{\partial}{\partial x}(D_I(t,x)\frac{\partial v_n}{\partial x}) - d_I(t,x)v_n,$$

where $D_I(t,x)$ and $d_I(t,x)$ are the random diffusion rate and the natural death rate, respectively. Let $T(t,s), t \geq s$, be the evolution family on $BC(\mathbb{R},\mathbb{R})$ associated with the linear reaction-diffusion equation

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}(D_I(t,x)\frac{\partial v}{\partial x}) - d_I(t,x)v.$$

This implies that the density of newly emerging matured individuals at location x and time $n+\tau$ of the n-th year is

$$R(x;N_n) = [T(\tau,0)g(\cdot,N_n)](x).$$

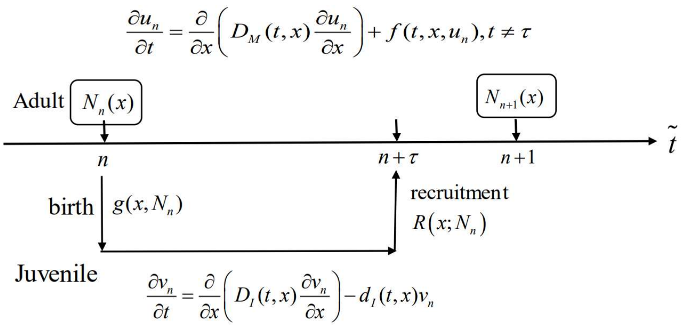


Figure 2.1: The schematic diagram of the evolution of population dynamics for the adult $u_n(t,x)$ and juvenile $v_n(t,x)$ at time $n+t$ and location x, here $t \in [0,1]$ and n represents the n-th year.

Accordingly, we have the following time-space periodic evolution system on the adult population density with an annual impulsive maturation emergence:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = \dfrac{\partial}{\partial x}(D_M(t,x)\dfrac{\partial u_n}{\partial x}) + f(t,x,u_n), \ 0 < t \leq 1, t \neq \tau, x \in \mathbb{R}, n \in \mathbb{N}, \\ u_n(t^+,x) = u_n(t,x) + R(x;N_n), \ t = \tau, \\ u_n(0,x) = N_n(x), \\ N_{n+1}(x) = u_n(1,x), \end{cases} \tag{2.1}$$

with the initial data $u_0(0,x) = N_0(x)$ for any $x \in \mathbb{R}$. The schematic diagram of the evolution of the adult and juvenile populations is described in Figure 2.1. We further assume that:

(H1) Both functions $D_M(t,x)$ and $D_I(t,x)$ are in $C^{\frac{\nu}{2},\nu+1}(\mathbb{R}_+ \times \mathbb{R})$, $d_I(t,x)$ is in $C^{\frac{\nu}{2},\nu}(\mathbb{R}_+ \times \mathbb{R})$ for some $\nu \in (0,1)$, and these three functions are 1-periodic in time t and L-periodic in x for some $L > 0$; there exists a number $\alpha > 0$ such that $D_i(t,x) \geq \alpha$ $(i = M, I)$ and $d_I(t,x) > 0$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.

(H2) $f(t,x,u)$ is in $C^{\frac{\nu}{2},\nu,1}(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+)$, 1-periodic in time t and L-periodic in x, $f(t,x,0) = 0 > \partial_u f(t,x,0)$, and $\frac{f(t,x,u)}{u}$ is strictly decreasing in u.

(H3) $g(x,N)$ is locally Lipschitz continuous on $\mathbb{R}\times\mathbb{R}_+$ and is L-periodic in x, $g(x,0) = 0 < \partial_N g(x,0)$, $g(x,N) > 0$ for all $N > 0$, and $\frac{g(x,N)}{N}$ is nonincreasing for N.

Clearly, (H1) is a standard assumption addressing the time-space periodicity and random diffusion. (H2) and (H3) imply that the functions f and g are subhomogeneous in $u > 0$, which are crucial for us to prove the existence and global attractivity of the positive periodic solution.

Let $M(t,s), t \geq s$, be the evolution family on $BC(\mathbb{R},\mathbb{R})$ associated with

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(D_M(t,x)\frac{\partial u}{\partial x}) + f(t,x,u).$$

For adult density $\phi_n(x)$ at location x at the beginning of the n-th year, the distribution at time τ in the same year is $[M(\tau,0)\phi_n](x)$, and at time τ^+, it becomes

$$[M(\tau,0)\phi_n + T(\tau,0)g(\cdot,\phi_n)](x).$$

After that, the population continues to evolve from time τ and to time 1, and the distribution at the end of the year becomes

$$[M(1,\tau)[M(\tau,0)\phi_n + T(\tau,0)g(\cdot,\phi_n)]](x).$$

Therefore, the time-one solution map of system (2.1) is

$$Q[\phi](x) = [M(1,\tau)[M(\tau,0)\phi + T(\tau,0)g(\cdot,\phi)]](x). \tag{2.2}$$

For each time instant $\tilde{t} \geq 0$, there is a unique decomposition $\tilde{t} = \lfloor \tilde{t} \rfloor + t$, where $t \in [0,1)$ and $\lfloor \tilde{t} \rfloor$ denotes the largest integer less than or equal to $\tilde{t}$. Thus, the time-$\tilde{t}$ solution map of system (2.1) can be expressed as

$$\begin{aligned}
&\Phi_{\tilde{t}}[\phi](x) \\
&= \begin{cases} \left[M(\tilde{t},\lfloor \tilde{t} \rfloor)Q^{\lfloor \tilde{t} \rfloor}[\phi]\right](x), 0 \leq \tilde{t} - \lfloor \tilde{t} \rfloor \leq \tau, \\ \left[M(\tilde{t},\lfloor \tilde{t} \rfloor + \tau)[M(\tau,0)Q^{\lfloor \tilde{t} \rfloor}[\phi] + T(\tau,0)(g(\cdot,Q^{\lfloor \tilde{t} \rfloor}[\phi]))]\right](x), \tau < \tilde{t} - \lfloor \tilde{t} \rfloor < 1, \end{cases}
\end{aligned}$$

with the initial data $\phi(x) = N_0(x), \forall x \in \mathbb{R}$. Clearly, $\Phi_{\tilde{t}} \circ \Phi_1 = \Phi_{\tilde{t}+1}$ for all $\tilde{t} \geq 0$. Note that $\Phi_{\tilde{t}}[\phi]$ is continuous in $\tilde{t} \in \mathbb{R}_+ \backslash \{n + \tau : n \in \mathbb{N}\}$ for each given ϕ, and $\Phi_{\tilde{t}}[\phi]$ is continuous in ϕ uniformly for $\tilde{t}$ in any bounded interval. Replacing the traditional joint continuity in (t,ϕ) with the aforementioned two continuity properties, we can regard $\{\Phi_{\tilde{t}}\}_{\tilde{t}\geq 0}$ as a time-one periodic semiflow in a weak sense.

Note that the Poincaré map of (2.1) is exactly $\Phi_1 = Q$. By [151, Section 3.1], it then follows that the evolution dynamics of system (2.1) can be investigated via the following discrete-time recursion

$$N_{n+1}(x) = Q[N_n](x),\ x \in \mathbb{R}, n \in \mathbb{N}. \tag{2.3}$$

Thus, we will focus on the evolution dynamics of the discrete-time semiflow $\{Q^n\}_{n\in\mathbb{N}}$ associated with system (2.3) in the unbounded and bounded domains, respectively.

2.3 Spreading speeds and traveling waves

In this section, we first study the spreading speeds and the spatially L-periodic traveling waves of system (2.3) in the monotone case, and then investigate the spreading speeds of this system in the non-monotone case by comparison arguments.

2.3.1 The monotone case of $g(x, N)$

We first present the threshold dynamics of system (2.3) with spatially periodic initial conditions, and then investigate the spreading speeds and spatially periodic traveling waves.

Let E^{per} be the set of all continuous and L-periodic functions from $\mathbb{R}$ to $\mathbb{R}$ with the maximum norm $\|\cdot\|_{E^{per}}$, and $E_+^{per} = \{\psi \in E^{per} : \psi(x) \geq 0, \forall x \in \mathbb{R}\}$. Then (E^{per}, E_+^{per}) is a strongly ordered Banach space. We consider the following discrete-time system associated with (2.3):

$$N_n(x) = Q^n[N_0](x),\ N_0(x) = \phi \in E_+^{per}, \tag{2.4}$$

where $\phi(x + L) = \phi(x), \forall x \in \mathbb{R}$. It follows from the conditions (H1)-(H3) and the monotonicity of function g that Q is monotone and strongly subhomogeneous (see [151]). By the definition of Q and the chain rule, we can compute the Fréchet derivative $DQ(0)$ on E^{per} as follows

$$\begin{aligned} DQ(0) &= DM(1,\tau)(M(\tau,0)(0) + T(\tau,0)g(\cdot,0))[DM(\tau,0)(0) + T(\tau,0)\partial_N g(\cdot,0)] \\ &= DM(1,\tau)(0)[DM(\tau,0)(0) + T(\tau,0)\partial_N g(\cdot,0)] \\ &= \bar{M}(1,\tau)[\bar{M}(\tau,0) + T(\tau,0)\partial_N g(\cdot,0)], \end{aligned}$$

where $\bar{M}(t,s), t \geq s$, is the evolution family on E^{per} associated with the linear reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(D_M(t,x)\frac{\partial u}{\partial x}) + \partial_u f(t,x,0)u.$$

It then follows that the linear discrete-time recursion is

$$N_{n+1}(x) = DQ(0)[N_n](x),\ N_0(x) = \phi \in E_+^{per}, x \in \mathbb{R}, n \in \mathbb{N}, \tag{2.5}$$

that is,

$$N_{n+1}(x) = DQ(0)[N_n](x) = \bar{M}(1,\tau)\left[\left[\bar{M}(\tau,0)(N_n)\right] + T(\tau,0)\left[\partial_N g(\cdot,0)N_n\right]\right](x), x \in \mathbb{R}.$$

Clearly, Q is compact in E_+^{per}, and $DQ(0)$ is compact and strongly positive in E^{per}. Let $r(DQ(0))$ be the spectral radius of $DQ(0)$.

As a straightforward consequence of Lemma 1.3.1 and Theorem 1.3.6, we have the following threshold-type result for system (2.3) with spatially periodic initial data.

Proposition 2.3.1. *Let $N_n(x,\phi)$ be the solution of (2.3) with $\phi \in E_+^{per}$. Then the following statements are valid:*

(i) If $r(DQ(0)) \leq 1$, then the fixed point zero is globally asymptotically stable in E_+^{per}.

(ii) If $r(DQ(0)) > 1$, then system (2.3) admits a unique positive L-periodic fixed point $N^*(x)$, and it is globally asymptotically stable in $E_+^{per}\backslash\{0\}$.

In order to study the propagation dynamics for system (2.3), we assume that $r(DQ(0)) > 1$ in the rest of this section so that the operator Q admits a globally stable positive L-periodic fixed point N^*.

Let $\mathcal{C}$ be the set of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}$. For any $\phi, \psi \in \mathcal{C}$, we write $\phi \geq \psi$ if $\phi(x) \geq \psi(x)$ for all $x \in \mathbb{R}$; and $\phi > \psi$ if $\phi \geq \psi$ and $\phi \neq \psi$. Define

$$\|\phi\|_{\mathcal{C}} = \sum_{k=1}^{\infty} \frac{\max_{|x|\leq k} |\phi(x)|}{2^k},\ \forall \phi \in \mathcal{C},$$

where $|\cdot|$ denotes the usual norm in $\mathbb{R}$. Then $(\mathcal{C}, \|\cdot\|_{\mathcal{C}})$ is a normed space. Let $d(\cdot,\cdot)$ be the distance induced by the norm $\|\cdot\|_{\mathcal{C}}$. It follows that the topology in the metric space $(\mathcal{C}, d)$ is the same as the compact open topology in $\mathcal{C}$, that is, a sequence of points ϕ^n converges to ϕ in $\mathcal{C}$ if the sequence $\phi^n(x)$ of functions converges to $\phi(x)$ uniformly for x in any compact subset of $\mathbb{R}$. Let $\beta \in int(E_+^{per})$ be fixed. Define

$$\mathcal{C}_\beta := \{\phi \in \mathcal{C} : 0 \leq \phi(x) \leq \beta(x), \forall x \in \mathbb{R}\},\ \mathcal{C}_\beta^{per} = \{\phi \in \mathcal{C}_\beta : \phi(x) = \phi(x+L), \forall x \in \mathbb{R}\}.$$

For $I = [a,b] \subset \mathbb{R}$ and $\psi \in \mathcal{C}$, we define $\psi_I \in C(I,\mathbb{R})$ by

$$\psi_I(x) := \psi(x),\ x \in I.$$

For any given subset U of $\mathcal{C}$, we define $U_I := \{\psi_I : \psi \in U\}$ and the norm $|\cdot|$ in U_I by $|\psi_I| = \max_{x\in I} |\psi_I(x)|$.

Theorem 2.3.1. *Assume that (H1)-(H3) hold and $r(DQ(0)) > 1$. Then there exist two real numbers c_+^* and c_-^*, called the rightward and leftward spreading speeds, such that the following statements are valid:*

(i) If $\phi \in \mathcal{C}_{N^*}$ satisfies $0 \le \phi \le \psi \ll N^*$ for some $\psi \in \mathcal{C}_{N^*}^{per}$ and $\phi(x) = 0$ for x outside a bounded interval, then

$$\lim_{n\to\infty, x \ge cn} Q^n[\phi](x) = 0, \forall c > c_+^*, \quad \lim_{n\to\infty, x\le -cn} Q^n[\phi](x) = 0, \forall c > c_-^*;$$

(ii) If $\phi \in \mathcal{C}_{N^*}$ and $\phi \not\equiv 0$, then for any c and $\tilde{c}$ satisfying $-c_-^* < -\tilde{c} < c < c_+^*$, there holds

$$\lim_{n\to\infty, -\tilde{c}n \le x \le cn} (Q^n[\phi](x) - N^*(x)) = 0.$$

Proof. We appeal to the theory developed by [69] to establish the existence of the asymptotic speeds of spread for (2.3) under assumptions (H1)-(H3) and $r(DQ(0)) > 1$. Under these assumptions, we see from Proposition 2.3.1 that N^* is the L-periodic fixed point of (2.3). For any $y \in \mathbb{R}$ and any function $h : \mathbb{R} \to \mathbb{R}$, we define the translation operator $\tilde{T}_y$ by

$$\tilde{T}_y[h](x) = h(x - y).$$

According to [69], we need to verify that the map Q defined in (2.2) satisfies the following conditions:

(E1) Q is spatially L-periodic in the sense that $Q \circ \tilde{T}_y(\phi) = \tilde{T}_y \circ Q(\phi)$ for any $y \in L\mathbb{Z}, \phi \in \mathcal{C}_{N^*}$, where $L\mathbb{Z} = \{..., -2L, -L, 0, L, 2L, ...\}$.

(E2) $Q[\mathcal{C}_{N^*}] \subset \mathcal{C}$ is uniformly bounded and $Q : \mathcal{C}_{N^*} \to \mathcal{C}$ is continuous.

(E3) For any closed interval $I \subset \mathbb{R}$ with endpoints 0 and $p \in L\mathbb{Z}$, there is a positive number $k(p) < 1$ such that $\alpha((Q[U])_I) \le k(p)\alpha(U_I)$ for any $U \subset \mathcal{C}_{N^*}$, where α denotes the Kuratowski measure of non-compactness on $\mathcal{C}_I$.

(E4) $Q : \mathcal{C}_{N^*} \to \mathcal{C}_{N^*}$ is monotone (order-preserving) in the sense that $Q[u] \ge Q[v]$ whenever $u \ge v$ in C_{N^*}.

(E5) Q admits exactly two L-periodic fixed points 0 and N^* in $\mathbb{R}$, and for any L-periodic function $\phi \in \mathcal{C}$ with $0 \ll \phi \le N^*$, we have $\lim_{n\to\infty} |Q^n[\phi](x) - N^*(x)| = 0$ uniformly for $x \in \mathbb{R}$.

Indeed, for any $y \in L\mathbb{Z}$, $u(\tilde{t}, x; \tilde{T}_y(\phi))$ and $u(\tilde{t}, x - y; \phi)$ are solutions of (2.1) with initial conditions $u(0, x; \tilde{T}_y(\phi)) = \tilde{T}_y(\phi)(x) = \phi(x - y)$ and $u(0, x - y; \phi) = \phi(x - y)$, respectively. Then the uniqueness of solutions of (2.1) implies that (E1) holds true. It follows from above analysis that Q satisfies (E2). Note that Q is compact on $\mathcal{C}_{N^*}$, and hence (E3) holds. Further, since the function g is monotonically increasing, Q

is order-preserving and (E4) holds. Under the condition $r(DQ(0)) > 1$, we see from Proposition 2.3.1 that (E5) holds. By [69, Theorem 5.1], it then follows that the map Q admits a rightward spreading speed c_+^* and a leftward spreading speed c_-^* such that above two statements hold true. Here we have used the property that $c_+^* + c_-^* > 0$, which will be proved in Lemma 2.3.2. □

To obtain computational formulas of $c_\pm^*$, we use the linear operators approach (see, e.g., [68, 131]). We then linearize system (2.1) at the zero solution to obtain the following linear system:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = \dfrac{\partial}{\partial x}(D_M(t,x)\dfrac{\partial u_n}{\partial x}) + \partial_u f(t,x,0)u_n, \ 0 < t \leq 1, t \neq \tau, x \in \mathbb{R}, n \in \mathbb{N}, \\ u_n(t^+,x) = u_n(t,x) + \bar{R}(x;N_n), \ t = \tau, \\ u_n(0,x) = N_n(x), \ N_{n+1}(x) = u_n(1,x), \\ N_0(x) = \phi \in \mathcal{C}^{per}, \end{cases} \tag{2.6}$$

where $\bar{R}(x;N_n) = T(\tau,0)[\partial_N g(\cdot,0)N_n](x)$. Let $\mathbb{L}(\tilde{t})$ be the time-$\tilde{t}$ map on $\mathcal{C}^{per}$ generated by (2.6), that is, $\mathbb{L}(\tilde{t})(\phi) = u(\tilde{t},\cdot;\phi)$, where $u(\tilde{t},x;\phi)$ is the unique solution of (2.6) with $u(0,x;\phi) = \phi \in \mathcal{C}^{per}$. For any given $\mu \in \mathbb{R}$, substituting $u(\tilde{t},x) = e^{-\mu x}w(\tilde{t},x)$ into (2.6) yields

$$\begin{cases} \dfrac{\partial w_n}{\partial t} = D_M(t,x)\dfrac{\partial^2 w_n}{\partial x^2} - \left[2\mu D_M(t,x) - \dfrac{\partial D_M(t,x)}{\partial x}\right]\dfrac{\partial w_n}{\partial x} \\ \qquad + \left[\mu^2 D_M(t,x) - \mu\dfrac{\partial D_M(t,x)}{\partial x}\right] w_n \\ \qquad + \partial_u f(t,x,0)w_n, \ 0 < t \leq 1, t \neq \tau, x \in \mathbb{R}, n \in \mathbb{N}, \\ w_n(t^+,x) = w_n(t,x) + \bar{R}(x;W_n), \ t = \tau, \\ w_n(0,x) = W_n(x), \ W_{n+1}(x) = w_n(1,x). \end{cases} \tag{2.7}$$

Let $L_\mu(\tilde{t})$ be the time-$\tilde{t}$ map on $\mathcal{C}^{per}$ generated by system (2.7), that is, $L_\mu(\tilde{t})\sigma = w(\tilde{t},\cdot;\sigma)$, where $w(\tilde{t},x;\sigma)$ is the unique solution of system (2.7) with $w(0,x;\sigma) = \sigma \in \mathcal{C}^{per}$. Then

$$\mathbb{L}(\tilde{t})[e^{-\mu\cdot}\sigma](x) = e^{-\mu x}L_\mu(\tilde{t})[\sigma](x), \ \sigma \in \mathcal{C}^{per}, x \in \mathbb{R}, \tilde{t} \geq 0.$$

Furthermore, substituting $w(\tilde{t},x) = e^{\Lambda\tilde{t}}r_1(\tilde{t},x)$ into (2.7) yields the following periodic eigenvalue problem

$$
\begin{cases}
\Lambda r_{1,n}(t,x) = -\dfrac{\partial r_{1,n}}{\partial t} + D_M(t,x)\dfrac{\partial^2 r_{1,n}}{\partial x^2} \\
\qquad\qquad - \left[2\mu D_M(t,x) - \dfrac{\partial D_M(t,x)}{\partial x}\right]\dfrac{\partial r_{1,n}}{\partial x} \\
\qquad\qquad + \left[\mu^2 D_M(t,x) - \mu\dfrac{\partial D_M(t,x)}{\partial x}\right] r_{1,n} \\
\qquad\qquad + \partial_u f(t,x,0) r_{1,n},\ 0 < t \le 1, t \ne \tau, x \in \mathbb{R}, n \in \mathbb{N}, \\
r_{1,n}(t^+,x) = r_{1,n}(t,x) + \bar{R}(x; Z_n),\ t = \tau, \\
r_{1,n}(0,x) = Z_n(x),\ Z_{n+1}(x) = r_{1,n}(1,x), \\
r_1(\tilde{t}, x+L) = r_1(\tilde{t},x),\ r_1(\tilde{t}+1,x) = r_1(\tilde{t},x),\ \forall(\tilde{t},x) \in \mathbb{R}_+ \times \mathbb{R}.
\end{cases}
\tag{2.8}
$$

Define the Poincaré map $P_\mu : \mathcal{C}^{per} \to \mathcal{C}^{per}$ by $P_\mu(\sigma) = L_\mu(1)(\sigma)$. Then $P_\mu^n(\sigma) = L_\mu(n)(\sigma)$ for any integer $n \ge 0$. Clearly, P_μ^n is compact and strongly positive in $\mathcal{C}^{per}$. It follows from Theorem 1.3.2 (the Krein-Rutman Theorem) that the spectral radius $r(P_\mu) > 0$ is an algebraically simple eigenvalue of P_μ with an eigenfunction $\sigma_\mu \gg 0$ ($\sigma_\mu \in \mathcal{C}^{per}$), that is, $P_\mu \sigma_\mu = r(P_\mu)\sigma_\mu$. Clearly, $r(P_\mu)$ and σ_μ are the corresponding principal eigenvalue and principal eigenfunction of P_μ, respectively.

Let $R_1(\tilde{t})$ be the time-$\tilde{t}$ solution map on $\mathcal{C}^{per}$ of the following system:

$$
\begin{cases}
\dfrac{\partial r_{1,n}}{\partial t} = D_M(t,x)\dfrac{\partial^2 r_{1,n}}{\partial x^2} - \left[2\mu D_M(t,x) - \dfrac{\partial D_M(t,x)}{\partial x}\right]\dfrac{\partial r_{1,n}}{\partial x} \\
\qquad + \left[\mu^2 D_M(t,x) - \mu\dfrac{\partial D_M(t,x)}{\partial x}\right] r_{1,n} \\
\qquad + \partial_u f(t,x,0) r_{1,n} - \Lambda r_{1,n}(t,x),\ 0 < t \le 1, t \ne \tau, x \in \mathbb{R}, n \in \mathbb{N}, \\
r_{1,n}(t^+,x) = r_{1,n}(t,x) + \bar{R}(x; Z_n),\ t = \tau, \\
r_{1,n}(0,x) = Z_n(x),\ Z_{n+1}(x) = r_{1,n}(1,x), \\
Z_0(x) = \sigma \in \mathcal{C}^{per}.
\end{cases}
\tag{2.9}
$$

Then we have

$$
R_1(1)(\sigma)(x) = r_1(1,x;\sigma) = e^{-\Lambda} w(1,x;\sigma),\ \forall x \in \mathbb{R},
$$

where $w(\tilde{t},x;\sigma)$ is the solution of (2.7), and hence,

$$
R_1(1)(\sigma_\mu) = e^{-\Lambda}[L_\mu(1)(\sigma_\mu)] = e^{-\Lambda} r(P_\mu)\sigma_\mu.
$$

Let $\Lambda(\mu) = \ln r(P_\mu)$. Then σ_μ is a positive fixed point of $R_1(1)$, and hence, the solution $r_1(\tilde{t},x) := r_1(\tilde{t},x;\sigma_\mu)$ of (2.9) is positive, 1-periodic in $\tilde{t}$ and L-periodic in x. It then follows that $\Lambda(\mu)$ and $r_1(\tilde{t},x)$ satisfy (2.8), and hence, $\Lambda(\mu)$ is the principal eigenvalue of problem (2.8). So we have the following observation.

Lemma 2.3.1. *Assume that $\mu \geq 0$ and (H1)-(H3) hold. Then $r(P_\mu) = r(L_\mu(1))$ is the principal eigenvalue of $P_\mu = L_\mu(1)$, and $\Lambda(\mu) = \ln r(P_\mu)$ is the principal eigenvalue of problem (2.8) with an eigenvector $r_1(\tilde{t}, x) := r_1(\tilde{t}, x; \sigma_\mu)$, which is positive, 1-periodic in $\tilde{t}$ and L-periodic in x.*

Remark 2.3.1. *In the case where $\mu = 0$, the solution map $L_0(\tilde{t}) \equiv \mathbb{L}(\tilde{t})$ on $\mathcal{C}^{per}$. Then the above arguments show that the spectral radius $r(\mathbb{L}(1)) = r(P_0) = e^{\Lambda(0)}$ is the principal eigenvalue of $\mathbb{L}(1)$. Namely, there is $\sigma_0 \gg 0$ ($\sigma_0 \in \mathcal{C}^{per}$) such that $\mathbb{L}(1)\sigma_0 = P_0\sigma_0 = e^{\Lambda(0)}\sigma_0$.*

Lemma 2.3.2. *Let c_+^* and c_-^* be the rightward and leftward spreading speeds of Q, respectively. Then*

$$c_+^* = \inf_{\mu>0} \frac{\Lambda(\mu)}{\mu}, \quad c_-^* = \inf_{\mu>0} \frac{\Lambda(-\mu)}{\mu}.$$

Furthermore, $c_+^* + c_-^* > 0$.

Proof. Denote $\Phi(\mu) := \frac{\ln[r(P_\mu)]}{\mu} = \frac{\Lambda(\mu)}{\mu}$. By Lemma 2.3.1, it follows that for any given $\mu > 0$, there exists the corresponding time-space periodic principal eigenfunction $r_1(\tilde{t}, x) > 0$ with the principal eigenvalue $\Lambda(\mu)$ satisfying (2.8). Due to the time-space periodicity of $r_1(\tilde{t}, x)$, for any given $\mu > 0$, there exists $(\tilde{t}_\mu, x_\mu) \in \mathbb{R}_+ \times \mathbb{R}$ such that $r_1(\tilde{t}_\mu, x_\mu) = \min_{(\tilde{t},x)\in\mathbb{R}_+\times\mathbb{R}} r_1(\tilde{t}, x)$, which implies that

$$\frac{\partial r_1}{\partial \tilde{t}}\bigg|_{(\tilde{t}_\mu, x_\mu)} = 0, \quad \frac{\partial r_1}{\partial x}\bigg|_{(\tilde{t}_\mu, x_\mu)} = 0, \quad \frac{\partial^2 r_1}{\partial x^2}\bigg|_{(\tilde{t}_\mu, x_\mu)} \geq 0. \tag{2.10}$$

Letting $(\tilde{t}, x) = (\tilde{t}_\mu, x_\mu)$ in (2.8), and the unique decomposition $\tilde{t}_\mu = n + t_\mu$, we see from (2.10) that

$$\begin{aligned}\Lambda(\mu) r_{1,n}(t_\mu, x_\mu) \geq{}& [\mu^2 D_M(t_\mu, x_\mu) - \mu \partial_x D_M(t_\mu, x_\mu)] r_{1,n}(t_\mu, x_\mu) \\ &+ \partial_u f(t_\mu, x_\mu, 0) r_{1,n}(t_\mu, x_\mu), \ 0 < t_\mu \leq 1, t_\mu \neq \tau, x_\mu \in \mathbb{R}, n \in \mathbb{N},\end{aligned}$$

and hence,

$$\frac{\Lambda(\mu)}{\mu} \geq [\mu D_M(t_\mu, x_\mu) - \partial_x D_M(t_\mu, x_\mu)] + \frac{\partial_u f(t_\mu, x_\mu, 0)}{\mu}.$$

This, together with the boundedness of $\partial_x D_M(t_\mu, x_\mu)$ and $\partial_u f(t_\mu, x_\mu, 0)$, implies that

$$\lim_{\mu\to\infty} \Phi(\mu) = \lim_{\mu\to\infty} \frac{\Lambda(\mu)}{\mu} = \infty. \tag{2.11}$$

Since $\Lambda(0) > 0$, we have

$$\lim_{\mu\to 0^+} \Phi(\mu) = \infty. \tag{2.12}$$

By virtue of (2.11) and (2.12), it is easy to see that $\Phi(\mu)$ attains its minimum at some finite μ^*. Since the solution of (2.3) is a lower solution of the linear system (2.6) under the conditions (H2) and (H3), we have

$$\Phi_{\tilde{t}}[\phi] \leq \mathbb{L}(\tilde{t})[\phi], \ \forall \phi \in \mathcal{C}^{per}, \tilde{t} \geq 0.$$

By the arguments similar to those for [131, Theorem 2.5] and [68, Theorem 3.10(i)], we obtain $c_+^* \leq \inf_{\mu>0} \Phi(\mu)$.

For any given $\varepsilon \in (0,1)$, there exists $\delta = \delta(\varepsilon)$ such that

$$g(x,N) \geq (1-\varepsilon)\partial_N g(x,0)N, \ f(t,x,u) \geq (1+\varepsilon)\partial_u f(t,x,0)u, \ \forall x \in \mathbb{R}, N, u \in [0,\delta].$$

Choose a constant number $\xi = \xi(\delta) > 0$ such that $0 \leq u(t,x;\xi) \leq \delta, \forall x \in \mathbb{R}, t \in [0,1]$. By the comparison principle, we have

$$u(t,x;\rho) \leq u(t,x;\xi) \leq \delta, \ \forall \rho \in \mathcal{C}_\xi, x \in \mathbb{R}, t \in [0,1].$$

It then follows that for any $\rho \in \mathcal{C}_\xi$, the solution $u(\tilde{t},x;\rho)$ of (2.3) satisfies

$$\begin{cases} \dfrac{\partial u_n}{\partial t} \geq \dfrac{\partial}{\partial x}(D_M(t,x)\dfrac{\partial u_n}{\partial x}) + (1+\varepsilon)\partial_u f(t,x,0)u_n, \ 0 < t \leq 1, t \neq \tau, x \in \mathbb{R}, n \in \mathbb{N}, \\ u_n(t^+,x) = u_n(t,x) + R(x;N_n), \ t = \tau, \\ u_n(0,x) = N_n(x), \ N_{n+1}(x) = u_n(1,x), \\ N_0(x) = \rho \in \mathcal{C}_\xi, \end{cases}$$

with

$$R(x;N_n) \geq T(\tau,0)\left[(1-\varepsilon)\partial_N g(x,0)N_n\right](x).$$

Consider the linear system

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = \dfrac{\partial}{\partial x}(D_M(t,x)\dfrac{\partial u_n}{\partial x}) + (1+\varepsilon)\partial_u f(t,x,0)u_n, \ 0 < t \leq 1, t \neq \tau, x \in \mathbb{R}, n \in \mathbb{N}, \\ u_n(t^+,x) = u_n(t,x) + \bar{R}_\varepsilon(x;N_n), \ t = \tau, \\ u_n(0,x) = N_n(x), \ N_{n+1}(x) = u_n(1,x), \\ N_0(x) = \rho \in \mathcal{C}, \end{cases}$$

where

$$\bar{R}_\varepsilon(x;N_n) = T(\tau,0)\left[(1-\varepsilon)\partial_N g(x,0)N_n\right](x).$$

Let $\mathbb{L}^\varepsilon(\tilde{t})$ be the time-$\tilde{t}$ map on $\mathcal{C}$ generated by the above linear system. Furthermore, the comparison principle implies that

$$\Phi_{\tilde{t}}[\phi] \geq \mathbb{L}^\varepsilon(\tilde{t})[\phi], \ \forall \phi \in \mathcal{C}_\xi, \tilde{t} \in [0,1],$$

and hence,

$$\Phi_1[\phi] \geq \mathbb{L}^\varepsilon(1)[\phi], \ \forall \phi \in \mathcal{C}_\xi.$$

For $\mu \geq 0$, let $L_\mu^\varepsilon(\tilde{t})$ be the time-$\tilde{t}$ map on $\mathcal{C}$ generated by the following system

$$\begin{cases} \dfrac{\partial w_n}{\partial t} = D_M(t,x)\dfrac{\partial^2 w_n}{\partial x^2} - \left[2\mu D_M(t,x) - \partial_x D_M(t,x)\right]\dfrac{\partial w_n}{\partial x} \\ \qquad + \left[\mu^2 D_M(t,x) - \mu \partial_x D_M(t,x)\right] w_n \\ \qquad + (1+\varepsilon)\partial_u f(t,x,0) w_n, \ 0 < t \leq 1, t \neq \tau, x \in \mathbb{R}, n \in \mathbb{N}, \\ w_n(t^+, x) = w_n(t,x) + \bar{R}_\varepsilon(x; K_n), \ t = \tau, \\ w_n(0,x) = K_n(x), \ K_{n+1}(x) = w_n(1,x), \\ K_0(x) =: \sigma \in \mathcal{C}. \end{cases} \tag{2.13}$$

Define

$$\Phi^\varepsilon(\mu) := \frac{\ln[r(L_\mu^\varepsilon(1))]}{\mu}, \ \forall \mu > 0,$$

where $r(L_\mu^\varepsilon(1))$ is the spectral radius of the Poincaré map associated with system (2.13). By the analysis on $L_\mu^\varepsilon(\tilde{t})$ similar to that for $L_\mu(\tilde{t})$ and the arguments similar to those in [131, Theorem 2.4] and [68, Theorem 3.10(ii)], it then follows that $c_+^* \geq \inf_{\mu>0} \Phi^\varepsilon(\mu)$. Letting $\varepsilon \to 0$, we have $c_+^* \geq \inf_{\mu>0} \Phi(\mu)$. Consequently, $c_+^* = \inf_{\mu>0} \Phi(\mu)$.

By a change of variables $\hat{u}(\tilde{t},x) = u(\tilde{t},-x)$ and $\hat{v}(\tilde{t},x) = v(\tilde{t},-x)$ and the similar arguments as above, it follows that $c_-^* = \inf_{\mu>0} \frac{\Lambda(-\mu)}{\mu}$ is the rightward spreading speed of the resulting equation for $\hat{u}$, and hence, $c_-^* = \inf_{\mu>0} \frac{\Lambda(-\mu)}{\mu}$ is the leftward asymptotic speed of spread for Q. We can further prove $c_+^* + c_-^* > 0$ by using the arguments similar to those for [74, Lemma 2.10] and [69, Proposition 7.4]. □

Definition 2.3.1. *Let $\{Q^n\}_{n\in\mathbb{N}}$ be the discrete-time semiflow on $\mathcal{C}_{N^*}$.*

(i) We say that $W(x, x-cn)$ is a L-periodic rightward traveling wave of $\{Q^n\}_{n\in\mathbb{N}}$ if $W(\cdot, \cdot + a) \in \mathcal{C}_{N^*}, \forall a \in \mathbb{R}$, $Q^n[Y](x) = W(x, x-cn), \forall n \in \mathbb{N}$, and $W(x,\xi)$ is a L-periodic function in x for any fixed $\xi \in \mathbb{R}$, where $Y(x) := W(x,x)$. Moreover, we say that $W(x, x-cn)$ connects N^* to 0 if $\lim_{\xi\to-\infty}(W(x,\xi) - N^*(x)) = 0$ and $\lim_{\xi\to+\infty} W(x,\xi) = 0$ uniformly for all $x \in \mathbb{R}$.

(ii) We say that $W(x, x+cn)$ is a L-periodic leftward traveling wave of $\{Q^n\}_{n\in\mathbb{N}}$ if $W(\cdot, \cdot + a) \in \mathcal{C}_{N^*}, \forall a \in \mathbb{R}$, $Q^n[Y](x) = W(x, x+cn), \forall n \in \mathbb{N}$, and $W(x,\xi)$ is a L-periodic function in x for any fixed $\xi \in \mathbb{R}$, where $Y(x) := W(x,x)$. Moreover, we say that $W(x, x+cn)$ connects 0 to N^* if $\lim_{\xi\to+\infty}(W(x,\xi) - N^*(x)) = 0$ and $\lim_{\xi\to-\infty} W(x,\xi) = 0$ uniformly for all $x \in \mathbb{R}$.

By [69, Theorem 5.3], as applied to $\{Q^n\}_{n\in\mathbb{N}}$, we have the following result on the L-periodic traveling waves of system (2.3), which shows that the spreading speeds given in Theorem 2.3.1 coincide with the minimal wave speeds of the L-periodic traveling waves of system (2.3).

Theorem 2.3.2. *Assume that (H1)-(H3) hold, $r(DQ(0)) > 1$, and $c^*_\pm$ is given in Theorem 2.3.1. Then the following statements are valid:*

(i) For any $c \geq c^*_+$, system (2.3) admits a L-periodic rightward traveling wave $U(x, x-cn)$ connecting N^* to 0 with the wave profile $U(x,\xi)$ being continuous and non-increasing in $\xi \in \mathbb{R}$. While for any $c \in (0, c^*_+)$, system (2.3) admits no periodic rightward traveling wave connecting N^* to 0;

(ii) For any $c \geq c^*_-$, system (2.3) admits a L-periodic leftward traveling wave $V(x, x+cn)$ connecting 0 to N^* with the wave profile $V(x,\xi)$ being continuous and non-decreasing in $\xi \in \mathbb{R}$. While for any $c \in (0, c^*_-)$, system (2.3) admits no periodic leftward traveling wave connecting 0 to N^*.

2.3.2 The non-monotone case of $g(x, N)$

In this section, we assume that the birth function $g(x, N)$ is not monotonically increasing with respect to N, which implies that the time-one solution map Q is not monotone on its domain. To overcome this difficulty, we construct two monotone auxiliary systems of (2.3) under an additional assumption on g:

(H4) There is a constant $a > 0$ such that $g(x, N)$ is non-decreasing with respect to N for $0 \leq N \leq a$.

For any $x \in \mathbb{R}$, we define

$$g^+(x, N) = \max_{0\leq M\leq N} g(x, M), \ \ N \in \mathbb{R}_+.$$

By the definition of g^+ and the assumption (H3) on the function g, it then easily follows that g^+ is non-decreasing with respect to N, $\partial_N g^+(\cdot, 0) = \partial_N g(\cdot, 0)$, and g^+ also satisfies (H3).

In the case of $r(DQ(0)) > 1$, Proposition 2.3.1(ii) implies that (2.3) with g replaced by g^+ has a positive L-periodic fixed point $N^{*+}(x)$. Then for any $x \in \mathbb{R}$, we define

$$g^-(x, N) = \min_{N\leq V\leq \max_{x\in[0,L]} N^{*+}(x)} g(x, V), \quad 0 \leq N \leq \max_{x\in[0,L]} N^{*+}(x).$$

Thus, g^- is non-decreasing with respect to N, $\partial_N g^-(\cdot, 0) = \partial_N g(\cdot, 0)$, and g^- also satisfies the assumption (H3). Similarly, system (2.3) with g replaced by g^- admits a positive L-periodic fixed point $N^{*-}(x)$. Note that $0 < N^{*-}(\cdot) \leq N^*(\cdot) \leq N^{*+}(\cdot)$. Moreover, $g^-(\cdot, N) \leq g(\cdot, N) \leq g^+(\cdot, N)$, $\partial_N g^\pm(\cdot, 0) = \partial_N g(\cdot, 0), g^\pm(\cdot, N) \leq \partial_N g(\cdot, 0)N$, and there exists $\tilde{a} \in (0, a]$ such that $g^\pm(\cdot, N) = g(\cdot, N)$ for all $N \in (0, \tilde{a}]$.

Theorem 2.3.3. *Assume that (H1)-(H4) hold and $r(DQ(0)) > 1$. Then the following statements are valid:*

(i) If $\phi \in \mathcal{C}_{N^{*+}}$ satisfies $0 \le \phi \le \psi \ll N^{*+}$ for some $\psi \in \mathcal{C}^{per}_{N^{*+}}$ and $\phi(x) = 0$ for x outside a bounded interval, then

$$\lim_{n\to\infty, x\ge cn} Q^n[\phi](x) = 0, \forall c > c^*_+, \quad \lim_{n\to\infty, x\le -cn} Q^n[\phi](x) = 0, \forall c > c^*_-;$$

(ii) If $\phi \in \mathcal{C}_{N^{*+}}$ and $\phi \not\equiv 0$, then for any c and $\tilde{c}$ satisfying $-c^*_- < -\tilde{c} < c < c^*_+$, there holds

$$\limsup_{n\to\infty, -\tilde{c}n\le x\le cn} (Q^n[\phi](x) - N^{*+}(x)) \le 0 \le \liminf_{n\to\infty, -\tilde{c}n\le x\le cn} (Q^n[\phi](x) - N^{*-}(x)).$$

Proof. We consider two auxiliary systems:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = \dfrac{\partial}{\partial x}(D_M(t,x)\dfrac{\partial u_n}{\partial x}) + f(t,x,u_n), \ 0 < t \le 1, t \ne \tau, x \in \mathbb{R}, n \in \mathbb{N}, \\ u_n(t^+, x) = u_n(t,x) + R^+(x; N^+_n), \ t = \tau, \\ u_n(0,x) = N^+_n(x), \\ N^+_{n+1}(x) = u_n(1,x), \end{cases} \tag{2.14}$$

and

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = \dfrac{\partial}{\partial x}(D_M(t,x)\dfrac{\partial u_n}{\partial x}) + f(t,x,u_n), \ 0 < t \le 1, t \ne \tau, x \in \mathbb{R}, n \in \mathbb{N}, \\ u_n(t^+, x) = u_n(t,x) + R^-(x; N^-_n), \ t = \tau, \\ u_n(0,x) = N^-_n(x), \\ N^-_{n+1}(x) = u_n(1,x), \end{cases} \tag{2.15}$$

where

$$R^\pm(x; N^\pm_n) = T(\tau, 0)(g^\pm(\cdot, N^\pm_n))(x).$$

Similar to the procedures of recursion operator Q in (2.2), and following systems (2.14) and (2.15), we can define two recurrence relations for $N^\pm_{n+1}(x)$ as

$$\begin{aligned} N^+_{n+1}(x) &= Q^+\left[N^+_n\right](x) \\ &= \left[M(1,\tau)\left[M(\tau,0)(N^+_n) + T(\tau,0)(g^+(\cdot, N^+_n))\right]\right](x), \ x \in \mathbb{R}, n \in \mathbb{N}, \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} N^-_{n+1}(x) &= Q^-\left[N^-_n\right](x) \\ &= \left[M(1,\tau)\left[M(\tau,0)(N^-_n) + T(\tau,0)(g^-(\cdot, N^-_n))\right]\right](x), \ x \in \mathbb{R}, n \in \mathbb{N}, \end{aligned} \tag{2.17}$$

respectively. Let $N^+_n(x)$, $N^-_n(x)$ and $N_n(x)$ be the solutions of (2.16), (2.17) and (2.3), respectively. Thus, the comparison arguments imply that if $N^-_0(x) \le N_0(x) \le N^+_0(x) \le N^{*+}(x)$, then

$$0 \le N^-_n(x) \le N_n(x) \le N^+_n(x), \ x \in \mathbb{R}, n \in \mathbb{N}.$$

Note that $c_\pm^*$ in section 2.3.1 are the spreading speeds of (2.3) with monotone birth function g, and the expressions of $c_\pm^*$ only depend on the linearized systems at $u = 0$. So we can define $c_\pm^{+,*}$ and $c_\pm^{-,*}$ for (2.16) and (2.17), respectively, in a similar way. Moreover, $c_+^{+,*} = c_+^{-,*}$ and $c_-^{+,*} = c_-^{-,*}$. By virtue of two auxiliary systems (2.14) and (2.15), we can use the comparison arguments similar to those in [46, Theorem 2.2] to prove the statements for (2.3) in the non-monotone case, which implies that $c_+^* = c_+^{+,*} = c_+^{-,*}$ and $c_-^* = c_-^{+,*} = c_-^{-,*}$ are also the spreading speeds for system (2.3). □

2.4 Global dynamics in a bounded domain

In this section, we study the evolution dynamics of system (2.1) in a bounded domain Ω under the boundary conditions

$$Bw = 0,\ (\tilde{t}, x) \in (0,\infty) \times \partial\Omega, w \in \{u, v\},$$

where $\Omega \subset \mathbb{R}^n$, $(n \geq 1)$, and if $n > 1$, we suppose that $\partial\Omega$ is a class of $C^{2+\theta}(0 < \theta \leq 1)$. We assume that $Bw = w$ (Dirichlet boundary condition) or $Bw = \partial w/\partial n + b(x)w$ (Robin type boundary condition) for some nonnegative function $b \in C^{1+\theta}(\partial\Omega)$, where $\partial/\partial n$ denotes the differentiation in the direction of the outward normal n to $\partial\Omega$. Similar to (H1)-(H3), we assume that:

(A1) Both functions $D_M(t,x)$ and $D_I(t,x)$ are in $C^{\frac{\theta}{2},\theta+1}(\mathbb{R}_+ \times \overline{\Omega})$, $d_I(t,x)$ is in $C^{\frac{\theta}{2},\theta}(\mathbb{R}_+ \times \overline{\Omega})$, and these three functions are 1-periodic in time t; there exists a number $\alpha > 0$ such that $D_i(t,x) \geq \alpha$ $(i = M, I)$ and $d_I(t,x) > 0$ for all $(t,x) \in \mathbb{R}_+ \times \overline{\Omega}$.

(A2) $f(t,x,u)$ is in $C^{\frac{\theta}{2},\theta,1}(\mathbb{R}_+ \times \overline{\Omega} \times \mathbb{R}_+)$, 1-periodic in time t, $f(t,x,0) = 0 > \partial_u f(t,x,0)$, and $\frac{f(t,x,u)}{u}$ is strictly decreasing in u.

(A3) $g(x,N)$ is locally Lipschitz continuous on $\overline{\Omega} \times \mathbb{R}_+$, $g(x,0) = 0 < \partial_N g(x,0)$, $g(x,N) > 0$ for all $N > 0$, and $\frac{g(x,N)}{N}$ is nonincreasing for N.

Let $p \in (n,\infty)$ be fixed, and $X := L^p(\Omega)$. For each $\gamma \in \left(\frac{1}{2} + \frac{n}{2p}, 1\right)$, let X^γ be the fractional power space (see, e.g., [43, Definition 1.4.7]) of X with respect to $\left(\frac{\partial}{\partial x}\left(D_M(t,x)\frac{\partial}{\partial x}\right), B\right)$. It is known that (X^γ, X_+^γ) is an ordered Banach space and $X^\gamma \subset C^{1+\lambda}(\overline{\Omega})$ with continuous inclusion for $\lambda \in [0, 2\gamma - 1 - \frac{n}{p}]$, see [11,43]. Let $\|\cdot\|_\gamma$ be the norm on X^γ. Then there exists a M_γ such that $\|\phi\|_\infty = \max_{x\in\overline{\Omega}} |\phi(x)| \leq M_\gamma \|\phi\|_\gamma$. It follows from [11, Theorem 1.11] that system (2.1) generates a local time-one periodic semiflow $\left\{\tilde{\Phi}_{\tilde{t}}\right\}_{\tilde{t}\geq 0}$ on X^γ in a weak sense, and bounded orbits in X^γ are precompact. From (A2)-(A3), we can establish the L^∞-boundedness of solutions of system (2.1), and consequently, we can infer the global existence of the solutions. By the arguments similar to those for the discrete-time recursion (2.3) with the bounded domain, we can investigate the following recursion

$$N_{n+1}(x) = \tilde{Q}[N_n](x), \ x \in \overline{\Omega}, n \in \mathbb{N}. \tag{2.18}$$

Thus, the time-one solution map $\tilde{Q} := \tilde{\Phi}_1$ has the following expression:

$$\tilde{Q}\left[N_n\right](x) = \left[\tilde{M}(1,\tau)\left[\tilde{M}(\tau,0)(N_n) + \tilde{T}(\tau,0)(g(\cdot,N_n))\right]\right](x), \ \forall x \in \overline{\Omega},$$

where $\tilde{T}(t,s), \tilde{M}(t,s), t \geq s$, are the evolution families on X^γ associated with the linear equations $\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}(D_I(t,x)\frac{\partial v}{\partial x}) - d_I(t,x)v$ and $\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(D_M(t,x)\frac{\partial u}{\partial x}) + f(t,x,u)$ with the boundary conditions $Bv = 0$ and $Bu = 0$, respectively. Moreover, the map $\tilde{Q}$ is compact in X^γ.

Next, we focus on the evolution dynamics of the discrete-time semiflow $\left\{\tilde{Q}^n\right\}_{n\in\mathbb{N}}$ associated with system (2.18) in the monotone and non-monotone cases, respectively.

2.4.1 The monotone case of $g(x,N)$

We consider the system (2.18) under assumptions (A1)-(A3), with the birth function g assumed to be monotone. Clearly, $\tilde{Q}$ is monotone in X^γ. With (A2)-(A3), we can easily show that $\tilde{Q}$ is strongly subhomogeneous in X^γ.

By the definition of $\tilde{Q}$ and the Chain rule, we compute the Fréchet derivative $D\tilde{Q}(0)$ on X^γ as follows

$$D\tilde{Q}(0) = M_l(1,\tau)[M_l(\tau,0) + \tilde{T}(\tau,0)\partial_N g(\cdot,0)],$$

where $M_l(t,s), t \geq s$, is the evolution family on X^γ associated with the linear reaction-diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}(D_M(t,x)\dfrac{\partial u}{\partial x}) + \partial_u f(t,x,0)u, \ x \in \Omega, \\ Bu = 0, \ x \in \partial\Omega. \end{cases}$$

Then we have the following linear discrete-time recursion

$$N_{n+1}(x) = D\tilde{Q}(0)[N_n](x), \ x \in \overline{\Omega}, n \in \mathbb{N}, \tag{2.19}$$

that is,

$$N_{n+1}(x) = D\tilde{Q}(0)[N_n](x) = M_l(1,\tau)\left[M_l(\tau,0)(N_n) + \tilde{T}\left(\tau,0\right)\left(\partial_N g(\cdot,0)N_n\right)\right](x), x \in \overline{\Omega}.$$

Denote the spectral radius of $D\tilde{Q}(0)$ as $r(D\tilde{Q}(0))$. Then we have the following threshold-type result.

Theorem 2.4.1. *Assume that (A1)-(A3) hold, and the birth function g is monotonically increasing, and let $N_n(x;\phi)$ be the solution of system (2.18) with $\phi \in X_+^\gamma$. Then the following statements are valid:*

(i) If $r(D\tilde{Q}(0)) < 1$, then the fixed point zero is globally asymptotically stable in X_+^γ.

(ii) If $r(D\tilde{Q}(0)) > 1$, then system (2.18) admits a unique positive fixed point $N^* \in X^\gamma$, and it is globally asymptotically stable in $X_+^\gamma \backslash \{0\}$.

Proof. By Theorem 1.3.6, we have the following result on the global attractivity:

(a) If $r(D\tilde{Q}(0)) \leq 1$, then $\lim_{n\to\infty} \|N_n(\cdot;\phi)\|_\gamma = 0$ for all $\phi \in X_+^\gamma$.

(b) If $r(D\tilde{Q}(0)) > 1$, then system (2.18) admits a unique positive fixed point $N^* \in X^\gamma$ such that $\lim_{n\to\infty} \|N_n(\cdot;\phi) - N^*\|_\gamma = 0$ for all $\phi \in X_+^\gamma \backslash \{0\}$.

In the case where $r(D\tilde{Q}(0)) < 1$, the fixed point zero is linearly asymptotically stable, and hence, (i) holds true. It remains to prove the Lyapunov stability of N^* in the case where $r(D\tilde{Q}(0)) > 1$. Letting $u^*(t,x) := \tilde{\Phi}_t[N^*](x), 0 < t \leq 1$, we consider the linear system

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = \dfrac{\partial}{\partial x}(D_M(t,x)\dfrac{\partial u_n}{\partial x}) + \partial_u f(t,x,u^*)u_n, \ 0 < t \leq 1, t \neq \tau, x \in \Omega, n \in \mathbb{N}, \\ u_n(t^+,x) = u_n(t,x) + R_l(x;N_n), \ t = \tau, \\ u_n(0,x) = N_n(x), N_{n+1}(x) = u_n(1,x), u_0(0,x) = \phi, \ \phi \in X^\gamma, \end{cases} \tag{2.20}$$

where

$$R_l(x;N_n) = \tilde{T}(\tau,0)\left(\partial_N g(\cdot,N^*)N_n\right)(x).$$

Note that system (2.20) generates a time-one periodic semiflow $\mathcal{M}(1) =: \mathcal{M}$. In particular,

$$\mathcal{M}\phi = D\tilde{Q}(N^*)\phi, \ \forall \phi \in X^\gamma.$$

We introduce an following auxiliary system:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = \dfrac{\partial}{\partial x}(D_M(t,x)\dfrac{\partial u_n}{\partial x}) + \dfrac{f(t,x,u^*)}{u^*} \times u_n, \ 0 < t \leq 1, t \neq \tau, x \in \Omega, n \in \mathbb{N}, \\ u_n(t^+,x) = u_n(t,x) + \hat{R}(x;N_n), \ t = \tau, \\ u_n(0,x) = N_n(x), N_{n+1}(x) = u_n(1,x), u_0(0,x) = \phi, \ \phi \in X^\gamma, \end{cases} \tag{2.21}$$

where

$$\hat{R}(x;N_n) = \tilde{T}(\tau,0)\left(\frac{g(\cdot,N^*)}{N^*} \times N_n\right)(x).$$

Let $\mathcal{N}(t)$ be the solution map of system (2.21), and denote $\mathcal{N} := \mathcal{N}(1)$. Clearly, $N^*(x)$ is a positive fixed point of $\mathcal{N}$, that is, $\mathcal{N}(N^*) = N^*$, which implies $r(\mathcal{N}) = 1$. In view of assumptions (A2)-(A3), it is easy to see that $\partial_u f(t,x,u)u < f(t,x,u)$ and $\partial_N g(x,N)N \leq g(x,N)$, $\forall x \in \overline{\Omega}$. Then $\mathcal{N}\phi \gg \mathcal{M}\phi, \forall \phi \in X_+^\gamma \backslash \{0\}$, which means that $\mathcal{N}\phi(x) > \mathcal{M}\phi(x), \forall x \in \overline{\Omega}$, and hence, $N^* > \mathcal{M}(N^*)$. Since $\mathcal{M}$ is a compact and strongly positive linear operator on X^γ, it follows from [44, Theorem 7.3 (i)] that $r(\mathcal{M}) < 1$. Therefore, we have $r(D\tilde{Q}(N^*)) < 1$, and hence, N^* is linearly asymptotically stable. This, together with the global attractivity of N^*, implies that N^* is globally asymptotically stable in $X_+^\gamma \backslash \{0\}$. ◻

2.4.2 The non-monotone case of $g(x, N)$

Now we consider system (2.18) under assumptions (A1)-(A3), with the birth function $g(x,N)$ assumed to be non-monotone with respect to N. Clearly, $\tilde{Q}$ is not monotone in X^γ. In this section, we need the following additional assumption:

(A4) There is $a > 0$ such that for each $x \in \overline{\Omega}$, $g(x,N)$ is non-decreasing with respect to $N \in [0,a]$.

Theorem 2.4.2. *Assume that (A1)-(A4) hold and let $N_n(x;\phi)$ be the solution of (2.18) with $\phi \in X_+^\gamma$. Then the following statements are valid:*

(i) If $r(D\tilde{Q}(0)) < 1$, then the fixed point zero is globally asymptotically stable in X_+^γ.

(ii) If $r(D\tilde{Q}(0)) > 1$, then the solution sequence $N_n(x)$ of system (2.18) with the initial data $N_0 = \phi \in X_+^\gamma \backslash \{0\}$ satisfies

$$\limsup_{n\to\infty} \max_{x\in\overline{\Omega}} \left(N_n(x;\phi) - N^{*+}(x)\right) \leq 0 \leq \liminf_{n\to\infty} \min_{x\in\overline{\Omega}} \left(N_n(x;\phi) - N^{*-}(x)\right),$$

where $N^{*+}(x)$ and $N^{*-}(x)$ are positive fixed points of systems (2.24) and (2.25) later, respectively. Further, system (2.18) admits at least one positive fixed point.

Proof. For any $x \in \overline{\Omega}$, we define $g^+(x,N)$ and $g^-(x,N)$ as in section 2.3.2. Then we have the following two auxiliary systems:

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = \dfrac{\partial}{\partial x}(D_M(t,x)\dfrac{\partial u_n}{\partial x}) + f(t,x,u_n), \ 0 < t \leq 1, t \neq \tau, x \in \overline{\Omega}, n \in \mathbb{N}, \\ u_n(t^+,x) = u_n(t,x) + \tilde{R}^+(x;N_n^+), \ t = \tau, \\ u_n(0,x) = N_n^+(x), \\ N_{n+1}^+(x) = u_n(1,x), \\ u(0,x) = N_0^+(x) = \phi \in X^\gamma, \end{cases} \tag{2.22}$$

and

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = \dfrac{\partial}{\partial x}(D_M(t,x)\dfrac{\partial u_n}{\partial x}) + f(t,x,u_n),\ 0 < t \le 1, t \ne \tau, x \in \overline{\Omega}, n \in \mathbb{N}, \\ u_n(t^+,x) = u_n(t,x) + \tilde{R}^-(x; N_n^-),\ t = \tau, \\ u_n(0,x) = N_n^-(x), \\ N_{n+1}^-(x) = u_n(1,x), \\ u(0,x) = N_0^-(x) = \phi \in X^\gamma, \end{cases} \tag{2.23}$$

where

$$\tilde{R}^{\pm}(x; N_n^{\pm}) = [\tilde{T}(\tau,0)g^{\pm}(\cdot, N_n^{\pm})](x).$$

Thus, the time-one solution maps of systems (2.22) and (2.23) give rise to the following discrete-time recursions:

$$\begin{aligned} N_{n+1}^+(x) &= \tilde{Q}^+\left[N_n^+\right](x) \\ &= \left[\tilde{M}(1,\tau)\left[\tilde{M}(\tau,0)(N_n^+) + \tilde{T}(\tau,0)(g^+(\cdot,N_n^+))\right]\right](x),\ x \in \overline{\Omega}, n \in \mathbb{N}, \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} N_{n+1}^-(x) &= \tilde{Q}^-\left[N_n^-\right](x) \\ &= \left[\tilde{M}(1,\tau)\left[\tilde{M}(\tau,0)(N_n^-) + \tilde{T}(\tau,0)(g^-(\cdot,N_n^-))\right]\right](x),\ x \in \overline{\Omega}, n \in \mathbb{N}. \end{aligned} \tag{2.25}$$

By the comparison arguments, it follows that if $0 < N_0^-(x) \le N_0(x) \le N_0^+(x) \le N^{*+}(x)$, then

$$0 \le N_n^-(x) \le N_n(x) \le N_n^+(x),\ x \in \overline{\Omega}, n \in \mathbb{N}. \tag{2.26}$$

Using the same arguments as in section 2.4.1, we see that systems (2.22) and (2.23), with the monotone birth functions g^+ and g^- respectively, admit the same linearized system, and the spectral radius of the time-one map of this linearized system is $r(D\tilde{Q}(0))$. Using (2.26) and the comparison arguments, we see that the desired s-tatements hold true. Furthermore, we define

$$G = \{w \in X^\gamma : N^{*-}(x) \le w(x) \le N^{*+}(x), x \in \overline{\Omega}\}.$$

Clearly, G is a closed, bounded, convex and nonempty subset of X^γ. By utilizing the compactness of $\tilde{Q}$ and Schauder's fixed point theorem, we can prove that system (2.18) admits at least one positive fixed point. □

2.5 Numerical simulations

In this section, we present some numerical simulations for system (2.1). We first investigate the influence of τ for the evolution of adult population. For simplicity, let the loss function $f(t, x, u) \equiv f(u) = -au - bu^2$, with $a = 1$ being the natural death rate of the adult population and $b = 0.01$ representing the strength of the density-dependent interspecific competition between individuals. Let $g(x, N) \equiv g(N) = \frac{pN}{q+N}$, with $p = 1.8$ and $q = 0.2$. The function $g(N)$ is the Beverton-Holt function, which is monotone with respect to N. The random diffusion functions are fixed at $D_M = 1$ and $D_I = 0.2$. Let the initial function $N_0(x) = \cos\left(\frac{\pi x}{20}\right)$ with a compact support on $[-10, 10]$ in the domain $[-100, 100]$.

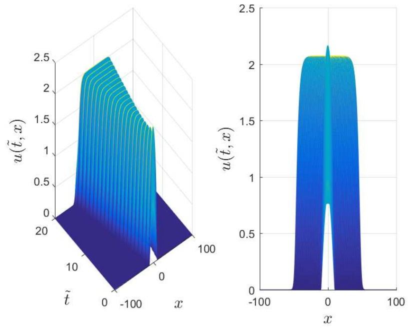
Based on the above parameters and different d_I and τ, the spreading of species is shown as in Figure 2.2, which demonstrates the effects of d_I and τ on the positive fixed point of the adult population. More precisely, Figures 2.2(a) and 2.2(b) indicate that the positive fixed point of the adult population will decrease with τ increasing as $d_I = 0.5$. Observing Figures 2.2(c) and 2.2(d) carefully, we can find that the negative correlation between the positive fixed point and τ as d_I decreases to $d_I = 0.05$. The fundamental reason is the relationship between functions f and d_I. In terms of ecology, when $|f| > d_I$, that is, the mortality rate of mature individuals is higher than that of immature individuals. Therefore, the longer it takes for immature individuals to grow into adult individuals (i.e., $\tau \nearrow$), the higher the positive fixed point of adult groups. When $|f| < d_I$, the correlation is the opposite.

Both of these situations can occur in ecology. In the first case, for example, some species will move into the wild after they mature, which makes them easy to be hunted; that is, $|f| > d_I$. In the second case, adult species are more likely to adapt to the environment and survive in some harsh environments; that is, $|f| < d_I$. This result improves the numerical simulations in [8].

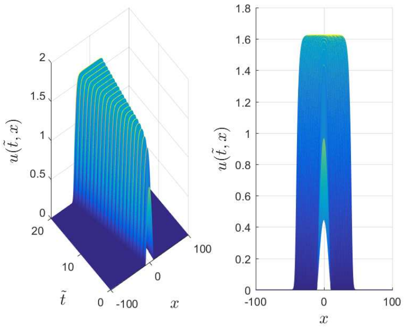
Next we give numerical experiments to show the long-time behaviour of the solution of system (2.1) with time-space periodic parameters in the monotone and non-monotone cases, respectively. Let $N_0(x) = 0.2\cos\left(\frac{\pi x}{20}\right)$ with a compact support on $[-10, 10]$ in the domain $[-50, 50]$, and $f(t, x, u) = w(t, x)f(u) = w(t, x)(-au - bu^2)$ with $w(t, x) = (1 + 0.3\cos t)(1 + 0.3\sin x)$, $a = 1$ and $b = 0.01$. In the monotone case, we adopt the birth function g as $g_1 = \frac{pN}{q+N}$ form, and other parameters remain the same as above. This result is demonstrated in Figure 2.3. In the non-monotone case, we set $g = g_2 = Ne^{r-kN}$ with positive constants $r = 2.5$ and $k = 1$. The function g_2 represents the Ricker function. The corresponding result is shown in Figure 2.4. In particular, the spreading speeds of system (2.1) are visually depicted in Figures 2.3 and 2.4 for monotone and non-monotone cases, respectively.

2.6 Conclusions and discussion

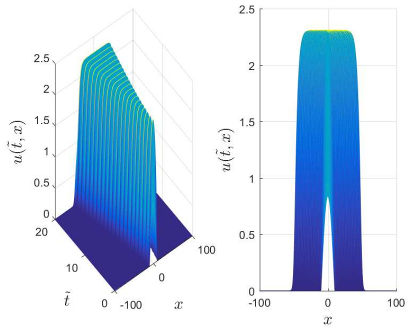
In this chapter, by incorporating the temporal and spatial variations into an impulsive system, we propose a time-space periodic reaction-diffusion model with an annual impulsive maturation emergence term to study the invasion dynamics in unbounded



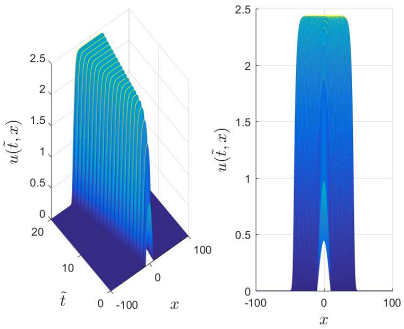
(a) $d_I = 0.5$ and $\tau = 0.2$.



(b) $d_I = 0.5$ and $\tau = 0.8$.

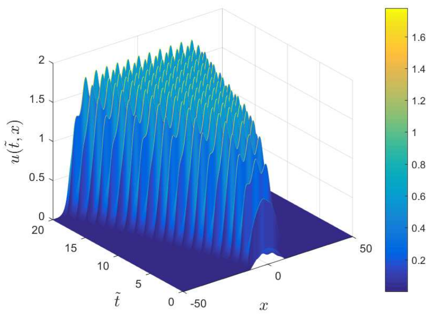


(c) $d_I = 0.05$ and $\tau = 0.2$.

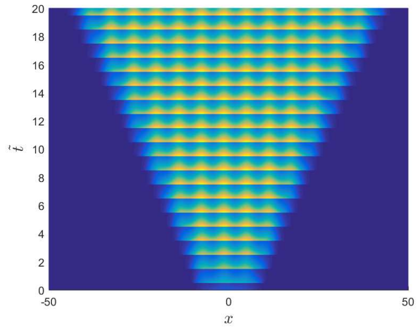


(d) $d_I = 0.05$ and $\tau = 0.8$.

Figure 2.2: If $d_I = 0.5$, then the positive fixed point will decrease when only parameter τ increases from 0.2 to 0.8, (from Figure 2.2(a) to Figure 2.2(b)). Nevertheless, if $d_I = 0.05$, then the positive fixed point will increase when only parameter τ increases from 0.2 to 0.8, (from Figure 2.2(c) to Figure 2.2(d)).

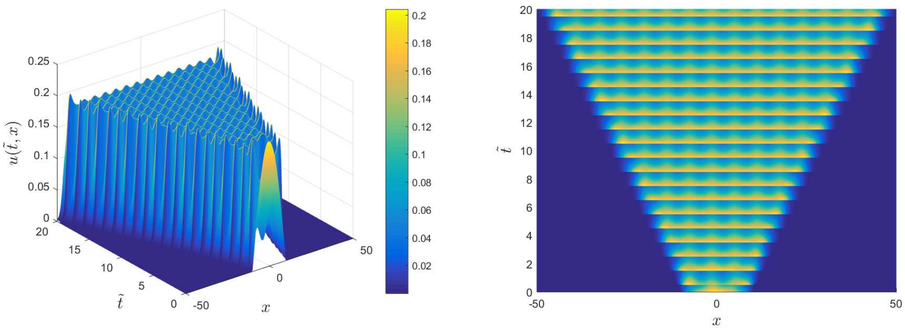


(a) The spatial and temporal evolution of adult population.



(b) The two-dimensional projection of Figure 2.3(a) on the $x\tilde{t}$ plane.

Figure 2.3: The long-time behaviour of the solution of system (2.1) with time-space periodic parameters and the monotone birth function.



(a) The spatial and temporal evolution of adult population.

(b) The two-dimensional projection of Figure 2.4(a) on the $x\tilde{t}$ plane.

Figure 2.4: The long-time behaviour of the solution of system (2.1) with time-space periodic parameters and the non-monotone birth function.

and bounded domains, respectively. We reduce the study of the evolution dynamics of such a model to that of a discrete-time system from the evolution viewpoint. When the habitat is unbounded, we obtain the existence of spreading speeds in both monotone and non-monotone cases and show that they are linearly determinate. We further prove that the spreading speeds in the monotone case coincide with the minimal speeds of spatially periodic traveling waves. When the habitat is bounded, we introduce the fractional power space to deal with general boundary conditions and establish the global stability results. Note that when the bounded domain is one-dimensional with Dirichlet boundary condition, we can also obtain the critical domain size to describe the persistence and extinction of species, which is similar to that in [8, Section 4] and [136, Section 2]. The numerical simulations reveal meaningful phenomena when $|f| > d_I$, the positive fixed point and time τ are positively correlated, while when $|f| < d_I$, the relationship is converse. In particular, we give basic ecological reasons for these two numerical results. We also use numerical experiments to illustrate the long-time behaviour of solutions of system (2.1) with time-space periodic parameters in both monotone and non-monotone cases. Simultaneously, we visually demonstrate the spreading speeds in both scenarios.

It is worth mentioning that in the non-monotone case, the nonexistence of spatially periodic traveling waves is a straightforward consequence of the spreading speeds (see, e.g., [136, Theorem 3.5(i)]). However, the existence of spatially periodic traveling waves in this case is still a challenging open problem, and we leave it for future investigation.

In plant and animal populations, reproductive synchrony is evident, as seen in events like salmon spawning in October, which allows us to designate the production season at the beginning of the year. Similarly, seasonal pest control and the harvesting of certain plants or animals during specific seasons can create pulse-like patterns that are suitable for analyzing dynamic behaviors.

Chapter 3

A time-delayed nonlocal reaction-diffusion model of within-host viral infections in a bounded domain

In this chapter, we study a time-delayed nonlocal reaction-diffusion model of within-host viral infections. We derive the basic reproduction number $\mathscr{R}_0$ and show that the infection-free steady state is globally asymptotically stable when $\mathscr{R}_0 \leq 1$, while the disease is uniformly persistent when $\mathscr{R}_0 > 1$. In the case where all coefficients and reaction terms are spatially homogeneous, we obtain an explicit formula of $\mathscr{R}_0$ and the global attractivity of the positive constant steady state. Numerically, we illustrate the analytical results, conduct sensitivity analysis, and investigate the impact of drugs on curtailing the spread of the viruses.

3.1 Introduction

In-host viral infections commonly involve hepatitis B virus (HBV), hepatitis C virus (HCV), and human immunodeficiency virus (HIV). These infections have been extensively investigated through mathematical models to understand their patterns of infection and transmission. The basic within-host viral infection model consists of uninfected target cells, infected target cells and free viruses (see, e.g., [59, 60, 64, 98, 107, 119, 142, 143]). In [142], the authors investigated the following within-host viral infection model with both virus-to-cell and cell-to-cell transmissions and three distributed delays:

$$\begin{cases} \frac{dT(t)}{dt} = n(T(t)) - \beta_1 T(t)V(t) - \beta_2 T(t)I(t), \\ \frac{dI(t)}{dt} = \int_0^\infty \beta_1 T(t-s)V(t-s)f_1(s)e^{-\mu_1 s}ds \\ \qquad\quad + \int_0^\infty \beta_2 T(t-s)I(t-s)f_2(s)e^{-\mu_1 s}ds - \mu_1 I(t), \\ \frac{dV(t)}{dt} = b\int_0^\infty e^{-\mu_2 s}f_3(s)I(t-s)ds - \mu V(t), \end{cases} \tag{3.1}$$

where $n(T)$ is a function describing the natural change (i.e., production and turnover) of healthy target cells. This compartmental model includes the concentrations of healthy target cells T which are susceptible to infection, infected cells I that produce viruses, and viruses V. Here β_1 and β_2 are the virus-to-cell infection rate and the cell-to-cell infection rate; μ_1 and μ are the death rates of actively infected cells and viruses, respectively; μ_2 is the death rate of immature viruses between viral RNA transcription and viral release and maturation; b is the average number of viruses that bud out from an infected cell; and $f_i(s), i = 1, 2, 3$, are the probability distributions.

Recent evidence suggests that spatial heterogeneity plays an essential role in the within-host viral infections [39]. To consider the influences of spatial structure on viral infection dynamics, a reaction-diffusion model of within-host HBV was introduced in [122], where the existence of the minimum wave speed was established when the diffusion ability of virions is very small. A general viral infection model inside a spherical organ, which includes the diffusion, penetration and proliferation of immune cells, was studied in [23]. Later, a reaction-diffusion model was proposed in [61] to study the virion repulsion effect by infected cells, where the global dynamics and spreading speed were discussed. In [98], the authors developed a spatially explicit within-host HIV model to investigate the influence of cell or virion mobility and spatial heterogeneity on HIV pathogenesis. To understand virus persistence in a heterogeneous space, they studied the bounded space scenario with the Neumann boundary condition. Furthermore, these authors explored the existence of traveling wave solutions and determined the virus spreading speed in vivo by addressing the traveling wave solution problem on an unbounded domain $\mathbb{R}$ with spatially homogeneous parameters. Recently, the authors of [107] investigated a comprehensive within-host model involving spatial heterogeneity and general cell-free and cell-to-cell modes. They analyzed the basic reproduction number, a critical parameter influencing the global dynamics of the model, and proved the global attractivity of the chronic-infection steady state by constructing a suitable Lyapunov functional.

Motivated by system (3.1), in this paper we present and analyze a time-delayed nonlocal reaction-diffusion model. We introduce the basic reproduction number $\mathscr{R}_0$, and show that the infection-free steady state is globally asymptotically stable when $\mathscr{R}_0 \leq 1$, whereas the disease is uniformly persistent when $\mathscr{R}_0 > 1$. More precisely, we use the comparison arguments to deal with the cases $\mathscr{R}_0 < 1$ and $\mathscr{R}_0 = 1$ simultaneously for the global attractivity of the infection-free steady state. By a simple observation on Lyapunov stability (see Lemma 3.3.7), we then obtain its global asymptotic stability. And we establish the uniform persistence of the disease and the existence of a positive steady state in the case where $\mathscr{R}_0 > 1$. For a simplified model with constant coefficients, we further prove the global attractivity of the positive constant steady state via the method of Lyapunov functionals. We note that the construction of such a Lyapunov functional is highly nontrivial in the presence of non-local terms and time delays. Finally, we employ the Matlab to conduct numerical simulations. Utilizing parameter values documented in the published literature, we perform a numerical calculation to determine the value of $\mathscr{R}_0$. The evolution of solutions is visually presented for both cases $\mathscr{R}_0 > 1$ and $\mathscr{R}_0 < 1$. We also incorporate sensitivity analysis to examine the impact of individual parameters on $\mathscr{R}_0$, investigate an optimal drug strategy within a spatially heterogeneous context, and provide pertinent control strategies.

The rest of the chapter is organized as follows. In section 3.2, we formulate a time-delayed nonlocal reaction-diffusion model and address the well-posedness of the solutions. In section 3.3, we study the global dynamics of the model in a bounded spatial domain. In section 3.4, we conduct some numerical simulations that support the analytical results and reveal relevant virus transmission strategies. A brief discussion concludes the chapter.

3.2 Model formulation

We assume that the infections occur in a spatially bounded domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$ of class $C^{2+\theta}(0 < \theta \leq 1)$. Let $\Gamma(t, x, y; D)$ be the Green function associated with the following linear equation:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \nabla \cdot (D(x)\nabla u) - m(x)u, \ t > 0, x \in \Omega, \\
\frac{\partial u}{\partial \nu} &= 0, t > 0, x \in \partial\Omega,
\end{aligned}$$

where ν is the outward normal vector to $\partial\Omega$, and m is the death rate. It follows that if a diffusive species has spatial density $\phi(y)$ at time s, then the integral $\int_\Omega \Gamma(t - s, x, y; D)\phi(y)dy$ gives the spatial density at time $t \geq s$ due to the diffusion.

Let $\Phi(t)$ be the solution semigroup on $C(\bar{\Omega}, \mathbb{R})$ associated with the linear reaction-diffusion equation:

$$\begin{aligned}
\frac{\partial I(t,x)}{\partial t} &= \nabla \cdot (D_I(x)\nabla I(t,x)) - \mu_1(x)I(t,x), \ t > 0, x \in \Omega, \\
\frac{\partial I(t,x)}{\partial \nu} &= 0, \ t > 0, x \in \partial\Omega.
\end{aligned}$$

Let $\Gamma_I(t, x, y; D_I)$ be the Green function associated with the above equation, that is,

$$[\Phi(t)\phi](x) = \int_\Omega \Gamma_I(t, x, y; D_I)\phi(y)dy.$$

We assume that the rates of new infections at location x and time t via two transmission modes are $f_1(x, T, V)$ and $f_2(x, T, I)$, which describe the virus-to-cell infection and the cell-to-cell infection, respectively. Then the newly infected cells produced by viruses and infected cells at time t and location x are

$$[\Phi(\tau_1)f_1(\cdot, T(t-\tau_1, \cdot), V(t-\tau_1, \cdot))](x) = \int_\Omega \Gamma_I(\tau_1, x, y; D_I)f_1(y, T(t-\tau_1, y), V(t-\tau_1, y))dy$$

and

$$[\Phi(\tau_2)f_2(\cdot, T(t-\tau_2, \cdot), I(t-\tau_2, \cdot))](x) = \int_\Omega \Gamma_I(\tau_2, x, y; D_I)f_2(y, T(t-\tau_2, y), I(t-\tau_2, y))dy,$$

respectively, where τ_1 and τ_2 are the average incubation periods for the healthy target cells infected by viruses and infected cells.

Let $\Psi(t)$ be the solution semigroup on $C(\bar{\Omega}, \mathbb{R})$ associated with the linear reaction-diffusion equation:

$$\begin{aligned}
\frac{\partial V(t,x)}{\partial t} &= \nabla \cdot (D_V(x)\nabla V(t,x)) - \mu_2(x)V(t,x), t > 0, x \in \Omega,\\
\frac{\partial V(t,x)}{\partial \nu} &= 0, t > 0, x \in \partial\Omega,
\end{aligned}$$

where $\mu_2(x)$ is the death rate of immature viruses between viral RNA transcription and viral release and maturation. Let $\Gamma_1(t, x, y; D_V)$ be the Green function associated with the above equation, that is,

$$[\Psi(t)\psi](x) = \int_\Omega \Gamma_1(t, x, y; D_V)\psi(y)dy.$$

It follows that the matured viruses produced by infected cells at time t and location x are

$$[\Psi(\tau_3)b(\cdot)I(t - \tau_3, \cdot)](x) = \int_\Omega \Gamma_1(\tau_3, x, y; D_V)b(y)I(t - \tau_3, y)dy,$$

where $b(x)$ is the number of viruses that bud from an infected cell at location x, and τ_3 is the average maturation time of the viruses.

Accordingly, the model (3.1) can be modified into the following time-delayed nonlocal reaction-diffusion system:

$$\begin{cases}
\frac{\partial T(t,x)}{\partial t} = \nabla \cdot (D_T(x)\nabla T(t,x)) + n(x, T(t,x)) - f_1(x, T(t,x), V(t,x)) \\
\qquad - f_2(x, T(t,x), I(t,x)), \\
\frac{\partial I(t,x)}{\partial t} = \nabla \cdot (D_I(x)\nabla I(t,x)) - \mu_1(x)I(t,x) \\
\qquad + [\Phi(\tau_1)f_1(\cdot, T(t-\tau_1, \cdot), V(t-\tau_1, \cdot))](x) \\
\qquad + [\Phi(\tau_2)f_2(\cdot, T(t-\tau_2, \cdot), I(t-\tau_2, \cdot))](x), \\
\frac{\partial V(t,x)}{\partial t} = \nabla \cdot (D_V(x)\nabla V(t,x)) + [\Psi(\tau_3)b(\cdot)I(t-\tau_3, \cdot)](x) \\
\qquad - \mu(x)V(t,x), t > 0, x \in \Omega,
\end{cases} \tag{3.2}$$

subject to the Neumann boundary condition

$$\frac{\partial T(t,x)}{\partial \nu} = \frac{\partial I(t,x)}{\partial \nu} = \frac{\partial V(t,x)}{\partial \nu} = 0, t > 0, x \in \partial\Omega.$$

Here we assume that all constant parameters are positive and that the functions $\mu_1(x)$, $\mu_2(x)$, and $\mu(x)$ are continuous and positive on $\bar{\Omega}$. Moreover, $D_T(x), D_I(x)$ and $D_V(x)$ are in $C^{1+\theta}(\bar{\Omega})$ $(0 < \theta \le 1)$, and they are positive on $\bar{\Omega}$.

We make the following assumptions for $n(x,T)$ and f_i $(i = 1,2)$:

(A1) $n(x,T) \in C^1(\bar{\Omega} \times \mathbb{R}_+)$, $\sup_{(x,T)\in\bar{\Omega}\times\mathbb{R}_+} \frac{\partial n(x,T)}{\partial T} < 0$, and there exists a function $T_0 \in C(\bar{\Omega}, (0,\infty))$ such that $n(x, T_0(x)) = 0$ for all $x \in \bar{\Omega}$.

(A2) $f_1(x,T,V), f_2(x,T,I) \in C^{1,1,2}(\bar{\Omega} \times \mathbb{R}^2_+)$, $\partial_T f_1(x,T,V) > 0$, $\partial_V f_1(x,T,V) > 0$, $\partial_T f_2(x,T,I) > 0$ and $\partial_I f_2(x,T,I) > 0$ for all $(x,T,I,V) \in \bar{\Omega} \times int(\mathbb{R}^3_+)$; $f_1(x,T,V) = 0$ whenever $TV = 0$, and $f_2(x,T,I) = 0$ whenever $TI = 0$; $\frac{\partial^2 f_1(x,T,V)}{\partial V^2} \le 0$ and $\frac{\partial^2 f_2(x,T,I)}{\partial I^2} \le 0$ for all $(x,T,I,V) \in \bar{\Omega} \times \mathbb{R}^3_+$.

Under assumption (A1), the mean value theorem implies that there exist two positive numbers s and h such that

$$n(x,T) \le s - hT, \quad \forall (x,T) \in \bar{\Omega} \times \mathbb{R}_+.$$

The prototypical examples of the functions f_i satisfying (A2) include the Holling type II functional response [121], Beddington-DeAngelis functional response [47], saturation infection rate [140], and Crowley-Martin functional response [121].

Let $\mathbb{X} := C(\bar{\Omega}, \mathbb{R}^3)$ be the Banach space with the supremum norm $\|\cdot\|_{\mathbb{X}}$, and let $\hat{\tau} = \max\{\tau_1, \tau_2, \tau_3\}$. Define $X := C([-\hat{\tau}, 0], \mathbb{X})$ with the norm $\|\phi\|_X = \max_{\theta\in[-\hat{\tau},0]} \|\phi(\theta)\|_{\mathbb{X}}$, $\forall \phi \in X$. Let $\mathbb{X}^+ := C(\bar{\Omega}, \mathbb{R}^3_+)$ and $X^+ := C([-\hat{\tau}, 0], \mathbb{X}^+)$. Then $(\mathbb{X}, \mathbb{X}^+)$ and (X, X^+) are ordered Banach spaces. We also identity $\phi \in X$ with an element in $C([-\hat{\tau}, 0] \times \bar{\Omega}, \mathbb{R}^3)$ by defining $\phi(\theta, x) = \phi(\theta)(x)$, $\forall \theta \in [-\hat{\tau}, 0]$, $\forall x \in \bar{\Omega}$. Given a function $u : [-\hat{\tau}, \sigma) \to \mathbb{X}$ for $\sigma > 0$, we define $u_t \in X$ by $u_t(\theta) = u(t+\theta)$, $\forall \theta \in [-\hat{\tau}, 0]$, for any $t \in [0, \sigma)$.

Let $\mathbb{Y} := C(\bar{\Omega}, \mathbb{R})$ and $\mathbb{Y}^+ := C(\bar{\Omega}, \mathbb{R}_+)$. Let $T_1(t)$ be the solution semigroup on $\mathbb{Y}$ generated by the linear reaction-diffusion equation:

$$\begin{aligned} \frac{\partial T(t,x)}{\partial t} &= \nabla \cdot (D_T(x) \nabla T(t,x)), \ t > 0, x \in \Omega, \\ \frac{\partial T(t,x)}{\partial \nu} &= 0, t > 0, x \in \partial\Omega, \end{aligned}$$

and let $\Gamma_T(t,x,y; D_T)$ be the Green function associated with the above linear equation. It then follows that

$$[T_1(t)\psi](x) = \int_\Omega \Gamma_T(t,x,y; D_T)\psi(y)dy.$$

Let $T_2(t)$ be the solution semigroup on $\mathbb{Y}$ of the linear reaction-diffusion equation:

$$\begin{aligned}\frac{\partial I(t,x)}{\partial t} &= \nabla \cdot (D_I(x)\nabla I(t,x)) - \mu_1(x)I(t,x), t>0, x\in\Omega,\\ \frac{\partial I(t,x)}{\partial \nu} &= 0, t>0, x\in\partial\Omega,\end{aligned}$$

with the Green function $\Gamma_I(t,x,y;D_I)$, that is,

$$[T_2(t)\psi](x) = [\Phi(t)\psi](x) = \int_\Omega \Gamma_I(t,x,y;D_I)\psi(y)dy.$$

Let $T_3(t)$ be the solution semigroup on $\mathbb{Y}$ of the linear reaction-diffusion equation:

$$\begin{aligned}\frac{\partial V(t,x)}{\partial t} &= \nabla \cdot (D_V(x)\nabla V(t,x)) - \mu(x)V(t,x), t>0, x\in\Omega,\\ \frac{\partial V(t,x)}{\partial \nu} &= 0, t>0, x\in\partial\Omega,\end{aligned}$$

with the Green function $\Gamma_2(t,x,y;D_V)$, that is,

$$[T_3(t)\psi](x) = \int_\Omega \Gamma_2(t,x,y;D_V)\psi(y)dy.$$

It then follows that for each $t>0$, $T_i(t):\mathbb{Y}\to\mathbb{Y}(i=1,2,3)$ is compact and strongly positive (see, e.g., [109, Theorems 7.3.1 and 7.4.1]). Furthermore, $\mathcal{T}(t) := (T_1(t), T_2(t), T_3(t))$ is a C_0 semigroup on $\mathbb{X}$.

Denote

$$\mathcal{F}\begin{pmatrix}T\\ I\\ V\end{pmatrix} = \begin{pmatrix} n(\cdot,T(0,\cdot)) - f_1(\cdot,T(0,\cdot),V(0,\cdot)) - f_2(\cdot,T(0,\cdot),I(0,\cdot))\\ [\Phi(\tau_1)f_1(\cdot,T(-\tau_1,\cdot),V(-\tau_1,\cdot))] + [\Phi(\tau_2)f_2(\cdot,T(-\tau_2,\cdot),I(-\tau_2,\cdot))]\\ [\Psi(\tau_3)b(\cdot)I(-\tau_3,\cdot)]\end{pmatrix}.$$

Let $u=(T,I,V)$ with $u_0=\phi$. Then we can rewrite system (3.2) as the following abstract integral form:

$$\begin{cases} u(t,\phi) = \mathcal{T}(t)\phi + \displaystyle\int_0^t \mathcal{T}(t-s)\mathcal{F}(u_s)ds, t>0,\\ u_0 = \phi \in X^+.\end{cases} \tag{3.3}$$

Lemma 3.2.1. *For any $\phi\in X^+$, system (3.2) has a unique non-continuable non-negative mild solution $u(t,\cdot,\phi) = (T(t,\cdot,\phi), I(t,\cdot,\phi), V(t,\cdot,\phi))$ with $u_0=\phi$ on the maximal interval of existence $t\in[0,t_\phi)$, where $t_\phi\le+\infty$. Moreover, $u(t,\cdot,\phi)$ is a classical solution of system (3.2) for $t\in(\hat\tau,t_\phi)$ whenever $\hat\tau\in[0,t_\phi)$.*

Proof. Clearly, $\mathcal{F}$ is locally Lipschitz continuous. By Corollary 1.4.1, it suffices to verify that

$$\lim_{k\to 0^+} \frac{1}{k}\text{dist}(\phi(0,\cdot)+k\mathcal{F}(\phi),\mathbb{X}^+)=0, \tag{3.4}$$

for any given $\phi \in X^+$. In view of (A2), we have $f_1(x,0,V)\equiv 0$ and $f_2(x,0,I)\equiv 0$. It then follows that there exists a sufficiently large number $c=c(\phi)>0$ such that

$$\mathcal{F}_1(\phi)(x)+c\phi_1(0,x)\geq 0, \quad \forall x\in\bar{\Omega},$$

and hence,

$$\phi(0,x)+k\mathcal{F}(\phi)(x)\geq \begin{pmatrix}\phi_1(0,x)(1-ck)\\ \phi_2(0,x)\\ \phi_3(0,x)\end{pmatrix}, \quad \forall x\in\bar{\Omega}, k\geq 0.$$

This implies that $\phi(0,\cdot)+k\mathcal{F}(\phi)\in\mathbb{X}^+$ for sufficiently small $k>0$, which proves (3.4). Furthermore, similar to the proof of [79, Theorem 1], we can conclude that $u(t,\cdot,\phi)$ is a classical solution of system (3.2) for $t\in(\hat{\tau},t_\phi)$ whenever $\hat{\tau}\in[0,t_\phi)$. □

We consider the following scalar reaction-diffusion equation:

$$\begin{aligned}\frac{\partial m(t,x)}{\partial t}&=\nabla\cdot(D_T(x)\nabla m(t,x))+n(x,m(t,x)),\ t>0,x\in\Omega,\\ \frac{\partial m(t,x)}{\partial \nu}&=0,t>0,x\in\partial\Omega.\end{aligned} \tag{3.5}$$

Lemma 3.2.2. *System (3.5) admits a globally asymptotically stable positive steady state $T^*(x)$ in $\mathbb{Y}^+$.*

Proof. Note that all sufficiently large positive constants are upper solutions of system (3.5). It then follows that for any $\varphi\in\mathbb{Y}^+$, system (3.5) has a unique solution $m(t,x,\varphi)$ on $[0,\infty)$ with $m(0,\cdot,\varphi)=\varphi$ and that solutions of (3.5) are uniformly bounded, and hence, the time-t map $Z(t)$ on $\mathbb{Y}^+$ of (3.5), defined by $Z(t)\varphi=m(t,\cdot,\varphi)$, is compact for each $t>0$. In view of (A1), the function $n(x,m)$ satisfies

$$n(x,\alpha m)>n(x,m)>\alpha n(x,m), \quad \forall x\in\bar{\Omega},\ m>0,\ \alpha\in(0,1).$$

Thus, the solution semiflow $Z(t)$ is strongly monotone and strictly subhomogeneous on $\mathbb{Y}^+$.

Since $n(x,0)>0,\forall x\in\bar{\Omega}$, the maximum principle implies that $m(t,\cdot,\varphi)\gg 0$ in $\mathbb{Y}$ for all $t>0$ and $\varphi\in\mathbb{Y}^+$. Let $\omega(\varphi)$ be the omega limit set of the forward orbit through $\varphi\in\mathbb{Y}^+$ for the semiflow $Z(t)$. Then $\omega(\varphi)$ is compact and invariant for $Z(t)$ in the sense that $Z(t)(\omega(\varphi))=\omega(\varphi),\forall t\geq 0$. For any $\psi\in\omega(\varphi)$, there exists $\hat{\psi}\in\omega(\varphi)\subset\mathbb{Y}^+$ such that $\psi=Z(1)\hat{\psi}=m(1,\cdot,\hat{\psi})\gg 0$. This implies that $\omega(\varphi)\subset int(\mathbb{Y}^+),\forall\varphi\in\mathbb{Y}^+$. By Theorem 1.3.5, as applied to the map $Z(1)$, it follows that there exists $T^*\in int(\mathbb{Y}^+)$

such that $\omega(\varphi) = T^*, \forall \varphi \in \mathbb{Y}^+$. Clearly, the invariance of $\omega(\varphi)$ for $Z(t)$ implies that $T^*(x)$ is a steady state of system (3.5). In view of Lemma 1.3.1, we see that $T^*(x)$ is Lyapunov stable. Consequently, $T^*(x)$ is a globally asymptotically stable steady state for system (3.5) in $\mathbb{Y}^+$. □

Lemma 3.2.3. *For any $\phi \in X^+$, system (3.2) has a unique nonnegative mild solution $u(t,\cdot,\phi)$ on $[0,+\infty)$ with $u_0 = \phi$, and solutions are ultimately bounded. Moreover, the solution semiflow $Q(t)$ generated by system (3.2) has a strong global attractor in X^+.*

Proof. Our arguments are motivated by the proof of [113, Theorem 2.1]. By the positivity of solutions of (3.2) (see Lemma 3.2.1) and (A2), there holds

$$\frac{\partial T(t,x)}{\partial t} \leq \nabla \cdot (D_T(x)\nabla T(t,x)) + n(x,T(t,x)). \tag{3.6}$$

It follows from Lemma 3.2.2 and the standard parabolic comparison theorem that $T(t,\cdot,\phi)$ is bounded on $t \in [0,t_\phi)$; that is, for any $\phi \in X^+$, there exists $B_1 = B_1(\phi) > 0$, such that

$$||T(t,x,\phi)||_{\mathbb{Y}} \leq B_1, \ \forall t \in [0,t_\phi), x \in \bar{\Omega}.$$

In view of (A2), we have

$$\begin{aligned} \frac{\partial I(t,x)}{\partial t} \leq\ & \nabla \cdot (D_I(x)\nabla I(t,x)) - \mu_1(x)I(t,x) \\ & + \left[\Phi(\tau_1)\frac{\partial f_1(\cdot,B_1,0)}{\partial V}V(t-\tau_1,\cdot)\right](x) \\ & + \left[\Phi(\tau_2)\frac{\partial f_2(\cdot,B_1,0)}{\partial I}I(t-\tau_2,\cdot)\right](x), \ \forall t \in [0,t_\phi). \end{aligned} \tag{3.7}$$

It follows that I- and V- equations of system (3.2) are dominated by the following cooperative and linear system with time delay:

$$\begin{aligned} \frac{\partial I(t,x)}{\partial t} =\ & \nabla \cdot (D_I(x)\nabla I(t,x)) - \mu_1(x)I(t,x) \\ & + \left[\Phi(\tau_1)\frac{\partial f_1(\cdot,B_1,0)}{\partial V}V(t-\tau_1,\cdot)\right](x) \\ & + \left[\Phi(\tau_2)\frac{\partial f_2(\cdot,B_1,0)}{\partial I}I(t-\tau_2,\cdot)\right](x), \ \forall t \in [0,t_\phi), x \in \Omega, \\ \frac{\partial V(t,x)}{\partial t} =\ & \nabla \cdot (D_V(x)\nabla V(t,x)) + [\Psi(\tau_3)b(\cdot)I(t-\tau_3,\cdot)](x) \\ & - \mu(x)V(t,x), \ \forall t \in [0,t_\phi), x \in \Omega, \end{aligned}$$

subject to the Neumann boundary conditions. By virtue of the global existence of solutions for linear systems (see, e.g., [133, Theorem 2.1.1]), it follows that $t_\phi = +\infty$

for each $\phi \in X^+$. Then there exists $B_2 > 0$, independent of ϕ, such that for any $\phi \in X^+$, there exists $t_1 = t_1(\phi) > 0$ such that

$$||T(t, x, \phi)||_{\mathbb{Y}} \leq B_2, \ \forall t \geq t_1, x \in \bar{\Omega}. \tag{3.8}$$

Integrating the first equation of (3.2) yields

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega T(t,y)dy = \int_\Omega \frac{\partial T(t,y)}{\partial t}dy \leq \int_\Omega n(y, T(t,y))dx - \int_\Omega f_1(y, T(t,y), V(t,y))dy \\
&\leq \int_\Omega [s - hT(t,y)]dy - \int_\Omega f_1(y, T(t,y), V(t,y))dy \\
&= s|\Omega| - h\int_\Omega T(t,y)dy - \int_\Omega f_1(y, T(t,y), V(t,y))dy, \ \forall t > 0,
\end{aligned}$$

and hence,

$$\int_\Omega f_1(y, T(t,y), V(t,y))dy \leq s|\Omega| - h\int_\Omega T(t,y)dy - \frac{d}{dt}\int_\Omega T(t,y)dy, \ \forall t > 0.$$

In a similar way, we obtain

$$\int_\Omega f_2(y, T(t,y), I(t,y))dy \leq s|\Omega| - h\int_\Omega T(t,y)dy - \frac{d}{dt}\int_\Omega T(t,y)dy, \ \forall t > 0.$$

It then follows that

$$\begin{aligned}
&\int_\Omega f_1(y, T(t-\tau_1, y), V(t-\tau_1, y))dy + \int_\Omega f_2(y, T(t-\tau_2, y), I(t-\tau_2, y))dy \\
&\leq 2s|\Omega| - h\int_\Omega T(t-\tau_1, y)dy - \frac{d}{dt}\int_\Omega T(t-\tau_1, y)dy \\
&\quad - h\int_\Omega T(t-\tau_2, y)dy - \frac{d}{dt}\int_\Omega T(t-\tau_2, y)dy, \ \forall t \geq \hat{\tau}.
\end{aligned}$$

This, together with the second equation of (3.2), implies that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega I(t,y)dy = \int_\Omega \frac{\partial I(t,y)}{\partial t}dy \\
&\leq -\underline{\mu_1}\int_\Omega I(t,y)dy + c_1\int_\Omega f_1(y, T(t-\tau_1, y), V(t-\tau_1, y))dy \\
&\quad + c_1\int_\Omega f_2(y, T(t-\tau_2, y), I(t-\tau_2, y))dy \\
&\leq -\underline{\mu_1}\int_\Omega I(t,y)dy + c_1\left[2s|\Omega| - h\int_\Omega T(t-\tau_1, y)dy\right. \\
&\quad \left. -h\int_\Omega T(t-\tau_2, y)dy - \frac{d}{dt}\int_\Omega T(t-\tau_1, y)dy - \frac{d}{dt}\int_\Omega T(t-\tau_2, y)dy\right], \ \forall t \geq \hat{\tau},
\end{aligned}$$

where $\underline{\mu_1} = \min_{x\in\bar{\Omega}} \mu_1(x)$, and c_1 is a positive constant independent of ϕ. Thus, we have

$$\frac{d}{dt}\left\{\int_\Omega I(t,y)dy + c_1\int_\Omega T(t-\tau_1,y)dy + c_1\int_\Omega T(t-\tau_2,y)dy\right\}$$
$$\leq -\min\{\underline{\mu_1},h\}\left\{\int_\Omega I(t,y)dy + c_1\int_\Omega T(t-\tau_1,y)dy + c_1\int_\Omega T(t-\tau_2,y)dy\right\}$$
$$+ 2c_1 s|\Omega|, \ \forall t \geq \hat{\tau}.$$

This implies that there exist a positive number c_2, independent of ϕ, and a positive number $c_3 = c_3(\phi)$ such that

$$\begin{aligned}\int_\Omega I(t,y)dy \leq \int_\Omega I(t,y)dy + c_1\int_\Omega T(t-\tau_1,y)dy + c_1\int_\Omega T(t-\tau_2,y)dy \\ \leq c_3(\phi)e^{-\min\{\underline{\mu_1},h\}t} + c_2, \ \forall t \geq \hat{\tau},\end{aligned} \tag{3.9}$$

and hence, $\lim_{t\to\infty}\int_\Omega I(t,y)dy \leq c_2$. It then follows that there exist a positive number c_4, independent of ϕ, and a sufficiently large number $t_2 = t_2(\phi) > 0$ such that

$$\frac{\partial V(t,x)}{\partial t} \leq \nabla\cdot(D_V(x)\nabla V(t,x)) + c_4 - \mu(x)V(t,x), \ \forall t \geq t_2, x \in \Omega,$$
$$\frac{\partial V(t,x)}{\partial \nu} = 0, t > 0, x \in \partial\Omega.$$

By the comparison principle, there exist a positive number $\hat{B}_2$, independent of ϕ, and a sufficiently large number $t_3 = t_3(\phi) > 0$ such that

$$||V(t,x,\phi)||_{\mathbb{Y}} \leq \hat{B}_2, \ \forall t \geq t_3, x \in \bar{\Omega}. \tag{3.10}$$

This, together with (3.7) and (3.9), implies that there exist a positive number c_5, independent of ϕ, and a sufficiently large number $t_4 = t_4(\phi) > 0$ such that

$$\frac{\partial I(t,x)}{\partial t} \leq \nabla\cdot(D_I(x)\nabla I(t,x)) - \mu_1(x)I(t,x) + c_5, \forall t \geq t_4, x \in \bar{\Omega}.$$

By the comparison principle again, there exist a positive $\tilde{B}_2 > 0$, independent of ϕ, and a sufficiently large number $t_5 = t_5(\phi) > 0$ such that

$$||I(t,x,\phi)||_{\mathbb{Y}} \leq \tilde{B}_2, \ \forall t \geq t_5, x \in \bar{\Omega}. \tag{3.11}$$

Thus, (3.8), (3.10) and (3.11) imply that solutions of system (3.2) are ultimately bounded.

Now we can define the solution semiflow $Q(t)$ on X^+ of system (3.2) by

$$Q(t)\phi = u_t(\phi), \ \forall \phi \in X^+, t \geq 0.$$

Clearly, the ultimate boundedness of solutions implies that $Q(t)$ is point dissipative on X^+. Furthermore, $Q(t)$ is compact for any $t > \hat{\tau}$ (see, e.g., [133, Theorem 2.1.8]). It then follows from the continuous-time version of Theorem 1.1.2(a) (see also [41, Theorem 3.4.8]) that $Q(t)$ has a strong global attractor in X^+. □

3.3 Threshold dynamics

In this section, we first focus on conditions for viral extinction or persistence in heterogeneous environments and then study the case where all the parameters are spatially homogenous.

3.3.1 The general system

We first introduce the basic reproduction number $\mathscr{R}_0$ by using the theory developed in [152]. Let $\mathbb{E} := C(\bar{\Omega}, \mathbb{R}^2)$ be the Banach space with the supremum norm $\|\cdot\|_{\mathbb{E}}$. Define $E := C([-\hat{\tau}, 0], \mathbb{E})$ with the norm $\|\psi\|_E = \max_{\theta\in[-\hat{\tau},0]} \|\psi(\theta)\|_{\mathbb{E}}$, $\forall \psi \in E$. Let $\mathbb{E}^+ := C(\bar{\Omega}, \mathbb{R}^2_+)$ and $E^+ := C([-\hat{\tau}, 0], \mathbb{E}^+)$. Then $(\mathbb{E}, \mathbb{E}^+)$ and (E, E^+) are ordered Banach spaces. We also identity $\psi \in E$ with an element in $C([-\hat{\tau}, 0] \times \bar{\Omega}, \mathbb{R}^2)$ by defining $\psi(\theta, x) = \psi(\theta)(x)$, $\forall \theta \in [-\hat{\tau}, 0]$, $\forall x \in \bar{\Omega}$. Given a function $w : [-\hat{\tau}, \sigma) \to \mathbb{E}$ for $\sigma > 0$, we define $w_t \in E$ by $w_t(\theta) = w(t + \theta)$, $\forall \theta \in [-\hat{\tau}, 0]$, for any $t \in [0, \sigma)$.

Let $\mathcal{L}(E, \mathbb{E})$ be the space of all bounded and linear operators from E to $\mathbb{E}$. For any $L \in \mathcal{L}(E, \mathbb{E})$, we define $\hat{L} \in \mathcal{L}(\mathbb{E}, \mathbb{E})$ by

$$\hat{L}y = L(\hat{y}), \forall y \in \mathbb{E},$$

where $\hat{y}(\theta) = y, \forall \theta \in [-\hat{\tau}, 0]$.

Setting $I = 0$ in system (3.2), we obtain the density of healthy cells satisfies (3.5). It follows from Lemma 3.2.2 that system (3.2) has an infection-free steady state $(T^*(x), 0, 0)$, where $T^*(x)$ satisfies (3.5) and is globally attractive in $\mathbb{Y}$. Linearizing system (3.2) at $(T^*(x), 0, 0)$, we obtain a cooperative time-delayed nonlocal system for the infectious compartments I and V:

$$\begin{aligned}
\frac{\partial I(t,x)}{\partial t} &= \nabla \cdot (D_I(x)\nabla I(t,x)) - \mu_1(x)I(t,x) \\
&\quad + \left[\Phi(\tau_1)\frac{\partial f_1(\cdot, T^*(\cdot), 0)}{\partial V} V(t-\tau_1, \cdot)\right](x) \\
&\quad + \left[\Phi(\tau_2)\frac{\partial f_2(\cdot, T^*(\cdot), 0)}{\partial I} I(t-\tau_2, \cdot)\right](x), \\
\frac{\partial V(t,x)}{\partial t} &= \nabla \cdot (D_V(x)\nabla V(t,x)) + [\Psi(\tau_3)b(\cdot)I(t-\tau_3, \cdot)](x) \\
&\quad - \mu(x)V(t,x), t > 0, x \in \Omega,
\end{aligned} \tag{3.12}$$

subject to the Neumann boundary condition $\frac{\partial I(t,x)}{\partial \nu} = \frac{\partial V(t,x)}{\partial \nu} = 0, t > 0, x \in \partial\Omega$.

Denote

$$Aw = \operatorname{diag}\left(\nabla \cdot (D_I(x)\nabla w_1), \nabla \cdot (D_V(x)\nabla w_2)\right) + \operatorname{diag}\left(-\mu_1(x)w_1, -\mu(x)w_2\right).$$

Clearly, A generates a semigroup $T_A(t)$, and $T_A(t)$ is compact on $\mathbb{E}$ for each $t > 0$. Define L and $\mathcal{F} \in \mathcal{L}(E, \mathbb{E})$ by

$$L\psi(x) = \begin{pmatrix} 0 \\ [\Psi(\tau_3)b(\cdot)\psi_1(-\tau_3, \cdot)](x) \end{pmatrix}$$

and

$$\mathcal{F}\psi(x) = \begin{pmatrix} \left[\Phi(\tau_1)\frac{\partial f_1(\cdot,T^*(\cdot),0)}{\partial V}\psi_2(-\tau_1,\cdot)\right](x) + \left[\Phi(\tau_2)\frac{\partial f_2(\cdot,T^*(\cdot),0)}{\partial I}\psi_1(-\tau_2,\cdot)\right](x) \\ 0 \end{pmatrix},$$

for any $\psi = (\psi_1, \psi_2) \in E$. Then we consider the following autonomous linear FDE system:

$$\frac{dw}{dt} = \mathcal{F}(w_t) - \mathcal{V}(w_t), \tag{3.13}$$

where $-\mathcal{V}\psi := A\psi(0) + L\psi$. It then follows that (3.13) generates a semigroup $U(t)$ on E, and the autonomous linear FDE

$$\frac{dw}{dt} = -\mathcal{V}(w_t) \tag{3.14}$$

generates a semigroup $\Upsilon(t)$ on E, respectively.

Recall that the exponential growth bound of the semigroup $\Upsilon(t)$ is defined as

$$\omega(\Upsilon) = \inf\{\tilde{\omega} : \exists M \geq 1 \text{ such that } \|\Upsilon(t)\| \leq Me^{\tilde{\omega}t},\ \forall t \geq 0\}.$$

Lemma 3.3.1. *The exponential growth bound $\omega(\Upsilon) < 0$.*

Proof. In view of [152, Theorem 3.2], $\omega(\Upsilon)$ and $s(A + \hat{L})$ have the same sign, where

$$A + \hat{L} = \begin{pmatrix} \nabla(D_I(\cdot)\nabla) - \mu_1(\cdot) & 0 \\ \Psi(\tau_3)b(\cdot) & \nabla(D_V(\cdot)\nabla) - \mu(\cdot) \end{pmatrix}.$$

Note that

$$\frac{dw}{dt} = (A + \hat{L})w$$

generates a semigroup $\hat{\Upsilon}(t)$ on $\mathbb{E}$ satisfying $\hat{\Upsilon}(t) = e^{(A+\hat{L})t}$. It then follows from the spectral mapping theorem that $r(e^{(A+\hat{L})t}) = e^{s(A+\hat{L})t}$ for any $t > 0$. In view of [112, Proposition A.2], there holds

$$\omega(\hat{\Upsilon}) = \frac{\ln r(\hat{\Upsilon}(t_0))}{t_0} = \frac{\ln r(e^{(A+\hat{L})t_0})}{t_0} = \frac{\ln e^{s(A+\hat{L})t_0}}{t_0} = s(A + \hat{L})$$

for any constant $t_0 > 0$. Thus, we have $sign(\omega(\Upsilon)) = sign(\omega(\hat{\Upsilon}))$. So it suffices to consider the following system without time delay:

$$\begin{aligned} \frac{\partial I(t,x)}{\partial t} &= \nabla \cdot (D_I(x)\nabla I(t,x)) - \mu_1(x)I(t,x), \\ \frac{\partial V(t,x)}{\partial t} &= \nabla \cdot (D_V(x)\nabla V(t,x)) + \Psi(\tau_3)b(x)I(t,x) - \mu(x)V(t,x), t > 0, x \in \Omega, \end{aligned}$$

subject to the Neumann boundary condition. Since $\mu_1(x)$ and $\mu(x)$ are positive on $\bar{\Omega}$, it easily follows that $\omega(\hat{\Upsilon}) < 0$. This implies that $\omega(\Upsilon) < 0$. $\square$

Since $\mathcal{F}$ and L are positive, it follows from Lemma 3.3.1 that $\mathcal{F}$ and $\mathcal{V}$ satisfy the following assumptions:

(H1) $\mathcal{F}$ is positive in the sense that $\mathcal{F}(E^+) \subset \mathbb{E}^+$.

(H2) $T_A(t)$ is a positive semigroup on $\mathbb{E}$, L is positive, and $\omega(\Upsilon) < 0$.

Note that the internal transition system (3.14) involves time delay. Regarding the linear system (3.13), we easily derive $\hat{\mathcal{F}}$ and $\hat{\mathcal{V}}$ as follows:

$$\hat{\mathcal{F}} = \begin{pmatrix} \Phi(\tau_2)\frac{\partial f_2(\cdot,T^*(\cdot),0)}{\partial I} & \Phi(\tau_1)\frac{\partial f_1(\cdot,T^*(\cdot),0)}{\partial V} \\ 0 & 0 \end{pmatrix},$$
$$\hat{\mathcal{V}} = \begin{pmatrix} -\nabla\left(D_I(\cdot)\nabla\right) + \mu_1(\cdot) & 0 \\ -\Psi(\tau_3)b(\cdot) & -\nabla\left(D_V(\cdot)\nabla\right) + \mu(\cdot) \end{pmatrix}.$$

According to [152], the basic reproduction number for system (3.13) is the spectral radius of the next generation operator $\hat{\mathcal{F}} \circ \hat{\mathcal{V}}^{-1}$, that is,

$$\mathscr{R}_0 = r\left(\hat{\mathcal{F}} \circ \hat{\mathcal{V}}^{-1}\right). \tag{3.15}$$

Lemma 3.3.2. [Theorem 1.5.4] *$\mathscr{R}_0 - 1$ has the same sign as $\omega(U)$.*

Let $U_\lambda(t)$ be the solution semigroup on E of the following linear system with parameter $\lambda \in [0, +\infty)$:

$$\frac{dw}{dt} = \lambda\mathcal{F}(w_t) - \mathcal{V}(w_t). \tag{3.16}$$

Lemma 3.3.3. [Theorem 1.5.5] *If $\mathscr{R}_0 > 0$, then $\lambda = \frac{1}{\mathscr{R}_0}$ is the unique positive solution of $r(U_\lambda(t_0)) = 1$ for any given $t_0 > 0$, and also the unique positive solution of $\omega(U_\lambda) = 0$.*

Lemma 3.3.4. *In the case where*

$$D_T(x) \equiv D_T,\ D_I(x) \equiv D_I,\ D_V(x) \equiv D_V,\ \mu_1(x) \equiv \mu_1,\ \mu_2(x) \equiv \mu_2,$$
$$b(x) \equiv b,\ \mu(x) \equiv \mu,\ f_1(x,T,V) \equiv f_1(T,V),\ f_2(x,T,I) \equiv f_2(T,I),$$

the basic reproduction number for the resulting system is

$$\mathscr{R}_0 = \frac{e^{-\mu_1\tau_2}\frac{\partial f_2(T^*,0)}{\partial I}}{\mu_1} + \frac{e^{-\mu_1\tau_1}\frac{\partial f_1(T^*,0)}{\partial V}e^{-\mu_2\tau_3}b}{\mu_1\mu}.$$

Proof. In this special case, it is easy to see that

$$\int_\Omega \Gamma_I(t,x,y;D_I)dy = e^{-\mu_1 t}, \int_\Omega \Gamma_{V1}(t,x,y;D_V)dy = e^{-\mu_2 t}, \int_\Omega \Gamma_{V2}(t,x,y;D_V)dy = e^{-\mu t}.$$

Following the above argument, we denote $\mathcal{F}_s$ and $\mathcal{V}_s$ for the simplified system as follows

$$\mathcal{F}_s\psi(x) = \begin{pmatrix} \left[e^{-\mu_1\tau_1}\frac{\partial f_1(T^*,0)}{\partial V}\psi_2(-\tau_1,x)\right] + \left[e^{-\mu_1\tau_2}\frac{\partial f_2(T^*,0)}{\partial I}\psi_1(-\tau_2,x)\right] \\ 0 \end{pmatrix}$$

and $-\mathcal{V}_s\psi := A_s\psi(0) + L_s\psi$ with

$$A_s w = \operatorname{diag}(D_I\triangle w_1, D_V\triangle w_2) + \operatorname{diag}(-\mu_1 w_1, -\mu w_2)$$

and

$$L_s\psi(x) = \begin{pmatrix} 0 \\ [e^{-\mu_2\tau_3}b\psi_1(-\tau_3,x)] \end{pmatrix},$$

for any $\psi = (\psi_1,\psi_2) \in E$.

Let $C = C([-\hat{\tau},0],\mathbb{R}^2)$. We define $F_s, V_s \in \mathcal{L}(C,\mathbb{R}^2)$ as follows

$$F_s(\phi) = \begin{pmatrix} \left[e^{-\mu_1\tau_1}\frac{\partial f_1(T^*,0)}{\partial V}\phi_2(-\tau_1)\right] + \left[e^{-\mu_1\tau_2}\frac{\partial f_2(T^*,0)}{\partial I}\phi_1(-\tau_2)\right] \\ 0 \end{pmatrix}$$

and

$$V_s(\phi) = \begin{pmatrix} \mu_1\phi_1(0) \\ [-e^{-\mu_2\tau_3}b\phi_1(-\tau_3)] + \mu\phi_2(0) \end{pmatrix}$$

for any $\phi = (\phi_1,\phi_2) \in C$. Further we have $\hat{\mathcal{F}}_s$, $\hat{\mathcal{V}}_s$, $\hat{F}_s$ and $\hat{V}_s$ are as follows

$$\hat{\mathcal{F}}_s = \begin{pmatrix} e^{-\mu_1\tau_2}\frac{\partial f_2(T^*,0)}{\partial I} & e^{-\mu_1\tau_1}\frac{\partial f_1(T^*,0)}{\partial V} \\ 0 & 0 \end{pmatrix}, \ \hat{\mathcal{V}}_s = \begin{pmatrix} -D_I\triangle + \mu_1 & 0 \\ -e^{-\mu_2\tau_3}b & -D_V\triangle + \mu \end{pmatrix},$$

$$\hat{F}_s = \begin{pmatrix} e^{-\mu_1\tau_2}\frac{\partial f_2(T^*,0)}{\partial I} & e^{-\mu_1\tau_1}\frac{\partial f_1(T^*,0)}{\partial V} \\ 0 & 0 \end{pmatrix}, \ \hat{V}_s = \begin{pmatrix} \mu_1 & 0 \\ -e^{-\mu_2\tau_3}b & \mu \end{pmatrix}.$$

It is easy to see that

$$\hat{\mathcal{F}}_s \circ \hat{\mathcal{V}}_s^{-1}(v) = \hat{F}_s \circ \hat{V}_s^{-1}(v), \ \forall v = (v_1,v_2) \in \mathbb{R}^2,$$

due to the fact $\triangle v_1 = \triangle v_2 = 0$ for any $v = (v_1,v_2) \in \mathbb{R}^2$. In view of [67, Lemma 2.4], we have $r\left(\hat{\mathcal{F}}_s \circ \hat{\mathcal{V}}_s^{-1}\right) = r\left(\hat{F}_s \circ \hat{V}_s^{-1}\right)$. According to (3.15), the basic reproduction number for the simplified system is

$$\mathscr{R}_0 = r\left(\hat{\mathcal{F}}_s \circ \hat{\mathcal{V}}_s^{-1}\right) = r\left(\hat{F}_s \circ \hat{V}_s^{-1}\right) = \frac{e^{-\mu_1\tau_2}\frac{\partial f_2(T^*,0)}{\partial I}}{\mu_1} + \frac{e^{-\mu_1\tau_1}\frac{\partial f_1(T^*,0)}{\partial V}e^{-\mu_2\tau_3}b}{\mu_1\mu}.$$

This completes the proof. □

As illustrated in [66, Lemma 4.7] and [106, Proposition 2.4], by using the comparison principle and Lemma 3.2.3, we have the following observation.

Lemma 3.3.5. *Let $u(t,\cdot,\phi)$ be the solution of system (3.2) with $u_0 = \phi \in X^+$. Then the following three statements are valid:*

(i) If there exists some $t_0 \geq 0$ such that $u_i(t_0,\cdot,\phi) \not\equiv 0$ for some $i \in \{2,3\}$, then $u_i(t,x,\phi) > 0$ for all $t > t_0$, $x \in \bar{\Omega}$.

(ii) For any $\phi \in X^+$, we have $u_1(t,x,\phi) > 0$, $t > 0, x \in \bar{\Omega}$, and $\liminf_{t\to\infty} u_1(t,x,\phi) \geq \bar{\eta}$ uniformly for $x \in \bar{\Omega}$, where $\bar{\eta}$ is a ϕ-independent positive constant.

(iii) If there exists some $t_0 \geq 0$ such that $u_i(t_0,\cdot,\phi) \not\equiv 0$ for some $i \in \{2,3\}$, then $u_i(t,x,\phi) > 0$ for both $i = 2,3$, with $t > t_0 + \hat{\tau}$, $x \in \bar{\Omega}$.

Substituting $(I(t,x), V(t,x)) = e^{\lambda t}(\psi_1(x), \psi_2(x))$ into (3.12), we obtain the following eigenvalue problem:

$$\begin{aligned}
\lambda\psi_1(x) &= \nabla\cdot(D_I(x)\nabla\psi_1(x)) - \mu_1(x)\psi_1(x)\\
&\quad + e^{-\lambda\tau_1}\left[\Phi(\tau_1)\frac{\partial f_1(\cdot,T^*(\cdot),0)}{\partial V}\psi_2(\cdot)\right](x)\\
&\quad + e^{-\lambda\tau_2}\left[\Phi(\tau_2)\frac{\partial f_2(\cdot,T^*(\cdot),0)}{\partial I}\psi_1(\cdot)\right](x),\\
\lambda\psi_2(x) &= \nabla\cdot(D_V(x)\nabla\psi_2(x)) + e^{-\lambda\tau_3}[\Psi(\tau_3)b(\cdot)\psi_1(\cdot)](x)\\
&\quad - \mu(x)\psi_2(x), t > 0, x \in \Omega,
\end{aligned} \tag{3.17}$$

subject to the Neumann boundary condition. By [112, Proposition A.2] and the arguments similar to those for [113, Theorem 2.2], we have the following result.

Lemma 3.3.6. *$\lambda^* := \omega(U)$ is the principal eigenvalue of problem (3.17) associated with a positive eigenfunction.*

Let $\{P(t)\}_{t\geq 0}$ be an autonomous continuous-time semiflow on a metric space (G,d). We say $e^* \in G$ is an equilibrium if $P(t)e^* = e^*$ for all $t \geq 0$, and e^* attracts a subset S of G if $\lim_{t\to\infty} d(P(t)x, e^*) = 0$ uniformly for $x \in S$.

Lemma 3.3.7. *Let e^* be an equilibrium of $P(t)$. If e^* attracts every compact subset of some open neighbourhood of itself, then e^* is Lyapunov stable.*

Proof. Let J be the open neighbourhood of e^* such that e^* attracts every compact subset of J, and let n_0 be a positive integer such that $B(e^*, \frac{1}{n_0}) \subset J$. Assume, by contradiction, that e^* is not Lyapunov stable. Then there exist $\epsilon_0 > 0$, $x_n \in B(e^*, \frac{1}{n_0+n})$ and $t_n \geq 0$ such that

$$d(P(t_n)x_n, e^*) \geq \epsilon_0,\ \forall n \geq 1. \tag{3.18}$$

Clearly, $\lim_{n\to\infty} x_n = e^*$ and the set $C = \{x_n : n \geq 1\} \cup \{e^*\}$ is a compact subset of J. Since e^* attracts C, there exists $T = T(\epsilon_0) > 0$ such that

$$P(t)x \in B(e^*, \epsilon_0),\ \forall x \in C,\ \forall t \geq T. \tag{3.19}$$

In view of (3.18) and (3.19), it follows that $t_n \in [0, T)$ for all $n \geq 1$. Thus, there exists $n_k \to \infty$ such that $\lim_{k\to\infty} t_{n_k} = t^* \in [0, T]$. Letting $k \to \infty$ in the inequality (3.18) with $n = n_k$, we obtain $\epsilon_0 \leq d(P(t^*)e^*, e^*) = d(e^*, e^*) = 0$, a contradiction. □

Now we are ready to prove a threshold type result on the global dynamics of system (3.2) in terms of $\mathscr{R}_0$.

Theorem 3.3.1. *The following statements are valid:*

(i) If $\mathscr{R}_0 \leq 1$, then the infection-free steady state $(T^*(x), 0, 0)$ is globally asymptotically stable for system (3.2) in X^+.

(ii) If $\mathscr{R}_0 > 1$, then there exists $\hat{\eta} > 0$ such that for any $\phi \in X^+$ with $\phi_2(0, \cdot) \not\equiv 0$ or $\phi_3(0, \cdot) \not\equiv 0$, the solution $u(t, x, \phi) = (u_i(t, x, \phi))(1 \leq i \leq 3)$ of system (3.2) satisfies

$$\liminf_{t\to\infty} \min_{x\in\bar{\Omega}} u_i(t, x, \phi) \geq \hat{\eta}, (1 \leq i \leq 3).$$

Further, system (3.2) admits at least one positive steady state.

Proof. (i) Let (ψ_1, ψ_2) be a positive eigenfunction corresponding to the principal eigenvalue λ^* of problem (3.17). In the case where $\mathscr{R}_0 \leq 1$, Lemma 3.3.2 implies that $\lambda^* \leq 0$. For any given $\phi = (\phi_1, \phi_2, \phi_3) \in X^+$, let

$$u(t, x, \phi) = (u_1(t, x, \phi), u_2(t, x, \phi), u_3(t, x, \phi))$$

be the unique solution of (3.2) with $u_0 = \phi$. Following the ideas of [13, 106], we define

$$c(t; \phi) = \max\left\{\max_{x\in\bar{\Omega}, \theta\in[-\max\{\tau_2,\tau_3\},0]} \frac{u_2(t+\theta, x, \phi)}{e^{\lambda^*(t+\theta)}\psi_1(x)}, \max_{x\in\bar{\Omega}, \theta\in[-\tau_1,0]} \frac{u_3(t+\theta, x, \phi)}{e^{\lambda^*(t+\theta)}\psi_2(x)}\right\},\ t \geq 0.$$

It is easy to see that $c(t; \phi)$ is continuous in $\phi \in X^+$.

Claim 1. For any $\phi \in D := \{\phi \in X^+ : \phi_1(\cdot, x) \leq T^*(x), \forall x \in \bar{\Omega}\}$, $c(t; \phi)$ is non-increasing in t, and $\lim_{t\to\infty} c(t; \phi) = 0$ whenever $\lambda^* = 0$.

Since $\phi_1(\cdot, x) \leq T^*(x)$, it follows that $u_1(t, x, \phi) \leq T^*(x), \forall x \in \bar{\Omega}, t \geq 0$. Fix $t_0 > 0$, and let $\bar{u}_2(t, x) = c(t_0; \phi)e^{\lambda^* t}\psi_1(x)$ and $\bar{u}_3(t, x) = c(t_0; \phi)e^{\lambda^* t}\psi_2(x)$ for $t \geq t_0, x \in \bar{\Omega}$.

Then we see from (A2) that

$$
\begin{aligned}
\frac{\partial u_2(t,x)}{\partial t} &\le \nabla\cdot(D_I(x)\nabla u_2(t,x)) - \mu_1(x)u_2(t,x)\\
&\quad + \Phi(\tau_1)\frac{\partial f_1(x,T^*(x),0)}{\partial V}u_3(t-\tau_1,x)\\
&\quad + \Phi(\tau_2)\frac{\partial f_2(x,T^*(x),0)}{\partial I}u_2(t-\tau_2,x),\ t\ge t_0, x\in\Omega,\\
\frac{\partial u_3(t,x)}{\partial t} &= \nabla\cdot(D_V(x)\nabla u_3(t,x)) + \Psi(\tau_3)b(x)u_2(t-\tau_3,x)\\
&\quad - \mu(x)u_3(t,x),\ t\ge t_0, x\in\Omega,\\
\frac{\partial u_2(t,x)}{\partial\nu} &= \frac{\partial u_3(t,x)}{\partial\nu} = 0,\ t\ge t_0, x\in\partial\Omega,\\
u_2(t_0+\theta_2,x) &\le \bar{u}_2(t_0+\theta_2,x),\ \forall x\in\bar{\Omega}, \theta_2\in[-\max\{\tau_2,\tau_3\},0],\\
u_3(t_0+\theta_3,x) &\le \bar{u}_3(t_0+\theta_3,x),\ \forall x\in\bar{\Omega}, \theta_3\in[-\tau_1,0].
\end{aligned}
$$

By the comparison principle, it follows that

$$
\begin{aligned}
(u_2(t,x),u_3(t,x)) &\le (\bar{u}_2(t,x),\bar{u}_3(t,x))\\
&= (c(t_0;\phi)e^{\lambda^* t}\psi_1(x), c(t_0;\phi)e^{\lambda^* t}\psi_2(x)), \forall x\in\bar{\Omega}, t\ge t_0,
\end{aligned}
$$

and hence, $c(t;\phi)\le c(t_0;\phi)$ for all $t\ge t_0$. This implies that $c(t;\phi)$ is non-increasing in $t\in[0,\infty)$. Since $c(t;\phi)\ge 0$, we have $\lim_{t\to+\infty}c(t;\phi)=c^*\ge 0$. In addition, if $\phi_2\not\equiv 0$ or $\phi_3\not\equiv 0$, we see from Lemma 3.3.5 (i) that $u_2(t,x,\phi)>0$ or $u_3(t,x,\phi)>0$ for all $x\in\bar{\Omega}$ and $t>0$. By Lemma 3.3.5 (iii) and the strong comparison principle, we further have

$$
u_2(t,x) < \bar{u}_2(t,x) = c(t_0;\phi)e^{\lambda^* t}\psi_1(x),\ u_3(t,x) < \bar{u}_3(t,x) = c(t_0;\phi)e^{\lambda^* t}\psi_2(x),
$$

for all $t>t_0+2\hat{\tau}$ and $x\in\bar{\Omega}$. It then follows that $c(t;\phi)<c(t_0;\phi)$ whenever $t>t_0+3\hat{\tau}$, and hence, $c(t;\phi)$ is not a constant function of t.

Now we consider the case where $\lambda^*=0$. For any given $H=(H_1,H_2,H_3)\in\omega(\phi)$, there exists $t_n\to\infty$ such that $\lim_{n\to\infty}Q(t_n)\phi=H$, and hence,

$$
\lim_{n\to\infty}Q(t+t_n)\phi = \lim_{n\to\infty}Q(t)(Q(t_n)\phi) = Q(t)H, \quad \forall t\ge 0.
$$

By the definition of $c(t;H)$, it then follows that

$$
c(t;H) = \lim_{n\to\infty}c(t+t_n;\phi) = c^*, \quad \forall t\ge 0.
$$

Assume, by contradiction, that $c^*>0$. Then we must have either $H_2\ne 0$, or $H_3\ne 0$. Otherwise, there holds $c(t;H)\equiv 0$. By our additional observation for $c(t;\phi)$, it follows that $c(t;H)$ is not a constant function of t, which contradicts the fact that $c(t;H)\equiv c^*$. This completes the proof of Claim 1.

Claim 2. $\lim_{t\to\infty} \|(u(t,\cdot,\phi) - (T^*,0,0)\|_{\mathbb{X}} = 0$ uniformly for ϕ in any compact subset J of D.

For any $\phi \in D$, the second and third equations in (3.2) are dominated by linear system (3.12). In the case where $\lambda^* < 0$, the standard comparison argument implies that

$$\lim_{t\to\infty} \|(u_2(t,\cdot,\phi), u_3(t,\cdot,\phi))\|_{\mathbb{E}} = 0$$

uniformly for $\phi \in J$. In the case where $\lambda^* = 0$, it follows from Claim 1 and Dini's theorem that $\lim_{t\to\infty} c(t;\phi) = 0$ uniformly for $\phi \in J$, and hence,

$$\lim_{t\to\infty} \|(u_2(t,\cdot,\phi), u_3(t,\cdot,\phi))\|_{\mathbb{E}} = 0$$

uniformly for $\phi \in J$. Since the solution semiflow $Q(t)$ has a global attractor (see Lemma 3.2.3), it easily follows that the orbit of any compact set is bounded. Thus, there exists $B = B(J) > 0$ such that $\|u(t,\cdot,\phi)\|_{\mathbb{X}} \le B$ for all $\phi \in J$ and $t \ge 0$. It then follows that

$$\lim_{t\to\infty} f_1(x, u_1(t,x,\phi), u_3(t,x,\phi)) = 0, \quad \lim_{t\to\infty} f_2(x, u_1(t,x,\phi), u_2(t,x,\phi)) = 0,$$

uniformly for $x \in \bar{\Omega}$ and $\phi \in J$. By using the first equation in (3.2) and comparison arguments, we can easily prove that $\lim_{t\to\infty} \|u_1(t,\cdot,\phi) - T^*\|_{\mathbb{Y}} = 0$ uniformly for $\phi \in J$. This proves Claim 2.

Let K be any given compact subset of X^+, and $\omega(K)$ be its omega limit set for $Q(t)$ on X^+. Since T^* attracts every compact sets for the solution semiflow of (3.5) on $\mathbb{Y}^+$, it follows from the comparison argument that $\omega(K) \subset D$. In view of Claim 2 above and the invariance of $\omega(K)$, we have $\omega(K) = (T^*,0,0)$. Since $Q(t)$ admits a strong global attractor on X^+, it follows from Lemma 1.1.2 that $\omega(K)$ attracts K, and hence, $(T^*,0,0)$ attracts K. This, together with Lemma 3.3.7, implies that $(T^*,0,0)$ is is globally asymptotically stable for system (3.2) in X^+.

(ii) In the case where $\mathscr{R}_0 > 1$, Lemma 3.3.2 implies that $\lambda^* > 0$. Let

$$Z_0 := \{\psi = (\psi_1, \psi_2, \psi_3) \in X^+ : \psi_2(0,\cdot) \not\equiv 0 \text{ and } \psi_3(0,\cdot) \not\equiv 0\},$$

and

$$\partial Z_0 := X^+ \setminus Z_0 = \{\psi \in X^+ : \psi_2(0,\cdot) \equiv 0 \text{ or } \psi_3(0,\cdot) \equiv 0\}.$$

For any $\psi \in Z_0$, it then follows from Lemma 3.3.5 (i) that $I(t,x,\psi) > 0$ and $V(t,x,\psi) > 0$, $t \ge 0$, $x \in \bar{\Omega}$. This implies that $Q(t)(Z_0) \subseteq Z_0, t \ge 0$. Now we prove that Q is uniformly persistent with respect to $(Z_0, \partial Z_0)$.

Let $M = (\hat{T}^*, 0, 0)$, where $\hat{T}^* \in Y$ and $\hat{T}^*(\theta) = T^*$, $\forall \theta \in [-\hat{\tau}, 0]$. We consider a perturbed linear system with parameter $\delta > 0$:

$$\begin{aligned}
\frac{\partial I(t,x)}{\partial t} &= \nabla \cdot (D_I(x)\nabla I(t,x)) - \mu_1(x)I(t,x) \\
&\quad + \left[\Phi(\tau_1)\frac{\partial f_1(\cdot,(T^*(\cdot)-\delta),\delta)}{\partial V}V(t-\tau_1,\cdot)\right](x) \\
&\quad + \left[\Phi(\tau_2)\frac{\partial f_2(\cdot,(T^*(\cdot)-\delta),\delta)}{\partial I}I(t-\tau_2,\cdot)\right](x), t>0, x\in\Omega, \\
\frac{\partial V(t,x)}{\partial t} &= \nabla \cdot (D_V(x)\nabla V(t,x)) + [\Psi(\tau_3)b(\cdot)I(t-\tau_3,\cdot)](x) \\
&\quad - \mu(x)V(t,x), t>0, x\in\Omega, \\
\frac{\partial I(t,x)}{\partial \nu} &= \frac{\partial V(t,x)}{\partial \nu} = 0, t>0, x\in\partial\Omega.
\end{aligned} \tag{3.20}$$

Let λ_δ^* be the principal eigenvalue associated with the perturbed system (3.20). Since $\lim_{\delta\to 0^+}\lambda_\delta^* = \lambda^* > 0$, we can fix a sufficiently small $\delta > 0$ such that

$$\delta < \min_{x\in\bar{\Omega}} T^*(x) \text{ and } \lambda_\delta^* > 0.$$

We further have the following claim.

Claim 3. $\limsup_{t\to\infty}\|Q(t)\psi - M\| \geq \delta$ for all $\psi \in Z_0$.

Assume, by contradiction, that there exists $\psi_0 \in Z_0$ such that $\limsup_{t\to\infty}\|Q(t)\psi_0 - M\| < \delta$. It follows from Lemma 3.3.5 that there exists the time $T_u > 0$ such that $T(t,x,\psi_0) > T^*(x) - \delta$, $0 < I(t,x,\psi_0) \leq \delta$, and $0 < V(t,x,\psi_0) \leq \delta$ for all $t \geq T_u$ and $x \in \bar{\Omega}$. This, together with the assumption (A2), implies that $I(t,x,\psi_0)$ and $V(t,x,\psi_0)$ satisfy

$$\begin{aligned}
\frac{\partial I(t,x)}{\partial t} &\geq \nabla \cdot (D_I(x)\nabla I(t,x)) - \mu_1(x)I(t,x) \\
&\quad + \left[\Phi(\tau_1)\frac{\partial f_1(\cdot,(T^*(\cdot)-\delta),\delta)}{\partial V}V(t-\tau_1,\cdot)\right](x) \\
&\quad + \left[\Phi(\tau_2)\frac{\partial f_2(\cdot,(T^*(\cdot)-\delta),\delta)}{\partial I}I(t-\tau_2,\cdot)\right](x),\ t\geq T_u, x\in\Omega, \\
\frac{\partial V(t,x)}{\partial t} &\geq \nabla \cdot (D_V(x)\nabla V(t,x)) + [\Psi(\tau_3)b(\cdot)I(t-\tau_3,\cdot)](x) \\
&\quad - \mu(x)V(t,x),\ t\geq T_u, x\in\Omega, \\
\frac{\partial I(t,x)}{\partial \nu} &= \frac{\partial V(t,x)}{\partial \nu} = 0,\ t\geq T_u, x\in\partial\Omega.
\end{aligned}$$

Let $\psi_\delta = (I_\delta, V_\delta)$ be a strongly positive eigenfunction corresponding to λ_δ^*. For any given $\psi_0 \in Z_0$, it then follows from Lemma 3.3.5 (i) that $I(t,x,\psi_0) > 0$ and $V(t,x,\psi_0) > 0$, $t \geq 0$, $x \in \bar{\Omega}$. Then we can choose a $\kappa > 0$ such that

$$(I(t,x,\psi_0), V(t,x,\psi_0)) \geq \kappa e^{\lambda_\delta^* t}\psi_\delta,\ \forall t\in[T_u-\hat{\tau}, T_u], x\in\bar{\Omega}.$$

Note that the linear system (3.20) admits a solution

$$(i_\delta(t,x,\psi_0), v_\delta(t,x,\psi_0)) = \kappa e^{\lambda_\delta^*(t-T_u)}\psi_\delta, \quad \forall t \geq T_u.$$

Employing the comparison principle, we then have

$$(I(t,x,\psi_0), V(t,x,\psi_0)) \geq (i_\delta(t,x,\psi_0), v_\delta(t,x,\psi_0)) := \kappa e^{\lambda_\delta^*(t-T_u)}\psi_\delta,\ \forall t \geq T_u, x \in \bar{\Omega}.$$

This inequality, together with $\lambda_\delta^* > 0$, implies that $I(t,\cdot,\psi_0) \to \infty$ and $V(t,\cdot,\psi_0) \to \infty$ as $t \to \infty$, a contradiction. This proves Claim 3.

In view of Claim 3, it follows that M is an isolated invariant set for $Q(t)$ in X^+, and $W^S(M) \bigcap Z_0 = \emptyset$, where $W^S(M)$ is the stable set of M for $Q(t)$. Define

$$M_\partial := \{\psi \in \partial Z_0 : Q(t)(\psi) \in \partial Z_0, \forall t \geq 0\},$$

and $\omega(\psi)$ be the omega limit set of the forward orbit $\gamma^+(\psi) := \{Q(t)(\psi) : \forall t \geq 0\}$.

Claim 4. $\omega(\psi) = M$ for any $\psi \in M_\partial$, and M is globally stable for $Q(t)$ in M_∂.

By the definition of M_∂, for any given $\psi \in M_\partial$, we have $Q(t)(\psi) \in \partial Z_0$, and hence, $I(t,\cdot,\psi) \equiv 0$ or $V(t,\cdot,\psi) \equiv 0$ for all $t \geq 0$. In fact, on the one hand, if $I(t_0,\cdot,\psi) \not\equiv 0$ for some $t_0 \geq 0$, then Lemma 3.3.5 (iii) implies that $I(t,\cdot,\psi) > 0$ and $V(t,\cdot,\psi) > 0$ for $t > t_0 + \hat{\tau}$, a contradiction; on the other hand, if $V(t_0,\cdot,\psi) \not\equiv 0$ for some $t_0 \geq 0$, then Lemma 3.3.5 (iii) ensures a similar contradiction. Therefore, for any given $\psi \in M_\partial$, we have $I(t,\cdot,\psi) \equiv 0$ and $V(t,\cdot,\psi) \equiv 0$ for all $t \geq 0$. It then follows from the T-equation in (3.2) and Lemma 3.2.2 that $\lim_{t\to\infty}(T(t,x,\psi) - T^*(x)) = 0$ uniformly for $x \in \bar{\Omega}$, and hence, $\omega(\psi) \subset M$ for any $\psi \in M_\partial$. As argued in the case of (i), we can show that $M \subset \omega(\psi)$. Therefore, $\omega(\psi) = M$ for any $\psi \in M_\partial$. This implies that M is globally attractive for $Q(t)$ in M_∂. By using Lemma 3.2.2 again, we conclude that M is locally Lyapunov stable for $Q(t)$ in M_∂. This proves Claim 4.

With Claim 4, we see that M cannot form a cycle for $Q(t)$ in ∂Z_0. Since $Q(t)$ admits a global attractor on X^+, it follows from the acyclicity theorem on uniform persistence for maps (see., e.g., Theorem 1.2.1) that $Q(t) : X^+ \to X^+$ is uniformly persistent with respect to $(Z_0, \partial Z_0)$ in the sense that there exists an $\tilde{\eta} > 0$ such that

$$\liminf_{t\to\infty} d(Q(t)(\psi), \partial Z_0) \geq \tilde{\eta},\ \forall \psi \in Z_0.$$

Now we prove the practical uniform persistence. By Theorem 1.2.2, we obtain that $Q(t) : Z_0 \to Z_0$ admits a global attractor A_0. Since $A_0 = Q(t)A_0$, we have $\psi_2(0,\cdot) > 0$ and $\psi_3(0,\cdot) > 0$ for all $\psi \in A_0$. Then $A_0 \subseteq Z_0$ and $\lim_{t\to\infty} d(Q(t)\psi, A_0) = 0$ for all $\psi \in Z_0$. Define a continuous function $p : X^+ \to \mathbb{R}_+$ by

$$p(\psi) = \min\left\{\min_{x\in\bar{\Omega}} \psi_2(0,x), \min_{x\in\bar{\Omega}} \psi_3(0,x)\right\}, \forall \psi = (\psi_1,\psi_2,\psi_3) \in X^+.$$

Clearly, $p(\psi) > 0$ for all $\psi \in A_0$. Since A_0 is a compact subset of Z_0, we have $\inf_{\psi \in A_0} p(\psi) = \min_{\psi \in A_0} p(\psi) > 0$. By the attractivity of A_0, it follows that there exists an $\check{\eta} > 0$ such that

$$\liminf_{t\to\infty} \min \left\{ \min_{x\in\bar{\Omega}} I(t,x,\psi), \min_{x\in\bar{\Omega}} V(t,x,\psi) \right\} \geq \check{\eta}, \forall \psi \in Z_0.$$

In view of Lemma 3.3.5 (ii), we further obtain

$$\liminf_{t\to\infty} \min_{x\in\bar{\Omega}} u_i(t,x,\psi) \geq \hat{\eta} := \min\left\{\bar{\eta}, \check{\eta}\right\}, (1 \leq i \leq 3).$$

For any given $\psi \in X^+$ with $\psi_2(0,\cdot) \not\equiv 0$ or $\psi_3(0,\cdot) \not\equiv 0$, it follows from Lemma 3.3.5 (iii) that there exists a time $n_0 = n_0(\psi) \geq 0$ such that $Q(n_0)(\psi) \in Z_0$. Since $Q(t)\psi = Q(t-n_0)\left(Q(n_0)(\psi)\right), \forall t \geq n_0$, we have $\liminf_{t\to\infty} \min_{x\in\bar{\Omega}} u_i(t,x,\psi) \geq \hat{\eta}, (1 \leq i \leq 3)$. Thus, the uniform persistence stated in the conclusion (ii) is proved.

In order to prove the existence of a positive steady state of system (3.2), we let $Q_0(t) : \mathbb{X}^+ \to \mathbb{X}^+, t \geq 0$, be the solution semiflow generated by the following nonlocal reaction-diffusion system:

$$\begin{aligned}
\frac{\partial T(t,x)}{\partial t} &= \nabla \cdot (D_T(x)\nabla T(t,x)) + n(x,T(t,x)) - f_1(x,T(t,x),V(t,x)) \\
&\quad - f_2(x,T(t,x),I(t,x)), \\
\frac{\partial I(t,x)}{\partial t} &= \nabla \cdot (D_I(x)\nabla I(t,x)) - \mu_1(x)I(t,x) + [\Phi(\tau_1)f_1(\cdot,T(t,\cdot),V(t,\cdot))](x) \\
&\quad + [\Phi(\tau_2)f_2(\cdot,T(t,\cdot),I(t,\cdot))](x), \\
\frac{\partial V(t,x)}{\partial t} &= \nabla \cdot (D_V(x)\nabla V(t,x)) + [\Psi(\tau_3)b(\cdot)I(t,\cdot)](x) \\
&\quad - \mu(x)V(t,x), t > 0, x \in \Omega,
\end{aligned} \tag{3.21}$$

subject to the Neumann boundary condition

$$\frac{\partial T(t,x)}{\partial \nu} = \frac{\partial I(t,x)}{\partial \nu} = \frac{\partial V(t,x)}{\partial \nu} = 0, t > 0, x \in \partial\Omega.$$

Let $\lambda_0 = s(\hat{\mathcal{F}} - \hat{\mathcal{V}})$, then $\mathscr{R}_0 > 1$ implies that $\lambda_0 > 0$. By the arguments similar to those for $Q(t) : X^+ \to X^+$, it follows that $Q_0(t) : \mathbb{X}^+ \to \mathbb{X}^+$ is point dissipative, compact for any $t > 0$, and uniformly persistent with respect to $(W_0, \partial W_0)$, where

$$W_0 := \{\phi = (\phi_1, \phi_2, \phi_3) \in \mathbb{X}^+ : \phi_2(\cdot) \not\equiv 0 \text{ and } \phi_3(\cdot) \not\equiv 0\}$$

and

$$\partial W_0 := \mathbb{X}^+ \setminus W_0 = \{\phi \in \mathbb{X}^+ : \phi_2(\cdot) \equiv 0 \text{ or } \phi_3(\cdot) \equiv 0\}.$$

By Theorem 1.2.6, $Q_0(t)$ has an equilibrium $\phi^* \in W_0$, and hence, $\phi^*(x)$ is a steady state of system (3.21). Clearly, $\phi^*(x)$ is also a steady state of system (3.2). Moreover, Lemma 3.3.5 implies that $\phi^*(x)$ is pointwise positive. □

3.3.2 A simplified system with constant coefficients

In this section, we consider the case where all coefficients in system (3.2) are positive constants and $f_1(x,T,V) \equiv \beta_1 TV$ and $f_2(x,T,I) \equiv \beta_2 TI$. Similar to [128, 141], we also assume that the two latency periods are equal for healthy target cells after exposure to infected cells and viruses, respectively; that is, $\tau_1 = \tau_2 := \tau$. We make the following assumption for $n(T)$:

(A1′) There exists a unique $T^* > 0$ such that $n(T^*) = 0$; $n'(T) < 0$ for all $T \geq 0$.

It is easy to see that T^* is the equilibrium concentration of the healthy target cells under the natural change.

For simplicity, let $\Gamma(t,x,y;D)$ be the Green function associated with $D\triangle$ subject to the Neumann boundary condition. Therefore, system (3.2) is reduced to the following autonomous reaction-diffusion system:

$$\begin{cases} \frac{\partial T(t,x)}{\partial t} = D_T \triangle T(t,x) + n(T(t,x)) \\ \qquad\qquad - \beta_1 T(t,x)V(t,x) - \beta_2 T(t,x)I(t,x), \\ \frac{\partial I(t,x)}{\partial t} = D_I \triangle I(t,x) - \mu_1 I(t,x) \\ \qquad\qquad + e^{-\mu_1\tau}\int_\Omega \Gamma(D_I\tau,x,y)\beta_1 T(t-\tau,y)V(t-\tau,y)dy \\ \qquad\qquad + e^{-\mu_1\tau}\int_\Omega \Gamma(D_I\tau,x,y)\beta_2 T(t-\tau,y)I(t-\tau,y)dy, \\ \frac{\partial V(t,x)}{\partial t} = D_V \triangle V(t,x) + be^{-\mu_2\tau_3}\int_\Omega \Gamma(D_V\tau_3,x,y)I(t-\tau_3,y)dy \\ \qquad\qquad - \mu V(t,x), t>0, x\in\Omega, \\ \frac{\partial T(t,x)}{\partial \nu} = \frac{\partial I(t,x)}{\partial \nu} = \frac{\partial V(t,x)}{\partial \nu} = 0, t>0, x\in\partial\Omega. \end{cases} \tag{3.22}$$

From the argument in Section 3.1 and Lemma 3.3.4, the basic reproduction number of system (3.22) is given by

$$\mathscr{R}_0 = \frac{e^{-\mu_1\tau}\beta_2 T^*}{\mu_1} + \frac{e^{-\mu_1\tau}\beta_1 T^* e^{-\mu_2\tau_3} b}{\mu_1\mu}.$$

Define $\mathbb{U} := C(\bar{\Omega},\mathbb{R})$, $\mathbb{U}^+ := C(\bar{\Omega},\mathbb{R}_+)$ and

$$U := C\left([-\tau,0],\mathbb{U}^+\right) \times C\left([-\max\{\tau,\tau_3\},0],\mathbb{U}^+\right) \times C\left([-\tau,0],\mathbb{U}^+\right).$$

Then we have the following result on the global attractivity for system (3.22).

Theorem 3.3.2. *Let $u(t,x,\phi)$ be the solution of system (3.22) with the initial value $u_0 = \phi \in U$. Then the following statements are valid:*

(i) If $\mathscr{R}_0 \leq 1$, then $e_0 = (T^*,0,0)$ is globally asymptotically stable for system (3.22) in U.

(ii) If $\mathscr{R}_0 > 1$, then system (3.22) has a unique constant equilibrium $u^* = (T_*, I_*, V_*)$ such that for any $\phi \in U$ with $\phi_2(0,\cdot) \not\equiv 0$ or $\phi_3(0,\cdot) \not\equiv 0$, we have $\lim_{t\to\infty} u(t,x,\phi) = u^*$ uniformly for all $x \in \bar{\Omega}$.

Proof. Note that statement (i) is the straightforward consequence of Theorem 3.3.1 (i). It remains to prove (ii). When $\mathscr{R}_0 > 1$, system (3.22) has a unique positive equilibrium $u^* = (T_*, I_*, V_*)$ with

$$T_* = \frac{\mu\mu_1 e^{\mu_1\tau}}{\beta_1 b e^{-\mu_2\tau_3} + \beta_2\mu}, \; I_* = \frac{n(T_*)}{\mu_1 e^{\mu_1\tau}}, \; V_* = \frac{b e^{-\mu_2\tau_3}}{\mu} I_*.$$

It follows from Theorem 3.3.1 (ii) that system (3.22) is uniformly persistent as $\mathscr{R}_0 > 1$; that is, there exists $\xi > 0$ such that for any $\phi = (\phi_1, \phi_2, \phi_3) \in U$ with $\phi_2(0,\cdot) \not\equiv 0$ or $\phi_3(0,\cdot) \not\equiv 0$, the solution $u = (t, x, \phi)$ satisfies

$$\liminf_{t\to\infty} \min_{x\in\bar{\Omega}} u_i(t,x,\phi) \geq \xi, (1 \leq i \leq 3). \tag{3.23}$$

Let $U_0 := \left\{\phi \in U : \phi_i(0,x) > 0, \forall x \in \bar{\Omega}, i = 1,2,3\right\}$. Next, we show that u^* is globally attractive for system (3.22) by using the method of Lyapunov functionals. Set $g(u) = u - 1 - \ln u, u > 0$. Clearly, $g(u) \geq 0$ for all $u > 0$ and $\min_{0<u<+\infty} g(u) = g(1) = 0$. Define a continuous functional $W : U_0 \to \mathbb{R}$:

$$W(\phi) = \int_\Omega [W_1(x,\phi) + W_2(x,\phi) + W_3(x,\phi) + W_4(x,\phi)]dx,$$

where

$$W_1 = \phi_1(0,x) - T_* - \int_{T_*}^{\phi_1(0,x)} \frac{T_*}{s} ds + e^{\mu_1\tau} I_* g\left(\frac{\phi_2(0,x)}{I_*}\right) + \frac{\beta_1 T_* V_*}{b e^{-\mu_2\tau_3} I_*} V_* g\left(\frac{\phi_3(0,x)}{V_*}\right),$$

$$W_2 = \beta_1 T_* V_* \int_{-\tau}^{0}\int_\Omega \Gamma\left(D_I(-\theta), x, y\right) g\left(\frac{\phi_1(\theta,y)\phi_3(\theta,y)}{T_* V_*}\right) dy d\theta,$$

$$W_3 = \beta_2 T_* I_* \int_{-\tau}^{0}\int_\Omega \Gamma\left(D_I(-\theta), x, y\right) g\left(\frac{\phi_1(\theta,y)\phi_2(\theta,y)}{T_* I_*}\right) dy d\theta,$$

$$W_4 = \beta_1 T_* V_* \int_{-\tau_3}^{0}\int_\Omega \Gamma\left(D_V(-\theta), x, y\right) g\left(\frac{\phi_2(\theta,y)}{I_*}\right) dy d\theta.$$

Next we fix $\phi = (\phi_1, \phi_2, \phi_3) \in U$ with $\phi_2(0,\cdot) \not\equiv 0$ or $\phi_3(0,\cdot) \not\equiv 0$. From (3.23), without loss of generality, we can assume that $u_t(\phi) \in U_0, \forall t \geq 0$. Let $\omega(\phi)$ be the omega limit set of the orbit $\gamma^+(\phi)$ for the semiflow $Q(t)$. Since (3.22) is uniformly persistent, we have $\omega(\phi) \subset U_0$. Note that

$$n(T_*) = \beta_1 T_* V_* + \beta_2 T_* I_*, \; e^{-\mu_1\tau}(\beta_1 T_* V_* + \beta_2 T_* I_*) = \mu_1 I_*, \; b e^{-\mu_2\tau_3} I_* = \mu V_*.$$

Now we calculate the time derivative of $W(u_t(\phi))$ along the solution of system (3.22). It then follows that

$$
\begin{aligned}
&\frac{\partial W_1(u_t(\phi))}{\partial t} \\
&= \frac{\partial T(t,x)}{\partial t} - \frac{T_*}{T}\frac{\partial T(t,x)}{\partial t} + e^{\mu_1\tau} I_* \left(1 - \frac{I_*}{I}\right)\frac{1}{I_*}\frac{\partial I(t,x)}{\partial t} \\
&\quad + \frac{\beta_1 T_* V_*}{b e^{-\mu_2\tau_3} I_*} V_* \left(1 - \frac{V_*}{V}\right)\frac{1}{V_*}\frac{\partial V(t,x)}{\partial t} \\
&= B + \left(1 - \frac{T_*}{T}\right)(n(T) - n(T_*)) + \beta_1 T_* V_* \left(1 - \frac{T_*}{T}\right)\left(1 - \frac{TV}{T_* V_*}\right) \\
&\quad + \beta_2 T_* I_* \left(1 - \frac{T_*}{T}\right)\left(1 - \frac{TI}{T_* I_*}\right) \\
&\quad + \int_\Omega \Gamma(D_I\tau, x, y)\beta_1 T(t-\tau, y) V(t-\tau, y)dy \\
&\quad + \int_\Omega \Gamma(D_I\tau, x, y)\beta_2 T(t-\tau, y) I(t-\tau, y)dy \\
&\quad - \beta_1 T_* V_* \int_\Omega \Gamma(D_I\tau, x, y)\frac{T(t-\tau, y)V(t-\tau, y)I_*}{T_* V_* I}dy \\
&\quad - \beta_2 T_* I_* \int_\Omega \Gamma(D_I\tau, x, y)\frac{T(t-\tau, y)I(t-\tau, y)I_*}{T_* I_* I}dy \\
&\quad + \beta_1 T_* V_* + \beta_2 T_* I_* - \beta_2 T_* I - \beta_1 T_* V_* \frac{I}{I_*} + \beta_1 T_* V_* - \beta_1 T_* V \\
&\quad + \frac{\beta_1 T_* V_*}{I_*}\int_\Omega \Gamma(D_V\tau_3, x, y) I(t-\tau_3, y)dy \\
&\quad - \beta_1 T_* V_* \int_\Omega \Gamma(D_V\tau_3, x, y)\frac{I(t-\tau_3, y)V_*}{I_* V}dy,
\end{aligned}
$$

where

$$
B := \left(1 - \frac{T_*}{T}\right) D_T \Delta T + e^{\mu_1\tau}\left(1 - \frac{I_*}{I}\right) D_I \Delta I + \frac{\beta_1 T_* V_*}{b e^{-\mu_2\tau_3} I_*}\left(1 - \frac{V_*}{V}\right) D_V \Delta V.
$$

Furthermore, we have

$$\begin{aligned}\frac{\partial W_2(u_t(\phi))}{\partial t} &= \beta_1 T_* V_* \left[\frac{TV}{T_*V_*} - \ln\frac{TV}{T_*V_*} - \int_\Omega \Gamma(D_I\tau, x, y)\frac{T(t-\tau,y)V(t-\tau,y)}{T_*V_*}dy\right.\\ &\quad \left. + \int_\Omega \Gamma(D_I\tau, x, y)\ln\frac{T(t-\tau,y)V(t-\tau,y)}{T_*V_*}dy\right],\\ \frac{\partial W_3(u_t(\phi))}{\partial t} &= \beta_2 T_* I_* \left[\frac{TI}{T_*I_*} - \ln\frac{TI}{T_*I_*} - \int_\Omega \Gamma(D_I\tau, x, y)\frac{T(t-\tau,y)I(t-\tau,y)}{T_*I_*}dy\right.\\ &\quad \left. + \int_\Omega \Gamma(D_I\tau, x, y)\ln\frac{T(t-\tau,y)I(t-\tau,y)}{T_*I_*}dy\right],\\ \frac{\partial W_4(u_t(\phi))}{\partial t} &= \beta_1 T_* V_* \left[\frac{I}{I_*} - \ln\frac{I}{I_*} - \int_\Omega \Gamma(D_V\tau_3, x, y)\frac{I(t-\tau_3,y)}{I_*}dy\right.\\ &\quad \left. + \int_\Omega \Gamma(D_V\tau_3, x, y)\ln\frac{I(t-\tau_3,y)}{I_*}dy\right].\end{aligned}$$

It then follows from the function $g(\cdot)$ that

$$\begin{aligned}&\frac{\partial W_1(u_t(\phi))}{\partial t} + \frac{\partial W_2(u_t(\phi))}{\partial t} + \frac{\partial W_3(u_t(\phi))}{\partial t} + \frac{\partial W_4(u_t(\phi))}{\partial t}\\ &= B + \left(1 - \frac{T_*}{T}\right)(n(T) - n(T_*))\\ &\quad - \beta_1 T_* V_* \left[\int_\Omega \Gamma(D_I\tau, x, y) g\left(\frac{T(t-\tau,y)V(t-\tau,y)I_*}{T_*V_*I}\right)dy\right.\\ &\quad \left. + \int_\Omega \Gamma(D_V\tau_3, x, y) g\left(\frac{I(t-\tau_3,y)V_*}{I_*V}\right)dy + g\left(\frac{T_*}{T}\right)\right]\\ &\quad - \beta_2 T_* I_* \left[g\left(\frac{T_*}{T}\right) + \int_\Omega \Gamma(D_I\tau, x, y) g\left(\frac{T(t-\tau,y)I(t-\tau,y)I_*}{T_*I_*I}\right)dy\right].\end{aligned}$$

By the assumption (A1′), we have

$$\left(1 - \frac{T_*}{T}\right)(n(T) - n(T_*)) \le 0,$$

and equality holds only in the case of $T = T_*$. Using the Neumann boundary conditions and Divergence Theorem, we obtain

$$\begin{aligned}0 &= \int_{\partial\Omega} \nabla w \cdot \nu dx = \int_\Omega \operatorname{div}(\nabla w) dx = \int_\Omega \Delta w dx,\\ 0 &= \int_{\partial\Omega} \frac{1}{w}\nabla w \cdot \nu dx = \int_\Omega \operatorname{div}\left(\frac{1}{w}\nabla w\right) dx = \int_\Omega \left(\frac{\Delta w}{w} - \frac{\|\nabla w\|^2}{w^2}\right) dx,\end{aligned}$$

for $w \in \{T, I, V\}$, and hence, $\int_\Omega \Delta w dx = 0$ and $\int_\Omega \frac{\Delta w}{w} dx = \int_\Omega \frac{\|\nabla w\|^2}{w^2} dx \ge 0$. It follows that $\int_\Omega B dx \le 0$. Thus, there holds

$$
\begin{aligned}
&\frac{dW(u_t(\phi))}{dt} \\
&= \int\limits_\Omega \left[\frac{\partial W_1(u_t(\phi))}{\partial t} + \frac{\partial W_2(u_t(\phi))}{\partial t} + \frac{\partial W_3(u_t(\phi))}{\partial t} + \frac{\partial W_4(u_t(\phi))}{\partial t}\right] dx \\
&\le -\beta_1 T_* V_* \left[\int_\Omega \int_\Omega \Gamma(D_I \tau, x, y) g\left(\frac{T(t-\tau, y)V(t-\tau, y)I_*}{T_* V_* I}\right) dy dx \right. \\
&\quad \left. + \int_\Omega \int_\Omega \Gamma(D_V \tau_3, x, y) g\left(\frac{I(t-\tau_3, y)V_*}{I_* V}\right) dy dx + \int_\Omega g\left(\frac{T_*}{T}\right) dx\right] \\
&\quad - \beta_2 T_* I_* \left[\int_\Omega g\left(\frac{T_*}{T}\right) dx + \int_\Omega \int_\Omega \Gamma(D_I \tau, x, y) g\left(\frac{T(t-\tau, y)I(t-\tau, y)I_*}{T_* I_* I}\right) dy dx\right] \\
&:= U_\phi(t).
\end{aligned} \tag{3.24}
$$

Note that $W(u_t(\phi))$ is nonincreasing and bounded below on $[0, \infty)$, hence, there is a real number $L \ge 0$ such that $\lim_{t\to\infty} W(u_t(\phi)) = L$. For any $\psi \in \omega(\phi)$, there exists a sequence $t_n \to \infty$ such that $\lim_{n\to\infty} u_{t_n}(\phi) = \psi$ in U_0. This implies that $W(\psi) = L, \forall \psi \in \omega(\phi)$. Since $u_t(\psi) \in \omega(\phi)$, it follows that $W(u_t(\psi)) = L, \forall t \ge 0$, and hence, $\frac{dW(u_t(\psi))}{dt} = 0$. Replacing ϕ in (3.24) with ψ, we obtain $0 = \frac{dW(u_t(\psi))}{dt} \le U_\psi(t) \le 0$. This shows that $U_\psi(t) = 0, \forall t \ge 0$. By the definition of the set U_0 and Lemma 3.3.5, we see that $u_t(\psi) \gg 0, \forall t \ge \max\{\tau, \tau_3\}$. It then follows from the expression of $U_\psi(t)$ in (3.24) that $u_t(\psi) = u^*, \forall t \ge \max\{\tau, \tau_3\}$. Since $\psi \in \omega(\phi)$ is arbitrary, there also holds $u_t(\omega(\phi)) = u^*, \forall t \ge \max\{\tau, \tau_3\}$. In view of the invariance of omega limit sets, it is easy to see that $\omega(\phi) = u_{\hat{\tau}}(\omega(\phi)) = u^*, \hat{\tau} = \max\{\tau, \tau_3\}$, which implies that $\lim_{t\to\infty} u_t(\phi) = u^*$. □

3.4 Numerical simulations

In this section, we illustrate our analytical results on the basic reproduction number and global dynamics of the model via numerical simulations, and investigate the role of drugs in controlling the spread of the viruses. Note that the estimated values of the parameters in model (3.2) are constant and independent of space in many studies and experiments, and these fixed values may reflect the average levels of these factors in the host body or due to the limitations of experimental conditions. For simplicity, we assume that $n(T) = \Lambda - d_T T$, $f_1(x, T, V) = \beta_1 TV$ and $f_2(x, T, I) = \beta_2 TI$. Therefore, we first collect these values and present them in Table 3.1 as the mean values of the parameters in model (3.2).

As a straightforward consequence of Lemma 1.5.1, we have the following observation.

Lemma 3.4.1. *For any given* $\lambda \in [0, +\infty)$*, we choose* $v_0 \in int(E^+)$ *and define*

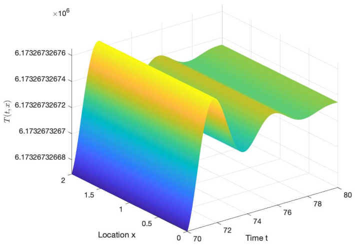
$$a_n = \|U_\lambda(t_0)v_{n-1}\|_E, \ v_n = \frac{U_\lambda(t_0)v_{n-1}}{a_n}, \ \forall n \geq 1,$$

for any given $t_0 > 0$*. If* $\lim_{n\to\infty} a_n$ *exists, then* $r(U_\lambda(t_0)) = \lim_{n\to\infty} a_n$*.*

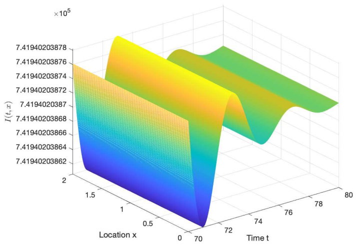
It then follows that we can use the bisection method and Lemma 3.4.1 to solve $r(U_\lambda(t_0)) = 1$ numerically, which provides an approximation for $\frac{1}{\mathscr{R}_0}$ (see Lemma 3.3.3). By employing the parameter values in Table 3.1 and the above numerical method, we compute the basic reproduction number as $\mathscr{R}_0 = 3.2034$.

Table 3.1: Model parameters and their mean values for system (3.2)

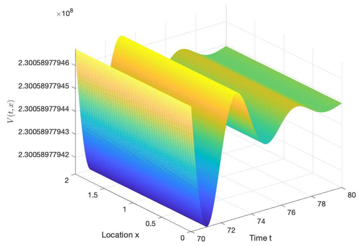
Parameters	Mean values	References
Λ	5×10^5 cells/(day mL)	[88]
d_T	0.01/day	[88]
μ_1	0.5/day	[90]
μ	3/day	[90]
μ_2	2.5/day	variable
b	1000 virions/(cell day)	[33]
β_1	1.2×10^{-10} mL/(virions cells day)	[98]
β_2	4.5×10^{-8} mL/(virions cells day)	[98]
D_T	0.09648 mm^2/day	[83]
D_I	0.05 mm^2/day	[98]
D_V	0.17 mm^2/day	[110]
τ_1	1 day	[87]
τ_2	0.82 day	[14]
τ_3	0.5 day	[40]



(a) $T(t,x)$



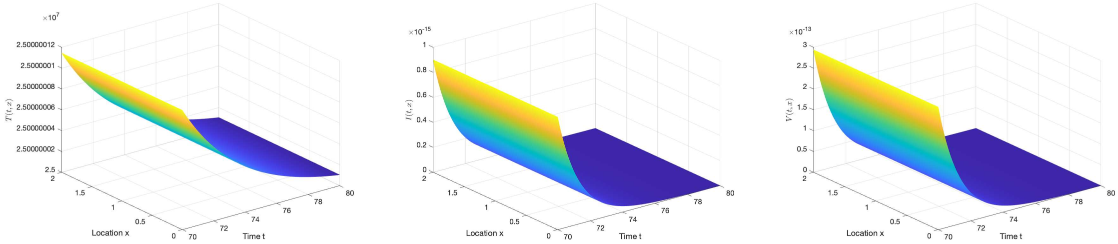
(b) $I(t,x)$



(c) $V(t,x)$

Figure 3.1: The evolution of the compartments T, I and V based on the parameters in Table 3.1 and initial data (3.25), corresponding to $\mathscr{R}_0 = 3.2034$.

We find that the parameters Λ, β_1 and β_2 in Table 3.1 are all related to volume. To fit them into a one-dimensional region and make them constant, we assume that the region is a thin wire with a cross-sectional area of $1mm \times 1mm$, simplified to



(a) $T(t,x)$ (b) $I(t,x)$ (c) $V(t,x)$

Figure 3.2: The evolution of the compartments T, I and V by reducing $\beta_1(x)$ to $0.2\beta_1(x)$ and $\beta_2(x)$ to $0.2\beta_2(x)$ under initial data (3.25), corresponding to $\mathscr{R}_0 = 0.7764$.

$\Omega = (0, 2)$. Since lymphoid tissue consists of lymph nodes, we set the environments periodic for convenience in the following simulations. Applying the difference method to the system with the Neumann boundary condition, we observe the evolution of each compartment in system (3.2) with the initial data

$$\begin{pmatrix} T(\theta,x) \\ I(\theta,x) \\ V(\theta,x) \end{pmatrix} = \begin{pmatrix} 5\times 10^7 \times (1+0.5\cos(\pi x)) \\ 1\times 10^2 \times (1+0.5\cos(\pi x)) \\ 200\times (1+0.5\cos(\pi x)) \end{pmatrix}, \forall \theta \in [-\hat{\tau}, 0], x \in [0,2], \tag{3.25}$$

and the results are shown in Figure 3.1. The numerical results are well consistent with Theorem 3.3.1 (ii); that is, the viruses will be uniformly persistent under $\mathscr{R}_0 = 3.2034 > 1$. If only $\beta_1(x)$ decreases to $0.2\beta_1(x)$, and $\beta_2(x)$ decreases to $0.2\beta_2(x)$, we obtain $\mathscr{R}_0 = 0.7764$. The evolution of the solution is indicated in Figure 3.2 with initial data (3.25), which corresponds to the conclusion in Theorem 3.3.1 (i); that is, the viruses will become extinct under $\mathscr{R}_0 = 0.7764 < 1$.

To facilitate a comprehensive elucidation of the impact of diverse input parameters and their respective fluctuations on the model outcomes, and to characterize the pivotal parameters, the sensitivity analysis is performed. This is achieved by computing the Partial Rank Correlation Coefficients (PRCCs) with respect to the basic reproduction number $\mathscr{R}_0$ for a range of parameters, as outlined in [78]. The sensitivity analysis results are shown in Figure 3.3, indicating that $\mathscr{R}_0$ is highly sensitive to μ_1 and τ_2. As a result, we implement appropriate measures to increase μ_1 and τ_2, which can effectively reduce the risk of virus transmission.

Finally, we study the optimal drug strategy in a spatially heterogeneous region. Let

$$\beta_1(x) = 1.2\times 10^{-10} \times x(2-x), \ \beta_2(x) = 4.5\times 10^{-8} \times x(2-x),$$

thereby indicating that β_1 and β_2 exhibit the highest infection coefficients at the central region of the spatial domain. We suppose that the distribution of drug is

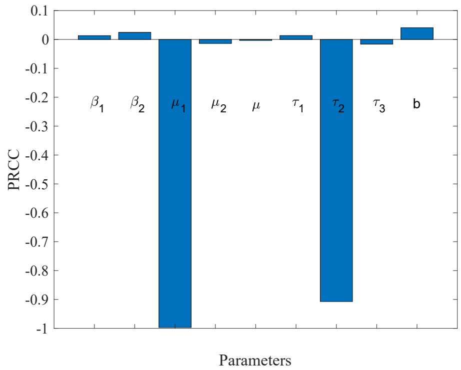
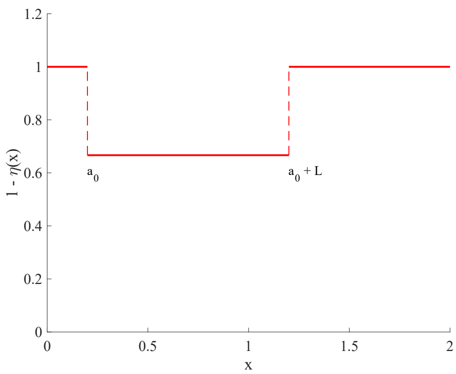
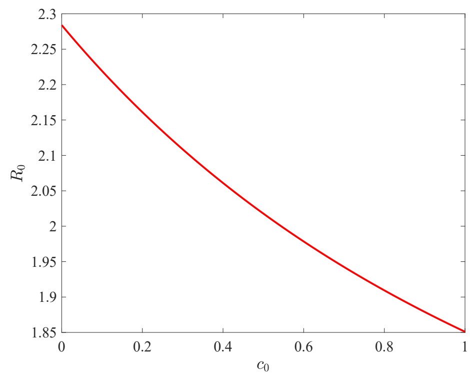


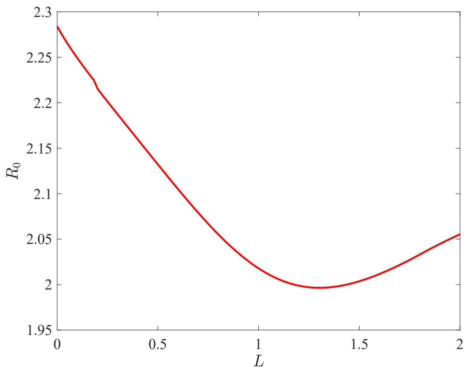
Figure 3.3: Sensitivity analysis of $\mathscr{R}_0$.



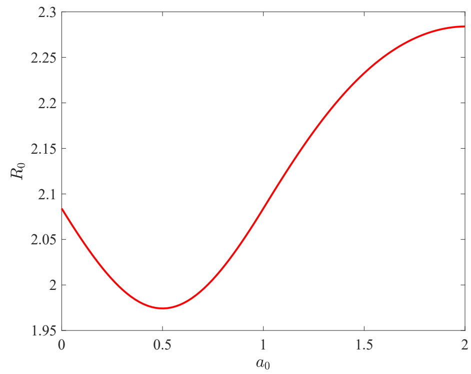
(a) $1 - \eta(x)$ as a function of x



(b) $\mathscr{R}_0$ as a function of c_0



(c) $\mathscr{R}_0$ as a function of L



(d) $\mathscr{R}_0$ as a function of a_0

Figure 3.4: The effects of the drug on the basic reproduction number $\mathscr{R}_0$.

described by

$$v_0(x) = \begin{cases} \frac{c_0}{L} & x \in [a_0, a_0 + L],\ a_0 \geq 0, a_0 + L \leq 2, \\ 0 & \text{otherwise}, \end{cases}$$

where $c_0 > 0$ is the total quantity of the drug, a_0 is the initial location of the drug effect, and L is the length of the interval in which the drug works. With the introduction of the drug, we assume that $\beta_1(x)$ and $\beta_2(x)$ are replaced by $\beta_1(x)(1 - \eta(x))$ and $\beta_2(x)(1 - \eta(x))$, respectively, where $\eta(x)$ is the drug efficacy. Here, we assume the drug has the same effect on cell-to-cell and cell-to-virus. For convenience, denote

$$\eta(x) = \frac{v_0(x)}{2 + v_0(x)}.$$

We aim to minimize the basic reproduction number $\mathscr{R}_0$ by carefully selecting the optimal values of a_0 and L while keeping c_0 fixed. This approach will yield the most effective strategy for drug distribution. Let $c_0 = 0.5$, $a_0 = 0.2$ and $L = 1$, then the relationship between $1 - \eta(x)$ and x is illustrated in Figure 3.4(a), which depicts the proportion of change in the infection rates β_1 and β_2 following the administration of the drug. When $a_0 = 0.2$ and $L = 1$ are fixed, there exists a negative correlation between $\mathscr{R}_0$ and c_0, as illustrated in Figure 3.4(b). Therefore, we can increase the drug dosage to reduce the risk of transmission. With fixed values of $a_0 = 0.2$ and $c_0 = 0.5$, the basic reproduction number $\mathscr{R}_0$ initially decreases as L increases, and subsequently rises after reaching its minimum point, as illustrated in Figure 3.4(c). In this context, the minimum value of $\mathscr{R}_0$ aligns with the optimal selection of L once the quantity of drugs and the initial effective location of the drug are determined. When $c_0 = 0.5$ and $L = 1$ are kept constants, $\mathscr{R}_0$ initially decreases as a_0 increases. After reaching its minimum value, it starts to increase with further increases in a_0, surpassing the initial value, and reaches its maximum value when a_0 reaches the boundary of the region $x = 2$, as illustrated in Figure 3.4(d).

3.5 Conclusions and discussion

In this chapter, we formulated a time-delayed nonlocal reaction-diffusion model of within-host disease transmission to investigate the influences of the mobility of cells or viruses and spatial heterogeneity on with-in host disease pathogenesis. In order to obtain a threshold condition for the disease transmission, we chose a bounded spatial domain and the Neumann boundary condition. Firstly, we presented the well-posedness of the model, and then we introduced the basic reproduction number $\mathscr{R}_0$ to show that the infection-free steady state is globally asymptotically stable when $\mathscr{R}_0 \leq 1$, while the disease is uniformly persistent when $\mathscr{R}_0 > 1$. In the case where all coefficients and reaction terms are spatially homogeneous, we used the method of

Lyapunov functionals to obtain the global attractivity of the positive constant steady state.

Numerically, our primary focus lies in elucidating the analytical results, performing the sensitivity analysis, delving into how parameters affect the basic reproduction number, and probing the efficacy of drugs in mitigating the spread of the virus. The numerical results reveal the following insights: (i) PRCC analysis demonstrates that variables μ_1 and τ_2 exhibit the highest sensitivity to $\mathscr{R}_0$, showing the negative correlation. This suggests that implementing measures to augment both μ_1 and τ_2 could effectively curb virus transmission. (ii) The simulations of the optimal drug strategy underscore the significance of judiciously selecting the location and duration of drug delivery, especially in scenarios where drug resources are limited. This plays a pivotal role in exerting control over virus transmission.

Chapter 4

A time-delayed nonlocal reaction-diffusion model of within-host viral infections in an unbounded domain

In this chapter, we study traveling waves for a time-delayed nonlocal reaction-diffusion model of within-host viral infections. Firstly, we establish the existence of semi-traveling waves that converge to an unstable infection-free equilibrium as the moving coordinate goes to $-\infty$, provided the wave speed $c > c^*$ for some positive number c^* and the basic reproduction number $\mathscr{R}_0 > 1$. Then we construct a Lyapunov functional to show that the semi-travelling waves converge to an endemic equilibrium as the moving coordinate goes to $+\infty$, and use a limiting argument to obtain the existence of the traveling wave connecting these two equilibria for $c = c^*$ and $\mathscr{R}_0 > 1$. We further employ a Laplace transform technique to prove the non-existence of bounded semi-traveling waves when $0 < c < c^*$ and $\mathscr{R}_0 > 1$. It turns out that c^* is the minimum wave speed for traveling waves connecting the infection-free equilibrium and the endemic equilibrium. Finally, we conduct numerical simulations to illustrate the long-time behavior of solutions and the dependence of c^* on parameters.

4.1 Introduction

The basic model for within-host viral infection comprises uninfected target cells, infected target cells and free virus (see, e.g., [59,60,64,98,107,119,142,143]). In Chapter

3, we investigated the following time-delayed nonlocal reaction-diffusion system:

$$\begin{cases} \frac{\partial T(t,x)}{\partial t} = D_T \triangle T(t,x) + n(T(t,x)) - \beta_1 T(t,x)V(t,x) \\ \qquad\qquad - \beta_2 T(t,x)I(t,x), \ t > 0, x \in \Omega, \\ \frac{\partial I(t,x)}{\partial t} = D_I \triangle I(t,x) - \mu_1 I(t,x) \\ \qquad\qquad + e^{-\mu_1\tau_1} \int_\Omega \Gamma(D_I\tau_1, x, y)\beta_1 T(t-\tau_1, y)V(t-\tau_1, y)dy \\ \qquad\qquad + e^{-\mu_1\tau_1} \int_\Omega \Gamma(D_I\tau_1, x, y)\beta_2 T(t-\tau_1, y)I(t-\tau_1, y)dy, \ t > 0, x \in \Omega, \\ \frac{\partial V(t,x)}{\partial t} = D_V \triangle V(t,x) + be^{-\mu_2\tau_2} \int_\Omega \Gamma(D_V\tau_2, x, y)I(t-\tau_2, y)dy \\ \qquad\qquad - \mu V(t,x), \ t > 0, x \in \Omega, \\ \frac{\partial T(t,x)}{\partial \nu} = \frac{\partial I(t,x)}{\partial \nu} = \frac{\partial V(t,x)}{\partial \nu} = 0, \ t > 0, x \in \partial\Omega, \end{cases} \tag{4.1}$$

where $D_T > 0, D_I > 0$ and $D_V > 0$ are the random diffusion rates, τ_1 is the average incubation period for the healthy target cells infected by viruses or infected cells, and τ_2 is the average maturation period of the viruses. For model (4.1), we make the following assumption for $n(T)$:

(A1) There exists a unique $T^* > 0$ such that $n(T^*) = 0$; $n'(T) < 0$ for all $T \geq 0$.

From Chapter 3, we see that system (4.1) has an infection-free equilibrium $e_0 = (T^*, 0, 0)$, and that the basic reproduction number of system (4.1) can be computed as

$$\mathscr{R}_0 = \frac{e^{-\mu_1\tau_1}\beta_2 T^*}{\mu_1} + \frac{e^{-\mu_1\tau_1}\beta_1 T^* e^{-\mu_2\tau_2} b}{\mu_1 \mu}.$$

Also, when $\mathscr{R}_0 > 1$, system (4.1) has a unique positive equilibrium $u^* = (T_*, I_*, V_*)$ with

$$T_* = \frac{\mu\mu_1 e^{\mu_1\tau_1}}{\beta_1 be^{-\mu_2\tau_2} + \beta_2\mu}, \ I_* = \frac{n(T_*)}{\mu_1 e^{\mu_1\tau_1}}, \ V_* = \frac{be^{-\mu_2\tau_2}}{\mu} I_*.$$

In this chapter, we will investigate the traveling wave solutions of this non-monotone system on the unbounded spatial domain $\mathbb{R}$. Firstly, we introduce the parameter c^*. As our system lacks monotonicity, those well-established techniques relying on the comparison principle, such as monotone iterative schemes (see, e.g., [17, 94, 134]) and theory of traveling waves for monotone semiflows (see, e.g., [26, 68]), are inapplicable. To overcome this difficulty, we employ Schauder's fixed-point theorem to establish the existence of bounded semi-traveling wave solutions. These solutions represent wave behavior that converges to the unstable disease-free equilibrium as the moving frame $z = x + ct$ approaches $-\infty$, provided that the wave speed $c > c^*$ and $\mathscr{R}_0 > 1$. This accomplishment is attained through meticulous construction of a pair of lower and upper solutions. Specifically, compared to the traditional methods (see, e.g., [58, 76, 80, 134, 153]), the bilinear incidences make it impossible to find a bounded upper solution, so we introduce a truncated problem and further use some estimates to show that the solutions are C^2. Next, we prove the convergence of the semi-traveling waves to the endemic equilibrium as z approaches $+\infty$ by constructing a Lyapunov functional. While the Lyapunov functional approach is commonly favored in non-monotone systems, as exemplified in [2, 3, 20, 32, 50, 64] and the references therein, it is important to note that devising a Lyapunov functional for our system is highly nontrivial. When $c = c^*$ and $\mathscr{R}_0 > 1$, we establish the existence of a traveling wave solution connecting the disease-free equilibrium e_0 and the endemic equilibrium u^* by using a limiting argument and a way of contradiction. In order to obtain the non-existence of bounded semi-traveling wave solutions when $0 < c < c^*$ and $\mathscr{R}_0 > 1$, we leverage several key factors. These include the non-existence of positive eigenvalues associated to the unstable steady state and the utilization of the two-sided Laplace transform. Finally, we employ the Matlab to conduct numerical simulations. The minimal wave speed c^* is an essential parameter of our system. Based on its analytical definition, we present a numerical method for c^*. In addition, we summarize the parameter values of the model from some published literature, numerically calculate the values of c^* and the basic production number $\mathscr{R}_0$ for the model, and further investigate the long-time behaviour of the solutions. Furthermore, we explore the dependence of c^* on the system parameters.

The remainder of this chapter is organized as follows. In section 4.2, we formulate a time-delayed nonlocal reaction-diffusion model, and recall a threshold dynamics result in terms of $\mathscr{R}_0$ for the model on a bounded spatial domain. In section 4.3, we study traveling waves connecting the infection-free equilibrium and the endemic equilibrium for such a non-monotone system on an unbounded domain, and prove the existence of the minimum wave speed. In section 4.4, we present numerical simulation results to illustrate the long-time behavior of solutions and the dependence of c^* on parameters.

4.2 The model

Since the lymphoid tissue consists of 1000-1200 lymph nodes and the size of the lymph nodes is 2-38 mm (see [97]), the sizes of HIV and CD4 T cells are much smaller than the lymphoid tissue. For this reason and for convenience, we assume that the domain is $\mathbb{R}$. In view of Theorem 3.3.2, we only need to consider the case of $\mathscr{R}_0 > 1$ in order to study the spread of viral infection. Then system (4.1) reduces to the following one on the unbounded spatial domain $\mathbb{R}$:

$$
\begin{cases}
\frac{\partial T(t,x)}{\partial t} = D_T \triangle T(t,x) + n(T(t,x)) - \beta_1 T(t,x)V(t,x) - \beta_2 T(t,x)I(t,x), \\
\frac{\partial I(t,x)}{\partial t} = D_I \triangle I(t,x) - \mu_1 I(t,x) \\
\qquad\qquad + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, x-y)\beta_1 T(t-\tau_1, y)V(t-\tau_1, y)dy \\
\qquad\qquad + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, x-y)\beta_2 T(t-\tau_1, y)I(t-\tau_1, y)dy, \\
\frac{\partial V(t,x)}{\partial t} = D_V \triangle V(t,x) + b e^{-\mu_2\tau_2}\int_{\mathbb{R}} \Gamma(D_V\tau_2, x-y)I(t-\tau_2, y)dy - \mu V(t,x),
\end{cases}
\tag{4.2}
$$

where $\Gamma(t,x) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$ is the fundamental solution associated with the operator $\partial_t - \frac{\partial^2}{\partial x^2}$.

4.3 Traveling waves

In this section, we study the existence and nonexistence of traveling wave solutions for system (4.2), and show that there is a minimum wave speed for traveling waves connecting the infection-free equilibrium and the endemic equilibrium in the case of $\mathscr{R}_0 > 1$.

Let $(T(t,x), I(t,x), V(t,x)) = (T(z), I(z), V(z))$, where $z = x + ct$, the parameter c is called the wave speed. Then system (4.2) gives rise to

$$\begin{cases} D_T T'' - cT' + n(T) - \beta_1 TV - \beta_2 TI = 0, \\ D_I I'' - cI' - \mu_1 I + e^{-\mu_1\tau_1}\int_{\mathbb{R}}\Gamma(D_I\tau_1, y)\beta_1 T(z-c\tau_1-y)V(z-c\tau_1-y)dy \\ \qquad\qquad + e^{-\mu_1\tau_1}\int_{\mathbb{R}}\Gamma(D_I\tau_1, y)\beta_2 T(z-c\tau_1-y)I(z-c\tau_1-y)dy = 0, \\ D_V V'' - cV' + be^{-\mu_2\tau_2}\int_{\mathbb{R}}\Gamma(D_V\tau_2, y)I(z-c\tau_2-y)dy - \mu V = 0. \end{cases} \tag{4.3}$$

For system (4.2), the traveling wave profile satisfies (4.3) and the boundary conditions

$$\lim_{z\to-\infty}(T(z), I(z), V(z)) = e_0, \quad \lim_{z\to+\infty}(T(z), I(z), V(z)) = u^*. \tag{4.4}$$

Furthermore, we can linearize system of (4.3) at $e_0 = (T^*, 0, 0)$ to obtain

$$\begin{aligned} &D_T T'' - cT' + n'(T^*)T - \beta_1 T^* V - \beta_2 T^* I = 0, \\ &D_I I'' - cI' - \mu_1 I + e^{-\mu_1\tau_1}\int_{\mathbb{R}}\Gamma(D_I\tau_1, y)\beta_1 T^* V(z-c\tau_1-y)dy \\ &\qquad + e^{-\mu_1\tau_1}\int_{\mathbb{R}}\Gamma(D_I\tau_1, y)\beta_2 T^* I(z-c\tau_1-y)dy = 0, \\ &D_V V'' - cV' + be^{-\mu_2\tau_2}\int_{\mathbb{R}}\Gamma(D_V\tau_2, y)I(z-c\tau_2-y)dy - \mu V = 0. \end{aligned} \tag{4.5}$$

Letting $(T(z), I(z), V(z))^T = e^{\lambda z}(u_1, u_2, u_3)^T$, we then have

$$M(u_1, u_2, u_3)^T - c\lambda(u_1, u_2, u_3)^T = 0, \tag{4.6}$$

where

$$M = \begin{pmatrix} D_T\lambda^2 + n'(T^*) & -\beta_2 T^* & -\beta_1 T^* \\ 0 & a_1 & a_2 \\ 0 & a_3 & a_4 \end{pmatrix}$$

with

$$\begin{aligned} a_1 &= D_I\lambda^2 - \mu_1 + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_2 T^* e^{\lambda(-c\tau_1 - y)} dy \\ &= D_I\lambda^2 - \mu_1 + e^{-\mu_1\tau_1}\beta_2 T^* e^{(D_I\lambda^2 - c\lambda)\tau_1}, \\ a_2 &= e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1 T^* e^{\lambda(-c\tau_1 - y)} dy = e^{-\mu_1\tau_1}\beta_1 T^* e^{(D_I\lambda^2 - c\lambda)\tau_1}, \\ a_3 &= be^{-\mu_2\tau_2}\int_{\mathbb{R}} \Gamma(D_V\tau_2, y) e^{\lambda(-c\tau_2 - y)} dy = be^{-\mu_2\tau_2} e^{(D_V\lambda^2 - c\lambda)\tau_2}, \\ a_4 &= D_V\lambda^2 - \mu. \end{aligned}$$

Let

$$\begin{aligned} \hat{D} &= \begin{bmatrix} D_I & 0 \\ 0 & D_V \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}, \quad \hat{S} = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu \end{bmatrix}, \\ \hat{F} &= \begin{bmatrix} e^{-\mu_1\tau_1}\beta_2 T^* e^{(D_I\lambda^2 - c\lambda)\tau_1} & e^{-\mu_1\tau_1}\beta_1 T^* e^{(D_I\lambda^2 - c\lambda)\tau_1} \\ be^{-\mu_2\tau_2} e^{(D_V\lambda^2 - c\lambda)\tau_2} & 0 \end{bmatrix}. \end{aligned}$$

Denote $H(\lambda, c) = \lambda^2\hat{D} - \lambda\hat{C} - \hat{S} + \hat{F}$. It follows that the I- and V- equations in (4.6) can be written as

$$H(\lambda, c)\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = 0.$$

Let $D = \hat{S}^{-1}\hat{D}$, $C = \hat{S}^{-1}\hat{C}$ and $F = \hat{S}^{-1}\hat{F}$. Then the above system becomes

$$(-D\lambda^2 + C\lambda + I)^{-1}F \cdot \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_3 \end{pmatrix},$$

where

$$(-D\lambda^2 + C\lambda + I)^{-1}F = \begin{bmatrix} \frac{e^{-\mu_1\tau_1}\beta_2 T^* e^{(D_I\lambda^2 - c\lambda)\tau_1}}{m_1(\lambda,c)} & \frac{e^{-\mu_1\tau_1}\beta_1 T^* e^{(D_I\lambda^2 - c\lambda)\tau_1}}{m_1(\lambda,c)} \\ \frac{be^{-\mu_2\tau_2} e^{(D_V\lambda^2 - c\lambda)\tau_2}}{m_2(\lambda,c)} & 0 \end{bmatrix}$$

with

$$m_1(\lambda, c) = -D_I\lambda^2 + c\lambda + \mu_1, \;\; m_2(\lambda, c) = -D_V\lambda^2 + c\lambda + \mu.$$

Denote $G(\lambda, c) = (-D\lambda^2 + C\lambda + I)^{-1}F$. Then we have

$$G(\lambda, c)\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}.$$

Let $\rho(\lambda, c)$ be the principle eigenvalue of $G(\lambda, c)$ and

$$\lambda(c) = \min\left\{\frac{c + \sqrt{c^2 + 4D_I\mu_1}}{2D_I}, \frac{c + \sqrt{c^2 + 4D_V\mu}}{2D_V}\right\}.$$

For $c \geq 0$ and $\lambda \in [0, \lambda(c))$, we get

$$\rho(\lambda, c) = \frac{\frac{e^{-\mu_1\tau_1}\beta_2 T^* e^{(D_I\lambda^2 - c\lambda)\tau_1}}{-D_I\lambda^2 + c\lambda + \mu_1}}{2} + \frac{\sqrt{\left(\frac{e^{-\mu_1\tau_1}\beta_2 T^* e^{(D_I\lambda^2 - c\lambda)\tau_1}}{-D_I\lambda^2 + c\lambda + \mu_1}\right)^2 + 4\frac{e^{-\mu_1\tau_1}\beta_1 T^* e^{(D_I\lambda^2 - c\lambda)\tau_1}}{-D_I\lambda^2 + c\lambda + \mu_1}\frac{be^{-\mu_2\tau_2}e^{(D_V\lambda^2 - c\lambda)\tau_2}}{-D_V\lambda^2 + c\lambda + \mu}}}{2}.$$

Proposition 4.3.1. *The following statements are valid:*

(i) $\lambda(c)$ *is strictly increasing in* $c \in [0, +\infty)$ *and* $\lim_{c\to+\infty} \lambda(c) = +\infty$*.*

(ii) $\frac{\partial}{\partial c}\rho(\lambda, c) < 0$ *for any* $\lambda \in (0, \lambda(c))$*.*

(iii) $\rho(\lambda, 0)$ *is strictly increasing in* $\lambda \in [0, \lambda(0))$*;* $\lim_{\lambda\to\lambda(c)-0} \rho(\lambda, c) = +\infty$ *for any* $c \geq 0$*; and* $\rho(0, c) > 1$ *for any* $c \geq 0$*.*

Proof. We will only prove (ii) and (iii) since (i) can be easily verified. For any given $\lambda \in (0, \lambda(c))$, differentiating ρ with respect to c, we obtain

$$\frac{\partial\rho}{\partial c} = \frac{e^{-\mu_1\tau_1}\beta_2 T^* \frac{\partial}{\partial c}\left(\frac{e^{(D_I\lambda^2 - c\lambda)\tau_1}}{-D_I\lambda^2 + c\lambda + \mu_1}\right)}{2} + \frac{p(\lambda, c)}{2\sqrt{\left(\frac{e^{-\mu_1\tau_1}\beta_2 T^* e^{(D_I\lambda^2 - c\lambda)\tau_1}}{-D_I\lambda^2 + c\lambda + \mu_1}\right)^2 + 4\frac{e^{-\mu_1\tau_1}\beta_1 T^* e^{(D_I\lambda^2 - c\lambda)\tau_1}}{-D_I\lambda^2 + c\lambda + \mu_1}\frac{be^{-\mu_2\tau_2}e^{(D_V\lambda^2 - c\lambda)\tau_2}}{-D_V\lambda^2 + c\lambda + \mu}}},$$

where

$$\begin{aligned} p(\lambda, c) = {} & \frac{e^{-\mu_1\tau_1}\beta_2 T^* e^{(D_I\lambda^2 - c\lambda)\tau_1}}{-D_I\lambda^2 + c\lambda + \mu_1} e^{-\mu_1\tau_1}\beta_2 T^* \frac{\partial}{\partial c}\left(\frac{e^{(D_I\lambda^2 - c\lambda)\tau_1}}{-D_I\lambda^2 + c\lambda + \mu_1}\right) \\ & + 2\frac{be^{-\mu_2\tau_2}e^{(D_V\lambda^2 - c\lambda)\tau_2}}{-D_V\lambda^2 + c\lambda + \mu} e^{-\mu_1\tau_1}\beta_1 T^* \frac{\partial}{\partial c}\left(\frac{e^{(D_I\lambda^2 - c\lambda)\tau_1}}{-D_I\lambda^2 + c\lambda + \mu_1}\right) \\ & + 2\frac{e^{-\mu_1\tau_1}\beta_1 T^* e^{(D_I\lambda^2 - c\lambda)\tau_1}}{-D_I\lambda^2 + c\lambda + \mu_1} be^{-\mu_2\tau_2} \frac{\partial}{\partial c}\left(\frac{e^{(D_V\lambda^2 - c\lambda)\tau_2}}{-D_V\lambda^2 + c\lambda + \mu}\right). \end{aligned}$$

Since $-D_I\lambda^2 + c\lambda + \mu_1 > 0$ and $-D_V\lambda^2 + c\lambda + \mu > 0$ for $\lambda \in [0, \lambda(c))$, it follows that

$$\begin{aligned} \frac{\partial}{\partial c}\left(\frac{e^{(D_I\lambda^2 - c\lambda)\tau_1}}{-D_I\lambda^2 + c\lambda + \mu_1}\right) &= \frac{-\lambda\tau_1 e^{(D_I\lambda^2 - c\lambda)\tau_1}(-D_I\lambda^2 + c\lambda + \mu_1) - e^{(D_I\lambda^2 - c\lambda)\tau_1}\lambda}{(-D_I\lambda^2 + c\lambda + \mu_1)^2} \\ &= \frac{-\lambda e^{(D_I\lambda^2 - c\lambda)\tau_1}(\tau_1(-D_I\lambda^2 + c\lambda + \mu_1) + 1)}{(-D_I\lambda^2 + c\lambda + \mu_1)^2} < 0, \\ \frac{\partial}{\partial c}\left(\frac{e^{(D_V\lambda^2 - c\lambda)\tau_2}}{-D_V\lambda^2 + c\lambda + \mu}\right) &= \frac{-\lambda e^{(D_V\lambda^2 - c\lambda)\tau_2}(\tau_2(-D_V\lambda^2 + c\lambda + \mu) + 1)}{(-D_V\lambda^2 + c\lambda + \mu)^2} < 0. \end{aligned}$$

And hence, $\frac{\partial}{\partial c}\rho(\lambda, c) < 0$, which implies that (ii) holds.

Differentiating $\rho(\lambda, 0)$ with respect to $\lambda \in (0, \lambda(0))$, we have

$$\frac{\partial}{\partial \lambda}\rho(\lambda,0) = \frac{e^{-\mu_1\tau_1}\beta_2 T^* \frac{\partial}{\partial\lambda}\left(\frac{e^{(D_I\lambda^2)\tau_1}}{-D_I\lambda^2+\mu_1}\right)}{2} + \frac{q(\lambda,0)}{2\sqrt{\left(\frac{e^{-\mu_1\tau_1}\beta_2 T^* e^{(D_I\lambda^2)\tau_1}}{-D_I\lambda^2+\mu_1}\right)^2 + 4\frac{e^{-\mu_1\tau_1}\beta_1 T^* e^{(D_I\lambda^2)\tau_1}}{-D_I\lambda^2+\mu_1}\frac{be^{-\mu_2\tau_2}e^{(D_V\lambda^2)\tau_2}}{-D_V\lambda^2+\mu}}},$$

where

$$\begin{aligned} q(\lambda,0) = {} & \frac{e^{-\mu_1\tau_1}\beta_2 T^* e^{(D_I\lambda^2)\tau_1}}{-D_I\lambda^2+\mu_1} e^{-\mu_1\tau_1}\beta_2 T^* \frac{\partial}{\partial\lambda}\left(\frac{e^{(D_I\lambda^2)\tau_1}}{-D_I\lambda^2+\mu_1}\right) \\ & + 2\frac{be^{-\mu_2\tau_2}e^{(D_V\lambda^2)\tau_2}}{-D_V\lambda^2+\mu} e^{-\mu_1\tau_1}\beta_1 T^* \frac{\partial}{\partial\lambda}\left(\frac{e^{(D_I\lambda^2)\tau_1}}{-D_I\lambda^2+\mu_1}\right) \\ & + 2\frac{e^{-\mu_1\tau_1}\beta_1 T^* e^{(D_I\lambda^2)\tau_1}}{-D_I\lambda^2+\mu_1} be^{-\mu_2\tau_2} \frac{\partial}{\partial\lambda}\left(\frac{e^{(D_V\lambda^2)\tau_2}}{-D_V\lambda^2+\mu}\right). \end{aligned}$$

Since $-D_I\lambda^2 + \mu_1 > 0$ and $-D_V\lambda^2 + \mu > 0$, we further obtain

$$\frac{\partial}{\partial\lambda}\left(\frac{e^{(D_I\lambda^2)\tau_1}}{-D_I\lambda^2+\mu_1}\right) = \frac{e^{(D_I\lambda^2)\tau_1}(2D_I\lambda\tau_1)(-D_I\lambda^2+\mu_1) + e^{(D_I\lambda^2)\tau_1}(2D_I\lambda)}{(-D_I\lambda^2+\mu_1)^2} > 0,$$

$$\frac{\partial}{\partial\lambda}\left(\frac{e^{(D_V\lambda^2)\tau_2}}{-D_V\lambda^2+\mu}\right) = \frac{e^{(D_V\lambda^2)\tau_2}(2D_V\lambda\tau_2)(-D_V\lambda^2+\mu) + e^{(D_V\lambda^2)\tau_2}(2D_V\lambda)}{(-D_V\lambda^2+\mu)^2} > 0.$$

Thus, $\frac{\partial}{\partial\lambda}\rho(\lambda, 0) > 0$.

Note that

$$\lim_{\lambda\to\lambda(c)-0} \max\left\{\frac{1}{-D_I\lambda^2+c\lambda+\mu_1}, \frac{1}{-D_V\lambda^2+c\lambda+\mu}\right\} = +\infty, \; c \geq 0,$$

it is easy to see that $\lim_{\lambda\to\lambda(c)-0}\rho(\lambda, c) = +\infty$.

By the definition of $\rho(\lambda, c)$, it follows that

$$\rho(0,c) = \frac{\frac{e^{-\mu_1\tau_1}\beta_2 T^*}{\mu_1} + \sqrt{\left(\frac{e^{-\mu_1\tau_1}\beta_2 T^*}{\mu_1}\right)^2 + 4\frac{e^{-\mu_1\tau_1}\beta_1 T^*}{\mu_1}\frac{be^{-\mu_2\tau_2}}{\mu}}}{2}.$$

As in Chapter 3, define $\mathbb{E} := C(\bar{\Omega}, \mathbb{R}^2)$ and $E := C([-\hat{\tau}, 0], \mathbb{E})$ with $\hat{\tau} = \max\{\tau_1, \tau_2\}$. Further, let $\mathcal{L}(E, \mathbb{E})$ be the space of all bounded and linear operators from E to $\mathbb{E}$. For any $L \in \mathcal{L}(E, \mathbb{E})$, we define $\hat{L} \in \mathcal{L}(\mathbb{E}, \mathbb{E})$ by

$$\hat{L}y = L(\hat{y}), \forall y \in \mathbb{E},$$

where $\hat{y}(\theta) = y, \forall\theta \in [-\hat{\tau}, 0]$.

Note that we can write the I- and V- equations of linear system (4.1) as following FDEs in two ways:

$$\frac{du}{dt} = \mathcal{F}_1(u_t) - \mathcal{V}_1(u_t) = \mathcal{F}_2(u_t) - \mathcal{V}_2(u_t), \tag{4.7}$$

where

$$\mathcal{F}_1\psi(x) = \begin{pmatrix} [e^{-\mu_1\tau_1}\beta_1 T^*\psi_2(-\tau_1, x)] + [e^{-\mu_1\tau_1}\beta_2 T^*\psi_1(-\tau_1, x)] \\ 0 \end{pmatrix}$$

and $-\mathcal{V}_1\psi := A_1\psi(0) + L_1\psi$, with

$$A_1 u = \mathrm{diag}\,(D_I\triangle u_1, D_V\triangle u_2) + \mathrm{diag}\,(-\mu_1 u_1, -\mu u_2)\,,\ L_1\psi(x) = \begin{pmatrix} 0 \\ [e^{-\mu_2\tau_2} b\psi_1(-\tau_2, x)] \end{pmatrix};$$

and

$$\mathcal{F}_2\psi(x) = \begin{pmatrix} [e^{-\mu_1\tau_1}\beta_1 T^*\psi_2(-\tau_1, x)] + [e^{-\mu_1\tau_1}\beta_2 T^*\psi_1(-\tau_1, x)] \\ [e^{-\mu_2\tau_2} b\psi_1(-\tau_2, x)] \end{pmatrix}$$

and $-\mathcal{V}_2\psi := A_1\psi(0)$, for any $\psi = (\psi_1, \psi_2) \in E$. It then follows that (4.7) generates a semigroup $U(t)$ on E. Furthermore, the corresponding $\hat{\mathcal{F}}_1$, $\hat{\mathcal{V}}_1$, $\hat{\mathcal{F}}_2$ and $\hat{\mathcal{V}}_2$ are as follows

$$\hat{\mathcal{F}}_1 = \begin{pmatrix} e^{-\mu_1\tau_1}\beta_2 T^* & e^{-\mu_1\tau_1}\beta_1 T^* \\ 0 & 0 \end{pmatrix},\ \hat{\mathcal{V}}_1 = \begin{pmatrix} -D_I\triangle + \mu_1 & 0 \\ -e^{-\mu_2\tau_2} b & -D_V\triangle + \mu \end{pmatrix},$$
$$\hat{\mathcal{F}}_2 = \begin{pmatrix} e^{-\mu_1\tau_1}\beta_2 T^* & e^{-\mu_1\tau_1}\beta_1 T^* \\ e^{-\mu_2\tau_2} b & 0 \end{pmatrix},\ \hat{\mathcal{V}}_2 = \begin{pmatrix} -D_I\triangle + \mu_1 & 0 \\ 0 & -D_V\triangle + \mu \end{pmatrix}.$$

Let $C = C([-\hat{\tau}, 0], \mathbb{R}^2)$. We define $F_1, V_1, F_2, V_2 \in \mathcal{L}(C, \mathbb{R}^2)$ as follows

$$F_1(\phi) = \begin{pmatrix} [e^{-\mu_1\tau_1}\beta_1 T^*\phi_2(-\tau_1)] + [e^{-\mu_1\tau_1}\beta_2 T^*\phi_1(-\tau_1)] \\ 0 \end{pmatrix},$$
$$V_1(\phi) = \begin{pmatrix} \mu_1\phi_1(0) \\ [-\Psi(\tau_2)b(\cdot)\phi_1(-\tau_2)] + \mu\phi_2(0) \end{pmatrix},$$
$$F_2(\phi) = \begin{pmatrix} [e^{-\mu_1\tau_1}\beta_1 T^*\phi_2(-\tau_1)] + [e^{-\mu_1\tau_1}\beta_2 T^*\phi_1(-\tau_1)] \\ [\Psi(\tau_2)b(\cdot)\phi_1(-\tau_2)] \end{pmatrix},$$
$$V_2(\phi) = \begin{pmatrix} \mu_1\phi_1(0) \\ \mu\phi_2(0) \end{pmatrix},$$

for any $\phi = (\phi_1, \phi_2) \in C$. Further we can determine that $\hat{F}_1$, $\hat{V}_1$, $\hat{F}_2$ and $\hat{V}_2$ are as follows

$$\hat{F}_1 = \begin{pmatrix} e^{-\mu_1\tau_1}\beta_2 T^* & e^{-\mu_1\tau_1}\beta_1 T^* \\ 0 & 0 \end{pmatrix},\ \hat{V}_1 = \begin{pmatrix} \mu_1 & 0 \\ -e^{-\mu_2\tau_2} b & \mu \end{pmatrix},$$
$$\hat{F}_2 = \begin{pmatrix} e^{-\mu_1\tau_1}\beta_2 T^* & e^{-\mu_1\tau_1}\beta_1 T^* \\ e^{-\mu_2\tau_2} b & 0 \end{pmatrix},\ \hat{V}_2 = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu \end{pmatrix}.$$

It is easy to see that

$$\hat{\mathcal{F}}_1 \circ \hat{\mathcal{V}}_1^{-1}(v) = \hat{F}_1 \circ \hat{V}_1^{-1}(v),\ \hat{\mathcal{F}}_2 \circ \hat{\mathcal{V}}_2^{-1}(v) = \hat{F}_2 \circ \hat{V}_2^{-1}(v),\ \forall v = (v_1, v_2) \in \mathbb{R}^2,$$

due to the fact $\triangle v_1 = \triangle v_2 = 0$ for any $v = (v_1, v_2) \in \mathbb{R}^2$. In view of [67, Lemma 2.4], we have $r\left(\hat{\mathcal{F}}_1 \circ \hat{\mathcal{V}}_1^{-1}\right) = r\left(\hat{F}_1 \circ \hat{V}_1^{-1}\right)$ and $r\left(\hat{\mathcal{F}}_2 \circ \hat{\mathcal{V}}_2^{-1}\right) = r\left(\hat{F}_2 \circ \hat{V}_2^{-1}\right)$. According to Section 1.4.2, we may define two basic reproduction numbers for system (4.1) as follows:

$$r\left(\hat{\mathcal{F}}_1 \circ \hat{\mathcal{V}}_1^{-1}\right) = r\left(\hat{F}_1 \circ \hat{V}_1^{-1}\right) = \mathscr{R}_0, \quad r\left(\hat{\mathcal{F}}_2 \circ \hat{\mathcal{V}}_2^{-1}\right) = r\left(\hat{F}_2 \circ \hat{V}_2^{-1}\right) = \rho(0, c).$$

By Theorem 1.5.4, we see that $sign(\rho(0, c) - 1) = sign(\omega(U))$ and $sign(\mathscr{R}_0 - 1) = sign(\omega(U))$, where $\omega(U)$ is the exponential growth bound of the semigroup $U(t)$. Since $\mathscr{R}_0 > 1$, we further have $\rho(0, c) > 1$. □

In view of Proposition 4.3.1, we define

$$\Lambda(c) = \min_{\lambda \in [0, \lambda(c))} \rho(\lambda, c),\ c \geq 0.$$

It follows that $\Lambda(0) = \rho(0, c)$. Since $\mathscr{R}_0 > 1$, we have $\Lambda(0) > 1$. Note that $\lim_{c\to\infty} \Lambda(c) = 0$, $\Lambda(c)$ is continuous and strictly decreasing in $c \in [0, \infty)$. This behavior is also depicted in Figure 4.1 of Section 4.4. It then follows that there exists a constant $c^* > 0$ such that $\Lambda(c^*) = 1$; $\Lambda(c) > 1$ for $c \in [0, c^*)$; and $\Lambda(c) < 1$ for $c \in (c^*, \infty)$. Denote

$$\lambda^* = \inf\{\lambda \in [0, \lambda(c^*)) : \rho(\lambda, c^*) = 1\}.$$

It follows that $\rho(\lambda^*, c^*) = 1$, and $\rho(\lambda^*, c) < 1$ for any $c > c^*$. Define

$$\hat{\lambda}(c) = \sup\{\lambda \in (0, \lambda^*) : \rho(\lambda, c) = 1,\ \rho(\lambda', c) \geq 1,\ \forall \lambda' \in (0, \lambda)\}.$$

Note that $\rho(\lambda^*, c) < 1$ for any $c > c^*$. So we have the following results.

Proposition 4.3.2. *There exist $c^* > 0$ and $\lambda^* \in (0, \lambda(c^*))$ such that:*

(i) $\rho(\lambda, c) > 1$ *for any* $c \in [0, c^*)$ *and* $\lambda \in (0, \lambda(c))$.

(ii) $\rho(\lambda^*, c^*) = 1$; $\rho(\lambda, c^*) > 1$ *for any* $\lambda \in (0, \lambda^*)$; *and* $\rho(\lambda, c^*) \geq 1$ *for any* $\lambda \in (0, \lambda(c^*))$.

(iii) for any $c > c^*$, *there exists* $\hat{\lambda}(c) \in (0, \lambda^*)$ *such that* $\rho(\hat{\lambda}(c), c) = 1$; $\rho(\lambda, c) \geq 1$ *for* $\lambda \in (0, \hat{\lambda}(c))$; *and* $\rho(\hat{\lambda}(c) + \epsilon_n(c), c) < 1$ *for some decreasing sequence* $\{\epsilon_n(c)\}$ *satisfying* $\lim_{n\to\infty} \epsilon_n = 0$ *and* $\epsilon_n + \hat{\lambda}(c) < \lambda^*$ *for any* $n \in \mathbb{N}$. *Moreover,* $\hat{\lambda}(c)$ *is strictly decreasing in* $c \in (c^*, \infty)$.

(iv) for any $c > c^*$, there exist positive unit vectors $(u_2(c), u_3(c))^T$ and $(w_2(c), w_3(c))^T$ such that

$$G(\hat{\lambda}(c), c)\begin{pmatrix} u_2(c) \\ u_3(c) \end{pmatrix} = \begin{pmatrix} u_2(c) \\ u_3(c) \end{pmatrix},$$
$$G(\hat{\lambda}(c) + \epsilon_n(c), c)\begin{pmatrix} w_2(c) \\ w_3(c) \end{pmatrix} = \rho(\hat{\lambda}(c) + \epsilon_n(c), c)\begin{pmatrix} w_2(c) \\ w_3(c) \end{pmatrix}.$$

Proof. We only need to prove (iv). Clearly, the matrix $G(\lambda, c)$ is nonnegative and irreducible for $\lambda \in [0, \lambda(c))$. By virtue of Theorem 1.3.1 (Perron-Frobenius theorem), it is easy to see that 1 is the principle eigenvalue of matrix $G(\hat{\lambda}(c), c)$ with the eigenvector $(u_2(c), u_3(c))^T$, and that $\rho(\hat{\lambda}(c) + \epsilon_n(c), c)$ is the principle eigenvalue of the matrix $G(\hat{\lambda}(c) + \epsilon_n(c), c)$ with eigenvector $(w_2(c), w_3(c))^T$. □

For each $c > c^*$, since $\rho(\hat{\lambda}(c) + \epsilon_n(c), c) < 1$, Proposition 4.3.2 implies the following observation.

Lemma 4.3.1. *For each $c > c^*$, there exist $\lambda_c > 0$, $\epsilon > 1$, and positive eigenvectors $w_c = (w_{1c}, w_{2c})^T \gg 0$ and $w_{\epsilon c} = (w_{1\epsilon c}, w_{2\epsilon c})^T \gg 0$, such that*

$$A_1(\lambda_c)w_c = c\lambda_c w_c, \quad A_1(\epsilon\lambda_c)w_{\epsilon c} < c\epsilon\lambda_c w_{\epsilon c},$$

where

$$A_1 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix},$$

with a_1, a_2, a_3 and a_4 being defined in (4.6).

In the following, we present the existence of traveling wave solutions, which connect the two steady states e_0 and u^* for $c > c^*$. Motivated by [10, 132], we define the following nonnegative continuous functions:

$$\bar{T} = T^*, \underline{T} = \max\{0, T^* - \sigma e^{\alpha z}\}, (\bar{I}, \bar{V})^T = w_c e^{\lambda_c z},$$
$$(\underline{I}, \underline{V})^T = \max\{w_c e^{\lambda_c z} - q w_{\epsilon c} e^{\epsilon\lambda_c z}, 0\},$$

where $\sigma, \alpha, q > 0$ will be defined later.

Lemma 4.3.2. *The functions $(\bar{I}, \bar{V})$ satisfy*

$$0 \geq D_I\bar{I}'' - c\bar{I}' - \mu_1\bar{I} + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1 T^*\bar{V}(z - c\tau_1 - y)dy$$
$$+ e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_2 T^*\bar{I}(z - c\tau_1 - y)dy,$$
$$0 \geq D_V\bar{V}'' - c\bar{V}' + be^{-\mu_2\tau_2}\int_{\mathbb{R}} \Gamma(D_V\tau_2, y)\bar{I}(z - c\tau_2 - y)dy - \mu\bar{V},$$

for any $z \in \mathbb{R}$. Moreover, $\bar{I}$ and $\bar{V}$ are twice continuously differentiable on $\mathbb{R}$.

Proof. Since $(\bar{I},\bar{V})^T = w_c e^{\lambda_c z}$ and $A_1(\lambda_c)w_c = c\lambda_c w_c$ (see Lemma 4.3.1), it follows that

$$\begin{pmatrix} \{D_I\bar{I}'' - c\bar{I}' - \mu_1\bar{I} + e^{-\mu_1\tau_1}\int_{\mathbb{R}}\Gamma(D_I\tau_1,y)\beta_1T^*\bar{V}(z-c\tau_1-y)dy \\ \qquad + e^{-\mu_1\tau_1}\int_{\mathbb{R}}\Gamma(D_I\tau_1,y)\beta_2T^*\bar{I}(z-c\tau_1-y)dy\} \\ D_V\bar{V}'' - c\bar{V}' + be^{-\mu_2\tau_2}\int_{\mathbb{R}}\Gamma(D_V\tau_2,y)\bar{I}(z-c\tau_2-y)dy - \mu\bar{V} \end{pmatrix}$$
$$= e^{\lambda_c z}(A_1(\lambda_c) - c\lambda_c I)\omega_c = 0.$$

Clearly, the properties of twice continuous differentiability hold true. □

Lemma 4.3.3. *For any* $0 < \alpha < \min\{\lambda_c, c/D_T\}$ *and*

$$\sigma > \max\left\{T^*, \frac{T^*(\beta_1 w_{2c} + \beta_2 w_{1c})}{\alpha(c - D_T\alpha)}\right\},$$

there holds

$$D_T\underline{T}'' - c\underline{T}' + n(\underline{T}) - \beta_1\underline{T}\bar{V} - \beta_2\underline{T}\bar{I} \geq 0, \quad \forall z \neq (\ln(T^*/\sigma))/\alpha := x_0.$$

Moreover, $\underline{T}$ *is twice continuously differentiable on* $\mathbb{R}$ *except* $z = x_0$*, and* $\underline{T}'(x_0+) \geq \underline{T}'(x_0-)$.

Proof. When $z > x_0$, $\underline{T} = 0$, it is easy to see that the statement is valid. Now we assume that $z < x_0 < 0$ (the definition of x_0 implies that $x_0 < 0$), then $\underline{T} = T^* - \sigma e^{\alpha z}$ and $0 < T^* - \sigma e^{\alpha z} < T^*$. It then follows from the condition (A1) for the function n that $n(T^* - \sigma e^{\alpha z}) > 0$. We further have

$$\begin{aligned} &D_T\underline{T}'' - c\underline{T}' + n(\underline{T}) - \beta_1\underline{T}\bar{V} - \beta_2\underline{T}\bar{I} \\ &= -e^{\alpha z}\sigma D_T\alpha^2 + c\sigma\alpha e^{\alpha z} + n(T^* - \sigma e^{\alpha z}) \\ &\quad - \beta_1(T^* - \sigma e^{\alpha z})w_{2c}e^{\lambda_c z} - \beta_2(T^* - \sigma e^{\alpha z})w_{1c}e^{\lambda_c z} \\ &\geq e^{\alpha z}T^*\left[\frac{\sigma}{T^*}\alpha(c - D_T\alpha) - (\beta_1 w_{2c} + \beta_2 w_{1c})e^{(\lambda_c - \alpha)z}\right] + n(T^* - \sigma e^{\alpha z}) \\ &> e^{\alpha z}T^*\left[\frac{\sigma}{T^*}\alpha(c - D_T\alpha) - (\beta_1 w_{2c} + \beta_2 w_{1c})\right] \\ &\geq 0. \end{aligned}$$

From the graph of $\underline{T}$, we see that $\underline{T}'(x_0+) \geq \underline{T}'(x_0-)$ holds true. □

Lemma 4.3.4. *For sufficiently large* q *and* $1 < \epsilon < 1 + \alpha/\lambda_c$*, we have*

$$0 \leq D_I\underline{I}'' - c\underline{I}' - \mu_1\underline{I} + e^{-\mu_1\tau_1}\int_{\mathbb{R}}\Gamma(D_I\tau_1,y)\beta_1\underline{T}(z-c\tau_1-y)\underline{V}(z-c\tau_1-y)dy$$
$$+ e^{-\mu_1\tau_1}\int_{\mathbb{R}}\Gamma(D_I\tau_1,y)\beta_2\underline{T}(z-c\tau_1-y)\underline{I}(z-c\tau_1-y)dy,$$

for all $z \neq \frac{\ln \frac{w_{1c}}{qw_{1\epsilon c}}}{\lambda_c(\epsilon-1)} := x_1$*. Moreover,* $\underline{I}$ *is twice continuously differentiable on* $\mathbb{R}$ *except* $z = x_1$*, and* $\underline{I}'(x_1+) \geq \underline{I}'(x_1-)$*. In addition,*

$$0 \leq D_V\underline{V}'' - c\underline{V}' + be^{-\mu_2\tau_2}\int_{\mathbb{R}} \Gamma(D_V\tau_2, y)\underline{I}(z - c\tau_2 - y)dy - \mu\underline{V},$$

for all $z \neq \frac{\ln \frac{w_{2c}}{qw_{2\epsilon c}}}{\lambda_c(\epsilon-1)} := x_2$*. Moreover,* $\underline{V}$ *is twice continuously differentiable on* $\mathbb{R}$ *except* $z = x_2$*, and* $\underline{V}'(x_2+) \geq \underline{V}'(x_2-)$*.*

Proof. We only consider the case of $x_1 \leq x_2$ since we can prove the opposite one similarly. Let

$$q > \max\left\{\frac{w_{1c}}{w_{1\epsilon c}}, \frac{w_{2c}}{w_{2\epsilon c}}, \frac{w_{1c}}{w_{1\epsilon c}}\left(\frac{T^*}{\sigma}\right)^{-\frac{\lambda_c(\epsilon-1)}{\alpha}}, \frac{w_{2c}}{w_{2\epsilon c}}\left(\frac{T^*}{\sigma}\right)^{-\frac{\lambda_c(\epsilon-1)}{\alpha}}, \frac{q_1}{q_2}\right\},$$

where

$$\begin{aligned}
q_1 &= e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1\sigma dy \cdot w_{2c} + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_2\sigma dy \cdot w_{1c} > 0,\\
q_2 &= w_{1\epsilon c}\left[c\epsilon\lambda_c - D_I(\epsilon\lambda_c)^2 - e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_2 T^* dy + \mu_1\right]\\
&\quad - e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1 T^* dy \cdot w_{2\epsilon c} > 0.
\end{aligned}$$

Clearly, q_1 is bounded, and $x_1, x_2 < \frac{\ln \frac{T^*}{\sigma}}{\alpha} = x_0$. From $x_1 \leq x_2$, we now address the following three scenarios.

(i) Clearly, the first inequality is valid for $z > x_1$, and the second is valid for $z > x_2$.

(ii) In the case of $x_1 \leq z < x_2 < 0$, we have $\underline{I} = 0$, $\underline{V} = w_{2c}e^{\lambda_c z} - qw_{2\epsilon c}e^{\epsilon\lambda_c z} > 0$, $\underline{T} = T^* - \sigma e^{\alpha z} > 0$. Note that $A_1(\lambda_c)w_c = c\lambda_c w_c$, $A_1(\epsilon\lambda_c)w_{\epsilon c} < c\epsilon\lambda_c w_{\epsilon c}$ and $w_{1c}e^{\lambda_c z} - qw_{1\epsilon c}e^{\epsilon\lambda_c z} \leq 0$. Then we obtain

$$\begin{aligned}
&D_V\underline{V}'' - c\underline{V}' + be^{-\mu_2\tau_2}\int_{\mathbb{R}} \Gamma(D_V\tau_2, y)\underline{I}(z - c\tau_2 - y)dy - \mu\underline{V},\\
&= D_V(\lambda_c)^2 w_{2c}e^{\lambda_c z} - D_V q(\epsilon\lambda_c)^2 w_{2\epsilon c}e^{\epsilon\lambda_c z} - c\lambda_c w_{2c}e^{\lambda_c z}\\
&\quad + cqw_{2\epsilon c}\epsilon\lambda_c e^{\epsilon\lambda_c z} - \mu w_{2c}e^{\lambda_c z} + \mu qw_{2\epsilon c}e^{\epsilon\lambda_c z}\\
&= qe^{\epsilon\lambda_c z}[-D_V w_{2\epsilon c}(\epsilon\lambda_c)^2 + c\epsilon\lambda_c w_{2\epsilon c} + \mu w_{2\epsilon c}] + (D_V(\lambda_c)^2 - c\lambda_c - \mu)w_{2c}e^{\lambda_c z}\\
&> qe^{\epsilon\lambda_c z}be^{-\mu_2\tau_2}\int_{\mathbb{R}} \Gamma(D_V\tau_2, y)e^{\epsilon\lambda_c(-c\tau_2 - y)}dy w_{1\epsilon c}\\
&\quad - be^{-\mu_2\tau_2}\int_{\mathbb{R}} \Gamma(D_V\tau_2, y)e^{\lambda_c(-c\tau_2 - y)}dy w_{1c}e^{\lambda_c z}\\
&\geq 0.
\end{aligned}$$

The first inequality is obviously valid in this case.

(iii) In the case of $z < x_1 \leq x_2$, there hold $\underline{I} = w_{1c}e^{\lambda_c z} - qw_{1\epsilon c}e^{\epsilon\lambda_c z} > 0$, $\underline{V} = w_{2c}e^{\lambda_c z} - qw_{2\epsilon c}e^{\epsilon\lambda_c z} > 0$ and $\underline{T} = T^* - \sigma e^{\alpha z} > 0$. For any $y \in \mathbb{R}$, we further obtain the following results:

(a) if $z - c\tau_1 - y < x_0$, i.e., $y > z - c\tau_1 - x_0$, then $\underline{T}(z - c\tau_1 - y) = T^* - \sigma e^{\alpha(z-c\tau_1-y)}$; if $y < z - c\tau_1 - x_0$, then $\underline{T}(z - c\tau_1 - y) = 0$.

(b) if $z - c\tau_1 - y < x_1$, i.e., $y > z - c\tau_1 - x_1$, then $\underline{I}(z - c\tau_1 - y) = w_{1c}e^{\lambda_c(z-c\tau_1-y)} - qw_{1\epsilon c}e^{\epsilon\lambda_c(z-c\tau_1-y)}$; if $y < z - c\tau_1 - x_1$, then $\underline{I}(z - c\tau_1 - y) = 0$.

(c) if $z - c\tau_1 - y < x_2$, i.e., $y > z - c\tau_1 - x_2$, then $\underline{V}(z - c\tau_1 - y) = w_{2c}e^{\lambda_c(z-c\tau_1-y)} - qw_{2\epsilon c}e^{\epsilon\lambda_c(z-c\tau_1-y)}$; if $y < z - c\tau_1 - x_2$, then $\underline{V}(z - c\tau_1 - y) = 0$.

Then we have

$$
\begin{aligned}
&e^{-\mu_1\tau_1}\int_{\mathbb{R}}\Gamma(D_I\tau_1, y)\beta_1\underline{T}(z - c\tau_1 - y)\underline{V}(z - c\tau_1 - y)dy\\
&= e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_2}^{+\infty}\Gamma(D_I\tau_1, y)\beta_1(T^* - \sigma e^{\alpha(z-c\tau_1-y)})(w_{2c}e^{\lambda_c(z-c\tau_1-y)} - qw_{2\epsilon c}e^{\epsilon\lambda_c(z-c\tau_1-y)})dy\\
&= e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_2}^{+\infty}\Gamma(D_I\tau_1, y)\beta_1T^*w_{2c}e^{\lambda_c(z-c\tau_1-y)}dy\\
&\quad - e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_2}^{+\infty}\Gamma(D_I\tau_1, y)\beta_1T^*qw_{2\epsilon c}e^{\epsilon\lambda_c(z-c\tau_1-y)}dy\\
&\quad - e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_2}^{+\infty}\Gamma(D_I\tau_1, y)\beta_1\sigma e^{\alpha(z-c\tau_1-y)}w_{2c}e^{\lambda_c(z-c\tau_1-y)}dy\\
&\quad + e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_2}^{+\infty}\Gamma(D_I\tau_1, y)\beta_1\sigma e^{\alpha(z-c\tau_1-y)}qw_{2\epsilon c}e^{\epsilon\lambda_c(z-c\tau_1-y)}dy
\end{aligned}
$$

and

$$
\begin{aligned}
&e^{-\mu_1\tau_1}\int_{\mathbb{R}}\Gamma(D_I\tau_1, y)\beta_2\underline{T}(z - c\tau_1 - y)\underline{I}(z - c\tau_1 - y)dy\\
&= e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_1}^{+\infty}\Gamma(D_I\tau_1, y)\beta_2(T^* - \sigma e^{\alpha(z-c\tau_1-y)})(w_{1c}e^{\lambda_c(z-c\tau_1-y)} - qw_{1\epsilon c}e^{\epsilon\lambda_c(z-c\tau_1-y)})dy\\
&= e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_1}^{+\infty}\Gamma(D_I\tau_1, y)\beta_2T^*w_{1c}e^{\lambda_c(z-c\tau_1-y)}dy\\
&\quad - e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_1}^{+\infty}\Gamma(D_I\tau_1, y)\beta_2T^*qw_{1\epsilon c}e^{\epsilon\lambda_c(z-c\tau_1-y)}dy\\
&\quad - e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_1}^{+\infty}\Gamma(D_I\tau_1, y)\beta_2\sigma e^{\alpha(z-c\tau_1-y)}w_{1c}e^{\lambda_c(z-c\tau_1-y)}dy\\
&\quad + e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_1}^{+\infty}\Gamma(D_I\tau_1, y)\beta_2\sigma e^{\alpha(z-c\tau_1-y)}qw_{1\epsilon c}e^{\epsilon\lambda_c(z-c\tau_1-y)}dy.
\end{aligned}
$$

Similar to $A_1(\lambda_c)w_c = c\lambda_c w_c$ and $A_1(\epsilon\lambda_c)w_{\epsilon c} < c\epsilon\lambda_c w_{\epsilon c}$, we have

$$\begin{aligned}
&\left[D_I(\lambda_c)^2 - \mu_1 + e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_1}^{+\infty}\Gamma(D_I\tau_1, y)\beta_2 T^* e^{\lambda_c(-c\tau_1-y)}dy\right]w_{1c} \\
&\quad + e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_2}^{+\infty}\Gamma(D_I\tau_1, y)\beta_1 T^* e^{\lambda_c(-c\tau_1-y)}dy w_{2c} \\
&= c\lambda_c w_{1c}
\end{aligned}$$

and

$$\begin{aligned}
&\left[D_I(\epsilon\lambda_c)^2 - \mu_1 + e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_1}^{+\infty}\Gamma(D_I\tau_1, y)\beta_2 T^* e^{\epsilon\lambda_c(-c\tau_1-y)}dy\right]w_{1\epsilon c} \\
&\quad + e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_2}^{+\infty}\Gamma(D_I\tau_1, y)\beta_1 T^* e^{\epsilon\lambda_c(-c\tau_1-y)}dy w_{2\epsilon c} \\
&< c\epsilon\lambda_c w_{1\epsilon c}.
\end{aligned}$$

Then we have

$$D_I\underline{I}'' - c\underline{I}' - \mu_1\underline{I} + e^{-\mu_1\tau_1}\int_{\mathbb{R}}\Gamma(D_I\tau_1,y)\beta_1\underline{T}(z-c\tau_1-y)\underline{V}(z-c\tau_1-y)dy$$

$$+ e^{-\mu_1\tau_1}\int_{\mathbb{R}}\Gamma(D_I\tau_1,y)\beta_2\underline{T}(z-c\tau_1-y)\underline{I}(z-c\tau_1-y)dy,$$

$$= D_I w_{1c}(\lambda_c)^2 e^{\lambda_c z} - D_I q w_{1\epsilon c}(\epsilon\lambda_c)^2 e^{\epsilon\lambda_c z} - c w_{1c}\lambda_c e^{\lambda_c z} + c q w_{1\epsilon c}(\epsilon\lambda_c)e^{\epsilon\lambda_c z}$$

$$- \mu_1 w_{1c} e^{\lambda_c z} + \mu_1 q w_{1\epsilon c} e^{\epsilon\lambda_c z} + e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_2}^{+\infty}\Gamma(D_I\tau_1,y)\beta_1 T^* w_{2c} e^{\lambda_c(z-c\tau_1-y)}dy$$

$$- e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_2}^{+\infty}\Gamma(D_I\tau_1,y)\beta_1 T^* q w_{2\epsilon c} e^{\epsilon\lambda_c(z-c\tau_1-y)}dy$$

$$- e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_2}^{+\infty}\Gamma(D_I\tau_1,y)\beta_1\sigma e^{\alpha(z-c\tau_1-y)} w_{2c} e^{\lambda_c(z-c\tau_1-y)}dy$$

$$+ e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_2}^{+\infty}\Gamma(D_I\tau_1,y)\beta_1\sigma e^{\alpha(z-c\tau_1-y)} q w_{2\epsilon c} e^{\epsilon\lambda_c(z-c\tau_1-y)}dy$$

$$+ e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_1}^{+\infty}\Gamma(D_I\tau_1,y)\beta_2 T^* w_{1c} e^{\lambda_c(z-c\tau_1-y)}dy$$

$$- e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_1}^{+\infty}\Gamma(D_I\tau_1,y)\beta_2 T^* q w_{1\epsilon c} e^{\epsilon\lambda_c(z-c\tau_1-y)}dy$$

$$- e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_1}^{+\infty}\Gamma(D_I\tau_1,y)\beta_2\sigma e^{\alpha(z-c\tau_1-y)} w_{1c} e^{\lambda_c(z-c\tau_1-y)}dy$$

$$+ e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_1}^{+\infty}\Gamma(D_I\tau_1,y)\beta_2\sigma e^{\alpha(z-c\tau_1-y)} q w_{1\epsilon c} e^{\epsilon\lambda_c(z-c\tau_1-y)}dy$$

$$= q e^{\epsilon\lambda_c z}\Bigg[c(\epsilon\lambda_c)w_{1\epsilon c} - D_I(\epsilon\lambda_c)^2 w_{1\epsilon c} - e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_2}^{+\infty}\Gamma(D_I\tau_1,y)\beta_1 T^* e^{\epsilon\lambda_c(-c\tau_1-y)}dy w_{2\epsilon c}$$

$$-e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_1}^{+\infty}\Gamma(D_I\tau_1,y)\beta_1 T^* e^{\epsilon\lambda_c(-c\tau_1-y)}dy w_{1\epsilon c} + \mu_1 w_{1\epsilon c}\Bigg]$$

$$- e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_2}^{+\infty}\Gamma(D_I\tau_1,y)\beta_1\sigma e^{\alpha(z-c\tau_1-y)} w_{2c} e^{\lambda_c(z-c\tau_1-y)}dy$$

$$+ e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_2}^{+\infty}\Gamma(D_I\tau_1,y)\beta_1\sigma e^{\alpha(z-c\tau_1-y)} q w_{2\epsilon c} e^{\epsilon\lambda_c(z-c\tau_1-y)}dy$$

$$- e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_1}^{+\infty}\Gamma(D_I\tau_1,y)\beta_2\sigma e^{\alpha(z-c\tau_1-y)} w_{1c} e^{\lambda_c(z-c\tau_1-y)}dy$$

$$+ e^{-\mu_1\tau_1}\int_{z-c\tau_1-x_1}^{+\infty}\Gamma(D_I\tau_1,y)\beta_2\sigma e^{\alpha(z-c\tau_1-y)} q w_{1\epsilon c} e^{\epsilon\lambda_c(z-c\tau_1-y)}dy$$

$$> e^{(\alpha+\lambda_c)z}(q\cdot q_2 - q_1)$$

$$\geq 0$$

since $z < 0$ and $1 < \epsilon < 1 + \alpha/\lambda_c$. The second inequality can be obtained similarly.

By the graphs of $\underline{I}$ and $\underline{V}$, we see that $\underline{I}'(x_1+) \geq \underline{I}'(x_1-)$ and $\underline{V}'(x_2+) \geq \underline{V}'(x_2-)$. □

Next, we consider a truncated problem and establish the existence of the semi-traveling wave solutions that connect e_0 for system (4.3) by limiting arguments.

Suppose $c > c^*$ and $l > \max_{j=0,1,2}\{|x_j|\}$. Let $I_l = [-l, l]$ and $X = C(I_l) \times C(I_l) \times C(I_l)$. Define

$$\Lambda = \big\{(T, I, V) \in X : \ \underline{T}(z) \leq T(z) \leq T^*, \ \underline{I}(z) \leq I(z) \leq \bar{I}(z), \ \underline{V}(z) \leq V(z) \leq \bar{V}(z), \\ T(\pm l) = \underline{T}(\pm l), \ I(\pm l) = \underline{I}(\pm l), \ V(\pm l) = \underline{V}(\pm l)\big\}.$$

Clearly, Λ is a closed convex set in X. For any $(u_{10}, u_{20}, u_{30}) \in \Lambda$, define

$$\hat{u}_{10}(z) = \begin{cases} u_{10}(z), & |z| < l, \\ \underline{T}(z), & |z| \geq l, \end{cases} \quad \hat{u}_{20}(z) = \begin{cases} u_{20}(z), & |z| < l, \\ \underline{I}(z), & |z| \geq l, \end{cases} \quad \hat{u}_{30}(z) = \begin{cases} u_{30}(z), & |z| < l, \\ \underline{V}(z), & |z| \geq l. \end{cases}$$

Then we consider the boundary value problem as follows:

$$\begin{aligned} & D_T u_1'' - c u_1' + n(u_1) - \beta_1 u_1 u_{30} - \beta_2 u_1 u_{20} = 0, \ z \in (-l, l), \\ & D_I u_2'' - c u_2' - \mu_1 u_2 + e^{-\mu_1 \tau_1} \int_{\mathbb{R}} \Gamma(D_I \tau_1, y) \beta_1 \hat{u}_{10}(z - c\tau_1 - y) \hat{u}_{30}(z - c\tau_1 - y) dy \\ & \qquad + e^{-\mu_1 \tau_1} \int_{\mathbb{R}} \Gamma(D_I \tau_1, y) \beta_2 \hat{u}_{10}(z - c\tau_1 - y) \hat{u}_{20}(z - c\tau_1 - y) dy = 0, \ z \in (-l, l), \\ & D_V u_3'' - c u_3' + b e^{-\mu_2 \tau_2} \int_{\mathbb{R}} \Gamma(D_V \tau_2, y) \hat{u}_{20}(z - c\tau_2 - y) dy - \mu u_3 = 0, \ z \in (-l, l), \\ & u_1(z) = \underline{T}(z), \ u_2(z) = \underline{I}(z), \ u_3(z) = \underline{V}(z), \ z \in \{-l, l\}. \end{aligned} \tag{4.8}$$

In view of [37, Corollary 9.18], it is easy to see that system (4.8) admits a unique solution (u_1, u_2, u_3) satisfying $u_1, u_2, u_3 \in W^{2,p}(I_l, \mathbb{R}) \cap C(I_l)$ for any $p > 1$. By the embedding theorem (see [37, Theorem 7.26]), we obtain that $u_1(\cdot), u_2(\cdot), u_3(\cdot) \in W^{2,p}(I_l) \hookrightarrow C^{1+\alpha}(I_l)$ for some $\alpha \in (0, 1)$. Note that system (4.8) is not a coupled system, we define $F = (F_1, F_2, F_3) : \Lambda \to C(I_l)$ by

$$u_1 = F_1(u_{10}, u_{20}, u_{30}), \ u_2 = F_2(u_{10}, u_{20}, u_{30}), \ u_3 = F_3(u_{10}, u_{20}, u_{30}),$$

where (u_1, u_2, u_3) is the solution of (4.8) for any $(u_{10}, u_{20}, u_{30}) \in \Lambda$.

Lemma 4.3.5. *$F(\Lambda) \subset \Lambda$.*

Proof. Let (u_1, u_2, u_3) be the solution of system (4.8) with any given $(u_{10}, u_{20}, u_{30}) \in \Lambda$. It is easy to see that 0 and T^* are the sub- and super-solutions of the first equation of (4.8) on $(-l, l)$, respectively. By using the maximum principle (see [37, Theorem 9.6]) and the fact

$$0 < u_1(l) = \underline{T}(l) < T^*, \ 0 = u_1(-l) = \underline{T}(-l) < T^*,$$

we obtain that $0 \le u_1(z) \le T^*$ for any $z \in I_l$. It then follows from Lemma 4.3.3 that $\underline{T}(z)$ satisfies

$$\begin{aligned}0 &\ge -D_T\underline{T}''(z) + c\underline{T}'(z) - n(\underline{T}(z)) + \beta_1\underline{T}(z)\bar{V}(z) + \beta_2\underline{T}(z)\bar{I}(z)\\ &\ge -D_T\underline{T}''(z) + c\underline{T}'(z) - n(\underline{T}(z)) + \beta_1\underline{T}(z)V(z) + \beta_2\underline{T}(z)I(z),\end{aligned}$$

on $[-l, x_0]$. Since $u_1(-l) = \underline{T}(-l)$ and $u_1(x_0) \ge \underline{T}(x_0) = 0$, it follows from the maximum principle that $\underline{T}(z) \le u_1(z)$ for $z \in [-l, x_0]$. As a result, we have $\underline{T}(z) \le u_1(z) \le T^*$ for all $z \in I_l$.

Note that

$$\underline{T}(z) \le \hat{u}_{10}(z) \le T^*,\ \underline{I}(z) \le \hat{u}_{20}(z) \le \bar{I}(z),\ \underline{V}(z) \le \hat{u}_{30}(z) \le \bar{V}(z),$$

for any $|z| \ge l$. From the second equation of (4.8), we then get

$$\begin{aligned}&D_I u_2'' - cu_2' - \mu_1 u_2\\ &= -e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1\hat{u}_{10}(z - c\tau_1 - y)\hat{u}_{30}(z - c\tau_1 - y)dy\\ &\quad - e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_2\hat{u}_{10}(z - c\tau_1 - y)\hat{u}_{20}(z - c\tau_1 - y)dy\\ &\le -e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1\underline{T}(z - c\tau_1 - y)\underline{V}(z - c\tau_1 - y)dy\\ &\quad - e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_2\underline{T}(z - c\tau_1 - y)\underline{I}(z - c\tau_1 - y)dy\\ &\le 0,\ z \in (-l, l),\end{aligned}$$

and $u_2(l) = 0, u_2(-l) = \underline{I}(-l) > 0$, since $l > \max_{j=0,1,2}\{|x_j|\}$. It follows from the comparison principle that $u_2(z) > 0$ for all $z \in (-l, l)$. By the arguments similar to those for u_2, we can prove that $u_3(z) > 0$ for all $z \in (-l, l)$.

For any $(u_{10}, u_{20}, u_{30}) \in \Lambda$, it follows from the second equation of (4.8) that

$$\begin{aligned}&D_I u_2'' - cu_2' - \mu_1 u_2 + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1 T^*\bar{V}(z - c\tau_1 - y)dy\\ &\qquad + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_2 T^*\bar{I}(z - c\tau_1 - y)dy \ge 0,\\ &D_I u_2'' - cu_2' - \mu_1 u_2 + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1\underline{T}(z - c\tau_1 - y)\underline{V}(z - c\tau_1 - y)dy\\ &\qquad + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_2\underline{T}(z - c\tau_1 - y)\underline{I}(z - c\tau_1 - y)dy \le 0,\end{aligned}$$

for all $z \in (-l, l)$. Let $w = \bar{I} - u_2$. It then follows from Lemma 4.3.2 that $D_I w'' - cw' - \mu_1 w \le 0$ for all $z \in (-l, l)$. Note that $w(\pm l) > 0$, then $w \ge 0$ for all $z \in I_l$ by the comparison principle, and hence, $u_2 \le \bar{I}$ for all $z \in I_l$. Let $p = u_2 - \underline{I}$. It then follows from Lemma 4.3.4 that $D_I p'' - cp' - \mu_1 p \leq 0$ for all $z \in (-l, x_1)$, $p(-l) = 0$, and $p(x_1) = u_2(x_1) > 0$. The comparison principle yields that $p \geq 0$ for all $z \in [-l, x_1]$. And hence, $u_2 \geq \underline{I}$ for all $z \in I_l$, since $\underline{I} = 0$ for all $z > x_1$. Therefore, $u_2 \in [\underline{I}, \bar{I}]$ for all $z \in I_l$.

By the similar argument as above and using the results in Lemmas 4.3.2 and 4.3.4, we can prove that $u_3 \in [\underline{V}, \bar{V}]$ for all $z \in I_l$. □

Lemma 4.3.6. *The operator $F : \Lambda \to \Lambda$ is completely continuous.*

Proof. We can initially obtain that F is compact on Λ by using the global elliptic estimate (see [37, Lemma 9.17]) and the embedding theorem.

Suppose $(u_{10}, u_{20}, u_{30}) \in \Lambda$ and $(v_{10}, v_{20}, v_{30}) \in \Lambda$. Now, we can define $(\hat{v}_{10}, \hat{v}_{20}, \hat{v}_{30})$ in a similar manner to how we defined $(\hat{u}_{10}, \hat{u}_{20}, \hat{u}_{30})$. Similar to system (4.8), we then define

$$(u_1, u_2, u_3) = F(u_{10}, u_{20}, u_{30}), \ (v_1, v_2, v_3) = F(v_{10}, v_{20}, v_{30}).$$

Let $w = u_1 - v_1$. Then $w(\pm l) = 0$. By making the difference between the equations satisfied by u_1 and v_1, we have

$$w'' - \frac{c}{D_T} w' + f(z)w(z) = h(z), \ z \in (-l, l),$$

where

$$f = -\frac{1}{D_T}(\beta_1 v_{30} + \beta_2 v_{20}), \ h = -\frac{1}{D_T}[n(u_1) - n(v_1)] + \frac{u_1}{D_T}[\beta_1(u_{30} - v_{30}) + \beta_2(u_{20} - v_{20})].$$

Since $v_1, u_{20}, v_{20}, u_{30}$ and v_{30} are bounded in $C(I_l)$, we obtain that F_1 is continuous on Λ by using the global elliptic estimate and the embedding theorem. Similarly, we can prove that F_2 and F_3 are continuous on Λ. □

Applying Schauder's fixed-point theorem to the operator F, we conclude that there exists $(u_1, u_2, u_3) \in \Lambda$ satisfying $(u_1, u_2, u_3) = F(u_1, u_2, u_3)$. More precisely, (u_1, u_2, u_3) satisfies:

$$\begin{aligned}
&D_T u_1'' - cu_1' + n(u_1) - \beta_1 u_1 u_3 - \beta_2 u_1 u_2 = 0, \ z \in (-l, l), \\
&D_I u_2'' - cu_2' - \mu_1 u_2 + e^{-\mu_1 \tau_1} \int_{\mathbb{R}} \Gamma(D_I \tau_1, y) \beta_1 \hat{u}_1(z - c\tau_1 - y) \hat{u}_3(z - c\tau_1 - y) dy \\
&\qquad + e^{-\mu_1 \tau_1} \int_{\mathbb{R}} \Gamma(D_I \tau_1, y) \beta_2 \hat{u}_1(z - c\tau_1 - y) \hat{u}_2(z - c\tau_1 - y) dy = 0, \ z \in (-l, l), \\
&D_V u_3'' - cu_3' + be^{-\mu_2 \tau_2} \int_{\mathbb{R}} \Gamma(D_V \tau_2, y) \hat{u}_2(z - c\tau_2 - y) dy - \mu u_3 = 0, \ z \in (-l, l), \\
&u_1(z) = \underline{T}(z), \ u_2(z) = \underline{I}(z), \ u_3(z) = \underline{V}(z), \ z \in \{-l, l\},
\end{aligned} \tag{4.9}$$

where

$$\hat{u}_1(z) = \begin{cases} u_1(z), & |z| < l, \\ \underline{T}(z), & |z| \geq l, \end{cases} \quad \hat{u}_2(z) = \begin{cases} u_2(z), & |z| < l, \\ \underline{I}(z), & |z| \geq l, \end{cases} \quad \hat{u}_3(z) = \begin{cases} u_3(z), & |z| < l, \\ \underline{V}(z), & |z| \geq l. \end{cases}$$

Next, by the arguments similar to those for [148, Theorem 3.9], we have some estimates for u_1, u_2, u_3.

Lemma 4.3.7. *For any given $\hat{l} \in (0, l)$, there exists a constant $M = M(\hat{l}) > 0$ such that*

$$\|u_1\|_{C^3[-\hat{l},\hat{l}]} \leq M, \quad \|u_2\|_{C^3[-\hat{l},\hat{l}]} \leq M, \quad \|u_3\|_{C^3[-\hat{l},\hat{l}]} \leq M.$$

It is noteworthy that Lemma 4.3.7 implies the following result:

$$u_1 \to T, \; u_2 \to I, \; u_3 \to V, \text{ in } C^2_{loc}(\mathbb{R}) \text{ as } l \to \infty.$$

Now we present the existence of the semi-traveling wave solutions that connect e_0, i.e., the wave solutions that converge to an unstable boundary (disease-free) equilibrium as the moving frame $z = x + ct \to -\infty$.

Proposition 4.3.3. *For any $c > c^*$, system (4.3) admits a solution $(T(z), I(z), V(z))$ satisfying $(T(-\infty), I(-\infty), V(-\infty)) = (T^*, 0, 0)$, and $0 < T(z) < T^*$, $I(z) > 0$, $V(z) > 0$ for any $z \in \mathbb{R}$.*

Proof. Let $\{l_n\}_{n\in\mathbb{N}}$ be an increasing sequence with $l_1 > \max_{j=0,1,2}\{|x_j|\}$ and $l_n \to +\infty$ as $n \to +\infty$. Let (T_n, I_n, V_n) be the solution of system (4.9) with $l = l_n$. It then follows from Lemma 4.3.7 and the Lebesgue's dominated convergence theorem that there exists a solution $(T, I, V) \in C^2(\mathbb{R}, \mathbb{R}^3)$ of (4.3) such that

$$\underline{T} \leq T \leq \bar{T}, \; \underline{I} \leq I \leq \bar{I}, \; \underline{V} \leq V \leq \bar{V}, \; \forall z \in \mathbb{R}.$$

And hence, $(T(-\infty), I(-\infty), V(-\infty)) = (T^*, 0, 0)$.

Assume, by contradiction, that there exists a z_1 such that $T(z_1)$=0, then $T'(z_1) = 0$ and $T''(z_1) \geq 0$ since $T(z) \geq 0$. But the first equation of system (4.3) implies that $D_T T''(z_1) = -n(T(z_1)) = -n(0) < 0$, a contradiction. Hence, $T(z) > 0$ for all $z \in \mathbb{R}$.

Suppose that there exists a z_2 such that $I(z_2) = 0$, then $I'(z_2) = 0$ and $I''(z_2) \geq 0$ since $I(z) \geq 0$. It follows from the second equation of (4.3) that

$$\begin{aligned} D_I I''(z_2) = &-e^{-\mu_1\tau_1} \int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1 T(z_2 - c\tau_1 - y)V(z_2 - c\tau_1 - y)dy \\ &- e^{-\mu_1\tau_1} \int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_2 T(z_2 - c\tau_1 - y)I(z_2 - c\tau_1 - y)dy \leq 0, \end{aligned}$$

which means that $I(z) \equiv V(z) \equiv 0$ for any $z \in \mathbb{R}$, a contradiction. Therefore, $I(z) > 0$ for all $z \in \mathbb{R}$. By the similar argument as above, we see that $V(z) > 0$ for all $z \in \mathbb{R}$.

Then we show that $T < T^*$. Suppose that there exists a z_3 such that $T(z_3) = T^*$. Then $T'(z_3) = 0$ and $T''(z_3) \leq 0$. It then follows from the first equation in (4.3) that $D_T T''(z_3) = \beta_1 T^* V(z_3) + \beta_2 T^* I(z_3) > 0$, a contradiction. Hence, $T(z) < T^*$ for all $z \in \mathbb{R}$. $\square$

We can further establish an upper bound for the infected cells I under the following additional assumption:

(A2) $D_T \geq D_I > 0$.

The assumption (A2) is reasonable biologically. It implies that the activity capacity of healthy cells should be stronger than that of infected cells. In the rest of this paper, we always assume that (A2) holds.

In the following, we prove that the semi-traveling wave solutions are indeed the traveling wave solutions connecting the two steady states by using the method of Lyapunov functionals and Lebesgue's dominated convergence theorem.

Lemma 4.3.8. *Let (T, I, V) be the solution of (4.3) obtained in Proposition 4.3.3. Then the following statements are valid:*

(i) The functions $I(z)$ and $V(z)$ are bounded on $\mathbb{R}$;

(ii) The limits $\liminf_{z\to+\infty} I(z) > 0$ and $\liminf_{z\to+\infty} V(z) > 0$. Moreover, there exists a constant $\delta > 0$ such that $T(z) \geq \delta$ for all $z \in \mathbb{R}$.

(iii) There exists a constant $k > 0$ such that

$$\max\left\{\max_{s\in[z-1,z+1]} I(s), \max_{s\in[z-1,z+1]} V(s)\right\} \leq k \min\left\{\min_{s\in[z-1,z+1]} I(s), \min_{s\in[z-1,z+1]} V(s)\right\}$$

for any $z \in \mathbb{R}$.

(iv) $|T'(z)|$, $|I'(z)|$, $|V'(z)|$, $|T'(z)/T(z)|$, $|I'(z)/I(z)|$ and $|V'(z)/V(z)|$ are bounded on $\mathbb{R}$.

Proof. (i). For any $z \in \mathbb{R}$, denote

$$\begin{aligned}
A(z) &= \beta_1 T(z)V(z) + \beta_2 T(z)I(z), \\
P(z) &= \int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\left[\beta_1 T(z - c\tau_1 - y)V(z - c\tau_1 - y) + \beta_2 T(z - c\tau_1 - y)I(z - c\tau_1 - y)\right] dy \\
&= \int_{\mathbb{R}} \Gamma(D_I\tau_1, y)A(z - c\tau_1 - y)dy, \\
Q(z) &= \int_{\mathbb{R}} \Gamma(D_V\tau_2, y)I(z - c\tau_2 - y)dy.
\end{aligned}$$

By the assumption (A1), we can find that there exist two positive numbers s and h such that $n(T) \leq s - hT$ for any $T \in \mathbb{R}_+$. For any $z \in \mathbb{R}$, we further have

$$\begin{cases} -D_T T''(z) + cT'(z) + hT(z) \leq s - A(z), \\ -D_I I''(z) + cI'(z) + \mu_1 I(z) \leq e^{-\mu_1\tau_1} P(z), \\ -D_V V''(z) + cV'(z) + \mu V(z) \leq be^{-\mu_2\tau_2} Q(z). \end{cases}$$

Now we can consider the following Cauchy problems:

$$\begin{cases} \frac{\partial}{\partial t} k_1(t,z) - D_T \frac{\partial^2}{\partial z^2} k_2(t,z) + c\frac{\partial}{\partial z} k_1(t,z) + hk_1(t,z) = s - A(z), \ t > 0, z \in \mathbb{R}, \\ k_1(0,z) = T(z), \ z \in \mathbb{R}, \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial t} k_2(t,z) - D_I \frac{\partial^2}{\partial z^2} k_2(t,z) + c\frac{\partial}{\partial z} k_2(t,z) + \mu_1 k_2(t,z) = e^{-\mu_1\tau_1} P(z), \ t > 0, z \in \mathbb{R}, \\ k_2(0,z) = I(z), \ z \in \mathbb{R}, \end{cases}$$

and

$$\begin{cases} \frac{\partial}{\partial t} k_3(t,z) - D_V \frac{\partial^2}{\partial z^2} k_3(t,z) + c\frac{\partial}{\partial z} k_3(t,z) + \mu k_3(t,z) = be^{-\mu_2\tau_2} Q(z), \ t > 0, z \in \mathbb{R}, \\ k_3(0,z) = V(z), \ z \in \mathbb{R}. \end{cases}$$

By the theory of Cauchy problems (see, e.g., [30, Chapter 1, Theorems 12 and 16]), we obtain that the solution

$$\begin{aligned} k_2(t,z) = & \frac{e^{-\mu_1 t}}{\sqrt{4\pi D_I t}} \int_{\mathbb{R}} e^{-\frac{(z-ct-y)^2}{4D_I t}} I(y)dy \\ & + \int_0^t \int_{\mathbb{R}} \frac{e^{-\mu_1 s}}{\sqrt{4\pi D_I s}} e^{-\frac{(z-cs-y)^2}{4D_I s}} P(y)dyds, \ t > 0, z \in \mathbb{R}. \end{aligned}$$

It follows from the comparison principle that $I(z) \leq k_2(t,x)$ for any $t > 0$ and $z \in \mathbb{R}$. Clearly, $(T(z), I(z), V(z))^T \leq (T^*, w_{1c}e^{\lambda_c z}, w_{2c}e^{\lambda_c z})^T$ for any $z \in \mathbb{R}$, and $m_1(\lambda_c, c) > 0$. Further, we obtain that

$$\begin{aligned} I(z) \leq & \lim_{t\to+\infty} k_2(t,z) \\ \leq & \lim_{t\to+\infty} \frac{e^{-\mu_1 t}}{\sqrt{4\pi D_I t}} \int_{\mathbb{R}} e^{-\frac{(z-ct-y)^2}{4D_I t}} I(y)dy + \int_0^{+\infty} \frac{e^{-\mu_1 t}}{\sqrt{4\pi D_I t}} \int_{\mathbb{R}} e^{-\frac{y^2}{4D_I t}} P(z-y-ct)dydt \\ \leq & \lim_{t\to+\infty} \frac{e^{-\mu_1 t}}{\sqrt{4\pi D_I t}} \int_{\mathbb{R}} e^{-\frac{(z-ct-y)^2}{4D_I t}} w_{1c}e^{\lambda_c y}dy \\ & + \int_0^{+\infty} \frac{e^{-\mu_1 t}}{\sqrt{4\pi D_I t}} \int_{\mathbb{R}} e^{-\frac{y^2}{4D_I t}} P(z-y-ct)dydt \\ = & \lim_{t\to+\infty} \frac{e^{-\mu_1 t}}{\sqrt{4\pi D_I t}} \int_{\mathbb{R}} e^{-\frac{[y-(z-ct+2D_I\lambda_c t)]^2}{4D_I t}} e^{D_I\lambda_c^2 t + \lambda_c(z-ct)} w_{1c}dy \\ & + \int_0^{+\infty} \frac{e^{-\mu_1 t}}{\sqrt{4\pi D_I t}} \int_{\mathbb{R}} e^{-\frac{y^2}{4D_I t}} P(z-y-ct)dydt \\ = & \lim_{t\to+\infty} e^{-m_1(\lambda_c,c)t} e^{\lambda_c z} w_{1c} + \int_0^{+\infty} \frac{e^{-\mu_1 t}}{\sqrt{4\pi D_I t}} \int_{\mathbb{R}} e^{-\frac{y^2}{4D_I t}} P(z-y-ct)dydt, \ \forall z \in \mathbb{R}. \end{aligned}$$

Under the assumption (A2), then we have

$$\begin{aligned}
&\int_0^{+\infty}\frac{e^{-\mu_1 t}}{\sqrt{4\pi D_I t}}\int_{\mathbb{R}} e^{-\frac{y^2}{4D_I t}}P(z-y-ct)dydt\\
&=\int_0^{+\infty}\frac{e^{-\mu_1 t}}{\sqrt{4\pi D_I t}}\int_{\mathbb{R}} e^{-\frac{y^2}{4D_I t}}\int_{\mathbb{R}}\Gamma(D_I\tau_1,y)A(z-r-c\tau_1-y-ct)drdydt\\
&=\frac{1}{\sqrt{D_I}}\int_{\mathbb{R}}\Gamma(D_I\tau_1,y)\int_0^{+\infty}\frac{e^{-\mu_1 t}}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{y^2}{4D_I t}}A(z-r-c\tau_1-y-ct)dydtdr\\
&\le\frac{1}{\sqrt{D_I}}\int_{\mathbb{R}}\Gamma(D_I\tau_1,y)\int_0^{+\infty}\frac{e^{-\mu_1 t}}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{y^2}{4D_T t}}A(z-r-c\tau_1-y-ct)dydtdr\\
&=\frac{\sqrt{D_T}}{\sqrt{D_I}}\int_{\mathbb{R}}\Gamma(D_I\tau_1,y)\int_0^{+\infty}\frac{e^{-\mu_1 t}}{\sqrt{4D_T\pi t}}\int_{\mathbb{R}} e^{-\frac{y^2}{4D_T t}}A(z-r-c\tau_1-y-ct)dydtdr\\
&\le\frac{\sqrt{D_T}}{\sqrt{D_I}}e^{-\mu_1\tau_1}T^*,\ \forall z\in\mathbb{R}.
\end{aligned}$$

And hence, $I(z)\le\frac{\sqrt{D_T}}{\sqrt{D_I}}e^{-\mu_1\tau_1}T^*$ for any $z\in\mathbb{R}$. Similarly, we can show that V is bounded on $\mathbb{R}$.

(ii). By the arguments similar to those for [50, Proposition 3.5], we have

$$\liminf_{z\to+\infty} I(z)>0,\ \liminf_{z\to+\infty} V(z)>0.$$

We further claim that there exists a constant $\delta>0$ such that $T(z)\ge\delta$ for all $z\in\mathbb{R}$. Assume on contrary that the claim is not true. Then the strict positivity of $T(z)$ and $\lim_{z\to-\infty}T(z)=T^*>0$ imply that there exist a positive sequence $z_j\to+\infty$ as $j\to+\infty$ and a constant M such that

(a) $T(z_j)\to 0$ as $j\to+\infty$;

(b) For any $z_j>M$, it holds that $T(z)\ge T(z_j)$ for any $z\in[M,z_j)$.

Property (b) and the assumption (A1) imply that for any $z_j>M$,

$$n(T(z_j))\ge n(T(z)),\ z\in[M,z_j).$$

Since $T(z_j)\to 0$ as $j\to+\infty$, $n(T)>0$ for all $T<T^*$, and I and V are bounded, it follows that there exists a sufficiently large constant $m>0$ such that $z_m>M$ and

$$n(T(z_m))-\beta_1 T(z_m)V(z_m)-\beta_2 T(z_m)I(z_m)\ge\frac{1}{2}n(T(z_m))>0.$$

It is easy to see that $T'(z_m)\le 0$ by above (b). Further, by the first equation of (4.3) and above inequality, it follows that $T''(z_m)<0$. So, $T(z)$ is decreasing in $z\in[z_m,z_m+\alpha]$ for some $\alpha>0$. By repeating the above argument, we can conclude that $T''(z) < 0$ for all $z \geq z_m$. Thus, $T(z) \to -\infty$ as $z \to +\infty$, which contradicts the positivity of $T(z)$.

(iii). Based on the definition of P, we can express I-equation of (4.3) as follows

$$D_I I''(z) - cI'(z) - \mu_1 I(z) + e^{-\mu_1\tau_1} P(z) = 0.$$

Since both $P(z)$ and $I(z)$ are bounded, $I(z)$ can be uniquely expressed as

$$I(z) = \varpi \left[\int_{-\infty}^{z} e^{\mu^-(z-s)} P(s) ds + \int_{z}^{+\infty} e^{\mu^+(z-s)} P(s) ds \right],$$

where

$$\mu^{\pm} = \frac{c \pm \sqrt{c^2 + 4D_I\mu_1}}{2D_I}, \ \varpi = \frac{e^{-\mu_1\tau_1}}{\sqrt{c^2 + 4D_I\mu_1}}.$$

Similarly, we have

$$V(z) = \omega^* \left[\int_{-\infty}^{z} e^{\hat{\mu}^-(z-s)} Q(s) ds + \int_{z}^{+\infty} e^{\hat{\mu}^+(z-s)} Q(s) ds \right],$$

where

$$\hat{\mu}^{\pm} = \frac{c \pm \sqrt{c^2 + 4D_V\mu}}{2D_V}, \ \omega^* = \frac{be^{-\mu_2\tau_2}}{\sqrt{c^2 + 4D_V\mu}}.$$

Let $\mu^* = \max\{-\mu^-, \mu^+, -\hat{\mu}^-, \hat{\mu}^+\}$. For any $z \in \mathbb{R}$, it is easy to see that $I(z+y)e^{\mu^* y}$ and $V(z+y)e^{\mu^* y}$ are increasing in $y \in \mathbb{R}$, while $I(z+y)e^{-\mu^* y}$ and $V(z+y)e^{-\mu^* y}$ are decreasing in $y \in \mathbb{R}$. It follows that

$$\begin{aligned}
&-D_I I''(z) + cI'(z) + \mu_1 I(z) \\
&= e^{-\mu_1\tau_1} \int_{-\infty}^{-c\tau_1} \Gamma(D_I\tau_1, y) e^{\mu^*(y+c\tau_1)} e^{-\mu^*(y+c\tau_1)} \left[\beta_1 TV + \beta_2 TI\right](z - c\tau_1 - y) dy \\
&\quad + e^{-\mu_1\tau_1} \int_{-c\tau_1}^{+\infty} \Gamma(D_I\tau_1, y) e^{\mu^*(y+c\tau_1)} e^{-\mu^*(y+c\tau_1)} \left[\beta_1 TV + \beta_2 TI\right](z - c\tau_1 - y) dy \\
&\geq e^{-\mu_1\tau_1} \left[\beta_1 V(z) + \beta_2 I(z)\right] \left[\int_{-\infty}^{-c\tau_1} \Gamma(D_I\tau_1, y) e^{\mu^*(y+c\tau_1)} T(z - c\tau_1 - y) dy \right. \\
&\quad \left. + \int_{-c\tau_1}^{+\infty} \Gamma(D_I\tau_1, y) e^{-\mu^*(y+c\tau_1)} T(z - c\tau_1 - y) dy \right], \forall z \in \mathbb{R}.
\end{aligned}$$

By [29, Theorem 3.9 and Lemma 3.10], we can find that there exist $\hat{k} > 0$ and $k > 0$ such that

$$\begin{aligned}
&\max\left\{ \sup_{s\in(z-1,z+1)} I(s), \sup_{s\in(z-1,z+1)} V(s) \right\} \\
&\leq \hat{k} \max\left\{ \|I\|_{L^p(z-2,z+2)}, \|V\|_{L^p(z-2,z+2)} \right\} \\
&\leq k \min\left\{ \inf_{s\in(z-2,z+2)} I(s), \inf_{s\in(z-2,z+2)} V(s) \right\} \\
&\leq k \min\left\{ \inf_{s\in(z-1,z+1)} I(s), \inf_{s\in(z-1,z+1)} V(s) \right\},
\end{aligned}$$

with some $p > 1$. Thus, the statement is now established.

(iv). By the expression of $I(z)$ in (iii), we can further obtain that

$$I'(z) = \varpi \left[\mu^- \int_{-\infty}^{z} e^{\mu^-(z-s)} P(s)ds + \mu^+ \int_{z}^{+\infty} e^{\mu^+(z-s)} P(s)ds \right].$$

And hence, $|I'(z)/I(z)| \leq |\mu^-| + |\mu^+| = -\mu^- + \mu^+$ for any $z \in \mathbb{R}$. Similarly, we can prove that $|T'(z)/T(z)|$ and $|V'(z)/V(z)|$ are bounded on $\mathbb{R}$. Moreover, it then follows from the boundedness of T, I and V (see Proposition 4.3.3 and (i)) that $|T'(z)|$, $|I'(z)|$ and $|V'(z)|$ are bounded on $\mathbb{R}$. □

Lemma 4.3.9. *For any $c > 0$, if there exists a nonnegative traveling wave solution (T, I, V) satisfying (4.3), then there exists a constant $\alpha > 0$ such that*

$$\sup_{z\in\mathbb{R}}\{I(z)e^{-\alpha z}\} < +\infty, \ \sup_{z\in\mathbb{R}}\{V(z)e^{-\alpha z}\} < +\infty, \ \sup_{z\in\mathbb{R}}\{|I'(z)|e^{-\alpha z}\} < +\infty,$$
$$\sup_{z\in\mathbb{R}}\{|V'(z)|e^{-\alpha z}\} < +\infty, \ \sup_{z\in\mathbb{R}}\{|I''(z)|e^{-\alpha z}\} < +\infty, \ \sup_{z\in\mathbb{R}}\{|V''(z)|e^{-\alpha z}\} < +\infty.$$

Proof. Our arguments are inspired by [149, Lemma 3.7]. For any given $c > 0$, assume that $(T(z), I(z), V(z))$ is a nonnegative traveling wave solution satisfying (4.3). It follows from the fact $T(-\infty) = T^*$ that there exists $\hbar > 0$ sufficiently large such that

$$\int_{-\hbar}^{+\infty} \Gamma(D_I \tau_1, y)dy > 1 - \varrho, \ T(z) > T^*(1 - \varrho), \ \forall z \in (-\infty, -2\hbar),$$

where $\varrho \in (0,1)$ is a small constant to be determined later. For $z < -2\hbar$, we get

$$\begin{aligned}
cI' &= D_I I'' - \mu_1 I + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1 T(z - c\tau_1 - y)V(z - c\tau_1 - y)dy \\
&\quad + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_2 T(z - c\tau_1 - y)I(z - c\tau_1 - y)dy \\
&\geq D_I I'' - \mu_1 I + e^{-\mu_1\tau_1}\int_{-\hbar}^{+\infty} \Gamma(D_I\tau_1, y)\beta_1 T(z - c\tau_1 + \hbar)V(z - c\tau_1 - y)dy \\
&\quad + e^{-\mu_1\tau_1}\int_{-\hbar}^{+\infty} \Gamma(D_I\tau_1, y)\beta_2 T(z - c\tau_1 + \hbar)I(z - c\tau_1 - y)dy \\
&\quad + e^{-\mu_1\tau_1}\int_{-\infty}^{-\hbar} \Gamma(D_I\tau_1, y)\beta_1 T(z - c\tau_1 - y)V(z - c\tau_1 - y)dy \\
&\quad + e^{-\mu_1\tau_1}\int_{-\infty}^{-\hbar} \Gamma(D_I\tau_1, y)\beta_2 T(z - c\tau_1 - y)I(z - c\tau_1 - y)dy \\
&> D_I I'' - \mu_1 I + e^{-\mu_1\tau_1}\beta_1 T^*(1-\varrho)\int_{-\hbar}^{+\infty} \Gamma(D_I\tau_1, y)V(z - c\tau_1 - y)dy \\
&\quad + e^{-\mu_1\tau_1}\beta_2 T^*(1-\varrho)\int_{-\hbar}^{+\infty} \Gamma(D_I\tau_1, y)I(z - c\tau_1 - y)dy \\
&\geq D_I I'' + e^{-\mu_1\tau_1}\beta_1 T^*(1-\varrho)\int_{-\hbar}^{+\infty} \Gamma(D_I\tau_1, y)[V(z - c\tau_1 - y) - V(z)]dy \\
&\quad + e^{-\mu_1\tau_1}\beta_2 T^*(1-\varrho)\int_{-\hbar}^{+\infty} \Gamma(D_I\tau_1, y)[I(z - c\tau_1 - y) - I(z)]dy \\
&\quad + e^{-\mu_1\tau_1}\beta_1 T^*(1-\varrho)^2 V + [e^{-\mu_1\tau_1}\beta_2 T^*(1-\varrho)^2 - \mu_1]I.
\end{aligned} \tag{4.10}$$

Denote $\widetilde{V} = \int_{-\infty}^{z} V(s)ds$, $\widetilde{I} = \int_{-\infty}^{z} I(s)ds$ for any $z \in \mathbb{R}$. Integrating both sides of inequality (4.10) from $-\infty$ to z with $z < -2\hbar$, we obtain

$$\begin{aligned}
&e^{-\mu_1\tau_1}\beta_1 T^*(1-\varrho)^2\widetilde{V} + [e^{-\mu_1\tau_1}\beta_2 T^*(1-\varrho)^2 - \mu_1]\widetilde{I} \\
&< -D_I I_z + cI - e^{-\mu_1\tau_1}\beta_1 T^*(1-\varrho)\int_{-\hbar}^{+\infty} \Gamma(D_I\tau_1, y)[\widetilde{V}(z - c\tau_1 - y) - \widetilde{V}(z)]dy \\
&\quad - e^{-\mu_1\tau_1}\beta_2 T^*(1-\varrho)\int_{-\hbar}^{+\infty} \Gamma(D_I\tau_1, y)[\widetilde{I}(z - c\tau_1 - y) - \widetilde{I}(z)]dy.
\end{aligned} \tag{4.11}$$

Note that

$$\begin{aligned}
&\int_{-\infty}^{z}\int_{-\hbar}^{+\infty}\Gamma(D_I\tau_1,y)[\widetilde{V}(s-c\tau_1-y)-\widetilde{V}(s)]dyds\\
&=\lim_{m\to-\infty}\int_{m}^{z}\int_{-\hbar}^{+\infty}\Gamma(D_I\tau_1,y)[\widetilde{V}(s-c\tau_1-y)-\widetilde{V}(s)]dyds\\
&=\lim_{m\to-\infty}-\int_{m}^{z}\int_{-\hbar}^{+\infty}(c\tau_1+y)\Gamma(D_I\tau_1,y)\int_0^1 V(s-\theta(c\tau_1+y))d\theta dyds\\
&=-\int_{-\hbar}^{+\infty}(c\tau_1+y)\Gamma(D_I\tau_1,y)\int_0^1 \widetilde{V}(z-\theta(c\tau_1+y))d\theta dy.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_{-\infty}^{z}\int_{-\hbar}^{+\infty}\Gamma(D_I\tau_1,y)[\widetilde{I}(s-c\tau_1-y)-\widetilde{I}(s)]dyds\\
&=-\int_{-\hbar}^{+\infty}(c\tau_1+y)\Gamma(D_I\tau_1,y)\int_0^1 \widetilde{I}(z-\theta(c\tau_1+y))d\theta dy.
\end{aligned}$$

Integrating both sides of (4.4) from $-\infty$ to z, we further have

$$\begin{aligned}
&e^{-\mu_1\tau_1}\beta_1T^*(1-\varrho)^2\int_{-\infty}^{z}\widetilde{V}(s)ds+[e^{-\mu_1\tau_1}\beta_2T^*(1-\varrho)^2-\mu_1]\int_{-\infty}^{z}\widetilde{I}(s)ds+D_II\\
&<c\widetilde{I}+e^{-\mu_1\tau_1}\beta_1T^*\int_{-\hbar}^{+\infty}(c\tau_1+y)\Gamma(D_I\tau_1,y)\int_0^1\widetilde{V}(z-\theta(c\tau_1+y))d\theta dy\\
&\quad+e^{-\mu_1\tau_1}\beta_2T^*\int_{-\hbar}^{+\infty}(c\tau_1+y)\Gamma(D_I\tau_1,y)\int_0^1\widetilde{I}(z-\theta(c\tau_1+y))d\theta dy.
\end{aligned}$$

Since $(c\tau_1+y)\widetilde{V}(z-\theta(c\tau_1+y))$ and $(c\tau_1+y)\widetilde{I}(z-\theta(c\tau_1+y))$ are non-increasing on $\theta\in[0,1]$, there holds

$$\begin{aligned}
&e^{-\mu_1\tau_1}\beta_1T^*(1-\varrho)^2\int_{-\infty}^{z}\widetilde{V}(s)ds+[e^{-\mu_1\tau_1}\beta_2T^*(1-\varrho)^2-\mu_1]\int_{-\infty}^{z}\widetilde{I}(s)ds+D_II\\
&<c\widetilde{I}+(c\tau_1+G_1)e^{-\mu_1\tau_1}\beta_1T^*\widetilde{V}+(c\tau_1+G_1)e^{-\mu_1\tau_1}\beta_2T^*\widetilde{I},
\end{aligned}\tag{4.12}$$

where $G_1=\int_0^{+\infty}\Gamma(D_I\tau_1,y)ydy>\int_{-\hbar}^{+\infty}\Gamma(D_I\tau_1,y)ydy$. By the argument similar to that for I, for $z<-2\hbar$, we have

$$\begin{aligned}
&be^{-\mu_2\tau_2}(1-\varrho)^2\int_{-\infty}^{z}\widetilde{I}(s)ds-\mu\int_{-\infty}^{z}\widetilde{V}(s)ds+D_VV\\
&<c\widetilde{V}+(c\tau_2+G_2)be^{-\mu_2\tau_2}\widetilde{I},
\end{aligned}\tag{4.13}$$

where $G_2=\int_0^{+\infty}\Gamma(D_V\tau_2,y)ydy>\int_{-\hbar}^{+\infty}\Gamma(D_V\tau_2,y)ydy$.

Next we show that there exist constants $r_1, r_2, r_3, r_4 \geq 0$, such that

$$r_1 \int_{-\infty}^{z} \widetilde{I}(s)ds + r_2 \int_{-\infty}^{z} \widetilde{V}(s)ds \leq r_3 \widetilde{I}(z) + r_4 \widetilde{T}(z), \ \forall z < -2\hbar. \tag{4.14}$$

Since

$$\mathscr{R}_0 = \frac{e^{-\mu_1\tau_1}\beta_2 T^*}{\mu_1} + \frac{e^{-\mu_1\tau_1}\beta_1 T^* e^{-\mu_2\tau_2} b}{\mu_1 \mu} > 1,$$

we proceed with four cases.

(i) $e^{-\mu_1\tau_1}\beta_2 T^* - \mu_1 > 0$. Let $\varrho \in (0,1)$ be sufficiently small such that $e^{-\mu_1\tau_1}\beta_2 T^*(1-\varrho)^2 - \mu_1 > 0$. Consider (4.12), we can take the values

$$r_1 = e^{-\mu_1\tau_1}\beta_2 T^*(1-\varrho)^2 - \mu_1, \ r_2 = e^{-\mu_1\tau_1}\beta_1 T^*(1-\varrho)^2,$$
$$r_3 = c + (c\tau_1 + G_1)e^{-\mu_1\tau_1}\beta_2 T^*, \ r_4 = (c\tau_1 + G_1)e^{-\mu_1\tau_1}\beta_1 T^*.$$

(ii) $e^{-\mu_1\tau_1}\beta_2 T^* - \mu_1 \leq 0$ and $e^{-\mu_1\tau_1}\beta_2 T^*\mu + e^{-\mu_1\tau_1}\beta_1 T^* e^{-\mu_2\tau_2} b - \mu_1\mu > 0$. Let $\varrho \in (0,1)$ be small enough such that

$$e^{-\mu_1\tau_1}\beta_2 T^*\mu(1-\varrho)^2 + e^{-\mu_1\tau_1}\beta_1 T^*(1-\varrho)^2 e^{-\mu_2\tau_2} b(1-\varrho)^2 - \mu_1\mu > 0.$$

By adding two sides of inequalities (4.12) $\times\ \mu$ and (4.13) $\times\ e^{-\mu_1\tau_1}\beta_1 T^*(1-\varrho)^2$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{z} \widetilde{I}(s)ds \\
&< \mu c\widetilde{I} + \mu(c\tau_1 + G_1)e^{-\mu_1\tau_1}\beta_1 T^*\widetilde{V} + \mu(c\tau_1 + G_1)e^{-\mu_1\tau_1}\beta_2 T^*\widetilde{I} \\
&\quad + e^{-\mu_1\tau_1}\beta_1 T^*(1-\varrho)^2 c\widetilde{V} + e^{-\mu_1\tau_1}\beta_1 T^*(1-\varrho)^2(c\tau_2 + G_2)be^{-\mu_2\tau_2}\widetilde{I},
\end{aligned}$$

for any $z < -2\hbar$. And hence, we can take

$$r_1 = \mu e^{-\mu_1\tau_1}\beta_2 T^*(1-\varrho)^2 + e^{-\mu_1\tau_1}\beta_1 T^*(1-\varrho)^2 be^{-\mu_2\tau_2}(1-\varrho)^2 - \mu_1\mu, \quad r_2 = 0,$$
$$r_3 = \mu c + \mu(c\tau_1 + G_1)e^{-\mu_1\tau_1}\beta_2 T^* + e^{-\mu_1\tau_1}\beta_1 T^*(1-\varrho)^2(c\tau_2 + G_2)be^{-\mu_2\tau_2},$$
$$r_4 = \mu(c\tau_1 + G_1)e^{-\mu_1\tau_1}\beta_1 T^* + e^{-\mu_1\tau_1}\beta_1 T^*(1-\varrho)^2 c.$$

(iii) $e^{-\mu_1\tau_1}\beta_1 T^* e^{-\mu_2\tau_2} b - \mu_1\mu > 0$. We can take $\varrho \in (0,1)$ satisfying

$$e^{-\mu_1\tau_1}\beta_1 T^* e^{-\mu_2\tau_2} b(1-\varrho)^4 - \mu_1\mu > 0.$$

We add two sides of inequalities (4.12) $\times\ e^{-\mu_2\tau_2} b(1-\varrho)^2$ and (4.13) $\times\ \mu_1$ to obtain

$$\begin{aligned}
&\int_{-\infty}^{z} \widetilde{I}(s)ds + [e^{-\mu_2\tau_2} be^{-\mu_1\tau_1}\beta_1 T^*(1-\varrho)^4 - \mu_1\mu] \int_{-\infty}^{z} \widetilde{V}(s)ds \\
&< e^{-\mu_2\tau_2} b(1-\varrho)^2 c\widetilde{I} + e^{-\mu_2\tau_2} b(1-\varrho)^2(c\tau_1 + G_1)e^{-\mu_1\tau_1}\beta_1 T^*\widetilde{V} \\
&\quad + e^{-\mu_2\tau_2} b(1-\varrho)^2(c\tau_1 + G_1)e^{-\mu_1\tau_1}\beta_2 T^*\widetilde{I} + \mu_1 c\widetilde{V} + \mu_1(c\tau_2 + G_2)be^{-\mu_2\tau_2}\widetilde{I},
\end{aligned}$$

for any $z < -2\hbar$. And hence, we can take

$$r_1 = e^{-\mu_2\tau_2} b e^{-\mu_1\tau_1}\beta_2 T^*(1-\varrho)^4, \qquad r_2 = e^{-\mu_2\tau_2} b e^{-\mu_1\tau_1}\beta_1 T^*(1-\varrho)^4 - \mu_1\mu,$$
$$r_3 = e^{-\mu_2\tau_2} b(1-\varrho)^2 c + e^{-\mu_2\tau_2} b(1-\varrho)^2(c\tau_1+G_1)e^{-\mu_1\tau_1}\beta_2 T^* + \mu_1(c\tau_2+G_2)be^{-\mu_2\tau_2},$$
$$r_4 = e^{-\mu_2\tau_2} b(1-\varrho)^2(c\tau_1+G_1)e^{-\mu_1\tau_1}\beta_1 T^* + \mu_1 c.$$

(iv) $e^{-\mu_1\tau_1}\beta_1 T^* e^{-\mu_2\tau_2} b - \mu_1\mu \le 0$ and $e^{-\mu_1\tau_1}\beta_2 T^*\mu + e^{-\mu_1\tau_1}\beta_1 T^* e^{-\mu_2\tau_2} b - \mu_1\mu > 0$. This case can be treated by the argument similar to that in (ii).

Now, let $\widetilde{W}(z) = \widetilde{I}(z) + \widetilde{V}(z)$. It then follows from (4.14) that there exist constants $r_5, r_6 > 0$ such that

$$r_5 \int_{-\infty}^{z} \widetilde{W}(s)ds \le r_6 \widetilde{W}(z), \ \forall z < -2\hbar.$$

Furthermore, we have

$$r_5 \int_{-\infty}^{0} \widetilde{W}(z+s)ds \le r_6 \widetilde{W}(z), \ \forall z < -2\hbar.$$

Since $\widetilde{W}(\cdot)$ is increasing, we have $r_5 s\widetilde{W}(z-s) \le r_6\widetilde{W}(z)$ for any $z < -2\hbar$ and any $s > 0$. Thus, there exist $s_0 > 0$ large enough and $\delta \in (0,1)$ such that

$$\widetilde{W}(z-s_0) \le \delta \widetilde{W}(z), \ \forall z < -2\hbar.$$

Denote $w(z) = \widetilde{W}(z)e^{-\frac{1}{s_0}\ln\frac{1}{\delta}z}$. Clearly, $\frac{1}{s_0}\ln\frac{1}{\delta} > 0$. It then follows that

$$w(z-s_0) = \widetilde{W}(z-s_0)e^{-\frac{1}{s_0}\ln\frac{1}{\delta}(z-s_0)} \le \delta\widetilde{W}(z)e^{-\frac{1}{s_0}\ln\frac{1}{\delta}(z-s_0)} = w(z), \ \forall z < -2\hbar.$$

Since $w(z) \to 0$ as $z \to +\infty$, there exists a constant $\hat{w}$ such that $w(z) \le \hat{w}$ for $z \in \mathbb{R}$. It follows that $\widetilde{W}(z) \le \hat{w}e^{\frac{1}{s_0}\ln\frac{1}{\delta}z}$ for any $z \in \mathbb{R}$. Accordingly, there exists $\gamma > 0$ such that

$$\int_{-\infty}^{z} \widetilde{I}(s)ds \le \gamma e^{\frac{1}{s_0}\ln\frac{1}{\delta}z}, \quad \int_{-\infty}^{z} \widetilde{V}(s)ds \le \gamma e^{\frac{1}{s_0}\ln\frac{1}{\delta}z}, \quad \forall z < 0.$$

In view of (4.12) and (4.13), there exists $q > 0$ such that

$$I(z) \le q e^{\frac{1}{s_0}\ln\frac{1}{\delta}z}, \quad V(z) \le q e^{\frac{1}{s_0}\ln\frac{1}{\delta}z}, \quad \forall z \in \mathbb{R}.$$

From (4.10) and (4.11), we then get

$$\sup_{z\in\mathbb{R}}\{I(z)e^{-\alpha z}\} < +\infty, \ \sup_{z\in\mathbb{R}}\{|I'(z)|e^{-\alpha z}\} < +\infty, \ \sup_{z\in\mathbb{R}}\{|I''(z)|e^{-\alpha z}\} < +\infty,$$

and similarly,

$$\sup_{z\in\mathbb{R}}\{V(z)e^{-\alpha z}\} < +\infty, \ \sup_{z\in\mathbb{R}}\{|V'(z)|e^{-\alpha z}\} < +\infty, \ \sup_{z\in\mathbb{R}}\{|V''(z)|e^{-\alpha z}\} < +\infty.$$

This completes the proof. □

Lemma 4.3.10. *For any $c > 0$, if system (4.3) has a bounded semi-traveling wave solution (T, I, V), then $I(z)$ and $V(z)$ decay exponentially as $z \to -\infty$. Furthermore, if $\lim_{z\to+\infty} I(z) = 0$ and $\lim_{z\to+\infty} V(z) = 0$, then $I(z)$ and $V(z)$ decay exponentially as $z \to +\infty$.*

Proof. The statement of the case where $z \to -\infty$ is directly followed by Lemma 4.3.9. By arguments similar to those in the proof of Theorem 4.3.2 later, we can conclude that $\lim_{z\to+\infty} I(z) = 0$ and $\lim_{z\to+\infty} V(z) = 0$ imply $\lim_{z\to+\infty} T(z) = T^*$. Thus, $\lim_{z\to+\infty}(T(z), I(z), V(z)) = (T^*, 0, 0)$. Furthermore, we can again use similar arguments to those in the proof of Lemma 4.3.9 to conclude the statements for the case of $z \to +\infty$. □

Theorem 4.3.1. *For any $c > c^*$, system (4.3) admits a traveling wave solution connecting the disease-free equilibrium e_0 and the endemic equilibrium u^*, i.e., it satisfies the boundary conditions (4.4).*

Proof. For any $c > c^*$, it is sufficient to show that $(T(+\infty), I(+\infty), V(+\infty)) = u^*$, where (T, I, V) is the solution obtained in Proposition 4.3.3.

Let $v_1 = T, v_2 = T', v_3 = I, v_4 = I', v_5 = V, v_6 = V'$. Then system (4.3) gives rise to

$$\begin{cases} v_1' = v_2, \\ D_T v_2' = cv_2 - n(v_1) + \beta_1 v_1 v_5 + \beta_2 v_1 v_3, \\ v_3' = v_4, \\ D_I v_4' = cv_4 + \mu_1 v_3 - e^{-\mu_1\tau_1} \int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1 v_1(z - c\tau_1 - y)v_5(z - c\tau_1 - y)dy \\ \qquad - e^{-\mu_1\tau_1} \int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_2 v_1(z - c\tau_1 - y)v_3(z - c\tau_1 - y)dy, \\ v_5' = v_6, \\ D_V v_6' = cv_6 - be^{-\mu_2\tau_2} \int_{\mathbb{R}} \Gamma(D_V\tau_2, y)v_3(z - c\tau_2 - y)dy + \mu v_5, \end{cases} \tag{4.15}$$

where $'$ denotes the derivative with respect to z. Let

$$f(x) = x - 1 - \ln x, \ x > 0,$$

and

$$g_1(x) = \int_x^{+\infty} \Gamma(D_I\tau_1, y)dy, \ g_2(x) = \int_{-\infty}^x \Gamma(D_I\tau_1, y)dy,$$
$$g_3(x) = \int_x^{+\infty} \Gamma(D_V\tau_2, y)dy, \ g_4(x) = \int_{-\infty}^x \Gamma(D_V\tau_2, y)dy.$$

It is easy to see that the functions $f(x)$, $g_1(x)$, $g_2(x)$, $g_3(x)$ and $g_4(x)$ have the following properties

$$f(x) = x - 1 - \ln x \geq 0, x > 0;\ f(x) = 0 \Leftrightarrow x = 1;$$
$$g_1(0) = g_2(0) = g_3(0) = g_4(0) = \frac{1}{2},\ g_2(-x) = g_1(x),\ g_4(-x) = g_3(x),$$
$$g_1'(x) = -\Gamma(D_I\tau_1, x),\ g_2'(x) = \Gamma(D_I\tau_1, x),\ g_3'(x) = -\Gamma(D_V\tau_2, x),\ g_4'(x) = \Gamma(D_V\tau_2, x).$$

Consider above system (4.15) and let $(v_1^*, v_3^*, v_5^*) = (T_*, I_*, V_*)$. Now we present our refinement of conventional Lyapunov functional $L(v(z))$ with

$$L = L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8, \tag{4.16}$$

where

$$\begin{aligned}
L_1 &= c\int_{v_1^*}^{v_1}\left[1 - \frac{v_1^*}{u}\right]du + e^{\mu_1\tau_1}cv_3^*f\left(\frac{v_3}{v_3^*}\right) + \frac{\beta_1 v_1^* v_5^*}{be^{-\mu_2\tau_2}v_3^*}cv_5^*f\left(\frac{v_5}{v_5^*}\right),\\
L_2 &= D_T\left[\frac{v_1^*}{v_1} - 1\right]v_2 + e^{\mu_1\tau_1}D_I\left[\frac{v_3^*}{v_3} - 1\right]v_4 + \frac{\beta_1 v_1^* v_5^*}{be^{-\mu_2\tau_2}v_3^*}D_V\left[\frac{v_5^*}{v_5} - 1\right]v_6,\\
L_3 &= \beta_1 v_1^* v_5^*\int_{z-c\tau_1}^{z} f\left(\frac{v_1(s)v_5(s)}{v_1^* v_5^*}\right)ds,\\
L_4 &= \beta_2 v_1^* v_3^*\int_{z-c\tau_1}^{z} f\left(\frac{v_1(s)v_3(s)}{v_1^* v_3^*}\right)ds,\\
L_5 &= \beta_1 v_1^* v_5^*\int_{z-c\tau_2}^{z} f\left(\frac{v_3(s)}{v_3^*}\right)ds,\\
L_6 &= \beta_1 v_1^* v_5^*\int_0^{+\infty} g_1(y)f\left(\frac{v_1(z-c\tau_1-y)v_5(z-c\tau_1-y)}{v_1^* v_5^*}\right)dy\\
&\quad - \beta_1 v_1^* v_5^*\int_{-\infty}^{0} g_2(y)f\left(\frac{v_1(z-c\tau_1-y)v_5(z-c\tau_1-y)}{v_1^* v_5^*}\right)dy,\\
L_7 &= \beta_2 v_1^* v_3^*\int_0^{+\infty} g_1(y)f\left(\frac{v_1(z-c\tau_1-y)v_3(z-c\tau_1-y)}{v_1^* v_3^*}\right)dy\\
&\quad - \beta_2 v_1^* v_3^*\int_{-\infty}^{0} g_2(y)f\left(\frac{v_1(z-c\tau_1-y)v_3(z-c\tau_1-y)}{v_1^* v_3^*}\right)dy,\\
L_8 &= \beta_1 v_1^* v_5^*\int_0^{+\infty} g_3(y)f\left(\frac{v_3(z-c\tau_2-y)}{v_3^*}\right)dy\\
&\quad - \beta_1 v_1^* v_5^*\int_{-\infty}^{0} g_4(y)f\left(\frac{v_3(z-c\tau_2-y)}{v_3^*}\right)dy.
\end{aligned}$$

Clearly,

$$\frac{\partial L_1}{\partial v_1} = c\left[1 - \frac{v_1^*}{v_1}\right],\ \frac{\partial L_1}{\partial v_3} = e^{\mu_1\tau_1}c\left[1 - \frac{v_3^*}{v_3}\right],\ \frac{\partial L_1}{\partial v_5} = \frac{\beta_1 v_1^* v_5^*}{be^{-\mu_2\tau_2}v_3^*}c\left[1 - \frac{v_5^*}{v_5}\right],$$

which imply that the point (v_1^*, v_3^*, v_5^*) is a stationary point of the function $L_1(v_1, v_3, v_5)$ and it is the unique stationary point and the global minimum of this function. So, L_1 is bounded from below, and hence, L_1 is bounded from below along the solution. In view of Lemma 4.3.8, we see that L_2 is bounded. It is easy to see that L_3, L_4, L_5, are bounded from below. Finally, combining the properties of f, g_1, g_2, g_3, g_4, the persistence property of v_1, v_3 and v_5 with $z \to +\infty$, the boundedness of v_1, and the exponential decay of v_3 and v_5 with $z \to \pm\infty$, we see that L_6, L_7, L_8 are also bounded from below. And hence, $L(v(z))$ is well defined for all $z \in \mathbb{R}$. Now we calculate the derivatives of $L_i (i = 1, 2, ..., 8)$ along the solution of system (4.15) as follows

$$\begin{aligned}
&\frac{d(L_1+L_2)}{dz}\\
&= c\left[1-\frac{v_1^*}{v_1}\right]\frac{1}{c}\left[D_T v_2' + n(v_1) - \beta_1 v_1 v_5 - \beta_2 v_1 v_3\right]\\
&\quad + e^{\mu_1\tau_1} c\left[1-\frac{v_3^*}{v_3}\right]\frac{1}{c}\left[D_I v_4' - \mu_1 v_3 + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1 v_1(z-c\tau_1-y)v_5(z-c\tau_1-y)dy\right.\\
&\quad \left. + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_2 v_1(z-c\tau_1-y)v_3(z-c\tau_1-y)dy\right]\\
&\quad + \frac{\beta_1 v_1^* v_5^*}{be^{-\mu_2\tau_2} v_3^*} c\left[1-\frac{v_5^*}{v_5}\right]\frac{1}{c}\left[D_V v_6' - \mu v_5 + be^{-\mu_2\tau_2}\int_{\mathbb{R}} \Gamma(D_V\tau_2, y)v_3(z-c\tau_2-y)dy\right]\\
&\quad + D_T\left[\frac{v_1^*}{v_1}-1\right]v_1'' - D_T v_1^*\left[\frac{v_1'}{v_1}\right]^2 + e^{\mu_1\tau_1} D_I\left[\frac{v_3^*}{v_3}-1\right]v_3'' - e^{\mu_1\tau_1} D_I v_3^*\left[\frac{v_3'}{v_3}\right]^2\\
&\quad + \frac{\beta_1 v_1^* v_5^*}{be^{-\mu_2\tau_2} v_3^*} D_V\left[\frac{v_5^*}{v_5}-1\right]v_5'' - \frac{\beta_1 v_1^* v_5^*}{be^{-\mu_2\tau_2} v_3^*} D_V v_5^*\left[\frac{v_5'}{v_5}\right]^2\\
&= \left[1-\frac{v_1^*}{v_1}\right]\left[n(v_1) - \beta_1 v_1 v_5 - \beta_2 v_1 v_3\right]\\
&\quad + e^{\mu_1\tau_1}\left[1-\frac{v_3^*}{v_3}\right]\left[-\mu_1 v_3 + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1 v_1(z-c\tau_1-y)v_5(z-c\tau_1-y)dy\right.\\
&\quad \left. + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_2 v_1(z-c\tau_1-y)v_3(z-c\tau_1-y)dy\right]\\
&\quad + \frac{\beta_1 v_1^* v_5^*}{be^{-\mu_2\tau_2} v_3^*}\left[1-\frac{v_5^*}{v_5}\right]\left[-\mu v_5 + be^{-\mu_2\tau_2}\int_{\mathbb{R}} \Gamma(D_V\tau_2, y)v_3(z-c\tau_2-y)dy\right]\\
&\quad - D_T v_1^*\left[\frac{v_1'}{v_1}\right]^2 - e^{\mu_1\tau_1} D_I v_3^*\left[\frac{v_3'}{v_3}\right]^2 - \frac{\beta_1 v_1^* v_5^*}{be^{-\mu_2\tau_2} v_3^*} D_V v_5^*\left[\frac{v_5'}{v_5}\right]^2.
\end{aligned}$$

Note that

$$n(v_1^*) = \beta_1 v_1^* v_5^* + \beta_2 v_1^* v_3^*, \quad e^{-\mu_1\tau_1}(\beta_1 v_1^* v_5^* + \beta_2 v_1^* v_3^*) = \mu_1 v_3^*, \quad be^{-\mu_2\tau_2} v_3^* = \mu v_5^*.$$

It then follows that

$$\begin{aligned}
&\frac{d(L_1+L_2)}{dz} \\
&= \left[1-\frac{v_1^*}{v_1}\right][n(v_1)-n(v_1^*)]+\beta_1 v_1^* v_5^* \left[1-\frac{v_1^*}{v_1}\right]\left[1-\frac{v_1 v_5}{v_1^* v_5^*}\right] \\
&\quad +\beta_2 v_1^* v_3^* \left[1-\frac{v_1^*}{v_1}\right]\left[1-\frac{v_1 v_3}{v_1^* v_3^*}\right] \\
&\quad -\beta_1 v_1^* v_5^* \frac{v_3}{v_3^*}-\beta_2 v_1^* v_3+\beta_1 v_1^* v_5^*+\beta_2 v_1^* v_3^* \\
&\quad +\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1 v_1(z-c\tau_1-y)v_5(z-c\tau_1-y)dy \\
&\quad -\beta_1 v_1^* v_5^* \int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\frac{v_1(z-c\tau_1-y)v_5(z-c\tau_1-y)v_3^*}{v_1^* v_5^* v_3(z)}dy \\
&\quad +\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_2 v_1(z-c\tau_1-y)v_3(z-c\tau_1-y)dy \\
&\quad -\beta_2 v_1^* v_3^* \int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\frac{v_1(z-c\tau_1-y)v_3(z-c\tau_1-y)v_3^*}{v_1^* v_3^* v_3(z)}dy \\
&\quad +\beta_1 v_1^* v_5^*-\beta_1 v_1^* v_5+\frac{\beta_1 v_1^* v_5^*}{v_3^*}\int_{\mathbb{R}} \Gamma(D_V\tau_2, y)v_3(z-c\tau_2-y)dy \\
&\quad -\beta_1 v_1^* v_5^* \int_{\mathbb{R}} \Gamma(D_V\tau_2, y)\frac{v_3(z-c\tau_2-y)v_5^*}{v_3^* v_5}dy \\
&\quad -D_T v_1^* \left[\frac{v_1'}{v_1}\right]^2-e^{\mu_1\tau_1}D_I v_3^* \left[\frac{v_3'}{v_3}\right]^2-\frac{\beta_1 v_1^* v_5^*}{be^{-\mu_2\tau_2}v_3^*}D_V v_5^* \left[\frac{v_5'}{v_5}\right]^2.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\frac{dL_3}{dz} &= \beta_1 v_1^* v_5^* f\left(\frac{v_1(z)v_5(z)}{v_1^* v_5^*}\right)-\beta_1 v_1^* v_5^* f\left(\frac{v_1(z-c\tau_1)v_5(z-c\tau_1)}{v_1^* v_5^*}\right), \\
\frac{dL_4}{dz} &= \beta_2 v_1^* v_3^* f\left(\frac{v_1(z)v_3(z)}{v_1^* v_3^*}\right)-\beta_2 v_1^* v_3^* f\left(\frac{v_1(z-c\tau_1)v_3(z-c\tau_1)}{v_1^* v_3^*}\right), \\
\frac{dL_5}{dz} &= \beta_1 v_1^* v_5^* f\left(\frac{v_3(z)}{v_3^*}\right)-\beta_1 v_1^* v_5^* f\left(\frac{v_3(z-c\tau_2)}{v_3^*}\right).
\end{aligned}$$

Since

$$\lim_{y\to+\infty} g_1(y) f\left(\frac{v_1(z-c\tau_1-y)v_5(z-c\tau_1-y)}{v_1^*v_5^*}\right)=0,$$
$$\lim_{y\to-\infty} g_2(y) f\left(\frac{v_1(z-c\tau_1-y)v_5(z-c\tau_1-y)}{v_1^*v_5^*}\right)=0,$$
$$\lim_{y\to+\infty} g_1(y) f\left(\frac{v_1(z-c\tau_1-y)v_3(z-c\tau_1-y)}{v_1^*v_3^*}\right)=0,$$
$$\lim_{y\to-\infty} g_2(y) f\left(\frac{v_1(z-c\tau_1-y)v_3(z-c\tau_1-y)}{v_1^*v_3^*}\right)=0,$$
$$\lim_{y\to+\infty} g_3(y) f\left(\frac{v_3(z-c\tau_2-y)}{v_3^*}\right)=0,$$
$$\lim_{y\to-\infty} g_4(y) f\left(\frac{v_3(z-c\tau_2-y)}{v_3^*}\right)=0,$$

using the integration by parts and the properties of g_1, g_2, we obtain

$$\begin{aligned}
\frac{dL_6}{dz} &= \beta_1 v_1^* v_5^* \int_0^{+\infty} g_1(y)\frac{d}{dz}\left\{f\left(\frac{v_1(z-c\tau_1-y)v_5(z-c\tau_1-y)}{v_1^*v_5^*}\right)\right\}dy\\
&\quad - \beta_1 v_1^* v_5^* \int_{-\infty}^{0} g_2(y)\frac{d}{dz}\left\{f\left(\frac{v_1(z-c\tau_1-y)v_5(z-c\tau_1-y)}{v_1^*v_5^*}\right)\right\}dy\\
&= -\beta_1 v_1^* v_5^* \int_0^{+\infty} g_1(y)\frac{d}{dy}\left\{f\left(\frac{v_1(z-c\tau_1-y)v_5(z-c\tau_1-y)}{v_1^*v_5^*}\right)\right\}dy\\
&\quad + \beta_1 v_1^* v_5^* \int_{-\infty}^{0} g_2(y)\frac{d}{dy}\left\{f\left(\frac{v_1(z-c\tau_1-y)v_5(z-c\tau_1-y)}{v_1^*v_5^*}\right)\right\}dy\\
&= \beta_1 v_1^* v_5^* f\left(\frac{v_1(z-c\tau_1)v_5(z-c\tau_1)}{v_1^*v_5^*}\right)\\
&\quad - \beta_1 v_1^* v_5^* \int_{\mathbb{R}} \Gamma(D_I\tau_1, y) f\left(\frac{v_1(z-c\tau_1-y)v_5(z-c\tau_1-y)}{v_1^*v_5^*}\right)dy.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{dL_7}{dz} &= \beta_2 v_1^* v_3^* f\left(\frac{v_1(z-c\tau_1)v_3(z-c\tau_1)}{v_1^*v_3^*}\right)\\
&\quad - \beta_2 v_1^* v_3^* \int_{\mathbb{R}} \Gamma(D_I\tau_1, y) f\left(\frac{v_1(z-c\tau_1-y)v_3(z-c\tau_1-y)}{v_1^*v_3^*}\right)dy,\\
\frac{dL_8}{dz} &= \beta_1 v_1^* v_5^* f\left(\frac{v_3(z-c\tau_2)}{v_3^*}\right) - \beta_1 v_1^* v_5^* \int_{\mathbb{R}} \Gamma(D_V\tau_2, y) f\left(\frac{v_3(z-c\tau_2-y)}{v_3^*}\right)dy.
\end{aligned}$$

Thus, there holds

$$
\begin{aligned}
&\frac{d(L_1+L_2+L_3+L_4+L_5+L_6+L_7+L_8)}{dz} \\
&= -\beta_1 v_1^* v_5^* \left[f\left(\frac{v_1^*}{v_1}\right) + \int_{\mathbb{R}} \Gamma(D_I\tau_1, y) f\left(\frac{v_1(z-c\tau_1-y)v_5(z-c\tau_1-y)v_3^*}{v_1^* v_5^* v_3(z)}\right) dy \right. \\
&\quad \left. + \int_{\mathbb{R}} \Gamma(D_V\tau_2, y) f\left(\frac{v_3(z-c\tau_2-y)v_5^*}{v_3^* v_5(z)}\right) dy \right] \\
&\quad - \beta_2 v_1^* v_3^* \left[f\left(\frac{v_1^*}{v_1}\right) + \int_{\mathbb{R}} \Gamma(D_I\tau_1, y) f\left(\frac{v_1(z-c\tau_1-y)v_3(z-c\tau_1-y)v_3^*}{v_1^* v_3^* v_3(z)}\right) dy \right] \\
&\quad + \left[1 - \frac{v_1^*}{v_1}\right][n(v_1) - n(v_1^*)] - D_T v_1^* \left[\frac{v_1'}{v_1}\right]^2 \\
&\quad - e^{\mu_1\tau_1} D_I v_3^* \left[\frac{v_3'}{v_3}\right]^2 - \frac{\beta_1 v_1^* v_5^*}{be^{-\mu_2\tau_2} v_3^*} D_V v_5^* \left[\frac{v_5'}{v_5}\right]^2 .
\end{aligned}
$$

By the assumption (A1), we have

$$\left[1 - \frac{v_1^*}{v_1}\right][n(v_1) - n(v_1^*)] \le 0,$$

and the equality holds only in the case of $v_1 = v_1^*$. Therefore, $L(v(z))$ is decreasing in z. This, together with the boundedness from below of L, implies that there exists a constant χ such that

$$\lim_{z\to+\infty} L(v(z)) = \chi.$$

Since $v_2 = v_1', v_4 = v_3', v_6 = v_5'$, we can write $L(v_1, v_2, v_3, v_4, v_5, v_6)(z) := L(v_1, v_3, v_5)(z)$. Next pick an increasing sequence $\{z_i\}_{i\ge0}$ of positive real numbers such that $z_i \to +\infty$ as $i \to +\infty$ as well as the sequences of functions

$$v_{1i}(z) = v_1(z+z_i),\ \ v_{3i}(z) = v_3(z+z_i),\ \ v_{5i}(z) = v_5(z+z_i).$$

It follows that

$$L(v_{1i}, v_{3i}, v_{5i})(z) = L(v_1, v_3, v_5)(z+z_i).$$

Furthermore, we have

$$\lim_{i\to+\infty} L(v_{1i}, v_{3i}, v_{5i})(z) = \chi,\ \ \forall z \in \mathbb{R}.$$

Utilizing elliptic estimates, we can establish that, up to extracting a subsequence, for simplicity, $\{v_{1i}\}, \{v_{3i}\}, \{v_{5i}\}$ converge to some non-negative functions $v_1^\diamond, v_3^\diamond, v_5^\diamond$ in $C^1_{loc}(\mathbb{R})$. Clearly, $(v_1^\diamond, v_3^\diamond, v_5^\diamond)$ is a solution of (4.3). Since $L(v(z))$ is bounded from below and decreasing in z, for large i,

$$\underline{L} \le L(v_{1i}, v_{3i}, v_{5i})(z) = L(v_1, v_3, v_5)(z+z_i) \le L(v_1, v_3, v_5)(z)$$

also holds. It then follows that there exists a constant number χ such that

$$\lim_{i\to+\infty} L(v_{1i}, v_{3i}, v_{5i})(z) = \lim_{z+z_i\to+\infty} L(v_1, v_3, v_5)(z + z_i) = \chi, \ \forall z \in \mathbb{R}.$$

Next we claim that $\lim_{i\to+\infty} L(v_{1i}, v_{3i}, v_{5i})(z) = L(v_1^\diamond, v_3^\diamond, v_5^\diamond)(z) = \chi$ for any $z \in \mathbb{R}$. Indeed, taking $i \to +\infty$ in (4.16), we have

$$\lim_{i\to+\infty} (L_1 + L_2)(v_{1i}, v_{3i}, v_{5i})(z) = (L_1 + L_2)(v_1^\diamond, v_3^\diamond, v_5^\diamond)(z), \ \forall z \in \mathbb{R}.$$

Since L_3, L_4 and L_5 are similar, we only focus on demonstrating L_3. Letting $s_1 = s-z_i$, we further have

$$L_3(v_1, v_5)(z) = \beta_1 v_1^* v_5^* \int_{z-z_i-c\tau_1}^{z-z_i} f\left(\frac{v_1(s_1 + z_i)v_5(s_1 + z_i)}{v_1^* v_5^*}\right) ds_1.$$

It follows that

$$\lim_{i\to+\infty} L_3(v_{1i}, v_{5i})(z) = \beta_1 v_1^* v_5^* \int_{z-c\tau_1}^{z} f\left(\frac{v_1^\diamond(s_1)v_5^\diamond(s_1)}{v_1^* v_5^*}\right) ds_1 = L_3(v_1^\diamond, v_5^\diamond)(z), \ \forall z \in \mathbb{R},$$

and hence,

$$\lim_{i\to+\infty} (L_3 + L_4 + L_5)(v_{1i}, v_{3i}, v_{5i})(z) = (L_3 + L_4 + L_5)(v_1^\diamond, v_3^\diamond, v_5^\diamond)(z), \ \forall z \in \mathbb{R}.$$

Since

$$\begin{aligned} L_6(v_1, v_5)(z + z_i) = {} & \beta_1 v_1^* v_5^* \int_0^{+\infty} g_1(y) f\left(\frac{v_1(z + z_i - c\tau_1 - y)v_5(z + z_i - c\tau_1 - y)}{v_1^* v_5^*}\right) dy \\ & - \beta_1 v_1^* v_5^* \int_{-\infty}^{0} g_2(y) f\left(\frac{v_1(z + z_i - c\tau_1 - y)v_5(z + z_i - c\tau_1 - y)}{v_1^* v_5^*}\right) dy, \end{aligned}$$

taking the limit as $i \to +\infty$ and applying Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{i\to+\infty} L_6(v_{1i}, v_{5i})(z) = {} & \beta_1 v_1^* v_5^* \int_0^{+\infty} g_1(y) f\left(\frac{v_1^\diamond(z - c\tau_1 - y)v_5^\diamond(z - c\tau_1 - y)}{v_1^* v_5^*}\right) dy \\ & - \beta_1 v_1^* v_5^* \int_{-\infty}^{0} g_2(y) f\left(\frac{v_1^\diamond(z - c\tau_1 - y)v_5^\diamond(z - c\tau_1 - y)}{v_1^* v_5^*}\right) dy, \\ = {} & L_6(v_1^\diamond, v_5^\diamond)(z), \ \forall z \in \mathbb{R}. \end{aligned}$$

Similarly, we can deduce the properties of L_7 and L_8. Therefore,

$$\lim_{i\to+\infty} L(v_{1i}, v_{3i}, v_{5i})(z) = L(v_1^\diamond, v_3^\diamond, v_5^\diamond)(z) = \chi, \ \forall z \in \mathbb{R}.$$

Note that $L(v(z))$ is a constant if and only if condition $\frac{dL(v(z))}{dz} \equiv 0$ holds, which is also equivalent to condition $v_1 \equiv v_1^*, v_2 \equiv 0, v_3 = v_3^*, v_4 \equiv 0, v_5 \equiv v_5^*, v_6 \equiv 0$, for all $z \in \mathbb{R}$. And hence, $v_1^\diamond \equiv v_1^*, v_3^\diamond \equiv v_3^*, v_5^\diamond \equiv v_5^*$. In view of the arbitrariness of the sequence $\{z_i\}$, we obtain $\lim_{z\to+\infty}(v_1(z), v_3(z), v_5(z)) = (v_1^*, v_3^*, v_5^*)$. □

Theorem 4.3.2. *For $c = c^*$, system (4.3) admits a traveling wave solution connecting the disease-free equilibrium e_0 and the endemic equilibrium u^*, i.e., it satisfies the boundary conditions (4.4).*

Proof. Our arguments are motivated by [98, Theorem 6] and [149, Theorem 2.14]. Let $\{c_i\}$ be a sequence satisfying $c^* < c_{i+1} < c_i < c^* + 1$ and $\lim_{i\to+\infty} c_i \to c^*$. From the proofs of Lemma 4.3.8 and Theorem 4.3.1 for the case $c > c^*$, then for any c_i, system (4.3) has a positive solution (T_i, I_i, V_i) satisfying $(T_i(-\infty), I_i(-\infty), V_i(-\infty)) = e_0$ and $(T_i(+\infty), I_i(+\infty), V_i(+\infty)) = u^*$; in addition,

$$T_i(z),\ I_i(z),\ V_i(z),\ |T_i'(z)|,\ |I_i'(z)|,\ |V_i'(z)|,\ |T_i'(z)/T_i(z)|,\ |I_i'(z)/I_i(z)|,\ |V_i'(z)/V_i(z)|$$

remain bounded on $\mathbb{R}$. Also, there holds

$$\max\left\{\max_{s\in[z-1,z+1]} I_i(s),\ \max_{s\in[z-1,z+1]} V_i(s)\right\} \le k\min\left\{\min_{s\in[z-1,z+1]} I_i(s),\ \min_{s\in[z-1,z+1]} V_i(s)\right\}, \tag{4.17}$$

for a constant $k > 0$ and for any $z \in \mathbb{R}$.

Note that $(T_i(\cdot + m), I_i(\cdot + m), V_i(\cdot + m))$ is a solution of (4.3) and satisfies the boundary conditions (4.4) for any $m \in \mathbb{R}$. Then we assume that

$$T_i(0) = \frac{T^* + T_*}{2}.$$

By using interior elliptic estimates as mentioned in Lemma 4.3.7, Arzelà-Ascoli theorem and a diagonalization argument, we have that there exists a subsequence of $\{(T_i, I_i, V_i)\}_{i\in\mathbb{N}}$, again denoted as $\{(T_i, I_i, V_i)\}_{i\in\mathbb{N}} \in C^2(\mathbb{R}, \mathbb{R}^3)$, satisfying $(T_i, I_i, V_i) \to (T, I, V)$ as $i \to \infty$ in $C^2_{loc}(\mathbb{R}, \mathbb{R}^3)$. Clearly, (T, I, V) satisfies (4.3) and

$$T(0) = \frac{T^* + T_*}{2}. \tag{4.18}$$

It follows from (4.18) that $(T, I, V) \not\equiv (T^*, 0, 0)$. Similar to the argument as that in the proof of Proposition 4.3.3, we can obtain the positiveness of (T, I, V) and the boundedness of T. Since the conclusions in Lemma 4.3.8 and Lemma 4.3.10 are independent of c, they are also true here. It then follows that $(T(+\infty), I(+\infty), V(+\infty)) = u^*$ holds by similar argument as that in Theorem 4.3.1. It remains to prove $(T(-\infty), I(-\infty), V(-\infty)) = (T^*, 0, 0)$.

By the proof of Theorem 4.3.1, we have $L'(v(z)) \le 0$ for any $z \in \mathbb{R}$, indicating that $L(v(z))$ is nonincreasing with respect to z for any $z \in \mathbb{R}$. This implies either

$$\lim_{z\to-\infty} L(v(z)) = \hat{L} < +\infty \tag{4.19}$$

or

$$\lim_{z\to-\infty} L(v(z)) = +\infty. \tag{4.20}$$

If (4.19) holds, then we can show that $(T(-\infty), I(-\infty), V(-\infty)) = u^*$ by the argument similar to that in Theorem 4.3.1. It follows that $L(v(z)) \to L(v^*)$ as $z \to -\infty$. Based on the proof of Theorem 4.3.1, we also observe that $L(v(z)) \to L(v(+\infty)) = L(v^*)$ as $z \to +\infty$. Due to the monotonicity, we conclude that $L(v(z)) \equiv L(v^*)$ for any $z \in \mathbb{R}$, and thus $L'(v(z)) \equiv 0$ for all $z \in \mathbb{R}$. It then follows from the argument similar to that in Theorem 4.3.1 that $T(z) \equiv T_*$, $I(z) \equiv I_*$, $V(z) \equiv V_*$ for each $z \in \mathbb{R}$, which contradicts with (4.18). Therefore, (4.20) holds true.

We first prove that
$$\liminf_{z\to-\infty} I(z) = 0.$$
We assume, by contradiction, that $\liminf_{z\to-\infty} I(z) > 0$. This, together with
$$\liminf_{z\to+\infty} I(z) = I_*$$
and $I(z) > 0$ for all $z \in \mathbb{R}$, implies that there exists a number $\delta > 0$ such that $I(z) > \delta$ for all $z \in \mathbb{R}$. By the definition of Q in Lemma 4.3.8, it follows from the boundedness of I that Q is also bounded. Thus, the V-equation of (4.3) yields that
$$D_V V'' - cV' + e^{-\mu_2\tau_2} bQ(z) - \mu V = 0.$$
Since both $Q(z)$ and $V(z)$ are bounded, $V(z)$ can be uniquely expressed as
$$V(z) = \hat{\varpi}\left[\int_{-\infty}^{z} e^{\hat{\mu}^-(z-s)} bQ(s)ds + \int_{z}^{+\infty} e^{\hat{\mu}^+(z-s)} bQ(s)ds\right],$$
where
$$\hat{\mu}^{\pm} = \frac{c \pm \sqrt{c^2 + 4D_V\mu}}{2D_V},\ \hat{\varpi} = \frac{e^{-\mu_2\tau_2}}{\sqrt{c^2 + 4D_V\mu}}.$$
And hence, there exists a number $\alpha > 0$ such that $V(z) > \alpha$ for any $z \in \mathbb{R}$. Accordingly, we have $\limsup_{z\to-\infty} L(v(z)) < +\infty$, which contradicts (4.20). Thus, $\liminf_{z\to-\infty} I(z) = 0$.

Next we show that $\lim_{z\to-\infty} I(z) = 0$. If $\limsup_{z\to-\infty} I(z) = \hat{\delta} > 0$ for some $\hat{\delta} > 0$, then there exists a sequence $\{z_i\}$ such that $z_i \to -\infty$ as $i \to +\infty$ and

$\lim_{i\to+\infty} I(z_i) = \hat{\delta}$. Let $N \in \mathbb{N}$ with $c^*\hat{\tau} \in [N, N+1)$. It follows from (4.17) that

$$\begin{aligned}
&\min\left\{\min_{z\in[z_i-n_0-1,z_i-n_0+1]} I(z), \min_{z\in[z_i-n_0-1,z_i-n_0+1]} V(z)\right\} \\
&\geq \frac{1}{k}\max\left\{\max_{z\in[z_i-n_0-1,z_i-n_0+1]} I(z), \max_{z\in[z_i-n_0-1,z_i-n_0+1]} V(z)\right\} \\
&\geq \frac{1}{k}\min\left\{\min_{z\in[z_i-n_0,z_i-n_0+2]} I(z), \min_{z\in[z_i-n_0,z_i-n_0+2]} V(z)\right\} \\
&\geq \frac{1}{k^2}\max\left\{\max_{z\in[z_i-n_0,z_i-n_0+2]} I(z), \max_{z\in[z_i-n_0,z_i-n_0+2]} V(z)\right\} \\
&\geq \cdots \\
&\geq \frac{1}{k^{n_0+1}}\max\left\{\max_{z\in[z_i-1,z_i+1]} I(z), \max_{z\in[z_i-1,z_i+1]} V(z)\right\} \\
&\geq \frac{\hat{\delta}}{2k^{n_0+1}},
\end{aligned}$$

for any $n_0 \in \{1, 2, \cdots, N\}$ and $i > K$, where $K \in \mathbb{N}$ with $I(z_i) > \frac{\hat{\delta}}{2}$ for $i > K$. It follows that

$$\min\left\{\min_{s\in[0,c^*\hat{\tau}]} I(z_i - s), \min_{s\in[0,c^*\hat{\tau}]} V(z_i - s)\right\} \geq \frac{\hat{\delta}}{2k^{N+1}}, \ \forall i > K.$$

Consequently, we have

$$\limsup_{i\to\infty} L(v(z_i)) < +\infty,$$

which contradicts (4.20). Therefore, $\lim_{z\to-\infty} I(z) = 0$. By a similar argument, we have $\lim_{z\to-\infty} V(z) = 0$.

Finally, we prove that $\lim_{z\to-\infty} T(z) = T^*$.

In the first step, we prove that $\lim_{z\to-\infty} T(z)$ exists. We assume, by contradiction, that $\lim_{z\to-\infty} T(z)$ does not exist. By the boundedness of $T(z)$ as in Proposition 4.3.3, it follows that

$$\liminf_{z\to-\infty} T(z) < \limsup_{z\to-\infty} T(z) \leq T^*.$$

Let $\{z_j\}$ be a decreasing sequence such that $z_j \to -\infty$ as $j \to +\infty$ and

$$\lim_{j\to+\infty} T(z_j) = \liminf_{z\to-\infty} T(z) < T^*, \quad \frac{dT(z_j)}{dz} = 0, \quad \frac{d^2T(z_j)}{dz^2} \geq 0. \tag{4.21}$$

Note that

$$\lim_{j\to+\infty} I(z_j) = 0, \ \lim_{j\to+\infty} V(z_j) = 0.$$

Thus, the T-equation in (4.3) with $c = c^*$ and assumption (A1) imply that

$$\lim_{j\to+\infty} T(z_j) \geq T^*,$$

which contradicts (4.21).

In the second step, we prove that $\lim_{z\to-\infty} T(z) = T^*$. We suppose that

$$\lim_{z\to-\infty} T(z) = T^{**}.$$

Write the T-equation of (4.3) as

$$D_T T'' - cT' + n(T) - \beta_1 TV - \beta_2 TI = 0. \tag{4.22}$$

By Lemma 4.3.8, we see that $|T'(z)|$ is bounded on $\mathbb{R}$. This, together with the boundedness of $n(T), T, I$ and V, implies that $|T''(z)|$ is bounded on $\mathbb{R}$, and hence, $T'(z)$ is uniformly continuous on $\mathbb{R}$. Since $\lim_{z\to-\infty} T(z)$ exists, it follows from Barbalat's lemma (see, e.g., [38]) that $\lim_{z\to-\infty} T'(z) = 0$. Further, taking the derivative of z on both sides of (4.22) and using the boundedness of $n'(T)$ for $T \le T^*$ along with the aforementioned bounds, we conclude that $|T'''(z)|$ is bounded on $\mathbb{R}$. It then follows that $\lim_{z\to-\infty} T''(z) = 0$. Note that $\lim_{z\to-\infty} I(z) = 0$ and $\lim_{z\to-\infty} V(z) = 0$. Then the boundedness of $T(z)$ on $\mathbb{R}$ implies that $\lim_{z\to-\infty} \beta_1 T(z)V(z) = 0$ and $\lim_{z\to-\infty} \beta_2 T(z)I(z) = 0$. Therefore, $\lim_{z\to-\infty} n(T(z)) = 0$. In view of the assumption (A1), it is easy to see that

$$T^{**} = \lim_{z\to-\infty} T(z) = T^*.$$

Consequently, $(T(-\infty), I(-\infty), V(-\infty)) = (T^*, 0, 0)$. □

By the definition of the matrix A_1 in Lemma 4.3.1, we have $|A_1 - c\lambda| = p_1(\lambda)p_2(\lambda) - a_2a_3$, where $p_1(\lambda) = a_1 - c\lambda$, $p_2(\lambda) = a_4 - c\lambda$. Then we introduce the following lemma to establish the non-existence of the traveling wave solutions for $0 < c < c^*$.

Lemma 4.3.11. *For any $0 < c < c^*$, $|A_1 - c\lambda| = 0$ does not admit positive real root λ such that $p_1(\lambda) < 0$ and $p_2(\lambda) < 0$.*

Proof. Our argument is motivated by [98, Lemma 17]. Assume, by contradiction, there exists a $\kappa > 0$ such that $|A_1 - c\lambda|_{\lambda=\kappa} = p_1(\kappa)p_2(\kappa) - a_2(\kappa)a_3(\kappa) = 0$, $p_1(\kappa) < 0$ and $p_2(\kappa) < 0$, where

$$a_2(\kappa) = e^{-\mu_1\tau_1} \int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1 T^* e^{\kappa(-c\tau_1 - y)} dy,$$

$$a_3(\kappa) = be^{-\mu_2\tau_2} \int_{\mathbb{R}} \Gamma(D_V\tau_2, y) e^{\kappa(-c\tau_2 - y)} dy.$$

Since $c < c^*$, we have

$$c\kappa < \frac{a_1(\kappa) + a_4(\kappa) + \sqrt{(a_1(\kappa) - a_4(\kappa))^2 + 4a_2(\kappa)a_3(\kappa)}}{2},$$

where

$$a_1(\kappa) = D_I\kappa^2 - \mu_1 + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_2 T^* e^{\kappa(-c\tau_1-y)}dy, \quad a_4(\kappa) = D_V\kappa^2 - \mu.$$

This gives rise to

$$0 > p_1(\kappa) = a_1(\kappa) - c\kappa > \frac{a_1(\kappa) - a_4(\kappa) - \sqrt{(a_1(\kappa) - a_4(\kappa))^2 + 4a_2(\kappa)a_3(\kappa)}}{2},$$
$$0 > p_2(\kappa) = a_4(\kappa) - c\kappa > \frac{a_4(\kappa) - a_1(\kappa) - \sqrt{(a_1(\kappa) - a_4(\kappa))^2 + 4a_2(\kappa)a_3(\kappa)}}{2}.$$

It follows that

$$\begin{aligned} p_1(\kappa)p_2(\kappa) < & \frac{a_1(\kappa) - a_4(\kappa) - \sqrt{(a_1(\kappa) - a_4(\kappa))^2 + 4a_2(\kappa)a_3(\kappa)}}{2} \\ & \times \frac{a_4(\kappa) - a_1(\kappa) - \sqrt{(a_1(\kappa) - a_4(\kappa))^2 + 4a_2(\kappa)a_3(\kappa)}}{2} \\ = & \, a_2(\kappa)a_3(\kappa), \end{aligned}$$

which contradicts $|A_1 - c\lambda|_{\lambda=\kappa} = 0$. ☐

Theorem 4.3.3. *If $0 < c < c^*$, then there do not exist nonnegative traveling wave solutions satisfying system (4.3) and the boundary conditions (4.4).*

Proof. Suppose, by contradiction, that there exists a nonnegative traveling wave solution (T, I, V) satisfying the conditions in Theorem 4.3.1 for some $c \in (0, c^*)$. It follows from Lemma 4.3.9 that there exists a $\alpha > 0$ such that

$$\sup_{z\in\mathbb{R}}\{I(z)e^{-\alpha z}\} < +\infty, \ \sup_{z\in\mathbb{R}}\{V(z)e^{-\alpha z}\} < +\infty, \ \sup_{z\in\mathbb{R}}\{|I'(z)|e^{-\alpha z}\} < +\infty,$$
$$\sup_{z\in\mathbb{R}}\{|V'(z)|e^{-\alpha z}\} < +\infty, \ \sup_{z\in\mathbb{R}}\{|I''(z)|e^{-\alpha z}\} < +\infty, \ \sup_{z\in\mathbb{R}}\{|V''(z)|e^{-\alpha z}\} < +\infty.$$

Integrating the first equation of (4.3) from s_1 to s_2 for fixed s_2, we have

$$T'(s_1) = e^{\frac{-c}{D_T}(s_2-s_1)}T_z(s_2) + \frac{1}{D_T}\int_{s_1}^{s_2} e^{\frac{-c}{D_T}(z-s_1)}P_1(z)dz,$$

where $P_1(z) = n(T(z)) - \beta_1 T(z)V(z) - \beta_2 T(z)I(z)$. By the boundedness of T, I and V in Lemma 4.3.8, we can obtain that $\lim_{s\to-\infty} T'(s) = 0$. Integrating the first equation of (4.3) from $-\infty$ to z, we obtain

$$D_T T'(z) - c(T(z) - T^*) = g_1(z),$$

where $g_1(z) = \int_{-\infty}^{z} [-n(T(s)) + \beta_1 T(s)V(s) + \beta_2 T(s)I(s)]\, ds$. Let $\hat{T}(z) = T^* - T(z)$. It then follows that

$$-D_T\hat{T}'(z) + c\hat{T}(z) = g_1(z).$$

Further,

$$\hat{T}(z) = \hat{T}(0)e^{\frac{cz}{D_T}} + \frac{1}{D_T}\int_z^0 e^{\frac{c(z-s)}{D_T}} g_1(s)ds$$
$$< \hat{T}(0)e^{\frac{cz}{D_T}} + \frac{1}{D_T}\int_z^0 e^{\frac{c(z-s)}{D_T}} g_2(s)ds,$$

where $g_2(z) = \int_{-\infty}^{z} [\beta_1 T(s)V(s) + \beta_2 T(s)I(s)]\, ds$, due to the condition (A1). Since

$$\sup_{z\in\mathbb{R}}\{I(z)e^{-\alpha z}\} < +\infty, \ \sup_{z\in\mathbb{R}}\{V(z)e^{-\alpha z}\} < +\infty,$$

there holds $g_2(z) \le Ce^{\alpha z}$ for some $C > 0$ for all $z \in \mathbb{R}$, and $g_2(z) = o(e^{\frac{\alpha}{2}z})$ as $z \to -\infty$. Therefore, $\hat{T}(z) = o(e^{\alpha' z})$ as $z \to -\infty$, where $\alpha' = \min\{\alpha/4, c/(2D_T)\}$. And hence, $\sup_{z\in\mathbb{R}}\{(T^* - T(z))e^{-\alpha' z}\} < +\infty$.

To complete the proof, we define the two-sided Laplace transform by

$$J_\phi(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda z}\phi(z)dz, \ \lambda \ge 0, \ \phi(z) \ge 0.$$

Clearly, $J_\phi(\lambda)$ is defined in $[0, \lambda_\phi^+)$ such that $\lambda_\phi^+ < +\infty$ satisfying $\lim_{\lambda\to\lambda_\phi^+} J_\phi(\lambda) = +\infty$ or $\lambda_\phi^+ = +\infty$. So, $\lambda_I^+ \ge \alpha, \lambda_V^+ \ge \alpha, \lambda_{I(T^*-T)}^+ \ge \lambda_I^+ + \alpha'$ and $\lambda_{V(T^*-T)}^+ \ge \lambda_V^+ + \alpha'$.

Consider the differential system

$$D_I I'' - cI' - \mu_1 I + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1 T(z - c\tau_1 - y)V(z - c\tau_1 - y)dy$$
$$+ e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_2 T(z - c\tau_1 - y)I(z - c\tau_1 - y)dy = 0,$$
$$D_V V'' - cV' + be^{-\mu_2\tau_2}\int_{\mathbb{R}} \Gamma(D_V\tau_2, y)I(z - c\tau_2 - y)dy - \mu V = 0.$$

Taking the two-sided Laplace transform of above system, we have

$$\begin{aligned} p_1(\lambda)J_I &= -e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1 T^* e^{\lambda(-c\tau_1-y)}dy J_V + h, \\ p_2(\lambda)J_V &= -be^{-\mu_2\tau_2}\int_{\mathbb{R}} \Gamma(D_V\tau_2, y)e^{\lambda(-c\tau_2-y)}dy J_I, \end{aligned} \tag{4.23}$$

where $h = e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)e^{\lambda(-c\tau_1-y)}dy J_{(\beta_1 V+\beta_2 I)(T^*-T)}$, and p_1, p_2 are mentioned in Lemma 4.3.11.

Now we show that $\lambda_I^+ = \lambda_V^+ < +\infty$. By the first equation of (4.23), we see that

$$\triangle(\lambda) =: (D_I\lambda^2 - \mu_1 - c\lambda)J_I + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)e^{\lambda(-c\tau_1-y)}dy J_{\beta_1 TV+\beta_2 TI} = 0.$$

Suppose that $\lambda_I^+ = +\infty$. Since $J_I > 0$ and $J_{\beta_1 TV+\beta_2 TI} > 0$ for $\lambda \in [0, \lambda_I^+)$, then $\triangle(+\infty) = +\infty$, a contradiction. $\lambda_V^+ < +\infty$ can be obtained similarly. Suppose

$\lambda_I^+ < \lambda_V^+ < +\infty$. Then $\lim_{\lambda\to\lambda_I^+} J_I = +\infty$ and $\lim_{\lambda\to\lambda_I^+} J_V < +\infty$, which contradicts to the second equation of (4.23). And hence, $\lambda_I^+ \geq \lambda_V^+$. Similarly, we can obtain $\lambda_I^+ \leq \lambda_V^+$. Therefore, $\lambda_I^+ = \lambda_V^+ := \lambda^+ < +\infty$.

Suppose that $p_1(\lambda^+) \geq 0$. Then for $\lambda = \lambda^+$, we have

$$\begin{aligned} +\infty &= \left[p_1(\lambda)J_I + e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1 T^* e^{\lambda(-c\tau_1 - y)} dy J_V \right]_{\lambda=\lambda^+} \\ &> \left[e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y) e^{\lambda(-c\tau_1 - y)} dy J_{(\beta_1 V + \beta_2 I)(T^* - T)} \right]_{\lambda=\lambda^+}, \end{aligned}$$

which contradicts to the first equation of (4.23). Hence, $p_1(\lambda^+) < 0$. By the similar argument as above, we can show that $p_2(\lambda^+) < 0$.

Multiplying the first equation by the second one of (4.23), we get

$$\begin{aligned} &\left[p_1(\lambda)p_2(\lambda) - e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1 T^* e^{\lambda(-c\tau_1 - y)} dy \cdot b e^{-\mu_2\tau_2}\int_{\mathbb{R}} \Gamma(D_V\tau_2, y) e^{\lambda(-c\tau_2 - y)} dy \right] J_I J_V \\ &= -b e^{-\mu_2\tau_2}\int_{\mathbb{R}} \Gamma(D_V\tau_2, y) e^{\lambda(-c\tau_2 - y)} dy \cdot h J_I. \end{aligned}$$

Since $h(\lambda^+) < +\infty$, it follows that

$$\begin{aligned} &\left[p_1(\lambda)p_2(\lambda) - e^{-\mu_1\tau_1}\int_{\mathbb{R}} \Gamma(D_I\tau_1, y)\beta_1 T^* e^{\lambda(-c\tau_1 - y)} dy \cdot b e^{-\mu_2\tau_2}\int_{\mathbb{R}} \Gamma(D_V\tau_2, y) e^{\lambda(-c\tau_2 - y)} dy \right]_{\lambda=\lambda^+} \\ &= \lim_{\lambda\to\lambda^+} \frac{-b e^{-\mu_2\tau_2}\int_{\mathbb{R}} \Gamma(D_V\tau_2, y) e^{\lambda(-c\tau_2 - y)} dy \cdot h J_I}{J_I J_V} = 0. \end{aligned}$$

And hence, there exists a $\lambda^+ > 0$ such that $p_i(\lambda^+) < 0$ $(i = 1, 2)$ satisfying $|A_1 - c\lambda^+| = 0$, which contradicts Lemma 4.3.11. □

4.4 Numerical simulations

In this section, we focus on the numerical calculation of the minimal wave speed c^*, the long-time behaviour of the solutions of (4.3), and the dependence of the minimal wave speed c^* on the parameters of system (4.3).

We assume that $n(T) = A - d_T T$ for simplicity. Furthermore, we first collect these values and present them in Table 4.1 as the values of the parameters in the model (4.3). Based on the parameter values in Table 4.1 and the numerical calculation method in [67], we obtain $\mathscr{R}_0 = 3.5882$.

Table 4.1: Parameters and their corresponding values for system (4.3).

Parameters	Values	References

A	5×10^5 cells/(day mL)	[88]
d_T	0.01/day	[88]
μ_1	0.5/day	[90]
μ	3/day	[90]
μ_2	2.5/day	variable
b	1000 virions/(cell day)	[33]
β_1	1.2×10^{-10} mL/(virions cells day)	[98]
β_2	4.5×10^{-8} mL/(virions cells day)	[98]
D_T	0.09648 mm^2/day	[83]
D_I	0.05 mm^2/day	[98]
D_V	0.17 mm^2/day	[110]
τ_1	1 day	[87]
τ_2	0.5 day	[40]

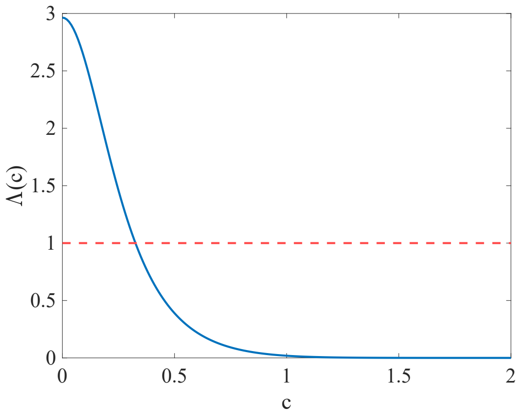
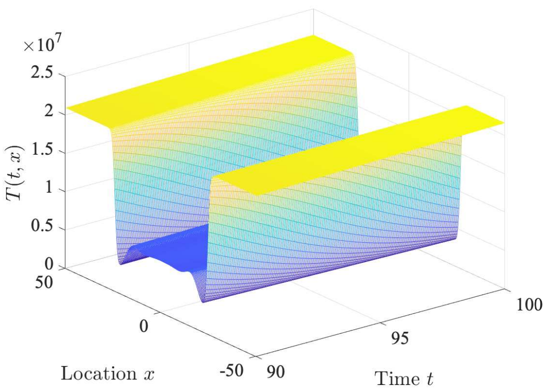


Figure 4.1: The determination of c^*

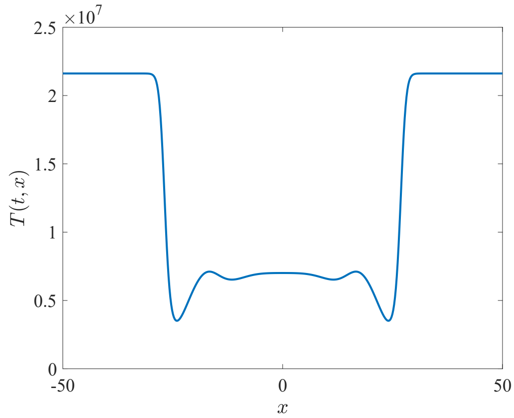
The minimal wave speed c^* is a significant parameter for system (4.3), so it is vital to provide its calculation method. We can numerically calculate the minimal wave speed c^* according to the definitions of $\lambda(c)$, $\rho(\lambda, c)$, and $\Lambda(c) = \min_{\lambda \in [0,\lambda(c))} \rho(\lambda, c), c \geq 0$. Based on the parameter values provided in Table 4.1, it is evident that $\Lambda(c)$ exhibits a continuous decrease until it reaches zero as c increases, as visually represented in Figure 4.1. In particular, the point of intersection between the curve and the horizontal line, with an ordinate coordinate of 1, serves to ascertain the minimal wave speed c^*. This implies that the abscissa value of this intersection point corresponds to the $c^* = 0.33$.

Since lymphoid tissue consists of lymph nodes, we set the environments periodic for convenience in the following simulations. Taking the initial data as

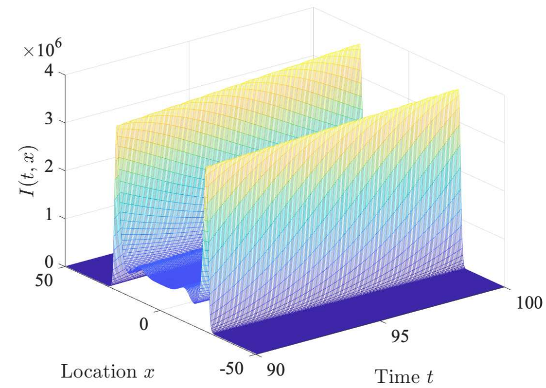
$$\begin{pmatrix} T(\theta, x) \\ I(\theta, x) \\ V(\theta, x) \end{pmatrix} = \begin{pmatrix} 5 \times 10^7 \times (1 + \cos(\frac{\pi}{20}x)) \\ 1 \times 10^2 \times (1 + \cos(\frac{\pi}{20}x)) \\ 200 \times (1 + \cos(\frac{\pi}{20}x)) \end{pmatrix}, \ \forall \theta \in [-\hat{\tau}, 0],$$



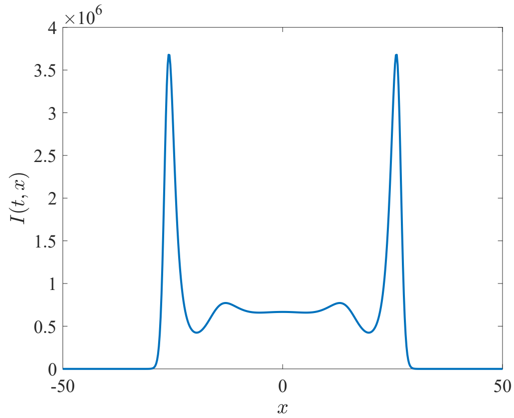
(a) The long-time behaviour of $T(t,x)$



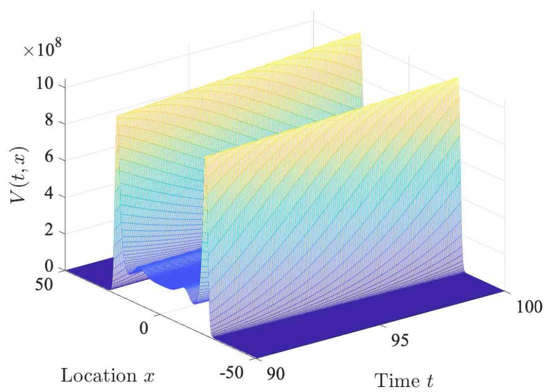
(b) The behaviour of $T(t,x)$ at $t = 100$



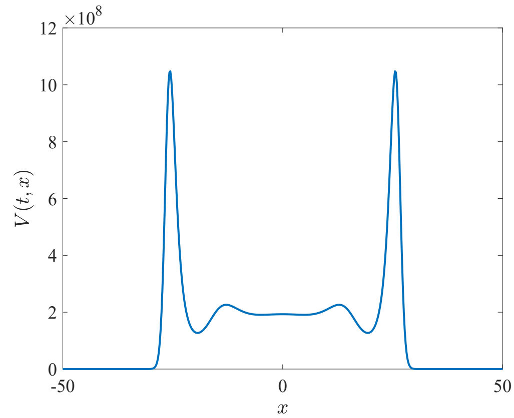
(c) The long-time behaviour of $I(t,x)$



(d) The behaviour of $I(t,x)$ at $t = 100$

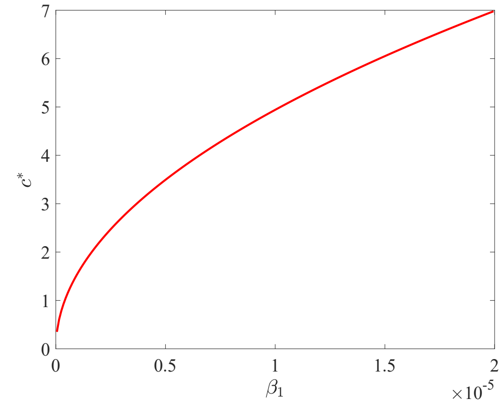


(e) The long-time behaviour of $V(t,x)$.

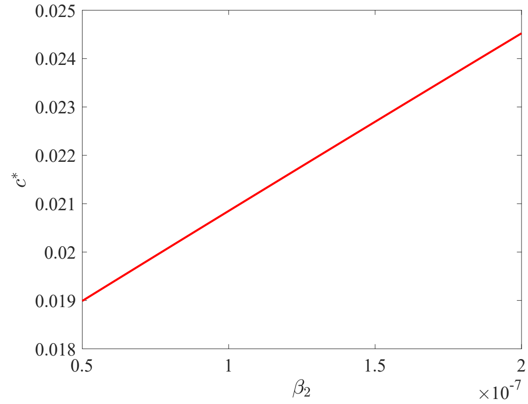


(f) The behaviour of $V(t,x)$ at $t = 100$

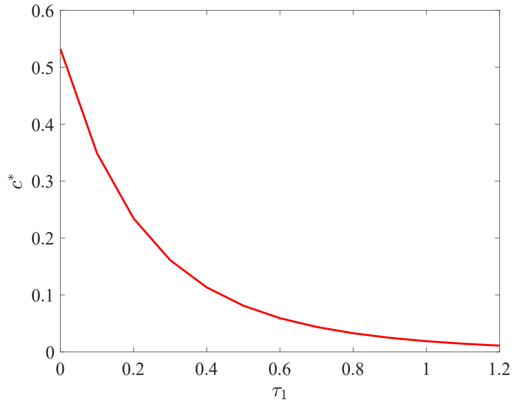
Figure 4.2: The long-time behaviour of the solution for (4.3)



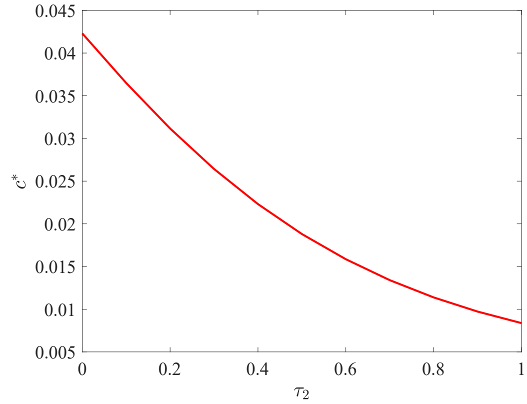
(a) The relationship between c^* and β_1



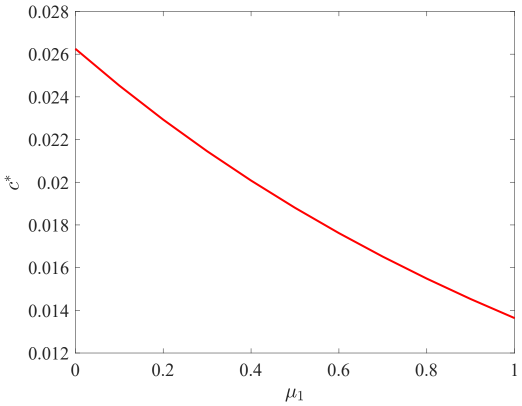
(b) The relationship between c^* and β_2



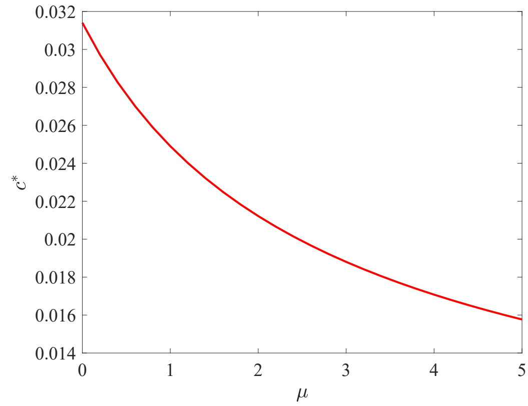
(c) The relationship between c^* and τ_1



(d) The relationship between c^* and τ_2



(e) The relationship between c^* and μ_1



(f) The relationship between c^* and μ

Figure 4.3: The continuous dependence of c^* on the parameters β_1, β_2, τ_1, τ_2, μ_1 and μ

with a compact support from -10 to 10 in the domain $[-50, 50]$, we obtain the spreading of species in system (4.3) by using the parameter values provided in Table 4.1, as depicted in Figure 4.2.

The influence of parameters on the minimal wave speed c^* is crucial for understanding the nature of the propagation properties of solutions in research. Utilizing the approach outlined in Figure 4.1 for numerical computation of c^*, while altering a single parameter and maintaining the remaining parameter values specified in Table 4.1, we observe continuous relationship between c^* and the parameters β_1, β_2, τ_1, τ_2, μ_1 and μ, as illustrated in Figure 4.3. This demonstrates that c^* is positively correlated with β_1 and β_2, whereas it is negatively correlated with τ_1, τ_2, μ_1 and μ, as illustrated in Figure 4.3. Upon examining the expression for $\mathscr{R}_0$:

$$\mathscr{R}_0 = \frac{e^{-\mu_1\tau_1}\beta_2 T^*}{\mu_1} + \frac{e^{-\mu_1\tau_1}\beta_1 T^* e^{-\mu_2\tau_2} b}{\mu_1\mu},$$

it becomes evident that $\mathscr{R}_0$ diminishes with decreasing values of β_1 and β_2, and conversely, increases with decreasing values of τ_1, τ_2, μ_1 and μ. Therefore, we can reduce β_1 and β_2, and increase τ_1, τ_2, μ_1 and μ through correlation strategies, which will result in a smaller value for c^* while ensuring $\mathscr{R}_0 > 1$. (That is to say, when $\mathscr{R}_0$ is very close to 1, c^* will become very small). Consequently, when the wave speed c exceeds a very small threshold c^*, a traveling wave solution will emerge, connecting the disease-free equilibrium and the positive equilibrium.

Chapter 5

A nonlocal periodic reaction-diffusion model of Chikungunya disease

This chapter is devoted to the study of a nonlocal reaction-diffusion model of Chikungunya disease with periodic time delays. We establish two threshold type results on the global dynamics for the growth of mosquitoes and the disease transmission, respectively. Further, we obtain the global attractivity of a positive steady state for a simplified nonlocal and time-delayed system with constant coefficients. We also conduct numerical simulations for the Chikungunya transmission in Ceará, Brazil to investigate the effects of spatial heterogeneity on the disease transmission.

5.1 Introduction

Mosquito-borne diseases are transmitted by the bite of infected mosquitoes, which include Zika virus, West Nile virus, Chikungunya virus, dengue, and malaria. Aedes aegypti and Aedes albopictus are the main vectors of Chikungunya virus [96, 130]. Affected by global warming, mass immune deficiency and the increase of floating population in epidemic areas, Chikungunya has a periodic outbreak trend in tropical and subtropical areas, and gradually develops into a global health problem [89, 104, 111]. In 2010, imported cases from Reunion Island, India and Indonesia were confirmed in the United States, France and Taiwan [99, 111]. In 2013, Chikungunya broke out on the island of St. Martin and then swept across the Americas. Since its transmission to Brazil in 2014 [57, 92], 45 countries have reported indigenous transmission cases [95]. It is noteworthy that about 1.3 billion people worldwide are at high risk of infection [91].

The life cycle of Aedes albopictus can be divided into four stages: egg, larva, pupa and adult. The first three stages are also called aquatic stage or juvenile stage. The climate change greatly affects the survival area of Aedes albopictus and exposes new areas to the risk of Chikungunya transmission. Among climatic factors, temperature change has significant impact on the growth process of Aedes albopictus, especially birth rate, maturity rate, mortality rate, bite rate and external incubation period (EIP) [85]. Another important climatic factor is rainfall. Unlike other mosquito vectors, Aedes aegypti usually lays eggs in areas without water, and the incubation of mosquito eggs requires exposure to water and immersion as a condition [115]. Humans are the main vector of Chikungunya virus during the outbreak of Chikungunya.

There have been various mathematical models on the transmission of mosquito-borne diseases. However, only a few models focus on the transmission of Chikungunya between humans and mosquitoes, and they are mainly ODE models [21, 22, 70] and time-delay models [71, 72]. In [21], Dumont et al. proposed an ODE vector-borne disease model and showed that some control strategies were effective via numerical simulations. In [22], Dumont et al. used a pulsed model to assess the efficacy of periodic release. In [70], Liu et al. considered the influence of rainy and dry seasons. Assuming that the birth rate of mosquito population is different and the contact rate between mosquitoes and humans also changes with time, they investigated an ODE mosquito-borne disease model with time-dependent parameters and pulse vaccination. In order to study the effect of temperature on mosquito population dynamics, Liu et al. [71] introduced a Chikungunya transmission model with time-dependent parameters and time-dependent extrinsic incubation period (EIP) and conducted a case study for the transmission of Chikungunya disease in Delhi, India. In [55], Kakarla et al. constructed a temperature-dependent R_0 model to study the disease transmission of Chikungunya virus in India. More recently, Liu et al. [72] further studied a periodic and time-delayed Chikungunya model with effects of temperature and rainfall. Up to now, there are many papers on periodic models with seasonal factors (see, e.g., [4, 51, 65, 101, 127, 129, 138, 144, 147]).

In this chapter, we incorporate the spatial heterogeneity into the model of [72] to formulate a nonlocal and time-delayed reaction-diffusion system. Since this spatial model involves the nonlocal terms and time-periodic delays, the study of its global dynamics is more challenging. Note that the diffusion coefficients of the hospitalized humans and mosquitoes in the egg stage are assumed to be zero, which makes the solution maps of this model non-compact. For the time-delayed reaction-diffusion system on the growth of mosquitoes, we first introduce a new phase space to prove that the solution maps are eventually strongly monotone and subhomogeneous, and its period map is α-contracting, and then employ the theory of monotone dynamical systems to obtain a threshold type result on the global stability in terms of the mosquito reproduction ratio $\mathscr{R}_m$. For the full system of the disease transmission, we first show that its period map has a global attractor that attracts any bounded set, and then use the comparison arguments and an abstract theorem on uniform persistence to establish the global dynamics in terms of $\mathscr{R}_m$ and the disease reproduction ratio $\mathscr{R}_0$. For a simplified nonlocal and time-delayed reaction-diffusion system with constant coefficients, we are able to construct a Lyapunov functional to prove the global attractivity of a positive steady state in the case where $\mathscr{R}_m > 1$ and $\mathscr{R}_0 > 1$. Further, we numerically investigate the impact of the temporal and spatial heterogeneities on the growth of mosquitoes and the spread of Chikungunya.

The rest of the chapter is organized as follows. In section 5.2, we present the model and study its well-posedness. In section 5.3, we prove the global stability for the mosquito growth model in terms of $\mathscr{R}_m$. In section 5.4, we first establish the global dynamics for the full system in terms of $\mathscr{R}_m$ and $\mathscr{R}_0$, and then prove the global attractivity of a positive steady state for a simplified reaction-diffusion system with time delay and constant coefficients. In section 5.5, we conduct some numerical simulations for the Chikungunya virus transmission in Ceará, Brazil, and give a brief discussion.

5.2 The model

In this section, we propose a spatial model for the spread of Chikungunya, and consider the dynamics of infection in hosts (humans) and vectors (mosquitoes). We assume that all populations remain in a bounded spatial habitat Ω with boundary $\partial\Omega$ of class $C^{2+\theta}(0 < \theta \leq 1)$. Since we focus on a local regional scale, the spatial spread of Chikungunya is assumed to be only due to active movements. Following [72], we divide the human population into susceptible (S_h), exposed (E_h), asymptomatic infectious (A_h), symptomatic infectious (I_h), hospitalized (P_h), and recovered (R_h). Aedes aegypti mosquitoes are divided into dry eggs (E_d), wet eggs (E_w), larvae (L_m), susceptible mature mosquitoes (S_m), exposed mature mosquitoes (E_m), and infectious mosquitoes (I_m). We assume that the diffusion coefficients of P_h, E_d and E_w are 0 because the hospitalized population will be managed and the eggs themselves cannot move.

Since humans are endotherms, we assume that all human-related coefficients are positive constants and that all mosquito-related coefficients are positive and ω-periodic functions. We consider the active movements of humans, larvae and mosquitoes, and assume that they perform an unbiased random walk and that the dispersion of the same species is homogeneous in space, that is, $D_h > 0$, $D_L > 0$ and $D_m > 0$, respectively. We do not consider immigration or emigration of individuals, that is, no population flux crosses the boundary $\partial\Omega$. Let Δ be the Laplacian operator, ν be the outward unit normal vector on $\partial\Omega$. Based on the above assumptions, the model parameters and their definitions are shown in Table 5.1, and the detailed transmission diagram is given in Figure 5.1. The mortality rate of this virus is shallow, so we do not consider human death due to the disease and assume that the human birth rate is equal to the natural mortality rate. That is, we assume that the total density of the human population (at the time t and any location x) $N_h(t,x)$ stabilizes at a steady state $N_h^*(x)$.

Table 5.1: Biological explanations for parameters of system (5.1)

Parameters	Description
Λ_h	recruitment rate
μ	natural mortality rate
ρ	fraction of exposed humans becoming asymptomatic
$\beta(t,x)$	biting rate of mosquitoes (i.e. the average number of bites per mosquito per unit time at time t and location x)
α_h	transmission probability per bite from infective mosquitoes to humans
α_m	transmission probability per bite from infective humans to mosquitoes
τ_h	intrinsic incubation period (IIP) of humans
$\tau_m(t)$	extrinsic incubation period (EIP) of mosquitoes
$\tau_w(t)$	maturation time of the larvae
q_1	recovery rate of asymptomatic infectious humans
q_2	recovery rate of symptomatic infectious humans
q_3	recovery rate of hospitalized humans
η_h	hospitalizing rate of symptomatic infectious humans
$K_d(t,x)$	the maximal dry eggs capacity
$\mu_b(t,x)$	birth rate of eggs
$k_d(t,x)$	fraction of dry eggs that become wet eggs
$\eta_w(t,x)$	maturation rate of wet eggs
$\mu_E(t,x)$	mortality rate of eggs
$\mu_L(t,x)$	mortality rate of larvae
$\mu_m(t,x)$	mortality rate of mature female mosquitoes

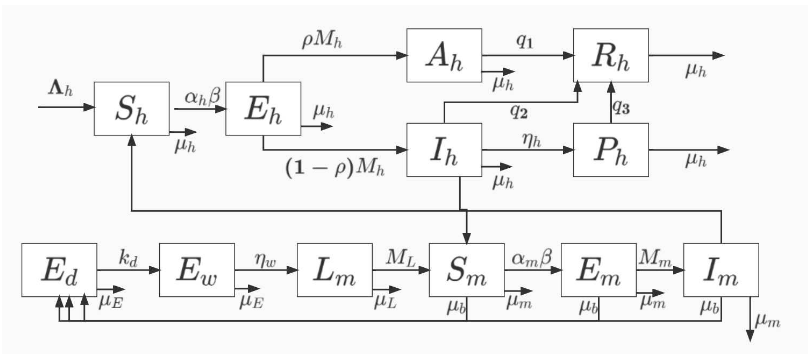


Figure 5.1: Schematic diagram for Chikungunya disease transmission.

Since $\beta(t,x)$ is the biting rate of mosquitoes, that is, $\beta(t,x)$ is the average number of bites per mosquito per unit time at time t and location x. Note that the total number of bites made by mosquitoes is equal to the total number of bites received by humans, that is $S_m + E_m + I_m$. Then the average number of bites per human receives per unit time at time t and location x is

$$\beta(t,x)\frac{S_m(t,x)+E_m(t,x)+I_m(t,x)}{N_h^*(x)}.$$

Since α_h and α_m are the transmission probabilities per bite from infective mosquitoes to humans and from infective humans to mosquitoes, respectively. Therefore, the infection rates per susceptible human and susceptible mosquito are given by

$$\begin{aligned}
&\beta(t,x)\frac{S_m(t,x)+E_m(t,x)+I_m(t,x)}{N_h^*(x)} \times \alpha_h \times \frac{I_m(t,x)}{S_m(t,x)+E_m(t,x)+I_m(t,x)} \\
&= \alpha_h\beta(t,x)\frac{I_m(t,x)}{N_h^*(x)}, \\
&\beta(t,x) \times \alpha_m \times \frac{A_h(t,x)+I_h(t,x)}{N_h^*(x)} = \alpha_m\beta(t,x)\frac{A_h(t,x)+I_h(t,x)}{N_h^*(x)},
\end{aligned}$$

respectively. The motivation for this comes from [74, 124].

Let $M_h(t,x)$, $M_L(t,x)$ and $M_m(t,x)$ be the density of newly occurred infectious people, newly occurred mature mosquitoes and newly occurred infectious mosquitoes per unit time at time t and location x. By the arguments similar to those in [136], we have

$$\begin{aligned}
M_h(t,x) &= \int_\Omega \Gamma_h(t,t-\tau_h,x,y)\alpha_h\beta(t-\tau_h,y)\frac{I_m(t-\tau_h,y)}{N_h^*(y)}S_h(t-\tau_h,y)dy, \\
M_L(t,x) &= (1-\tau_w'(t))\int_\Omega \Gamma_L(t,t-\tau_w(t),x,y)\eta_w\left(t-\tau_w(t),y\right)E_w\left(t-\tau_w(t),y\right)dy, \\
M_m(t,x) &= (1-\tau_m'(t))\int_\Omega \Gamma_m(t,t-\tau_m(t),x,y)\alpha_m\beta\left(t-\tau_m(t),y\right) \\
&\qquad \times\frac{A_h\left(t-\tau_m(t),y\right)+I_h\left(t-\tau_m(t),y\right)}{N_h^*(y)}S_m\left(t-\tau_m(t),y\right)dy,
\end{aligned}$$

where $\Gamma_i(t,t_0,x,y)$ with $t \geq t_0$ and $x, y \in \Omega$, is the Green function associated with $\frac{\partial u}{\partial t} = D_i\Delta u - \mu_i(t,\cdot)u$, $(i = h, L, m)$, subject to the Neumann boundary conditions.

Let $\beta_h(t,x) := \alpha_h\beta(t,x)\frac{1}{N_h^*(x)}$ and $\beta_m(t,x) := \alpha_m\beta(t,x)\frac{1}{N_h^*(x)}$. It follows that $\beta_h(t-\tau_h,x) = \alpha_h\beta(t-\tau_h,x)\frac{1}{N_h^*(x)}$ and $\beta_m(t-\tau_m(t),x) = \alpha_m\beta(t-\tau_m(t),x)\frac{1}{N_h^*(x)}$. Therefore, we have the following nonlocal reaction-diffusion system with temperature and rainfall effects:

$$
\begin{cases}
\frac{\partial S_h}{\partial t} = D_h \Delta S_h + \Lambda_h - \beta_h(t,x) I_m S_h - \mu_h S_h, \\
\frac{\partial E_h}{\partial t} = D_h \Delta E_h + \beta_h(t,x) I_m S_h - \int_\Omega \Gamma_h(t,t-\tau_h,x,y)\beta_h(t-\tau_h,y) \\
\qquad \times I_m(t-\tau_h,y) S_h(t-\tau_h,y) dy - \mu_h E_h, \\
\frac{\partial A_h}{\partial t} = D_h \Delta A_h + \rho \int_\Omega \Gamma_h(t,t-\tau_h,x,y)\beta_h(t-\tau_h,y) \\
\qquad \times I_m(t-\tau_h,y) S_h(t-\tau_h,y) dy - (q_1+\mu_h) A_h, \\
\frac{\partial I_h}{\partial t} = D_h \Delta I_h + (1-\rho) \int_\Omega \Gamma_h(t,t-\tau_h,x,y)\beta_h(t-\tau_h,y) \\
\qquad \times I_m(t-\tau_h,y) S_h(t-\tau_h,y) dy - (\eta_h + q_2 + \mu_h) I_h, \\
\frac{\partial P_h}{\partial t} = \eta_h I_h - (q_3+\mu_h) P_h, \\
\frac{\partial R_h}{\partial t} = D_h \Delta R_h + q_1 A_h + q_2 I_h + q_3 P_h - \mu_h R_h, \\
\frac{\partial E_d}{\partial t} = \mu_b(t,x) \left(1 - \frac{E_d}{K_d}\right) (S_m + E_m + I_m) - (k_d(t,x) + \mu_E(t,x)) E_d, \\
\frac{\partial E_w}{\partial t} = k_d(t,x) E_d - (\eta_w(t,x) + \mu_E(t,x)) E_w, \\
\frac{\partial L_m}{\partial t} = D_L \Delta L_m + \eta_w(t,x) E_w - (1-\tau_w'(t)) \int_\Omega \Gamma_L(t,t-\tau_w(t),x,y) \\
\qquad \times \eta_w(t-\tau_w(t),y) E_w(t-\tau_w(t),y) dy - \mu_L(t,x) L_m, \\
\frac{\partial S_m}{\partial t} = D_m \Delta S_m + (1-\tau_w'(t)) \int_\Omega \Gamma_L(t,t-\tau_w(t),x,y) \eta_w(t-\tau_w(t),y) \\
\qquad \times E_w(t-\tau_w(t),y) dy - \beta_m(t,x)(A_h + I_h) S_m - \mu_m(t,x) S_m, \\
\frac{\partial E_m}{\partial t} = D_m \Delta E_m + \beta_m(t,x)(A_h + I_h) S_m \\
\qquad - (1-\tau_m'(t)) \int_\Omega \Gamma_m(t,t-\tau_m(t),x,y) \beta_m(t-\tau_m(t),y) \\
\qquad \times (A_h(t-\tau_m(t),y) + I_h(t-\tau_m(t),y)) S_m(t-\tau_m(t),y) dy - \mu_m(t,x) E_m, \\
\frac{\partial I_m}{\partial t} = D_m \Delta I_m + (1-\tau_m'(t)) \int_\Omega \Gamma_m(t,t-\tau_m(t),x,y) \beta_m(t-\tau_m(t),y) \\
\qquad \times (A_h(t-\tau_m(t),y) + I_h(t-\tau_m(t),y)) S_m(t-\tau_m(t),y) dy - \mu_m(t,x) I_m, \\
\frac{\partial S_h}{\partial \nu} = \frac{\partial E_h}{\partial \nu} = \frac{\partial A_h}{\partial \nu} = \frac{\partial I_h}{\partial \nu} = \frac{\partial R_h}{\partial \nu} = \frac{\partial L_m}{\partial \nu} = \frac{\partial S_m}{\partial \nu} = \frac{\partial E_m}{\partial \nu} = \frac{\partial I_m}{\partial \nu} = 0, t > 0, x \in \partial\Omega.
\end{cases}
\tag{5.1}
$$

Let $\rho_m(r)$ be the developmental proportion of Chikungunya virus in mosquitoes at time r, and let $\rho_w(r)$ be the developmental proportion in larvaes at time r. When the accumulative proportion from $t-\tau_i(t)(i=m,w)$ to t reaches 1, the individual moves to the next stage (see, e.g., [73]). It then follows that

$$
\int_{t-\tau_m(t)}^{t} \rho_m(r) dr = 1, \quad \int_{t-\tau_w(t)}^{t} \rho_w(r) dr = 1. \tag{5.2}
$$

Differentiating both sides of (5.2) with respect to t, we obtain

$$
1 - \tau_m'(t) = \frac{\rho_m(t)}{\rho_m(t-\tau_m(t))} > 0, \quad 1 - \tau_w'(t) = \frac{\rho_w(t)}{\rho_w(t-\tau_w(t))} > 0.
$$

Here we assume that $\tau_m(t)$ and $\tau_w(t)$ are ω-periodic in t, and $\tau_m'(t)$ and $\tau_w'(t)$ are Hölder continuous on $\mathbb{R}$. All constant parameters are positive, and functions $K_d(t,x)$, $\beta_h(t,x)$, $\mu_h(t,x)$, $q_1(t,x)$, $\eta_h(t,x)$, $q_2(t,x)$ and $q_3(t,x)$ are Hölder continuous and nonnegative nontrivial on $\mathbb{R}\times\bar{\Omega}$, and ω-periodic in t. The functions $\mu_b(t,x)$, $k_d(t,x)$, $\mu_E(t,x)$, $\eta_w(t,x)$, $\mu_L(t,x)$, $\beta_m(t,x)$ and $\mu_m(t,x)$ are Hölder continuous and positive on $\mathbb{R}\times\bar{\Omega}$, and ω-periodic in t. According to [72, 120], we always impose the following condition on $K_d(t,x)$:

$$\frac{\partial K_d(t,x)}{\partial t}\geqslant-\left(k_d(t,x)+\mu_E(t,x)\right)K_d(t,x),\ \forall x\in\bar{\Omega},t\in\mathbb{R}. \tag{5.3}$$

Since $E_h(t,x)$, $P_h(t,x)$, $R_h(t,x)$, and $L_m(t,x)$ do not appear in the other equations of system (5.1), it suffices to study the following reduced system:

$$\begin{cases}\frac{\partial S_h}{\partial t}=D_h\Delta S_h+\Lambda_h-\beta_h(t,x)I_mS_h-\mu_hS_h,\\ \frac{\partial A_h}{\partial t}=D_h\Delta A_h+\rho\int_\Omega\Gamma_h(t,t-\tau_h,x,y)\beta_h(t-\tau_h,y)\\ \qquad\times I_m(t-\tau_h,y)S_h(t-\tau_h,y)dy-\left(q_1+\mu_h\right)A_h,\\ \frac{\partial I_h}{\partial t}=D_h\Delta I_h+(1-\rho)\int_\Omega\Gamma_h(t,t-\tau_h,x,y)\beta_h(t-\tau_h,y)\\ \qquad\times I_m(t-\tau_h,y)S_h(t-\tau_h,y)dy-\left(\eta_h+q_2+\mu_h\right)I_h,\\ \frac{\partial E_d}{\partial t}=\mu_b(t,x)\left(1-\frac{E_d}{K_d}\right)\left(S_m+E_m+I_m\right)-\left(k_d(t,x)+\mu_E(t,x)\right)E_d,\\ \frac{\partial E_w}{\partial t}=k_d(t,x)E_d-\left(\eta_w(t,x)+\mu_E(t,x)\right)E_w,\\ \frac{\partial S_m}{\partial t}=D_m\Delta S_m+\left(1-\tau_w'(t)\right)\int_\Omega\Gamma_L(t,t-\tau_w(t),x,y)\eta_w\left(t-\tau_w(t),y\right)\\ \qquad\times E_w\left(t-\tau_w(t),y\right)dy-\beta_m(t,x)\left(A_h+I_h\right)S_m-\mu_m(t,x)S_m,\\ \frac{\partial E_m}{\partial t}=D_m\Delta E_m+\beta_m(t,x)\left(A_h+I_h\right)S_m\\ \qquad-\left(1-\tau_m'(t)\right)\int_\Omega\Gamma_m(t,t-\tau_m(t),x,y)\beta_m\left(t-\tau_m(t),y\right)\\ \qquad\times\left(A_h\left(t-\tau_m(t),y\right)+I_h\left(t-\tau_m(t),y\right)\right)S_m\left(t-\tau_m(t),y\right)dy-\mu_m(t,x)E_m,\\ \frac{\partial I_m}{\partial t}=D_m\Delta I_m+\left(1-\tau_m'(t)\right)\int_\Omega\Gamma_m(t,t-\tau_m(t),x,y)\beta_m\left(t-\tau_m(t),y\right)\\ \qquad\times\left(A_h\left(t-\tau_m(t),y\right)+I_h\left(t-\tau_m(t),y\right)\right)S_m\left(t-\tau_m(t),y\right)dy-\mu_m(t,x)I_m,\\ \frac{\partial S_h}{\partial\nu}=\frac{\partial A_h}{\partial\nu}=\frac{\partial I_h}{\partial\nu}=\frac{\partial S_m}{\partial\nu}=\frac{\partial E_m}{\partial\nu}=\frac{\partial I_m}{\partial\nu}=0,t>0,x\in\partial\Omega.\end{cases} \tag{5.4}$$

Let $\mathbb{X}:=C(\bar{\Omega},\mathbb{R}^8)$ be the Banach space with the supremum norm $\|\cdot\|_{\mathbb{X}}$, and let $\hat{\tau}=\max\left\{\tau_h,\max_{t\in[0,\omega]}\tau_m(t),\max_{t\in[0,\omega]}\tau_w(t)\right\}$. Define $X:=C([-\hat{\tau},0],\mathbb{X})$ with the norm $\|\phi\|=\max_{\theta\in[-\hat{\tau},0]}\|\phi(\theta)\|_{\mathbb{X}}$, $\forall\phi\in X$. Let $\mathbb{X}^+:=C(\bar{\Omega},\mathbb{R}^8_+)$ and $X^+:=C([-\hat{\tau},0],\mathbb{X}^+)$. Then $(\mathbb{X},\mathbb{X}^+)$ and (X,X^+) are ordered Banach spaces. We also identity $\phi\in X$ with an element in $C([-\hat{\tau},0]\times\bar{\Omega},\mathbb{R}^8)$ by defining $\phi(\theta,x)=\phi(\theta)(x)$, $\forall\theta\in[-\hat{\tau},0]$, $\forall x\in\bar{\Omega}$. Given a function $z:[-\hat{\tau},\sigma)\to\mathbb{X}$ for $\sigma>0$, we define $z_t\in X$ by $z_t(\theta)=z(t+\theta)$, $\forall\theta\in[-\hat{\tau},0]$, for any $t\in[0,\sigma)$.

Let $\mathbb{Y}:=C(\bar{\Omega},\mathbb{R})$ and $\mathbb{Y}^+:=C(\bar{\Omega},\mathbb{R}_+)$. Define the linear evolution operators $T_i(t,s)$, $i=4,5$, on $\mathbb{Y}$ by $T_4(t,s)\phi_4=e^{-\int_s^t(k_d(r,\cdot)+\mu_E(r,\cdot))dr}\phi_4$ and $T_5(t,s)\phi_5=e^{-\int_s^t(\eta_w(r,\cdot)+\mu_E(r,\cdot))dr}\phi_5$, $\forall t\geq s$, respectively. Let $T_i(t,s):\mathbb{Y}\to\mathbb{Y}(i=1,2,3,6)$ be the evolution operator associated with $\frac{\partial u}{\partial t}=D_h\Delta u-\mu_hu$, $\frac{\partial u}{\partial t}=D_h\Delta u-(q_1+\mu_h)u$, $\frac{\partial u}{\partial t}=D_h\Delta u-(\eta_h+q_2+\mu_h)u$ and $\frac{\partial u}{\partial t}=D_m\Delta u-\mu_m(t,x)u$, $t\geq s$, $x\in$

$\bar{\Omega}$, subject to the Neumann boundary condition, respectively. Since $\mu_m(t,\cdot)$ is ω-periodic in t, [15, Lemma 6.1] implies that $T_6(t+\omega, s+\omega) = T_6(t,s)$ for $(t,s) \in \mathbb{R}^2$ with $t \geq s$. Moreover, for any $(t,s) \in \mathbb{R}^2$ with $t > s$, $T_6(t,s)$ is compact and strongly positive (see, e.g., [44, Chaper II] and [109, Theorem 7.3.1 and Theorem 7.4.1]). Clearly, $T_i(t,s)$ $(i = 1,2,3)$ are compact and strongly positive. Let $T(t,s) = \text{diag}\{T_1(t,s), T_2(t,s), T_3(t,s), T_4(t,s), T_5(t,s), T_6(t,s), T_6(t,s), T_6(t,s)\} : \mathbb{X} \to \mathbb{X}$. Define $F = (F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8) : [0, +\infty) \times X^+ \to \mathbb{X}$ by

$$
\begin{aligned}
F_1(t,\phi) &= \Lambda_h - \beta_h(t,\cdot)\phi_8(0,\cdot)\phi_1(0,\cdot),\\
F_2(t,\phi) &= \rho \int_\Omega \Gamma(t, t-\tau_h, \cdot, y)\beta_h(t-\tau_h, y)\phi_8(-\tau_h, y)\phi_1(-\tau_h, y)dy,\\
F_3(t,\phi) &= (1-\rho) \int_\Omega \Gamma(t, t-\tau_h, \cdot, y)\beta_h(t-\tau_h, y)\phi_8(-\tau_h, y)\phi_1(-\tau_h, y)dy,\\
F_4(t,\phi) &= \mu_b(t,\cdot)\left(1 - \frac{\phi_4(0,\cdot)}{K_d}\right)\left(\phi_6(0,\cdot) + \phi_7(0,\cdot) + \phi_8(0,\cdot)\right),\\
F_5(t,\phi) &= k_d(t,\cdot)\phi_4(0,\cdot),\\
F_6(t,\phi) &= (1-\tau_w'(t)) \int_\Omega \Gamma(t, t-\tau_w(t), \cdot, y)\eta_w\left(t-\tau_w(t), y\right)\phi_5\left(-\tau_w(t), y\right)dy\\
&\quad -\beta_m(t,\cdot)\left(\phi_2(0,\cdot) + \phi_3(0,\cdot)\right)\phi_6(0,\cdot),\\
F_7(t,\phi) &= \beta_m(t,\cdot)\left(\phi_2(0,\cdot) + \phi_3(0,\cdot)\right)\phi_6(0,\cdot) - (1-\tau_m'(t)) \int_\Omega \Gamma(t, t-\tau_m(t), \cdot, y)\\
&\quad \times \beta_m\left(t-\tau_m(t), y\right)\left(\phi_2\left(-\tau_m(t), y\right) + \phi_3\left(-\tau_m(t), y\right)\right)\phi_6\left(-\tau_m(t), y\right)dy,\\
F_8(t,\phi) &= (1-\tau_m'(t)) \int_\Omega \Gamma(t, t-\tau_m(t), \cdot, y)\beta_m\left(t-\tau_m(t), y\right)\\
&\quad \times \left(\phi_2\left(-\tau_m(t), y\right) + \phi_3\left(-\tau_m(t), y\right)\right)\phi_6\left(-\tau_m(t), y\right)dy,
\end{aligned}
$$

for all $t \geq 0$, and $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8) \in X^+$. By the abstract results in [79], we have the integral form of system (5.4) as

$$
\begin{cases}
u(t,\phi) = T(t,0)\phi + \displaystyle\int_0^t T(t,s)F(s,u_s)ds, t > 0,\\
u_0 = \phi \in X^+.
\end{cases}
$$

From the expression of F, we see that F satisfies locally Lipschitz continuous. Clearly, we have $\phi(0,\cdot) + kF(t,\phi) \in \mathbb{X}$ for any $(t,\phi) \in \mathbb{R}_+ \times X^+$, and hence

$$
\lim_{k \to 0^+} \frac{1}{k}\text{dist}(\phi(0,\cdot) + kF(t,\phi), \mathbb{X}) = 0.
$$

Since $T_1(t,s), T_2(t,s), T_3(t,s), T_4(t,s), T_5(t,s)$ and $T_6(t,s)$ are positive for $t \geqslant s$, we see that $T(t,s) : \mathbb{X} \to \mathbb{X}$, $\forall t \geq s \geq 0$. By Corollary 1.4.1 with $K = \mathbb{X}$, it then follows that for any $\phi \in X^+$, system (5.4) has a unique non-continuable mild solution $u(t,\cdot,\phi) =$

$(S_h(t,\cdot), A_h(t,\cdot), I_h(t,\cdot), E_d(t,\cdot), E_w(t,\cdot),\ S_m(t,\cdot),\ E_m(t,\cdot), I_m(t,\cdot))$ with $u_0 = \phi$ on its maximal interval of existence $t \in [0, t_\phi)$, where $t_\phi \leq +\infty$, and $u(t,\cdot,\phi) \in \mathbb{X}$, $\forall t \in [0, t_\phi)$. Moreover, it follows from the semigroup theory arguments similar to those in [79, Theorem 1] that $u(t,\cdot,\phi)$ is a classical solution of system (5.4) for $t > \hat{\tau}$.

Choose

$$K = \{\psi \in \mathbb{X} : \psi_i \geq 0, \forall i \neq 7,\ \psi_6 + \psi_7 + \psi_8 \geq 0\}.$$

For any given $(t,\phi) \in \mathbb{R}_+ \times C([-\hat{\tau}, 0], K)$, we have $\phi(0,\cdot) + kF(t,\phi) \in K$ for sufficiently small $k > 0$, and hence, $\lim_{k\to 0^+} \frac{1}{k}\mathrm{dist}(\phi(0,\cdot) + kF(t,\phi), K) = 0$. Clearly, $T(t,s) : K \to K$, $\forall t \geq s \geq 0$. It follows from Corollary 1.4.1 that for any $\phi \in C([-\hat{\tau}, 0], K)$, the solution $u(t,\cdot,\phi)$ with $u_0 = \phi$ satisfies $S_h(t,\cdot) \geq 0$, $A_h(t,\cdot) \geq 0$, $I_h(t,\cdot) \geq 0$, $E_d(t,\cdot) \geq 0$, $E_w(t,\cdot) \geq 0$, $S_m(t,\cdot) \geq 0$ and $I_m(t,\cdot) \geq 0$ for all $t \in [0, t_\phi)$. In view of the biological meaning of $\tau_m(t)$, we impose the following compatibility condition:

$$E_m(0,\cdot) = \int_{-\tau_m(0)}^{0} T_6(0,s)\beta_m(s,\cdot)\left(A_h(s,\cdot) + I_h(s,\cdot)\right) S_m(s,\cdot)ds. \tag{5.5}$$

Define

$$D := \left\{ \phi \in X^+ : \phi_7(0,\cdot) = \int_{-\tau_m(0)}^{0} T_6(0,s)\beta_m(s,\cdot)\left(\phi_2(s,\cdot) + \phi_3(s,\cdot)\right)\phi_6(s,\cdot)ds \right\}.$$

By the uniqueness of solutions of E_m on system (5.4) and the compatibility condition (5.5), it follows that

$$E_m(t,\cdot) = \int_{t-\tau_m(t)}^{t} T_6(t,s)\beta_m(s,\cdot)\left(A_h(s,\cdot) + I_h(s,\cdot)\right) S_m(s,\cdot)ds, \tag{5.6}$$

and hence, $E_m(t,\cdot) \geq 0$ for all $t \in [0, t_\phi)$. Therefore, for any $\phi \in D$, $u(t,\cdot,\phi)$ with $u_0 = \phi$ is nonnegative for all $t \in [0, t_\phi)$.

Consider system (5.1), denote $N_h(t,x) = S_h(t,x) + E_h(t,x) + A_h(t,x) + I_h(t,x)$, then we have

$$\begin{aligned} \frac{\partial N_h}{\partial t} &= D_h \Delta N_h + \Lambda_h - \mu_h N_h - q_1 A_h - (q_2 + \eta_h) I_h \\ &\leqslant D_h \Delta N_h + \Lambda_h - \mu_h N_h, \forall t > 0, x \in \Omega, \end{aligned} \tag{5.7}$$

subject to the Neumann boundary condition. Then $S_h(t,\cdot)$, $E_h(t,\cdot)$, $A_h(t,\cdot)$ and $I_h(t,\cdot)$ are bounded on $t \in [0, t_\phi)$.

Then denote $N_m(t,x) = S_m(t,x) + E_m(t,x) + I_m(t,x)$, it follows from system (5.1) that $E_d(t,x)$, $E_w(t,x)$, $L_m(t,x)$ and $N_m(t,x)$ satisfy

$$
\begin{aligned}
\frac{\partial E_d}{\partial t} &= \mu_b(t,x)\left(1-\frac{E_d}{K_d}\right)N_m - \left(k_d(t,x)+\mu_E(t,x)\right)E_d,\\
\frac{\partial E_w}{\partial t} &= k_d(t,x)E_d - \left(\eta_w(t,x)+\mu_E(t,x)\right)E_w,\\
\frac{\partial L_m}{\partial t} &= D_L\Delta L_m + \eta_w(t,x)E_w - (1-\tau_w'(t))\int_\Omega \Gamma(t,t-\tau_w(t),x,y)\\
&\quad \times \eta_w\left(t-\tau_w(t),y\right)E_w\left(t-\tau_w(t),y\right)dy - \mu_L(t,x)L_m,\\
\frac{\partial N_m}{\partial t} &= D_m\Delta N_m + (1-\tau_w'(t))\int_\Omega \Gamma(t,t-\tau_w(t),x,y)\eta_w\left(t-\tau_w(t),y\right)\\
&\quad \times E_w\left(t-\tau_w(t),y\right)dy - \mu_m(t,x)N_m,\\
\frac{\partial L_m}{\partial \nu} &= \frac{\partial N_m}{\partial \nu} = 0, t>0, x\in\partial\Omega.
\end{aligned}
\tag{5.8}
$$

Note that $E_d(t,\cdot) \leqslant \max\left\{\max_{\theta\in[-\hat{\tau},0],x\in\bar{\Omega}} E_d(\theta,x), \overline{K_d}\right\} := \overline{E_d}$ for all $t\in[0,t_\phi)$, where $E_d(\theta,\cdot)$ is the initial data and $\overline{K_d} = \max_{t\in[0,\omega],x\in\bar{\Omega}} K_d(t,x)$, and hence, it follows from (5.8) that

$$
\begin{aligned}
\frac{\partial E_w}{\partial t} &\leqslant \overline{k_d E_d} - \left(\underline{\eta_w}+\underline{\mu_E}\right)E_w,\\
\frac{\partial N_m}{\partial t} &\leqslant D_m\Delta N_m + (1-\tau_w'(t))\int_\Omega \Gamma(t,t-\tau_w(t),x,y)\overline{\eta_w}\,\overline{E_w}dy - \underline{\mu_m}N_m,\\
\frac{\partial N_m}{\partial \nu} &= 0, t>0, x\in\partial\Omega.
\end{aligned}
\tag{5.9}
$$

where $\overline{E_w} = \max\left\{\max_{\theta\in[-\hat{\tau},0],x\in\bar{\Omega}} E_w(\theta,x), \frac{\overline{k_d E_d}}{\underline{\eta_w}+\underline{\mu_E}}\right\}$, $\overline{k_d} = \max_{t\in[0,\omega],x\in\bar{\Omega}} k_d(t,x)$, $\underline{\eta_w} = \min_{t\in[0,\omega],x\in\bar{\Omega}} \eta_w(t,x)$, $\overline{\eta_w} = \max_{t\in[0,\omega],x\in\bar{\Omega}} \eta_w(t,x)$, $\underline{\mu_E} = \min_{t\in[0,\omega],x\in\bar{\Omega}} \mu_E(t,x)$, $\underline{\mu_m} = \min_{t\in[0,\omega],x\in\bar{\Omega}} \mu_m(t,x)$, and $E_w(\theta,\cdot)$ is the initial data. Therefore, $S_h(t,\cdot)$, $A_h(t,\cdot)$, $I_h(t,\cdot)$, $E_d(t,\cdot)$, $E_w(t,\cdot)$, $S_m(t,\cdot)$, $E_m(t,\cdot)$, and $I_m(t,\cdot)$ are bounded on $t\in[0,t_\phi)$, which implies $t_\phi=\infty$.

In view of (5.7), it is easy to see that $S_h(t,\cdot)$, $E_h(t,\cdot)$, $A_h(t,\cdot)$ and $I_h(t,\cdot)$ are ultimately bounded. By (5.3) and (5.8), one implies that for any $\phi\in X^+$, there exists $\tilde{t}=\tilde{t}(\phi)>0$ such that

$$E_d(t,x,\phi) \leqslant \overline{K_d},\ \forall t\geq \tilde{t}, x\in\bar{\Omega}.$$

Then we have $\frac{\partial E_w}{\partial t} \leqslant \overline{k_d K_d} - \left(\underline{\eta_w}+\underline{\mu_E}\right)E_w, \forall t\geq\tilde{t}$, it follows that $E_w(t,\cdot)$ is ultimately bounded. Furthermore, similar to (5.9) for N_m-equation, by using the comparison principle, we can conclude that $S_m(t,\cdot)$, $E_m(t,\cdot)$, and $I_m(t,\cdot)$ are ultimately bounded. Therefore, the solutions of system (5.4) with initial data in D and also in X^+, exist globally on $[0,\infty)$ and are also ultimately bounded.

From (5.7), it is also easy to see that $S_h(t,\cdot)$, $E_h(t,\cdot)$, $A_h(t,\cdot)$ and $I_h(t,\cdot)$ are uniformly bounded. By (5.3) and (5.8), for any given $B > 0$ with $\|\phi\|_X \le B$, we have

$$E_d(t,x,\phi) \leqslant \max\{B, \overline{K}_d\},\ \forall t \ge 0, x \in \bar{\Omega}.$$

It then follows from (5.9) and the comparison principle that there exists $\hat{B} = \hat{B}(B) \ge \max\{B, \overline{K}_d\}$, such that

$$\|u(t,x,\phi)\|_{\mathbb{X}} \le \hat{B},\ \forall t \ge 0, x \in \bar{\Omega},$$

whenever $\phi \in D$ with $\|\phi\|_X \le B$. This implies that solutions of system (5.4) are uniformly bounded. So we have the following result on the well-posedness of system (5.4).

Lemma 5.2.1. *For any $\phi \in D$, system (5.4) has a unique nonnegative mild solution $u(t,\cdot,\phi)$ with $u_0 = \phi$ and $u_t(\cdot,\phi) \in D$ for all $[0,+\infty)$, and solutions are ultimately bounded and uniformly bounded.*

5.3 The growth of mosquitoes

If Chikungunya disease dose not exist, then the subsystem for mosquito population is as follows:

$$\begin{aligned}
\frac{\partial E_d}{\partial t} &= \mu_b(t,x)\left(1 - \frac{E_d}{K_d}\right) S_m - \left(k_d(t,x) + \mu_E(t,x)\right) E_d,\\
\frac{\partial E_w}{\partial t} &= k_d(t,x) E_d - \left(\eta_w(t,x) + \mu_E(t,x)\right) E_w,\\
\frac{\partial S_m}{\partial t} &= D_m \Delta S_m + (1 - \tau_w'(t)) \int_\Omega \Gamma_L(t, t-\tau_w(t), x, y)\eta_w\left(t - \tau_w(t), y\right)\\
&\quad \times E_w\left(t - \tau_w(t), y\right) dy - \mu_m(t,x) S_m,
\end{aligned} \tag{5.10}$$

where S_m is subject to Neumann boundary condition.

Let $\mathbb{E} := C(\bar{\Omega}, \mathbb{R}^3)$, $\mathbb{E}^+ := C(\bar{\Omega}, \mathbb{R}^3_+)$, and $C_\omega(\mathbb{R}, \mathbb{E})$ be the Banach space consisting of all ω-periodic and continuous functions from $\mathbb{R}$ to $\mathbb{E}$, where $\|\phi\|_{C_\omega(\mathbb{R},\mathbb{E})} := \max_{\theta\in[0,\omega]} \|\phi(\theta)\|_{\mathbb{E}}$ for any $\phi \in C_\omega(\mathbb{R}, \mathbb{E})$. Let $E := C([-\hat{\tau}, 0], \mathbb{E})$ and $E^+ := C([-\hat{\tau}, 0], \mathbb{E}^+)$. By the theory of abstract functional differential equations developed in [79], we can easily prove the following result.

Lemma 5.3.1. *For any $\phi \in E^+$, system (5.10) has a unique nonnegative solution $u(t,\cdot,\phi)$ on $[0,+\infty)$ with $u_0 = \phi$, and solutions are ultimately bounded and uniformly bounded. Moreover, system (5.10) generates an ω-periodic semiflow $Q(t) := w_t(\cdot) : E^+ \to E^+$.*

Linearizing system (5.3.1) at $(0,0,0)$, we obtain a cooperative periodic system:

$$\begin{aligned}\frac{\partial E_d}{\partial t} &= \mu_b(t,x)S_m - (k_d(t,x)+\mu_E(t,x))\,E_d,\\ \frac{\partial E_w}{\partial t} &= k_d(t,x)E_d - (\eta_w(t,x)+\mu_E(t,x))\,E_w,\\ \frac{\partial S_m}{\partial t} &= D_m\Delta S_m + (1-\tau_w'(t))\int_\Omega \Gamma_L(t,t-\tau_w(t),x,y)\eta_w\,(t-\tau_w(t),y)\\ &\quad \times E_w\,(t-\tau_w(t),y)\,dy - \mu_m(t,x)S_m.\end{aligned} \tag{5.11}$$

We define the operator $\widetilde{F}(t): E \to \mathbb{E}$ by

$$\widetilde{F}(t)\phi = (\widetilde{F}_1(t)\phi_3, \widetilde{F}_2(t)\phi_1, \widetilde{F}_3(t)\phi_2), \forall \phi = (\phi_1,\phi_2,\phi_3) \in E, t \in \mathbb{R},$$

where

$$\begin{aligned}\widetilde{F}_1(t)\phi_3 &= \mu_b(t,\cdot)\phi_3(0,\cdot),\\ \widetilde{F}_2(t)\phi_1 &= k_d(t,\cdot)\phi_1(0,\cdot),\\ \widetilde{F}_3(t)\phi_1 &= (1-\tau_w'(t))\int_\Omega \Gamma_L(t,t-\tau_w(t),\cdot,y)\eta_w\,(t-\tau_w(t),\cdot)\,\phi_2\,(-\tau_w(t),y)\,dy.\end{aligned}$$

Let

$$-\widetilde{V}(t)v = \mathbf{D}\Delta v - W(t)v,$$

where $\mathbf{D} = diag(0,0,D_m)$ and

$$-[W(t)](x) = \begin{pmatrix} -(k_d(t,x)+\mu_E(t,x)) & 0 & 0\\ 0 & -(\eta_w(t,x)+\mu_E(t,x)) & 0\\ 0 & 0 & -\mu_m(t,x)\end{pmatrix}, \forall x \in \bar{\Omega}.$$

Then system (5.11) can be written as

$$\frac{dv}{dt} = \widetilde{F}(t)v_t - \widetilde{V}(t)v, t \geq 0.$$

Then

$$\Phi(t,s) = \mathrm{diag}(T_4(t,s), T_5(t,s), T_6(t,s)), t \geq s,$$

is the evolution family on $\mathbb{E}$ associated with the following linear system:

$$\frac{dv}{dt} = -\widetilde{V}(t)v.$$

Recall that the exponential growth bound of $\Phi(t,s)$ is defined as

$$\tilde{\omega}(\Phi) = \inf\{\bar{\omega} : \exists M_0 \geq 1 \text{ such that } \|\Phi(t+s,s)\| \leq M_0 e^{\bar{\omega}t},\ \forall s \in \mathbb{R},\ t \geq 0\}.$$

By [112, Proposition A.2], we have

$$\tilde{\omega}(\Phi) = \frac{\ln r(\Phi(\omega, 0))}{\omega}.$$

With Theorem 1.3.2 (Krein-Rutman Theorem), we see from [44, Lemma 14.2] that $0 < r(T_6(\omega, 0)) < 1$. Since $r(T_4(\omega, 0)) \leq \|T_4(\omega, 0)\| < 1$ and $r(T_5(\omega, 0)) \leq \|T_5(\omega, 0)\| < 1$, it follows that

$$r(\Phi(\omega, 0)) = \max\{r(T_4(\omega, 0)), r(T_5(\omega, 0)), r(T_6(\omega, 0))\} < 1,$$

and hence, $\tilde{\omega}(\Phi) < 0$. Note that $\Phi(t, s)$ is a positive operator in the sense that $\Phi(t, s)\mathbb{E}^+ \subseteq \mathbb{E}^+$ for all $t \geq s$. Therefore, $\widetilde{F}(t)$ and $\Phi(t, s)$ satisfy the following assumptions:

(H1) For any $t \geqslant 0$, $\widetilde{F}(t)$ is a positive operator on $\mathbb{E}$.

(H2) For any $t \geqslant s$, $\Phi(t, s)$ is a positive operator on $\mathbb{E}$, and $\tilde{\omega}(\Phi) < 0$.

Following Section 1.5.1, we define the linear next generation operator on $C_\omega(\mathbb{R}, \mathbb{E})$ by

$$[Lv](t) := \int_0^{+\infty} \Phi(t, t-s)\widetilde{F}(t-s)v(t-s+\cdot)ds, \ \forall t \in \mathbb{R}, \ v \in C_\omega(\mathbb{R}, \mathbb{E}).$$

Let A and B be two bounded linear operators on $C_\omega(\mathbb{R}, \mathbb{E})$ defined by

$$[Av](t) = \int_0^{+\infty} \Phi(t, t-s)v(t-s)ds, \ [Bv](t) = \widetilde{F}(t)v_t, \ \forall t \in \mathbb{R}, \ v \in C_\omega(\mathbb{R}, \mathbb{E}).$$

Further,

$$Av = (A_1v_1, A_2v_2, A_3v_3), \ Bv = (B_1v_3, B_2v_1, B_3v_2), \ \forall v = (v_1, v_2, v_3) \in C_\omega(\mathbb{R}, \mathbb{E}),$$

where

$$\begin{aligned}
(t) &= \int_0^{+\infty} T_4(t, t-s)v(t-s)ds, \\
[A_2v_2](t) &= \int_0^{+\infty} T_5(t, t-s)v(t-s)ds, \\
[A_3v_3](t) &= \int_0^{+\infty} T_6(t, t-s)v(t-s)ds, \ \forall t \in \mathbb{R}, \\
B_1v_3(t) &= \widetilde{F}_1(t)v_{3t}, B_2v_1(t) = \widetilde{F}_2(t)v_{1t}, \\
B_3v_2(t) &= \widetilde{F}_3(t)v_{2t}, \forall v_t = (v_{1t}, v_{2t}, v_{3t}) \in E(t), \forall t \in \mathbb{R}.
\end{aligned}$$

Clearly, $L = A \circ B$. According to Section 1.5.1, we define the mosquito reproduction number as $\mathscr{R}_m := r(L)$, where $r(L)$ is the spectral radius of L.

For any given $t \geq 0$, let $\hat{P}(t)$ be the solution map of system (5.11) on E; that is, $\hat{P}(t)\phi = v_t(\phi)$, where $v_t(\phi)(\theta, x) = v(t+\theta, x, \phi) = (v_1(t+\theta, x, \phi), v_2(t+\theta, x, \phi), v_3(t+\theta, x, \phi))$, $\forall(\theta, x) \in [-\hat{\tau}, 0] \times \bar{\Omega}$, and $v(t, x, \phi)$ is the unique solution of system (5.11) with $v(\theta, x, \phi) = \phi(\theta, x)$, $\forall(\theta, x) \in [-\hat{\tau}, 0] \times \bar{\Omega}$. Then $\hat{P} := \hat{P}(\omega)$ is the Poincaré map associated with system (5.11). Let $r(\hat{P})$ be the spectral radius of $\hat{P}$.

Note that the periodic semiflow $\hat{P}(t)$ is monotone but not strongly monotone on E. In order to apply Theorem 1.3.4 (Generalized Krein-Rutman Theorem), we define

$$\mathcal{E} := C\left([-\tau_w(0), 0], \mathbb{Y}\right) \times \mathbb{Y} \times \mathbb{Y},$$
$$\mathcal{E}^+ := C\left([-\tau_w(0), 0], \mathbb{Y}^+\right) \times \mathbb{Y}^+ \times \mathbb{Y}^+.$$

Then $(\mathcal{E}, \mathcal{E}^+)$ is an ordered Banach space. Given a function $w : [-\tau_w(0), +\infty) \times [0, +\infty) \times [0, +\infty) \to \mathbb{E}$, we define $w_t \in \mathcal{E}$ by $w_t(\theta) = (w_1(t+\theta), w_2(t), w_3(t))$, $\forall\theta \in [-\tau_w(0), 0]$, $\forall t \geq 0$.

Lemma 5.3.2. *For any $\psi \in \mathcal{E}^+$, system (5.11) has a unique nonnegative solution $w(t, \cdot, \psi)$ on $[0, +\infty)$ with $w_0 = \psi$.*

Proof. Our arguments are motivated by [63]. Let $\bar{\tau} = \min_{t\in[0,\omega]} \tau_w(t)$. For any $t \in [0, \bar{\tau}]$, since $t-\tau_w(t)$ is strictly increasing in t, we have $-\tau_w(0) = 0-\tau_w(0) \leq t-\tau_w(t) \leq \bar{\tau} - \tau_w(\bar{\tau}) \leq \bar{\tau} - \bar{\tau} = 0$, and hence, $E_w(t - \tau_w(t), \cdot) = \psi_2(t - \tau_w(t), \cdot)$. Thus,

$$\begin{aligned}
\frac{\partial E_d}{\partial t} &= \mu_b(t, x)S_m - (k_d(t, x) + \mu_E(t, x))\, E_d, \\
\frac{\partial E_w}{\partial t} &= k_d(t, x)E_d - (\eta_w(t, x) + \mu_E(t, x))\, E_w, \\
\frac{\partial S_m}{\partial t} &= D_m\Delta S_m + (1 - \tau_w'(t)) \int_\Omega \Gamma(t, t-\tau_w(t), x, y)\eta_w\left(t - \tau_w(t), y\right) \\
&\quad \times\psi_2\left(t - \tau_w(t), y\right) dy - \mu_m(t, x)S_m,
\end{aligned}$$

subject to the Neumann boundary condition. Given $\psi \in \mathcal{E}^+$, the solution

$$(w_1(t, \cdot), w_2(t, \cdot), w_3(t, \cdot))$$

of the above linear system exists uniquely for $t \in [0, \bar{\tau}]$. This implies that $\psi_1(\theta, \cdot) := w_1(\theta, \cdot)$, $\forall\theta \in [-\tau(0), \bar{\tau}]$, $\psi_2(\theta, \cdot) := w_2(\theta, \cdot)$, $\forall\theta \in [0, \bar{\tau}]$ and $\psi_3(\theta, \cdot) := w_3(\theta, \cdot)$, $\forall\theta \in [0, \bar{\tau}]$.

We can repeat this procedure to $[n\bar{\tau}, (n+1)\bar{\tau}]$ for all $n \in \mathbb{N}$ by the method of steps. Thus, for any $\psi \in \mathcal{E}^+$, the solution $w(t, \cdot, \psi)$ exists uniquely and is nonnegative for all $t \geq 0$. □

Remark 5.3.1. *By the uniqueness of solutions in Lemmas 5.3.1 and 5.3.2, it follows that for any $\varphi \in E^+$ and $\psi \in \mathcal{E}^+$ with $\varphi_1(\theta, \cdot) = \psi_1(\theta, \cdot)$, $\forall\theta \in [-\tau_w(0), 0]$, $\varphi_2(0, \cdot) = \psi_2(\cdot)$ and $\varphi_3(0, \cdot) = \psi_3(\cdot)$, then $v(t, \cdot, \varphi) = w(t, \cdot, \psi)$, $t \geq 0$, where $v(t, \cdot, \varphi)$ and $w(t, \cdot, \psi)$ are solutions of system (5.11) satisfying $v_0 = \varphi$ and $w_0 = \psi$, respectively.*

For any given $t \geq 0$, let $P(t)$ be the solution map of system (5.11) on $\mathcal{E}$; that is, $P(t)\phi = w_t(\phi)$, where

$$w_t(\phi)(\theta, x) = w(t+\theta, x, \phi) = (w_1(t+\theta, x, \phi), w_2(t, x, \phi), w_3(t, x, \phi)),$$

for all $(\theta, x) \in [-\tau_w(0), 0] \times \bar{\Omega}$, and $w(t, x, \phi)$ is the unique solution of system (5.11) with $w(\theta, x, \phi) = \phi(\theta, x)$, $\forall(\theta, x) \in [-\tau_w(0), 0] \times \bar{\Omega}$. Then $P := P(\omega)$ is the Poincaré map associated with system (5.11). Let $r(P)$ be the spectral radius of P. By the arguments similar to those in [63], we can prove the following two results.

Lemma 5.3.3. *For any φ and ψ in $\mathcal{E}$ with $\varphi > \psi$ (that is, $\varphi \geq \psi$, but $\varphi \not\equiv \psi$), the solutions $\bar{w}(t, \cdot, \varphi)$ and $w(t, \cdot, \psi)$ of system (5.11) with $\bar{w}_0 = \varphi$ and $w_0 = \psi$, respectively, satisfy $\bar{w}_i(t, \cdot, \varphi) \gg w_i(t, \cdot, \psi)$ for all $t > \hat{\tau}$, $i = 1, 2, 3$, and hence, $P(t)\varphi \gg P(t)\psi$ in $\mathcal{E}$ for all $t > 2\hat{\tau}$.*

Lemma 5.3.4. *If $r(\hat{P}) \geq 1$, then $r(\hat{P})$ is an eigenvalue of $\hat{P}$ with a strongly positive eigenvector on E. Moreover, $r(\hat{P}) = r(P)$.*

Let $\{U(t, s, \lambda) : t \geq s\}$ be the evolution family on E of the following linear periodic system with parameter $\lambda \in [0, +\infty)$:

$$\frac{dv(t)}{dt} = \lambda \widetilde{F}(t)v_t - \widetilde{V}(t)v(t), t \geq 0. \tag{5.12}$$

In order to obtain the relationship between the signs of $\mathscr{R}_m - 1$ and $r(\hat{P}) - 1$, we need to verify the following additional assumptions:

(H3) The positive linear operator L possesses the principal eigenvalue.

(H4) The positive linear operator $U(\omega, 0, \lambda)$ possesses the isolated principal eigenvalue with finite multiplicity for any $\lambda \in [0, +\infty)$ whenever $r(U(\omega, 0, \lambda)) \geq 1$.

(H5) Either the principal eigenvalue of L is isolated, or there exists an integer $n_0 > 0$ such that L^{n_0} is strongly positive.

Lemma 5.3.5. *$\mathscr{R}_m - 1$ has the same sign as $r(\hat{P}) - 1$.*

Proof. Our arguments are motivated by [67, Lemma 4.7]. We have known that (H1) and (H2) hold. Next we prove that (H3)-(H5) are valid.
Claim 1. L^3 is compact on $C_\omega(\mathbb{R}, \mathbb{E})$.

Clearly, A_1, B_1, A_2, B_2, A_3 and B_3 are bounded in $C_\omega(\mathbb{R}, \mathbb{E})$. By [109, Chapter 7], A_3 is compact on $C_\omega(\mathbb{R}, \mathbb{E})$. Since $A_1B_1A_3B_3A_2B_2$, $A_2B_2A_1B_1A_3B_3$ and $A_3B_3A_2B_2A_1B_1$ are compact on $C_\omega(\mathbb{R}, \mathbb{Y})$, we have L^3 is compact on $C_\omega(\mathbb{R}, \mathbb{E})$ due to $L^3v = ABABABv = (A_1B_1A_3B_3A_2B_2v_1, A_2B_2A_1B_1A_3B_3v_2, A_3B_3A_2B_2A_1B_1v_3)$.
Claim 2. L admits the principal eigenvalue.

Obviously, A_1, B_1, A_2, B_2, A_3 and B_3 are strictly positive and map $\mathrm{Int}(Y^+)$ to $\mathrm{Int}(Y^+)$. Note that $T_6(t,s)$ is strongly positive on $\mathbb{Y}$ for any $t > s$. Then for any $v_3 \in C_\omega(\mathbb{R}, \mathbb{Y}^+) \setminus \{0\}$, $\int_0^{+\infty} T_6(t, t-s)v_3(t-s)ds$, $t \in \mathbb{R}$ is strongly positive on $\mathbb{Y}$, that is, A_3 is strongly positive on $C_\omega(\mathbb{R}, \mathbb{Y})$, and hence, $A_3B_3A_2B_2A_1B_1$ are strongly positive on $C_\omega(\mathbb{R}, \mathbb{Y})$.

Now we prove that $r(L) > 0$. For a fixed $v_3 \in C_\omega(\mathbb{R}, \mathbb{Y}^+) \setminus \{0\}$, there exists $r > 0$ such that $A_3B_3A_2B_2A_1B_1v_3 \geq rv_3$ in $\mathbb{Y}$. Then $L^3v \geq rv$, where $v = (0, 0, v_3)$, and hence, $r(L^3) > 0$ due to the Gelfand's formula. It follows from Theorem 1.3.2 (Krein-Rutman Theorem) that L^3 possesses the principal eigenvalue with an eigenfunction $\tilde{v} \in C_\omega(\mathbb{R}, \mathbb{E}^+)\setminus\{0\}$. Since $r^3(L) = r(L^3)$ and $(r^3(L)-L^3)\tilde{v} = 0$, we have $(r(L)-L)\hat{v} = 0$, where $\hat{v} = (r^2(L) + L^2 + r(L)L)\tilde{v} \in C_\omega(\mathbb{R}, \mathbb{E}^+) \setminus \{0\}$. This implies that L possesses the principal eigenvalue with positive eigenfunction in $C_\omega(\mathbb{R}, \mathbb{E}^+)$. Therefore, (H3) and (H5) hold true.

It remains to prove (H4). Let $\hat{P}_\lambda := U(\omega, 0, \lambda)$ be the Poincaré map on E associated by system (5.12). We repeat the arguments in Lemma 5.3.3 to obtain that $r(\hat{P}_\lambda)$ is the principal eigenvalue whenever $r(\hat{P}_\lambda) \geq 1$. Then (H4) holds true. Thus, Theorem 1.5.1 implies that $\mathscr{R}_m - 1$ has the same sign as $r(\hat{P}) - 1$. □

By Lemma 5.3.1, we can define the solution maps $Q(t) : E^+ \to E^+$ associated with system (5.10) by $Q(t)\phi = u_t(\phi)$, where $u_t(\phi)(\theta, x) = u(t+\theta, x, \phi)$, $t \geq 0$, $(\theta, x) \in [-\hat{\tau}, 0]\times\bar{\Omega}$, and $u(t, x, \phi)$ is the unique solution of system (5.10) with $u(\theta, x) = \phi(\theta, x)$, $(\theta, x) \in [-\hat{\tau}, 0] \times \bar{\Omega}$. Then $Q := Q(\omega)$ is the Poincaré map associated with system (5.10). Since the first two equations in system (5.10) have no diffusion terms, its solution map $Q(t)$ is not compact. But we can use the arguments similar to those in [63] to prove the following result.

Lemma 5.3.6. *Q is α-contracting in the sense that $\lim_{n\to\infty} \alpha(Q^n(B)) = 0$ for any bounded set $B \subseteq E^+$.*

Proof. Let B be a given bounded subset of E^+. Motivated by [45, Lemma 4.1], we first show that $Q^n = Q(n\omega)$ is asymptotically compact on B in the sense that for any sequences $\psi_k = (\psi_{k1}, \psi_{k2}, \psi_{k3}) \in B$ and $n_k \to \infty$, there exist subsequences $k_j \to \infty$ and $\psi_{k_j} \in B$ such that $Q^{n_{k_j}}(\psi_{k_j})$ converges in E as $j \to \infty$. By Lemma 5.3.1, the solution $\{u(t, \cdot, \psi_k)\}_{k\geq 1}$ is uniformly bounded on $\bar{\Omega}$ for all $k \geq 1$, and hence, there exists an $\eta > 0$ such that

$$|E_d(t, x, \psi_k)| < \eta, |E_w(t, x, \psi_k)| < \eta, |S_m(t, x, \psi_k)| < \eta, \forall k \geq 1, t \geq 0, x \in \bar{\Omega}.$$

Since $n_k\omega > \hat{\tau}$ for all sufficiently large k, in view of Theorem 1.4.1 (Generalized Arzela-Ascoli Theorem) for space $E^+ := C([-\hat{\tau}, 0], \mathbb{E}^+)$, it suffices to prove that (i) for each $\theta \in [-\hat{\tau}, 0]$, the set $\{Q^{n_k}(\psi_k)(\theta)\}_{k\geq 1}$ is precompact in $\mathbb{E}^+$; (ii) the sequence $\{Q^{n_k}(\psi_k)\}_{k\geq 1}$ is equicontinuous in $\theta \in [-\hat{\tau}, 0]$.

Now we prove the statement (i). By the Arzela-Ascoli theorem, it suffices to prove for any given $\theta \in [-\hat{\tau}, 0]$, $\{Q^{n_k}(\psi_k)(\theta, x)\}_{k\geq 1}$ is equicontinuous in $x \in \bar{\Omega}$ for all $k \geq 1$. Note that $\{S_m(n_k\omega+\theta, x, \psi_k)\}_{k\geq 1}$ is equicontinuous in $x \in \bar{\Omega}$ for all $k \geq 1$, $\theta \in [-\hat{\tau}, 0]$. Then we first show that $\{E_d(n_k\omega + \theta, x, \psi_k)\}_{n\geq 1}$ is equicontinuous in $x \in \bar{\Omega}$ for all $k \geq 1$, $\theta \in [-\hat{\tau}, 0]$. Let

$$g_k(x,t) := \mu_b(t,x)S_m(t,x,\psi_k),$$

and hence, for each $k \geq 1$, $g_k(x,t)$ is a continuous function on $\bar{\Omega} \times \mathbb{R}_+$. Let $v_k(t,x) = E_d(t,x,\psi_k)$, $t \geq 0$, $x \in \bar{\Omega}$. Define $\overline{v}_k(t,x) := v_k(t + n_k\omega, x)$, $\forall t \geq -n_k\omega$, $x \in \bar{\Omega}$. Clearly,

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[(\bar{v}_k(t+\theta,x_1) - \bar{v}_k(t+\theta,x_2))^2\right] \\
&= 2\left(\bar{v}_k(t+\theta,x_1) - \bar{v}_k(t+\theta,x_2)\right)\frac{\partial}{\partial t}\left[\bar{v}_k(t+\theta,x_1) - \bar{v}_k(t+\theta,x_2)\right] \\
&\leqslant 2\left|(\bar{v}_k(t+\theta,x_1) - \bar{v}_k(t+\theta,x_2))\right| \times [g_k(x_1, t+n_k\omega+\theta) \\
&\quad - \left(k_d(x_1,t+n_k\omega+\theta) + \mu_E(x_1,t+n_k\omega+\theta)\right)\bar{v}_k(t+\theta,x_1) \\
&\quad -g_k(x_2,t+n_k\omega+\theta) - \left(k_d(x_2,t+n_k\omega+\theta) + \mu_E(x_2,t+n_k\omega+\theta)\right)\bar{v}_k(t+\theta,x_2)] \\
&= 2\left|(\bar{v}_k(t+\theta,x_1) - \bar{v}_k(t+\theta,x_2))\right| \times [g_k(x_1,t+n_k\omega+\theta) - g_k(x_2,t+n_k\omega+\theta) \\
&\quad - \left(k_d(x_1,t+n_k\omega+\theta) + \mu_E(x_1,t+n_k\omega+\theta)\right)\bar{v}_k(t+\theta,x_1) \\
&\quad - \left(k_d(x_2,t+n_k\omega+\theta) + \mu_E(x_2,t+n_k\omega+\theta)\right)\bar{v}_k(t+\theta,x_2)] \\
&\leqslant 4\eta|g_k(x_1,t+n_k\omega+\theta) - g_k(x_2,t+n_k\omega+\theta)| - 2m\left[\bar{v}_k(t+\theta,x_1) - \bar{v}_k(t+\theta,x_2)\right]^2,
\end{aligned}$$

where

$$\begin{aligned}
m = \min\{&[k_d(x_1,t+n_k\omega+\theta) + \mu_E(x_1,t+n_k\omega+\theta)], \\
&[k_d(x_2,t+n_k\omega+\theta) + \mu_E(x_2,t+n_k\omega+\theta)], x_i \in \overline{\Omega}, i = 1,2\big\},
\end{aligned}$$

for all $t \geq -n_k\omega-\theta$, $\theta \in [-\hat{\tau}, 0]$. Set $h_k(t,x,y) := |g_k(x, t+n_k\omega+\theta)-g_k(y, t+n_k\omega+\theta)|$. By the constant variation formula and the comparison argument, we obtain

$$\begin{aligned}
|\overline{v}_k(t+\theta,x_1) - \overline{v}_k(t+\theta,x_2)|^2 \leq\ & e^{-2m(t-s)}|\overline{v}_k(s+\theta,x_1) - \overline{v}_k(s+\theta,x_2)|^2 \\
&+ 4\eta\int_s^t e^{-2m(t-r)}h_k(r,x_1,x_2)dr, \qquad (5.13)
\end{aligned}$$

for all $t \geq s \geq -n_k\omega - \theta$. Letting $t = 0$ and $s = -n_k\omega - \theta$ in (5.13), we have

$$\begin{aligned}
|\overline{v}_k(\theta,x_1) - \overline{v}_k(\theta,x_2)|^2 \leq\ & e^{-2m(n_k\omega+\theta)}|\overline{v}_k(-n_k\omega,x_1) - \overline{v}_k(-n_k\omega,x_2)|^2 \\
&+ 4\eta\int_{-n_k\omega-\theta}^0 e^{2mr}h_k(r,x_1,x_2)dr,
\end{aligned}$$

and hence,

$$\begin{aligned}
|E_d(n_k\omega+\theta,x_1,\psi_k) - E_d(n_k\omega+\theta,x_2,\psi_k)|^2 \leq\ & e^{-2m(n_k\omega+\theta)}|\psi_{k1}(0,x_1) - \psi_{k1}(0,x_2)|^2 \\
&+ 4\eta\int_{-n_k\omega-\theta}^0 e^{2mr}h_k(r,x_1,x_2)dr, \quad (5.14)
\end{aligned}$$

for all $k \geq 1$, $x_1, x_2 \in \bar{\Omega}$. We further prove that for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|E_d(n_k\omega + \theta, x_1, \psi_k) - E_d(n_k\omega + \theta, x_2, \psi_k)| < \epsilon, \forall k \geq 1, \forall x_1, x_2 \in \bar{\Omega} \text{ with } |x_1 - x_2| < \delta.$$

Suppose, by contradiction, that there exist $\epsilon_0 > 0$, $k_j \to \infty$, $x_j, y_j \in \bar{\Omega}$ with $|x_j - y_j| < 1/j$ such that

$$|E_d(n_{k_j}\omega + \theta, x_j, \psi_{n_{k_j}}) - E_d(n_{k_j}\omega + \theta, y_j, \psi_{n_{k_j}})| \geq \epsilon_0, \forall j \geq 1.$$

It then follows from (5.14) that

$$\begin{aligned}
\epsilon_0^2 &\leq \limsup_{j \to \infty} |E_d(n_{k_j}\omega + \theta, x_j, \psi_{n_{k_j}}) - E_d(n_{k_j}\omega + \theta, y_j, \psi_{n_{k_j}})|^2 \\
&\leq 4\eta \limsup_{j \to \infty} \int_{-n_{k_j}\omega - \theta}^{0} e^{2mr} h_{n_{k_j}}(r, x_j, y_j) dr. \qquad (5.15)
\end{aligned}$$

For each $r \leq 0$, we can choose an integer $k_0 > 0$ such that $\{S_m(r + n_k\omega + \theta, x, \psi_k)\}_{k \geq k_0}$ is equicontinuous in $x \in \bar{\Omega}$ for all $k \geq k_0$. It is easy to see that for each $k \geq 1$, $g_k(x, t)$ is uniformly continuous in $(x, t) \in \bar{\Omega} \times \mathbb{R}_+$. Since $\lim_{j \to \infty} |S_m(r + n_{k_j}\omega + \theta, x_j, \psi_{n_{k_j}}) - S_m(r + n_{k_j}\omega + \theta, y_j, \psi_{n_{k_j}})| = 0$, for any given $r \leq 0$, we have $\lim_{j \to \infty} h_{n_{k_j}}(r, x_j, y_j) = 0$. By Fatou's lemma, (5.15) becomes

$$\epsilon_0^2 \leq 4\eta \int_{-\infty}^{0} e^{2mr} \limsup_{j \to \infty} h_{n_{k_j}}(r, x_j, y_j) dr = 0,$$

which is a contradiction. Similarly, we can verify that $\{E_w(n_k\omega + \theta, x, \psi_k)\}_{k \geq 1}$ is also equicontinuous in $x \in \bar{\Omega}$ for all $k \geq 1$. This shows that statement (i) holds true.

Next we prove statement (ii). Since $n_k \to \infty$, without loss of generality, we assume that $n_1\omega > \hat{\tau}$ and $n_k \geq n_1$, $\forall k \geq 1$. By the mean value theorem, we easily see that $E_d(n_k\omega + \theta, \cdot, \psi_k)$ and $E_w(n_k\omega + \theta, \cdot, \psi_k)$ are equicontinuous in $\theta \in [-\hat{\tau}, 0]$ on $C(\bar{\Omega}, \mathbb{R})$. Since $Q^{n_k}(\psi_k) = Q^{n_1}(Q^{n_k - n_1}(\psi_k)) = Q(n_1\omega)(Q^{n_k - n_1}(\psi_k))$, $\forall k \geq 1$, it follows that the sequence $\{(S_m)_{n_k\omega}(\psi_k)\}_{k \geq 1}$ is precompact in $C([-\hat{\tau}, 0], C(\bar{\Omega}, \mathbb{R}))$ (see, e.g., the proof of [151, Theorem 3.5.1]). This implies that the sequence of functions $\{S_m(n_k\omega + \theta, \cdot, \psi_k)\}_{k \geq 1}$ is equicontinuous in $\theta \in [-\hat{\tau}, 0]$ on $C(\bar{\Omega}, \mathbb{R})$. Consequently, the sequence $\{E_d(n_k\omega + \theta, \cdot, \psi_k), E_w(n_k\omega + \theta, \cdot, \psi_k), S_m(n_k\omega + \theta, \cdot, \psi_k)\}_{k \geq 1}$ is equicontinuous in $\theta \in [-\hat{\tau}, 0]$ on $C(\bar{\Omega}, \mathbb{R}^3)$. Thus, Q^n is asymptotically compact on B.

Now we consider the omega limit set of B for Q on E^+, defined by

$$\omega(B) = \{\psi \in E^+ : \lim_{k \to \infty} Q^{n_k}(\psi_k) = \psi \text{ for some sequence } \psi_k \in B \text{ and } n_k \to \infty\}.$$

Since Q^n is asymptotically compact on B, it follows that $\omega(B)$ is a nonempty, compact and invariant set for Q in E^+, and $\omega(B)$ attracts B (see, e.g., the proof of [103, Lemma 23.1(2)]). By Lemma 1.1.1(b), we have

$$\alpha(Q^n(B)) \leq \alpha(\omega(B)) + \delta(Q^n(B), \omega(B)) = \delta(Q^n(B), \omega(B)) \to 0 \text{ as } n \to \infty,$$

where $\delta(B, A) = \sup_{x\in B} d(x, A) = \sup_{x\in B} \inf_{y\in A} d(x, y)$ for any subsets A, B of Banach space. □

From Lemma 5.3.1, we see that Q is point dissipative on E^+, the positive orbits of bounded subsets for Q are bounded, and Q is α-contracting on E^+ by Lemma 5.3.6. It follows from Theorem 1.1.2(b) that Q has a global attractor that attracts each bounded set in E^+.

Lemma 5.3.7. *For any $v \gg 0$ in E^+ and any $\lambda \in (0, 1)$, we have $u_i(t, \cdot, \lambda v) > \lambda u_i(t, \cdot, v)$ for all $t > 0$, $i \in \{1, 2, 3\}$, and hence, $Q^n(\omega)(\lambda v) \gg \lambda Q^n(\omega)(v)$ in E^+ for all integers n with $n\omega > \hat{\tau}$.*

Proof. For any $v \gg 0$ in E^+ and any $\lambda \in (0, 1)$, let $u(t, \cdot, v)$ and $u(t, \cdot, \lambda v)$ be the solution of system (5.10) satisfying $u_0 = v$ and $u_0 = \lambda v$, respectively. Denote $\phi(t, \cdot) = \lambda u(t, \cdot, v)$ and $\varphi(t, \cdot) = u(t, \cdot, \lambda v)$. From Lemma 5.3.2, we have $\phi(t, \cdot) > 0$ and $\varphi(t, \cdot) > 0$ for all $t \geqslant 0$. Besides, $\phi_1(0, \cdot) = \lambda v_1 = \varphi_1(0, \cdot), \phi_2(\theta, \cdot) = \lambda v_2 = \varphi_2(\theta, \cdot), \phi_3(0, \cdot) = \lambda v_3 = \varphi_3(0, \cdot), \forall \theta \in [-\tau_w(0), 0]$.

Since $-\tau_w(0) = 0 - \tau_w(0) \leq t - \tau_w(t) \leq \bar{\tau} - \tau_w(\bar{\tau}) \leq \bar{\tau} - \bar{\tau} = 0$, $\forall t \in [0, \bar{\tau}]$, we have $\phi_2(t - \tau_w(t), \cdot) = \lambda v_2(t - \tau_w(t), \cdot) = \varphi_2(t - \tau_w(t), \cdot)$, and hence,

$$
\begin{aligned}
\frac{\partial \phi_1}{\partial t} &< \mu_b(t, x) \left(1 - \frac{\phi_1}{K_d}\right) \phi_3 - (k_d(t, x) + \mu_E(t, x))\, \phi_1, \\
\frac{\partial \phi_2}{\partial t} &< k_d(t, x)\phi_1 - (\eta_w(t, x) + \mu_E(t, x))\, \phi_2, \\
\frac{\partial \phi_3}{\partial t} &< D_m \Delta \phi_3 + (1 - \tau_w'(t)) \int_\Omega \Gamma_L(t, t - \tau_w(t), x, y) \eta_w (t - \tau_w(t), y) \\
&\quad \times \phi_2 (t - \tau_w(t), y)\, dy - \mu_m(t, x)\phi_3,
\end{aligned}
$$

subject to the Neumann boundary condition, for all $t \in [0, \bar{\tau}]$. By the comparison principle and $\phi(0) = \varphi(0)$, we obtain $\phi_i(t, \cdot) < \varphi_i(t, \cdot)$ for each $i = 1, 2, 3$ and $t \in [0, \bar{\tau}]$. Similarly, for any interval $(n\bar{\tau}, (n + 1)\bar{\tau}), n = 1, 2, 3, ...$, we have $\phi_i(t, \cdot) < \varphi_i(t, \cdot)$ for all $t > 0$ and each $i = 1, 2, 3$, namely, $u(t, \lambda v) \geqslant \lambda u(t, v)$, $\forall t > 0$. Thus, $u_t(\lambda v) \geqslant \lambda u_t(v)$, $\forall t > \tau_w(0)$. It follows that for all integers n satisfying $n\omega > \hat{\tau}$, there holds $Q^n(\omega)(\lambda v) \gg \lambda Q^n(\omega)(v)$ in E^+. □

It follows from Lemma 5.3.1 that system (5.10) generates an ω-periodic semiflow $Q(t) : E^+ \to E^+$. Note that $\hat{P}(t)$ is an ω-periodic semiflow of the linear system (5.11) on E. Now we are ready to prove a threshold type result on the global dynamics of system (5.10) in terms of $\mathscr{R}_m$.

Theorem 5.3.1. *The following statements are valid:*

(i) If $\mathscr{R}_m \leqslant 1$, then $(0, 0, 0)$ is globally asymptotically stable for system (5.10) in E^+.

(ii) If $\mathscr{R}_m > 1$, then system (5.10) admits a positive ω-periodic solution $(E_d^*(t,x), E_w^*(t,x), S_m^*(t,x))$, and it is globally asymptotically stable for system (5.10) in $E^+\backslash\{(0,0,0)\}$.

Proof. In the case where $\mathscr{R}_m < 1$, Lemmas 5.3.4 and 5.3.5 imply that $r(\hat{P}) < 1$. For any $\varphi \in E$, let $v(t,s,\varphi)$ be the unique solution of system (5.11) with $v_s(s,\varphi) = \varphi$, where $v_t(s,\varphi)(\theta,x) = v(t+\theta,s,x,\varphi)$, $\theta \in [-\hat{\tau},0]$. Note that $V(t,s)$, $t \geq s$, is the evolution family of system (5.11), and $V(t,s)\varphi = v_t(s,\varphi)$. Since $r(V(\omega,0)) = r(\hat{P}) < 1$, the exponential growth bound $\tilde{\omega}(V) < 0$, and hence, there exists $\gamma > 0$ such that $\tilde{\omega}(V) + \gamma < 0$. By the definition of $\tilde{\omega}(V)$, there exists $M_0 > 0$ such that

$$\|V(t+s,s)\varphi\|_E \leq M_0 e^{(\tilde{\omega}(V)+\gamma)t}\|\varphi\|_E, \forall t \geq 0, \forall s \in \mathbb{R}, \varphi \in E.$$

It then follows that $\|V(t+s,s)\varphi\|_E \to 0$ as $t \to \infty$, $\forall s \in \mathbb{R}$, and hence, $\|v_{t+s}(s,\cdot,\varphi)\|_E \to 0$ as $t \to \infty$, $\forall s \in \mathbb{R}$.

Note that the solution $(E_d(t,x,\phi), E_w(t,x,\phi), S_m(t,x,\phi))$ of system (5.10) satisfies

$$\begin{aligned}
&\tfrac{\partial E_d}{\partial t} \leq \mu_b(t,x)S_m - (k_d(t,x) + \mu_E(t,x))\,E_d,\\
&\tfrac{\partial E_w}{\partial t} \leq k_d(t,x)E_d - (\eta_w(t,x) + \mu_E(t,x))\,E_w,\\
&\tfrac{\partial S_m}{\partial t} \leq D_m\Delta S_m + (1-\tau_w'(t))\textstyle\int_\Omega \Gamma_L(t,t-\tau_w(t),x,y)\eta_w\,(t-\tau_w(t),y)\\
&\qquad\quad \times E_w\,(t-\tau_w(t),y)\,dy - \mu_m(t,x)S_m,\\
&\tfrac{\partial S_m}{\partial \nu} = 0, t > N_0\omega, x \in \partial\Omega.
\end{aligned}$$

For any given $\phi = (\phi_1,\phi_2,\phi_3) \in E^+$, the comparison principle implies that

$$(E_d(t,x,\phi), E_w(t,x,\phi), S_m(t,x,\phi)) \leq v(t,s,x,\varphi_0), \forall t \geq s > 0, x \in \bar{\Omega}, \varphi_0 = (\phi_1,\phi_2,\phi_3).$$

Then $\lim_{t\to\infty}(E_d(t,x,\phi), E_w(t,x,\phi), S_m(t,x,\phi)) = (0,0,0)$ uniformly for $x \in \bar{\Omega}$, $\forall \phi \in Z^+$, and hence, $(0,0,0)$ is globally attractive for system (5.10) in E^+. It follows from Lemma 1.3.1 that $(0,0,0)$ is locally stable. This shows that $(0,0,0)$ is globally asymptotically stable in E^+.

In the case where $\mathscr{R}_m \geqslant 1$, Lemmas 5.3.4 and 5.3.5 imply that $r(\hat{P}) \geqslant 1$. We fix an integer n_0 such that $n_0\omega > 2\hat{\tau}$. Clearly, $Q^{n_0}(\omega) = Q(n_0\omega)$. By Lemmas 5.3.3 and 5.3.7, $Q(n_0\omega)$ is a strongly monotone and strictly subhomogeneous map on E^+. Note that $Q(n_0\omega)(0) = 0$, and the Fréchet derivative $DQ(n_0\omega)(0) = \hat{P}(n_0\omega)$. It easily follows from Lemma 5.3.4 that $r(DQ(n_0\omega)(0))$ is the principal eigenvalue of $DQ(n_0\omega)(0)$. Further, Lemma 5.3.5 implies that $sign(\mathscr{R}_m - 1) = sign(r(DQ(n_0\omega)(0)) - 1)$. Applying Theorem 1.3.6 and Lemma 1.3.1 to the map $Q(n_0\omega)$, we have the following threshold type result:

(a) If $\mathscr{R}_m = 1$, then $(0,0,0)$ is globally asymptotically stable for system (5.10) in E^+;

(b) If $\mathscr{R}_m > 1$, then system (5.10) admits a positive $n_0\omega-$periodic solution

$$(E_d^*(t,x), E_w^*(t,x), S_m^*(t,x)),$$

and it is globally asymptotically stable for system (5.10) in $E^+\backslash\{(0,0,0)\}$.

Therefore, it suffices to show that $(E_d^*(t,x), E_w^*(t,x), S_m^*(t,x))$ is also ω-periodic in the above case (b). Let $\psi^* = u_0^*(\cdot) \in E$ with $u^*(t,\cdot) = (E_d^*(t,\cdot), E_w^*(t,\cdot), S_m^*(t,\cdot))$. Clearly, $Q(n_0\omega)\psi^* = \psi^*$. Since

$$Q(n_0\omega)\left(Q(\omega)\psi^*\right) = Q(\omega)\left(Q(n_0\omega)\psi^*\right) = Q(\omega)\left(\psi^*\right),$$

$Q(\omega)\left(\psi^*\right)$ is also a fixed point of $Q(n_0\omega)$. By the uniqueness of the positive fixed point of $Q(n_0\omega)$, it follows that $Q(\omega)\left(\psi^*\right) = \psi^*$, and hence, $(E_d^*(t,\cdot), E_w^*(t,\cdot), S_m^*(t,\cdot)) = u\left(t,\cdot,\psi^*\right)$ is an ω-periodic solution of system (5.10). □

5.4 The disease transmission

Let $\mathbb{H} := C(\bar{\Omega}, \mathbb{R}^4)$, $\mathbb{H}^+ := C(\bar{\Omega}, \mathbb{R}_+^4)$, and $C_\omega(\mathbb{R}, \mathbb{H})$ be the Banach space consisting of all ω-periodic and continuous functions from $\mathbb{R}$ to $\mathbb{H}$, where $\|\phi\|_{C_\omega(\mathbb{R},\mathbb{H})} := \max_{\theta\in[0,\omega]} \|\phi(\theta)\|_{\mathbb{H}}$. Let $H := C([-\hat{\tau}, 0], \mathbb{H})$ and $H^+ := C([-\hat{\tau}, 0], \mathbb{H}^+)$. Setting $A_h = I_h = E_m = I_m = 0$ in system (5.4), we have the following system:

$$\begin{aligned}
&\frac{\partial S_h}{\partial t} = D_h \Delta S_h + \Lambda_h - \mu_h S_h, \\
&\frac{\partial E_d}{\partial t} = \mu_b(t,x)\left(1 - \frac{E_d}{K_d}\right) S_m - \left(k_d(t,x) + \mu_E(t,x)\right) E_d, \\
&\frac{\partial E_w}{\partial t} = k_d(t,x) E_d - \left(\eta_w(t,x) + \mu_E(t,x)\right) E_w, \\
&\frac{\partial S_m}{\partial t} = D_m \Delta S_m + (1 - \tau_w'(t)) \int_\Omega \Gamma_L(t, t-\tau_w(t), x, y) \eta_w\left(t - \tau_w(t), y\right) \\
&\qquad \times E_w\left(t - \tau_w(t), y\right) dy - \mu_m(t,x) S_m, \\
&\frac{\partial S_h}{\partial \nu} = \frac{\partial S_m}{\partial \nu} = 0, t > 0, x \in \partial\Omega.
\end{aligned} \tag{5.16}$$

By Theorem 5.3.1 (ii), we see that when $\mathscr{R}_m > 1$, system (5.16) admits a globally asymptotically stable positive ω-periodic solution $(S_h^*, E_d^*(t,\cdot), E_w^*(t,\cdot), S_m^*(t,\cdot))$ in $\mathbb{H}^+$, where $S_h^* = \frac{\Lambda_h}{\mu_h}$.

Let $\mathbb{A} := C(\bar{\Omega}, \mathbb{R}^3)$, $\mathbb{A}^+ := C(\bar{\Omega}, \mathbb{R}_+^3)$ and $C_\omega(\mathbb{R}, \mathbb{A})$ be the Banach space consisting of all ω-periodic and continuous functions from $\mathbb{R}$ to $\mathbb{A}$, where $\|\phi\|_{C_\omega(\mathbb{R},\mathbb{A})} := \max_{\theta\in[0,\omega]} \|\phi(\theta)\|_{\mathbb{A}}$. Linearizing system (5.4) at the disease-free periodic solution

$$(S_h^*, 0, 0, E_d^*(t,\cdot), E_w^*(t,\cdot), S_m^*(t,\cdot), 0, 0),$$

we obtain the following linear system for the infectious compartments A_h, I_h and I_m:

$$\begin{aligned}
\frac{\partial A_h}{\partial t} &= D_h \Delta A_h + \rho \int_\Omega \Gamma_h(t, t-\tau_h, x, y)\beta_h(t-\tau_h, y) \times I_m(t-\tau_h, y) S_h^* dy - (q_1 + \mu_h) A_h, \\
\frac{\partial I_h}{\partial t} &= D_h \Delta I_h + (1-\rho) \int_\Omega \Gamma_h(t, t-\tau_h, x, y)\beta_h(t-\tau_h, y) \times I_m(t-\tau_h, y) S_h^* dy \\
&\quad - (\eta_h + q_2 + \mu_h) I_h, \\
\frac{\partial I_m}{\partial t} &= D_m \Delta I_m + (1-\tau_m'(t)) \int_\Omega \Gamma_m(t, t-\tau_m(t), x, y)\beta_m(t-\tau_m(t), y) \\
&\quad \times (A_h(t-\tau_m(t), y) + I_h(t-\tau_m(t), y)) S_m^*(t-\tau_m(t), y)\, dy - \mu_m(t,x) I_m, \\
\frac{\partial A_h}{\partial \nu} &= \frac{\partial I_h}{\partial \nu} = \frac{\partial I_m}{\partial \nu} = 0, t > 0, x \in \partial\Omega.
\end{aligned} \tag{5.17}$$

Let $A := C([-\hat{\tau}, 0], \mathbb{A})$ and $A^+ := C([-\hat{\tau}, 0], \mathbb{A}^+)$. Define the operator $F(t) : A \to \mathbb{A}$ by

$$F(t)\phi = (F_1(t)\phi_3, F_2(t)\phi_3, F_3(t)(\phi_1 + \phi_2)), \forall \phi = (\phi_1, \phi_2, \phi_3) \in A, t \in \mathbb{R},$$

where

$$\begin{aligned}
F_1(t)\phi_3 &= \rho \int_\Omega \Gamma_h(t, t-\tau_h, \cdot, y)\beta_h(t-\tau_h, y) \times \phi_3(-\tau_h, y) S_h^* dy, \\
F_2(t)\phi_3 &= (1-\rho) \int_\Omega \Gamma_h(t, t-\tau_h, \cdot, y)\beta_h(t-\tau_h, y) \times \phi_3(-\tau_h, y) S_h^* dy, \\
F_3(t)(\phi_1 + \phi_2) &= (1-\tau_m'(t)) \int_\Omega \Gamma_m(t, t-\tau_m(t), \cdot, y)\beta_m(t-\tau_m(t), y) \\
&\quad \times (\phi_1(-\tau_m(t), y) + \phi_2(-\tau_m(t), y)) S_m^*(-\tau_m(t), y)\, dy.
\end{aligned}$$

Let

$$-V(t)v := \mathbf{D}\Delta v - W(t)v,$$

where $\mathbf{D} = diag(D_h, D_h, D_m)$ and

$$-[W(t)](x) = \begin{pmatrix} -(q_1 + \mu_h) & 0 & 0 \\ 0 & -(\eta_h + q_2 + \mu_h) & 0 \\ 0 & 0 & -\mu_m(t,x) \end{pmatrix}, \forall x \in \bar{\Omega}.$$

Then system (5.17) can be written as

$$\frac{dv}{dt} = F(t)v_t - V(t)v, t \geq 0.$$

Let $T_i(t,s) : \mathbb{Y} \to \mathbb{Y}$ $(i = 2, 3, 8)$ be the evolution family associated with $\frac{\partial u}{\partial t} = D_h \Delta u - (q_1 + \mu_h)u$, $\frac{\partial u}{\partial t} = D_h \Delta u - (\eta_h + q_2 + \mu_h)u$ and $\frac{\partial u}{\partial t} = D_m \Delta u - \mu_m(t, \cdot)u$ subject to the Neumann boundary condition, respectively. Then $\Phi(t,s) = \mathrm{diag}(T_2(t,s), T_3(t,s), T_8(t,s))$,

$t \geq s$, is the evolution family on $\mathbb{A}$ associated with the following linear system:

$$\frac{dv}{dt} = -V(t)v.$$

As in Section 5.3, we can verify that $F(t)$ and $\Phi(t,s)$ satisfy the following assumptions:

(H1) For any $t \geqslant 0$, $F(t)$ is a positive operator on $\mathbb{A}$.

(H2) For any $t \geqslant s$, $\Phi(t,s)$ is a positive operator on $\mathbb{A}$, and $\tilde{\omega}(\Phi) < 0$.

Define the next generation operator on $C_\omega(\mathbb{R}, \mathbb{A})$ by

$$[Lv](t) := \int_0^{+\infty} \Phi(t, t-s)F(t-s)v(t-s+\cdot)ds, \ \forall t \in \mathbb{R}, \ v \in C_\omega(\mathbb{R}, \mathbb{A}).$$

Motivated by Section 1.5.1, we define the basic reproduction number as $\mathscr{R}_0 := r(L)$, where $r(L)$ is the spectral radius of L.

For any given $t \geq 0$, let $\widetilde{P}(t)$ be the solution map of system (5.17) on A; that is, $\widetilde{P}(t)\phi = v_t(\phi)$, where $v_t(\phi)(\theta,x) = v(t+\theta,x,\phi) = (v_1(t+\theta,x,\phi), v_2(t+\theta,x,\phi), v_3(t+\theta,x,\phi))$, $\forall(\theta,x) \in [-\hat{\tau},0] \times \bar{\Omega}$, and $v(t,x,\phi)$ is the unique solution of system (5.17) with $v(\theta,x,\phi) = \phi(\theta,x)$, $\forall(\theta,x) \in [-\hat{\tau},0] \times \bar{\Omega}$. Then $\widetilde{P} := \widetilde{P}(\omega)$ is the Poincaré map associated with system (5.17). Let $r(\widetilde{P})$ be the spectral radius of $\widetilde{P}$. In view of Theorem 1.5.1, we have the following observation.

Lemma 5.4.1. *$\mathscr{R}_0 - 1$ has the same sign as $r(\widetilde{P}(\omega)) - 1$.*

Let

$$\mathcal{A} := C([-\tau_h, 0], \mathbb{Y}) \times C([-\tau_h, 0], \mathbb{Y}) \times C([-\tau_m(0), 0], \mathbb{Y}),$$
$$\mathcal{A}^+ := C([-\tau_h, 0], \mathbb{Y}^+) \times C([-\tau_h, 0], \mathbb{Y}^+) \times C([-\tau_m(0), 0], \mathbb{Y}^+).$$

Then $(\mathcal{A}, \mathcal{A}^+)$ is an ordered Banach space. To study the global dynamic of system (5.4) in terms of $\mathscr{R}_0$, we first show that system (5.17) generates an (eventually) strongly monotone periodic semiflow on $\mathcal{A}$. Given a function $w : [-\tau_m(0), +\infty) \times [-\tau_m(0), +\infty) \times [-\tau_h, +\infty) \to \mathbb{A}$, we define $w_t \in \mathcal{A}$ by $w_t(\theta) = (w_1(t+\theta_1), w_2(t+\theta_2), w_3(t+\theta_3))$, $\forall \theta = (\theta_1,\theta_2,\theta_3) \in [-\tau_m(0),0] \times [-\tau_m(0),0] \times [-\tau_h,0]$, $\forall t \geq 0$. By the method of steps, we have the following observation.

Lemma 5.4.2. *For any $\psi \in \mathcal{A}^+$, system (5.17) has a unique nonnegative solution $w(t,\cdot,\psi)$ on $[0,+\infty)$ with $w_0 = \psi$.*

Remark 5.4.1. *By the uniqueness of solutions in Lemmas 5.2.1 and 5.4.2, it follows that for any $\varphi \in X^+$ and $\psi \in \mathcal{A}^+$ with $\varphi_1(\theta_1,\cdot) = \psi_1(\theta_1,\cdot)$, $\forall \theta_1 \in [-\tau_m(0),0]$, $\varphi_2(\theta_2,\cdot) = \psi_2(\theta_2,\cdot)$, $\forall \theta_2 \in [-\tau_m(0),0]$ and $\varphi_3(\theta_3,\cdot) = \psi_3(\theta_3,\cdot)$, $\forall \theta_3 \in [-\tau_h,0]$, there holds $v(t,\cdot,\varphi) = w(t,\cdot,\psi)$, $\forall t \geq 0$, where $v(t,\cdot,\varphi)$ and $w(t,\cdot,\psi)$ are solutions of system (5.17) satisfying $v_0 = \varphi$ and $w_0 = \psi$, respectively.*

For any given $t \geq 0$, let $P(t) : \mathcal{A} \to \mathcal{A}$ be the solution map of system (5.17); that is, $P(t)\phi = w_t(\phi)$, where $w_t(\phi)(\theta, x) = w(t+\theta, x, \phi) = (w_1(t+\theta_1, x, \phi), w_2(t+\theta_2, x, \phi), w_3(t+\theta_3, x, \phi))$, $\forall \theta_1 \in [-\tau_m(0), 0]$, $\theta_2 \in [-\tau_m(0), 0]$, $\theta_3 \in [-\tau_h, 0]$ and $x \in \bar{\Omega}$, and $w(t, x, \phi)$ is the unique solution of system (5.17) with $w(\theta, x, \phi) = \phi(\theta, x)$, $\forall (\theta, x) \in ([-\tau_m(0), 0], [-\tau_m(0), 0], [-\tau_h, 0])^T \times \bar{\Omega}$. Then $P := P(\omega)$ is the Poincaré map associated with system (5.17). Let $r(P)$ be the spectral radius of P. Now we prove that the periodic semiflow $P(t)$ is eventually strongly positive.

Lemma 5.4.3. *For any ψ in $\mathcal{A}^+$ with $\psi \not\equiv 0$, the solutions $w(t, \cdot, \psi)$ of system (5.17) with $w_0 = \psi$ satisfies $w_i(t, \cdot, \psi) > 0$ for all $t > 2\hat{\tau}$, $i = 1, 2, 3$, and hence, $P(t)\psi \gg 0$ in $\mathcal{A}^+$ for all $t > 3\hat{\tau}$.*

Proof. Using a simple comparison argument on each interval $[n\bar{\tau}, (n+1)\bar{\tau}]$, $n \in \mathbb{N}$, we can prove that $w_i(t, \cdot, \psi) \geq 0$ for all $t \geq 0$, $i = 1, 2, 3$.

Next we can choose a large number

$$K > \max\left\{(q_1 + \mu_h), (\eta_h + q_2 + \mu_h), \max_{t \in [0,\omega], x \in \bar{\Omega}} \mu_m(t, x)\right\},$$

such that for each $t \in \mathbb{R}$, $g_1(t, \cdot, w_1) := -(q_1 + \mu_h) w_1 + K w_1$ is increasing in w_1, and

$$g_2(t, \cdot, w_2) := -(\eta_h + q_2 + \mu_h) w_2 + K w_2$$

is increasing in w_2, and $g_3(t, \cdot, w_3) := -\mu_m(t, \cdot) w_3 + K w_3$ is increasing in w_3. It then follows that w_1, w_2 and w_3 satisfy the following system

$$\begin{aligned}
\frac{\partial w_1}{\partial t} &= D_h \Delta w_1 - K w_1 + g_1(t, x, w_1) + \rho \int_\Omega \Gamma_h(t, t-\tau_h, x, y) \beta_h(t-\tau_h, y) w_3(t-\tau_h, y) S_h^* dy, \\
\frac{\partial w_2}{\partial t} &= D_h \Delta w_2 - K w_2 + g_2(t, x, w_2) \\
&\quad + (1-\rho) \int_\Omega \Gamma_h(t, t-\tau_h, x, y) \beta_h(t-\tau_h, y) w_3(t-\tau_h, y) S_h^* dy, \\
\frac{\partial w_3}{\partial t} &= D_m \Delta w_3 - K w_3 + g_3(t, x, w_3) + \left(1 - \tau_m'(t)\right) \int_\Omega \Gamma_m(t, t-\tau_m(t), x, y) \beta_m(t-\tau_m(t), y) \\
&\quad \times (w_1(t-\tau_m(t), y) + w_2(t-\tau_m(t), y)) S_m^*(t-\tau_m(t), y) dy, \\
\frac{\partial w_1}{\partial \nu} &= \frac{\partial w_2}{\partial \nu} = \frac{\partial w_3}{\partial \nu} = 0, t > 0, x \in \partial\Omega.
\end{aligned}$$

Hence, for any given ψ in $\mathcal{A}^+$, we have

$$
\begin{aligned}
w_1(t,\cdot,\psi) &= \widetilde{T}_1(t,0)\psi_1(0) + \int_0^t \widetilde{T}_1(t,s)g_1(s,\cdot,w_1(s,\cdot))ds \\
&\quad + \int_0^t \widetilde{T}_1(t,s)\left[\rho\int_\Omega \Gamma_h(s,s-\tau_h,\cdot,y)\beta_h(s-\tau_h,y)w_3(s-\tau_h,y)S_h^* dy\right] ds, \\
w_2(t,\cdot,\psi) &= \widetilde{T}_2(t,0)\psi_2(0) + \int_0^t \widetilde{T}_2(t,s)g_2(s,\cdot,w_2(s,\cdot))ds \\
&\quad + \int_0^t \widetilde{T}_2(t,s)\left[(1-\rho)\int_\Omega \Gamma_h(s,s-\tau_h,\cdot,y)\beta_h(s-\tau_h,y)w_3(s-\tau_h,y)S_h^* dy\right] ds, \\
w_3(t,\cdot,\psi) &= \widetilde{T}_3(t,0)\psi_3(0) + \int_0^t \widetilde{T}_3(t,s)g_3(s,\cdot,w_3(s,\cdot))ds \\
&\quad + \int_0^t \widetilde{T}_3(t,s)\left[(1-\tau_m'(s))\int_\Omega \Gamma_m(s,s-\tau_m(s),\cdot,y)\beta_m\left(s-\tau_m(s),y\right)\right. \\
&\quad \left.\times\left(w_1\left(s-\tau_m(s),y\right)+w_2\left(s-\tau_m(s),y\right)\right)S_m^*\left(s-\tau_m(s),y\right)dy\right] ds,
\end{aligned}
\tag{5.18}
$$

where $\widetilde{T}_1(t,s), \widetilde{T}_2(t,s), \widetilde{T}_3(t,s) : \mathbb{Y} \to \mathbb{Y}$ are the evolution families associated with $\frac{\partial w_1}{\partial t} = D_h\Delta w_1 - Kw_1$, $\frac{\partial w_2}{\partial t} = D_h\Delta w_2 - Kw_2$, and $\frac{\partial w_3}{\partial t} = D_m\Delta w_3 - Kw_3$ subject to the Neumann boundary condition, respectively. Since $m_1(t) := t - \tau_m(t)$ is increasing in $t \in \mathbb{R}$, it is easily follows that $[-\tau_m(0), 0] \subset m_1([0,\hat{\tau}])$, and clearly, let $m_2(t) := t - \tau_h$, we also have $[-\tau_h, 0] \subset m_2([0,\hat{\tau}])$. Without loss of generality, we assume that $\psi_3 > 0$, then there exists $(\theta_3, x_0) \in [-\tau_h, 0] \times \Omega$ such that $w_3(\theta_3, x_0) > 0$. In view of the first two equations of system (5.18), we have $w_i(t,\cdot,\psi) > 0$ for all $t > \hat{\tau}$, $i = 1, 2$. Note that if $s > 2\hat{\tau}$, then $s - \tau_h > 2\hat{\tau} - \hat{\tau} = \hat{\tau}$. From the third equation of system (5.18), it follows that $w_3(t,\cdot,\psi) > 0$ for all $t > 2\hat{\tau}$. This shows that $w_i(t,\cdot,\psi) > 0$ for all $t > 2\hat{\tau}$, $i = 1, 2, 3$, and hence, the linear map $P(t)$ is strongly positive for all $t > 3\hat{\tau}$. □

We fix an integer n_0 such that $n_0\omega > 3\hat{\tau}$. By the proof of Lemma 5.4.3, we see that $P(\omega)^{n_0} = P(n_0\omega)$ is strongly positive. Further, by the arguments similar to those in [54, Lemma 2.6], one can prove that $P(\omega)^{n_0}$ is compact. By Theorem 1.3.2 (the Krein-Rutman Theorem), as applied to the linear operator $P(\omega)^{n_0}$, together with the fact that $r(P(\omega)^{n_0}) = (r(P(\omega)))^{n_0}$, we have $\lambda = r(P(\omega)) > 0$, where λ is a simple eigenvalue of $P(\omega)$ having a strongly positive eigenvector $\tilde{\psi} \in \mathrm{int}(\mathcal{A}^+)$. Therefore, the arguments similar to those in [73, Lemma 3.8] imply the following result.

Lemma 5.4.4. *Two Poincaré maps $\widetilde{P}(\omega) : A \to A$ and $P(\omega) : \mathcal{A} \to \mathcal{A}$ have the same spectral radius, that is, $r(\widetilde{P}(\omega)) = r(P(\omega))$, and hence, $\mathscr{R}_0 - 1$ has the same sign as $r(P(\omega)) - 1$.*

By Lemma 5.2.1, we can define the solution maps $Q(t)$ associated with system (5.4) on D by $Q(t)\phi = u_t(\phi)$, where $u_t(\phi)(\theta,x) = u(t+\theta,x,\phi)$, $t \geq 0$, $(\theta,x) \in [-\hat{\tau},0] \times \bar{\Omega}$, and $u(t,x,\phi)$ is the unique solution of system (5.4) with $u(\theta,x) = \phi(\theta,x)$,

$(\theta, x) \in [-\hat{\tau}, 0] \times \bar{\Omega}$. Then $Q := Q(\omega)$ is the Poincaré map associated with system (5.4). By the arguments similar to those in [63], we can prove the following result.

Lemma 5.4.5. *Q is α-contracting in the sense that $\lim_{n\to\infty} \alpha(Q^n(B)) = 0$ for any bounded set $B \subseteq D$.*

In view of Lemma 5.2.1, we see that Q is point dissipative on D, the positive orbits of bounded subsets for Q are bounded, and Q is α-contracting on D by Lemma 5.4.5. It follows from Theorem 1.1.2(b) that Q has a global attractor that attracts each bounded set in D.

Lemma 5.4.6. *Let $u(t, \cdot, \phi)$ be the solution of system (5.4) with $u_0 = \phi \in D$. Then the following three statements are valid:*

(i) If there exists some $t_0 \geq 0$ such that $u_i(t_0, \cdot, \phi) \not\equiv 0$ for some $i \in \{2, 3, 4, 5, 8\}$, then $u_i(t, x, \phi) > 0$ for all $t > t_0$, $x \in \bar{\Omega}$.

(ii) For any $\phi \in D$, we have $u_i(t, x, \phi) > 0$, $i = 1, 6$, $t > 0$, $x \in \bar{\Omega}$, and

$$\liminf_{t\to\infty} u_i(t, x, \phi) \geq \bar{\eta}, i = 1, 6,$$

uniformly for $x \in \bar{\Omega}$, where $\bar{\eta}$ is a ϕ-independent positive constant.

(iii) If there exists some $t_0 \geq 0$ such that $u_i(t_0, \cdot, \phi) \not\equiv 0$ for some $i \in \{2, 3, 8\}$, then $u_i(t, x, \phi) > 0$ for all $i = 2, 3, 8$ with $t > t_0 + 2\hat{\tau}$, $x \in \bar{\Omega}$.

Proof. Statements (i) and (ii) can be proved by the arguments as shown in [63], it remains to prove the statement (iii) holds true. If $A_h(t_0, \cdot, \phi) \not\equiv 0$ for some $t_0 \geq 0$, then $A_h(t, x, \phi)$ satisfies

$$\begin{aligned} \frac{\partial A_h}{\partial t} &\geqslant D_h \Delta A_h - (q_1 + \mu_h) A_h, \\ \frac{\partial A_h}{\partial \nu} &= 0, t > t_0, x \in \partial\Omega. \end{aligned}$$

By the arguments in case (i), we obtain that $A_h(t, x, \phi) > 0$ for $\forall t > t_0, x \in \bar{\Omega}$. Note that $I_m(t, x, \phi)$ satisfies

$$I_m(t, x, \phi) = T_6(t, 0)\phi_8(0, \cdot)(x) + \int_0^t [T_6(t, s)F_8(s, u(s))](x)ds, \forall t > 0, x \in \bar{\Omega},$$

where

$$\begin{aligned} F_8(s, u(s)) = (1 - \tau_m'(s)) \int_\Omega &\Gamma_m(s, s - \tau_m(s), \cdot, y)\beta_m(s - \tau_m(s), y) \\ &\times (A_h(-\tau_m(s), y) + I_h(-\tau_m(s), y)) S_m(-\tau_m(s), y)\, dy. \end{aligned}$$

It follows that $I_m(t,x,\phi)>0$ for all $t>t_0+\hat{\tau}, x\in\bar{\Omega}$, here we have used the strong positivity of $T_6(t,s)$ with $t>s$ and the positivity of S_m. By the expression of I_h in system (5.4) and the positivity of S_h and $I_m(t,x,\phi)>0$, we see that $I_h(t,x,\phi)>0$ for $\forall t>t_0+2\hat{\tau}, x\in\bar{\Omega}$. Similar to the above arguments, $I_h(t_0,\cdot,\phi)\not\equiv 0$ for some $t_0\geq 0$ implies that $I_m(t,x,\phi)>0$ for all $t>t_0+\hat{\tau}, x\in\bar{\Omega}$, and that $A_h(t,x,\phi)>0$ for all $t>t_0+2\hat{\tau}, x\in\bar{\Omega}$. Similarly, $I_m(t_0,\cdot,\phi)\not\equiv 0$ for some $t_0\geq 0$ implies that $A_h(t,x,\phi)>0$ for all $t>t_0+\hat{\tau}, x\in\bar{\Omega}$, and $I_h(t,x,\phi)>0$ for all $t>t_0+\hat{\tau}, x\in\bar{\Omega}$. □

Now we are ready to prove a threshold type result on the global dynamics of system (5.4) in terms of $\mathscr{R}_m$ and $\mathscr{R}_0$.

Theorem 5.4.1. *The following statements are valid:*

(i) If $\mathscr{R}_m\leqslant 1$, then $E_0=(S_h^*,0,0,0,0,0,0,0)$ is globally attractive for system (5.4) in D.

(ii) If $\mathscr{R}_m>1$ and $\mathscr{R}_0<1$, then the disease-free $\omega-$periodic solution $(S_h^*,0,0,E_d^*(t,x), E_w^*(t,x),S_m^*(t,x),0,0)$ is globally attractive for system (5.4) in $D\backslash\{E_0\}$.

(iii) If $\mathscr{R}_m>1$ and $\mathscr{R}_0>1$, then there exists $\hat{\eta}>0$ such that for any $\phi\in D$ with $\phi_2(0,\cdot)\not\equiv 0$ or $\phi_3(0,\cdot)\not\equiv 0$ or $\phi_8(0,\cdot)\not\equiv 0$, the solution $u(t,x,\phi)=(u_i(t,x,\phi))(1\leqslant i\leqslant 8)$ of system (5.4) satisfies

$$\liminf_{t\to\infty}\min_{x\in\bar{\Omega}} u_i(t,x,\phi)\geq\hat{\eta},(1\leqslant i\leqslant 8).$$

Proof. Since the proofs for case (i) and (ii) are similar, we only prove the latter.

(ii) In the case where $\mathscr{R}_m>1$ and $\mathscr{R}_0<1$, Lemmas 5.4.1 and 5.4.4 imply that $r(\widetilde{P}(\omega))<1$. Consider the following system with parameter $\varepsilon>0$:

$$\begin{aligned}
\frac{\partial A_h}{\partial t}&=D_h\Delta A_h+\rho\int_\Omega\Gamma_h(t,t-\tau_h,x,y)\beta_h(t-\tau_h,y)\\
&\quad\times I_m(t-\tau_h,y)(S_h^*+\varepsilon)dy-(q_1+\mu_h)A_h,\\
\frac{\partial I_h}{\partial t}&=D_h\Delta I_h+(1-\rho)\int_\Omega\Gamma_h(t,t-\tau_h,x,y)\beta_h(t-\tau_h,y)\\
&\quad\times I_m(t-\tau_h,y)(S_h^*+\varepsilon)dy-(\eta_h+q_2+\mu_h)I_h,\\
\frac{\partial I_m}{\partial t}&=D_m\Delta I_m+(1-\tau_m'(t))\int_\Omega\Gamma_m(t,t-\tau_m(t),x,y)\beta_m(t-\tau_m(t),y)\\
&\quad\times(A_h(t-\tau_m(t),y)+I_h(t-\tau_m(t),y))(S_m^*(t-\tau_m(t),y)+\varepsilon)dy-\mu_m(t,x)I_m,\\
\frac{\partial A_h}{\partial\nu}&=\frac{\partial I_h}{\partial\nu}=\frac{\partial I_m}{\partial\nu}=0,t>0,x\in\partial\Omega.
\end{aligned}\tag{5.19}$$

For any $\varphi\in A$, let $v^\varepsilon(t,s,\varphi)$ be the unique solution of system (5.19) with $v_s^\varepsilon(s,\varphi)=\varphi$, where $v_t^\varepsilon(s,\varphi)(\theta,x)=v^\varepsilon(t+\theta,s,x,\varphi)$, $\theta\in[-\hat{\tau},0]$. Let $V_\varepsilon(t,s)$, $t\geq s$, be the evolution family of system (5.19) on A, where $V_\varepsilon(t,s)\varphi = v_t^\varepsilon(s,\varphi)$. Since $\lim_{\varepsilon\to 0} r(V_\varepsilon(\omega,0)) = r(\widetilde{P}(\omega)) < 1$, we can fix a sufficiently small number $\varepsilon > 0$ such that $r(V_\varepsilon(\omega,0)) < 1$. It follows that the exponential growth bound $\tilde{\omega}(V_\varepsilon) < 0$, and hence, there exists $\gamma > 0$ such that $\tilde{\omega}(V_\varepsilon) + \gamma < 0$. By the definition of $\tilde{\omega}(V_\varepsilon)$, there exists $M_0 > 0$ such that

$$\|V_\varepsilon(t+s,s)\varphi\|_A \le M_0 e^{(\tilde{\omega}(V_\varepsilon)+\gamma)t}\|\varphi\|_A, \forall t \ge 0, \forall s \in \mathbb{R}, \varphi \in A.$$

Then $\|V_\varepsilon(t+s,s)\varphi\|_A \to 0$ as $t \to \infty$, $\forall s \in \mathbb{R}$, and hence, $\|v_{t+s}^\varepsilon(s,\cdot,\varphi)\|_A \to 0$ as $t \to \infty$, $\forall s \in \mathbb{R}$.

Since $\mathscr{R}_m > 1$, the global attractivity of $(S_h^*, S_m^*(t,\cdot))$ for system (5.16) and the comparison principle imply that there exists a sufficiently large integer $N_0 > 0$ such that $N_0\omega > \hat{\tau}$ and $S_h(t,x) \le S_h^* + \varepsilon$ and $S_m(t,x) \le S_m^*(t,x) + \varepsilon$, $\forall t \ge N_0\omega - \hat{\tau}$, $x \in \bar{\Omega}$. Clearly, the solution $(A_h(t,x,\phi), I_h(t,x,\phi), I_m(t,x,\phi))$ of system (5.4) satisfies

$$\begin{aligned}
\frac{\partial A_h}{\partial t} &\le D_h \Delta A_h + \rho \int_\Omega \Gamma_h(t,t-\tau_h,x,y)\beta_h(t-\tau_h,y) \\
&\quad \times I_m(t-\tau_h,y)(S_h^*+\varepsilon)dy - (q_1+\mu_h)\, A_h, \\
\frac{\partial I_h}{\partial t} &\le D_h \Delta I_h + (1-\rho) \int_\Omega \Gamma_h(t,t-\tau_h,x,y)\beta_h(t-\tau_h,y) \\
&\quad \times I_m(t-\tau_h,y)(S_h^*+\varepsilon)dy - (\eta_h+q_2+\mu_h)\, I_h, \\
\frac{\partial I_m}{\partial t} &\le D_m \Delta I_m + (1-\tau_m'(t)) \int_\Omega \Gamma_m(t,t-\tau_m(t),x,y)\beta_m\left(t-\tau_m(t),y\right) \\
&\quad \times \left(A_h\left(t-\tau_m(t),y\right) + I_h\left(t-\tau_m(t),y\right)\right)\left(S_m^*\left(t-\tau_m(t),y\right)+\varepsilon\right)dy - \mu_m(t,x) I_m, \\
\frac{\partial A_h}{\partial \nu} &= \frac{\partial I_h}{\partial \nu} = \frac{\partial I_m}{\partial \nu} = 0, t > N_0\omega, x \in \partial\Omega.
\end{aligned}$$

For any given $\phi = (\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6,\phi_7,\phi_8) \in D$, there exists $m_0 > 0$ such that

$$(A_h(t,x,\phi), I_h(t,x,\phi), I_m(t,x,\phi)) \le m_0 v^\varepsilon(t, N_0\omega, x, \varphi_0), \forall t \in [N_0\omega, N_0\omega+\hat{\tau}], x \in \bar{\Omega},$$

with $\varphi_0 = (\phi_2,\phi_3,\phi_8)$. By the comparison principle, we have

$$(A_h(t,x,\phi), I_h(t,x,\phi), I_m(t,x,\phi)) \le m_0 v^\varepsilon(t, N_0\omega+\hat{\tau}, x, \varphi_0), \forall t \ge N_0\omega+\hat{\tau}, x \in \bar{\Omega}.$$

Then $\lim_{t\to\infty}(A_h(t,x,\phi), I_h(t,x,\phi), I_m(t,x,\phi)) = (0,0,0)$ uniformly for $x \in \bar{\Omega}$, $\forall \phi \in D$. By Lemma 5.2.1, there exists a constant C such that

$$\begin{aligned}
\frac{\partial E_m}{\partial t} &= D_m \Delta E_m + \beta_m(t,x)\left(A_h+I_h\right) S_m \\
&\quad - \tau_m(t),x,y)\beta_m\left(t-\tau_m(t),y\right) \\
&\quad \times \left(A_h\left(t-\tau_m(t),y\right) + I_h\left(t-\tau_m(t),y\right)\right) S_m\left(t-\tau_m(t),y\right) dy - \mu_m(t,x) E_m \\
&\leqslant D_m \Delta E_m + \beta_m(t,x)\left(A_h+I_h\right) C - \mu_m(t,x) E_m.
\end{aligned}$$

It then follows from $E_m(t,\cdot) \geqslant 0$ that $\lim_{t\to\infty} E_m(t,x,\phi) = 0$ uniformly for $x \in \bar{\Omega}$. Thus, the S_h, E_d, E_w and S_m equations in system (5.4) are asymptotic to system

(5.16). Now we use the theory of internal chain transitive sets (see Section 1.1) to prove that

$$\lim_{t\to\infty} \|(S_h(t,x,\phi), A_h(t,x,\phi), I_h(t,x,\phi), E_d(t,x,\phi), E_w(t,x,\phi), S_m(t,x,\phi),$$
$$E_m(t,x,\phi), I_m(t,x,\phi)) - (S_h^*, 0, 0, E_d^*(t,x), E_w^*(t,x), S_m^*(t,x), 0, 0)\| = 0$$

uniformly for $x \in \bar{\Omega}$.

For any $\varphi \in A^+$, let $\nu(t,x,\varphi(0,\cdot))$ be the solution of system (5.17) with $\nu(0,x) = \varphi(0,x)$. Define a solution semiflow of system (5.17) on A^+ by

$$\nu_t(\theta,x,\varphi) = \begin{cases} \nu(t+\theta, x, \varphi(0,x)), & \text{if } t+\theta > 0, t > 0, \theta \in [-\hat{\tau}, 0], \\ \varphi(t+\theta, x), & \text{if } t+\theta \le 0, t > 0, \theta \in [-\hat{\tau}, 0]. \end{cases}$$

Let $\bar{P}(\varphi) = \nu_\omega(\varphi)$. For convenience, we rewrite the solution map $Q(t)$ for system (5.4) as $\hat{Q}(t)$ in the following way:

$$\left[\hat{Q}(t)\phi\right](\theta) = (S_h(t+\theta,\cdot,\phi), A_h(t+\theta,\cdot,\phi), I_h(t+\theta,\cdot,\phi), E_d(t+\theta,\cdot,\phi),$$
$$E_w(t+\theta,\cdot,\phi), S_m(t+\theta,\cdot,\phi), E_m(t+\theta,\cdot,\phi), I_m(t+\theta,\cdot,\phi)),$$

for any $\theta \in [-\hat{\tau}, 0]$, $t \ge 0$. Since $\mathscr{R}_m > 1$, Theorem 5.3.1 (ii) implies that

$$\lim_{n\to\infty} \left(\left(\hat{Q}^n\phi\right)_1, \left(\hat{Q}^n\phi\right)_4, \left(\hat{Q}^n\phi\right)_5, \left(\hat{Q}^n\phi\right)_6 \right) = (S_h^*, E_d^*(0), E_{w0}^*, S_{m0}^*),$$

where $(S_h^*, E_d^*(0), E_{w0}^*, S_{m0}^*) \in H$ and $E_d^*(0) = E_d^*(0,\cdot), E_{w0}^*(\theta,\cdot) = E_w^*(\theta,\cdot), S_{m0}^*(\theta,\cdot) = S_m^*(\theta,\cdot)$, $\theta \in [-\hat{\tau}, 0]$. Let $\mathcal{W} = \omega(\phi)$ be the omega limit set of $\phi \in D$ for the Poincaré map $\hat{Q}$. Since $\lim_{t\to\infty}(A_h(t,x,\phi), I_h(t,x,\phi), E_m(t,x,\phi), I_m(t,x,\phi)) = (0,0,0,0)$ uniformly for $x \in \bar{\Omega}$, there holds $\mathcal{W} = \bar{\omega} \times \{(\hat{0},\hat{0},\hat{0},\hat{0})\}$, where $\hat{0}(\theta,\cdot) = 0, \forall\theta \in [-\hat{\tau}, 0]$.

By the proof of Lemma 5.4.5, it follows that the discrete forward orbit $\gamma^+(\phi) = \{\hat{Q}^{n\omega}(\phi) : n \ge 0\}$ is asymptotically compact. Thus, its omega limit set $\omega(\phi)$ is nonempty, compact and invariant for $\hat{Q}$. It then follows from Lemma 1.1.3 that $\mathcal{W}$ is an internally chain transitive set for $\hat{Q}$, and hence, $\bar{\omega}$ is an internally chain transitive set for $\bar{P}$. Since $\bar{\omega} \ne \{(\hat{0},\hat{0},\hat{0},\hat{0})\}$ due to Lemma 5.4.6, and $(S_h^*, E_d^*(0), E_{w0}^*, S_{m0}^*)$ is globally attractive in H^+, we have $\bar{\omega} \bigcap W^S((S_h^*, E_d^*(0), E_{w0}^*, S_{m0}^*)) \ne \emptyset$, where $W^S((S_h^*, E_d^*(0), E_{w0}^*, S_{m0}^*))$ is the stable set of $(S_h^*, E_d^*(0), E_{w0}^*, S_{m0}^*)$. By Theorem 1.1.1, we get $\bar{\omega} = \{(S_h^*, E_d^*(0), E_{w0}^*, S_{m0}^*)\}$. Thus, $\mathcal{W} = \{(S_h^*, E_d^*(0), E_{w0}^*, S_{m0}^*, \hat{0},\hat{0},\hat{0},\hat{0})\}$. This implies that

$$\lim_{t\to\infty} \|(S_h(t,\cdot,\phi), A_h(t,\cdot,\phi), I_h(t,\cdot,\phi), E_d(t,\cdot,\phi), E_w(t,\cdot,\phi), S_m(t,\cdot,\phi),$$
$$E_m(t,\cdot,\phi), I_m(t,\cdot,\phi)) - (S_h^*(\cdot), 0, 0, E_d^*(t,\cdot), E_w^*(t,\cdot), S_m^*(t,\cdot), 0, 0)\| = 0.$$

(iii) In the case where $\mathscr{R}_m > 1$ and $\mathscr{R}_0 > 1$, we have $r(\hat{P}) > 1$ and $r(\tilde{P}) > 1$, and hence, $\tilde{\omega}(\hat{P}) > 0$ and $\tilde{\omega}(\tilde{P}) > 0$. Let

$$Z_0 = \{\psi = (\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6,\psi_7,\psi_8) \in D : \psi_2(0,\cdot) \not\equiv 0, \psi_3(0,\cdot) \not\equiv 0 \text{ and } \psi_8(0,\cdot) \not\equiv 0\},$$

and

$$\partial Z_0 := D \setminus Z_0 = \{\psi \in D : \psi_2(0,\cdot) \equiv 0 \text{ or } \psi_3(0,\cdot) \equiv 0 \text{ or } \psi_8(0,\cdot) \equiv 0\}.$$

For any $\psi \in Z_0$, it then follows from Lemma 5.4.6 that $A_h(t,x,\psi) > 0$, $I_h(t,x,\psi) > 0$ and $I_m(t,x,\psi) > 0$, $t \geq 0$, $x \in \bar{\Omega}$. This implies that $Q^n(Z_0) \subseteq Z_0$, $\forall n \in \mathbb{N}$. Now we prove that Q is uniformly persistent with respect to $(Z_0, \partial Z_0)$.

Let $M_1 = (S_h^*, 0, 0, 0, 0, 0, 0, 0)$ and $M_2 = (S_h^*, 0, 0, E_d^*(0), E_{w0}^*, S_{m0}^*, 0, 0)$, where $E_{w0}^*(\theta) = E_w^*(\theta)$, $S_{m0}^*(\theta) = S_m^*(\theta)$, $\forall \theta \in [-\hat{\tau}, 0]$. For each $i = 1, 2$, since $\lim_{\psi \to M_i} \|Q(t)\psi - Q(t)M_i\| = 0$ uniformly for $t \in [0, \omega]$, for any given $\delta_i > 0$, there exists a $\delta_{0i} > 0$ such that for any $\psi \in Z_0$ with $\|\psi - M_i\| < \delta_{0i}$, we have $\|Q(t)\psi - Q(t)M_i\| < \delta_i$ for all $t \in [0, \omega]$.

Claim 1. For each $i = 1, 2$, $\limsup_{n\to\infty} \|Q(n\omega)\psi - M_i\| \geq \delta_{0i}$ for all $\psi \in Z_0$.

We just prove this claim for $i = 2$, and the other case can be proved in a similar way. Suppose, by contradiction, that there exists $\psi_0 \in Z_0$ such that $\limsup_{n\to\infty} \|Q(n\omega)\psi_0 - M_2\| < \delta_{02}$. Then there exists an integer $n_0 \geq 1$ such that $\|Q(n\omega)\psi_0 - M_2\| < \delta_{02}$ for all $n \geq n_0$. For any $t \geq n_0\omega$, we have $t = t' + n\omega$ with $n \geq n_0$, $t' \in [0, \omega)$, and

$$\|Q(t)\psi_0 - Q(t)M_2\| = \|Q(t')Q(n\omega)\psi_0 - Q(t')M_2\| < \delta_2.$$

Therefore, $S_h(t,x,\psi_0) > S_h^*(x) - \delta_2$ and $S_m(t,x,\psi_0) > S_m^*(t,x) - \delta_2$ for all $t \geq n_0\omega$ and $x \in \bar{\Omega}$. Let $\tilde{P}_{\delta_2} : A \to A$ be the Poincaré map of the following perturbed linear system:

$$\begin{aligned}
\frac{\partial A_h}{\partial t} &= D_h \Delta A_h + \rho \int_\Omega \Gamma_h(t, t-\tau_h, x, y)\beta_h(t-\tau_h, y) \\
&\quad \times I_m(t-\tau_h, y)(S_h^*(t-\tau_h, y) - \delta_2)dy - (q_1 + \mu_h) A_h, \\
\frac{\partial I_h}{\partial t} &= D_h \Delta I_h + (1-\rho) \int_\Omega \Gamma_h(t, t-\tau_h, x, y)\beta_h(t-\tau_h, y) \\
&\quad \times I_m(t-\tau_h, y)(S_h^*(t-\tau_h, y) - \delta_2)dy - (\eta_h + q_2 + \mu_h) I_h, \\
\frac{\partial I_m}{\partial t} &= D_m \Delta I_m + (1 - \tau_m'(t)) \int_\Omega \Gamma_m(t, t-\tau_m(t), x, y)\beta_m(t-\tau_m(t), y) \\
&\quad \times (A_h(t-\tau_m(t), y) + I_h(t-\tau_m(t), y))(S_m^*(t-\tau_m(t), y) - \delta_2)dy - \mu_m(t,x) I_m, \\
\frac{\partial A_h}{\partial \nu} &= \frac{\partial I_h}{\partial \nu} = \frac{\partial I_m}{\partial \nu} = 0, t > 0, x \in \partial\Omega.
\end{aligned} \tag{5.20}$$

Since $\lim_{\delta_2 \to 0^+} r(\tilde{P}_{\delta_2}) = r(\tilde{P}) > 1$, we can fix a sufficiently small $\delta_2 > 0$ such that

$$\delta_2 < \min\{\min_{x\in\bar{\Omega}} S_h^*(x), \min_{t\in[0,\omega], x\in\bar{\Omega}} S_m^*(t,x)\} \text{ and } r(\tilde{P}_{\delta_2}) > 1.$$

Thus, $A_h(t,x,\psi_0)$, $I_h(t,x,\psi_0)$, and $I_m(t,x,\psi_0)$ in system (5.4) satisfy

$$\begin{aligned}
\frac{\partial A_h}{\partial t} &\geq D_h\Delta A_h + \rho\int_\Omega \Gamma_h(t,t-\tau_h,x,y)\beta_h(t-\tau_h,y)\\
&\quad\times I_m(t-\tau_h,y)(S_h^*(t-\tau_h,y)-\delta_2)dy-(q_1+\mu_h)A_h,\\
\frac{\partial I_h}{\partial t} &\geq D_h\Delta I_h + (1-\rho)\int_\Omega \Gamma_h(t,t-\tau_h,x,y)\beta_h(t-\tau_h,y)\\
&\quad\times I_m(t-\tau_h,y)(S_h^*(t-\tau_h,y)-\delta_2)dy-(\eta_h+q_2+\mu_h)I_h,\\
\frac{\partial I_m}{\partial t} &\geq D_m\Delta I_m + (1-\tau_m'(t))\int_\Omega \Gamma_m(t,t-\tau_m(t),x,y)\beta_m(t-\tau_m(t),y)\\
&\quad\times(A_h(t-\tau_m(t),y)+I_h(t-\tau_m(t),y))(S_m^*(t-\tau_m(t),y)-\delta_2)dy-\mu_m(t,x)I_m,\\
\frac{\partial A_h}{\partial \nu} &= \frac{\partial I_h}{\partial \nu}=\frac{\partial I_m}{\partial \nu}=0, t>0, x\in\partial\Omega.
\end{aligned} \tag{5.21}$$

Repeating the arguments in Lemma 5.4.4, we can obtain that $\tilde{P}_{\delta_2}$ possesses the principal eigenvalue with strongly positive vector on A. By the arguments similar to those for [9, Lemma 5], it follows that there exists a positive ω-periodic function $v_{\delta_2}^*(t,x)$ such that $e^{\mu_{\delta_2}t}v_{\delta_2}^*(t,x)$ is a solution of system (5.21), where $\mu_{\delta_2}=\frac{\ln r(\tilde{P}_{\delta_2})}{\omega}$.

Since $\psi_0\in Z_0$, $A_h(t,x,\psi_0)>0$, $I_h(t,x,\psi_0)>0$ and $I_m(t,x,\psi_0)>0$ for all $t\geq 0$ and $x\in\bar{\Omega}$, and hence, there exists a $\kappa>0$ such that

$$(A_h(t,x,\psi_0),I_h(t,x,\psi_0),I_m(t,x,\psi_0))\geq \kappa e^{\mu_{\delta_2}t}v_{\delta_2}^*(t,x), \forall t\in[n_0\omega-\hat{\tau},n_0\omega], x\in\bar{\Omega}.$$

By the comparison theorem, we have

$$(A_h(t,x,\psi_0),I_h(t,x,\psi_0),I_m(t,x,\psi_0))\geq \kappa e^{\mu_{\delta_2}t}v_{\delta_2}^*(t,x), \forall t\geq n_0\omega, x\in\bar{\Omega}.$$

Since $\mu_{\delta_2}=\tilde{\omega}(\tilde{P})>0$, it follows that $A_h(t,\cdot,\psi_0)\to\infty$, $I_h(t,\cdot,\psi_0)\to\infty$ and $I_m(t,\cdot,\psi_0)\to\infty$ as $t\to\infty$, a contradiction.

This claim implies that M_i $(i=1,2)$, are two isolated invariant sets for Q in D, and $W^S(M_i)\bigcap Z_0=\emptyset$, where $W^S(M_i)$ is the stable set of M_i for Q. Let

$$M_\partial := \{\psi\in\partial Z_0 : Q^n(\psi)\in\partial Z_0, \forall n\in\mathbb{N}\},$$

and $\omega(\psi)$ be the omega limit set of the forward orbit $\gamma^+(\psi)=\{Q^n(\psi):\forall n\in\mathbb{N}\}$.
Claim 2. $\omega(\psi)=M_1\cup M_2$ for any $\psi\in M_\partial$, and M_2 is globally stable for Q in $M_\partial\backslash J$, where

$$J=\{\psi=(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6,\psi_7,\psi_8)\in D:\psi_i\equiv 0, \forall i=1,2,3,4,5,6,7,8\}.$$

By the definition of M_∂, we obtain that either $A_h(n\omega,\cdot,\psi)\equiv 0$, $I_h(n\omega,\cdot,\psi)\equiv 0$ or $I_m(n\omega,\cdot,\psi)\equiv 0$ for $n\in\mathbb{N}$. It follows from Lemma 5.4.6 that for each $t\geqslant 0$, either $A_h(t,\cdot,\psi) \equiv 0$, $I_h(t,\cdot,\psi) \equiv 0$ or $I_m(t,\cdot,\psi) \equiv 0$. By the contraction argument with Lemma 5.4.6, we suppose that $A_h(t,\cdot,\psi) \equiv 0$, $I_h(t,\cdot,\psi) \equiv 0$ and $I_m(t,\cdot,\psi) \equiv 0$ for all $\psi \in M_\partial$ and $t \geqslant 0$. Consider the system (5.4) and the integral form (5.6) for $E_m(t,\cdot,\psi)$. Clearly, we obtain $E_m(t,\cdot,\psi) \equiv 0$ for all $t > 0$, and hence, $(S_h, E_d, E_w, S_m)(t,x,\psi)$ satisfies system (5.16). Since $\mathscr{R}_m > 1$, it follows from Theorem 5.3.1 (ii) that either $\lim_{t\to\infty} (S_h, E_d, E_w, S_m)(t,x,\psi) = (S_h^*, 0, 0, 0)$ or

$$\lim_{t\to\infty} \left((S_h, E_d, E_w, S_m)(t,x,\psi) - (S_h^*, E_d^*(t,x), E_w^*(t,x), S_m^*(t,x))\right) = (0,0,0,0)$$

uniformly for $x \in \overline{\Omega}$. Thus, $\omega(\psi) = M_1 \cup M_2$ for any $\psi \in M_\partial$. This implies that M_2 is globally attractive for Q in $M_\partial \backslash J$. Since system (5.16) is cooperative, M_2 is locally Lyapunov stable for Q in $M_\partial \backslash J$ due to Lemma 1.3.1. This proves Claim 2.

By Claim 2, we see that M_1 and M_2 cannot form a cycle for Q in ∂Z_0, and so is M_2. Also, it follows from Theorem 5.3.1 that M_1 cannot form a cycle for Q in ∂Z_0. That is, there is no subset of $\{M_1, M_2\}$ forms a cycle for Q in ∂Z_0. Since Q admits a global attractor on X^+, it follows from the acyclicity theorem on uniform persistence for maps (see., e.g., Theorem 1.2.1) that $Q : D \to D$ is uniformly persistent with respect to $(Z_0, \partial Z_0)$ in the sense that there exists an $\tilde{\eta} > 0$ such that

$$\liminf_{n\to\infty} d(Q^n(\psi), \partial Z_0) \geq \tilde{\eta}, \ \forall \psi \in Z_0.$$

Next we prove the practical uniform persistence. By Theorem 1.2.2, we obtain that $Q : Z_0 \to Z_0$ admits a global attractor A_0. Since $A_0 = Q(\omega)A_0 = Q(A_0)$, we have $\psi_2(0,\cdot) > 0$, $\psi_3(0,\cdot) > 0$ and $\psi_8(0,\cdot) > 0$ for all $\psi \in A_0$. Let $B_0 := \bigcup_{t\in[0,\omega]} Q(t)A_0$. Then $B_0 \subseteq Z_0$ and $\lim_{t\to\infty} d(Q(t)\psi, B_0) = 0$ for all $\psi \in Z_0$. Define a continuous function $p : D \to \mathbb{R}_+$ by

$$p(\psi) = \min\{\min_{x\in\bar{\Omega}} \psi_2(0,x), \min_{x\in\bar{\Omega}} \psi_3(0,x), \min_{x\in\bar{\Omega}} \psi_8(0,x)\},$$

for any $\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8) \in D$. Clearly, $p(\psi) > 0$ for all $\psi \in B_0$. Since B_0 is a compact subset of Z_0, we have $\inf_{\psi\in B_0} p(\psi) = \min_{\psi\in B_0} p(\psi) > 0$. By the attractiveness of B_0, it follows that there exists an $\check{\eta} > 0$ such that

$$\liminf_{t\to\infty} \min(\min_{x\in\bar{\Omega}} A_h(t,x,\psi), \min_{x\in\bar{\Omega}} I_h(t,x,\psi), \min_{x\in\bar{\Omega}} I_m(t,x,\psi)) \geq \check{\eta}, \forall \psi \in Z_0.$$

In view of Lemma 5.4.6 (ii), we further obtain

$$\liminf_{t\to\infty} u_i(t,\cdot,\psi) \geqslant \min\{\bar{\eta}, \check{\eta}\}, i = 1,2,3,6,8. \tag{5.22}$$

We can easily see from (5.22) that u_4, u_5 and u_7 are also uniformly persistent. Thus, there is $\hat{\eta} \in (0, \min\{\bar{\eta}, \check{\eta}\})$, such that

$$\liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u_i(t,x,\psi) \geqslant \hat{\eta}, (1 \leqslant i \leqslant 8).$$

For any given $\psi \in D$ with $\psi_2(0,\cdot) \not\equiv 0$ or $\psi_3(0,\cdot) \not\equiv 0$ or $\psi_8(0,\cdot) \not\equiv 0$, it follows from Lemma 5.4.6 (iii) that there exists an integer $n_0 = n_0(\psi) \geqslant 0$ such that $Q^{n_0}(\omega)\psi \in Z_0$. Since $Q(t)\psi = Q(t - n_0\omega)\left(Q^{n_0}(\omega)\psi\right), \forall t \geqslant n_0\omega$, we have $\liminf_{t\to\infty} \min_{x\in\overline{\Omega}} u_i(t,x,\psi) \geqslant \hat{\eta}, (1 \leqslant i \leqslant 8)$. □

Remark 5.4.2. *As argued for E_m, we have the corresponding expressions for E_h and L_m. Together with the expressions of P_h and R_h in (5.1), we can lift the threshold type result on system (5.4) to system (5.1).*

In the rest of this section, we consider the case where all coefficients in system (5.4) are positive constants, the infectious people are not classified as symptomatic or asymptomatic (i.e., $\rho = 0$), and mosquitoes have symptoms directly after being infected (i.e., remove E_m). Thus, system (5.4) is reduced to the following autonomous reaction-diffusion system:

$$\begin{cases} \frac{\partial S_h}{\partial t} = D_h \Delta S_h + \Lambda_h - \beta_h I_m S_h - \mu_h S_h, \\ \frac{\partial I_h}{\partial t} = D_h \Delta I_h + e^{-\mu_h \tau_h} \int_\Omega \Gamma(D_h\tau_h, x, y)\beta_h \\ \qquad \times I_m(t-\tau_h, y) S_h(t-\tau_h, y) dy - (\eta_h + q + \mu_h) I_h, \\ \frac{\partial E_d}{\partial t} = \mu_b \left(1 - \frac{E_d}{K_d}\right)(S_m + I_m) - (k_d + \mu_E) E_d, \\ \frac{\partial E_w}{\partial t} = k_d E_d - (\eta_w + \mu_E) E_w, \\ \frac{\partial S_m}{\partial t} = D_m \Delta S_m + e^{-\mu_L \tau_w} \int_\Omega \Gamma(D_m\tau_w, x, y)\eta_w \\ \qquad \times E_w(t-\tau_w, y) dy - \beta_m I_h S_m - \mu_m S_m, \\ \frac{\partial I_m}{\partial t} = D_m \Delta I_m + e^{-\mu_m \tau_m} \int_\Omega \Gamma(D_m\tau_m, x, y)\beta_m \\ \qquad \times I_h(t-\tau_m, y) S_m(t-\tau_m, y) dy - \mu_m I_m, \\ \frac{\partial S_h}{\partial \nu} = \frac{\partial I_h}{\partial \nu} = \frac{\partial S_m}{\partial \nu} = \frac{\partial I_m}{\partial \nu} = 0, t > 0, x \in \partial\Omega, \end{cases} \tag{5.23}$$

where $\Gamma(t,x,y)$ is the Green function associated with $\frac{\partial u}{\partial t} = \Delta u$ subject to the Neumann boundary condition.

Let

$$\tilde{F} = \begin{bmatrix} 0 & 0 & \mu_b \\ k_d & 0 & 0 \\ 0 & \eta_w e^{-\mu_L\tau_w} & 0 \end{bmatrix}, \quad \tilde{V} = \begin{bmatrix} k_d + \mu_E & 0 & 0 \\ 0 & \eta_w + \mu_E & 0 \\ 0 & 0 & \mu_m \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & \beta_h S_h^* e^{-\mu_h\tau_h} \\ \beta_m S_m^* e^{-\mu_m\tau_m} & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \eta_h + q + \mu_h & 0 \\ 0 & \mu_m \end{bmatrix},$$

where $S_h^* = \frac{\Lambda_h}{\mu_h}, S_m^* = \frac{K_d(k_d\eta_w e^{-\mu_L\tau_w})}{(\eta_w+\mu_E)\mu_m - \frac{k_d+\mu_E}{\mu_b}}$. By the arguments in Section 1.5.1 (also, see, [125, Theorem 3.4] and [150, Corollary 2.1]), it follows that $\mathscr{R}_m$ and $\mathscr{R}_0$ are the spectral radii of two matrixes $\tilde{F}\tilde{V}^{-1}$ and FV^{-1}, respectively, and hence,

$$\mathscr{R}_m = \frac{\mu_b k_d \eta_w e^{-\mu_L\tau_w}}{(k_d + \mu_E)(\eta_w + \mu_E)\mu_m}, \quad \mathscr{R}_0 = \sqrt{\frac{\beta_h S_h^* e^{-\mu_h\tau_h} \beta_m S_m^* e^{-\mu_m\tau_m}}{\mu_m(\eta_h + q + \mu_h)}}.$$

Clearly, $S_m^* = K_d(\mathscr{R}_m - 1)(\frac{k_d+\mu_E}{\mu_b})$.

It is easy to verify that when $\mathscr{R}_m > 1$ and $\mathscr{R}_0 < 1$, system (5.23) has a unique disease-free equilibrium $(S_h^*, 0, E_d^*, E_w^*, S_m^*, 0)$ with $E_d^* = \frac{K_d(\mathscr{R}_m-1)}{\mathscr{R}_m}$ and $E_w^* = \frac{E_d^* k_d}{\eta_w+\mu_E}$; and when $\mathscr{R}_m > 1$ and $\mathscr{R}_0 > 1$, system (5.23) has a unique positive equilibrium $u^* = (S_{h*}, I_{h*}, E_{d*}, E_{w*}, S_{m*}, I_{m*})$ with

$$
\begin{aligned}
S_{h*} &= \frac{\Lambda_h}{\beta_h I_{m*} + \mu_h}, I_{h*} = \frac{e^{-\mu_h \tau_h}\beta_h I_{m*} S_{h*}}{\eta_h + q + \mu_h}, E_{d*} = \frac{K_d(\mathscr{R}_m - 1)}{\mathscr{R}_m}, \\
E_{w*} &= \frac{k_d E_{d*}}{\eta_w + \mu_E}, S_{m*} = \frac{e^{-\mu_L \tau_w}\eta_w E_{w*}}{\beta_m I_{h*} + \mu_m},
\end{aligned}
$$

where I_{m*} is the unique positive fixed point of function

$$
g(x) = \frac{e^{-\mu_m \tau_m} e^{-\mu_L \tau_w}\eta_w k_d K_d(\mathscr{R}_m - 1)}{\mu_m(\eta_w + \mu_E)\mathscr{R}_m} - \frac{\frac{e^{-\mu_m \tau_m} e^{-\mu_L \tau_w}\eta_w k_d K_d(\mathscr{R}_m-1)}{(\eta_w+\mu_E)\mathscr{R}_m}}{\frac{\beta_m e^{-\mu_h \tau_h}\beta_h \Lambda_h}{\left(\beta_h + \frac{\mu_h}{x}\right)(\eta_h+q+\mu_h)} + \mu_m}.
$$

Define $\mathbb{U} := C(\bar{\Omega}, \mathbb{R})$, $\mathbb{U}^+ := C(\bar{\Omega}, \mathbb{R}_+)$ and

$$
\begin{aligned}
U := & C\left([-\tau_h, 0], \mathbb{U}^+\right) \times C\left([-\tau_m, 0], \mathbb{U}^+\right) \times \mathbb{U}^+ \times C\left([-\tau_w, 0], \mathbb{U}^+\right) \\
& \times C\left([-\tau_m, 0], \mathbb{U}^+\right) \times C\left([-\tau_h, 0], \mathbb{U}^+\right).
\end{aligned}
$$

In general, the global attractivity of the positive equilibrium can be demonstrated using the method of fluctuations (see, e.g., [145]) or the method of Lyapunov functionals. Combining Theorem 5.4.1 and the method of Lyapunov functionals, we have the following result on the global attractivity for system (5.23).

Theorem 5.4.2. *Let $u(t, x, \phi)$ be the solution of system (5.23) with the initial value $u_0 = \phi \in U$. Then the following statements are valid:*

(i) If $\mathscr{R}_m \leqslant 1$, then $e_0 = (S_h^*, 0, 0, 0, 0, 0)$ is globally attractive for system (5.23) in U.

(ii) If $\mathscr{R}_m > 1$ and $\mathscr{R}_0 < 1$, then the disease-free equilibrium $(S_h^*, 0, E_d^*, E_w^*, S_m^*, 0)$ is globally attractive for system (5.23) in $U \backslash \{e_0\}$.

(iii) If $\mathscr{R}_m > 1$ and $\mathscr{R}_0 > 1$, then system (5.23) has a unique constant equilibrium $u^* = (S_{h*}, I_{h*}, E_{d*}, E_{w*}, S_{h*}, I_{h*})$ such that for any $\phi \in U$ with $\phi_2(0, \cdot) \not\equiv 0$ or $\phi_6(0, \cdot) \not\equiv 0$, we have $\lim_{t\to\infty} u(t, x, \phi) = u^*$ uniformly for all $x \in \bar{\Omega}$.

Proof. From the analysis on the ultimate boundedness for system (5.4), we see that the set

$$H := \Big\{\phi \in U : \phi_1(\theta_1, x) \leqslant \frac{\Lambda_h}{\mu_h}, \phi_2(\theta_2, x) \leqslant \frac{\Lambda_h}{\mu_h}, \phi_3(0, x) \leqslant K_d, \phi_4(\theta_3, x) \leqslant \frac{k_d K_d}{\eta_w + \mu_E},$$

$$\phi_5(\theta_2, x) \leqslant \frac{e^{-\mu_L \tau_w} \eta_w \left(\frac{k_d K_d}{\eta_w + \mu_E}\right)}{\mu_m}, \phi_6(\theta_1, x) \leqslant \frac{e^{-\mu_L \tau_w} \eta_w \left(\frac{k_d K_d}{\eta_w + \mu_E}\right)}{\mu_m},$$

$$\forall \theta_1 \in [-\tau_h, 0], \forall \theta_2 \in [-\tau_m, 0], \forall \theta_3 \in [-\tau_w, 0], x \in \bar{\Omega}\Big\}$$

is positively invariant for the solution semiflow $Q(t)$ of system (5.23), and every forward orbit of system (5.23) from U enters H eventually. Therefore, it suffices to study the dynamics of system (5.23) on H. Note that cases (i) and (ii) are the straightforward consequences of Theorem 5.4.1 (i) and (ii) with $\rho = 0$. It remains to prove (iii). When $\mathscr{R}_m > 1$ and $\mathscr{R}_0 > 1$, it follows from Theorem 5.4.1 (iii) that system (5.23) is uniformly persistent, that is, there exists $\xi > 0$ such that for any $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) \in H$ with $\phi_2(0, \cdot) \not\equiv 0$ or $\phi_6(0, \cdot) \not\equiv 0$, the solution $u = (t, x, \phi)$ satisfies

$$\liminf_{t \to \infty} \min_{x \in \overline{\Omega}} u_i(t, x, \phi) \geqslant \xi, (1 \leqslant i \leqslant 6). \tag{5.24}$$

Let $H_0 := \left\{\phi \in H : \phi_i(0, x) > 0, \forall x \in \bar{\Omega}, i = 1, 2, 3, 4, 5, 6\right\}$. Next, we show that u^* is globally attractive for system (5.23) by the method of Lyapunov functionals. Set $f(u) = u - 1 - \ln u, u \in (0, \infty)$. Clearly, $f(u) \geqslant 0$ for all $u \in (0, \infty)$ and $\min_{0<u<+\infty} f(u) = f(1) = 0$. Define a continuous functional $V : H_0 \to \mathbb{R}$:

$$V(\phi) = \int_\Omega [V_1(x, \phi) + V_2(x, \phi)] dx,$$

where

$$V_1 = \frac{1}{\beta_h I_{m*}} f\left(\frac{\phi_1(0, x)}{S_{h*}}\right) + \frac{I_{h*}}{e^{-\mu_h \tau_h} \beta_h I_{m*} S_{h*}} \left(f\left(\frac{\phi_2(0, x)}{I_{h*}}\right)\right)$$
$$+ \int_{-\tau_h}^{0} \int_\Omega \Gamma\left(D(-s), x, y\right) f\left(\frac{\phi_6(s, y)\phi_1(s, y)}{I_{m*} S_{h*}}\right) dy ds,$$

and

$$V_2 = \frac{1}{k_d} f\left(\frac{\phi_3(0, x)}{E_{d*}}\right) + \frac{E_{w*}}{k_d E_{d*}} f\left(\frac{\phi_4(0, x)}{E_{w*}}\right) + \frac{1}{\eta_w} f\left(\frac{\phi_4(0, x)}{E_{w*}}\right)$$
$$+ \frac{S_{m*}}{e^{-\mu_L \tau_w} \eta_w E_{w*}} f\left(\frac{\phi_5(0, x)}{S_{m*}}\right) + \int_{-\tau_w}^{0} \int_\Omega \Gamma\left(D(-s), x, y\right) f\left(\frac{\phi_4(s, y)}{E_{w*}}\right) dy ds$$
$$+ \frac{1}{\beta_m I_{h*}} f\left(\frac{\phi_5(0, x)}{S_{m*}}\right) + \frac{I_{m*}}{e^{-\mu_m \tau_m} \beta_m I_{h*} S_{m*}} f\left(\frac{\phi_6(0, x)}{I_{m*}}\right)$$
$$+ \int_{-\tau_m}^{0} \int_\Omega \Gamma\left(D(-s), x, y\right) f\left(\frac{\phi_2(s, y)\phi_4(s, y)}{I_{h*} S_{m*}}\right) dy ds.$$

Next we fix $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) \in H$ with $\phi_2(0,\cdot) \not\equiv 0$ or $\phi_6(0,\cdot) \not\equiv 0$. From (5.24), without loss of generality, we can assume that $u_t(\phi) \in H_0, \forall t \geqslant 0$. Let $\omega(\phi)$ be the omega limit set of the orbit $\gamma^+(\phi)$ for the semiflow $Q(t)$. Since (5.23) is uniformly persistent, we have $\omega(\phi) \subset H_0$. Now we calculate the time derivative of $V(u_t(\phi))$ along the solution of system (5.23). It then follows that

$$
\begin{aligned}
\frac{\partial V_1}{\partial t} = & \frac{1}{\beta_h I_{m*}}\left(1-\frac{S_{h*}}{S_h}\right)\frac{1}{S_{h*}}\frac{\partial S_h}{\partial t} + \frac{I_{h*}}{e^{-\mu_h\tau_h}\beta_h I_{m*} S_{h*}}\left(1-\frac{I_{h*}}{I_h}\right)\frac{1}{I_{h*}}\frac{\partial I_h}{\partial t} \\
& + \left[f\left(\frac{I_m(t,x)S_h(t,x)}{I_{m*}S_{h*}}\right) - \int_\Omega \Gamma\left(D\tau_h, x, y\right) f\left(\frac{I_m(t-\tau_h,y)S_h(t-\tau_h,y)}{I_{m*}S_{h*}}\right) dy \right] \\
= & -\frac{\mu_h\left(S_h - S_{h*}\right)^2}{\beta_h I_{m*} S_{h*} S_h} + \frac{1}{I_{m*}S_{h*}}\left(1-\frac{S_{h*}}{S_h}\right)\left(I_{m*}S_{h*} - I_m S_h\right) \\
& + \frac{1}{\beta_h I_{m*}}\left(1-\frac{S_{h*}}{S_h}\right)\frac{1}{S_{h*}} D_h \Delta S_h + \frac{I_{h*}}{e^{-\mu_h\tau_h}\beta_h I_{m*} S_{h*}}\left(1-\frac{I_{h*}}{I_h}\right)\frac{1}{I_{h*}} D_h \Delta I_h \\
& + \left(\int_\Omega \Gamma\left(D\tau_h, x, y\right) \frac{I_m(t-\tau_h,y)S_h(t-\tau_h,y)}{I_{m*}S_{h*}} dy - \frac{I_h}{I_{h*}}\right)\left(1-\frac{I_{h*}}{I_h}\right) \\
& + \frac{S_h I_m}{S_{h*} I_{m*}} - \ln\frac{S_h I_m}{S_{h*} I_{m*}} \\
& - \int_\Omega \Gamma\left(D\tau_h, x, y\right)\left(\frac{I_m(t-\tau_h,y)S_h(t-\tau_h,y)}{I_{m*}S_{h*}} - \ln\frac{I_m(t-\tau_h,y)S_h(t-\tau_h,y)}{I_{m*}S_{h*}}\right) dy
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\mu_h \left(S_h - S_{h*}\right)^2}{\beta_h I_{m*} S_{h*} S_h} + \frac{1}{\beta_h I_{m*}}\left(1 - \frac{S_{h*}}{S_h}\right)\frac{1}{S_{h*}} D_h \Delta S_h \\
&\quad + \frac{I_{h*}}{e^{-\mu_h \tau_h}\beta_h I_{m*} S_{h*}}\left(1 - \frac{I_{h*}}{I_h}\right)\frac{1}{I_{h*}} D_h \Delta I_h + 2 - \frac{S_{h*}}{S_h} + \frac{I_m}{I_{m*}} - \frac{I_h}{I_{h*}} - \ln\frac{S_h I_m}{S_{h*} I_{m*}} \\
&\quad - \int_\Omega \Gamma\left(D\tau_h, x, y\right)\left(\frac{I_m(t-\tau_h, y) S_h(t-\tau_h, y) I_{h*}}{I_{m*} S_{h*} I_h}\right. \\
&\quad \left. - \ln\frac{I_m(t-\tau_h, y) S_h(t-\tau_h, y)}{I_{m*} S_{h*}}\right) dy \\
&= -\frac{\mu_h \left(S_h - S_{h*}\right)^2}{\beta_h I_{m*} S_{h*} S_h} + \frac{1}{\beta_h I_{m*}}\left(1 - \frac{S_{h*}}{S_h}\right)\frac{1}{S_{h*}} D_h \Delta S_h \\
&\quad + \frac{I_{h*}}{e^{-\mu_h \tau_h}\beta_h I_{m*} S_{h*}}\left(1 - \frac{I_{h*}}{I_h}\right)\frac{1}{I_{h*}} D_h \Delta I_h \\
&\quad + 2 - 1 - \frac{S_{h*}}{S_h} + \frac{I_m}{I_{m*}} - \frac{I_h}{I_{h*}} - \ln\frac{S_h I_m}{S_{h*} I_{m*}} - \ln\frac{I_{h*}}{I_h} \\
&\quad - \int_\Omega \Gamma\left(D\tau_h, x, y\right)\left(\frac{I_m(t-\tau_h, y) S_h(t-\tau_h, y) I_{h*}}{I_{m*} S_{h*} I_h}\right. \\
&\quad \left. - \ln\frac{I_m(t-\tau_h, y) S_h(t-\tau_h, y) I_{h*}}{I_{m*} S_{h*} I_h} - 1\right) dy \\
&= -\frac{\mu_h \left(S_h - S_{h*}\right)^2}{\beta_h I_{m*} S_{h*} S_h} + \frac{1}{\beta_h I_{m*}}\left(1 - \frac{S_{h*}}{S_h}\right)\frac{1}{S_{h*}} D_h \Delta S_h \\
&\quad + \frac{I_{h*}}{e^{-\mu_h \tau_h}\beta_h I_{m*} S_{h*}}\left(1 - \frac{I_{h*}}{I_h}\right)\frac{1}{I_{h*}} D_h \Delta I_h - f\left(\frac{S_{h*}}{S_h}\right) + \frac{I_m}{I_{m*}} - \frac{I_h}{I_{h*}} - \ln\frac{I_{h*}}{I_h} - \ln\frac{I_m}{I_{m*}} \\
&\quad - \int_\Omega \Gamma\left(D\tau_h, x, y\right) f\left(\frac{I_m(t-\tau_h, y) S_h(t-\tau_h, y) I_{h*}}{I_{m*} S_{h*} I_h}\right) dy,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial V_2}{\partial t} &= \frac{1}{k_d}\left(1 - \frac{E_{d*}}{E_d}\right)\frac{1}{E_{d*}}\frac{\partial E_d}{\partial t} + \frac{E_{w*}}{k_d E_{d*}}\left(1 - \frac{E_{w*}}{E_w}\right)\frac{1}{E_{w*}}\frac{\partial E_w}{\partial t} \\
&\quad + \frac{1}{\eta_w}\left(1 - \frac{E_{w*}}{E_w}\right)\frac{1}{E_{w*}}\frac{\partial E_w}{\partial t} + \frac{S_{m*}}{e^{-\mu_L \tau_w}\eta_w E_{w*}}\left(1 - \frac{S_{m*}}{S_m}\right)\frac{1}{S_{m*}}\frac{\partial S_m}{\partial t} \\
&\quad + \left[f\left(\frac{E_w(t,x)}{E_{w*}}\right) - \int_\Omega \Gamma\left(D\tau_w, x, y\right) f\left(\frac{E_w(t-\tau_w, y)}{E_{w*}}\right) dy\right] \\
&\quad + \frac{1}{\beta_m I_{h*}}\left(1 - \frac{S_{m*}}{S_m}\right)\frac{1}{S_{m*}}\frac{\partial S_m}{\partial t} + \frac{I_{m*}}{e^{-\mu_m \tau_m}\beta_m I_{h*} S_{m*}}\left(1 - \frac{I_{m*}}{I_m}\right)\frac{1}{I_{m*}}\frac{\partial I_m}{\partial t} \\
&\quad + \left[f\left(\frac{I_h(t,x) S_m(t,x)}{I_{h*} S_{m*}}\right) - \int_\Omega \Gamma\left(D\tau_m, x, y\right) f\left(\frac{I_h(t-\tau_m, y) S_m(t-\tau_m, y)}{I_{h*} S_{m*}}\right) dy\right]
\end{aligned}
$$

$$= -\frac{\mu_E (E_d - E_{d*})^2}{k_d E_{d*} E_d} + \frac{1}{E_{d*}}\left(1 - \frac{E_{d*}}{E_d}\right)(E_{d*} - E_d) + \frac{1}{E_{d*}}\left(1 - \frac{E_{w*}}{E_w}\right)\left(E_d - \frac{E_{d*}}{E_{w*}}E_w\right)$$
$$- \frac{(\eta_w + \mu_E)(E_w - E_{w*})^2}{\eta_w E_{w*} E_w} + \frac{1}{E_{w*}S_{m*}}\left(1 - \frac{S_{m*}}{S_m}\right)(E_{w*}S_{m*} - E_w S_m)$$
$$+ \frac{S_{m*}}{e^{-\mu_L \tau_w}\eta_w E_{w*}}\left(1 - \frac{S_{m*}}{S_m}\right)\frac{1}{S_{m*}} D_m \Delta S_m + \frac{1}{\beta_m I_{h*}}\left(1 - \frac{S_{m*}}{S_m}\right)\frac{1}{S_{m*}} D_m \Delta S_m$$
$$+ \frac{I_{m*}}{e^{-\mu_m \tau_m}\beta_m I_{h*} S_{m*}}\left(1 - \frac{I_{m*}}{I_m}\right)\frac{1}{I_{m*}} D_m \Delta I_m$$
$$+ \left(\int_\Omega \Gamma(D\tau_w, x, y)\frac{E_w(t-\tau_w, y)}{E_{w*}}dy - \frac{S_m}{S_{m*}}\right)\left(1 - \frac{S_{m*}}{S_m}\right)$$
$$+ \frac{E_w}{E_{w*}} - \ln\frac{E_w}{E_{w*}} - \int_\Omega \Gamma(D\tau_w, x, y)\left(\frac{E_w(t-\tau_w, y)}{E_{w*}} - \ln\frac{E_w(t-\tau_w, y)}{E_{w*}}\right)dy$$
$$- \frac{\mu_m (S_m - S_{m*})^2}{\beta_m I_{h*} S_{m*} S_m} + \frac{1}{I_{h*}S_{m*}}\left(1 - \frac{S_{m*}}{S_m}\right)(I_{h*}S_{m*} - I_h S_m) + \frac{I_h S_m}{I_{h*}S_{m*}} - \ln\frac{I_h S_m}{I_{h*}S_{m*}}$$
$$+ \left(\int_\Omega \Gamma(D\tau_m, x, y)\frac{I_h(t-\tau_m, y)S_m(t-\tau_m, y)}{I_{h*}S_{m*}}dy - \frac{I_m}{I_{m*}}\right)\left(1 - \frac{I_{m*}}{I_m}\right)$$
$$- \int_\Omega \Gamma(D\tau_m, x, y)\left(\frac{I_h(t-\tau_m, y)S_m(t-\tau_m, y)}{I_{h*}S_{m*}}\right.$$
$$\left.- \ln\frac{I_h(t-\tau_m, y)S_m(t-\tau_m, y)}{I_{h*}S_{m*}}\right)dy$$
$$= -\frac{\mu_E (E_d - E_{d*})^2}{k_d E_{d*} E_d} + 2 + 2 + 3 - \frac{E_{d*}}{E_d} - \frac{E_w}{E_{w*}} - \frac{E_{w*}E_d}{E_w E_{d*}} - \frac{(\eta_w + \mu_E)(E_w - E_{w*})^2}{\eta_w E_{w*} E_w}$$
$$+ \frac{1}{e^{-\mu_L \tau_w}\eta_w E_{w*}}\left(1 - \frac{S_{m*}}{S_m}\right) D_m \Delta S_m + \frac{1}{\beta_m I_{h*} S_{m*}}\left(1 - \frac{S_{m*}}{S_m}\right) D_m \Delta S_m$$
$$+ \frac{1}{e^{-\mu_m \tau_m}\beta_m I_{h*} S_{m*}}\left(1 - \frac{I_{m*}}{I_m}\right) D_m \Delta I_m - \frac{S_{m*}}{S_m} + \frac{E_w}{E_{w*}} - \frac{S_m}{S_{m*}} - \ln\frac{E_w}{E_{w*}}$$
$$- \int_\Omega \Gamma(D\tau_w, x, y)\left(\frac{E_w(t-\tau_w, y)S_{m*}}{E_{w*}S_m} - \ln\frac{E_w(t-\tau_w, y)}{E_{w*}}\right)dy$$
$$- \frac{\mu_m (S_m - S_{m*})^2}{\beta_m (A_{h*} + I_{h*}) S_{m*} S_m} - \frac{S_{m*}}{S_m} + \frac{I_h}{I_{h*}} - \frac{I_m}{I_{m*}} - \ln\frac{I_h S_m}{I_{h*}S_{m*}}$$
$$- \int_\Omega \Gamma(D\tau_m, x, y)\left(\frac{I_h(t-\tau_m, y)S_m(t-\tau_m, y)I_{m*}}{I_{h*}S_{m*}I_m}\right.$$
$$\left.- \ln\frac{I_h(t-\tau_m, y)S_m(t-\tau_m, y)}{I_{h*}S_{m*}}\right)dy$$

$$
\begin{aligned}
&= -\frac{\mu_E\left(E_d-E_{d*}\right)^2}{k_d E_{d*} E_d} + 7 - 2 - \frac{E_{d*}}{E_d} - \frac{E_w}{E_{w*}} - \frac{E_{w*}E_d}{E_w E_{d*}} - \frac{\left(\eta_w+\mu_E\right)\left(E_w-E_{w*}\right)^2}{\eta_w E_{w*} E_w} \\
&\quad + \frac{1}{e^{-\mu_L \tau_w}\eta_w E_{w*}}\left(1-\frac{S_{m*}}{S_m}\right) D_m \Delta S_m + \frac{1}{\beta_m I_{h*} S_{m*}}\left(1-\frac{S_{m*}}{S_m}\right) D_m \Delta S_m \\
&\quad + \frac{1}{e^{-\mu_m \tau_m}\beta_m I_{h*} S_{m*}}\left(1-\frac{I_{m*}}{I_m}\right) D_m \Delta I_m - \frac{S_{m*}}{S_m} + \frac{E_w}{E_{w*}} - \frac{S_m}{S_{m*}} - \ln\frac{E_w}{E_{w*}} - \ln\frac{S_{m*}}{S_m} \\
&\quad - \int\limits_{\Omega} \Gamma\left(D\tau_w, x, y\right)\left(\frac{E_w(t-\tau_w,y)S_{m*}}{E_{w*}S_m} - \ln\frac{E_w(t-\tau_w,y)S_{m*}}{E_{w*}S_m} - 1\right) dy \\
&\quad - \frac{\mu_m\left(S_m-S_{m*}\right)^2}{\beta_m I_{h*} S_{m*} S_m} - \frac{S_{m*}}{S_m} + \frac{I_h}{I_{h*}} - \frac{I_m}{I_{m*}} - \ln\frac{I_h S_m}{I_{h*} S_{m*}} - \ln\frac{I_{m*}}{I_m} \\
&\quad - \int\limits_{\Omega} \Gamma\left(D\tau_m, x, y\right)\left(\frac{I_h(t-\tau_m,y)S_m(t-\tau_m,y)I_{m*}}{I_{h*}S_{m*}I_m}\right. \\
&\quad \left. - \ln\frac{I_h(t-\tau_m,y)S_m(t-\tau_m,y)I_{m*}}{I_{h*}S_{m*}I_m} - 1\right) dy \\
&= -\frac{\mu_E\left(E_d-E_{d*}\right)^2}{k_d E_{d*} E_d} + 5 - \frac{E_{d*}}{E_d} - \frac{E_w}{E_{w*}} - \frac{E_{w*}E_d}{E_w E_{d*}} - \frac{\left(\eta_w+\mu_E\right)\left(E_w-E_{w*}\right)^2}{\eta_w E_{w*} E_w} \\
&\quad + \frac{1}{e^{-\mu_L \tau_w}\eta_w E_{w*}}\left(1-\frac{S_{m*}}{S_m}\right) D_m \Delta S_m + \frac{1}{\beta_m I_{h*} S_{m*}}\left(1-\frac{S_{m*}}{S_m}\right) D_m \Delta S_m \\
&\quad + \frac{1}{e^{-\mu_m \tau_m}\beta_m I_{h*} S_{m*}}\left(1-\frac{I_{m*}}{I_m}\right) D_m \Delta I_m - \frac{S_{m*}}{S_m} + \frac{E_w}{E_{w*}} - \frac{S_m}{S_{m*}} - \ln\frac{E_w}{E_{w*}} - \ln\frac{S_{m*}}{S_m} \\
&\quad - \int\limits_{\Omega} \Gamma\left(D\tau_w, x, y\right) f\left(\frac{E_w(t-\tau_w,y)S_{m*}}{E_{w*}S_m}\right) dy \\
&\quad - \frac{\mu_m\left(S_m-S_{m*}\right)^2}{\beta_m (A_{h*}+I_{h*}) S_{m*} S_m} - \frac{S_{m*}}{S_m} + \frac{I_h}{I_{h*}} - \frac{I_m}{I_{m*}} - \ln\frac{I_h S_m}{I_{h*} S_{m*}} - \ln\frac{I_{m*}}{I_m} \\
&\quad - \int\limits_{\Omega} \Gamma\left(D\tau_m, x, y\right) f\left(\frac{I_h(t-\tau_m,y)S_m(t-\tau_m,y)I_{m*}}{I_{h*}S_{m*}I_m}\right) dy.
\end{aligned}
$$

Clearly, $\ln v \leqslant \frac{v}{u} + \ln u - 1, \forall u, v > 0$, and hence, $2-u-\frac{v}{u}+\ln v \leqslant 0$ and $2-u-\frac{v}{u}+\ln v = 0$ if and only if $u = v = 1$. In particular, $2 - u - \frac{1}{u} \leqslant 0$ and $1 - v + \ln v \leqslant 0$. Note that $\int_\Omega \Delta u dx = 0$ and $\int_\Omega \frac{\Delta u}{u} dx = \int_\Omega \frac{\|\nabla u\|^2}{u^2} dx$. Thus, we have

$$
\begin{aligned}
&\frac{dV\left(u_t(\phi)\right)}{dt} \\
&= -\int\limits_\Omega \frac{\mu_h\left(S_h - S_{h*}\right)^2}{\beta_h I_{m*} S_{h*} S_h} dx - \int\limits_\Omega \frac{\mu_E\left(E_d - E_{d*}\right)^2}{k_d E_{d*} E_d} dx - \int\limits_\Omega \frac{\left(\eta_w + \mu_E\right)\left(E_w - E_{w*}\right)^2}{\eta_w E_{w*} E_w} dx \\
&\quad - \int\limits_\Omega \frac{\mu_m\left(S_m - S_{m*}\right)^2}{\beta_m (A_{h*} + I_{h*}) S_{m*} S_m} dx - \frac{D_h}{\beta_h I_{m*}} \int\limits_\Omega \frac{\|\nabla S_h(t,x)\|^2}{S_h^2(t,x)} dx \\
&\quad - \frac{I_{h*} D_h}{e^{-\mu_h \tau_h} \beta_h I_{m*} S_{h*}} \int\limits_\Omega \frac{\|\nabla I_h(t,x)\|^2}{I_h^2(t,x)} dx - \frac{S_{m*} D_m}{e^{-\mu_L \tau_w} \eta_w E_{w*}} \int\limits_\Omega \frac{\|\nabla S_m(t,x)\|^2}{S_m^2(t,x)} dx \\
&\quad - \frac{S_{m*} D_m}{\beta_m I_{h*} S_{m*}} \int\limits_\Omega \frac{\|\nabla S_m(t,x)\|^2}{S_m^2(t,x)} dx - \frac{I_{m*} D_m}{e^{-\mu_m \tau_m} \beta_m I_{h*} S_{m*}} \int\limits_\Omega \frac{\|\nabla I_m(t,x)\|^2}{I_m^2(t,x)} dx \\
&\quad - \int\limits_\Omega \int\limits_\Omega \Gamma\left(D\tau_h, x, y\right) f\left(\frac{I_m(t-\tau_h, y) S_h(t-\tau_h, y) I_{h*}}{I_{m*} S_{h*} I_h}\right) dy dx \\
&\quad - \int\limits_\Omega \int\limits_\Omega \Gamma\left(D\tau_w, x, y\right) f\left(\frac{E_w(t-\tau_w, y) S_{m*}}{E_{w*} S_m}\right) dy dx \\
&\quad - \int\limits_\Omega \int\limits_\Omega \Gamma\left(D\tau_m, x, y\right) f\left(\frac{I_h(t-\tau_m, y) S_m(t-\tau_m, y) I_{m*}}{I_{h*} S_{m*} I_m}\right) dy dx \\
&\quad - \int\limits_\Omega f\left(\frac{S_{h*}}{S_h}\right) dx + \int\limits_\Omega \left(2 - \frac{E_{d*}}{E_d} - \frac{E_{w*} E_d}{E_w E_{d*}} - \ln \frac{E_w}{E_{w*}}\right) dx \\
&\quad + \int\limits_\Omega \left(2 - \frac{S_{m*}}{S_m} - \frac{S_m}{S_{m*}}\right) dx - \int\limits_\Omega \frac{S_{m*}}{S_m} dx + \int\limits_\Omega \left(1 - \frac{I_{m*}}{I_m} - \ln \frac{I_m}{I_{m*}}\right) dx \\
&\leqslant -\int\limits_\Omega \frac{\mu_h\left(S_h - S_{h*}\right)^2}{\beta_h I_{m*} S_{h*} S_h} dx - \int\limits_\Omega \frac{\mu_E\left(E_d - E_{d*}\right)^2}{k_d E_{d*} E_d} dx - \int\limits_\Omega \frac{\left(\eta_w + \mu_E\right)\left(E_w - E_{w*}\right)^2}{\eta_w E_{w*} E_w} dx \\
&\quad - \int\limits_\Omega \frac{\mu_m\left(S_m - S_{m*}\right)^2}{\beta_m (A_{h*} + I_{h*}) S_{m*} S_m} dx - \int\limits_\Omega \int\limits_\Omega \Gamma\left(D\tau_w, x, y\right) f\left(\frac{E_w(t-\tau_w, y) S_{m*}}{E_{w*} S_m}\right) dy dx \\
&\quad - \int\limits_\Omega \int\limits_\Omega \Gamma\left(D\tau_h, x, y\right) f\left(\frac{I_m(t-\tau_h, y) S_h(t-\tau_h, y) I_{h*}}{I_{m*} S_{h*} I_h}\right) dy dx \\
&\quad - \int\limits_\Omega \int\limits_\Omega \Gamma\left(D\tau_m, x, y\right) f\left(\frac{I_h(t-\tau_m, y) S_m(t-\tau_m, y) I_{m*}}{I_{h*} S_{m*} I_m}\right) dy dx \\
&\quad - \int\limits_\Omega f\left(\frac{S_{h*}}{S_h}\right) dx + \int\limits_\Omega \left(2 - \frac{E_{d*}}{E_d} - \frac{E_{w*} E_d}{E_w E_{d*}} - \ln \frac{E_w}{E_{w*}}\right) dx \\
&\quad + \int\limits_\Omega \left(2 - \frac{S_{m*}}{S_m} - \frac{S_m}{S_{m*}}\right) dx - \int\limits_\Omega \frac{S_{m*}}{S_m} dx + \int\limits_\Omega \left(1 - \frac{I_{m*}}{I_m} - \ln \frac{I_m}{I_{m*}}\right) dx \\
&:= U_\phi(t).
\end{aligned}
\tag{5.25}
$$

Note that $V(u_t(\phi))$ is nonincreasing and bounded below on $[0,\infty)$, so there is a real number $L \geqslant 0$ such that $\lim_{t\to\infty} V(u_t(\phi)) = L$. For any $\psi \in \omega(\phi)$, there is a sequence $t_n \to \infty$ such that $\lim_{n\to\infty} u_{t_n}(\phi) = \psi$ in H_0. This shows that $V(\psi) = L, \forall \psi \in \omega(\phi)$. Since $u_t(\psi) \in \omega(\phi)$, it follows that $V(u_t(\psi)) = L, \forall t \geqslant 0$, and hence, $\frac{dV(u_t(\psi))}{dt} = 0$. Replacing ϕ in (5.25) with ψ, we have $0 = \frac{dV(u_t(\psi))}{dt} \leqslant U_\psi(t) \leqslant 0$. This implies that $U_\psi(t) = 0, \forall t \geqslant 0$. By the definition of the set H_0 and Lemma 5.4.6, we see that $u_t(\psi) \gg 0, \forall t \geqslant \max\{\tau_h, \tau_m, \tau_w\}$. It then follows from the expression of $U_\psi(t)$ in (5.23) that $u_t(\psi) = u^*, \forall t \geqslant \max\{\tau_h, \tau_m, \tau_w\}$. Since $\psi \in \omega(\phi)$ is arbitrary, there also holds $u_t(\omega(\phi)) = u^*, \forall t \geqslant \max\{\tau_h, \tau_m, \tau_w\}$. In view of the invariance of omega limit sets, it is easy to see that $\omega(\phi) = u_\tau(\omega(\phi)) = u^*, \tau = \max\{\tau_h, \tau_m, \tau_w\}$, which implies that $\lim_{t\to\infty} u_t(\phi) = u^*$. □

5.5 Numerical simulations

In this section, we use model (5.4) to study the Chikungunya transmission in Brazil. From 2014 to the end of September 2018, there were 697,564 reported Chikungunya cases in Brazil, and this is the highest number of Chikungunya cases among 51 countries in the Americas [12, 89]. The state of Ceará is located at the northeast of Brazil and has an area of 148, 862 km^2 with more than 80,000 cases [28, 108] from January 2016 to December 2017. Aedes aegypti is the main vector of the disease, and the transmission of the disease is greatly affected by climate factors such as temperature and rainfall. We mainly focus on the impact of these two climatic factors and spatial heterogeneity on the spread of the disease in Ceará, as well as the effectiveness of different control strategies.

For the sake of convenience, we assume that the spatial units are kilometres (km) and that the spatial domain Ω is one dimensional. Without loss of generality, we set $\Omega = (0, \pi)$. According to the Census Organization of Brazil 2010, the population of Ceará is 8,452,381. We assume that the geographical human population density function is

$$N_h^*(x) = \frac{8452381}{148862}(1.0 - 0.4\cos(0.4 - 2x)) = 56.78\,(1.0 - 0.4\cos(0.4 - 2x)).$$

From the CIA WorldFact Book, the life expectancy of Brazil is 74.3 years, and hence, the natural mortality rate of humans is estimated as $\mu_h = \frac{1}{74.3\times 12} = 0.001122$ month^{-1}. So we can estimate the recruitment rate Λ_h as $\Lambda_h = \mu_h \times N_h^*(x) = 0.0637\,(1.0 - 0.4\cos(0.4 - 2x))$. From [21], we have

$$K_d = 2N_h^*(x) = 113.56\,(1.0 - 0.4\cos(0.4 - 2x)).$$

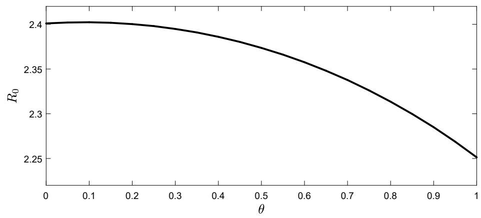
Liu et al. [72] presented the monthly average temperature and the rainfall of Ceará and fitted the temperature-dependent parameters τ_m, β, μ_m, τ_w, μ_b, η_w and

μ_L, and the rain-dependent parameter k_d, that is, they obtained the time-dependent expressions of these parameters with $\omega = 12$. These time-dependent parameters and other constant parameters are directly used in our numerical simulations except for parameters $D_h = 0.1\text{km}^2\ \text{month}^{-1}$ and $D_m = 0.0125\text{km}^2\ \text{month}^{-1}$ from [136].

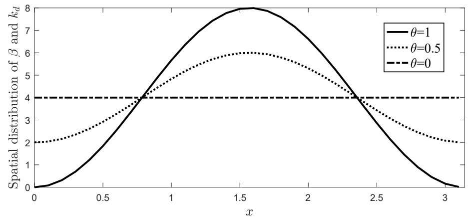
With above parameters, we obtain $\mathscr{R}_m = 2.056 > 1$ and $\mathscr{R}_0 = 3.6087 > 1$ by numerical calculation. Note that the above parameters are exactly the same as in [72], except for the random diffusion parameters and spatial heterogeneity. The numerical results in [72] indicate that $\mathscr{R}_m = 1.8583 < 2.056$ and $\mathscr{R}_0 = 5.2020 > 3.6087$, which implies that if the diffusion and spatial heterogeneity are not considered, the growth of mosquito population will be underestimated, but the spread of the epidemic will be overestimated. In order to explore the spatial heterogeneity effect on $\mathscr{R}_0$ and $\mathscr{R}_m$. We assume that the bite rate of mosquitoes is $\beta(x) = 4\,(1 - \theta\cos(2x))$ and that the ratio of dry eggs to wet eggs is $k_d(x) = 4\,(1 - \theta\cos(2x))$ with $\theta \in [0, 1)$. Note that if $\theta = 0$, then these two rates are homogeneous distribution in Ceará. When θ increases from 0 to 1 and approaches 1, there will be a highest rate around the center of the spatial domain (i.e., $x = \frac{\pi}{2}$) and a lowest bite rate around two boundaries (i.e., $x = 0$ and $x = \pi$) (see Figure 5.2(b)). Since the spatial average of $\beta(x)$ and $k_d(x)$ do not change for all $\theta \in [0, 1)$, the total bite rate and the ratio of dry eggs to wet eggs remain unchanged. It turns out that $\mathscr{R}_0$ and $\mathscr{R}_m$ will eventually decrease to the lowest as θ increases to 1. These results are shown in Figures 5.2(a), 5.2(c) and 5.2(d). In particular, Figure 5.2(d) illustrates that $\mathscr{R}_m$ will initially increase and then decrease with the increase of θ. This, together with Figure 5.2(a), also explains the underestimation of $\mathscr{R}_m$ and overestimation of $\mathscr{R}_0$ when the spatial heterogeneity is ignored. Thus, the spatial heterogeneity is an important factor in the evaluation of the epidemic outbreak and the control of disease transmission.

Define the average of a continuous ω-periodic function $f(t)$ by $[f] := \frac{1}{\omega}\int_0^\omega f(t)dt$. It follows from the parameter expressions in [72] that $[\tau_w] = 0.3205$ and $[\tau_m] = 0.3529$. Then we can use the above parameter values to calculate $\mathscr{R}_m = 2.0568$ and $\mathscr{R}_0 = 3.6509$, which are larger than $\mathscr{R}_m = 2.056$ and $\mathscr{R}_0 = 3.6087$ in the case of periodic delays. In order to further investigate the impact of periodic time delays on the reproduction numbers, we obtained the curve of $\mathscr{R}_m$ versus μ_b and the curve of $\mathscr{R}_0$ versus β by numerical simulations under periodic time delays and time-averaged delays, respectively (see Figure 5.3). This indicates that the use of time-averaged delays may overestimate the mosquitos production and the disease transmission risk.

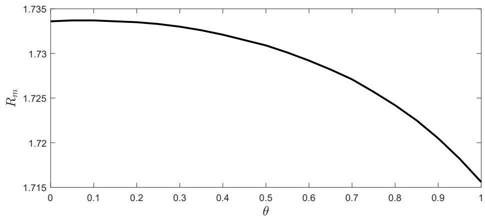
Applying the difference method to the system with the Neumann boundary condition, we obtain Figure 5.4 for the evolution of each compartment in system (5.4)



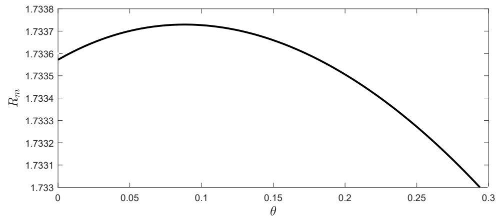
(a) Relationship between $\mathscr{R}_0$ and θ.



(b) Spatial distribution of β and k_d.

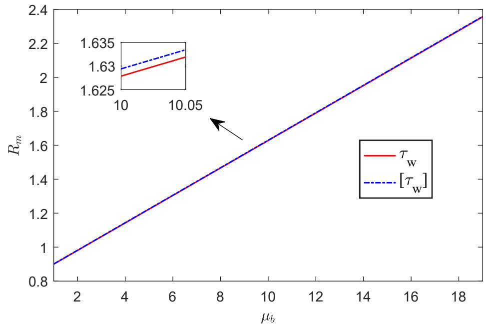


(c) Relationship between $\mathscr{R}_m$ and θ.

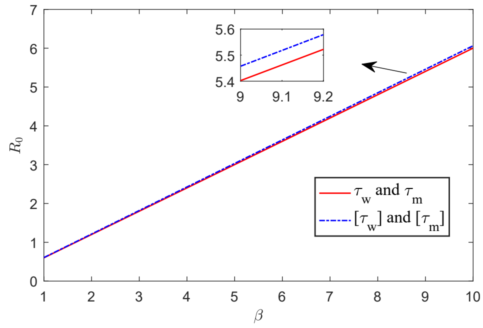


(d) The local enlargement of Figure 5.2(c) shows the relationship between $\mathscr{R}_m$ and θ.

Figure 5.2: Influence of spatial heterogeneity.



(a) $\mathscr{R}_m$ as the function of μ_b.



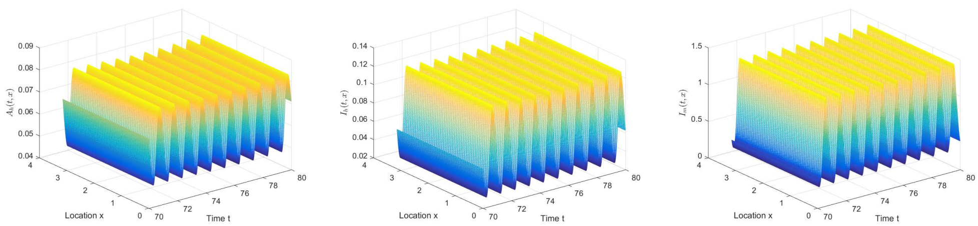
(b) $\mathscr{R}_0$ as the function of β.

Figure 5.3: Influence of periodic delays on the reproduction numbers $\mathscr{R}_m$ and $\mathscr{R}_0$.

with the initial data

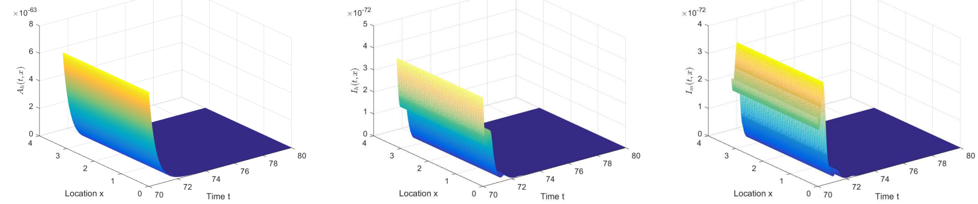
$$\begin{pmatrix} S_h(\theta,x) \\ A_h(\theta,x) \\ I_h(\theta,x) \\ E_d(\theta,x) \\ E_w(\theta,x) \\ S_m(\theta,x) \\ E_m(\theta,x) \\ I_m(\theta,x) \end{pmatrix} = \begin{pmatrix} 446100/148862 - 1\cos 2x \\ 7170/148862 - 0.5\cos 2x \\ 46230/148862 - 0.5\cos 2x \\ 15870000/148862 - 10\cos 2x \\ 17987000/148862 - 10\cos 2x \\ 38961000/148862 - 10\cos 2x \\ 33808/148862 - 0.2\cos 2x \\ 175000/148862 - 1\cos 2x \end{pmatrix}, \forall \theta \in [-\hat{\tau}, 0], x \in [0,\pi].$$

The numerical results are well consistent with Theorem 5.4.1 (iii), that is, the disease will be uniformly persistent. Moreover, our numerical results show that a positive periodic solution exists when $\mathscr{R}_m > 1$ and $\mathscr{R}_0 > 1$. However, we cannot use Theorem 1.2.3 to prove this result because we have only established that the Poincaré map of system (5.4) is α-contracting rather than α-condensing. Additionally, we are unable to apply Theorem 1.2.4 to prove it due to the non-convex nature of the set D.



(a) (b) (c)

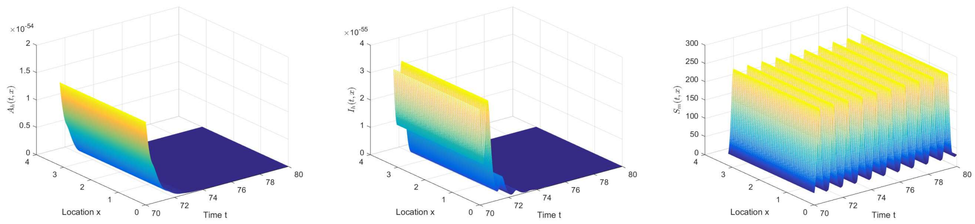
Figure 5.4: The evolution of human and mosquito populations when $\mathscr{R}_m = 2.056 > 1$ and $\mathscr{R}_0 = 3.6087 > 1$.



(a) (b) (c)

Figure 5.5: The evolution of human and mosquito populations when $\mathscr{R}_m = 0.9891 < 1$.

If $k_d(t)$ decreases to $0.4k_d(t)$, $\mu_b(t)$ decreases to $0.3\mu_b(t)$, $\mu_m(t)$ increases to $1.8\mu_m(t)$, and other parameters remain unchanged, we obtain $\mathscr{R}_m = 0.9891 < 1$. Figure 5.5



(a) (b) (c)

Figure 5.6: The evolution of human and mosquito populations when $\mathscr{R}_m = 1.6498 > 1$ and $\mathscr{R}_0 = 0.8221 < 1$.

illustrates the evolution of solution, which is exactly the same as the conclusion in Theorem 5.4.1 (i), that is, the mosquito population and the disease will go extinct. If only $\beta(t)$ decreases to $0.4\beta(t)$, $k_d(t)$ decreases to $0.9k_d(t)$, $\mu_b(t)$ decreases to $0.9\mu_b(t)$, $\mu_m(t)$ increases to $1.5\mu_m(t)$, we have $\mathscr{R}_m = 1.6498 > 1$ and $\mathscr{R}_0 = 0.8221 < 1$. The evolution of solution is shown in Figure 5.6, which corresponds to the conclusion in Theorem 5.4.1 (ii), that is, the mosquito population will survive, but the disease will become extinct. From the above analysis and numerical results, we can reduce $\beta(t)$, $k_d(t)$, $\mu_b(t)$ and increase $\mu_m(t)$ at the same time to make $\mathscr{R}_0 > 1$ become $\mathscr{R}_0 < 1$. If we continue to reduce $k_d(t)$, $\mu_b(t)$ and increase $\mu_m(t)$ at the same time, then we can further get $\mathscr{R}_m < 1$. Therefore, the above numerical results provide some strategies to control the spread of the disease and the growth of the mosquito population by reducing $\beta(t)$, $k_d(t)$, $\mu_b(t)$ while increasing $\mu_m(t)$.

5.6 Conclusions and discussion

In this chapter, we investigated a nonlocal reaction-diffusion model of Chikungunya disease that incorporates seasonality (temperature and rainfall), spatial heterogeneity, periodic maturation delay, and time-periodic extrinsic incubation period (EIP). Note that the diffusion coefficients of hospitalized humans and mosquitoes in the egg stage are assumed to be zero, which makes the solution maps of this model non-compact By the theory developed in [67, 151], we derived the basic reproduction ratio $\mathscr{R}_m$ for the vector and the basic reproduction ratio $\mathscr{R}_0$ for the disease. We showed that $\mathscr{R}_m$ as the threshold value for the extinction and persistence of the mosquito population by the global convergence theory for periodic semiflows, and $\mathscr{R}_0$ as the threshold value for the extinction and persistence of the disease by the persistence theory for periodic semiflows. In detail, when $\mathscr{R}_m \le 1$, both the mosquito population and the disease become extinct; when $\mathscr{R}_m > 1$ and $\mathscr{R}_0 < 1$, the mosquito population survives but the disease becomes extinct; and when $\mathscr{R}_m > 1$ and $\mathscr{R}_0 > 1$, both the mosquito population and the disease persist. For a simplified system, we derived explicit expressions for

$\mathscr{R}_m$ and $\mathscr{R}_0$, and we can demonstrate the global attractivity of the positive steady state by using the method of Lyapunov functionals when $\mathscr{R}_m > 1$ and $\mathscr{R}_0 > 1$.

As an application of the system, we numerically analyze the Chikungunya transmission in Ceará, Brazil with some feasible coefficients, temperature-dependent and rainfall-dependent parameters, which are derived from published works. The numerical simulations reveal the following insights: (i) The numerical results suggest that without considering diffusion and spatial heterogeneity, the growth of the mosquito population will be underestimated, while the spread of the epidemic will be overestimated. Hence, spatial heterogeneity plays a crucial role in both assessing epidemic outbreaks and controlling disease transmission. (ii) Numerical simulations show that using time-averaged delays leads to an overestimation of mosquito production and the risk of disease transmission. (iii) The numerical results show that a positive periodic solution exists when $\mathscr{R}_m > 1$ and $\mathscr{R}_0 > 1$. However, we are unable to prove it analytically, leaving this as an open problem for future investigation. (iv) The numerical results indicate that controlling the spread of the disease and the growth of the mosquito population can be achieved by decreasing $\beta(t)$, $k_d(t)$, and $\mu_b(t)$ while increasing $\mu_m(t)$.

Note that the transmission mechanisms of mosquito-borne diseases are very similar, we can use this model or modify it to investigate other mosquito-borne diseases, such as Zika virus, West Nile virus, dengue, and malaria.

Chapter 6

Summary and future works

In this chapter, we first briefly summarize the main results in this thesis, and then present some possible future works.

6.1 Research summary

In Chapter 2, by incorporating the temporal and spatial variations into an impulsive system, we proposed a time-space periodic reaction-diffusion model with an annual impulsive maturation emergence term to study the invasion dynamics in unbounded and bounded domains, respectively. When the habitat is unbounded, we obtained the existence of the spreading speeds in both monotone and non-monotone cases and showed they are linearly determinate. We further proved that the spreading speeds in the monotone case coincide with the minimal speeds of spatially periodic traveling waves. When the habitat is bounded, we introduced the fractional power space to deal with general boundary conditions and establish global stability results.

The numerical simulations reveal meaningful phenomena when $|f| > d_I$, the positive steady state and time τ are positively correlated, while when $|f| < d_I$, the relationship is converse. The fundamental reason is the relationship between functions f and d_I. In ecological terms, when $|f| > d_I$, indicating that the mortality rate of mature individuals is higher than that of immature individuals, the longer it takes for immature individuals to grow into adults (i.e., $\tau \nearrow$), the higher the positive steady-state of adult groups. Conversely, when $|f| < d_I$, the correlation is reversed. We also use numerical experiments to illustrate the long-time behaviour of solutions of system with time-space periodic parameters in both monotone and non-monotone cases. Simultaneously, we visually demonstrate the spreading speeds in both scenarios.

In Chapter 3, we formulated and analyzed a time-delayed nonlocal reaction-diffusion model of within-host disease transmission to investigate the influences of the mobility of cells or viruses and spatial heterogeneity on with-in host disease pathogenesis.

In order to obtain the threshold condition for the disease in a heterogeneous space, we chose a bounded spatial domain and the Neumann boundary condition. We introduced the basic reproduction number $\mathscr{R}_0$, which was shown as a threshold: the infection-free steady state is globally asymptotically stable as $\mathscr{R}_0 \leq 1$ while the disease is uniformly persistent as $\mathscr{R}_0 > 1$. More precisely, we utilized the comparison arguments to address both cases $\mathscr{R}_0 < 1$ and $\mathscr{R}_0 = 1$ simultaneously for the global attractivity of the infection-free steady state. We then established its global asymptotic stability through a simple observation on Lyapunov stability (see Lemma 3.3.7). Moreover, we confirmed the uniform persistence of the disease and the existence of a positive steady state in the case where $\mathscr{R}_0 > 1$. In a homogeneous space case, the global stability of the positive constant steady state was established via the method of Lyapunov functionals.

Numerically, our primary focus is on elucidating the analytical results, conducting sensitivity analysis, exploring how parameters affect the basic reproduction number, and investigating the efficacy of drugs in mitigating the spread of the virus. The numerical results reveal the following insights: (i) PRCC analysis demonstrates that variables μ_1 and τ_2 exhibit the highest sensitivity to $\mathscr{R}_0$, showing a negative correlation. This suggests that implementing measures to augment both μ_1 and τ_2 could effectively curb virus transmission. (ii) The simulations of the optimal drug strategy underscore the significance of carefully selecting the location and duration of drug delivery, especially in scenarios where drug resources are limited. This plays a pivotal role in exerting control over virus transmission.

In Chapter 4, for a time-delayed nonlocal reaction-diffusion model established in Chapter 3, we investigated the traveling wave solutions of this non-monotone system. Firstly, we employed a pair of lower and upper solutions and Schauder's fixed-point theorem to establish the existence of bounded semi-traveling wave solutions. These solutions represent wave behavior that converges to the unstable disease-free equilibrium as the moving frame $z = x + ct$ approaches $-\infty$, provided the wave speed $c > c^*$ and $\mathscr{R}_0 > 1$. Next, we proved the convergence of the semi-traveling waves to the endemic equilibrium as z approaches $+\infty$ by constructing a Lyapunov functional. When $c = c^*$ and $\mathscr{R}_0 > 1$, we established the existence of a traveling wave solution connecting the disease-free equilibrium e_0 and the endemic equilibrium u^* by using a limiting argument and a way of contradiction. In order to obtain the non-existence of bounded semi-traveling wave solutions when $0 < c < c^*$ and $\mathscr{R}_0 > 1$, we leverage several key factors, these include the non-existence of positive eigenvalues associated with the unstable steady state and the utilization of the two-sided Laplace transform.

Numerically, based on the analytical definition in Chapter 4, we presented a numerical method for c^*. Furthermore, we summarized the parameter values in the model in some published literature, numerically calculated the values of c^* and the basic production number $\mathscr{R}_0$ for the model, and investigated the long-time behaviour of the solutions. In addition, we explored the dependence of c^* on the system parameters. This demonstrates that c^* is positively correlated with β_1 and β_2, whereas it is negatively correlated with τ_1, τ_2, μ_1 and μ. Given the expression for $\mathscr{R}_0$:

$$\mathscr{R}_0 = \frac{e^{-\mu_1\tau_1}\beta_2 T^*}{\mu_1} + \frac{e^{-\mu_1\tau_1}\beta_1 T^* e^{-\mu_2\tau_2} b}{\mu_1\mu},$$

it is easy to see that $\mathscr{R}_0$ diminishes with decreasing values of β_1 and β_2, and conversely, increases with decreasing values of τ_1, τ_2, μ_1 and μ. Therefore, we can reduce β_1 and β_2, and increase τ_1, τ_2, μ_1 and μ through correlation strategies, which will result in a smaller value for c^* while ensuring $\mathscr{R}_0 > 1$. (That is to say, when $\mathscr{R}_0$ is very close to 1, c^* will become very small). Consequently, when the wave speed c exceeds a minimal threshold c^*, a traveling wave solution will emerge, connecting the disease-free equilibrium and the positive equilibrium.

In Chapter 5, we developed a nonlocal and time-delayed reaction-diffusion system. Note that the diffusion coefficients of the hospitalized humans and mosquitoes in the egg stage are assumed to be zero, making this model's solution maps non-compact. For the time-delayed reaction-diffusion system on the growth of mosquitoes, we first introduced a new phase space to prove that the solution maps are eventually strongly monotone and subhomogeneous, and its period map is α-contracting, and then employed the theory of monotone dynamical systems to obtain a threshold type result on the global stability in terms of the mosquito reproduction ratio $\mathscr{R}_m$. For the full system of the disease transmission, we first showed that its period map has a global attractor that attracts any bounded set, and then used the comparison arguments and an abstract theorem on uniform persistence to establish the global dynamics in terms of $\mathscr{R}_m$ and the disease reproduction ratio $\mathscr{R}_0$. For a simplified nonlocal and time-delayed reaction-diffusion system with constant coefficients, we proved the global attractivity of the positive steady state in the case where $\mathscr{R}_m > 1$ and $\mathscr{R}_0 > 1$ by using the method of Lyapunov functionals.

Finally, we conducted numerical simulations to investigate the effects of temporal and spatial heterogeneities on the growth of mosquitoes and the spread of Chikungunya transmission in Ceará, Brazil. The numerical results imply that without considering diffusion and spatial heterogeneity, the growth of the mosquito population will be underestimated, while the spread of the epidemic will be overestimated. Furthermore, we conclude that the use of time-averaged delays may overestimate mosquito production and the risk of disease transmission. In addition, the evolution of each compartment implies certain strategies to control the spread of the disease and the growth of the mosquito population. Specifically, our numerical results suggest the existence of a positive periodic solution when $\mathscr{R}_m > 1$ and $\mathscr{R}_0 > 1$, yet we cannot utilize Theorem 1.2.3 for the proof because we only proved that the Poincaré map of system (5.4) is α-contracting rather than α-condensing.

6.2 Future works

Related to the projects in this thesis, there are some open and challenging issues for future investigation.

In Chapter 2, we proposed a time-space periodic reaction-diffusion model with an annual impulsive maturation emergence term. Note that the nonexistence of spatially periodic traveling waves is a straightforward consequence of the spreading speeds (see, e.g., [136, Theorem 3.5(i)]) in the non-monotonic case. However, the existence of spatially periodic traveling waves, in this case, is still a challenging open problem. We leave it for future investigation. The model in Chapter 2 is an extension of [8]. While specific ecological backgrounds have been established and assumed in constructing the models in both Chapter 2 and [8], practical examples are yet to be provided. Consequently, we aim to apply our model to a specific example, such as a natural ecological model, and offer relevant environmental explanations for this concrete example. In addition, some birds, like bats, have hibernation and birth pulse happens at different times of the year [36]. This motivates us to incorporate the hibernation into our current model system.

In Chapter 3, we studied a time-delayed nonlocal reaction-diffusion model of within-host viral infections. When the basic reproduction number $\mathscr{R}_0 > 1$, we proved that the disease is uniformly persistent for the main system, and the system admits at least one positive steady state. We further showed the global attractivity of the positive constant steady state in the case where all coefficients and reaction terms are spatially homogeneous. However, the global stability of the positive steady state is still a significant challenging problem for the main system when $\mathscr{R}_0 > 1$. In addition, our model is only autonomous. In contrast, typically, these parameters, such as infection rates, mortality rates and incubation periods, are affected by factors such as temperature and climate (see, e.g., [51, 86, 101, 127, 129, 147]). Therefore, we may further study the extended model with the time-period parameters (see, e.g., [4, 51, 65, 101, 127, 129, 138, 144, 147]) and even the more general non-autonomous model (see, e.g., [146]). More specifically, our current model can be applied to the actual within-host transmission of viruses, including but not limited to HBV, HCV, and HIV.

In Chapter 4, we thoroughly investigated the traveling wave solutions for $c > c^*$, $c = c^*$ and $0 < c < c^*$ when $\mathscr{R}_0 > 1$. Our model originates from Section 3.3.2 and serves as a simplified version of the primary model presented in Chapter 3. That is, on the one hand, similar to [128, 141], we assumed that it takes equal time for healthy target cells to become infected after exposure to infected cells and viruses, i.e., $\tau_1 = \tau_2 := \tau$. On the other hand, the bilinear incidence rates are the simplest case of f_i. The prototypical examples of the functions f_i satisfying (A2) include the Holling type II functional response, Beddington-DeAngelis functional response, saturation infection rate, and Crowley-Martin functional response. Therefore, we can continue to study more general models. Moreover, we further study the conjecture that the minimum wave speed c^* is also the spreading speed. In addition, the diffusion rates of target and infected cells are minimal, or they may not move at all while the virus particles spread (see, e.g., [61, 107, 122, 139]). Therefore, we can assume $D_T = D_I = 0$ and $D_V > 0$ based on system (4.3). We further investigate the dynamics of this partially degenerate system.

In Chapter 5, our numerical results indicate the existence of a positive periodic solution when $\mathscr{R}_m > 1$ and $\mathscr{R}_0 > 1$. However, it remains an open problem to establish this result through mathematical proof. We developed a nonlocal reaction-diffusion model of Chikungunya disease with temperature and rainfall effects in this chapter. Aedes aegypti and Aedes albopictus are the main vectors of Chikungunya virus [96, 130]. Unlike other mosquito vectors, Aedes aegypti usually lays eggs in areas without water. As a necessary condition, mosquito eggs require immersion in water during incubation [115]. Hence, we can study the transmission of Chikungunya disease along a theoretical river to characterize environmental heterogeneity more accurately. So we may extend the model formulated in Chapter 5 by incorporating location-dependent diffusion coefficients and general boundary conditions at the downstream end of the river.

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