Applications of Quantum Stochastic Calculus and Open Quantum Systems, in the Modelling of the Financial Market

by

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Abstract

In this thesis we develop methods that enable the study of the phenomenon of noncommutativity in financial markets. In many cases, for example the purchasing of an item from a supermarket, prices behave in an essentially commutative fashion. Whilst the price may increase or decrease over time, when one comes to execute a transaction, the price is generally known in advance, and is not impacted (within reason) by the size of the transaction or what else is going on in the supermarket.

Often financial markets do not operate in this simple fashion. The execution price for a transaction cannot always be known in advance, and factors such as the size of the trade, and what other trades happen in the lead up to our transaction, impact the price we will achieve. This motivates us to study financial markets using quantum probability.

We start by applying the techniques of quantum stochastic calculus, and building on the quantum Black-Scholes approach of Accardi and Boukas. We then investigate the use of the open quantum systems method. We define an interaction between the financial market and its external environment, before using the singular coupling limit to derive a Markovian approximation to the master equation for the market dynamics. We go onto consider the role of the information entropy in the modelling of the financial market, and consider the relation between the classical and quantum approaches in this regard.

We also show how to incorporate financial effects such as trade size, market risk appetite and bid-offer spread, that arguably lie behind noncommutativity, into the modelling approach. In each case, we investigate the impact of introducing noncommutative like behaviour to classical models, using analytical and numerical methods.

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Chapter 1

Introduction

1.1 The Financial Market as a Non-commutative System

The objective in this thesis is to apply the ideas of non-commutativity to the modelling of the financial market. In particular, we seek to model the financial market price as a random variable, in a non-commutative space, and to derive what the impact of non-commutativity is on the resulting probability distributions and on how the market prices may evolve in the future.

Since the articles by Black & Scholes (see [12]), and Merton (see [47]), there have been a wide variety of models of the dynamics of traded financial market prices using stochastic calculus. These models seek to build the probability distributions for the long term behaviour of market prices from the ground up, by specifying in detail the random walk.

Alternatively, some authors have focused on the statistical properties of the daily changes in asset prices, and sought to fit known probability distributions. For example, in [40], Mandelbrot looks at the notion of "wild randomness" and the properties of the so called "stable distributions".

In both of these approaches, the assumption is made, either explicitly or implicitly, that the financial

market exists in a *commutative* probability space. One makes the assumptions that:

- One can measure the current state of the financial market with an unlimited degree of precision.
- One can carry out a measurement of the state of the financial system without impacting the state of the system.

In fact, one can view the financial market as a key example of a non-commutative system. In general, it is not possible to trade at exactly the price one observes on a trading platform. Not only is this price generally an estimate based on more granular data regarding potential buyers & sellers within an exchange, but in order to actually execute a trade, one must submit a buy or sell order to the exchange mechanism, thus changing the current state of the market on which the estimate is based. Indeed, investment firms are often paid large commissions to execute trading orders at the best possible price, and complex execution algorithms are designed with this in mind. This would of course not be necessary in a purely commutative world, where one could execute at the exact price one observes on a trading platform.

In the commutative approach to the modelling of the financial market, one simply models the market price as a random variable with an associated probability distribution, and models the dynamics (for example) by using a stochastic process. In a non-commutative approach, one will generally incorporate the following components:

- Information representing the state of the financial market. For example, one might want to quantify the depth of liquidity, or the current best available bid and offer prices.
- Operations defined on the market that both have an impact on its' current state, and return some real world information. For example executing a trade at the best available price.

Models based on commutative probability space, such as those that apply classical stochastic calculus, building on the ideas in [12] and [47], are widely used by practitioners in the financial industry (such as hedge funds, investment banks etc). These approaches are often very successful, for example to the extent that they allow trading desks to price & hedge the risk that they hold on the balance sheets of financial institutions. Indeed, especially in highly liquid markets, the assumption of commutativity may hold approximately true. Therefore, one crucial requirement in this research is to ensure that we develop frameworks that:

- Incorporate models with dynamics that match or approximate the dynamics implied from existing commutative approaches.
- Also allow for the introduction of non-commutativity in varying amounts. For example, by varying input model parameters. We show how to do this using a non-zero bid-offer spread, together with a rotation operator, in section 3.5.1 and 3.5.2.

It is by allowing the gradual introduction of non-commutativity to models that otherwise closely match standard models, that one can observe whether any of the anomalous effects that arise as a result, match behaviours of real financial markets.

1.2 Applications of Quantum Stochastic Calculus:

1.2.1 Introducing Non-commutative Effects:

In the non-commutative approach to financial modelling, the market is generally described by a quantum state acting on a Hilbert space, and traded prices by self-adjoint operators acting on that Hilbert space. Quantum stochastic calculus is a mathematical technique that allows the introduction of random noise to these operators.

Following the quantum stochastic approach of Hudson & Parthasarathy (see for example [34]), Accardi & Boukas show how to use this framework to design a risk neutral pricing equation in [16]. In many ways this equation is the non-commutative equivalent of the type of parabolic partial differential equations that result from classical stochastic calculus (for example the Black-Scholes partial differential equation), and indeed the authors go onto show how the classical Black-Scholes can be derived from their approach. Using this as the starting point we extend the modelling in [16] as follows. We first suggest how to construct a Hilbert space that allows for the modelling of both the market bid & market offer prices. We then go on to show how this allows us to distinguish between different types of trade execution. For example in section 2.5.5 we discuss the difference between a market order, where one simply purchases at the best available market offer, or sells at the best available market bid price. This enables the participant to execute immediately, whilst paying extra for the quick execution.

We show how this setup enables us to define price operators with a parameter that controls the size of the trade, and in equation 2.13 show how the expected execution price depends on the size of the trade. Furthermore, in section 2.5.7, we develop an uncertainty relation based on the non-zero commutator for two trades of differing sizes.

In section 3.6 we introduce some of the anomalous impacts of the setup, and in particular the impact of parameters including the size of the bid-offer spread and trade size on the moments of the resulting random process, and contrast these against effects observed in real market prices.

1.2.2 Technical Model Development:

Chapter 2 focuses on proposing a new Hilbert space structure that enables the capture of non-commutative effects. Then in sections 3.3 and 3.4, we show how to apply quantum stochastic calculus techniques from [16] and [34] to the development of a risk neutral pricing equation. Classically speaking the intuition

behind an arbitrage is the possibility of generating financial gain, without the possibility of experiencing losses. In the quantum context, the gain or loss will depend on the quantum state of the market. Therefore, in this thesis we suggest two new alternative definitions of non-arbitrage prices in the quantum context:

- The weak non-arbitrage price, definition 3.4.4, which represents a price operator together with a market state, such that no arbitrage exists.
- The strong non-arbitrage price, definition 3.4.5, whereby we have a price operator such that there is no market state whereby an arbitrage exists.

The classical Black-Scholes partial differential equation leads to a strong non-arbitrage price, in the sense that a solution represents a unique non-arbitrage price, and that the quantum state of the market is irrelevant.

In section 3.5.1, we show that by incorporating the type of non-commutative effects discussed above, into the model, we end up in the situation whereby there is no strong non-arbitrage price. In fact we develop the model further by allowing the observables to handle both the bid and offer prices. We go onto show how, from a mathematical perspective, this allows sufficient structure for the development of non-commutative operators, which then leads to the anomalous effects discussed. For example the impact of trade size, and non-zero bid-offer spread, on the resulting variance, as highlighted in equation 3.46.

1.2.3 Non-Gaussian Models:

The classical Black-Scholes, as well as the models discussed in section 3.5, are based on creation & annihilation operators acting on a Boson Fock space. This introduces random noise that is essentially Gaussian in nature. That is, the random processes used represent a quantum equivalent to classical Brownian motion.

In fact, the unitary evolution of a quantum stochastic process allows for more general Levy processes. In sections 3.5.2, and 3.7, we develop models whereby the size and frequency of jumps in the market price are characterized by the width of the bid-offer spread. Where the bid-offer spread reduces to zero, one finds an infinite number of infinitesimal jumps, and one gets back to Brownian motion. Where the bid-offer spread gets larger, characteristic jumps get bigger. From a financial perspective, this indicates the onset of illiquidity in the underlying price.

From a mathematical perspective, the resulting partial differential equations contain an infinite series of singular terms, whereby the impact of these terms scales with the ratio of the bid-offer spread (denoted ϵ) with the classical volatility over a time-step δt: $\epsilon/\sigma^2\delta t$. As the bid-offer spread increases, the singular terms in the partial differential equations have a larger effect. Similarly, as we look further into the future, the impact of these terms reduces gradually to a negligable level.

We develop a number of results to help study these models. For example, in proposition 3.6.2 we showed that a widening in the bid-offer spread is associated with a non-zero skew, and excess Kurtosis over and above the standard Gaussian kurtosis.

In chapter 4 we investigate solutions to the partial differential equations describing these models, using asymptotic methods. These include the associated forward Kolmogorov equations which we is derived in proposition 3.5.6. We derive approximation methods based on power series expansions, and derive results regarding the accuracy & convergence, before illustrating where the approximations break down.

1.2.4 Two Factor Models:

In section 3.5.1, whilst the bid-offer spread is modelled as a system coordinate, in addition to the market mid price, we only include a single observable, to represent the action of executing a trade (whereby there will be both a buyer and a seller). We also only include a single stochastic process.

In section 3.5.2 we simplify things further, with a view to studying solutions to partial differential equations with non-zero bid-offer spread, and introduce a more general Levy process, beyond Brownian motion. In fact there are many instances where one may wish to monitor the market beyond just the evolution of the mid-price. Therefore, in section 3.7, we define an observable that allows the measurement of the bid-offer spread, and a second stochastic process that controls the time-evolution of this second variable. In practice, the width of the bid-offer spread is often determined by the balance between competing factors. For example:

- The need for market makers, or liquidity providers in general, to quote a sufficiently wide bid-offer so that the profit per trade compensates them for risks such as the risk of adverse selection.
- The incentive for market makers, or liquidity providers in general, to narrow the bid-offer spread, thereby increasing the volume of business.

With this in mind we discuss how to incorporate the economics of the bid-offer spread into a potential function, and in propositions 3.7.5 and 3.7.6 we outline a two factor model. This model incorporates a stochastic bid-offer spread, whereby the model can randomly diffuse into periods of illiquidity, where the financial market develops non-commutative behaviour, and periods of high liquidity, where the market more closely represents classical Black-Scholes type behaviour. This model can be applied to the simulation of both the execution prices for trades, and the bid-offer spread.

1.3 The Open Quantum Systems Approach:

1.3.1 Financial Market Entropy:

In chapter 5 we propose an alternative means by which the time evolution of the financial market, and related uncertainty, can be modelled. This approach focuses on the interaction between the financial

market and its' environment, and allows us to consider the role of the information entropy.

Classically, when modelling market prices one typically focuses on properties of the probability distribution such as the variance and other statistical moments. In fact, as one looks further into the future, the fundamental change is an increased uncertainty in the future price, and a resulting increase in the information entropy associated with the probability distribution.

In a classical random walk, both the total variance and the information entropy increase monotonically with time. If one models the financial market as a quantum state acting on a Hilbert space, then one can have arbitrarily high variance in a state with zero entropy. For example in the case of unitary time evolution of a pure quantum state, where the entropy remains at zero.

It is known that (see also proposition 5.3.6), given the set of quantum states that fix the probability distribution for a real world observable, it is the state that equates to the classical random walk that maximises the Von-Neumann entropy. Thus, we can consider the classical approach as that which minimises our knowledge regarding the state of the market.

In fact, we give examples of where there may be a varying degree of information regarding the state of the market. For example we may know the price of listed option contracts and the implied volatility smile. We may know how the operation of the exchange mechanism impacts price formation, together with information regarding different market participants and their motivations. Crucially the quantum approach illustrated in chapter 5 allows one to study models with varying levels of growth in the information content of the market, and considers the anomalous behaviour that results from non-classical modelling.

1.3.2 Market Interaction and The Changing Risk Appetite:

In the open quantum systems approach we model the interaction between the market space, which we carry out measurements on by executing trades, and its' environment. Crucially, the approach (see for example [17]) applies the partial trace to reflect the fact that when one executes a trade, one gains information on the price itself, but not on the factors and interactions that lead to the formation of the price. Whilst the time evolution of the full system of market and environment is unitary, the evolution of operators that act only on the market, and provide no information regarding the state of the environment, is generally non-unitary, and is associated with an increase in the Von-Neumann entropy of the market. In section 5.4.1, we start by defining the nature of the interaction between the market and the environment. We choose to focus on changes in the general risk appetite (in the environment space) and the associated increase/decrease in price (in the market space). In order to model changes in the risk appetite, we introduce a finite dimensional Hilbert space: $\mathbb{C}^K$, whereby the set of eigenvectors represent increasing levels of risk appetite. By switching levels the environment interacts with the market by shifting the traded price up/down. For example a shift to a more bullish market state, with a higher risk appetite, would be associated with an increase in the price.

We then take the tensor product of this finite dimensional space, with the space of square integrable functions on a closed subset of the real line. From a technical perspective, this ensures the system Hamiltonian returns a continuous spectrum, which is required for convergence of the resulting integrals. From a financial perspective, states acting on this space configure the energy required in switching between the different levels of market risk appetite.

1.3.3 Lindblad Master Equation:

The key to the open quantum system approach is the calculation of a reduced density matrix, denoted ρ_{mkt}, by taking the partial trace over the environment. It is this state that returns the expected results of measurements on the market. The time evolution of ρ_{mkt} is described by proposition 5.4.7, which is the most general equation in this chapter, from which different models of the financial market can be derived.

In general the behaviour described by equation 5.4.7 is non-Markovian. In fact, there is evidence that real financial market prices are also non-Markovian. However, most classical models used by practitioners are Markovian in nature, and this property is generally considered a requirement for the non-arbitrage approach to financial modelling (see [10]). Indeed, many financial assets are at the very least Markovian to a reasonable approximation. Therefore, in proposition 5.4.6 and 5.4.9, we show how to use the strong coupling limit to derive a Markovian approximation to the general equation. We discuss the financial and technical reasons why the strong coupling limit is more applicable in this instance than the weak coupling limit (which can also be used to derive a Markovian approximation).

1.3.4 Classical and Non-Classical Illustrations:

The Markovian approximation can be used to describe many different models, depending on the precise choice of operators and environment state. We first show in sections 5.4.5 (finite dimensional case) and 5.4.6 (infinite dimensional case) how one can represent classical equations using proposition 5.4.9. This is an important step, first because in many cases financial markets are approximately commutative in nature, and secondly because it acts a starting point for the gradual introduction of non-commutativity, via more complex examples.

Two different ways in which we can introduce non-commutative behaviour are investigated in sections

5.5 and 5.6. In order to understand these, it is first helpful to consider the finite dimensional case. Ie $\mathcal{H}_{mkt} = \mathbb{C}^N$. In general, one cannot discuss non-commutative state or non-commutative observables without mentioning the choice of basis: $|e_i\rangle$. One will generally be able to diagonalise a matrix, and therefore one can only consider whether the state and observable commute relative to each other. However, the precise interpretation of the observables as returning tradeable prices, fixes the choice of basis. For example, one can fix the eigenvector $|e_i\rangle$ as returning the fixed price x_i with certainty, given the traded price observable X. Importantly this allows us to consider:

- In section 5.5, we look at *non-commutative observables*. For example particularly large trades where there is a fundamental limitation to the precision with which one can know the price in advance. Even where the market is in a price eigenstate such as $|e_i\rangle\langle e_i|$.

- In section 5.6, we look at *non-commutative states*. If the market reduced density matrix is diagonal, this represents the (essentially classical) case whereby we know the market will be found in an eigenstate, however we don't know which one. In section 5.6 we look at the use of non-diagonal states, and show that, as is the case for non-commutative observables, this leads to non-classical terms in the resulting master equations.

1.3.5 Numerical Results:

In section 5.7 we run some illustrative numerical simulations of both the Gaussian cases and the cases outlined in section 5.5 and 5.6. The key objective is to understand how the introduction of non-commutativity impacts the growth in both the Von-Neumann entropy and the variance over time. With this in mind we derive propositions 5.7.2 and 5.7.4 that quantify the variance in both cases. From studying the resulting equations, we are able to isolate the standard classical terms, and the terms that explicitly arise from the quantum nature of the model. Crucially, this type of analysis allows us to not only visually compare

the charts between those models with a classical representation, and the non-commutative models with no classical representation, but also to calculate the difference numerically.

We discuss the interaction between the growth in the total variance and the growth in the Von-Neumann entropy. We also consider the impact that different choices for the initial state make, before going on to illustrate how the excess kurtosis grows over time for the non-Gaussian cases.

1.4 Literature Review:

We begin by reviewing the work of Bouchaud (see [37], and [38]) which applies methods from statistical physics to the modelling of the financial market. This is a good place to start since the research seeks to apply a more general approach to financial modelling, and in particular derivative pricing, than is often found in existing classical models based on stochastic calculus. Bouchaud seeks to clarify the purpose of the modelling, before applying methods from the world of physics. We then go on to review some of the existing quantum approaches in the literature. Specifically, we look at:

i) The quantum binomial approach of Chen [21], which is discussed further in [27]/ [61], and [43]/[44].

ii) The Path Integral approach to problems in finance, which has been developed by Baaquie in [1], [2], and [3], and by Linetsky in [39].

iii) The approach initiated by Haven in [28]/[29], and developed further by Contreras et al, in [22]/[42].

iv) The continuous quantum measurement approach, introduced in a financial context by Melnyk & Tuluzov in [45].

v) The quantum harmonic oscillator model, described by Orrell in [54] (see also [51], [52] and [53]).

vi) The operator focused approach suggested by Segal & Segal in [58], which was developed using quantum stochastic calculus, by Accardi & Boukas in [16].

In each of the above cases, much research has been carried out, and it will not be possible to comprehensively cover everything. However, in this section we aim to highlight the core principals and to clarify how it relates to this current project.

1.4.1 Financial Modelling using Statistical Physics:

The starting point to the research presented in [38], is that whilst the future returns for a particular financial asset may be random (and therefore hard to model), consistency in the mechanisms underlying the formation of prices allows the application of probability distributions to assess the likelihood of returns being at different levels.

1.4.1.1 Random Walk Models

The next key assumption is that the probability distribution for the future price returns is the result of a random walk: that one can observe a price at one point in time, prior to it making a random change to a new value after a specified time interval: δt. In other words that the resulting distribution is the result of a sum of identically distributed random variables.

Finally, the starting assumption is made that the returns are independent from each other over most time scales. Thus one is lead to looking at the n-fold convolution of independent and identically distributed (i.i.d) returns ($*$ denotes the convolution):

$$P(x) = [\phi]^{*N}$$

If the moments for $\phi(x)$ are defined, then the Central Limit Theorem applies. If the mean for $\phi(x)$ is

given by m, and the variance by σ^2, then we have:

$$\lim_{N\to\infty} P\Big(u_1 \leq \frac{x - mN}{\sigma\sqrt{N}} \leq u_2\Big) = \frac{1}{\sqrt{2\pi}} \int_{u_1}^{u_2} exp(-u^2/2)du \tag{1.1}$$

Whilst this result seems to suggest that the use of Brownian motion as the building block for conventional models of the financial market, is the obvious choice, it is important to note the following:

- Depending on the choice of ϕ, the convergence to a Gaussian distribution can be slow. Therefore, the approach outlined allows for a wide variety of models with different short & medium term behaviour.

- Equation 1.1 applies locally and does not prescribe the behaviour in the extreme tails of the distribution.

There has been much empirical research (see for example [38], [40], [41]) to support the fact that the tails of the distribution of financial returns are governed by a power law distribution, rather than a Gaussian distribution:

$$P(x) \sim \frac{\mu A^\mu}{|x|^{1+\mu}}, x \to \pm\infty$$

In fact, if one relaxes the assumption that all the moments for the marginal distribution of returns: $\phi(x)$ are defined, then the N fold convolution: $[\phi]^{*N}$ tends to the Levy-stable family of distributions, which can lead to power law tails.

Regardless, the approaches investigated in [38] allow one to match the statistical properties of the financial market to the choice of the distribution: ϕ. Different choices for $\phi(x)$ will lead to different properties. For example the behaviour of the distribution in the tail, the kurtosis & skew, and the rate at which the distribution converges to a Gaussian or Levy-Stable distribution.

Conventional approaches to the modelling of the financial market, and in particular derivative pricing, are based on the assumption that one can completely eliminate risk together with the fact that by applying

models based on Brownian motion, one can construct solvable partial differential equations. A fact which is of considerable convenience.

However, in [38], Bouchaud outlines an alternative method for derivative pricing, that is closer to the reality of hedging a derivative position, and is capable of adapting to a wider variety of choices for the probability distribution of returns. This is outlined in the next section.

1.4.1.2 A General Approach to Derivative Pricing:

The concept of creating a risk-free portfolio is often used in derivations of partial differential equations, such as the Black-Scholes equation, that are used for derivative pricing. For example, in some derivations of the Black-Scholes model, an option position is perfectly hedged by buying or selling an appropriate amount of the risky underlying at each point in time. The resulting price is then the unique risk neutral price for the derivative, assuming the underling model (in the Black-Scholes case, the financial market is assumed to follow Geometric Brownian Motion) is a true description of reality. In [38], Bouchaud points out that:

- The risk neutral price is model dependent. If the market does not follow the assumed dynamics, then there will be residual gains or losses. Indeed, the identification of bias in the daily PL of a hedged derivative position is a key determining factor that can lead to trading desks updating the model being used.

- The reality of discrete re-hedging intervals means that fully eliminating risk is not possible. The fact that hedges cannot be continuously rebalanced means the daily PL will be a probability distribution, with mean of zero if the model is correct, and where the variance can be reduced by increasing the frequency of re-hedging.

Therefore, in the absence of perfect information Bouchaud proceeds as follows:

1) If the overall gain/loss in wealth over the life of the product is written: ΔW, then the fair price should be, for a chosen probability law P, $E^P[\Delta W] = 0$.

2) The hedging strategy should be chosen to minimise: $E^P[\Delta W^2] - E^P[\Delta W]^2$. In this sense, the choice of hedging strategy can be framed as a variational problem.

Using this approach, Bouchaud goes on to make two crucial points, both of which are highly relevant in respect to the current project.

- If one is happy to restrict oneself to probability laws derived from Ito processes, then it is possible to eliminate risk completely. However, in the general case, for example where there is excess (possibly even infinite) kurtosis, this is not necessarily possible.

- Again, if one is happy to restrict oneself to Ito processes, then the drift (defined as the expected return in the risky asset in a given time interval) does not impact the price. This is a key principal underlying the idea of risk neutral pricing, and is linked to the point above. However, in the general (non-Ito) case, this no longer holds. Bouchaud shows that the level of drift will impact the fair price of the option.

1.4.1.3 Conclusions:

The general approach to derivative pricing that Bouchaud presents is highly relevant to the current project. It is not at all clear that it is possible to construct a "risk free" portfolio in a noncommutative market. Therefore, whilst one can derive a risk neutral price, it is not at all clear that this is the price one should use in practice.

For example, how much should a trader charge for issuing an option in a noncommutative market? Should they charge the risk neutral, non-arbitrageable price? If the trader cannot, with full precision, determine

the exact market price, is it possible in practice to execute the perfect delta hedging strategy? Even if one can calculate a unique arbitrage free price, since the trader cannot eliminate risk, it is questionable whether this price is appropriate.

The primary objective of this project is to derive appropriate probability laws that a noncommutative market obeys. In other words, a first step is to identify an appropriate probability law for an observable X, in which we have: $E^P[X] = x_0$, where x_0 represents the current price. As a second step, the approach described in 1.4.1.2, seems much easier to generalise to the noncommutative case, than the existing risk neutral pricing approach.

1.4.2 Quantum Binomial Approach:

This approach was suggested by Chen in [21], and is further discussed in [27], [61], [43], [44]. The market state is represented by vector $\psi \in \mathbb{C}^n$, such that we have:

- $\psi = (q_1, q_2, ..., q_n)^T$
- By normalisation we have: $\sum_i |q_i|^2 = 1$.
- For our model to be arbitrage free, we require $|q_i|^2 > 0$.
- We can define a self-adjoint price operator to represent the stock price: $S = diag(s_1, s_2, ..., s_n)$, so that $S\psi = (s_1 q_1, s_2 q_2, ..., s_n q_n)$.
- In this case, we use the standard scalar product in $\mathbb{C}^n$ so that the expected value for the stock price is given by:

$$E^{\psi}[S] = \langle \psi | S\psi \rangle = \sum_i |q_i|^2 s_i \tag{1.2}$$

- By the spectral Theorem, we can form derivative payouts: $F(S) = diag(f(s_1), f(s_2), ..., f(s_n))$, for some complex function: f.

We can also represent the market state function as a self-adjoint density matrix: ρ, such that ρ is a linear sum of projection matrices that project into the eigenspace of a series of orthonormal basis vectors. Our pricing equation becomes (for derivative payout F):

$$E^{\rho}[F] = Tr[\rho F] \tag{1.3}$$

The key then becomes how to define ρ. For example, one might choose a self-adjoint Hamiltonian and evolve the quantum state from its value today (say ρ_0) to the value at maturity (say ρ_T).

In this setup, if we start with a commutative (or diagonal) state, and a commutative derivative payout operator, then we end up with:

$$E^{\rho}[F] = \sum_i p_i f(s_i) \tag{1.4}$$

$$p_i = |q_i|^2$$

In the continuous limit, this converges to the expectation function on a classical probability space:

$$E^{\rho}[F] = \int_{\Omega} f(\omega) dP(\omega)$$

We can generate non-commutative states, and payouts, using unitary transformations:

$$F = U_1^* diag(f(s_1), f(s_2), ..., f(s_n)) U_1$$

$$\rho = U_2^* diag(p_1, p_2, ..., p_n) U_2$$

$$p_i = |q_i|^2$$

1.4.2.1 Quantum vs Classical Volatility

We now illustrate the impact of non-commutativity, using a simple example, of the type considered in [43]. Consider a 2 state 'classical' model:

- We assume that the current price is given by: $S_0 = 1$, and that future values for the stock price are given by: $S_T = \begin{bmatrix} 1.1 & 0 \\ 0 & 0.7 \end{bmatrix}$
- State given by: $\rho = \begin{bmatrix} 0.75 & 0 \\ 0 & 0.25 \end{bmatrix}$, so that there is 75% chance of the stock price being $S_T = 1.1$, and 25% chance of the stock price being 0.7.
- Now, the ATM call option is given by: $\begin{bmatrix} 0.1 & 0 \\ 0 & 0] \end{bmatrix}$.
- We find that our model is a Martingale, and that:

$$mean(S_T) = 1 = S_0$$

$$variance(S_T) = 0.03$$

$$OptionPrice = Tr[\rho F] = 0.075$$

Alternatively, consider the following model:

$$S_T = \begin{bmatrix} 1.1732 & 0 \\ 0 & 0.8268 \end{bmatrix}, \rho = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, F = \begin{bmatrix} 0.1732 & 0 \\ 0 & 0 \end{bmatrix}$$

- In this model we find:

$$mean(S_T) = 1 = S_0$$

$$variance(S_t) = 0.03$$

$$OptionPrice = Tr[\rho F] = 0.0866$$

Therefore, despite the fact that the mean & variance are unchanged from above, the ATM call option has a higher price. Note further that the state ρ has eigenvalues 0 and 1, meaning that it is a 'pure' quantum state. In other words, unlike the first model, we know the current market state with complete certainty.

1.4.2.2 Noncommutativity in the Binomial Approach:

This approach highlights the fact that quantum formalism can be used to introduce non-trivial extensions of classical models of tradeable prices.

In the first example of the previous section, there was uncertainty with regards to what the future quantum state of the market would be. There was a 75% chance of finding a quantum state such that the stock price was 1.1, and 25% chance of finding a quantum state such that the stock price was 0.7. Therefore, we have essentially 'classical" variance in the stock price. In the future the stock price could go up or go down, depending on uncertain influences from outside the system. In the second example, the future quantum state is known with certainty. The variance in the future stock price is a purely quantum effect internal to the system.

The crucial question now arises, with regards to time evolution. Classically this could be defined through a binomial random walk. At each time-step the stock price jumps up or down. After m time-steps, we have $n = 2^m$ different possible values for the stock price, and the quantum state is a vector in $\mathbb{C}^n$.

From a quantum perspective, we could write the state as an m fold tensor product of the space $\mathbb{C}^2$. Then a Hamiltonian operator, $\hat{H}$, could be defined using the m fold tensor product of a 2 by 2 matrix. The time-evolution of the state would then be assumed to be defined via the Schrödinger equation, where $\hat{H}$ is self-adjoint:

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= -i\hat{H}\psi \\ \psi_T &= e^{-i\hat{H}T}\psi_0 \end{aligned} \tag{1.5}$$

Classical intuition tells us that as the time to maturity increases, and therefore the variance in the underlying price increases, the expected value of a derivative with strictly positive and convex pay-off will increase. This is indeed what is seen in classical models, where the variance increases linearly with time, and vanilla options have positive time-value.

Time-evolution operators of the form used in equation 1.5 form a unitary group, and are therefore reversible. There is no associated increase in the entropy of a system undergoing such time-evolution. In other words, if the system starts in a pure quantum state, it will remain in a pure quantum state. The time-evolution does not introduce random uncertainty in the state, although a system which started in an eigenstate of the traded underlying will not in general remain in such as eigenstate.

In [21], [27], [61], the time evolution follows an alternative equation (where $\hat{H}$ is assumed to be self-adjoint):

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= -\hat{H}\psi \\ \psi_T &= e^{-\hat{H}t}\psi_0 \end{aligned} \tag{1.6}$$

Time-evolution operators of the form used in equation 1.6 form a semi-group rather than a unitary group. One can postulate that by rotating to complex time:

$$t \to -it \tag{1.7}$$

we move to a system where the time-evolution is irreversible, and there is an increase in the system entropy, that would usually be associated with an external source of noise. We discuss this question further in section 1.3. There we propose extending the analysis discussed above in the following ways:

- We contrast between the uncertainty in the price associated with the fact that the market is not in a price eigenstate, and uncertainty to a lack of information regarding the future market state. Essentially due to external noise.

- We examine the mechanisms by which irreversible increases in entropy are linked to option time value by modelling the interaction of the closed quantum system, which represents the financial market, with an external environment.

1.4.3 Path Integral Methods:

The path integral approach developed by Baaquie in [2]-[3], and by Linetsky in [39] are natural extensions of the methods discussed in section 1.4.2 to the continuous domains.

In this framework, ψ is modelled in the Hilbert space: $L^2(\mathbb{R})$. We start by representing the price today by a wavefunction: $\psi_0 \in L^2(\mathbb{R})$. The time-evolution is then given by:

$$\psi_T = e^{-i\hat{H}t}\psi_0 \tag{1.8}$$

In the general case, the analysis in [24] shows how to construct an integral over all paths: "$\mathcal{D}x$", so that the state vector at time T can be derived from the initial condition: ψ_0, and the Hamiltonian: $\hat{H}$. This approach has a number of benefits. For example, the form of the Hamiltonian offers an alternative means to introduce different dynamics into the model.

- By starting with a conventional Hamiltonian given by:

$$\hat{H} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}$$

- Note that the pseudo-Hermiticity of the conventional Black-Scholes Hamiltonian:

$$\hat{H}_{BS} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \Big(\frac{\sigma^2}{2} - r\Big)\frac{\partial}{\partial x} + r$$

  is discussed by Jana & Roy in [36] with further analysis by Bagarello in [5].

- Developing the idea further, in [1], Baaquie shows how the Path Integral method can be used to obtain exact results for certain stochastic volatility models.

- Alternatively, in [3], Baaquie shows how the impact of supply-demand equilibrium can be incorporated into the dynamics of commodity prices.

- In [33], it is shown how to adapt the path integral method to the Quantum Black-Scholes framework of Accardi-Boukas ([16]).

1.4.3.1 Non-commutativity in Path Integral Methods:

When using the time-evolution unitary operator: $e^{-i\hat{H}t}$ in practice, for example to calculate the probability density: $p(x_t, t|x_0, 0)$, one must integrate over all potential paths starting at x_0, and finishing at x_T. To start with we break down each path into discrete steps, and then take the infinitesimal limit of step size to zero.

Starting with the step from x_{n-1} to x_n, over time interval δt, after integrating over possible values for the momentum p we get:

$$\begin{aligned}\langle x_n|e^{-i\hat{H}\delta t}|x_{n-1}\rangle &= \int_{\mathbb{R}} \langle x_n e^{-i\hat{H}\delta t}|p\rangle\langle p|x_{n-1}\rangle dp \\ &= \int_{\mathbb{R}} exp\Big(- i\delta t\Big(p\frac{dx}{dt} - \hat{H}\Big)\Big)dp \end{aligned} \tag{1.9}$$

When we carry out the momentum integral 1.9, by applying the method of stationary phase (for example see [8]), we find that we must integrate over those paths where the value: $\left(p\frac{dx}{dt} - \hat{H}\right)$ is minimised. This therefore leads to the Legendre transformation, and the standard form for the path integral method, involving the classical Lagrangian:

$$\langle x_n|e^{-i\hat{H}\delta t}|x_{n-1}\rangle = e^{-iL(x,\frac{dx}{dt})\delta t} \tag{1.10}$$

When applying this method to problems in finance, one again rotates to complex time, as per 1.7, leading to the semi-group: $e^{-L(x,\frac{dx}{dt})\delta t}$.

Crucially therefore, the path integral method is a powerful tool for developing semi-analytic results. However, by taking the rotation to complex time, the quantum nature of the system is lost. As discussed above the crucial aims for extending this approach are:

- Building the uncertainty intrinsic to a financial asset price, and market mechanism, into a quantum state.

- Examine the means by which interaction with an external system, introduces irreversible changes & associated entropy increases.

1.4.4 Arbitrage Potential Functions:

Another interesting application for the quantum probability framework is the arbitrage potential function suggested originally by Haven in [29], and developed by Contreras, for example in [22], [42].

As is the case in section 1.4.3, the derivative price is interpreted as a Schrödinger wave function. Haven assumes that every time a trade occurs, the derivative state function collapses to the price given by the Black-Scholes equation. If the the time between trades is infinitesimally small, then the quantum state function does not exist, and the traded price is simply given by the classical Black-Scholes (non-arbitrage) price.

However, if the time between trading activity is finite, then the derivative quantum state exists, alongside the potential for arbitrage. Haven shows that the possibility for arbitrage can be modelled using an interaction potential, and discusses some of the implications. As noted in [29], and [42], the notion of an arbitrage interaction potential can be used to model a number of effects, such as transaction costs, imperfect information etc.

In fact, one can translate the idea of an arbitrage potential function to the proposed setup for this project. For example, consider the market quantum state where the Hamiltonian function is defined by:

$$\hat{H} = \frac{-f(x)\sigma^2}{2}\frac{\partial^2}{\partial x^2} + V(x)$$

Proceeding informally, and writing $\hat{X}$ as the quantized variable for the traded price x (acting as usual by pointwise multiplication), then if $\hat{X}$ is a Martingale we have:

$$E^T[\hat{X}] = E^0[\hat{X}]$$

From the Heisenberg equation of motion we have:

$$\frac{d\hat{X}}{dt} = i[\hat{H}, \hat{X}] = f(\hat{X})\sigma^2\hat{P}$$
$$\hat{P} = -i\frac{\partial}{\partial x}$$

Further we have:

$$\frac{d\hat{P}}{dt} = i[\hat{H}, \hat{P}] = -\frac{dV}{d\hat{X}} + \frac{\sigma^2}{2}\frac{df}{d\hat{X}}\hat{H}$$

So it is clear, that in the event that we have a non-zero potential function in this setup, our traded price represented by operator: $\hat{X}$, is not a Martingale, and the model is not arbitrage free. The application of such a potential function, to model market imperfections, is a useful avenue for future research.

1.4.5 Continuous Measurement Approach:

1.4.5.1 Quantum Analogue of the Black-Scholes under continuous weak measurement:

One approach to the frequent measurement of a quantum system is the field of continuous quantum measurement (see [35] for an overview). The starting point is that frequent measurements of the system are taken with time interval δt. The key idea is that the amount of information obtained from the measurement goes to zero, as the time interval goes to zero. With this in mind, the measurement returns a probability distribution for the observable, which widens with reducing δt. The analysis presented in [35], shows that the resulting observable can be represented as an Ito stochastic process. Where y represents the observed measurement for operator $\hat{X}$, we have:

$$dy = \langle\hat{X}\rangle dt + \frac{dW}{\sqrt{8k}}$$

The state also evolves according to a stochastic differential equation:

$$d|\psi(t)\rangle = \Big(- k(\hat{X} - \langle\hat{X}\rangle)^2 dt + \sqrt{2k}(\hat{X} - \langle\hat{X}\rangle)dW\Big)|\psi(t)\rangle$$

This approach has been developed by Melnyk & Tuluzov in [45], where they look at a quantum analogue of the Black-Scholes, based on the noise introduced by a continuous weak measurement process. After including the system Hamiltonian, the stochastic process for $|\psi(t)\rangle$ becomes:

$$d|\psi(t)\rangle = \Big(\big(-i\hat{H} - k(\hat{X} - \langle \hat{X} \rangle)^2 \big) dt + \sqrt{2k}(\hat{X} - \langle \hat{X} \rangle) dW \Big) |\psi(t)\rangle \tag{1.11}$$

In order to develop the quantum analogue of the Black-Scholes equation, the next required step is to re-interpret the delta hedging argument applied in the Black-Scholes classical derivation.

Classically, for a derivative: $f(S)$ of the traded underlying S, one would hold a portfolio:

$$V = f - \left(\frac{\partial f}{\partial S} \right) S$$

In order to ensure the portfolio was hedged with respect to the underlying. Ie we required $dV = O(dS^2)$. To translate this into the quantum framework, following a similar approach to that outlined by Bouchaud in [38], Melnyk & Tuluzov choose the hedge in order to minimise the variance in the portfolio: $\langle \hat{V}^2 \rangle - \langle \hat{V} \rangle^2 \to \min$. They show the optimal hedge is given by:

$$\begin{aligned} \hat{V} &= \hat{f} - k_{opt} \hat{S} \\ k_{opt} &= \frac{\langle \hat{f} \cdot \hat{S} \rangle - \langle \hat{f} \rangle \langle \hat{S} \rangle}{\langle \hat{S}^2 \rangle - \langle \hat{S} \rangle^2} \\ \langle \hat{f} \cdot \hat{S} \rangle &= \langle \hat{f}\hat{S} + \hat{S}\hat{f} \rangle \end{aligned} \tag{1.12}$$

The final ingredient in their derivation involves the assumption that whatever is considered to be "measured", should have have an expected return given by a classical 'risk free' interest rate r. For example, if one considers that the price of the underlying S is continuously measured (through trading activity), we get:

$$\frac{1}{\langle \hat{S} \rangle} \cdot \frac{d\langle \hat{S} \rangle}{dt} = r \tag{1.13}$$

So in summary we have:

- The stochastic process for $\langle \hat{f} \rangle$, $\langle \hat{S} \rangle$ can be derived from equation 1.11.
- The make up of the optimally hedged portfolio is given by: 1.12.
- The drift of measured variables is given by the risk free rate: 1.13

This leads to the following quantum analogue of the Black-Scholes equation:

$$\left\langle \frac{d\hat{f}}{dt} \right\rangle + r\langle \hat{S} \rangle \frac{\langle \hat{f}\hat{S} \rangle - \langle \hat{f} \rangle \langle \hat{S} \rangle}{\langle \hat{S}^2 \rangle - \langle \hat{S} \rangle^2} + \frac{\langle \hat{f}\hat{B} \rangle \langle \hat{S}\hat{C} \rangle - \langle \hat{S}\hat{B} \rangle \langle \hat{f}\hat{C} \rangle}{\langle \hat{S}\hat{C} \rangle} = r\hat{f} \tag{1.14}$$

$$\hat{C} = k_{opt}(\hat{S} - \langle \hat{S} \rangle)$$

$$\hat{B} = -i\hat{H} - \frac{\hat{C}^2}{2}$$

1.4.5.2 Discussion of the Continuous Measurement Approach:

In the continuous measurement approach, the amount of information regarding the state of the system accrues gradually over time. This does not have an easy financial interpretation, since a trade settles effectively instantaneously at a fixed price. This leads us to consider alternative discrete approaches to measurement.

Under the conventional approach used in quantum physics, the measurement process leads to a collapse of the wave-function into an eigenstate of the relevant observable. Immediately after measuring a particle at position x_0, the wave-function will collapse to a function tightly bound around x_0 (for example depending on the precision of the measuring device). For the purposes of this project we note that we are not restricted in making the same assumptions about the measurement process applied to finance. For example, if I execute a trade with price given by x_0, then it is not clear that if I immediately execute a second trade, I will still be able to execute at x_0.

For much of the project, the objective is to apply the framework of quantum probability in order to investigate the future probability distribution for, and relevant statistical properties of, tradeable securities.

For this purpose, the notion of measurement is not necessarily required.

1.4.6 Quantum Harmonic Oscillator Approach:

In this section we give a brief overview of the model discussed by David Orrell in [54]. This work is partly motivated by the observation that classical approaches often fail to predict certain anomalous features of the dynamics observed in financial market prices, and by the fact that in general one cannot simultaneously observe a financial market price, together with its' rate of change. This in turn leads the author to investigating a quantum approach.

Similar to the model we introduce in chapter 2, an additional motivation for the work is the uncertainty regarding the execution price for a tradable asset. That is, an external buyer or seller can never know in advance what final execution price they will be able to achieve. In the approach discussed in chapters 2, 3, and 4 we tackle this question by considering the random evolution of the market bid & offer prices separately, and give an example of how one can construct different types of orders that test the best price available for an external buyer (or seller) wishing to execute immediately. In [51] Orrell instead considers a potential function made up of supply and demand forces:

$$F_o(X) = k_o(x - \mu_o) \tag{1.15}$$

$$F_b(x) = k_b(x - \mu_b)$$

These forces tend to create a linear restoring force that pulls/pushes the price back to an equilibrium:

$$F_o(x_{eqm}) = -F_b(x_{eqm})$$

Taken together this leads to a harmonic oscillator. Classically one could then proceed by adding a Brownian motion, which would lead to an Ornstein-Uhlenbeck stochastic process. In the model discussed in [54], a quantum approach is used. In other words, the market represents a quantum harmonic oscillator

system with the quadratic potential function 1.15. Each time a trade occurs, the market wave function will collapse to a certain value (the execution price), before evolving deterministically under the influence of the harmonic oscillator Hamiltonian.

In the ground state, the wave-function for the log-price will be normally distributed, and the model will return a flat Black-Scholes implied volatility. However, at higher energy levels the model will return a natural volatility smile similar to that implied by the market prices of listed option contracts. In [54] Orrell investigates the case whereby the energy state is determined by a Poisson process, with a parameter $\lambda = \langle n \rangle$, that determines the expected energy state number, and finds close agreement between the resulting volatility levels, and the implied volatility data for S& P 500 listed options over a period from 1992 to 2022.

1.4.6.1 Discussion of the Quantum Harmonic Oscillator Approach:

The approach outlined in [51]-[54] is powerful in the sense that it shows how to link economic factors, for example relating to the structure of the market, the width of the bid-offer spread, market energy level and the expected rate of transactions, to the price dynamics. However, the approach suggested in this research project follows a different path. In chapters 2, 3, and 4, we build an approach based on the quantum stochastic approach introduced in [34], which has the following benefits:

- The quantum stochastic approach is operator focused, in the sense that random noise is introduced from outside the system, to observables acting on a Hilbert space.
- We then go on to show how this approach can be extended such that non-commutativity starts to take effect.
- We also seek to show how, by extending the market Hilbert space, one can build in noncommutativity, not just in terms of the price and the instantaneous rate of change in the price, but additionally

noncommutativity in different types of trades. For example the impact of executing trades of differing sizes, in differing orders.

Finally, in section 5 we look at how the open quantum systems approach can be applied to finance. The focus in this chapter is on:

- How the interaction of the market with its' external environment (macro-economic events, geopolitics etc) can lead to a gradual loss of information, and thus to diffusion in the current price.
- How to extend simple models by incorporating non-commutative effects.

1.4.7 Quantum Stochastic Calculus:

Segal & Segal comment on the noncommutative nature of financial markets in [58], where they point out that many pieces of information associated with the financial market, lack simultaneous observability. They go on to propose an extension to the classical models, based on a Wiener process: W_t, via the addition of a second process: X_t, which represents the influence of factors that are not simultaneously observable with those factors that are represented by W_t.

The processes W_t and X_t are presented as operators acting on a Hilbert space. Crucially, this enables the extension of the model such that X_t and W_t do not commute, cannot be simultaneously observed, and cannot necessarily be considered as a simple case of adding a second Brownian motion.

Inspired by the work of Segal & Segal, Accardi & Boukas show how to apply the quantum stochastic calculus developed by Hudson & Parthasarathy. In [16], they show how one can construct a self financing hedge portfolio to a derivative position, and use quantum stochastic calculus to derive a general quantum Black-Scholes equation.

As mentioned above, one of the key objectives of this research is to introduce noncommutativity to the existing suite of models used for simulating the random evolution of financial market prices. Accardi &

Boukas show how to derive the classical Black-Scholes equation by making suitable modelling assumptions. Indeed, Hudson & Parthasarathy show how to realise the quantum equivalent of classical Brownian motion using the methods they develop.

In this respect, the work of Accardi & Boukas is the perfect starting point, and in chapters 2, 3, and 4, we show how to extend this framework to give results that go beyond the results of classical approaches. Specifically we show how these methods can be used for modelling some of the anomalous effects associated with the financial market.

Chapter 2

Modelling Illiquid Stocks Using Quantum Stochastic Calculus: Introduction and Hilbert Space Setup

2.1 Chapter Introduction:

Models of the financial market are often concerned with analysing the evolution of a traded market price using probability theory and the mathematics of random processes, rather than using fundamental analysis of the nature of the asset in question.

For example one may choose to model discounted future cash-flows for a particular company, and use these to predict either what the traded price should be, or whether the traded price should rise/fall, and at what rate/to what level. However, in [37] the authors argue that empirical evidence shows the link between external information signals and price changes in the market can be weaker than one might expect, and that the probabilistic approach is therefore justified in certain circumstances.

The fundamental objective of the alternative probabilistic approach to modelling the financial market is to derive probability distributions at future points in time, for a traded price. There are different ways of going about this. For example the statistical analysis of time-series data, and the fitting of different distributions (see for example [40]). Alternatively, encoding changes in the traded price into a particular random process, and using stochastic calculus (or other Mathematical techniques) to determine the probability distribution that results (see for example [10]). Regardless of the relative merits of either approach, both make the assumption that the traded price exists at all times, and that it can be determined with infinite precision.

The reality of financial markets is that most consist of a number of orders to buy a traded asset: *bids*, a number of orders to sell the asset: *offers*, and an automated mechanism designed to match the bids & offers into a single transaction. It is only at the instant a transaction occurs, that a traded price has been determined. Specifically:

- The traded price does not exist at all times. In fact the traded price emerges as a result of a complex process.
- In general, when a market participant enters an order they will not know in advance what their best achievable price is, and what price they will eventually trade at.

Another approach to modelling the financial market that seeks to capture the reality of price formation is

that of Limit Order Book modelling (see [37]). Here a single order has (for example) the following data:

$$X = (\epsilon_x, p_x, v_x, t_x) \tag{2.1}$$

$\epsilon_x = \pm 1$, +1 for a bid, -1 for an offer

$p_x =$ the order price

$v_x =$ the volume, or number of shares etc

$t_x =$ the order time

Limit Order Book models use a variety of strategies for modelling the arrival of orders to the exchange, and match bids & offers, and track financial market statistics such as the price, bid-offer spread etc. The quantum models we go on to discuss in chapters 3 and 4, sit somewhere between the two extremes of modelling the full limit order book, and simply treating the price as a real variable. We do not attempt to model the full granular detail, however we would like to:

- Capture the reality that the market, and the traded asset price are 2 separate entities. In fact, the quantum approach seems to match reality in the sense that the traded price is only realised when a trader makes an observation on the market, by executing a trade/entering a market order.
- Build a model that is capable of handling the fundamental uncertainty around the execution price for a particular order: X.
- Build a model which incorporates some of the non-commutative effects which we discuss below.

In this chapter, we discuss some of the non-commutative effects that can be captured in the model, and propose a Hilbert space structure that can achieve this.

2.2 General Chapter Overview:

In this chapter, prior to adding random noise or time evolution, we must first describe the Hilbert space structure that we use to represent the financial market.

In this thesis, we propose using a new structure based on the direct sum of two Hilbert spaces representing market buyers & market sellers. The choice of taking the direct sum, rather than tensor product, of two Hilbert spaces is based on:

- Taking the Hilbert-Schmidt norm (per [26] definition 19.4) of a vector representing the market as a whole involves taking the sum of the norms for buyers and sellers, rather than the product of these two norms.
- This in turn provides an easy way to represent the case whereby the volume of sellers outweighs the volume of buyers (and vice versa).
- The use of the direct sum allows us to build operators on the full market space using the matrix algebra of operators on each of the two individual components.

Specifically, we assume that we have a wave-function: $\psi_b \in \mathcal{H}_b$ representing potential buyers, and a second wave-function: $\psi_b \in \mathcal{H}_b$ representing potential sellers. The normalizing condition in the direct sum Hilbert space: $\mathcal{H}_b \oplus \mathcal{H}_o$, is given by:

$$||\psi_o||^2 + ||\psi_b||^2 = 1$$

This can be interpreted as meaning that, absent any further information, the probability of any of the individual current market participants (buyers or sellers) being the next to close out (through a transaction occurring) is equal. The probability that a waiting seller is the next to execute a trade is given by $||\psi_o||^2$, and the probability that a waiting buyer is the next to execute a trade is $||\psi_b||^2$.

In section 2.5.5 we show how one can build different operators to measure the expected traded price based on whether one is liquidity provider (or market maker) or a liquidity taker. We start with an operator that returns a higher price when acting on ψ_o, and a lower price when acting on ψ_b. This could represent a liquidity provider who earns the bid-offer spread, or an investor trading in small size, willing to wait in order to obtain a good price.

We go on to show how rotations can be applied to generate alternative operators that return a lower price when acting on ψ_o, and a higher price when acting on ψ_b. This could represent a liquidity taker, or an impatient investor trading in large size, and willing to pay the bid-offer spread for a quick execution. In section 2.5.6 we use this approach to consider the impact of factors such as trade size and market imbalance, and finally in section 2.5.7 we discuss how these different price operators no longer necessarily commute. This leads to the situation, for example, where the price obtained by executing a small trade *before* a large trade will potentially differ from the price obtained *after* a large trade.

Overall in this chapter we study how the proposed choice of Hilbert space structure can give some of the desired modelling properties, without yet adding any random noise, or time evolution into the picture. This is covered in chapter 3.

2.3 Modelling Non-Commutative Effects:

In this section, we consider a simplified setup to illustrate why a Hilbert space framework can be useful for modelling the financial market. We show how this setup can be used to extend previous approaches in the modelling of some non-commutative effects that can be observed in the real financial market. The quantum models of the financial market that we develop have the following key components:

Market Hilbert Space:

We assume that the information regarding the full set of market participants is represented by a quantum

density matrix (in the sense of [26] definition 19.7) acting on a Hilbert space: $\mathcal{H}_{mkt}$. Informally speaking, we assume that $\mathcal{H}_{mkt}$ contains all the information about the available bid & offer prices and the maximum volumes that can be traded at that price. We return to this question in section 2.4.

Expected Market Price:

As mentioned, one central outcome for the current model development is probability distributions that describe what the future price of a tradable asset may be. For this we require a self-adjoint operator that acts on the market Hilbert space to return what the expected price of a transaction is. We go on to discuss this further in section 2.5.5.

Trade Execution:

We also assume that market participants can execute the following:

- Sell a traded equity by entering a sell order or offer. External investors coming to the market can execute a transaction by purchasing at the best available offer price: X^o.
- Purchase a traded equity by entering a purchase order or bid. External investors coming to the market can execute a transaction by selling at the best available bid price: X^b.

This is discussed further in section 2.5.6.

2.4 Hilbert Space Representation of the Financial Market:

We consider the market as being made up of a number of buyers who would like to buy at the lower *bid* price, and sellers who would like to sell at the higher *offer* price.

We therefore assume that the state of the market for potential buyers (and sellers) is determined by wave

functions in the Hilbert space of complex valued square integrable functions on $\mathbb{R}$:

$$\psi_o(x) \in L^2(\mathbb{R}, \mathbb{C})$$

$$\psi_b(x) \in L^2(\mathbb{R}, \mathbb{C})$$

Alternatively, we can write: $|\psi_o\rangle, |\psi_b\rangle$. These wave functions represent the uncertainty in the bid and offer prices at any point in time, prior to a trade occurring. We assume $\psi_o(x, \epsilon)$ represents those market participants who have entered an offer to sell a particular traded equity asset, and $\psi_b(x, \epsilon)$ to represent those who have entered a bid to purchase the same equity.

The question now arises as to how to combine the two into a Hilbert space for the market.

2.4.1 Tensor Product Space:

We could consider 2 variables (bid price, offer price) as being coordinates in a Euclidean space. In which case the Hilbert space would be: $L^2(\mathbb{R}^2)$. It follows directly from [26] Proposition 19.12 that:

$$(\psi_o \otimes \psi_b)(x, y) \to \psi_o(x)\psi_b(y)$$

is an isomorphism between $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ and $L^2(\mathbb{R}^2)$. A point in Euclidean space: $(x, y) \in \mathbb{R}^2$ always has an x coordinate and a y coordinate. Crucially, we would need to specify both to know the state of the system.

Bounded operators on the tensor product space: $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ can be defined as the algebraic tensor product: $B(L^2(\mathbb{R})) \otimes B(L^2(\mathbb{R}))$.

$$(A \otimes B)(\psi_o \otimes \psi_b) = A\psi_o \otimes B\psi_b$$

Thus whilst interaction between the two spaces is possible, we have one set of operators that acts on the first space, and a second that acts on the second space.

In fact, unlike wave functions of coordinates in Euclidean space, the bid and offer wave functions act on the same axis, and we would like to be able to define operators that act on both:

$$A\psi_o$$

$$A\psi_b$$

Furthermore, whilst the buyers that make up a market may wish to transact at a low price and the sellers that make up a market may wish to transact at a higher price, the 2 may occasionally meet in the middle. In other words, we would like inner products like: $\langle\psi_o|\psi_b\rangle$ to have meaning.

Finally, the question arises as to how we wish the state to be normalised. If we have: $\psi_o \otimes \psi_b$, then the state is normalised via:

$$\begin{aligned} ||\psi_o \otimes \psi_b||^2 &= \langle\psi_o \otimes \psi_b|\psi_o \otimes \psi_b\rangle \\ &= \langle\psi_o|\psi_o\rangle\langle\psi_b|\psi_b\rangle \\ &= ||\psi_o||^2||\psi_b||^2 \\ &= 1 \end{aligned}$$

We can also consider the situation whereby there is a tradeable asset, such as shares in a more or less defunct company, where there are no buyers. In this extreme case, we would still like to be able to define a normalisable market state.

For this reason, we look to representing the Hilbert state using the direct sum, rather than the tensor product.

2.4.2 Hilbert Space Direct Sum:

We define:

$$\psi = \psi_o \oplus \psi_b \tag{2.2}$$

$$\psi_o,\, \psi_b \in L^2(\mathbb{R})$$

For $\phi = \phi_1 \oplus \phi_2$ and $\psi = \psi_1 \oplus \psi_2$, we have:

$$\langle \phi | \psi \rangle = \langle \phi_1 | \psi_1 \rangle + \langle \phi_2 | \psi_2 \rangle$$

So it follows that the normalisation condition becomes:

$$\begin{aligned} ||\psi_0 \oplus \psi_b||^2 &= \langle \psi_o \oplus \psi_b | \psi_o \oplus \psi_b \rangle \\ &= \langle \psi_o | \psi_o \rangle + \langle \psi_b | \psi_b \rangle \\ &= ||\psi_o||^2 + ||\psi_b||^2 \\ &= 1 \end{aligned} \tag{2.3}$$

For example, we may have an even balance of buyers & sellers, in which case:

$$||\psi_o||^2 = ||\psi_b||^2 = 1/2$$

In general, as long as the normalization condition, given by equation 2.3, is met then we can have:

- More buyers than sellers: $||\psi_b||^2 > ||\psi_o||^2$.
- More sellers than buyers: $||\psi_o||^2 > ||\psi_b||^2$.

2.4.3 Dealing With Unbounded Operators:

The operators we will be using (for example the conventional position & momentum operators: X & P), are generally unbounded operators, so that $X, P \notin B\big(L^2(\mathbb{R}) \oplus L^2(\mathbb{R})\big)$.

We note (see [26] corollary 9.9) that a symmetric operator A acting on functions in a Hilbert space: $\mathcal{H}$, with domain $Dom(A) = \mathcal{H}$, must be a bounded operator. Therefore, if X & P are to be self-adjoint operators, then we must restrict the domain. Ie $Dom(X) \subsetneq \mathcal{H}$, and $Dom(P) \subsetneq \mathcal{H}$.

However, we would still like to construct operators of the form: $X^k P^l$ and $P^m X^n$, and must therefore ensure we can define a consistent domain, such that X maps functions into $Dom(P)$ and vice versa.

Furthermore, we would like $\mathcal{D} = Dom(X) = Dom(P)$ to be dense in $\mathcal{H}$, in the sense that, for all $f \in \mathcal{H}$ there exists a sequence $f_n \in \mathcal{D}$ such that:

$$||f_n - f|| \to 0$$

We can achieve this for $L^2(\mathbb{R})$ by defining the domain (denoted above as $\mathcal{D}$) for X and P as the Schwartz space $\mathcal{S}(\mathbb{R})$. This consists of functions: $\psi(x)$, such that for all positive integers j, k we have:

$$\lim_{x \to \pm\infty} |x^j \partial_x^k \psi| = 0$$

Since $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, this meets the key requirements.

In our case, we can define an operator: A on $\mathcal{S}(\mathbb{R}) \oplus \mathcal{S}(\mathbb{R})$ by using 4 operators $A_{11}, A_{12}, A_{21}, A_{22}$ each acting on $\mathcal{S}(\mathbb{R})$, and setting:

$$A(\psi_o \oplus \psi_b) = (A_{11}\psi_o + A_{12}\psi_b) \oplus (A_{21}\psi_o + A_{22}\psi_b)$$

Remark 2.4.1. *Going forward, we make use of matrix notation (see for example [48] Theorem 3.4.1 and 3.4.2), so that for $\psi \in \mathcal{S}(\mathbb{R}) \oplus \mathcal{S}(\mathbb{R})$ we write:*

$$|\psi\rangle = \begin{pmatrix} \psi_0 \\ \psi_b \end{pmatrix}$$

$$A\psi = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_b \end{pmatrix}$$

Note that, we also apply the following abuse of notation, by writing:

$$\langle\psi| = \begin{pmatrix} \psi_o & \psi_b \end{pmatrix}$$

So that we write:

$$\begin{aligned} E^{\psi}[A] &= \langle\psi|A|\psi\rangle \\ &= \begin{pmatrix} \psi_o & \psi_b \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_b \end{pmatrix} \\ &= \int_{\mathbb{R}} \overline{\psi_o(x)}\big(A_{11}\psi_o)(x) + (A_{12}\psi_b)(x)\big)dx + \int_{\mathbb{R}} \overline{\psi_b(x)}\big((A_{21}\psi_o)(x) + (A_{22}\psi_b)(x)\big)dx \end{aligned}$$

2.5 Defining Key Operators:

2.5.1 The Realised Price Operator:

The realised price operator, returning the price for a trade execution, can be defined as:

$$\begin{aligned} X \begin{pmatrix} \psi_o \\ \psi_b \end{pmatrix} &= \begin{pmatrix} x + \epsilon/2 & 0 \\ 0 & x - \epsilon/2 \end{pmatrix} \begin{pmatrix} \psi_o \\ \psi_b \end{pmatrix} \\ &= \begin{pmatrix} (x + \epsilon/2)\psi_o \\ (x - \epsilon/2)\psi_b \end{pmatrix} \end{aligned} \tag{2.4}$$

Here, "x" represents the underlying price, and "ϵ" represents the bid-offer spread. We assume ϵ is a constant for the time being.

To illustrate the action of this operator, we consider the following example market, which we set by

choosing $\psi_o(x)$ and $\psi_b(x)$:

$$\psi = \psi_o \oplus \psi_b \tag{2.5}$$
$$|\psi_o(x)|^2 = \frac{1}{2\sigma\sqrt{\pi}} exp\Big(-\frac{1}{2}\Big(\frac{x - x_{mid}}{\sigma}\Big)^2\Big)$$
$$|\psi_b(x)|^2 = \frac{1}{2\sigma\sqrt{\pi}} exp\Big(-\frac{1}{2}\Big(\frac{x - x_{mid}}{\sigma}\Big)^2\Big)$$

In this example, we have deliberately chosen the case whereby $||\psi_o(x)||^2 = ||\psi_b(x)||^2 = 1/2$, reflecting the even balance of buyers and sellers. We can calculate the weighted average expected price, for the market defined by 2.5, as:

$$\begin{aligned} E^{\psi}[X] &= \langle\psi|X|\psi\rangle \\ &= \begin{pmatrix} \psi_o & \psi_b \end{pmatrix} \begin{pmatrix} x + \epsilon/2 & 0 \\ 0 & x - \epsilon/2 \end{pmatrix} \begin{pmatrix} \psi_o \\ \psi_b \end{pmatrix} \\ &= \int_{\mathbb{R}} (x + \epsilon/2)|\psi_o(x)|^2 dx + \int_{\mathbb{R}} (x - \epsilon/2)|\psi_b(x)|^2 dx \\ &= x_{mid} \end{aligned}$$

So, we can see in this case that the weighted average price for a trade is the mid-price. This is a result of the balance between buyers and sellers reflected in 2.5.

2.5.2 Representing Risk Off Market Sentiment:

Now consider a bear market, where existing stock holders are driven to sell at the the lower bid price to close out their position, and potential future buyers of the stocks decide to wait. In common terminology

a "Risk Off" market sentiment. We choose to represent this eventuality using a rotation: $\psi \to R(\theta)\psi$

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$\begin{pmatrix} \psi_o \\ \psi_b \end{pmatrix} \to \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \psi_o \\ \psi_b \end{pmatrix}$$

The effective position/trade price operator now becomes:

$$\begin{aligned} R(\theta)^* X R(\theta) &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x + \epsilon/2 & 0 \\ 0 & x - \epsilon/2 \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} x + \cos(2\theta)\epsilon/2 & -\sin(2\theta)\epsilon/2 \\ -\sin(2\theta)\epsilon/2 & x - \cos(2\theta)\epsilon/2 \end{pmatrix} \end{aligned} \tag{2.6}$$

If we calculate the weighted average price again, we find:

$$\begin{aligned} E^{\psi}[R(\theta)^* X R(\theta)] &= \begin{pmatrix} \psi_o & \psi_b \end{pmatrix} \begin{pmatrix} x + \cos(2\theta)\epsilon/2 & -\sin(2\theta)\epsilon/2 \\ -\sin(2\theta)\epsilon/2 & x - \cos(2\theta)\epsilon/2 \end{pmatrix} \begin{pmatrix} \psi_o \\ \psi_b \end{pmatrix} \\ &= x_{mid} - \sin(2\theta)\epsilon/2 \end{aligned}$$

Thus, the weighted average expected trade price is now lower, reflecting the fact that more market sellers are willing to sell at the available bid prices, rather than holding out for a higher price. For example, where they wish to close out their position in a hurry.

2.5.3 Derivative Operator:

We can define a derivative operator: L as:

$$L \begin{pmatrix} \psi_o \\ \psi_b \end{pmatrix} = \begin{pmatrix} -i\sigma\partial_x & 0 \\ 0 & -i\sigma\partial_x \end{pmatrix} \begin{pmatrix} \psi_o \\ \psi_b \end{pmatrix} \tag{2.7}$$

If we were to define a Hamiltonian operator, representing deterministic drift in the system, then we could interpret this operator as representing the momentum of the stock price scaled by the volatility (σ). The application of the volatility σ will be convenient later on, when we look at constructing a classical model using the framework of quantum stochastic calculus.

2.5.4 Imbalance of Buyers vs Sellers:

If we assume that the probability density functions:

$$p_o(x,\epsilon) = \frac{1}{||\psi_o||^2}|\psi_o(x,\epsilon)|^2 \tag{2.8}$$
$$p_b(x,\epsilon) = \frac{1}{||\psi_o||^2}|\psi_b(x,\epsilon)|^2$$

are represented by the bivariate normal distribution: $N(\mu,\nu)$. Where:

$$\mu = \begin{pmatrix} x_{mid} \\ \\ \epsilon_0 \end{pmatrix}, \ \nu = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_\epsilon \\ \\ \rho\sigma_x\sigma_\epsilon & \sigma_\epsilon^2 \end{pmatrix} \tag{2.9}$$

x_{mid} and ϵ_0 are constants that define the mid price & bid mean bid-offer spread respectively. σ_x, σ_ϵ, and ρ define the covariance matrix ν for the mid-price and bid offer spread. Then we find that the expected trade execution price is given by:

$$E[X] = x_{mid} + \eta_0\epsilon_0 \tag{2.10}$$
$$\eta_0 = ||\psi_o||^2 - ||\psi_b||^2$$

So if the market is made up of an even balance of bids & offers, then we find $\eta_0 = 0$. Since we don't know whether the next liquidity taker will be a buyer or a seller there is an even chance of both, and the expected execution price is the mid price x_{mid}.

If the market only consists of offers, then $\eta_0 = 1$. This represents the situation whereby none of the

liquidity providers are willing to purchase the equity. Therefore, if there is a transaction, it will be a liquidity taker buying at the current offer price.

2.5.5 Expected Market Price: Market Orders vs Limit Orders

A bid that is entered into an electronic market mechanism, which matches one of the offers that is already part of the exchange, represents a *market order*. For example, if the mid price is given by x_{mid}, and the bid offer spread by ϵ_0, then if a bid of $x+\epsilon_0$ is entered, then this trade is executed immediately. Alternatively a *limit order* is a bid of $x-\epsilon_0$, which will simply contribute to the existing stack of liquidity providers: ψ_b.

We now consider a operator that adjusts the probability that a market transaction will have to cross bid-offer spread in order to be executed. As shown in chapter 3, we can represent this by a rotation:

$$\begin{aligned}
|\psi_\theta\rangle &= R(\theta)|\psi\rangle \\
&= \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \psi_o(x,\epsilon) \\ \psi_b(x,\epsilon) \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta)\psi_o(x,\epsilon) - \sin(\theta)\psi_b(x,\epsilon) \\ \cos(\theta)\psi_b(x,\epsilon) + \sin(\theta)\psi_o(x,\epsilon) \end{pmatrix}
\end{aligned}$$

Writing the expected price when the market is in state: $|\psi\rangle$ as: $E^{\psi}[X]$, we have:

$$
\begin{aligned}
E^{\psi_\theta}[X] &= \langle\psi|R(\theta)^\dagger|X|R(\theta)|\psi\rangle \\
&= \begin{pmatrix} \psi_o(x,\epsilon) & \psi_b(x,\epsilon) \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x+\epsilon/2 & 0 \\ 0 & x-\epsilon/2 \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \psi_o(x,\epsilon) \\ \psi_b(x,\epsilon) \end{pmatrix} \\
&= \begin{pmatrix} \psi_o(x,\epsilon) & \psi_b(x,\epsilon) \end{pmatrix} \begin{pmatrix} x+\cos(2\theta)\epsilon/2 & -\sin(2\theta)\epsilon/2 \\ -\sin(2\theta)\epsilon/2 & x-\cos(2\theta)\epsilon/2 \end{pmatrix} \begin{pmatrix} \psi_o(x,\epsilon) \\ \psi_b(x,\epsilon) \end{pmatrix} \\
&= \int_{\mathbb{R}^2} |\psi_o(x,\epsilon)|^2 (x+\cos(2\theta)\epsilon/2) dx d\epsilon + \int_{\mathbb{R}^2} |\psi_b(x,\epsilon)|^2 (x-\cos(2\theta)\epsilon/2) dx d\epsilon \\
&- \frac{\sin(2\theta)}{2} \int_{\mathbb{R}^2} (\overline{\psi_b(x,\epsilon)}\psi_o(x,\epsilon) + \overline{\psi_o(x,\epsilon)}\psi_b(x,\epsilon))\epsilon dx d\epsilon
\end{aligned} \tag{2.11}
$$

Therefore, we can define a general price operator:

$$
X_\theta = \begin{pmatrix} x+\cos(2\theta)\epsilon/2 & -\sin(2\theta)\epsilon/2 \\ -\sin(2\theta)\epsilon/2 & x-\cos(2\theta)\epsilon/2 \end{pmatrix} \tag{2.12}
$$

So for a small trade, where there is no urgency, the participant may be able to sell at the higher price $x_{mid}+\epsilon_0/2$ and purchase at the lower price $x_{mid}-\epsilon_0/2$, by entering a limit order. This is represented by the case $\theta = 0$:

$$
X_0 = \begin{pmatrix} x+\epsilon/2 & 0 \\ 0 & x-\epsilon/2 \end{pmatrix}
$$

Similarly a large trade, where there is a high degree of urgency, the participant will have to execute a market order, paying at the higher price $x_{mid}+\epsilon_0/2$, or selling at the lower price $x_{mid}-\epsilon_0/2$. This is represented by the case $\theta = \pi/2$:

$$
X_{\pi/2} = \begin{pmatrix} x-\epsilon/2 & 0 \\ 0 & x+\epsilon/2 \end{pmatrix}
$$

2.5.6 Trade Execution:

First let us assume there is a large sell order. We represent this as follows. First we project onto the market offers, since to execute the trade one must submit an offer/sell order to the market. Furthermore, we represent the relative size & urgency of the trade using the parameter $\theta \in [0, \pi/2]$, and the operator given by 2.12. We write the operator to return the price: $\mathcal{P}_O X_\theta \mathcal{P}_O$.

$$E^\psi[\mathcal{P}_O X_\theta \mathcal{P}_O] = \frac{1}{||\psi_0||^2} \begin{pmatrix} \psi_o(x,\epsilon) & 0 \end{pmatrix} \begin{pmatrix} x + \cos(2\theta)\epsilon/2 & -\sin(2\theta)\epsilon/2 \\ -\sin(2\theta)\epsilon/2 & x - \cos(2\theta)\epsilon/2 \end{pmatrix} \begin{pmatrix} \psi_o(x,\epsilon) \\ 0 \end{pmatrix}$$

Where the market state is defined by equations 2.8 and 2.9, we get:

$$E^\psi[\mathcal{P}_O X_\theta \mathcal{P}_O] = x_{mid} + \cos(2\theta)\epsilon_0/2,\ \theta \in [0, \pi/2] \tag{2.13}$$

So that for a particularly large & urgent sell order we should set $\theta = \pi/2$, and expect to sell at $x_{mid} - \epsilon_0/2$. Similarly for a smaller less urgent sell order, we set $\theta = 0$ and can expect to receive the higher price: $x_{mid} + \epsilon_0/2$.

2.5.7 Non-commuting Prices:

We now consider the case where there are 2 trades, each with different values for θ, and consider the commutator and the circumstances in which the order in which the trades are executed matters. We label the trades:

$$X_1 = \begin{pmatrix} x + \cos(2\theta_1)\epsilon/2 & -\sin(2\theta_1)\epsilon/2 \\ -\sin(2\theta_1)\epsilon/2 & x - \cos(2\theta_1)/2\epsilon/2 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} x + \cos(2\theta_2)\epsilon/2 & -\sin(2\theta_2)\epsilon/2 \\ -\sin(2\theta_2)\epsilon/2 & x - \cos(2\theta_2)\epsilon/2 \end{pmatrix}$$

Taking the commutator, we find that:

$$[X_1, X_2] = \begin{pmatrix} 0 & \sin(2\theta_1 - 2\theta_2)\epsilon^2/4 \\ -\sin(2\theta_1 - 2\theta_2)\epsilon^2/4 & 0 \end{pmatrix} \tag{2.14}$$

The uncertainty principal gives us (for example see [26] Theorem 12.4):

$$(\Delta_\psi X_1)^2(\Delta_\psi X_2)^2 \geq \frac{1}{4}\Big|E^\psi\big[[X_1, X_2]\big]\Big|^2$$

$$E^\psi[X_i] = \langle\psi|X_1|\psi\rangle,\ i = 1, 2$$

$$(\Delta_\psi X_1)^2 = E^\psi\Big[X_1 - E^\psi[X_1]^2\mathbb{I}\Big]$$

Therefore, inserting from the commutator: 2.14, we get:

$$(\Delta_\psi X_1)^2(\Delta_\psi X_2)^2 \geq \frac{1}{4}\left|\frac{\sin(2\delta\theta)}{8}\int_{\mathbb{R}^2}\big(\overline{\psi_o(x,\epsilon)}\psi_b(x,\epsilon) - \overline{\psi_b(x,\epsilon)}\psi_o(x,\epsilon)\big)\epsilon^2 dx d\epsilon\right|^2 \tag{2.15}$$

$$\delta\theta = \theta_1 - \theta_2$$

First consider the case whereby the relative phase difference between $\psi_o(x,\epsilon)$, and $\psi_b(x,\epsilon)$ is zero. In this case we have:

$$\overline{\psi_o(x,\epsilon)}\psi_b(x,\epsilon) - \overline{\psi_b(x,\epsilon)}\psi_o(x,\epsilon) = 0$$

$$(\Delta_\psi X_1)^2(\Delta_\psi X_2)^2 \geq 0 \tag{2.16}$$

In this instance, whilst X_1 & X_2 do not commute, the expected difference in prices that arises from the order in which we execute the trades is zero. Put another way, whilst it is not possible to fix the execution price for both trades simultaneously, this does not lead to a minimum value in the price uncertainty, which at least according to 2.15, can be arbitrarily small.

Now consider the case whereby the relative phase difference between $\psi_o(x,\epsilon)$, and $\psi_b(x,\epsilon)$ is not zero. In

this instance we write:

$$\psi_o(x,\epsilon) = |\psi_o(x,\epsilon)|e^{i\phi_o}$$

$$\psi_o(x,\epsilon) = |\psi_b(x,\epsilon)|e^{i\phi_b}$$

So that:

$$\begin{aligned}\int_{\mathbb{R}^2} \big(\overline{\psi_o(x,\epsilon)}\psi_b(x,\epsilon) - \overline{\psi_b(x,\epsilon)}\psi_o(x,\epsilon)\big)\epsilon^2 dxd\epsilon &= \left(e^{i(\phi_b-\phi_o)} - e^{-i(\phi_b-\phi_o)}\right)\int_{\mathbb{R}^2} |\psi_0(x,\epsilon)||\psi_b(x,\epsilon)|\epsilon^2 dxd\epsilon \\ &= 2i\sin(\delta\phi)\int_{\mathbb{R}^2} |\psi_0(x,\epsilon)||\psi_b(x,\epsilon)|\epsilon^2 dxd\epsilon \\ \delta\phi &= \phi_b - \phi_o\end{aligned}$$

Inserting this into 2.15, we get:

$$(\Delta_\psi X_1)^2(\Delta_\psi X_2)^2 \geq \left(\frac{\sin(2\delta\theta)\sin(2\delta\phi)}{8}\int_{\mathbb{R}^2} |\psi_o(x,\epsilon)||\psi_b(x,\epsilon)|\epsilon^2 dxd\epsilon\right)^2 \tag{2.17}$$

Remark 2.5.1 (Bid-Offer Spread)**.** *We have not defined an operator that can extract the expected value for the bid-offer spread. This may be useful, for example in modelling the profitability of a market making strategy. Where we have a market such that the bid-offer spread was small:*

$$\int_{-\infty}^{\infty}\int_0^K \epsilon^2(|\psi_o(x,\epsilon)|^2 + |\psi_b(x,\epsilon)|^2)dxd\epsilon \approx 0, \text{ for small } K$$

Then, as is the case for a zero phase difference, whilst the operators would still not commute, we would find the minimum value in the price uncertainty given by 2.15, to be small.

So in summary we find that the minimum price uncertainty is determined by:

- The size difference between the 2 non-commuting trades, represented by $\delta\theta$.
- The phase difference between the functions representing market sellers: $\psi_o(x,\epsilon)$, and market buyers: $\psi_b(x,\epsilon)$.
- The expected level of the market bid-offer spread.

Chapter 3

Modelling Illiquid Stocks Using Quantum Stochastic Calculus: Theoretical Model Development

3.1 Chapter Introduction

In this chapter, we model the time evolution of the financial market by applying Quantum Stochastic Calculus to the market Hilbert space introduced in chapter 2. The objective is to investigate how the properties differ from conventional models built using classical stochastic calculus, and to evaluate whether these match behaviours observed in the real financial market.

Classical models of the random behaviour of the financial market are often used for 2 important purposes:

i) Quantifying financial risk in a portfolio of traded securities, and optimising portfolio selection to minimise risk.

ii) Pricing of derivative positions that reference traded securities.

The second purpose also requires one to define what is meant by "price". For example, it is conventional to calculate a risk neutral price using the Fundamental Theorem of Asset Pricing (for example see [10], Theorem 10.14).

In fact, the ability to hedge derivatives requires in turn the ability to identify the current price, with full precision, without impacting the state of the market. Whilst this will not generally be possible in a non-commutative market, in section 3.4 we focus on deriving a Martingale price process for a derivative, with the assumption that this is as good a place as any to start for ii), and will be sufficient for some key applications relating to i).

3.2 General Chapter Overview:

In many cases, financial markets are highly liquid and the spread between the bid & offer price is such that the assumption that one can trade at a "mid price" between the two is reasonable. For example, for the most liquid foreign exchange markets (GBPUSD for instance), the spread between the bid & offer price is generally around 1bps (0.01%). In this case, a single factor classical model may be sufficient.

In this chapter we show that the quantum stochastic approach provides a way to model the impact of a lack of liquidity on the dynamics. For example:

i) The fact that it may not be possible to determine the precise level of the bid or offer price.

ii) Furthermore, there may be an imbalance of buyers & sellers.

iii) Finally, the measurable observables may operate in a non-commutative manner. For example, one might assume that knowledge of the rate of change of the security price would impact the amount one was willing to pay for the security.

When we apply the quantum approach to the modelling of the bid & offer prices, we find that in general,

even if the bid & offer prices are driven by the same stochastic process, interaction between the 2 variables drives changes in the dynamics of the mid-price. In other words, one finds that the onset of illiquidity in the stock price impacts its' dynamics.

The quantum stochastic method is described in section 3.3 and section 3.4. This involves unitary operators that apply creation and annihilation operators to a Boson Fock space attached to the market Hilbert space.

The crucial point is that the resulting Fock space vectors add the random noise element to the time evolution of the operator under consideration. Further explanation of the method can be found at [14], where the author shows how to build quantum stochastic processes that can be identified as the quantum equivalent of classical Brownian motion, and the Poisson process.

In section 3.4 we derive the Quantum Black Scholes equation, originally given by Accardi and Boukas in [16]. There it is shown that one can use the rules of quantum stochastic calculus to expand a self-financing trading strategy that replicates a particular derivative payout, using a power series approach. The Martingale condition is then enforced by taking the expectation over both the market Hilbert space, and the Boson Fock space containing the random noise.

In section 3.5 we extend this analysis by showing how one can derive two different categories of solutions to the resulting quantum Black-Scholes equation. Strictly speaking the derivative price operator, as a solution to the quantum Black-Scholes equation, is an operator value function of the underlying price (and time). However, if one can solve the quantum Black-Scholes equation as a classical partial differential equation (whereby the operators are functions of price, acting on the market wave-function by pointwise multiplication), then one has a solution that is not dependent on the market state.

In this research, we also show how one can generate non-classical behaviour using the case whereby the solution of the quantum Black-Scholes *does* depend on the market wave-function. This is first introduced in section 3.5.1. Then in section 3.6, we develop the idea further by showing how the resulting variance

(as well as skew & kurtosis) depends on parameters that define the market state (such as imbalance of buyers vs sellers), and factors that depend on the nature of the precise operator (such as the trade size). In section 3.5.2, we show how the incorporation of a Quantum Poisson process, leads to a Fokker-Planck equation (proposition 3.5.6) that equates the moments of the probability distribution for the price to parameters such as the bid-offer spread, and imbalance of buyers and sellers.

Finally, in section 3.7, we extend the model further by building a stochastic model for the bid-offer spread. This allows us to consider the role of market makers in setting the bid-offer spread, and to suggest how the use of a Hamiltonian, incorporating a bid-offer potential function, can capture the relevant motivations for these liquidity providers.

3.3 Time Evolution:

In section 2.4, we have defined the basic structure of the market we are dealing with. The next objective is to define the time evolution. Since the focus of the approach is on the operators, rather than the quantum state, we work in the Heisenberg interpretation.

3.3.1 Quantum Stochastic Process

Following the approach outlined in [16] (see also [34]), we take the tensor product of $\mathcal{H}$ with the symmetric Fock space: $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}^+;\mathbb{C}))$, and use a unitary time evolution operator to build the price operator at $t = T$. We start with $X \otimes \mathbb{I}$, where X is a linear operator on $\mathcal{H}$. We write: $X \in \mathcal{L}(\mathcal{H})$. Then the pricing operator at $t = T$ is given by: $j_T(X) = U_T^*(X \otimes \mathbb{I})U_T$. U_t is defined by the process (see [34] proposition 7.1):

$$dU_t = -\left(\left(iH + \frac{L^*L}{2}\right) \otimes dt + L^*S \otimes dA_t - L \otimes dA_t^\dagger + (\mathbb{I} - S) \otimes d\Lambda_t\right)U_t \tag{3.1}$$

-	$dA_t^\dagger$	$d\Lambda_t$	dA_t	dt
$dA_t^\dagger$	0	0	0	0
$d\Lambda_t$	$dA_t^\dagger$	$d\Lambda_t$	0	0
dA_t	dt	dA_t	0	0
dt	0	0	0	0

Table 3.1: Ito multiplication operators for the basic operators of quantum stochastic calculus.

Whereby H, L, and S act on $\mathcal{H}$, and $dA_t, dA_t^\dagger$, and $d\Lambda_t$ act on the Fock space. By writing out (see [34] Theorem 4.5):

$$\begin{aligned} dj_t(X) &= d(U_t^*(X \otimes \mathbb{I})U_t) \\ &= dU_t^*(X \otimes \mathbb{I})U_t + U_t^*(X \otimes \mathbb{I})dU_t + dU_t^*(X \otimes \mathbb{I})dU_t \end{aligned}$$

and using the Itô multiplication table from [34] (see table 3.1) we can define a stochastic process for $dj_t(X)$, and $dj_t(X_t)^k, k \geq 2$:

$$\begin{aligned} dj_t(X) &= j_t(\alpha^\dagger)dA_t^\dagger + j_t(\alpha)dA_t + j_t(\lambda)d\Lambda_t + j_t(\theta)dt \qquad (3.2) \\ k \geq 2 : dj_t(X)^k &= j_t(\lambda^{k-1}\alpha^\dagger)dA_t^\dagger + j_t(\alpha\lambda^{k-1})dA_t + j_t(\lambda^k)d\Lambda_t + j_t(\alpha\lambda^{k-2}\alpha^\dagger)dt \\ \theta &= i[H, X] - \frac{1}{2}\Big(L^*LX + XL^*L - 2L^*XL\Big) \\ \alpha &= [L^*, X]S \\ \alpha^\dagger &= S^*[X, L] \\ \lambda &= S^*XS - X \end{aligned}$$

Remark 3.3.1. *Note that for an operator A acting on the system space: $\mathcal{H}$, $j_t(A) = U_t^*(A \otimes \mathbb{I})U_t$, acts on $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}^+; \mathbb{C}))$. Furthermore, in equation 3.2, and throughout, we follow standard notation, and*

write $dA_t, dA_t^\dagger, d\Lambda_t$ in place of $\mathbb{I} \otimes dA_t, \mathbb{I} \otimes dA_t^\dagger, \mathbb{I} \otimes d\Lambda_t$ for brevity. However, it should be understood that these act only on the Fock space.

Note also that for equations such as 3.1, we retain the notation (for example) $L \otimes dA_t^\dagger$, since the operators H, L, S act on $\mathcal{H}$, rather than $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}^+; \mathbb{C}))$.

Applying 2.4 and 2.7, we get:

$$[L^*, X] = \begin{pmatrix} i\sigma & 0 \\ 0 & i\sigma \end{pmatrix} \tag{3.3}$$

$$XL^*L + L^*LX - 2L^*XL = 0$$

If we insert 3.3, into 3.2, alongside $S = \mathbb{I}$ and $H = 0$, we get:

$$dj_t(X) = \begin{pmatrix} i\sigma & 0 \\ 0 & i\sigma \end{pmatrix} dA_t - \begin{pmatrix} i\sigma & 0 \\ 0 & i\sigma \end{pmatrix} dA_t^\dagger \tag{3.4}$$

$$dj_t(X)^2 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} dt$$

For detailed derivations of 3.2, 3.3, and 3.4, see appendix A.

3.3.2 A Classical Approach to Option Pricing:

In this section we briefly summarise the classical approach to derivative pricing, so that we can contrast the quantum approach in later sections. For a more detailed description of the arbitrage free pricing of derivatives, see [10].

The classical traded price for an underlying is modelled as a real valued function of an underlying probability space:

$$S : \mathbb{R}^+ \times \Omega \to \mathbb{R}$$

Where $\mathbb{R}^+$, represents the time axis, and each $\omega \in \Omega$ encodes a different possible probabilistic outcome. In order to model the probability of different possible outcomes, we specify a σ-algebra $\mathcal{F}$ on Ω, and a probability measure:

$$P : \mathcal{F} \to [0, 1]$$

$$P(\emptyset) = 0$$

$$P(\Omega) = 1$$

We wish to calculate the price at time $t < T$, for a payout that is defined by the value of the traded price observed at T. We write: $\chi(S)$, so that at time T, the value of the payout is given by:

$$V(T, S) = \chi(S)$$

We have (see for example [10] Theorem 10.18) that the arbitrage free price, $V(t, S)$, is given by:

$$V(t, S) = B(t)E^Q\Big[\frac{\chi(S)}{B(T)}\Big] \tag{3.5}$$

Where Q is the the probability measure such that:

$$E^Q\Big[\frac{S(T)}{B(T)}\Big] = \frac{S(0)}{B(0)} \tag{3.6}$$

$B(t)$ represents a risk free investment, such as a money market account paying interest rate r:

$$B(t) = e^{\int_0^t r(s)ds}$$

In fact, for equity underlyings, one must also adjust the price for cash dividends paid out whilst we hold a position in the stock. In practice, one can trade forward contracts in the market for the majority of liquid underlyings. Ie contracts to purchase the relevant asset at a future time, at a fixed price. Therefore, it is often easier to incorporate the future forecast dividend payments & interest rates by using the market price for the forward F, rather than the spot price S.

If we enter a forward contract at t, with final expiry T, then at T we pay the forward price $F(t,T)$ and receive the stock price: $S(T)$. Therefore, the value at T is given by: $V = S(T) - F(t,T)$. For both parties to enter into this contract at t, the expected value must be zero. Therefore, we must have:

$$F(t,T) = E^Q[S(T)] \tag{3.7}$$

Note that, by the tower law of probability:

$$\begin{aligned} E^Q[F(t+dt,T)|F(t,T)] &= E^Q\big[E^Q[S(T)]\big|F(t,T)\big] \\ &= E^Q[S(T)] \\ &= F(t,T) \end{aligned}$$

Therefore, $F(t,T)$ is a Martingale, and we can represent it using a drift free Ito process:

$$dF = \sigma F dW$$

We are free to write V as a function of F (which is also a traded market observable) rather than S. We can also make a change of variable, and solve the Kolmogorov backward equation in log space. Using Ito's lemma we write:

$$\begin{aligned} x &= log(F) \\ dx &= \sigma dW - \frac{\sigma^2}{2} dt \end{aligned}$$

Furthermore, we define the normalized price process (see for example [10] definition 10.11, 10.12) as:

$$V_B(t) = \frac{V(t)}{B(t)}$$

If we model V_B as a function of classical variables x, t, and expand using Ito's lemma, we get:

$$dV_B = \left(\frac{\partial V_B}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V_B}{\partial x^2} - \frac{\sigma^2}{2}\frac{\partial V_B}{\partial x}\right)dt + \frac{\partial V_B}{\partial x} dW \tag{3.8}$$

By taking expectations, and noting from 3.5 that the normalized price process: V_B, is also a Q-Martingale, we get:

$$\frac{\partial V_B}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V_B}{\partial x^2} - \frac{\sigma^2}{2}\frac{\partial V_B}{\partial x} = 0 \tag{3.9}$$

3.4 Outline of Quantum Approach to Option Pricing:

3.4.1 Step 1: Defining the Price Process

We define the derivative price process: $V(j_t(X), t)$, as a self-adjoint operator valued function:

$$V : \mathcal{L}(\mathcal{H} \otimes \Gamma) \times \mathbb{R}^+ \to \mathcal{L}(\mathcal{H} \otimes \Gamma)$$

Where $\mathcal{H}$ denotes the initial Hilbert space, and Γ our choice for the symmetric Fock space. $\mathcal{L}(\mathcal{H})$ represents the space of linear operators on the Hilbert space $\mathcal{H}$.

At each time t, the operator $V(j_t(X), t)$ acts on the market state function, returning real eigenvalues that represent possible values for the derivative price.

Classically, one proceeds on the basis that any derivative payout can be replicated using a self-financing trading strategy. After the initial investment, no further money need be invested to replicate the payout. One simply buys & sells the risky underlying at zero cost at the prevailing market price, and ends up with the same outcome as if one had purchased the derivative. This provides the financial rationale for why the discounted price of the derivative should be a Martingale (equation 3.5). Essentially, since one can re-create the payout at zero cost, the expected return on the initial investment should be zero after discounting.

In the quantum case, it is not clear that one can replicate derivative payouts in the same way. In a given market state, both the quantum version of the traded underlying and the derivative have uncertain

prices. Therefore, even if there exists a formula that outputs the required position (to replicate) based on the current price, since one doesn't know the current price, this is not sufficient.

However, for the time being, we work on the basis that the Martingale price process represents a fair price.

Definition 3.4.1. *A Martingale Price Process, under the normalised vector $|\psi\rangle \in \mathcal{H}$, is a self-adjoint operator valued map:*

$$V : \mathcal{L}(\mathcal{H} \otimes \Gamma) \times \mathbb{R}^+ \to \mathcal{L}(\mathcal{H} \otimes \Gamma)$$

Such that for all $t \in \mathbb{R}^+$:

$$\begin{aligned} E^{(\psi,\varepsilon)}[V(j_t(X), t)] &= \langle \psi \otimes \varepsilon | V(j_t(X), t) | \psi \otimes \varepsilon \rangle \\ &= E^{(\psi,\varepsilon)}[V(X_0, 0)] \\ &= V_0 \\ \psi &\in \mathcal{H} \\ \varepsilon &\in \Gamma \end{aligned}$$

Remark 3.4.2. *Note that in some cases we take expectations over the initial space: $\mathcal{H}$ say. In these cases, we write:*

$$E^{\psi}[\dots] = \langle \psi | \dots | \psi \rangle$$

After a stochastic process has been introduced, we take expectations over the tensor product with the symmetric Fock space: $\mathcal{H} \otimes \Gamma$. In this case we write:

$$E^{(\psi,\varepsilon)}[\dots] = \langle \psi \otimes \varepsilon | \dots | \psi \otimes \varepsilon \rangle$$

For a fixed time, we can define $V(j_t(X), t)$ using the spectral theorem for self-adjoint operators: [26] Theorem 10.4, and the associated functional calculus: [26] Definition 10.5. In other words, we write:

$$V_t(j_t(X), t) = f(j_t(X))$$

By the Spectral Theorem, there exists a unique projection valued measure: μ^X, such that:

$$\begin{aligned} j_t(X) &= \int_{\mathbb{R}} \lambda d\mu^X(\lambda) \\ f(j_t(X)) &= \int_{\mathbb{R}} f(\lambda) d\mu^X(\lambda) \end{aligned} \tag{3.10}$$

Where we have assumed the spectrum for the real valued, and unbounded, operator $j_t(X)$ is $\mathbb{R}$. Importantly, we can define a derivative: $\frac{\partial V}{\partial j_t(X)}$ using 3.10:

$$\frac{\partial V}{\partial j_t(X)} = \int_{\mathbb{R}} f'(\lambda) d\mu^X(\lambda)$$

Furthermore, assuming sufficient smoothness in the function V, we define:

$$\frac{\partial V}{\partial t} = \lim_{dt \to 0} \frac{V(j_t(X), t + dt) - V(j_t(X), t)}{dt}$$

3.4.2 Arbitrage in the Quantum Framework:

The question of the nature of quantum arbitrage has been addressed in [7], where the authors develop a Theorem of asset pricing based on equivalence classes of density operators. In this article we give two different definitions for arbitrage in a quantum model. We use an approach derived by extending the classical definition given in [10].

Classically, we can define an arbitrage as a derivative price: $V(S, t)$ and a probability measure P, such

that:

$$V(S,0) = V_0$$

$$P(V(S,T) > V_0) > 0$$

$$P(V(S,T) \geq V_0) = 1$$

The financial intuition behind this being that at $t = 0$ we can execute a trade for free that has the possibility of generating financial gain without the possibility of generating losses.

In the quantum case, under definition 3.4.1, $V(j_t(X), t)$ is an essentially self-adjoint operator (in the context of [26] definition 9.7) acting on: $\mathcal{H} \otimes \Gamma$. In order to define what we mean by arbitrage in a quantum model, we first define the following operators, where $f(\lambda)$ is defined by the Spectral Theorem, equation 3.10:

$$V_{>0} = \int_{\mathbb{R}} 1_{f(\lambda)>0}(\lambda) d\mu^X(\lambda) \tag{3.11}$$
$$V_{\geq 0} = \int_{\mathbb{R}} 1_{f(\lambda)\geq 0}(\lambda) d\mu^X(\lambda)$$

Where $1_{f(\lambda)>0}(\lambda)$ represents the indicator function. We can now extend the notion of arbitrage to operators on a Hilbert space:

Definition 3.4.3. *Let $V(j_t(X), t)$ be defined by:*

$$j_t(X) = \int_{\mathbb{R}} \lambda d\mu^X(\lambda)$$
$$V(j_t(X), t) = \int_{\mathbb{R}} f(\lambda) d\mu^X(\lambda)$$

For some function $f(\lambda)$. Furthermore, let $|\psi\rangle$ be a normalised vector in the Hilbert space: $\mathcal{H}$. Then

$V(j_t(X), t)$ is a quantum arbitrage under $|\psi\rangle$ if:

$$\begin{aligned} P(V(j_t(X), t) > 0) &= E^{(\psi,\varepsilon)}[V_{>0}] \\ &> 0 \\ P(V(j_t(X), t) \geq 0) &= E^{(\psi,\varepsilon)}[V_{\geq 0}] \\ &= 1 \end{aligned}$$

Where $P(V(j_t(X), t) > 0)$ and $P(V(j_t(X), t) \geq 0)$ now represent the probability of the results of a measurement being respectively greater than zero and greater than, or equal to, zero.

We can now give a definition for non-arbitrage price processes in the quantum framework:

Definition 3.4.4. *A weak non-arbitrage price process is a self-adjoint operator valued map:*

$$V : \mathcal{L}(\mathcal{H} \otimes \Gamma) \times \mathbb{R}^+ \to \mathcal{L}(\mathcal{H} \otimes \Gamma)$$

together with a quantum state represented by the vector, $|\psi\rangle \in \mathcal{H}$, such that there is no value of t, whereby $V(j_t(X), t)$ is a quantum arbitrage under $|\psi\rangle$.

Definition 3.4.5. *A strong non-arbitrage price process is a self-adjoint operator valued map:*

$$V : \mathcal{L}(\mathcal{H} \otimes \Gamma) \times \mathbb{R}^+ \to \mathcal{L}(\mathcal{H} \otimes \Gamma)$$

Such that there is no normalised vector $|\psi\rangle$, such that $V(j_t(X), t)$ is a weak non-arbitrage price process under $|\psi\rangle$.

3.4.3 Step 2: Deriving a Partial Differential Equation for V

Proposition 3.4.6 (See [16])**.** *Let $V(j_t(X), t)$ represent a Martingale Price Process for a derivative payout. Let U_t be the general time evolution operator defined by 3.1 with X defined by equation 3.2. Then*

we have:

$$E^{(\psi,\varepsilon)}\left[\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}j_t(\theta)+\frac{\partial^2 V}{\partial x^2}\frac{j_t(\alpha\alpha^\dagger)}{2}\right]=0 \tag{3.12}$$

With L defined by equation 2.7, $S=\mathbb{I}$, and $H=0$. Then we have:

$$E^{(\psi,\varepsilon)}\left[\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2}\right]=0 \tag{3.13}$$

By setting:

$$H=\begin{pmatrix} i(\sigma^2/2)\partial_x & 0 \\ 0 & i(\sigma^2/2)\partial_x \end{pmatrix}$$

We get:

$$E^{(\psi,\varepsilon)}\left[\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2}-\frac{\sigma^2}{2}\frac{\partial V}{\partial x}\right]=0 \tag{3.14}$$

Proof. We write out dV as a power series expansion. We denote the arguments of the function V as x and t:

$$\begin{aligned} dV &\coloneqq V(j_t(X)+dj_t(X),t+dt)-V(j_t(X),t) \\ &=\sum_{n,k}\frac{1}{n!k!}\frac{\partial^{(n+k)}V}{\partial x^n\partial t^k}(dj_t(X)^n)(dt^k) \end{aligned} \tag{3.15}$$

Expanding dV using equation 3.2, we have:

$$\begin{aligned} dV=&\left(\frac{\partial V}{\partial t}+j_t(\theta)+\frac{\partial^2 V}{\partial x^2}\frac{j_t(\alpha\alpha^\dagger)}{2}\right)dt \\ &+\left(\frac{\partial V}{\partial x}j_t(\alpha^\dagger)\right)dA_t^\dagger \\ &+\left(\frac{\partial V}{\partial x}j_t(\alpha)\right)dA_t \end{aligned} \tag{3.16}$$

Since the processes: dA_t, and $dA_t^\dagger$ are Martingales, after taking expectations and equating to zero, we are left with the dt terms. Equating these to zero leaves:

$$E^{(\psi,\varepsilon)}\left[\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}j_t(\theta)+\frac{\partial^2 V}{\partial x^2}\frac{j_t(\alpha\alpha^\dagger)}{2}\right]=0$$

Finally, using 3.3, and 3.4, we get:

$$E^{(\psi,\varepsilon)}\left[\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2}\right]=0 \tag{3.17}$$

Note now the difference between 3.17, and the classical result 3.9. Ie the missing term:

$$-\frac{\sigma^2}{2}\frac{\partial V}{\partial x}$$

This arises, since in section 3.3.1 we modelled a Gaussian process: $dj_t(X)$ with no drift terms. In the equivalent classical model, it is the forward contract that is drift free, which means that the classical variable x has a drift term $-\frac{\sigma^2}{2}dt$.

Proceeding along similar lines of thought we would like: $F = exp(j_t(X))$ to be drift free. Expanding using the quantum Ito's multiplication table, we can calculate the required dt terms. Starting with the power series expansion:

$$dF=\frac{\partial F}{\partial x}dj_t(X)+\sum_{k\geq 2}\frac{\partial^k F}{\partial x^k}dj_t(X)^k$$

Then applying equation 3.2 to calculate the required dt terms, we get:

$$dF=\left(Fj_t(\theta)+\frac{Fj_t(\alpha\alpha^\dagger)}{2}\right)dt+... \tag{3.18}$$

So, if we wish the forward contract to be drift free, we require:

$$j_t(\theta)=-\frac{j_t(\alpha\alpha^\dagger)}{2}$$

We can achieve this, by setting the operator H. For example:

$$H=\begin{pmatrix} i(\sigma^2/2)\partial_x & 0 \\ 0 & i(\sigma^2/2)\partial_x \end{pmatrix} \tag{3.19}$$

In which case, we get (see appendix A for details):

$$dj_t(X) = \begin{pmatrix} i\sigma & 0 \\ 0 & i\sigma \end{pmatrix} dA_t^\dagger - \begin{pmatrix} i\sigma & 0 \\ 0 & i\sigma \end{pmatrix} dA_t - \begin{pmatrix} \sigma^2/2 & 0 \\ 0 & \sigma^2/2 \end{pmatrix} dt \tag{3.20}$$

$$dj_t(X)^2 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} dt$$

Inserting this into 3.16, taking expectations and equating the terms in dt to zero (as for equation 3.12 above) we get:

$$E^{(\psi,\varepsilon)}\left[\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} - \frac{\sigma^2}{2}\frac{\partial V}{\partial x}\right] = 0 \tag{3.21}$$

□

Remark 3.4.7. *Note that whilst equation 3.13 is dependent on the quantum state, if we can find a solution to the classical PDE:*

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} - \frac{\sigma^2}{2}\frac{\partial V}{\partial x} = 0 \tag{3.22}$$

Then we find that 3.17 is met **regardless of the quantum state:** ψ.

Proposition 3.4.8. *The solution to equation 3.21, is a weak non-arbitrage price process. Where there is a solution to the underlying PDE: 3.22, this is a strong non-arbitrage price process.*

Proof. Assume, $V(j_t(X), t)$ is a quantum arbitrage under $|\psi\rangle$. Then by definition:

$$E^{(\psi,\varepsilon)}[V(j_0(X), 0)] = 0$$

From equation 3.16, we have:

$$dV = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} j_t(\theta) + \frac{\partial^2 V}{\partial x^2}\frac{j_t(\alpha\alpha^\dagger)}{2}\right)dt + \left(\frac{\partial V}{\partial x} j_t(\alpha^\dagger)\right)dA_t^\dagger + \left(\frac{\partial V}{\partial x} j_t(\alpha)\right)dA_t$$

Now take expectations over the Hilbert space: $\mathcal{H}$ to get:

$$\begin{aligned} E^{(\psi,\varepsilon)}[dV] &= E^{\psi}\left[\left(\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}j_t(\theta)+\frac{\partial^2 V}{\partial x^2}\frac{j_t(\alpha\alpha^{\dagger})}{2}\right)\right]dt \\ &+ E^{\psi}\left[\left(\frac{\partial V}{\partial x}j_t(\alpha^{\dagger})\right)\right]dA_t^{\dagger}+E^{\psi}\left[\frac{\partial V}{\partial x}\left(j_t(\alpha)\right)\right]dA_t \\ &= E^{\psi}\left[\left(\frac{\partial V}{\partial x}j_t(\alpha^{\dagger})\right)\right]dA_t^{\dagger}+E^{\psi}\left[\left(\frac{\partial V}{\partial x}j_t(\alpha)\right)\right]dA_t \end{aligned} \tag{3.23}$$

In fact, since $V(j_t(X),t)$ is a Martingale price process, we have $E^{\psi}[dV]=0$. Thus, after taking expectations over the Hilbert space $\mathcal{H}$, $V(j_t(X),t)$ is the zero process, with no stochastic noise. Therefore we have: $E^{\psi,\varepsilon}[V_{>0}]=0$, which contradicts the requirements for a quantum arbitrage.

Finally, if the PDE 3.22 is met, then this conditions holds regardless of the quantum state, and $V(j_t(X),t)$ must therefore be a strong non-arbitrage process. □

3.4.4 Remarks on the Fundamental Theorem of Asset Pricing:

In this section we have shown how to derive a non-arbitrage price process in the quantum approach, as a function of the underlying traded asset: $V(j_t(X),t)$. If we can find a suitable process that meets the initial conditions and/or boundary conditions as defined by the financial problem we are modelling, then the resulting price is a non-arbitrage price as defined in section 3.4.2. In particular, the only possible arbitrage results from the quantum uncertainty in the initial space: $\mathcal{H}$. Once we take expectation over $\mathcal{H}$, there is no arbitrage. In order to generalise we would consider general self-financing value processes of the form:

$$V(t,h)=\int_0^t h(u)dj_u(X) \tag{3.24}$$

$$h(t)\in\mathcal{L}(\mathcal{H}\otimes\Gamma)$$

The next step would be to prove the quantum equivalent of the Fundamental Theorem of Asset Pricing (see [10] Theorem 10.5). This states that there are no self-financing arbitrage strategies if and only if

there exists a Martingale measure for the traded asset.

Classically, we would proceed as follows:

1) Assume that the only non-negative derivative payout that can be replicated with a self-financing trading strategy is the zero payout.

2) Apply the Kreps-Yan Separation Theorem ([10] Theorem 10.7) that guarantees for a bounded random variable $X \in L^{\infty}(\Omega; dP)$, the existence of a Radon-Nikodym derivative: $L \in L^{1}(\Omega; dP)$, such that:

$$E^{P}[LV] \geq 0, \text{ for non-negative } V$$

$$E^{P}[LV] \leq 0, \text{ for self-financing } V$$

3) The required Martingale measure can then be derived from the probability measure P, using this Radon-Nikodym derivative:

$$dQ = LdP$$

4) The next step would involve applying a Martingale representation Theorem. For example where X_t represents a traded asset price, V_t a self-financing Martingale price process, and dZ_t a stochastic integral, we write:

$$\begin{aligned} dV_t &= g(t) \otimes dZ_t \\ dX_t &= \sigma(t) \otimes dZ_t \\ dV_t &= g(t)\sigma(t)^{-1} \otimes dX_t \end{aligned} \tag{3.25}$$

Thus showing that any such Martingale price process can be replicated by trading in the asset X.

There are a number of complications with extending this analysis to the quantum framework discussed here:

- The Kreps-Yan separation Theorem applies to bounded $X \in L^{\infty}(\Omega; dP)$. These random variables can be considered as operators acting on the Hilbert space: $L^2(\Omega; dP)$ by pointwise multiplication: $(Xf)(\omega) = X(\omega)f(\omega)$. However in general, and in particular for some of the non-commutative price operators discussed below, we have $X \notin L^{\infty}(\Omega; dP)$.

- Under the quantum framework, both $g(t)$ and $\sigma(t)$ are defined as operators on the Hilbert space: $\mathcal{H}$, and so $g(t)\sigma(t)^{-1}$ is not necessarily defined. For example, we require the operator $\sigma(t)$ to be invertible, and for the image of $\sigma^{-1}(t)$ to be in the domain for the operator: $g(t)$.

We defer consideration of these questions to a future study.

3.5 Extended Quantum Approaches:

3.5.1 First Extended Quantum Approach:

3.5.1.1 Defining a new Market Hilbert Space:

Consider now the case where the operator: X is defined on a dense subspace of $\mathcal{H} \oplus \mathcal{H}$ as before.

We still have 2 wavefunctions, representing the buyers, and sellers respectively:

$$\psi_o \in \mathcal{H}$$

$$\psi_b \in \mathcal{H}$$

However, now we set: $\mathcal{H} = L^2(\mathbb{R}^2)$, and treat both x and ϵ as Euclidean coordinates:

$$\psi_o(x, \epsilon) \in L^2(\mathbb{R}^2)$$

$$\psi_b(x, \epsilon) \in L^2(\mathbb{R}^2)$$

3.5.1.2 Introducing a Stochastic Process:

The functional form for the stochastic process, derived in [16], is given in equation 3.2 (reproduced below):

$$\begin{aligned}
dj_t(X) &= j_t(\alpha^\dagger)dA_t^\dagger + j_t(\alpha)dA_t + j_t(\lambda)d\Lambda_t + j_t(\theta)dt \\
k \geq 2 : dj_t(X)^k &= j_t(\lambda^{k-1}\alpha^\dagger)dA_t^\dagger + j_t(\alpha\lambda^{k-1})dA_t + j_t(\lambda^k)d\Lambda_t + j_t(\alpha\lambda^{k-2}\alpha^\dagger)dt \\
\theta &= i[H, X] - \frac{1}{2}\Big(L^*LX + XL^*L - 2L^*XL\Big) \\
\alpha &= [L^*, X]S \\
\alpha^\dagger &= S^*[X, L] \\
\lambda &= S^*XS - X
\end{aligned}$$

The first question is what to use for the linear operator: L. Some possibilities:

- A single factor stochastic process, with ∂_x operators, in which case the model essentially reduces to the simple quantum model above. Whilst ϵ is a Euclidean coordinate for the system, it is a static one that does not evolve.
- A 2 factor stochastic process, with separate operators; L_x and L_ϵ.
- A single factor stochastic process, which incorporates volatility in both the x and ϵ variables. In this case, L must incorporate both ∂_x and ∂_ϵ operators.

Whilst there is merit in the second approach (2 factor approach), in this section we stick with a 1 factor stochastic process, and set:

$$L = \begin{pmatrix} -i\sigma_x\partial_x - i\sigma_\epsilon\partial_\epsilon & 0 \\ 0 & -i\sigma_x\partial_x - i\sigma_\epsilon\partial_\epsilon \end{pmatrix} \tag{3.26}$$

Furthermore, to keep the approach as general as possible, we set:

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \tag{3.27}$$

and incorporate $R(\theta)$ directly into the X operator, which we define using equation 2.6:

$$X = \begin{pmatrix} x + \cos(2\theta)\epsilon/2 & -\sin(2\theta)\epsilon/2 \\ -\sin(2\theta)\epsilon/2 & x - \cos(2\theta)\epsilon/2 \end{pmatrix} \tag{3.28}$$

Proposition 3.5.1. *Let the stochastic process: $dj_t(X)$ be defined by equation 3.2, with L given by: 3.26, X given by: 3.28, $H = 0$, and $S = \mathbb{I}$. Then, the variance is given by:*

$$\begin{aligned} E^{(\psi,\varepsilon)}\big[j_t(X)^2\big] &= \big(\sigma_x^2 t + (\sigma_\epsilon^2/4)t + \cos(2\theta)\sigma_x\sigma_\epsilon t\big)||\psi_o||^2 \\ &+ \big(\sigma_x^2 t + (\sigma_\epsilon^2/4)t - \cos(2\theta)\sigma_x\sigma_\epsilon t\big)||\psi_b||^2 \\ &+ \sin(2\theta)\big(\langle\psi_b|\psi_o\rangle + \langle\psi_o|\psi_b\rangle\big)\sigma_x\sigma_\epsilon t \end{aligned} \tag{3.29}$$

Proof. We must calculate α, and θ. Applying 3.26 and 3.28, we get (see appendix A for details):

$$[L^*, X] = \begin{pmatrix} i\sigma_x + i\cos(2\theta)\sigma_\epsilon & -i\sin(2\theta)\sigma_\epsilon \\ -i\sin(2\theta)\sigma_\epsilon & i\sigma_x - i\cos(2\theta)\sigma_\epsilon \end{pmatrix}$$

$$XL^*L + L^*LX - 2L^*XL = 0$$

Inserting this into 3.2, with $S = \mathbb{I}$ (see appendix A), we get:

$$dj_t(X) = \begin{pmatrix} i\sigma_x + i\cos(2\theta)\sigma_\epsilon/2 & -i\sin(2\theta)\sigma_\epsilon/2 \\ -i\sin(2\theta)\sigma_\epsilon/2 & i\sigma_x - i\cos(2\theta)\sigma_\epsilon/2 \end{pmatrix} dA_t^\dagger \tag{3.30}$$

$$- \begin{pmatrix} i\sigma_x + i\cos(2\theta)\sigma_\epsilon/2 & -i\sin(2\theta)\sigma_\epsilon/2 \\ -i\sin(2\theta)\sigma_\epsilon/2 & i\sigma_x - i\cos(2\theta)\sigma_\epsilon/2 \end{pmatrix} dA_t$$

$$dj_t(X)^2 = \begin{pmatrix} \sigma_x^2 + \sigma_\epsilon^2/4 + \cos(2\theta)\sigma_x\sigma_\epsilon & \sin(2\theta)\sigma_x\sigma_\epsilon \\ \sin(2\theta)\sigma_x\sigma_\epsilon & \sigma_x^2 + \sigma_\epsilon^2/4 - \cos(2\theta)\sigma_x\sigma_\epsilon \end{pmatrix} dt$$

The stochastic integrals for $j_t(X)$, and $j_t(X)^2$ can be defined using definition 4.1 in [34]:

$$j_t(X) = \int_0^t \begin{pmatrix} i\sigma_x + i\cos(2\theta)\sigma_\epsilon/2 & -i\sin(2\theta)\sigma_\epsilon/2 \\ -i\sin(2\theta)\sigma_\epsilon/2 & i\sigma_x + i\cos(2\theta)\sigma_\epsilon/2 \end{pmatrix} dA_s^\dagger$$

$$- \int_0^t \begin{pmatrix} i\sigma_x + i\cos(2\theta)\sigma_\epsilon/2 & -i\sin(2\theta)\sigma_\epsilon/2 \\ -i\sin(2\theta)\sigma_\epsilon/2 & i\sigma_x + i\cos(2\theta)\sigma_\epsilon/2 \end{pmatrix} dA_s$$

$$j_t(X)^2 = \int_0^t \begin{pmatrix} \sigma_x^2 + \sigma_\epsilon^2/4 + \cos(2\theta)\sigma_x\sigma_\epsilon & \sin(2\theta)\sigma_x\sigma_\epsilon \\ \sin(2\theta)\sigma_x\sigma_\epsilon & \sigma_x^2 + \sigma_\epsilon^2/4 - \cos(2\theta)\sigma_x\sigma_\epsilon \end{pmatrix} ds$$

The operators: dA_t and $dA_t^\dagger$ act on exponential vectors in the Fock space: $\Gamma(L^2(\mathbb{R}^+;\mathbb{C}))$. For example:

$$e(u) = \bigoplus_{n=0}^{\infty} (n!)^{-1/2} u^{\otimes n}$$

$$u \in L^2(\mathbb{R}^+;\mathbb{C})$$

We write the Hilbert space vector for the market as:

$$|\psi \otimes ce(u)\rangle$$

Where c is a normalising constant:

$$c^2 = \frac{1}{\langle e(u)|e(u)\rangle}$$

Therefore we can write:

$$E^{(\psi,e(u))}[j_t(X)] = c^2\langle\big(\psi\otimes e(u)\big)|j_t(X)\big(\psi\otimes e(u)\big)\rangle$$

$$E^{(\psi,e(u))}[j_t(X)^2] = c^2\langle\big(\psi\otimes e(u)\big)|j_t(X)^2\big(\psi\otimes e(u)\big)\rangle$$

$$\psi = \begin{pmatrix}\psi_o\\ \psi_b\end{pmatrix}$$

In fact, we have (see for example [34]):

$$\Big\langle ce(u)\Big|\Big(\int_0^t ds\Big)ce(u)\Big\rangle = c^2\int_0^t 1ds\langle e(u)|e(u)\rangle$$

$$= t$$

Furthermore, we have:

$$E^{(\psi,e(u))}[j_t(X)] = X_0$$

So, conditional on $X_0 = 0$, we have:

$$\begin{aligned}
Var(X,t) &= E^{(\psi,\varepsilon)}[j_t(X)^2]\\
&= \begin{pmatrix}\psi_0 & \psi_b\end{pmatrix}\begin{pmatrix}\sigma_x^2+\sigma_\epsilon^2/4+\cos(2\theta)\sigma_x\sigma_\epsilon & \sin(2\theta)\sigma_x\sigma_\epsilon\\ \sin(2\theta)\sigma_x\sigma_\epsilon & \sigma_x^2+\sigma_\epsilon^2/4-\cos(2\theta)\sigma_x\sigma_\epsilon\end{pmatrix}\begin{pmatrix}\psi_o\\ \psi_b\end{pmatrix}t\\
&= \big(\sigma_x^2t+(\sigma_\epsilon^2/4)t+\cos(2\theta)\sigma_x\sigma_\epsilon t\big)||\psi_o||^2 \qquad (3.31)\\
&+ \big(\sigma_x^2t+(\sigma_\epsilon^2/4)t-\cos(2\theta)\sigma_x\sigma_\epsilon t\big)||\psi_b||^2\\
&+ \sin(2\theta)\big(\langle\psi_b|\psi_o\rangle+\langle\psi_o|\psi_b\rangle\big)\sigma_x\sigma_\epsilon t
\end{aligned}$$

□

Remark 3.5.2. *Importantly, due to the particular choices for the L and X operators (equations 3.26 and 3.28), we end in a situation where it is no longer possible to define V such that:*

$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}j_t(\theta)+\frac{\partial^2 V}{\partial x^2}\frac{j_t(\alpha\alpha^\dagger)}{2}$$

is the zero operator. Crucially, we need to take account of the quantum state to solve:

$$E^{(\psi,\varepsilon)}\left[\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}j_t(\theta)+\frac{\partial^2 V}{\partial x^2}\frac{j_t(\alpha\alpha^\dagger)}{2}\right]=0$$

3.5.1.3 Remarks on Quantum State Dependent Market Volatility:

In proposition 3.5.1, we showed how the quantum approach can be used to introduce a market volatility that depends on the quantum state. In this section, we use simple examples to illustrate how alternative choices for the operator L, in equation 3.12, can also be used to give similar effects.

Proposition 3.5.3. *Assume the Hilbert space is given by:*

$$\mathcal{H}=L^2(\mathbb{R})$$

Furthermore, in the definition of the evolution operator, equation 3.1, we set $S=\mathbb{I}$, and:

$$L=-i\sigma\frac{\partial}{\partial x}-\nu\frac{\partial^2}{\partial x^2}$$

Then, if $V(j_t(X),t)$ represents a Martingale Price Process for a derivative payout, we have:

$$E^{(\psi,\varepsilon)}\left[\frac{\partial V}{\partial t}+\frac{\partial^2 V}{\partial x^2}A_{\sigma\nu}\right]=0 \tag{3.32}$$
$$A_{\sigma\nu}=\frac{\sigma^2}{2}+2\nu^2\frac{\partial^2}{\partial x^2}+2i\sigma\nu\frac{\partial}{\partial x}$$

Proof. See appendix A, section A.1.4. □

Remark 3.5.4. *First of all, where $\nu=0$, we can solve the equation:*

$$\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2}=0$$

to find a strong non-arbitrage solution, which does not depend on the quantum state. However, assuming $\nu\neq 0$, and that:

$$\psi(x)=\int_{\mathbb{R}}f(k)e^{ikx}dk$$

Then, the variance for the process: $j_t(X)$ is given by:

$$\begin{aligned} Var(X,t) &= E^{\psi}[j_t(X)^2] \\ &= \sigma^2 + \int_{\mathbb{R}} \overline{\psi(x)} \Bigg(\int_{\mathbb{R}} f(k)(2\nu^2 k^2 + 2\sigma\nu k) e^{ikx} dk \Bigg) dx \end{aligned}$$

So again, we see that the variance of the stochastic process is dependent on the quantum state, and we cannot solve for a strong non-arbitrage price.

Note also that whilst the market volatility is now operator valued, rather than a constant, there is still no stochastic element, since $[L, A_{\sigma\nu}] = 0$. Other choices for L, such that $[L, A_{\sigma\nu}] \neq 0$, will lead to a stochastic component to the volatility. However, we defer this investigation for a future work.

3.5.2 Second Extended Quantum Approach:

3.5.2.1 Introducing a Non-Gaussian Stochastic Process:

In section 3.5.1, we outlined a stochastic process for a Gaussian observable: $j_t(X)$. We then introduced a bear market by applying the operator, $R(\theta)$:

$$\begin{pmatrix} \psi_o \\ \psi_b \end{pmatrix} \rightarrow \begin{pmatrix} \cos(\theta)\psi_o - \sin(\theta)\psi_b \\ \cos(\theta)\psi_b + \sin(\theta)\psi_o \end{pmatrix}$$

However, whilst this operator impacts the market state, and the resulting variance, the stochastic process is still Gaussian in nature. In this section, we show how the incorporation of $R(\theta)$ into the unitary time evolution operator defined by 3.1, leads to a non-gaussian process.

First set $S = R(\pi/2)$, in equation 3.1. We have:

$$
\begin{aligned}
S &= R(\pi/2) \\
\lambda &= S^* X S - X \\
&= \begin{pmatrix} x + \cos(\pi)\epsilon/2 & -\sin(\pi)\epsilon/2 \\ -\sin(\pi)\epsilon/2 & x - \cos(\pi)\epsilon/2 \end{pmatrix} - \begin{pmatrix} x + \epsilon/2 & 0 \\ 0 & x - \epsilon/2 \end{pmatrix} \\
&= \begin{pmatrix} x - \epsilon/2 & 0 \\ 0 & x + \epsilon/2 \end{pmatrix} - \begin{pmatrix} x + \epsilon/2 & 0 \\ 0 & x - \epsilon/2 \end{pmatrix} \\
&= \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}
\end{aligned}
$$

We apply the derivative operator from the simple quantum approach outlined in section 3.4:

$$
L \begin{pmatrix} \psi_o \\ \psi_b \end{pmatrix} = \begin{pmatrix} -i\sigma\partial_x & 0 \\ 0 & -i\sigma\partial_x \end{pmatrix} \begin{pmatrix} \psi_o \\ \psi_b \end{pmatrix}
$$

Inserting this into equation 3.2, we get the following stochastic process for $j_t(X)$, $j_t(X)^k$ (see appendix

A for details):

$$dj_t(X) = \begin{pmatrix} 0 & i\sigma \\ -i\sigma & 0 \end{pmatrix} dA_t + \begin{pmatrix} 0 & i\sigma \\ -i\sigma & 0 \end{pmatrix} dA_t^\dagger + j_t \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} d\Lambda_t \tag{3.33}$$

$$\begin{aligned} dj_t(X)^k = & \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} j_t \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}^{k-2} dt \\ & + \begin{pmatrix} 0 & i\sigma \\ -i\sigma & 0 \end{pmatrix} j_t \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}^{k-1} dA_t \\ & + j_t \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}^{k-1} \begin{pmatrix} 0 & i\sigma \\ -i\sigma & 0 \end{pmatrix} dA_t^\dagger \\ & + j_t \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}^{k} d\Lambda_t \end{aligned}$$

3.5.2.2 Deriving The Price Process:

As above, we model the derivative price as an operator valued function:

$$V : \mathcal{L}(\mathcal{H} \otimes \Gamma) \times \mathbb{R}^+ \to \mathcal{L}(\mathcal{H} \otimes \Gamma)$$

The goal is to solve for a derivative price $V(j_t(X), t)$ such that:

$$\begin{aligned} E^{(\psi,\varepsilon)}[V(j_t(X), t)] &= \langle (\psi \otimes \varepsilon) | V(j_t(X), t) | (\psi \otimes \varepsilon) \rangle \\ &= E^{(\psi,\varepsilon)}[V(j_0(X), 0)] \\ &= V_0 \end{aligned}$$

Proposition 3.5.5. *Let $V(j_t(X), t)$ represent a Martingale Price Process for a derivative payout. Let U_t be the time evolution operator defined by 3.1 with X defined by equation 3.28, L defined by equation*

3.26, $S = R(\pi/2)$, and $H = 0$. Then we have:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \sigma^2 \sum_{k\geq 2} \frac{\epsilon^{(2k-2)}}{(2k)!}\frac{\partial^{2k} V}{\partial x^{2k}} \tag{3.34}$$
$$+ \sigma^2(||\psi_b||^2 - ||\psi_o||^2) \sum_{k\geq 2} \frac{(-\epsilon)^{(2k-3)}}{(2k-1)!}\frac{\partial^{(2k-1)} V}{\partial x^{(2k-1)}} = 0$$

Proof. Expanding incremental changes in V as a power series we get:

$$\begin{aligned} dV &:= V(j_t(X) + dj_t(X), t + dt) - V(j_t(X), t) \\ &= \sum_{n,k} \frac{1}{n!k!}\frac{\partial^{(n+k)} V}{\partial x^n \partial t^k}(dj_t(X)^n)(dt^k) \end{aligned}$$

We then insert from 3.33 to get:

$$\begin{aligned} dV = &\left(\frac{\partial V}{\partial t} + \sum_{k\geq 2}\frac{1}{k!}\frac{\partial^k V}{\partial x^k}\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}\begin{pmatrix}-\epsilon & 0\\ 0 & \epsilon\end{pmatrix}^{k-2}\right)dt \\ &+ \left(\sum_{k\geq 1}\frac{\partial^k V}{\partial x^k}\begin{pmatrix}0 & i\sigma\\ -i\sigma & 0\end{pmatrix}\begin{pmatrix}-\epsilon & 0\\ 0 & \epsilon\end{pmatrix}^{k-1}\right)dA_t \\ &- \left(\sum_{k\geq 1}\frac{\partial^k V}{\partial x^k}\begin{pmatrix}-\epsilon & 0\\ 0 & \epsilon\end{pmatrix}^{k-1}\begin{pmatrix}0 & i\sigma & 0\\ -i\sigma & 0\end{pmatrix}\right)dA_t^\dagger \\ &+ \left(\sum_{k\geq 1}\frac{\partial^k V}{\partial x^k}\begin{pmatrix}0 & i\sigma\\ -i\sigma & 0\end{pmatrix}\begin{pmatrix}-\epsilon & 0\\ 0 & \epsilon\end{pmatrix}^{k}\begin{pmatrix}0 & i\sigma\\ -i\sigma & 0\end{pmatrix}\right)d\Lambda_t \end{aligned} \tag{3.35}$$

As above, we make use of the fact that the processes: dA_t, $dA_t^\dagger$, and $d\Lambda_t$ are Martingales (see {*add rereference*}). Therefore, taking expectations and equating to zero, we find that:

$$E^{(\psi,\varepsilon)}\left[\frac{\partial V}{\partial t} + \sum_{k\geq 2}\frac{1}{k!}\frac{\partial^k V}{\partial x^k}\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}\begin{pmatrix}(-\epsilon)^{k-2} & 0\\ 0 & \epsilon^{k-2}\end{pmatrix}\right] = 0$$

Setting:

$$|\psi\rangle = \begin{pmatrix} \psi_0 \\ \\ \psi_b \end{pmatrix}$$

We get:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + \sigma^2 \sum_{k\geq 2} \frac{\epsilon^{(2k-2)}}{(2k)!}\frac{\partial^{2k} V}{\partial x^{2k}} \tag{3.36}$$
$$+ \sigma^2(||\psi_b||^2 - ||\psi_o||^2) \sum_{k\geq 2} \frac{(-\epsilon)^{(2k-3)}}{(2k-1)!}\frac{\partial^{(2k-1)} V}{\partial x^{(2k-1)}} = 0$$

□

3.5.2.3 Associated Fokker-Planck Equation:

In this section, we calculate the Fokker-Planck equation associated to the partial differential equation 3.36. We follow the basic strategy outlined in [50]. See also [32] proposition 3.1.

First of all note that, since: $E^{(\psi,\varepsilon)}[dV] = 0$, we have that:

$$V_0 = E^{(\psi,\varepsilon)}[V(j_T(X), T]$$

If we are working with a derivative that pays out at final maturity T, then $V_(j_T(X), T)$ is defined by the contractual payout:

$$V_T(j_T(X), T) = \chi(j_T(X))$$

Thus we have:

$$V_0 = E^{(\psi,\varepsilon)}[\chi(j_T(X)]$$

For this reason, one can associate to equation 3.36 a Fokker-Planck equation for the underlying probability function.

Proposition 3.5.6. *The Fokker-Planck equation associated to the partial differential equation 3.36 is given by:*

$$\frac{\partial p}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} + \sigma^2 \sum_{k\geq 2} \frac{\epsilon^{(2k-2)}}{(2k)!}\frac{\partial^{2k} p}{\partial x^{2k}} + \sigma^2\big(||\psi_o||^2 - ||\psi_b||^2\big)\sum_{k\geq 2}\frac{(-\epsilon)^{(2k-3)}}{(2k-1)!}\frac{\partial^{(2k-1)} p}{\partial x^{(2k-1)}} \tag{3.37}$$

Proof. Applying the Spectral Theorem, we have (for some probability density function $p(y,t)$):

$$\begin{aligned} V_0 &= E^{(\psi,\varepsilon)}[V(j_T(X),T)] \\ &= \int_{\mathbb{R}} \chi(y)p(y,T)dy \end{aligned} \tag{3.38}$$

First expand the function:

$$d\chi \coloneqq \chi(j_t(X) + dj_t(X)) - \chi(j_t(X))$$

as a power series in $dj_t(X)$:

$$d\chi = \sum_k \frac{1}{k!}\frac{\partial^k \chi}{\partial x^k} dj_t(X)^k \tag{3.39}$$

After taking expectations over the symmetric Fock space, we find (for some unknown probability density function $p(x,t)$):

$$E^{(\psi,\varepsilon)}[d\chi] = \int_{\mathbb{R}} E^{\psi}\big[d\chi(y)\big]p(y,t)dy$$

Now, applying equation 3.33 to equation 3.39, then integrating from 0 to T, we get:

$$E^{(\psi,\varepsilon)}[\chi(j_T(X))] = \chi_0 + \int_0^T \int_{\mathbb{R}} E^{\psi}\Big[\sum_{k\geq 2}\frac{1}{k!}\frac{\partial^k \chi}{\partial x^k}\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}\begin{pmatrix}-\epsilon & 0\\ 0 & \epsilon\end{pmatrix}^{k-2}\Big]p(y,t)dydt \tag{3.40}$$

So we have:

$$\begin{aligned} E^{\psi}[\chi(j_T(X))] &= \chi_0 \\ &+ \int_0^T \int_{\mathbb{R}} \left(\sigma^2 \sum_{k\geq 2}\frac{\epsilon^{(2k-2)}}{(2k)!}\frac{\partial^{2k}\chi}{\partial x^{2k}} + \sigma^2\big(||\psi_b||^2 - ||\psi_0||^2\big)\sum_{k\geq 2}\frac{(-\epsilon)^{(2k-3)}}{(2k-1)!}\frac{\partial^{(2k-1)}\chi}{\partial x^{(2k-1)}}\right)p(t,y)dydt \end{aligned}$$

Inserting from equation 3.38, we get:

$$\int_{\mathbb{R}} \chi(y)p(y,T)dy = \int_0^T \int_{\mathbb{R}} \left(\sigma^2 \sum_{k\geq 2} \frac{\epsilon^{(2k-2)}}{(2k)!} \frac{\partial^{2k}\chi}{\partial x^{2k}} + \sigma^2 \big(||\psi_b||^2 - ||\psi_0||^2 \big) \sum_{k\geq 2} \frac{(-\epsilon)^{(2k-3)}}{(2k-1)!} \frac{\partial^{(2k-1)}\chi}{\partial x^{(2k-1)}} \right) p(t,y)dydt$$

Differentiating with respect to T:

$$\int_{\mathbb{R}} \chi(y) \frac{\partial p(y,T)}{\partial T} dy = \int_{\mathbb{R}} \left(\sigma^2 \sum_{k\geq 2} \frac{\epsilon^{(2k-2)}}{(2k)!} \frac{\partial^{2k}\chi}{\partial x^{2k}} + \sigma^2 \big(||\psi_b||^2 - ||\psi_0||^2 \big) \sum_{k\geq 2} \frac{(-\epsilon)^{(2k-3)}}{(2k-1)!} \frac{\partial^{(2k-1)}\chi}{\partial x^{(2k-1)}} \right) p(T,y)dy$$

If we truncate the right hand side at $k = N$, then the result follows by integrating by parts $2N$ times. Then taking the limit $N \to \infty$, gives us equation 3.37. □

3.5.2.4 Moment Generating Function:

Taking the Fourier transform of 3.37, and applying [23] Theorem 8.4.4, we get the following result:

Proposition 3.5.7. *The kth moment of the probability density function described by equation 3.37 is given by:*

$$\mu_k = \sum_n \frac{k!}{\#OP^n(k)!} \prod_{j \in OP^n(k)} \sigma^2 t a_j \tag{3.41}$$

$$OP^n(k) = \textit{nth ordered partition of k, not including 1}$$

$$\#OP^n(k) = \textit{number of elements in the nth ordered partition, not including 1}$$

In particular we have for the skew & kurtosis:

- $\mu_3 = \sigma^2 t\epsilon(||\psi_b||^2 - ||\psi_o||^2)$
- $\mu_4 = 3(\sigma^2 t)^2 + \sigma^2 t\epsilon^2$

Proof. Taking the Fourier transform of 3.37, we get:

$$\frac{\mathcal{F}(p)}{\partial t} = \sum_{k\geq 2} a_k (iz)^k \mathcal{F}(p)$$

$$a_k = \begin{cases} \frac{\sigma^2 \epsilon^{k-2}}{k!}, \text{ for k even} \\ \frac{\sigma^2 (||\psi_b||^2 - ||\psi_o||^2)\epsilon^{k-2}}{k!}, \text{ for k odd} \end{cases}$$

Thus, we get:

$$\mathcal{F}(p) = exp\Big(t \sum_{k\geq 2} a_k (iz)^k\Big) \tag{3.42}$$

Therefore, we can write the Moment Generating function for p as:

$$\begin{aligned} M_p(z) &= exp\Big(t \sum_{k\geq 2} a_k z^k\Big) \\ &= 1 + \sum_{k\geq 2} z^k \sum_n \prod_{j \in OP^n(k)} \frac{a_j t}{\#OP^n(k)!} \\ OP^n(k) &= \text{nth ordered partition of k, not including 1} \\ \#OP^n(k) &= \text{number of elements in the nth ordered partition, not including 1} \end{aligned}$$

□

For example, the only partition of 3, not including 1, is $\{3\}$. Therefore, the coefficient of z^3 in 3.41 is given by:

$$a_3 t z^3 = \frac{\sigma^2 t \epsilon (||\psi_b||^2 - ||\psi_o||^2) z^3}{6}$$

For 4, we have 2 ordered partitions that do not include 1: $\{4\}$ and $\{2, 2\}$. Therefore, the coefficient of z^4 in 3.41 is given by:

$$\frac{(a_2 t)^2}{2!} z^4 + a_4 z^4 = \frac{(\sigma^2 t)^2 z^4}{8} + \frac{(\sigma^2 t)\epsilon^2 z^4}{24}$$

The result then follows from the definition of the Moment generating function.

Remark 3.5.8 (Interpretation of the Quantum Stochastic Process)**.** *In addition to being a useful practical tool, the Forward Kolmogorov equation provides a useful way to visualise the quantum stochastic processes. For example, note that (see for example [55], [15]), if we assume* $||\psi_o||^2 = ||\psi_b||^2$ *in equation 3.42, we get:*

$$\mathcal{F}(p) = exp\Big(\frac{\sigma^2}{\epsilon^2}\big(e^{iz\epsilon} - iz\epsilon - 1\big)\Big) \tag{3.43}$$

In fact, equation 3.43 is the characteristic equation for a Lévy process that makes σ^2/ϵ^2 *jumps per second with a fixed jump size:* ϵ*. As we take* $\epsilon \to 0$ *we get an infinite number of infinitely small jumps, each with infinitely small size, and the process reduces to Brownian motion. Indeed, in the limit* $\epsilon \to 0$*, equation 3.37 becomes a conventional Gaussian equation:*

$$\frac{\partial p}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2}$$

In chapter 4 we examine solutions to equation 3.37, and show that as $t \to \infty$ *the solutions asymptotically approach the standard Gaussian solution, as is reflected also by the moments (equation 3.41) whereby for example the ratio of the non-Gaussian kurtosis to the Gaussian tends to one for large* t*:*

$$\frac{3(\sigma^2 t)^2 + \sigma^2 t\epsilon^2}{3(\sigma^2 t)^2} \to 1, \text{ as } t \to \infty$$

3.6 Quantum Volatility Effects

Using the price operator given by equation 2.12, propositions 3.5.1 shows that the expected variance (conditional on the market state) is given by:

$$E^{\psi}[Var(X_\theta, t)] = \begin{pmatrix} \psi_0 & \psi_b \end{pmatrix} \begin{pmatrix} \sigma_x^2 + \sigma_\epsilon^2/4 + \cos(2\theta)\sigma_x\sigma_\epsilon & \sin(2\theta)\sigma_x\sigma_\epsilon \\ \sin(2\theta)\sigma_x\sigma_\epsilon & \sigma_x^2 + \sigma_\epsilon^2/4 - \cos(2\theta)\sigma_x\sigma_\epsilon \end{pmatrix} \begin{pmatrix} \psi_o \\ \psi_b \end{pmatrix} t \tag{3.44}$$

Note first that equation 3.44 assumes that the variable X_θ, is a Martingale, in the sense that we have:

$$E^{\psi,t}[X_\theta] = E^{\psi,0}[X_\theta]$$
$$= X_0$$

Furthermore, note that equation describes the variance of a Gaussian process. That is, the variance of X_θ does not depend on the level of X_θ itself. From a financial perspective, it is therefore more natural to map X_θ to the natural log of a particular price, rather than the price itself. As such we require the variable: $F = exp(X_\theta)$ to be a Martingale. As we go on to show in proposition 3.4.6, the necessary drift adjustment required so that: F is a Martingale, rather than X, can easily be achieved using a Hamiltonian operator:

$$H = \begin{pmatrix} i(\sigma_x^2/2)\partial_x & 0 \\ 0 & i(\sigma_x^2/2)\partial_x \end{pmatrix}$$

Finally note that by incorporating both $-i\partial_x$, and $-i\partial_\epsilon$ in a single operator that introduces noise has the following significance:

- Firstly it ensures that both dimensions that determine the price operator: x, and ϵ will evolve stochastically.
- Secondly, since the noise is introduced through a single operator, the model can be thought of as a one factor model. One cannot discuss the correlation between the price: X_θ and any the bid-offer spread, without defining a quantum operator for the Bid-Offer spread.

The important information in this case is obtained in the variance: equation 3.44. In this subsection we highlight some of the unique properties of the model, and discuss how these map to the granular behaviours not generally captured in the top down probabilistic approach (for example [10]). First of all

we write:

$$\eta_0 = \int_{\mathbb{R}^2} (|\psi_o(x,\epsilon)|^2 - |\psi_b(x,\epsilon)|^2) dx d\epsilon \tag{3.45}$$
$$\gamma_0 = \int_{\mathbb{R}^2} (\overline{\psi_o(x,\epsilon)}\psi_b(x,\epsilon) + \overline{\psi_b(x,\epsilon)}\psi_o(x,\epsilon)) dx d\epsilon$$

Inserting this into equation 3.44, we get:

$$Var(X_\theta, t) = (\sigma_x^2 + \frac{\sigma_\epsilon^2}{4} + \eta_0 \cos(2\theta)\sigma_x\sigma_\epsilon + \gamma_0 \sin(2\theta)\sigma_x\sigma_\epsilon)t \tag{3.46}$$

We note that:

- As discussed previously, $\eta_0 = ||\psi_o||^2 - ||\psi_b||^2$ can be viewed as a parameter that controls the balance between buyers & sellers. Where the market consists exclusively of offers, we would have $\eta_0 = 1$. Where the market consists exclusively of bids, we would have $\eta_0 = -1$, and for a balanced market we would have $\eta_0 = 0$. In other words, we can view η_0 as a measure of the *market imbalance*. The impact of market imbalance has been widely discussed in the financial literature (for example see [19].

- We have also discussed above the relation of θ to the trade size. In particular, in section 2.5.6 we have discussed how θ can be used to incorporate the fact that for a small trade one would generally expect to be able to trade at a more favourable price, that for a large trade.

- γ_0 can be viewed as a measure of the relative phase between the wave-functions ψ_0 and ψ_b. Whilst this is difficult to interpret from a financial perspective, as discussed in section 2.5.7, and illustrated below, this can have a substantial impact on the model behaviour. Therefore, for the time being we view γ_0 as an additional model parameter that can be calibrated using historical data, in order to tune the model properties.

- Finally note that for a highly liquid market, one might expect the bid-offer spread: ϵ to remain at or near zero, and therefore that $\sigma_\epsilon \approx 0$. In this case, the model reduces to a representation of the a standard classical Brownian motion of the type that the log prices follow in a standard Black-Scholes approach:

$$Var(X_\theta, t) \approx \sigma_x^2 t$$

3.6.1 Modelling the Impact of Trade Size:

We now set $\sigma_x = 0.9\%$, being an estimate of the volatility for the daily log returns over 2 years, for the FTSE 100 index. Furthermore, we set σ_ϵ to a value of 0.05% for illustration purposes. Figure 3.1 shows the resulting variance for X_θ for $\eta_0 = -1$ to 1, $\gamma_0 = 0, 1$, and $\theta \in [0, \pi/2]$.

$$\begin{aligned}\sigma_x &= \sqrt{\frac{1}{502}\sum_{i=2}^{503} \log\left(\frac{FTSE_i}{FTSE_{i-1}}\right)^2} \\ &\approx 0.9\% \\ \sigma_\epsilon &= 0.05\%\end{aligned}$$

In [19], the authors carry out empirical analysis on the relationship between trade size, which we represent using θ, and the market volatility, and also between the order imbalance, which we represent using η, and the market volatility. They carry out the analysis using stocks listed on the NYSE and NASDAQ over a 6 month sample period.

The authors investigate the regression coefficients between the number of trades of different sizes and the daily market volatility. They find that the number of trades is a strong driver of the daily volatility and that the number of medium sized trades has the largest impact. Specifically they split trades by the number of shares exchanged: size bucket 1$\leq$ 500 shares, size bucket 2 $\leq$ 1000 shares, size bucket 3 $\leq$ 5000 shares, size bucket 4 $\leq$ 9999 shares, and size bucket 5 $\geq$ 10000 shares. They find that trades in

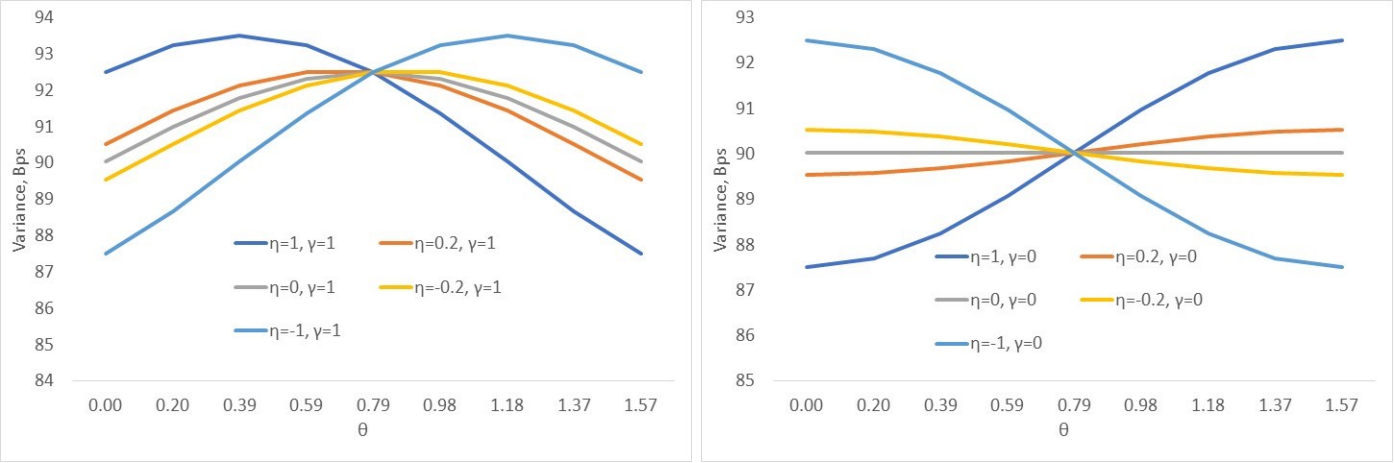

Figure 3.1: Variance in Bps, vs $\theta \in [0, \pi/2]$, for $\eta_0 \in [-1, 1]$, $\gamma_0 = 0, 1$.

size bucket 4 have the largest impact for the NYSE, and trades in bucket 2 have the largest impact for the NASDAQ. This in turn is consistent with the results in figure 3.1 for $\gamma_0 = 1$, where for smaller levels of the order imbalance, the largest volatility is seen for $\theta \approx \pi/4$.

The authors then go on to look at the impact of order imbalance using a 2 stage regression. In the first stage they calculate the number of buyer initiated trades minus the number of seller initiated trades. They then calculate the regression coefficients between the daily returns and the net number of trades in the different size buckets. This enables them to demonstrate the strong relationship between order imbalance and the return size. For example as we find in equation 2.10.

In the second stage of the regression the authors calculate the regression coefficient between the absolute residual returns from the first stage regression against the number of trades in the five size buckets discussed above. These are then compared against the original volatility regression coefficients.

The analysis shows that after correcting for the return impact due to a non-zero order imbalance, the relationship between the number of trades in the different size buckets, and the daily volatility is similar if slightly smaller.

Overall the study clearly establishes the importance of the order imbalance and trade size in determining

the level of market volatility.

3.6.2 Re-introducing Variable ϵ:

In the derivation of equation 3.42, we have made 2 simplifying assumptions:

1) Firstly we removed the operator $-i\sigma_\epsilon\partial_\epsilon$ from the stochastic process.

2) Next we simplified the Hilbert space to $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, so that ϵ is a fixed parameter.

This results in a process, where the properties are configured by the choice of the parameters:

- σ, which will control the variance of the process.
- $\eta_0 = ||\psi_b||^2 - ||\psi_o||^2$, which will control the skew of the process.
- ϵ, which will control the level of the skew, and kurtosis of the process.

In this section, we briefly consider the impact of relaxing these assumptions, and discuss some of the quantum effects that result.

First, we set the Hilbert space back to $L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$, thus allowing variable ϵ. The quantum Black-Scholes equation associated to the process 3.33, for $\mathcal{H}_{mkt} = L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$ is given by proposition 3.5.5:

$$E^{(\psi,\varepsilon)}\left[\frac{\partial V}{\partial t} + \sum_{k\geq 2}\frac{1}{k!}\frac{\partial^k V}{\partial x^k}\begin{pmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{pmatrix}\begin{pmatrix}(-\epsilon)^{k-2} & 0\\ 0 & \epsilon^{k-2}\end{pmatrix}\right] = 0 \tag{3.47}$$

Whilst it will not be generally possible to define a simple classical representation of equation 3.47, equation 3.43 allows us to interpret ϵ as a variable that controls the characteristic jump size. If we set:

$$\psi = \begin{pmatrix}\psi_o(x,\epsilon)\\ \psi_b(x,\epsilon)\end{pmatrix}$$

Then 3.47 becomes:

$$\frac{\partial V}{\partial t} + \sum_{k \geq 2} \frac{\sigma^2 b_k}{k!} \frac{\partial^k V}{\partial x^k} = 0 \tag{3.48}$$

$$b_k = \begin{cases} \int_{\mathbb{R}^2} \epsilon^{k-2}(|\psi_b(x,\epsilon)|^2 + |\psi_o(x,\epsilon)|^2) dx d\epsilon, \text{ for } k \text{ even} \\ \int_{\mathbb{R}^2} \epsilon^{k-2}(|\psi_b(x,\epsilon)|^2 - |\psi_o(x,\epsilon)|^2) dx d\epsilon, \text{ for } k \text{ odd} \end{cases}$$

So we find that we can configure the nature of the stochastic process, by the market quantum state.

Remark 3.6.1 (One Factor vs Two Factor Processes)**.** *Note that whilst equation 3.33 has two underlying variables: x and ϵ, there is only a single stochastic process: that described by equation 3.1, and 3.26. When we apply classical stochastic processes, we directly introduce random fluctuations into the price. Under quantum stochastic calculus, this is no longer the case. In fact the random noise is introduced via a unitary time evolution operator. Nonetheless, there is still only one stochastic process. Moreover, in section 3.7 we define an operator to pull out the bid-offer spread and show that the resulting measurable values are 100% correlated with the price operator X.*

In section 3.7, we go on to explore the impact of introducing a second stochastic process. Doing this allows the simulation of the bid-offer spread, independently of the price. For example, when modelling the share price for a listed company, one can model situations where a previously liquid stock with small bid-offer spread, encounters future periods where liquidity dries up, leading to a larger bid-offer spread.

3.6.3 Quantum Evolution of Skew & Kurtosis:

In equation 3.44, and equation 3.46, we have shown that by modelling in $L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$, and incorporating ϵ as a variable, rather than a fixed parameter, we derive a variance that depends on:

- The parameter θ, which defines the traded price operator (equation 2.12), and which we have taken to be a proxy for trade size.

- The market wave function, which is in turn defined by $\psi_o(x,\epsilon)$, and $\psi_b(x,\epsilon)$.

In this section, we derive similar relations for the excess skew & kurtosis for the Non-Gaussian moments. We make the simplifying assumption that:

$$E^{\psi}[X_\theta] = 0$$

and that the process: $j_t(X_\theta)$ is a Martingale in the sense that:

$$E^{\psi,\varepsilon}[X_\theta] = E^{\psi}[j_t(X_\theta)]$$
$$= 0$$

Thus to study the moments of the process, we look at the evolution of the operator: $j_t(X_\theta^k)$, using the following expansion:

$$dj_t(X_\theta^k) = \sum_{l=1}^{k} \binom{k}{l} X_\theta^{k-l} dj_t(X_\theta)^l \tag{3.49}$$

So that in the Gaussian case, where in equation 3.1 we have: $S = \mathbb{I}$, this becomes:

$$dj_t(X_\theta^3) = 3X_\theta^2 dj_t(X_\theta) + 3X_\theta dj_t(X_\theta)^2$$
$$dj_t(X_\theta^4) = 4X_\theta^3 dj_t(X_\theta) + 6X_\theta^2 dj_t(X_\theta)^2$$

We have assumed that the stochastic process: $j_t(X_\theta)$ is a Martingale in the sense that, after taking expectations over both the Hilbert space and the Boson Fock space, we have:

$$E^{\psi,\varepsilon}[dj_t(X_\theta)] = 0$$
$$E^{\psi,\varepsilon}[X_\theta] = 0$$

So that, after applying the quantum version of Ito's lemma, and taking expectations we get:

$$E^{\psi,\varepsilon}[dj_t(X_\theta^3)] = 0$$

In the Non-Gaussian case, equation 3.49 becomes:

$$\begin{aligned} dj_t(X_\theta^3) &= 3X_\theta^2 dj_t(X_\theta) + 3X_\theta dj_t(X_\theta)^2 + dj_t(X_\theta)^3 \\ &= dj_t(X_\theta)^3 \end{aligned}$$

Similarly, for the Kurtosis we have:

$$\begin{aligned} dj_t(X_\theta^4) &= 4X_\theta^3 dj_t(X_\theta) + 6X_\theta^2 dj_t(X_\theta)^2 + 4X_\theta dj_t(X_\theta)^3 + dj_t(X_\theta)^4 \\ &= 6X_\theta^2 dj_t(X_\theta)^2 + dj_t(X_\theta)^4 \end{aligned}$$

Proposition 3.6.2. *Assume:* $\mathcal{H}_{mkt} = L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$*, and that the market initial state is defined by:*

$$|\psi\rangle = \begin{pmatrix} \psi_o(x,\epsilon) \\ \\ \psi_b(x,\epsilon) \end{pmatrix}$$

Furthermore, in the time evolution operator given by equation: 3.1, we set:

$$L = \begin{pmatrix} -i\sigma_x\partial_x - i\sigma_\epsilon\partial_\epsilon & 0 \\ \\ 0 & -i\sigma_x\partial_x - i\sigma_\epsilon\partial_\epsilon \end{pmatrix},\ S = R(\pi/2),\ H = 0$$

Then we have the following:

$$\begin{aligned} E^{\psi,\varepsilon}[dj_t(X_\theta)^3] &= \left(\left(\sigma_x^2\cos(2\theta) + \frac{\sigma_\epsilon^2}{4}\cos(6\theta)\right)\eta_1 + \sigma_x\sigma_\epsilon\cos(4\theta)\epsilon_1 + \left(\frac{\sigma_\epsilon^2}{4}\sin(6\theta) - \sigma_x^2\sin(2\theta)\right)\gamma_1\right)dt \\ E^{\psi,\varepsilon}[dj_t(X_\theta)^4] &= \left(\left(\sigma_x^2 + \frac{\sigma_\epsilon^2}{4}\right)\epsilon_2 + \eta_2\cos(2\theta)\sigma_x\sigma_\epsilon + \gamma_2\sin(2\theta)\sigma_x\sigma_\epsilon\right)dt \\ \eta_k &= \int_{\mathbb{R}^2} \epsilon^k\left(|\psi_o(x,\epsilon)|^2 - |\psi_b(x,\epsilon)|^2\right)dxd\epsilon \\ \gamma_k &= \int_{\mathbb{R}^2} \epsilon^k\left(\overline{\psi_o(x,\epsilon)}\psi_b(x,\epsilon) + \overline{\psi_b(x,\epsilon)}\psi_o(x,\epsilon)\right)dxd\epsilon \\ \epsilon_k &= \int_{\mathbb{R}^2} \epsilon^k\left(|\psi_o(x,\epsilon)|^2 + |\psi_b(x,\epsilon)|^2\right)dxd\epsilon \end{aligned} \tag{3.50}$$

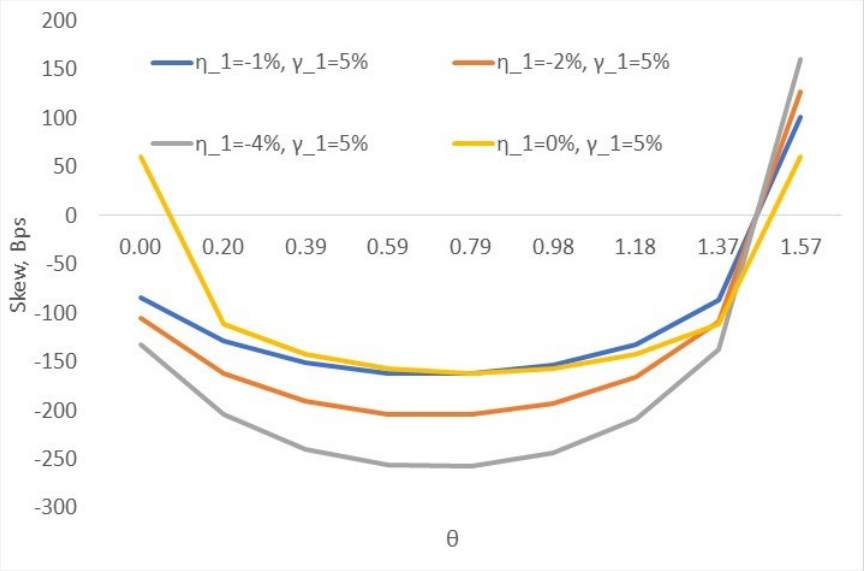

Figure 3.2: Skew in Bps for $\gamma_1 = 5\%$, and $\eta_1 = 0\%, -1\%, -2\%$, and -4%. $\theta \in [0, \pi/2]$

Proof. See appendix A. □

Proposition 3.6.2 shows the skew of a stochastic process, measured by $E^{\psi,\varepsilon}[dj_t(X_\theta)^3]$ parameters, and the excess Kurtosis, measured by $E^{\psi,\varepsilon}[dj_t(X_\theta)^4]$, can be parameterized by the choice of the η_1, γ_1, and η_2, γ_2 respectively. In section 3.6.1, we looked at the relationship between trade size & variance. There we noted that the choice of $\gamma_0 = 1$ matched the statistical analysis of market data carried out in [19]. In figure 3.2 we carry over $\sigma_x = 0.9\%$, and $\sigma_\epsilon = 0.05\%$, from section 3.6.1. Furthermore, we set $\gamma_1 = 5\%$, and look at results for $\eta_1 = 0\%$ to -4%. As is the case for the results on variance, the results show that medium sized trades have the largest negative skew.

3.6.4 Some Historical Data:

In this section we seek to provide simple algorithms by which one can fit some of the models we go onto develop in chapter 3, based on historical data. For this purpose we return to the simplified approach with fixed ϵ introduced in section 3.5.2, and defer a similar exercise for the more complex models discussed in

section 3.6.2 to a future research exercise.

3.6.4.1 Estimators for η_0, and ϵ_1:

To estimate the values for η_0, and ϵ_1 (per equation 3.50), using the approach outlined in section 3.5.2, and specifically the relations given in equation 3.41. We proceed as follows:

1) We collect a time-series of end of day close prices. The close price generally reflects the price set during the end of day auction process.

2) Using 20 years of historical data, we take non-overlapping log returns: $log(F_{t+n}/F_t)$. $n = 1$day to $n = 100$days. As described in section 3.6, we set:

$$F_t = \exp(X_\theta)$$

$$\log(F_{t+1}/F_t) = \log(F_{t+1}) - \log(F_t)$$

$$= \delta X_\theta$$

3) We then shift the start of the time-series by 1 day to calculate the non-overlapping log returns: $\log(F_{t+1+n}/F_{t+1})$. In this way, we have n sets of non-overlapping time-series: $\log(F_{t+k+n}/F_{t+k})$, for $k \in [0, \ldots, n-1]$. To clarify, if F_j represents the jth point in the time-series, we write:

$$\delta X_\theta(n, k, j) = \log(F_{j+k+n}/F_{j+k})$$

We label the time-series of non-overlapping n-day log returns, with shift of k-days, as TS_{nk}.

4) We make the assumption that one can participate in the end of trade auction (and therefore trade at the official close price) without impacting the market. In other words, we assume that $\theta = 0$. For this reason, the theoretical skew & kurtosis are given by equations 3.41. In fact from equation

3.41, we have:

$$\epsilon = \sqrt{\frac{\mu_4 - (3\sigma^2 t)^2}{\sigma^2 t}} \tag{3.51}$$
$$\eta = \frac{\mu_3}{\sqrt{\sigma^2 t(\mu_4 - (3\sigma^2 t)^2)}}$$

5) With equation 3.51 in mind, we measure the following estimators for $\epsilon(n)$ and $\eta(n)$, based on the set of n day non-overlapping log returns:

$$\epsilon(n) \approx \frac{1}{n}\sum_{k=0}^{n-1}\sqrt{\frac{Kurtosis(TS_{nk}) - 3 * Variance(TS_{nk})}{Variance(TS_{nk})}}$$
$$\eta(n) \approx \frac{1}{n}\sum_{k=0}^{n-1} Skew(TS_{nk})\sqrt{\frac{Variance(TS_{nk}))^3}{Kurtosis(TS_{nk})}}$$

Remark 3.6.3 (Expected Results)**.** *Note that, as $t \to \infty$, we have that the Gaussian contribution to the kurtosis in equation 3.41 grows with t^2, whereas the excess kurtosis grows only with t. In our calculation t represents the time interval over which the log returns are calculated, which we have labelled n to represent n steps on the time-series. Therefore, if the underlying assumptions for the model are correct, we should observe a flat line for $\epsilon(n)$, versus n. A similar result should apply for the skew parameter: $\eta(n)$.*
This is a crucial point. In fact, as is shown in [57], for any model to be free of arbitrage, the skew & excess kurtosis implied by the model should decay to zero. This in turn is a problem for conventional one factor models, because the skew implied by short dated options has been a constant factor over many years. Therefore, to address this discrepancy the market participant has two solutions:

- *Re-calibrate the model every day, re-introducing the small amount of skew that has fallen out due to these non-arbitrage considerations.*
- *Consider a 2 factor model, at the cost of analytical/numerical complexity.*

As we go on to show in chapter 4, for the model discussed in section 3.5.2, skew & fat-tails that persist only over the short term is natural feature.

3.6.4.2 Initial Results:

We show the results for a sample of 6 stocks selected from the NYSE and NASDAQ indices:

- Apple Inc
- Dish network Corp
- Exxon Mobil Corp
- Lincoln National Corp
- Microsoft Corp
- Newell Brands Inc

Figure 3.3 shows the estimated values for ϵ and η versus the number of days in the log returns: n.

3.7 Two Factor Quantum Approach:

3.7.1 Outline of the Approach:

In section 2.5, the bid-offer spread is modelled as a constant in the eigenvalues: $x + \epsilon/2$, and $x - \epsilon/2$, that define the position operator. The underlying price: x is the single underlying coordinate variable in $L^2(\mathbb{R})$.

Then in sections 3.5.1, and 3.5.2, we included the bid-offer spread as a second Euclidean coordinate variable in $L^2(\mathbb{R}^2)$. In this case, one can define an associated (unbounded) operator: E, such that for $\psi \in L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$, we have:

$$E\psi = \begin{pmatrix} \epsilon\psi_o \\ \epsilon\psi_b \end{pmatrix}$$

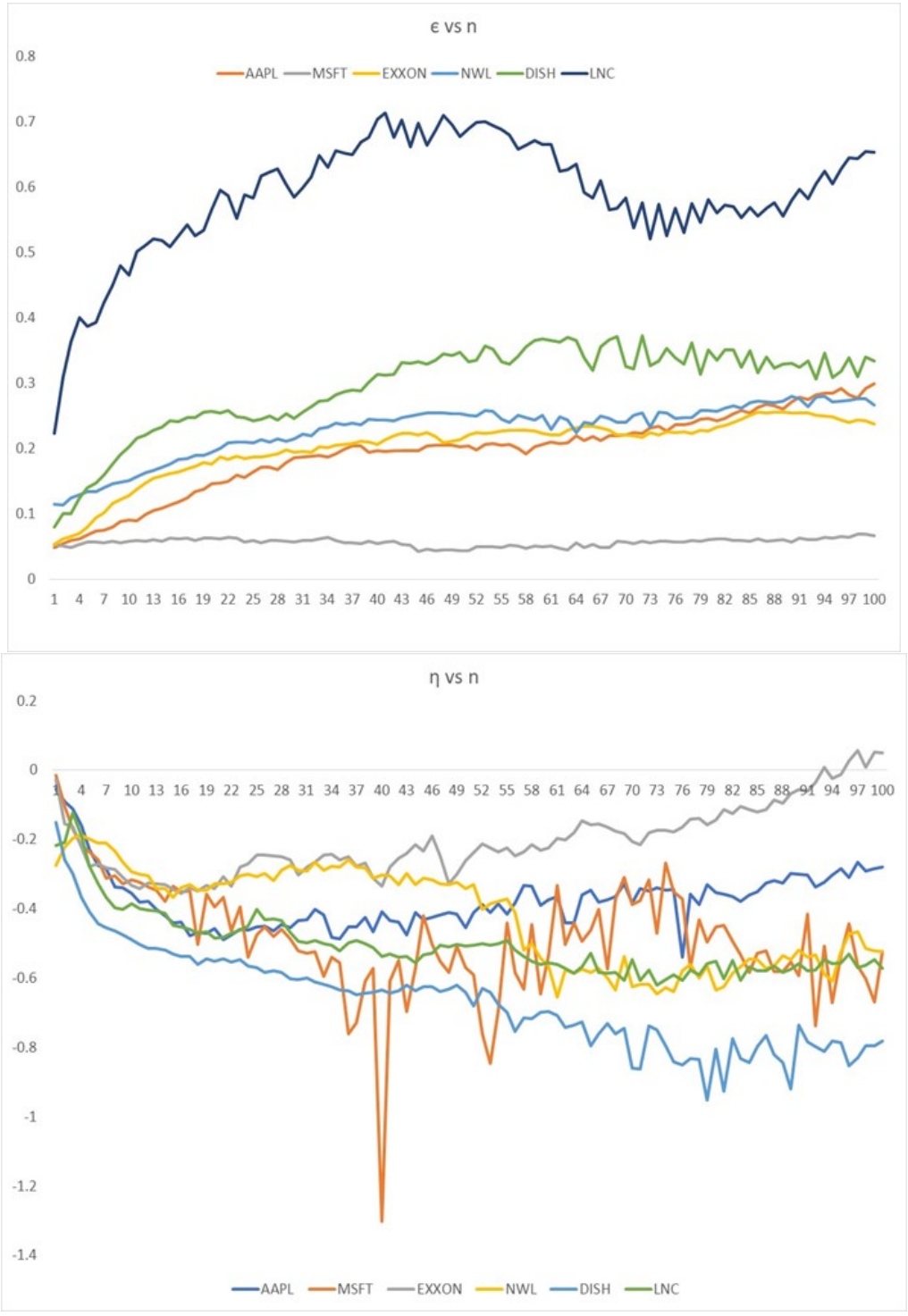

Figure 3.3: Estimate for ϵ versus n (left hand chart) and η versus n (right hand chart), for 6 stocks listed on the NYSE and NASDAQ. Timeseries data from June 2003 to June 2023

Thus, we can represent $E \in M_2\big(\mathcal{L}(\mathcal{S}(\mathbb{R}^2))\big)$:

$$E = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$$

We can in principle evolve the bid-offer spread using the same stochastic noise that impacts the price operator: X. In other words, we could define:

$$j_t(E) = U_t^*(E \otimes \mathbb{I})U_t$$

Where U_t is defined by 3.1. However, this approach does not appear to make financial sense. In fact, we find:

$$dj_t(XE) = d\big(U_t^*(XE \otimes \mathbb{I})U_t\big) \tag{3.52}$$

Furthermore, we have:

$$[L, X] = \begin{pmatrix} i\sigma_x & 0 \\ 0 & i\sigma_x \end{pmatrix}$$

$$[L, E] = \begin{pmatrix} i\sigma_\epsilon & 0 \\ 0 & i\sigma_\epsilon \end{pmatrix}$$

Therefore, collecting together terms in dt from 3.52, we get:

$$E^{(\psi,\varepsilon)}\big[dj_t(XE)\big] = \begin{pmatrix} \sigma_x\sigma_\epsilon & 0 \\ 0 & \sigma_x\sigma_\epsilon \end{pmatrix} dt$$

$$E^{(\psi,\varepsilon)}\big[dj_t(X^2)\big] = \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_x^2 \end{pmatrix} dt$$

$$E^{(\psi,\varepsilon)}\big[dj_t(E^2)\big] = \begin{pmatrix} \sigma_\epsilon^2 & 0 \\ 0 & \sigma_\epsilon^2 \end{pmatrix} dt$$

Thus one can see, that with a single stochastic process, the observables X and E will be 100% correlated, with a covariance of $\sigma_x \sigma_\epsilon t$. Since in reality these variables are not perfectly correlated, we look to introduce a second stochastic process.

Therefore the alternative, explored in [32], [33], is to extend the symmetric Fock space: $\Gamma\big(L^2(\mathbb{R}^+;\mathbb{C})\big) \rightarrow \Gamma\big(L^2(\mathbb{R}^+;\mathbb{C}^2)\big)$.

We can extend 3.1 by setting:

$$dA_{x,t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} dA_t,\ dA^\dagger_{x,t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} dA^\dagger_t,\ d\Lambda_{x,t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} d\Lambda_t$$

We re-write; U_t as $U_{x,t}$, and the analysis in section 3.5 proceeds exactly as previously. However, now we can define a second stochastic process that adds bid-offer noise. For example, if we wish the correlation between the 2 stochastic processes to be ρ, we could define:

$$dU_{\epsilon,t} = -\bigg(\Big(iH_\epsilon + \frac{L^*_\epsilon L_\epsilon}{2}\Big) \otimes dt + L^*_\epsilon S_\epsilon \otimes dA_{\epsilon,t} - L_\epsilon \otimes dA^\dagger_{\epsilon,t} + (\mathbb{I} - S_\epsilon) \otimes d\Lambda_{\epsilon,t}\bigg) U_{\epsilon,t} \tag{3.53}$$

$$dA_{\epsilon,t} = \begin{pmatrix} \rho \\ \sqrt{1-\rho^2} \end{pmatrix} dA_t,\ dA^\dagger_{\epsilon,t} = \begin{pmatrix} \rho \\ \sqrt{1-\rho^2} \end{pmatrix} dA^\dagger_t,\ d\Lambda_{\epsilon,t} = \begin{pmatrix} \rho \\ \sqrt{1-\rho^2} \end{pmatrix} d\Lambda_t$$

$$dU_{x,t} = -\bigg(\Big(iH_x + \frac{L^*_x L_x}{2}\Big) \otimes dt + L^*_x S_x \otimes dA_{x,t} - L_x \otimes dA^\dagger_{x,t} + (\mathbb{I} - S_x) \otimes d\Lambda_{x,t}\bigg) U_{x,t} \tag{3.54}$$

$$dA_{x,t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} dA_t,\ dA^\dagger_{x,t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} dA^\dagger_t,\ d\Lambda_{x,t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} d\Lambda_t$$

We now extend the single factor model, to incorporate both sources of noise. First set: $U_t = U_{x,t} U_{\epsilon,t}$,

then expand:

$$
\begin{aligned}
dj_t(X) &= d(U_t^*(X \otimes \mathbb{I})U_t) \qquad (3.55)\\
&= dU_t^*(X \otimes \mathbb{I})U_t + U_t^*(X \otimes \mathbb{I})dU_t + dU_t^*(X \otimes \mathbb{I})dU_t \\
&= \big(dU_{\epsilon,t}^* U_{x,t}^* + U_{\epsilon,t}^* dU_{x,t}^* + dU_{\epsilon,t}^* dU_{x,t}^*\big)(X \otimes \mathbb{I})U_{x,t}U_{\epsilon,t} \\
&+ U_{\epsilon,t}^* U_{x,t}^*(X \otimes \mathbb{I})\big(dU_{x,t}U_{\epsilon,t} + U_{x,t}dU_{\epsilon,t} + dU_{x,t}dU_{\epsilon,t}\big) \\
&+ \big(dU_{\epsilon,t}^* U_{x,t}^* + U_{\epsilon,t}^* dU_{x,t}^* + dU_{\epsilon,t}^* dU_{x,t}^*\big)(X \otimes \mathbb{I})\big(dU_{x,t}U_{\epsilon,t} + U_{x,t}dU_{\epsilon,t} + dU_{x,t}dU_{\epsilon,t}\big)
\end{aligned}
$$

In fact we can simplify 3.55 as follows:

- First set $\rho = 0$, so that the stochastic process driving the bid-offer spread is uncorrelated with the spot price.
- Secondly, we set:

$$L_x = \begin{pmatrix} -i\sigma_x\partial_x & 0 \\ 0 & -i\sigma_x\partial_x \end{pmatrix},\ L_\epsilon = \begin{pmatrix} -i\sigma_\epsilon\partial_\epsilon & 0 \\ 0 & -i\sigma_\epsilon\partial_\epsilon \end{pmatrix}$$

- We still would like $j_t(X)$ to be a Martingale, so we set $H_x = 0$ as above.
- Since it is not directly traded, we don't require $j_t(E)$ to be a Martingale. However, for now we leave H_ϵ unspecified and return to the question of what to use in section 3.7.3.
- We set $S_x = R(\pi/2)$ as in section 3.5.2, and $S_\epsilon = \mathbb{I}$.

Remark 3.7.1. *Since we model the log prices: $x = log(S)$, the offer price is given by: $S_o = exp(x)exp(\epsilon)$, and the bid price is given by: $S_b = exp(x)exp(-\epsilon)$. Thus, if $S_o > S_b$, then we require: $\epsilon > 0$.*
On the other hand, if we wish $exp(\epsilon)$ to be broadly lognormal in the sense that the volatility is proportional to the current value, then ϵ should be normally distributed, which leaves the possibility for $\epsilon < 0$.

The current choice of L_ϵ specified above will lead to a Gaussian process for the bid-offer spread. We will look to minimise the possibility of a negative value for ϵ by introducing a mean reversion. Furthermore, we note that there may well be rare (& brief) periods where buyers come into the market willing to pay the current offer or above, in the hope of a quick execution.

3.7.2 New Stochastic Process for the Log Price:

Proposition 3.7.2. *Let U_ϵ and U_x be defined by 3.53 and 3.54, with:*

- $\rho = 0$
- $S_x = R(\pi/2)$, $S_\epsilon = \mathbb{I}$.
- $H_x = 0$, $H_\epsilon \in \mathcal{L}\big(\mathcal{S}(\mathbb{R}^2) \oplus \mathcal{S}(\mathbb{R}^2)\big)$.
- $L_x = \begin{pmatrix} -i\sigma_x\partial_x & 0 \\ 0 & -i\sigma_x\partial_x \end{pmatrix}$, $L_\epsilon = \begin{pmatrix} -i\sigma_\epsilon\partial_\epsilon & 0 \\ 0 & -i\sigma_\epsilon\partial_\epsilon \end{pmatrix}$

Then the stochastic process: $j_t(X) = U^*_{\epsilon,t}U^*_{x,t}(X \otimes \mathbb{I})U_{x,t}U_{\epsilon,t}$, is defined by:

$$\begin{aligned} dj_t(X) = &\begin{pmatrix} 0 & i\sigma_x \\ -i\sigma_x & 0 \end{pmatrix} dA_{x,t} + \begin{pmatrix} 0 & i\sigma_x \\ -i\sigma_x & 0 \end{pmatrix} dA^\dagger_{x,t} \\ &+ \begin{pmatrix} i\sigma_\epsilon/2 & 0 \\ 0 & -i\sigma_\epsilon/2 \end{pmatrix} dA^\dagger_{\epsilon,t} + \begin{pmatrix} -i\sigma_\epsilon/2 & 0 \\ 0 & i\sigma_\epsilon/2 \end{pmatrix} dA_{\epsilon,t} \\ &+ j_t\Bigg(\begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \Bigg) d\Lambda_{x,t} + i[H_\epsilon, X]dt \end{aligned} \tag{3.56}$$

$$
\begin{aligned}
dj_t(X)^k = &\left(\begin{pmatrix} 0 & i\sigma_x \\ -i\sigma_x & 0 \end{pmatrix} j_t \begin{pmatrix} (-\epsilon)^{(k-1)} & 0 \\ 0 & \epsilon^{(k-1)} \end{pmatrix} \right) dA_{x,t} \\
&+ \left(j_t \begin{pmatrix} (-\epsilon)^{(k-1)} & 0 \\ 0 & \epsilon^{(k-1)} \end{pmatrix} \begin{pmatrix} 0 & i\sigma_x \\ -i\sigma_x & 0 \end{pmatrix} \right) dA^\dagger_{x,t} \\
&+ \left(j_t \begin{pmatrix} (-\epsilon)^{k} & 0 \\ 0 & \epsilon^{k} \end{pmatrix} \right) d\Lambda_{x,t} + \left(\begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_x^2 \end{pmatrix} j_t \begin{pmatrix} (-\epsilon)^{(k-2)} & 0 \\ 0 & \epsilon^{(k-2)} \end{pmatrix} \right) dt
\end{aligned} \tag{3.57}
$$

Proof. First note that, since $\rho = 0$, we have: $dA_{x,t}dA^\dagger_{\epsilon,t} = dA_{\epsilon,t}dA^\dagger_{x,t} = 0$. Therefore, 3.55 becomes:

$$
\begin{aligned}
dj_t(X) = &\big(dU^*_{\epsilon,t}U^*_{x,t} + U^*_{\epsilon,t}dU^*_{x,t}\big)(X \otimes \mathbb{I})U_{x,t}U_{\epsilon,t} \\
&+ U^*_{\epsilon,t}U^*_{x,t}(X \otimes \mathbb{I})\big(dU_{x,t}U_{\epsilon,t} + U_{x,t}dU_{\epsilon,t}\big) \\
&+ \big(dU^*_{\epsilon,t}U^*_{x,t} + U^*_{\epsilon,t}dU^*_{x,t}\big)(X \otimes \mathbb{I})\big(dU_{x,t}U_{\epsilon,t} + U_{x,t}dU_{\epsilon,t}\big)
\end{aligned}
$$

Inserting 3.53 and 3.54, and applying the Ito multiplication table, we get:

$$
\begin{aligned}
dj_t(X) &= j_t(\alpha^\dagger_x)dA^\dagger_{x,t} + j_t(\alpha_x)dA_{x,t} + j_t(\lambda_x)d\Lambda_{x,t} \\
&= j_t(\alpha^\dagger_\epsilon)dA^\dagger_{\epsilon,t} + j_t(\alpha_\epsilon)dA_{\epsilon,t} \\
&+ \big(j_t(\theta_\epsilon) + j_t(\theta_x)\big)dt
\end{aligned} \tag{3.58}
$$

Where since $S_\epsilon = \mathbb{I}$, the terms in $d\Lambda_\epsilon$ drop out. Furthermore we have:

$$\begin{aligned}
\alpha_x &= [L_x^*, X]S_x \\
&= \begin{pmatrix} 0 & i\sigma_x \\ -i\sigma_x & 0 \end{pmatrix} \\
\lambda_x &= S_X^* X S_x - X \\
&= \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \\
\alpha_\epsilon &= [L_\epsilon^*, X] \\
&= \begin{pmatrix} -i\sigma_\epsilon/2 & 0 \\ 0 & i\sigma_\epsilon/2 \end{pmatrix} \\
\theta_x &= -\frac{1}{2}\Big(L_x^* L_x X + X L_x^* L_x - 2L_x^* X L_x\Big) \\
&= 0 \\
\theta_\epsilon &= i[H_\epsilon, X] - \frac{1}{2}\Big(L_\epsilon^* L_\epsilon X + X L_\epsilon^* L_\epsilon - 2L_\epsilon^* X L_\epsilon\Big) \\
&= i[H_\epsilon, X]
\end{aligned}$$

The result follows by feeding this into 3.58.

For the higher moments, equation 3.57 follows by calculating powers of 3.56, using the Ito multiplication from [34]. □

3.7.3 Role of the Potential Function and choices for the Hamiltonian:

3.7.3.1 Role of the Potential Function:

In this section, we consider the impact of including a non-zero potential function, first in general terms, before going on to consider possible choices for the bid-offer Hamiltonian: H_ϵ.

First note, that since we see changes in the price variable: X, it is tempting to think that one could introduce drift effects, such as mean reversion, through the introduction of a potential function.

For example we could set a quadratic potential well, with a minimum of zero:

$$
\begin{aligned}
dU_t &= -\left(\left(iH + \frac{L^*L}{2}\right) \otimes dt + L^* \otimes dA_t - L \otimes dA_t^\dagger\right) U_t \\
dj_t(X) &= j_t(i[H_x, X])dt + \begin{pmatrix} i\sigma_x & 0 \\ 0 & i\sigma_x \end{pmatrix} dA_t^\dagger - \begin{pmatrix} i\sigma_x & 0 \\ 0 & i\sigma_x \end{pmatrix} dA_t \qquad (3.59) \\
H_x &= \begin{pmatrix} kx^2 & 0 \\ 0 & kx^2 \end{pmatrix}, \; L = \begin{pmatrix} -i\sigma\partial_x & 0 \\ 0 & -i\sigma\partial_x \end{pmatrix}
\end{aligned}
$$

However, with X defined by 2.4 and H_x defined by 3.59, we have: $[H_x, X] = 0$. Therefore, the immediate incremental change in the price: $dj_t(X)$, is unaffected.

Incorporating a kinetic energy term (with arbitrary constant m):

$$
H_x = \begin{pmatrix} (-1/2m)\partial_x^2 + kx^2 & 0 \\ 0 & (-1/2m)\partial_x^2 + kx^2 \end{pmatrix}
$$

Leads to:

$$
\begin{aligned}
i[H_x, X] &= \begin{pmatrix} (-i/2m)\partial_x & 0 \\ 0 & (-i/2m)\partial_x \end{pmatrix} \qquad (3.60) \\
&\neq 0
\end{aligned}
$$

The meaning of 3.60, from a financial perspective, is that to specify the value of $E[dj_t(X)]$, one needs to specify the initial rate of change relative to the constant: m.

In fact, the value of $E[dj_t(X)]$ is still not impacted by the presence of the potential function. However, if we label the "momentum":

$$P = \begin{pmatrix} -i\partial_x & 0 \\ 0 & -i\partial_x \end{pmatrix} \tag{3.61}$$

then since $[L, P] = 0$, we find 3.59 becomes:

$$\begin{aligned} dj_t(X) &= j_t((1/2m)P)dt + \begin{pmatrix} 0 & i\sigma_x \\ -i\sigma_x & 0 \end{pmatrix} dA_t + \begin{pmatrix} 0 & i\sigma_x \\ -i\sigma_x & 0 \end{pmatrix} dA_t^\dagger \\ dj_t(P) &= j_t(i[H_x, P])dt \\ &= -2k \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} dt \end{aligned} \tag{3.62}$$

Remark 3.7.3. *We have:*

$$\begin{aligned} [X, P] &= -i \begin{pmatrix} [x, \partial_x] & 0 \\ 0 & [x, \partial_x] \end{pmatrix} \\ &\neq 0 \end{aligned} \tag{3.63}$$

From 3.62, we see that in order to simulate the path of the price X, one needs to track both the price & momentum. However, since these 2 variables do not commute, this will not be possible.

Thus, whilst it may be possible to define a Martingale Price Process for a particular derivative, with $H_x \neq 0$, there is a fundamental limit to the accuracy with which we can simulate the path of X.

Knowing the price of X with full precision implies we cannot fully specify the drift (see for example [31]).

Remark 3.7.4. *In practice, as stated above, we assume that the Hamiltonian: H_x in equation* 3.54 *is zero, since we wish the process $j_t(X)$ to be a Martingale. For this reason we do not require the simultaneous*

tracking of both P and X. Whether or not the simulation of X with full precision is possible will depend on the choice for H_ϵ in equation 3.53. We discuss this in the next section.

3.7.3.2 A Potential Function for the Bid-Offer Spread:

A discussion of the economics of the bid-offer spread is given in [37] chapter 16. Financial market participants generally split into 2 camps:

- Liquidity providers, or market makers, who will quote a bid & offer price at which they are willing to trade.
- Liquidity takers, who trade with the market makers, by trading at the prices quoted.

The analysis in [37] suggests that the profitability of a market making strategy is largely determined by:

1) The profit per trade owing to the fact that the market maker is buying at a lower price, whilst hopefully selling at a higher price.
2) The key risk is that of adverse selection. This refers to the intrinsic bias owing to the fact that liquidity takers will only trade at a price where they think it is economically favourable. For example where they may know something that may impact the price.

The analysis in [37] suggests that the equilibrium bid-offer spread is such that these 2 competing effects balance out.

If the bid-offer spread is too high, then there is an incentive for new market makers to charge a lower spread in order to increase volume in what is a profitable business.

If the bid-offer spread is too low, then market making will likely be loss making, owing to the adverse bias. Thus the incentive is to charge more, or get out of the business altogether.

This suggests that, whatever stochastic process one chooses in 3.53, there should be an element of mean reversion to an equilibrium value. In other words, where the bid-offer spread has randomly diffused to very high levels, there should be forces that drag it back to the equilibrium.

Building on from the analysis presented above, the simplest way to reflect this, is to use a quadratic potential well:

$$H_\epsilon = \begin{pmatrix} -\partial_\epsilon^2 + (\nu/2)\epsilon^2 & 0 \\ 0 & -\partial_\epsilon^2 + (\nu/2)\epsilon^2 \end{pmatrix} \tag{3.64}$$

We investigate this choice further in the next section.

3.7.4 New Stochastic Process for the Bid-Offer Spread:

Proposition 3.7.5. *Let U_ϵ and U_x be defined by 3.53 and 3.54, with:*

- $\rho = 0$
- $S_x = R(\pi/2)$, $S_\epsilon = \mathbb{I}$.
- $H_x = 0$, $H_\epsilon \in \mathcal{L}\big(\mathcal{S}(\mathbb{R}^2) \oplus \mathcal{S}(\mathbb{R}^2)\big)$.
- $L_x = \begin{pmatrix} -i\sigma_x\partial_x & 0 \\ 0 & -i\sigma_x\partial_x \end{pmatrix}$, $L_\epsilon = \begin{pmatrix} -i\sigma_\epsilon\partial_\epsilon & 0 \\ 0 & -i\sigma_\epsilon\partial_\epsilon \end{pmatrix}$, $P_\epsilon = \begin{pmatrix} -i\partial_\epsilon & 0 \\ 0 & -i\partial_\epsilon \end{pmatrix}$
- $E = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$

Then the stochastic process: $j_t(E) = U^*_{\epsilon,t}U^*_{x,t}(E\otimes\mathbb{I})U_{x,t}U_{\epsilon,t}$*, is defined by:*

$$dj_t(E) = j_t(P_\epsilon)dt + \begin{pmatrix} i\sigma_\epsilon & 0 \\ 0 & i\sigma_\epsilon \end{pmatrix} dA^\dagger_{\epsilon,t} - \begin{pmatrix} i\sigma_\epsilon & 0 \\ 0 & i\sigma_\epsilon \end{pmatrix} dA_{\epsilon,t} \tag{3.65}$$

$$dj_t(P_\epsilon) = -\begin{pmatrix} -\nu\epsilon & 0 \\ 0 & -\nu\epsilon \end{pmatrix} dt$$

Proof. Applying 3.64, we have:

$$[H_\epsilon, E] = \begin{pmatrix} -i\partial_\epsilon & 0 \\ 0 & -i\partial_\epsilon \end{pmatrix} \tag{3.66}$$

$$(L^*_\epsilon L_\epsilon E + EL^*_\epsilon L_\epsilon - 2L^*_\epsilon E L_\epsilon) = 0 \tag{3.67}$$

$$[L_x, E] = 0 \tag{3.68}$$

$$[L_\epsilon, E] = \begin{pmatrix} -i\sigma_\epsilon & 0 \\ 0 & -i\sigma_\epsilon \end{pmatrix} \tag{3.69}$$

Applying 3.68, we have:

$$\begin{aligned} dj_t(E) &= \big(dU^*_{\epsilon,t}U^*_{x,t} + U^*_{\epsilon,t}dU^*_{x,t}\big)(E\otimes\mathbb{I})U_{x,t}U_{\epsilon,t} \\ &+ U^*_{\epsilon,t}U^*_{x,t}(E\otimes\mathbb{I})\big(dU_{x,t}U_{\epsilon,t} + U_{x,t}dU_{\epsilon,t}\big) \\ &+ \big(dU^*_{\epsilon,t}U^*_{x,t} + U^*_{\epsilon,t}dU^*_{x,t}\big)(E\otimes\mathbb{I})\big(dU_{x,t}U_{\epsilon,t} + U_{x,t}dU_{\epsilon,t}\big) \\ &= \big(dU^*_{\epsilon,t}U^*_{x,t}\big)(E\otimes\mathbb{I})U_{x,t}U_{\epsilon,t} + U^*_{\epsilon,t}U^*_{x,t}(E\otimes\mathbb{I})\big(U_{x,t}dU_{\epsilon,t}\big) + \big(dU^*_{\epsilon,t}U^*_{x,t}\big)(E\otimes\mathbb{I})\big(U_{x,t}dU_{\epsilon,t}\big) \end{aligned}$$

Applying 3.66, and 3.69, we get:

$$dj_t(E) = \begin{pmatrix} -i\partial_\epsilon & 0 \\ 0 & -i\partial_\epsilon \end{pmatrix} dt + \begin{pmatrix} i\sigma_\epsilon & 0 \\ 0 & i\sigma_\epsilon \end{pmatrix} dA^\dagger_{\epsilon,t} - \begin{pmatrix} i\sigma_\epsilon & 0 \\ 0 & i\sigma_\epsilon \end{pmatrix} dA_{\epsilon,t}$$

as required. Furthermore, we have:

$$\begin{aligned} dj_t(P_\epsilon) = &\big(dU^*_{\epsilon,t}U^*_{x,t} + U^*_{\epsilon,t}dU^*_{x,t}\big)(P_\epsilon \otimes \mathbb{I})U_{x,t}U_{\epsilon,t} \\ &+ U^*_{\epsilon,t}U^*_{x,t}(P_\epsilon \otimes \mathbb{I})\big(dU_{x,t}U_{\epsilon,t} + U_{x,t}dU_{\epsilon,t}\big) \\ &+ \big(dU^*_{\epsilon,t}U^*_{x,t} + U^*_{\epsilon,t}dU^*_{x,t}\big)(P_\epsilon \otimes \mathbb{I})\big(dU_{x,t}U_{\epsilon,t} + U_{x,t}dU_{\epsilon,t}\big) \end{aligned}$$

Since, $[L_\epsilon, P_\epsilon] = [L_x, P_\epsilon] = 0$, only the terms in dt survive, and we have:

$$dj_t(P_\epsilon) = \begin{pmatrix} -\nu\epsilon & 0 \\ 0 & -\nu\epsilon \end{pmatrix} dt$$

□

Finally, note that we can feed the Hamiltonian choice: 3.64, into proposition 3.7.2 to give the following:

Proposition 3.7.6. *Under the conditions of proposition 3.7.2, and with Hamiltonian: 3.64, we have:*

$$\begin{aligned} dj_t(X) = &\begin{pmatrix} 0 & i\sigma_x \\ -i\sigma_x & 0 \end{pmatrix} dA_{x,t} + \begin{pmatrix} 0 & i\sigma_x \\ -i\sigma_x & 0 \end{pmatrix} dA^\dagger_{x,t} \\ &+ \begin{pmatrix} i\sigma_\epsilon/2 & 0 \\ 0 & -i\sigma_\epsilon/2 \end{pmatrix} dA^\dagger_{\epsilon,t} + \begin{pmatrix} -i\sigma_\epsilon/2 & 0 \\ 0 & i\sigma_\epsilon/2 \end{pmatrix} dA_{\epsilon,t} \\ &+ j_t\Bigg(\begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}\Bigg) d\Lambda_{x,t} + j_t(P_\epsilon/2)dt \end{aligned} \tag{3.70}$$

$$dj_t(P_\epsilon) = \begin{pmatrix} -\nu\epsilon & 0 \\ 0 & -\nu\epsilon \end{pmatrix} \otimes dt$$

Proof. We feed 3.66 into 3.56, alongside the fact that:

$$X = \begin{pmatrix} x + \epsilon/2 & 0 \\ 0 & x - \epsilon/2 \end{pmatrix}$$

The result then follows from proposition 3.7.5. □

3.7.5 Financial Interpretation of the 2 Factor Model:

The non-Gaussian Fokker-Planck equation: 3.37, gives us an equation for the probability density function of the log price: x. Under this equation, those markets where one can buy & sell at a similar price are represented by the case $\epsilon \approx 0$. Where the bid-offer spread grows, and the best offer price is appreciably higher that the best bid price, then $\epsilon \neq 0$, and the market dynamics for the log price are affected.

The key objective in this section is to build on this, by incorporating the idea that the bid-offer spread is driven by its' own independent stochastic process. Therefore, in the long term, a particular asset may have periods of high liquidity, where $\epsilon \to 0$, and periods where the market starts to break down, and ϵ grows.

With this in mind the key results are:

- Proposition: 3.7.2, where we give the stochastic process for the log price under a 2 factor model. Note that using similar methods to those employed for proposition: 3.5.6, we can define a non-Gaussian Fokker-Planck partial differential equation. Similarly using methods developed for proposition 3.5.7, we can derive equations for the moments, and study how they depend on the bid-offer variable: ϵ.

- Proposition: 3.7.5, where we define a mean reverting stochastic process for the bid-offer spread. Actually the true bid-offer spread is given by:

$$\begin{aligned} \exp(x)\big(exp(\epsilon/2) - exp(-\epsilon/2)\big) &= exp(x)2\sinh(\epsilon/2) \\ &\approx \epsilon \exp(x), \text{ for small } \epsilon \end{aligned}$$

- Proposition: 3.7.6, where we highlight the impact of the mean reversion in ϵ on the bid & offer prices respectively.

Chapter 4

Modelling Illiquid Stocks Using Quantum Stochastic Calculus: Asymptotic Methods

4.1 Chapter Introduction:

In this chapter, we investigate asymptotic solutions to the theoretical models developed in chapter 3, section 3.5.2.

This model involves developing a Fokker-Planck equation whereby a non-zero bid-offer spread: ϵ impacts the dynamics of the asset mid-price: x, by giving the probability density function excess kurtosis (aka "fatter tails").

Furthermore, where the volume of potential sellers that make up the market, outweighs the volume of potential buyers (in this case represented by: $||\psi_o||^2 > ||\psi_b||^2$) we get a negative skew effect. We represent

the skew using a parameter: η. Quoting from the previous chapter, we have:

$$\frac{\partial p}{\partial t} = \sigma^2 \sum_{k\geq 1} \frac{\epsilon^{(2k-2)}}{(2k)!} \frac{\partial^{2k} p}{\partial x^{2k}} + \sigma^2 \eta \sum_{k\geq 2} \frac{(-\epsilon)^{(2k-3)}}{(2k-1)!} \frac{\partial^{(2k-1)} p}{\partial x^{(2k-1)}} \tag{4.1}$$

$$\eta = \left(||\psi_o||^2 - ||\psi_b||^2 \right)$$

We first investigate power series solutions to 4.1 in section 4.3, before considering the convergence of the power series in section 4.4, and carrying out numerical analysis in section 4.5.

4.2 General Chapter Overview:

The important new results developed in this chapter are as follows:

- In proposition 4.3.1 we present a power series solution to the Fokker-Planck equation, developed in section 3.5.2.
- In proposition 4.4.2, we show that this power series is a divergent asymptotic expansion. That is, at a fixed time to maturity, the solution diverges as more and more terms are added. However, the solution is asymptotic in the large time limit. In practice, this means that for sufficiently large time, a finite number of terms in the power series can generate an accurate solution to the partial differential equation. However, the inclusion of too many terms will lead to divergence. This is further demonstrated using numerical analysis in section 4.5.
- Finally in proposition 4.4.6 we develop an error bound for the power series. That is we develop a criterion that can be used to decide on a minimum limit in the time to maturity, to ensure a sufficient degree of convergence.

4.3 Power Series Solutions:

4.3.1 Deriving a Power Series Solution:

To find a solution to 4.1, motivated by the heat kernel solution to the classical heat equation (for example see [49] appendix C), we use a trial function:

$$p(x,t) = \frac{a_{00}}{\sqrt{t}} + \sum_{n\geq 1}\sum_{m=2}^{2n} \frac{a_{nm}}{\sqrt{t}}\left(\frac{x^m}{t^n}\right) \tag{4.2}$$

We substitute 4.2 into 4.1, and attempt to match the right & left hand side, thereby generating a sequence relation for the coefficients: a_{nm}.

Proposition 4.3.1. *Subject to convergence of the infinite series, equation 4.2 is a solution to 4.1, if the coefficients a_{nk} are given by:*

$$a_{12} = -\frac{a_{00}}{2\sigma^2}$$
$$\left(\frac{1}{2}-n\right)a_{(n-1)m} = \sigma^2 \sum_{l=1}^{\lfloor \frac{2n-m}{2} \rfloor} \binom{m+2l}{2l} \epsilon^{2l-2} a_{n(m+2l)} - \sigma^2\eta \sum_{l=2}^{\lfloor \frac{2n+1-m}{2} \rfloor} \binom{m+2l-1}{2l-1} \epsilon^{2l-3} a_{n(m+2l-1)}$$

Proof. Inserting 4.2 into the left hand side of 4.1, gives:

$$\frac{\partial p}{\partial t} = \sum_{n\geq 0}\left(-\frac{1}{2}-n\right)\frac{1}{\sqrt{t}}\sum_{m=0}^{2n} a_{nm}\left(\frac{x^m}{t^{n+1}}\right)$$

Similarly, inserting 4.2 into the right hand side of 4.1, gives:

$$\sigma^2 \sum_{k\geq 1} \frac{\epsilon^{(2k-2)}}{(2k)!}\frac{\partial^{2k} p}{\partial x^{2k}} = \sum_{n\geq 0}\frac{\sigma^2}{\sqrt{t}}\sum_{m=2}^{2n}\sum_{l=1}^{\lfloor m/2 \rfloor} \frac{m!\epsilon^{(2l-2)}a_{nm}}{(m-2l)!(2l)!}\left(\frac{x^{(m-2l)}}{t^n}\right)$$
$$\sigma^2\eta \sum_{k\geq 2} \frac{(-\epsilon)^{(2k-3)}}{(2k-1)!}\frac{\partial^{(2k-1)} p}{\partial x^{(2k-1)}} = \sum_{n\geq 0}\frac{\sigma^2\eta}{\sqrt{t}}\sum_{m=2}^{2n}\sum_{l=1}^{m} \frac{m!\epsilon^{(2l-3)}a_{nm}}{(m-2l+1)!(2l-1)!}\left(\frac{x^{(m-2l+1)}}{t^n}\right)$$

Combining the two, we get:

$$\sum_{n\geq 0}\Big(-\frac{1}{2}-n\Big)\frac{1}{\sqrt{t}}\sum_{m=0}^{n}a_{nm}\Big(\frac{x^m}{t^{n+1}}\Big)=\sum_{n\geq 0}\frac{\sigma^2}{\sqrt{t}}\sum_{m=2}^{2n}\sum_{l=1}^{\lfloor m/2\rfloor}\binom{m}{2l}\epsilon^{(2l-2)}a_{nm}\Big(\frac{x^{(m-2l)}}{t^n}\Big) \tag{4.3}$$
$$+\sum_{n\geq 0}\frac{\sigma^2\eta}{\sqrt{t}}\sum_{m=2}^{2n}\sum_{l=2}^{\lfloor (m+1)/2\rfloor}\binom{m}{2l-1}\epsilon^{(2l-3)}a_{nk}\Big(\frac{x^{(m-2l+1)}}{t^n}\Big)$$

In order to derive a series to calculate the coefficients a_{nm} we compare coefficients of:$\frac{x^m}{t^n}$ on each side of 4.3.

From the left hand side we have:

$$\Big(\frac{1}{2}-n\Big)\frac{a_{(n-1)m}}{\sqrt{t}}\Big(\frac{x^m}{t^n}\Big)$$

Similarly, from the right hand side we have:

$$\Big(\frac{\sigma^2}{\sqrt{t}}\sum_{l=1}^{\lfloor\frac{2n-m}{2}\rfloor}\binom{m+2l}{2l}\epsilon^{2l-2}a_{n(m+2l)}\Big(\frac{x^m}{t^n}\Big)\Big)-\Big(\frac{\sigma^2\eta}{\sqrt{t}}\sum_{l=2}^{\lfloor\frac{2n+1-m}{2}\rfloor}\binom{m+2l-1}{2l-1}\epsilon^{2l-3}a_{n(m+2l-1)}\Big(\frac{x^m}{t^n}\Big)\Big)$$

Therefore, equating the coefficients for both sides, we find:

$$\Big(\frac{1}{2}-n\Big)a_{(n-1)m}=\sigma^2\sum_{l=1}^{\lfloor\frac{2n-m}{2}\rfloor}\binom{m+2l}{2l}\epsilon^{2l-2}a_{n(m+2l)}-\sigma^2\eta\sum_{l=2}^{\lfloor\frac{2n+1-m}{2}\rfloor}\binom{m+2l-1}{2l-1}\epsilon^{2l-3}a_{n(m+2l-1)} \tag{4.4}$$

Finally, we can solve for the coefficients: a_{nk} in escalating powers of t. For $n=0$ we have:

$$-\frac{a_{00}}{2}=\sigma^2 a_{12}$$

We assume $a_{11}, a_{10}=0$.

Assume now we know the coefficients for a_{im} for all m, for $i\leq(n-1)$.

We start with the equation involving: $a_{(n-1)(2n-2)}$. We have $m=2n-2$. Therefore, $\frac{2n-m}{2}=1$, and we have only one term on the right hand side of 4.4:

$$\Big(\frac{1}{2}-n\Big)a_{(n-1)(2n-2)}=\sigma^2\binom{2n}{2}a_{n(2n)}$$

Therefore, from the value of $a_{(n-1)(2n-2)}$ we can calculate the value of $a_{n(2n)}$.

Now assume, as well as knowing all the coefficients a_{ij} with $i \leq (n-1)$, we know those with $i = n$ and $j = 2n$ down to $j = m + 4$. Then in equation 4.4, there is only one unknown coefficient: $a_{n(m+2)}$.

Thus by the second induction, we can calculate the rest of the coefficients a_{nj} for all j, and by the first induction, we can calculate all coefficients: a_{ij}, with $i \geq n$. □

4.4 Convergence Properties:

In order to apply proposition 4.3.1, we investigate the solution to the truncated partial differential equation. For example, with zero skew (number of buyers & sellers is balanced) we would have:

$$\frac{\partial p_K}{\partial t} = \sigma^2 \sum_{k=1}^{K} \frac{\epsilon^{(2k-2)}}{(2k)!} \frac{\partial^{2k} p_K}{\partial x^{2k}} \tag{4.5}$$

Proposition 4.4.1. *The power series $p_K(x,t)$, given by:*

$$p_K(x,t) = \frac{a_{00}}{\sqrt{t}} + \sum_{n \geq 1} \sum_{m=2(n-K+1)}^{2n} \frac{a_{nm}}{\sqrt{t}} \left(\frac{x^m}{t^n} \right) \tag{4.6}$$

is a solution to the truncated partial differential equation: 4.5, where the coefficients are given by:

$$\left(\frac{1}{2} - n \right) a_{(n-1)m} = \sigma^2 \sum_{l=1}^{\min(\lfloor \frac{2n-m}{2} \rfloor, K)} \binom{m+2l}{2l} \epsilon^{2l-2} a_{n(m+2l)} \tag{4.7}$$

Proof. Each term on the right hand side of 4.4, derives from a partial derivative: $\partial^{2l}/\partial x^{2l}$. Equation 4.7, follows by restricting $l \leq K$.

As described in the proof to proposition 4.3.1, we can proceed as follows:

- By setting $n = 0, m = 0$, we can calculate the value for a_{12}. Since $\frac{2n-m}{2} = 1$, then this is the only non-zero term for $n = 1$.
- For $n = 2$, we first calculate the value for a_{24} by setting $m = 2$.

- If $\epsilon = 0$, then equations 4.4 and 4.7 are the same. The only nonzero terms are of the form: $a_{n(2n)}$, and the resulting series is the Taylor expansion for the normal distribution probability density.
- At each value for n, we start by setting, $m = 2n - 2$. This yields the value for $a_{n(2n)}$. Then proceeding as described, the known value for $a_{(n-1)m}$ determines the value for $a_{n(m+2)}$.
- The left hand side of 4.7 gives K equations: $a_{(n-1)(2n-2)}$, $a_{(n-1)(2n-4)}$, etc down to $a_{(n-1)(2n-4-2K)}$.
- From these, we determine in turn the non-zero values for $a_{n(2n)}$ down to $a_{n(2n-2K-2)}$, as shown in the proof of proposition 4.3.1.

□

For $K = 1$, from 4.4.1, we get:

$$p_1(x,t) = \sum_{n\geq 0} \frac{a_{n(2n)}}{\sqrt{t}} \left(\frac{x^{2n}}{t^n} \right)$$

$$a_{n(2n)} = -\frac{a_{(n-1)(2n-2)}}{(2n)\sigma^2}$$

which, modulo a normalising constant, is the Taylor series expansion (about $x = 0$) for the standard Gaussian probability density.

When one moves from $K = 1$ to $K = 2$, one includes an additional series:

$$\phi_2(x,t) = \sum_{n\geq 2} \frac{a_{n(2n-2)}}{\sqrt{t}} \left(\frac{x^{(2n-2)}}{t^n} \right)$$

Similarly, when moving from $K = 2$ to $K = 3$ we add a third term:

$$\phi_3(x,t) = \sum_{n\geq 3} \frac{a_{n(2n-4)}}{\sqrt{t}} \left(\frac{x^{(2n-4)}}{t^n} \right)$$

Now, consider the power series 4.6 as a function of the variable $y = 1/t$:

$$p(x,y) = a_{00}\sqrt{y} + \sum_{j\geq 1} \phi_j(x,y) \tag{4.8}$$

$$p_K(x,y) = a_{00}\sqrt{y} + \sum_{j=1}^{K} \phi_j(x,y)$$

$$\phi_j(x,y) = \sum_{n\geq j} a_{n(2n-2j+2)}(x^{(2n-2j+2)})y^{n+0.5}$$

Proposition 4.4.2. *The series from proposition 4.4.1 is a divergent asymptotic expansion for the solution to equation 4.1, with $\eta = 0$.*

Remark 4.4.3. *In this proposition, we show that the series from proposition 4.4.1 is an asymptotic expansion in the sense of definition 10.1.1 from [23]. That is we show that in equation 4.8, we have:*

$$\phi_j(x,y) = o(\phi_{j-1}(x,y)),\ as\ y \to 0$$

Thus, for a fixed (and arbitrarily high) value for K, the truncation error (from ignoring $\phi_j(x,y)$ for $j \geq K+1$) tends to zero for $y \to 0$. In other words, the approximation becomes more and more accurate for higher values of t.

However, for a fixed value of x and t, the series diverges as $K \to \infty$. In section 4.5, we show that in practical applications it will be sufficient to include a small number of terms in approximating the solution.

Proposition 4.4.6 is then crucial in the sense that this enables us to calculate a cut-off time (dependent on x), in order to ensure the approximation error remains below a specified level. The solution should then only be applied for times above this cutoff time.

Proof of Proposition 4.4.2. We have from equation 4.4, that:

$$\Big(\frac{1}{2} - n\Big)a_{(n-1)2} = \sigma^2 \sum_{l=1}^{n-1} \binom{2l+2}{2l} \epsilon^{2l-2} a_{n(2+2l)} \tag{4.9}$$

If $\sum_{j\geq 1}\phi_j(x,y)$ is a convergent series, then we must have: $\sum_{j\geq 1}\phi_j(1,1)$ is also a convergent series. Therefore, we have that: $\sum_{j\geq 1}\sum_{k=0}^{2j}a_{jk}$ also converges. Let us write the series by ordering the a_{jk} first by j and then by k. We write:

$$S_N = \sum_{n=1}^{N} b_n$$

Where $b_1 = a_{00}$, $b_2 = a_{10}$, $b_3 = a_{11}$, $b_4 = a_{12}$, etc. Since we assume that the sequence: S_N converges as $N \to \infty$, we must have that $b_n \to 0$ as $n \to \infty$. Therefore, we can choose N such that: $|b_m| < |b_n|$, for $n > N$ and $m > n$.

Therefore, we can choose $n > N$, such that:

$$\begin{aligned} max_{(k\leq n)}a_{n(2k)} &= a_{n,max} \\ &< a_{(n-1)2} \end{aligned} \tag{4.10}$$

Now we have:

$$\begin{aligned} \sigma^2\sum_{l=1}^{n-1}\binom{2l+2}{2l}\epsilon^{2l-2}a_{n(2+2l)} &= \frac{\sigma^2}{\epsilon^4}\sum_{l=1}^{n-1}\binom{2l+2}{2l}\epsilon^{2l+2}a_{n(2+2l)} \\ &\leq \frac{\sigma^2}{\epsilon^4}\sum_{l=1}^{n-1}\left|\binom{2l+2}{2l}\epsilon^{2l+2}a_{n(2+2l)}\right| \\ &\leq \frac{a_{n,max}\sigma^2}{\epsilon^4}\sum_{l=1}^{n-1}\left|\binom{2l+2}{2l}\epsilon^{2l+2}\right| \end{aligned}$$

Now the sequence represented by:

$$R_n(\epsilon) = \sum_{l=1}^{n-1}\left|\binom{2l+2}{2l}\epsilon^{2l+2}\right|$$

converges as $n \to \infty$, for $|\epsilon| < 1$, by the ratio test. Therefore, we have for $n > N+1$:

$$a_{n,max} \geq \left(\frac{\epsilon^4\left(n-\frac{1}{2}\right)}{\sigma^2 R_\infty(\epsilon)}\right)a_{(n-1)2} \tag{4.11}$$

However, for large enough n, we have that equation 4.11 contradicts equation 4.10. Therefore the series: $\sum_{j\geq 1}\sum_{k=0}^{2j} a_{jk}$ is not convergent.

To show that $p_K(x,y)$ is asymptotic to $p(x,y)$ in equation 4.8, as $y \to 0$, note that for all $j \geq 1$ we have:

$$\phi_j(x,y) = O(y^{j+0.5}), \text{ as } j \to 0$$

$$y^j = O(\phi_{j-1}(x,y)), \text{ as } j \to 0$$

Therefore as $y \to 0$ we have:

$$\phi_j(x,y) \leq K_1 y^{j+0.5}, \text{ for some constant } K_1$$

$$y^j \leq K_2 \phi_{j-1}(x,y), \text{ for some constant } K_2$$

So:

$$\phi_j(x,y) \leq K_1 K_2 y^{0.5} \phi_{j-1}(x,y)$$

Which in turn implies:

$$\phi_j(x,y) = o(\phi_{j-1}(x,y)), \text{ as } y \to 0$$

□

Remark 4.4.4. *Note that, since $\epsilon^4 = O(R_\infty(\epsilon))$ as $\epsilon \to 1$, the contradiction given by equation 4.11 is met at smaller values for n as ϵ increases, and gets closer to 1. Thus we expect more rapid divergence as ϵ gets larger (increases from $\epsilon = 0$), and that the series will get closer to the Gaussian solution as $\epsilon \to 0$.*

We now show that, whilst the series given by equation 4.8:

$$S_K(x,y) = \sum_{j=1}^{K} \phi_j(x,y)$$

is divergent for large x, and y (small t), each individual term: $\phi_j(x,y)$ does converge for all x, and y.

Proposition 4.4.5. *The series defined by:*

$$\phi_j(x,y) = \sum_{n \geq j} a_{n(2n-2j+2)}(x^{(2n-2j+2)})y^{n+0.5}$$

converges for all x and y.

Proof. We write:

$$b_n^k = a_{n(2n-k)}$$

Note that:

$$b_n^0 = -\frac{1}{(2\sigma^2)^n n!}$$

So that it is clear that the sequence: b_n^0 converges with $O(e^n/n!)$ as $n \to \infty$. We now assume that this also applies for b_n^j for all $j \leq (k-1)$.

Now, from equation 4.4, we have:

$$b_n^k = \left[\left(\frac{1}{2} - n\right)b_{n-1}^k - \sigma^2 \sum_{l=2}^{k+1} \binom{2n-2k-2+2l}{2l} \epsilon^{2l-2} b_n^{k+1-l}\right] \left(\sigma^2 \binom{2n-2}{2}\right)^{-1}$$

In the summation, we have $k-1$ individual terms, which by assumption, each converge at least to $O(e^n/n!)$. For the first term, we have:

$$\frac{\left(\frac{1}{2} - n\right)b_{n-1}^k}{\sigma^2 \binom{2n-2}{2}} = -\frac{(2n-1)}{\sigma^2(2n-2)(2n-3)} b_{n-1}^k$$

From which it follows that the b_n^k term also converges with $O(e^n/n!)$. Since we have:

$$\sum_{n=1}^{\infty} y^n x^k \frac{e^n}{n!}$$

converges for all x, y, it follows that the series:

$$\sqrt{y} \sum_{n=1}^{\infty} b_n^k x^k y^n$$

converges, and that therefore: $\phi_j(x,y)$ converges in n for all x, y, j. □

As mentioned in remark 4.4.3, we now apply propositions 4.4.2 and 4.4.5, to show how to calculate upper bounds for y, based on the value for K, to ensure the series is convergent and the relative error remains small.

Proposition 4.4.6. *For the series defined in proposition 4.4.1, we have:*

$$\frac{\phi_j(x,y)}{\phi_{j-1}(x,y)} \approx c_1 y + c_2 x^2 y^2 + O(y^3) \tag{4.12}$$

$$c_1 = a_{j2}/a_{(j-1)2}$$
$$c_2 = \frac{\left(a_{(j+1)4} - (a_{j2}/a_{(j-1)2})a_{j4}\right)}{a_{(j-1)2}}$$

Therefore, to ensure that: $|\phi_j(x,y)| < \varepsilon|\phi_{j-1}(x,y)|$*, we must have:*

$$|c_1 y + c_2 y^2 x^2| < \varepsilon \tag{4.13}$$

$$c_1 = a_{j2}/a_{(j-1)2}$$
$$c_2 = \frac{\left(a_{(j+1)4} - (a_{j2}/a_{(j-1)2})a_{j4}\right)}{a_{(j-1)2}}$$

Proof. We first write out the ratio of subsequent terms in the series $(\phi_j(x,y)/\phi_{j-1}(x,y))$ and invert the Padé approximation technique outlined in [8] section 8.3. We first write:

$$\begin{aligned}\frac{\phi_j(x,y)}{\phi_{j-1}(x,y)} &= \frac{\sum_{n=j}^{\infty} a_{n(2n-2j+2)} y^{n+0.5} x^{(2n-2j+2)}}{\sum_{n=j-1}^{\infty} a_{n(2n-2j+4} y^{n+0.5} x^{(2n-2j+4)}} \\ &= \frac{a_{j2} y^{j+0.5} x^2 + a_{(j+1)4} y^{j+1.5} x^4 + \ldots}{a_{(j-1)2} y^{j-0.5} x^2 + a_{j4} y^{j+0.5} x^4 + \ldots}\end{aligned}$$

We first divide through top & bottom by $y^{j-0.5}$ to get:

$$\frac{\phi_j(x,y)}{\phi_{j-1}(x,y)} = \frac{\sum_{k=1}^{\infty} A_k(x) y^k}{\sum_{l=0}^{\infty} B_l(x) y^l} \tag{4.14}$$

$$A_k(x) = a_{(j+k-1)(2k)} x^{2k}$$

$$B_l(x) = a_{(j+l-1)(2l+2)} x^{2l+2}$$

We now equate the quotient 4.14, to a power series in y:

$$\sum_{i=1}^{\infty} c_i(x)y^i = \frac{\sum_{k=1}^{\infty} A_k(x)y^k}{\sum_{l=0}^{\infty} B_l(x)y^l}$$

We can calculate the coefficients: a_i by equating powers of y. We have:

$$\Big(\sum_{i=1}^{\infty} c_i(x)y^i\Big)\Big(\sum_{l=0}^{\infty} B_l(x)y^l\Big) = \sum_{k=1}^{\infty} A_k(x)y^k$$

So that:

$$c_1(x)B_0(x) = A_1(x)$$

$$c_2(x)B_0(x) + c_1(x)B_1(x) = A_2(x)$$

From this we get:

$$c_1(x) = a_{j2}/a_{(j-1)2}$$

$$c_2(x) = \frac{\big(a_{(j+1)4} - (a_{j2}/a_{(j-1)2})a_{j4}\big)x^4}{a_{(j-1)2}x^2}$$

So that for small y we get:

$$\frac{\phi_j(x,y)}{\phi_{j-1}(x,y)} \approx c_1 y + c_2 x^2 y^2 + O(y^3)$$

$$c_1 = a_{j2}/a_{(j-1)2}$$

$$c_2 = \frac{\big(a_{(j+1)4} - (a_{j2}/a_{(j-1)2})a_{j4}\big)}{a_{(j-1)2}}$$

□

4.5 Numerical Results:

4.5.1 First Results with $\eta = 0$:

In this section, we truncate the trial solution power series to a maximum number of terms in n, as well as truncating the partial differential equation:

$$p(x,t) = \frac{a_{00}}{\sqrt{t}} + \sum_{n=1}^{N} \sum_{m=2(n-K+1)}^{2n} \frac{a_{nm}}{\sqrt{t}} \left(\frac{x^m}{t^n} \right) \tag{4.15}$$

Starting, with a value $N = 100$, we plot the solutions for $K = 1$ to $K = 5$ (terms up to and including ϵ^{10}).

First, figure 4.1 shows the 1 day solutions ($t = 0.004$), with $\sigma = 10\%$, $\epsilon = 0.005$.

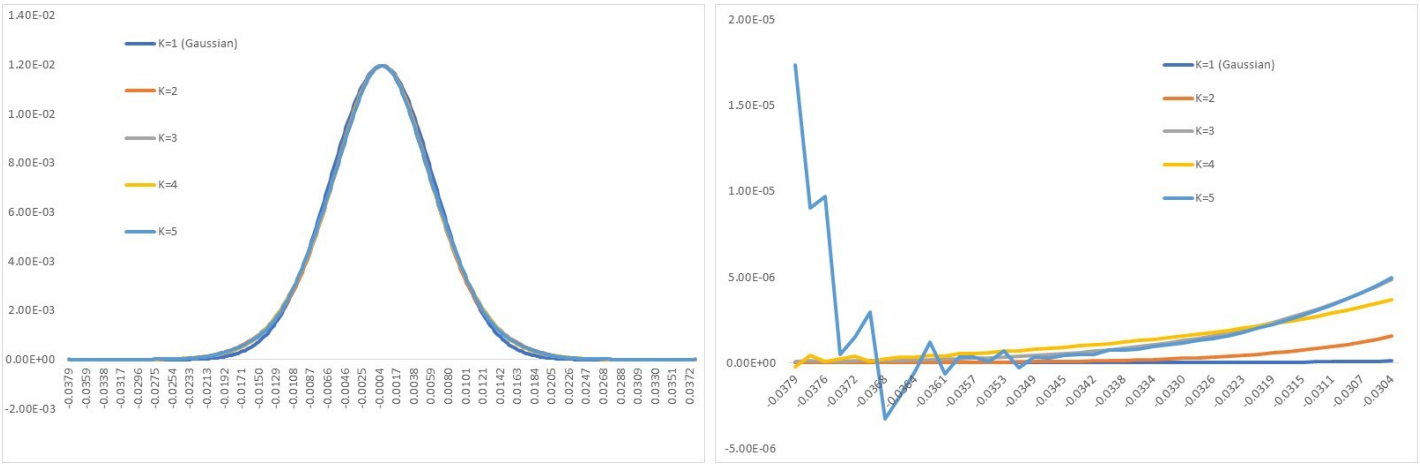

Figure 4.1: Approximate solutions for $N = 100$, $K = 1$ to 5, t=0.004. The first chart shows the full distribution, the left chart focuses on the left tail.

Next, figure 4.2 shows the same solutions for after 1 month has elapsed.

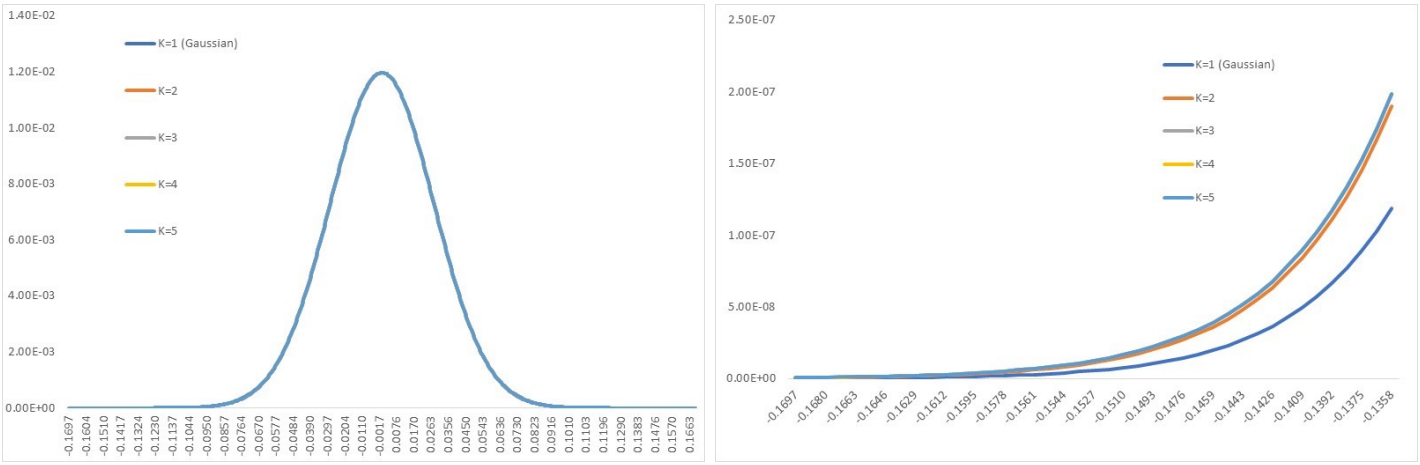

Figure 4.2: Approximate solutions for $N = 100$, $K = 1$ to 5, t=0.08

4.5.2 Convergence in N:

Figure 4.3 shows the convergence in the tail, for $K = 3$. This shows the series has converged for $k \geq 70$.

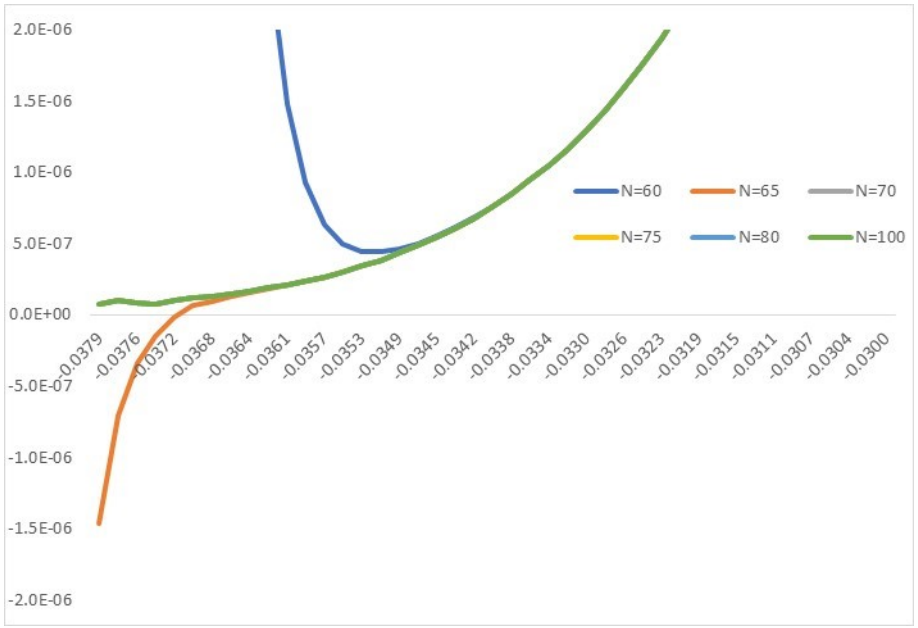

Figure 4.3: Convergence of the tail probabilities, t=0.004, epsilon=0.005, K=3

Similarly, figure 4.4 shows the convergence in the tail, for $K = 5$.

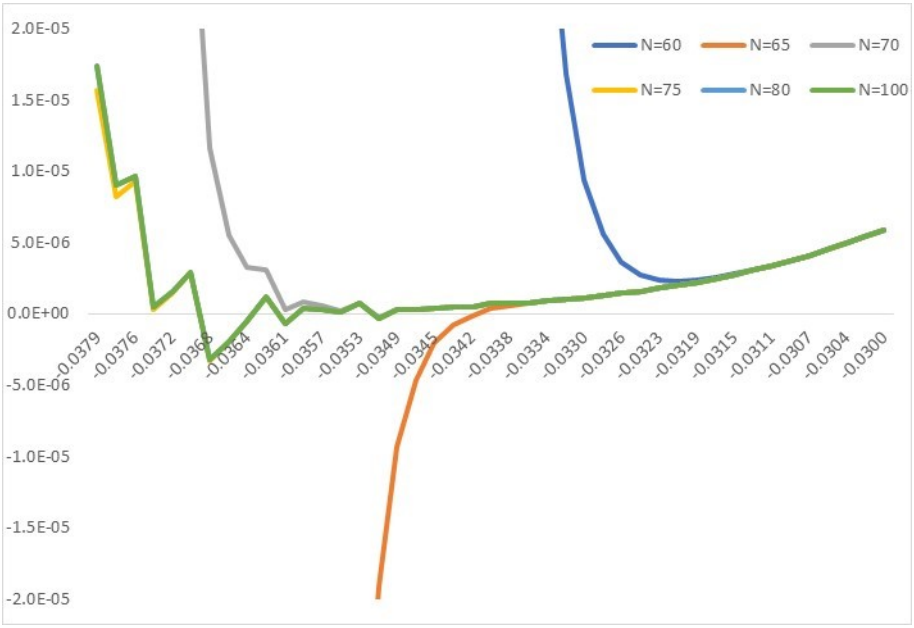

Figure 4.4: Convergence of the tail probabilities, t=0.004, epsilon=0.005,K=5

We note that, in this case, the power series has converged for $N \geq 75$. However, there is instability in tail for $K = 5$, and above. As K increases, the power-series coefficients get larger and larger, the final solution involves subtracting very large numbers from each other.

This is reflected in the table below, which shows the maximum value of the contributing monomials, and the ratio of the final sum to the maximum contributing monomial.

The values are taken at 6 standard deviations, and so the final sum of all monomials should be near zero. However, for $K = 7$, this involves subtracting monomials with a value of $O\left(10^{+16}\right)$ from each other.

As the size of the individual monomials increases, the number of digits required to capture accuracy to $O\left(10^{-16}\right)$, increases. Thus, eventually the limitations of floating point arithmetic restrict the accuracy of the final result.

k ($x = 6$ std deviations, $t = 0.004$)	$\text{Max}\left(\frac{a_{n(n-k)}}{\sqrt{t}}\left(\frac{x^{2(n-k)}}{t^n}\right)\right)$	Final Sum/$\text{Max}\left(\frac{a_{n(n-k)}}{\sqrt{t}}\left(\frac{x^{2(n-k)}}{t^n}\right)\right)$
1 (Gaussian)	9.72e+7	2.46e−15
2	3.54e+9	2.18e−15
3	8.87e+10	1.10e−15
4	2.00e+12	−1.37e−16
5	4.19e+13	5.49e−16
6	8.31e+14	3.71e−16
7	1.70e+16	2.66e−16

4.5.3 Divergence in K:

The analysis above shows that for fixed K, pending sufficient data retention in the floating point arithmetic used, one can use proposition 4.3.1 to calculate solutions.

In this section, we show however that these series diverge for fixed N, as $K \to \infty$. This effect is exacerbated for large ϵ. We show the results in figure 4.5 below, for the mid-tail probabilities. We set $\epsilon = 0.005$, $t = 0.004$ (1 day), and $N = 100$.

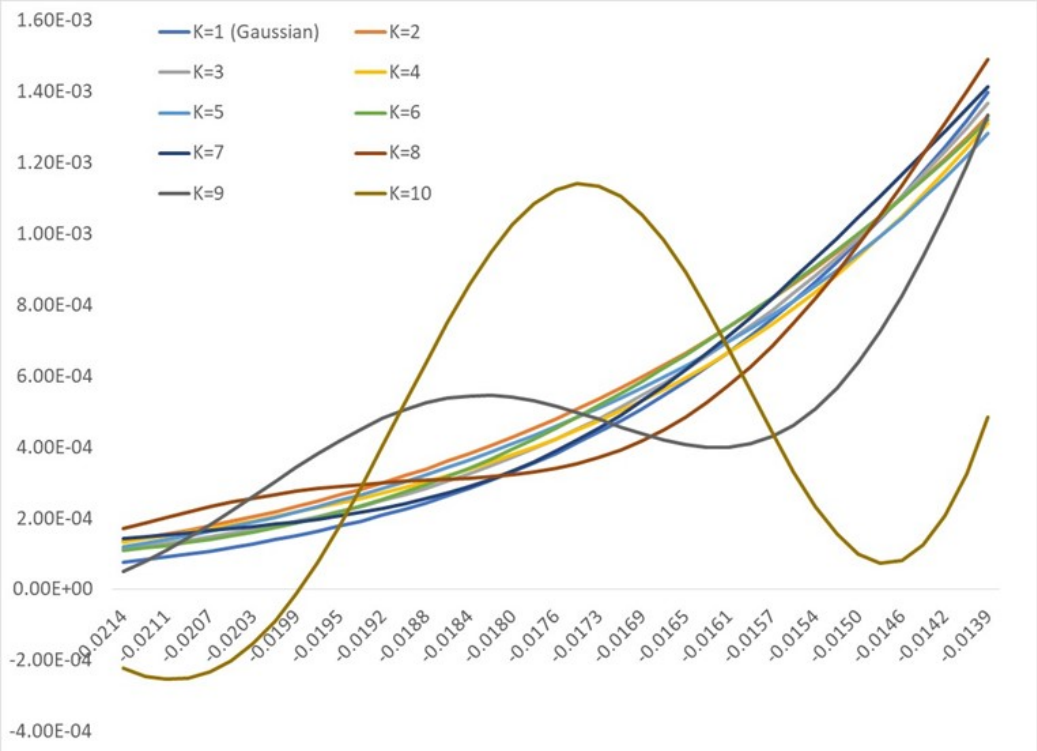

Figure 4.5: Divergence of the mid-tail probabilities, t=0.004, $\epsilon = 0.005$

Figure 4.6 shows the same model after a time frame of 1M has ellapsed. As time goes, the relative of impact of ϵ^2 versus the total variance: $\sigma^2 t$ reduces, and the probability distribution gets closer and closer to the Gaussian distribution. For $t = 0.08$, the divergence seen in figure 4.5 is no longer apparent.

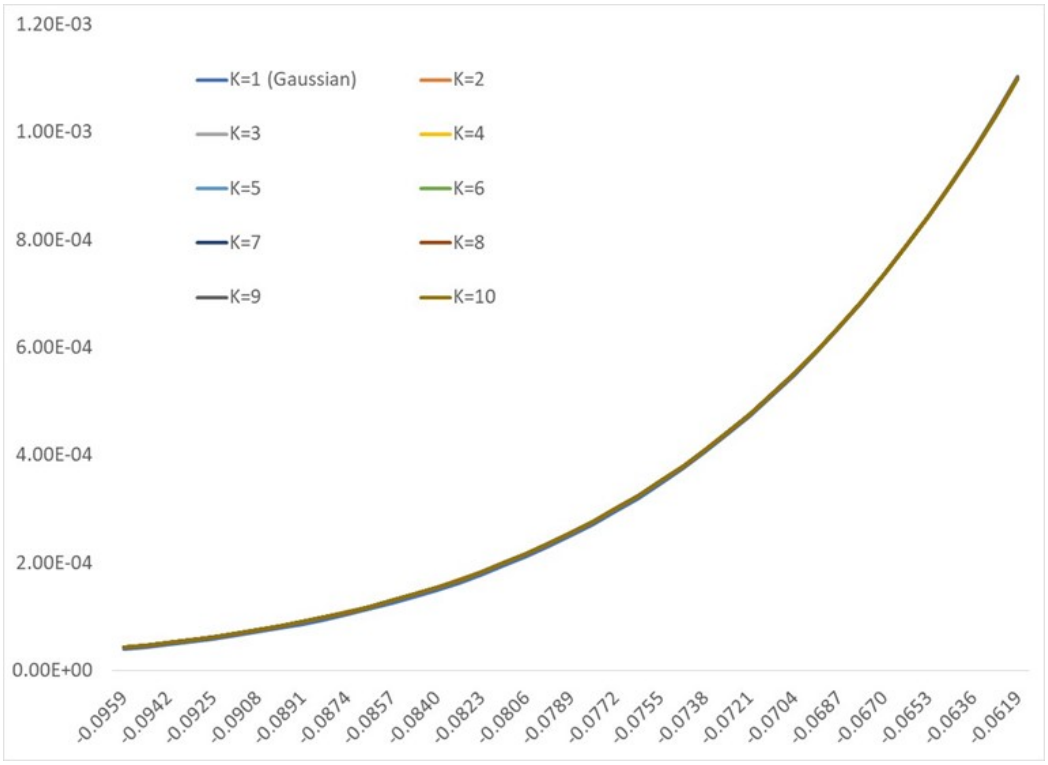

Figure 4.6: Mid-tail probabilities, t=0.08, $\epsilon = 0.005$

4.5.4 Results with $\eta \neq 0$:

In figure 4.7, we show the 1 day simulation from figure 4.1: $\sigma = 0.1$, $\epsilon = 0.005$) with $\eta = 0$ to $\eta = -0.5$.

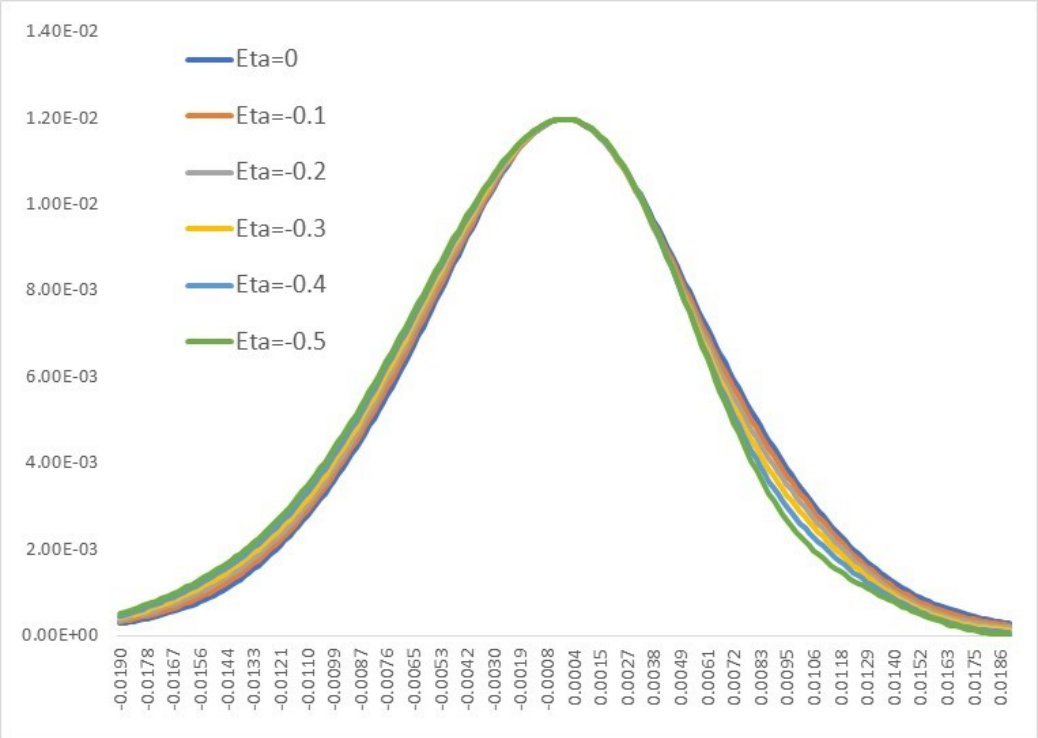

Figure 4.7: Approximate solutions for $N = 100$, $K = 7$, t=0.004, $\epsilon = 0.005$, $\sigma = 0.1$, $\eta = 0$ to $\eta = -0.5$.

The negative skew parameter of $\eta = -0.5$, reflects the situation whereby the volume of sellers represented by $||\psi_o||^2$, is greater than the volume of buyers. We have:

$$||\psi_o||^2 + ||\psi_b||^2 = 1$$

$$||\psi_o||^2 - ||\psi_b||^2 = -0.5$$

Which implies, in this case, that 75% of market participants are sellers, and 25% of market participants are buyers. The resulting approximate solution has $\mu_3 = -8.3\text{e}{-08}$, versus a theoretical skew of: $\sigma^2 t\epsilon\eta = -10.0\text{e}{-08}$.

4.5.5 Analysis and Conclusion:

The table below shows a comparison between the approximation based on the truncated partial differential equation (finite K) and the theoretical kurtosis, based on the full partial differential equation.

First of all it must be considered that, whilst proposition 4.3.1 may well represent a solution to the truncated partial differential equation, there is no guarantee that it will not differ substantially from the true solution, or even that it represents a valid probability desnsity function for a stochastic process.

However, there are a number of reasons that suggest that this is a useful way to generate approximate solutions. First of all, the solutions converge to the Gaussian solution for small ϵ, and/or long time frames t. In addition, the excess kurtosis observed is of a similar order of magnitude to the theoretical results calculated in the previous chapter.

Finally, these solutions have the benefit that they can be calculated in a quick & stable manner. With this in mind, we summarise some key results below.

ϵ	σ	t	K	Theoretical Kurtosis	Truncated Solution Kurtosis	Diff%
0.005	0.1	0.004	0 (Gaussian)	4.80e−09	4.80e−09	0.00%
0.005	0.1	0.004	2	5.80e−09	5.54e−09	−4.23%
0.005	0.1	0.004	4	5.80e−09	5.57e−09	−3.59%
0.005	0.1	0.08	0 (Gaussian)	1.92e−09	1.92e−09	0.00%
0.005	0.1	0.08	2	1.97e−09	1.97e−09	−0.01%
0.005	0.1	0.08	4	1.97e−09	1.97e−09	−0.01%

The final table shows the percentiles for x values in excess of 2, 3, and 4 standard deviations. The table shows that the probability of a 1 day move in excess of 4 standard deviations is increased by a factor of 8. Ie, 1 day every 17 years, rather than 1 day every 134 years.

By contrast, the probability of a 1 month return in excess of 4 standard deviations is impacted to a much

lower degree.

Percentile vs Tail Event	ϵ	σ	t	$K = 0$ (Gaussian)	$K = 4$
-3sd	0.005	0.1	0.004 (1 day)	0.1374%	0.2758%
-4sd	0.005	0.1	0.004 (1 day)	0.0030%	0.0240%
-3sd	0.005	0.1	0.08 (1M)	0.1417%	0.1577%
-4sd	0.005	0.1	0.08 (1M)	0.0031%	0.0042%

Chapter 5

Modelling Financial Markets Using an Open Quantum Systems Approach

5.1 Chapter Introduction:

The overall objective of this research is to build models for the noncommutative behaviour of financial markets. In chapters 2, 3 and 4, we have applied the techniques of quantum stochastic calculus for this purpose. In this chapter, we investigate an alternative approach based on the interaction between the financial market, which we represent as a quantum system, with its' external environment.
With any model of the financial market one must decide:

a) How we plan to represent the current state of the financial market.

b) How we plan to model the time evolution of the financial market. In particular so that we can use the model to make financial decisions.

In section 5.4, we answer these key questions, and attempt to show how this determines the functional form for the key equations.

Classically, when modelling the uncertainty in the market price of tradeable securities, one thinks of the variance in the price. For example, the variance implied by a historical time-series of price returns, the variance implied by a probability distribution used to model future prices, or alternatively the market quoted Black-Scholes implied volatility for listed option prices. However, the uncertainty in future prices is also associated with a lack of information about what the price might be. Conversely, how much information would be gained in finding out the future price of a traded security in advance. For this reason, we discuss the concept of Entropy in section 5.3.

5.1.1 Writing Classical Random Variables as Self-Adjoint Operators:

Financial asset prices are often modelled using random variables defined on a classical probability space: $(\Omega, \mathcal{F}, P)$. The price we are modelling is represented as an $\mathcal{F}$ measurable function:

$$X : \Omega \to \mathbb{R} \tag{5.1}$$

In fact, the $\mathcal{F}$ measurability of X means that it induces a probability measure on $\mathbb{R}$. For all $B \in \mathcal{B}(\mathbb{R})$ (where $\mathcal{B}(\mathbb{R})$ represents the Borel σ-algebra):

$$\begin{aligned} \mu(B) &= P(X^{-1}(B)) \\ E[X] &= \int_{\mathbb{R}} x d\mu(x) \end{aligned} \tag{5.2}$$

Following [20], we write the classical random variable X as a self-adjoint operator on the Hilbert space:

$$\begin{aligned} \mathcal{H} &= L^2(\Omega, dP) \\ \text{for } f \in \mathcal{H},\ (Xf)(\omega) &= X(\omega) f(\omega) \end{aligned} \tag{5.3}$$

We apply the following definition ([26] definition 7.10):

Definition 5.1.1. *Let Ω be a set, and $\mathcal{F}$ be a σ-algebra in Ω. A map $\mu : \mathcal{F} \to \mathcal{B}(\mathcal{H})$ (where $\mathcal{B}(\mathcal{H})$ represents the space of bounded linear maps from $\mathcal{H}$ to $\mathcal{H}$) is called a projection valued measure if the following properties are satisfied:*

- *For each $E \in \mathcal{F}$, $\mu(E)$ is an orthogonal projection.*
- *$\mu(\emptyset) = 0$, and $\mu(\Omega) = 1$.*
- *If $E_1, E_2, E_3, \cdots \in \mathcal{F}$ are disjoint, then for all $\psi \in \mathcal{H}$, we have have:*

$$\mu\Big(\cup_{j=1}^{\infty} E_j \Big)\psi = \sum_{j=1}^{\infty} \mu(E_j)\psi$$

  where the convergence of the sum is in the norm topology in $\mathcal{H}$.

- *For all $E_1, E_2 \in \mathcal{F}$, we have $\mu(E_1 \cap E_2) = \mu(E_1)\mu(E_2)$.*

We can now apply the Spectral Theorem for self-adjoint operators ([26] Theorem 10.4, re-written out here as theorem 5.1.2) to write X as an integral over the projection valued measures from definition 5.1.1.

Theorem 5.1.2. *Suppose X is a self-adjoint operator on the Hilbert space $\mathcal{H}$. Then there is a unique projection-valued measure μ^X on the spectrum of X: $\sigma(X)$, such that:*

$$\int_{\sigma(X)} \lambda d\mu^X(\lambda) = X$$

Here, the integral is the operator defined integration defined by [26] proposition 7.11.

Remark 5.1.3. *It is as a consequence of equations 5.1, 5.3 and definition 5.1.1, that we can write the classical random variable: X as a self-adjoint operator on $\mathcal{H}$. Furthermore, we can apply Theorem 5.1.2 to measure the probability of the event that the classical random variable X falls in the set $E \in \mathcal{F}$ as:*

$$P[X \in E] = \int_E \lambda d\mu^X(\lambda)$$

Note that, since in general X will be unbounded, it is defined on the subspace (see [26] proposition 10.1):

$$S_X = \left\{ \psi \in \mathcal{H} \middle| \int_\Omega |X(\omega)|^2 \mu^\psi(d\omega) < \infty \right\}$$

Whereby, for $E \in \mathcal{F}$, $\mu^\psi(E) = \langle \psi | \mu(E) \psi \rangle$

Therefore, for unbounded X, one would wish to further show that S_X is a dense subset of $\mathcal{H}$. If this is not the case, then there will be valid market wave functions upon which X is not defined. From a financial perspective, one could argue this would represent a financial market whereby determining the fair market price was not possible. However, in this chapter, whilst we look at infinite dimensional cases for illustrative purposes, whenever we apply these methods in practice, we do so by using a finite dimensional Hilbert space, with bounded operators.

5.1.2 Quantum State:

Equation 5.2 defines an expectation value on the set of self-adjoint operators on $\mathcal{H}$, which in turn defines a unique quantum state:

Theorem 5.1.4. *For the expectation value defined by equation 5.2, there is a unique bounded non-negative operator: ρ, acting on the Hilbert space $\mathcal{H} = L^2(\Omega, dP)$, such that $Tr[\rho] = 1$, and we have:*

$$E[X] = Tr[\rho X]$$

We call ρ the quantum density matrix for the system in question.

Proof. See [26], Theorem 19.9. □

Therefore, we can see that the classical probability framework defined by $(\Omega, \mathcal{F}, P)$, and the $\mathcal{F}$ measurable random variable X, can equally be represented by the quantum probability space defined by:

- $\mathcal{H} = L^2(\Omega, dP)$.

- The set of projection valued measures: $\mu : \mathcal{F} \to \mathcal{B}(\mathcal{H})$.

- The quantum density matrix ρ, that defines the expectation value.

Thus the quantum probability framework can be used as an alternative approach to the standard classical framework. One of the objectives of this chapter is to highlight the benefits of doing so.

5.2 General Chapter Overview:

The key contributions presented in this chapter are as follows. In section 5.3 we start by showing how the Von-Neumann entropy of a quantum state, representing the financial market, matters and how the open quantum systems approach allows for the incorporation of differing levels of Von-Neumann entropy without impacting the probability distribution for the market price.

We argue that one can consider a 'classical' market as one in which the density matrix that represents the market commutes with the price operator, and that this represents the state which maximizes the Von-Neumann entropy, without impacting the actual probability distribution for the price.

In section 5.4 we propose a Hilbert space structure for the external environment that can incorporate differing levels of risk appetite. We show how an interaction between the market & its' environment leads to diffusion and show how to apply the strong coupling limit to derive a Markovian approximation for the dynamics.

We then go on to extend this method in two ways. Firstly by applying different market responses to a change in the environment risk appetite. For example if the market jumps by a fixed amount in response to an increase in risk appetite, then the result is a Gaussian diffusion process. If the market jumps by an uncertain amount, then the result is generally more complex. We investigate the impact of such an approach on the market entropy and variance in section 5.7, where we develop proposition 5.7.2 that

links the variance to the choice of market response.

The second way we look to extend the method is by allowing for a non-diagonal market state. In the finite dimensional case, states that are not diagonalized relative to the eigenstates of the price operator can be thought of as 'non-classical' states, with a lower level of Von-Neumann entropy. We investigate the exotic properties of these states in section 5.6, and develop a result that links the variance to the choice of market state in proposition 5.7.4.

5.3 Entropy of the Financial Market:

Before developing the mechanism for time evolution, we briefly discuss the concept of Entropy, and explain why it is linked to the nature of the time evolution we wish to incorporate.

5.3.1 Entropy Of a Classical Random Walk:

We have (see [59] Theorem 1):

Theorem 5.3.1. *Let X_1, X_2,... be independent and identically distributed square integrable random variables. Furthermore we assume we have an expectation value given by:*

$$E[X_i] = \int_{\mathbb{R}} xp(x)dx, \text{ for all } i$$

Where $p(x) \in L^\infty(\mathbb{R})$, $p(x) \geq 0$, and:

$$\int_{\mathbb{R}} p(x)dx = 1$$

Then we have:

$$H\Big(\frac{X_1 + ... + X_n}{\sqrt{n}}\Big) \leq H\Big(\frac{X_1 + ... + X_{n+1}}{\sqrt{n+1}}\Big) \tag{5.4}$$

$$H(X) = -\int_{\mathbb{R}} p(x)\log(p(x))dx$$

We now consider the example of an n step random walk:

Definition 5.3.2. *Let Ω_N be a finite space with cardinality N, and X_i classical random variables defined on Ω_N:*

$$X_i : \Omega_N \to \mathbb{R}$$

Then we define an n step random walk as follows:

$$W_n = \sum_{i=1}^{n} X_i(\omega_i) \text{ for } \omega_i \in \Omega_N$$

There are N^n different possible n step random walks in Ω_N, which we label using the index k: $\{W_n^k : k = 1 \text{ to } N^n\}$. The ith step of the kth random walk is determined by: $\omega_{ik} \in \Omega_N$. The discrete probability of observing the precise random walk W_n^k, determined by: $\{\omega_{ik} : i = 1 \text{ to } n\}$, is labelled p_k:

$$E[W_n] = \sum_{k=1}^{N^n} p_k W_n^k$$

Then since we are dealing with discrete probability distributions, we can interpret the function H in 5.4, as the Shannon entropy.

Theorem 5.3.3. *Let the total variance for the random walk be defined by:*

$$Var(W_n) = E[W_n^2] - E[W_n]^2$$

Then if each of the classical random variables in definition 5.3.2: X_i, are independent in the sense that:

$$E[X_i X_j] = 0, \text{ for } i \neq j$$

Then we have, for $m \geq n$:

$$Var(W_m) \geq Var(W_n) \tag{5.5}$$

Proof. First note that by the tower law of probabilities we have:

$$\begin{aligned} Var(W_k) &= E[W_k^2] - E[W_k]^2 \\ &= E[W_k^2] - 2E[W_k]E[W_k] + E[W_k]E[W_k] \\ &= E[W_k^2] - E[2W_k E[W_k]] + E[E[W_k]^2] \\ &= E[(W_k - E[W_k])^2] \end{aligned}$$

So we have:

$$Var(W_k) = Var(\overline{W_k})$$

$$\overline{W_k} = W_k - E[W_k]$$

So without loss of generality, we can assume: $E[W_n] = 0$. Furthermore, we have:

$$\begin{aligned} Var(W_{n+1}) &= E[W_{n+1}^2] - E[W_{n+1}]^2 \\ &= E[W_n^2] + E[X_{n+1}^2] + 2E[W_n X_{n+1}] - E[X_{n+1}]^2, \text{ since } E[W_n] = 0 \\ &= E[W_n^2] + E[X_{n+1}^2] - E[X_{n+1}]^2, \text{ by independence} \\ &= E[W_n^2] + E[(X_{n+1} - E[X_{n+1}])^2], \text{ by the Tower Law} \\ &> 0, \text{ since } X_{n+1} \text{ is real valued.} \end{aligned}$$

□

It follows from equations 5.4 and 5.5, that both the entropy and total variance increase monotically as one looks further and further into the future. Once one has fixed the specific nature of the random walk (as defined by the probability law for the independent and identically distributed steps: X_i), then one can calculate the entropy from knowing the total variance, and vice versa. One does not need to distinguish between the two concepts.

If one were to specify a random walk as the underlying model for the random fluctuations in the price for a tradeable financial asset then one can argue that:

- The total variance of the possible future price arises because of our lack of information about the future, *OR:*
- We lack information about what the future price will be because the traded price is random in nature and the total variance of possible future prices increases the further you look into the future.

Crucially for a quantum model of the financial market, there exists observables that have arbitrarily high variance, even in a system described by a pure state with zero entropy. We discuss this further in section 5.3.2.

5.3.2 Entropy in the Quantum Case:

It follows from definition 5.3.2, that W_n is a function from a finite probability space of dimension N^n (representing n steps of dimension N):

$$W_n : \Omega_{N^n} \to \mathbb{R}$$

Thus, applying theorem 5.1.2 ([26] theorem 10.4), we can write W_n as an operator on the Hilbert space: $\mathcal{H} = \mathbb{C}^{N^n}$. Then following from Theorem 5.1.4, we can use a bounded non-negative operator (with trace 1): ρ to define an expectation value:

$$E[W_n] = Tr[\rho W_n]$$

The generalization of the Shannon entropy to the quantum case is given by the Von Neumann entropy:

Definition 5.3.4. *The Von-Neumann entropy of a density matrix ρ is given by:*

$$S(\rho) = -Tr[\rho \ln(\rho)]$$

Similarly, the variance for an operator is defined by:

Definition 5.3.5. *The variance of an operator A is given by:*

$$\begin{aligned} Var(A) &= E[A^2] - E[A]^2 \\ &= Tr[\rho A^2] - (Tr[\rho A])^2 \end{aligned}$$

One can prepare a quantum state with zero Von Neumann entropy, where the variance is arbitrarily large. For example, if we label the orthonormal eigenvectors for $\mathcal{H}$ as: $|e_i\rangle$ for $i = 1 \dots N^n$, then we could set:

$$\rho = \Big(\sum_{i=1}^{N^n} q_i |e_i\rangle\Big)\Big(\sum_{i=1}^{N^n} \overline{q_i}\langle e_i|\Big)$$

$$\sum_{i=1}^{N^n} |q_i|^2 = 1$$

$$W_n|e_i\rangle = \lambda_i|e_i\rangle$$

Then we would have:

$$S(\rho) = 0$$

$$Var(W_n) = \sum_{i=1}^{N^n} \lambda_i^2 |q_i|^2 - \Big(\sum_{i=1}^{N^n} \lambda_i |q_i|^2\Big)^2$$

Under the unitary evolution of the state: ρ in a closed quantum system, the variance may increase with time, even though the Von-Neumann entropy remains at zero.

Thus, in the quantum case one can distinguish between:

- Increases in total variance that result from the unitary evolution of a closed system.
- Increases in total variance of an open system that result from interaction with an external environment, that also drive an increase in the Von-Neumann entropy of the system.

5.3.3 Modelling Price Uncertainty, and Information Content of the Financial Market:

In this section, we use the finite dimensional Hilbert space, and associated price operator:

$$\mathcal{H} = \mathbb{C}^N \tag{5.6}$$
$$X = \sum_{i=1}^{N} x_i |e_i\rangle\langle e_i|$$

Where $\{x_i : i = 1, .., N\}$ represent the possible price eigenvalues. As discussed above a 'classical' probability model for the financial market assigns a probability: p_i for each price: x_i, and we can write a 'classical' state as a diagonal matrix:

$$E[X] = \sum_{i=1}^{N} p_i x_i$$
$$= Tr[\rho X]$$
$$\rho = \sum_{i=1}^{N} p_i |e_i\rangle\langle e_i|$$

The following proposition shows, that given a known probability distribution for a price variable, the classical state, as in equation 5.6, has the maximum entropy.

Proposition 5.3.6. *Let the operator X be given by equation 5.6, and define the set of projection operators:*

$$P_i = |e_i\rangle\langle e_i| \tag{5.7}$$

Finally, consider the set of density matrices $\rho \in \mathcal{A}$ for which we have:

$$E[P_i] = Tr[\rho P_i]$$
$$= p_i$$

In other words $\mathcal{A}$ is the set of density matrices which fixes the probability of finding the price x_i, for each $i = 1$ to N.

Then the classical density matrix:

$$\rho_{classical} = \sum_{i=1}^{N} p_i |e_i\rangle\langle e_i|$$

Maximises the Von-Neumann entropy within $\mathcal{A}$.

Proof. First of all, we consider a state $\rho \in \mathcal{A}$, and write out the spectral resolution in some orthonormal basis: $|\phi_i\rangle$:

$$\rho = \sum_{j=1}^{N} q_j |\phi_j\rangle\langle\phi_j|$$

Where, $q_j \geq 0$ (some q_j could be zero). Then by assumption:

$$\begin{aligned} p_i &= Tr[\rho P_i] \\ &= \sum_{j=1}^{N} q_j |\langle\phi_j|e_i\rangle|^2 \end{aligned}$$

Then we have:

$$\begin{aligned} S(\rho_{classical}) &= -\sum_{i=1}^{N} p_i \log(p_i) \\ &= -\sum_{i=1}^{N} \Big(\sum_{j=1}^{N} q_j |\langle e_i|\phi_j\rangle|^2 \Big) \log \Big(\sum_{j=1}^{N} q_j |\langle e_i|\phi_j\rangle|^2 \Big) \end{aligned}$$

We now label: $a_{ij} = |\langle e_i|\phi_j\rangle|^2$, and note that:

$$\begin{aligned} a_{ij} &\geq 0 \\ \sum_{i=1}^{N} a_{ij} &= \sum_{j=1}^{N} a_{ij} = 1 \end{aligned}$$

Therefore, we have:

$$\begin{aligned} S(\rho_{classical}) &= -\sum_{i=1}^{N}\Big(\sum_{j=1}^{N} q_j a_{ij}\Big)\log\Big(\sum_{j=1}^{N} q_j a_{ij}\Big) \\ &= \sum_{i=1}^{N} f\Big(\sum_{j=1}^{N} q_j a_{ij}\Big), \text{ where } f(x) = -x\log(x) \\ &= \sum_{i=1}^{N} f(q_1 a_{i1} + q_2 a_{i2} + ... + q_N a_{iN}) \end{aligned}$$

We have: $f''(x) \leq 0$, for $x \geq 0$. So therefore, $f(x)$ is a concave function, and we have from Jensen's inequality that:

$$\begin{aligned} S(\rho_{classical}) &\geq \sum_{i=1}^{N} a_{i1} f(q_1) + a_{i2} f(q_2) + ... + a_{iN} f(q_N) \\ &= \Big(\sum_{i=1}^{N} a_{i1}\Big) f(q_1) + \Big(\sum_{i=1}^{N} a_{i2}\Big) f(q_2) + ... + \Big(\sum_{i=1}^{N} a_{iN}\Big) f(q_N) \\ &= \sum_{j=1}^{N} f(q_j) \\ &= S(\rho) \end{aligned}$$

□

5.3.4 Entropy Example: Listed Stock Price vs Listed Options Price

Proposition 5.3.6 shows that given a specific finite dimensional probability distribution for a traded financial market price, the classical state represents the state about which we have the least information. Alternatively, it represents the case where the most information is gained from finding out what the price will be with certainty, having previously only known the probability distribution.

In many circumstances, there may be uncertainty regarding a particular traded price, but where additional information is available to the market. For example the specific market mechanisms that go into

determining a trade execution price, or the official end of day close price. Alternatively the size & motivation of market participants. Therefore, the quantum probability framework discussed in this chapter enables a way to distinguish between situations where the probability law for the price is the same, but the overall information available to the market is different. With a view to illustrating the point, we consider the following toy example:

- Due to imperfections in market price fixing mechanism (for example non-zero bid-offer spread). There are 3 possible prices for the traded price of an asset. The market Hilbert space is therefore set to: $\mathcal{H} = \mathbb{C}^3$.

- The 3 possible prices are x_1, x_2, x_3, associated to the eigenvectors $|e_i\rangle$, $i = 1, 2, 3$.

$$X = \sum_{i=1}^{3} x_i |e_i\rangle\langle e_i|$$

- We also consider the traded price operator: O, for a Strangle option consisting of an 'at the money' listed call option and an 'at the money' listed put option.

- Since the listed put and call options are both 'at the money', the option has the lowest value o_- if the market is in the middle eigenstate: $|e_2\rangle$.

- We assume the option has the value $o_{1,+}$ in the eigenstate: $|v_1\rangle = \frac{|e_1\rangle + |e_3\rangle}{\sqrt{2}}$, and the value $o_{2,+}$ in the eigenstate $|v_2\rangle = \frac{|e_1\rangle - |e_3\rangle}{\sqrt{2}}$.

- Note that $|e_1\rangle$, $|v_1\rangle$, and $|v_2\rangle$ are an alternative orthonormal basis and we can write:

$$O = o_-|e_2\rangle\langle e_2| + o_{1,+}|v_1\rangle\langle v_1| + o_{2,+}|v_2\rangle\langle v_2|$$

We first consider the case that the market state is given by:

$$\rho_{classical} = 0.25|e_1\rangle\langle e_1| + 0.5|e_2\rangle\langle e_2| + 0.25|e_3\rangle\langle e_3| \tag{5.8}$$

This has a 25% chance of finding the X price of x_1, a 50% chance of finding the X price of x_2, and a 25% chance of finding the X price of x_3. Similarly we find a 25% chance each of finding the O price of $o_{1,+}$ or $o_{2,+}$, and a 50% chance of finding the O price of o_-. The Von-Neumann entropy is given in this case by:

$$S(\rho_{classical}) \approx 1.04$$

Next consider the case:

$$\rho_{quantum} = 0.25|e_1\rangle\langle e_1| + 0.5|e_2\rangle\langle e_2| + 0.25|e_3\rangle\langle e_3| + 0.25|e_1\rangle\langle e_3| + 0.25|e_3\rangle\langle e_1| \tag{5.9}$$

This has a lower value for the Von-Neumann entropy:

$$S(\rho_{quantum}) \approx 0.69$$

even though the discrete probability distribution for the traded price: X is the same. In this case the lower Von-Neumann entropy reflects the fact we have additional information regarding the market price of the Strangle option O, that doesn't effect the probability distribution for X, whereas, for the state 5.8, we have no more information about the traded Option price O, than we do about the traded stock price X.

In the state 5.9, we can eliminate the possibility of finding the traded price $o_{2,+}$. In fact we have:

$$\rho_{quantum} = 0.5|v_1\rangle\langle v_1| + 0.5|e_2\rangle\langle e_2|$$

meaning that there is a 50% chance of finding the value $o_{1,+}$, and a 50% chance of finding the value o_-. For example, if $o_{2,+} > o_{1,+}$, this could reflect the possibility that investors will not pay more than $o_{1,+}$ for this Strangle option. This additional information is reflected in the lower entropy.

5.4 Time Evolution of the Financial Market:

5.4.1 Defining the System Hilbert Space:

In this section we outline the open quantum systems method to the modelling of the financial market, following the basic approach outlined in [17] section 3. The full system we are now modelling is represented by the tensor product of the Hilbert space that represents the state of the financial market (labelled $\mathcal{H}_{mkt}$), with the Hilbert space that represents the external environment (labelled $\mathcal{H}_{env}$):

$$\mathcal{H} = \mathcal{H}_{mkt} \otimes \mathcal{H}_{env} \tag{5.10}$$

Let the environment Hilbert space be given by:

$$\mathcal{H}_{env} = \mathbb{C}^K \otimes L^2[\mathcal{K}],\ K \geq 2 \tag{5.11}$$

Where, $\mathcal{K}$ is a bounded subset of $\mathbb{R}$. For example: $\mathcal{K} = [-L, L]$ for some $L > 0$. Let the system Hamiltonian be given by:

$$H = H_I + (\mathbb{I} \otimes H_{env}) \tag{5.12}$$

$$H_I = \sqrt{\kappa\gamma} \sum_{\alpha \in \{u,d\}} A_\alpha \otimes B_\alpha$$

H_I models the interaction between the financial market and its environment. In this equation, the operator tensor product acts on the Hilbert space tensor product in equation 5.10, and not the tensor product that defines the environment space: equation 5.11. A_u and A_d act on the market Hilbert space: $\mathcal{H}_{mkt}$, γ, κ are constants. B_u and B_d are ladder operators that act on the environment space (see for example [4]). In this context, they are defined by:

$$B_u = \sum_{i=1}^{K-1} |e_{i+1}\rangle\langle e_i| \otimes \mathbb{I},\ B_d = \sum_{i=1}^{K-1} |e_i\rangle\langle e_{i+1}| \otimes \mathbb{I} \tag{5.13}$$

H_{env} is the environment Hamiltonian, which we assume has the form:

$$H_{env} = \gamma \sum_{l=1}^{K} l|e_l\rangle\langle e_l| \otimes H' \tag{5.14}$$

Where H' acts on the space: $L^2[\mathcal{K}]$. We return to the financial rational behind these definitions in section 5.4.2. Meanwhile, we now establish some further results that we will apply later on, when outlining the time evolution in section 5.4.3. The first result relates to the quantum state at time t:

Proposition 5.4.1. *Assume ρ_B remains in stationary at all times, and in particular assume that $[H', \rho_B] = 0$. Then the quantum state at time t, acting on the Hilbert space 5.10, with the environment space being given by 5.11, can be represented as the following sum:*

$$\rho(t) = \sum_{l,m=1}^{K} \rho_{mkt}^{lm}(t) \otimes |e_l\rangle\langle e_m| \otimes \rho_B \tag{5.15}$$

Proof. If $|f_i\rangle$, $\{i = 1 \ldots \infty\}$ is a basis for $\mathcal{H}_{mkt}$, then $\rho(t)$ can be written:

$$\rho(t) = \sum_{i,j=1}^{\infty} \sum_{l,m=1}^{K} a_{ijlm}(t)|f_i\rangle\langle f_j| \otimes |e_l\rangle\langle e_m| \otimes \rho_B \tag{5.16}$$

Equation 5.16 is of the form 5.15, with:

$$\rho_{mkt}^{lm}(t) = \sum_{i,j=1}^{\infty} a_{ijlm}(t)|f_i\rangle\langle f_j|$$

□

We apply the following definition for operators in the interaction picture:

Definition 5.4.2. *Let A be an operator on the Hilbert space $\mathcal{H}$ given by 5.10, with the system Hamiltonian given by 5.12. We also let the environment Hamiltonian: H_{env}, be defined by equation 5.14. Then we define the interaction picture operators as follows:*

$$A^I(t) = e^{i(\mathbb{I}\otimes H_{env})t} A e^{-i(\mathbb{I}\otimes H_{env})t}$$

Under representation 5.15, and definition 5.4.2, we have the following result for the interaction picture state.

Proposition 5.4.3. *The interaction picture state for 5.15 is given by:*

$$\rho^I(t) = \sum_{l,m=1}^{K} \rho^{lm}_{mkt}(t) \otimes |e_l\rangle\langle e_m| \otimes e^{i\gamma(l-m)tH'}\rho_B$$

Proof. We have:

$$\begin{aligned}\rho^I(t) &= e^{i(\mathbb{I}\otimes\ H_{env})t}\big(\rho(t)\big)e^{-i(\mathbb{I}\otimes\ H_{env})t} \\ &= \sum_{l,m=1}^{K} \rho^{lm}_{mkt} \otimes \big(e^{iH_{env}t}\big[|e_l\rangle\langle e_m| \otimes \rho_B\big]e^{-iH_{env}t}\big) \end{aligned} \tag{5.17}$$

Using equation 5.14 for the environment Hamiltonian, we get:

$$\begin{aligned} e^{iH_{env}t} &= \exp\Big(i\gamma t \sum_{l=1}^{K} l|e_l\rangle\langle e_l| \otimes H'\Big) \\ &= \sum_{l=1}^{K} |e_l\rangle\langle e_l| \otimes e^{i\gamma ltH'} \end{aligned}$$

Applying this to 5.17, we get (since $[H', \rho_B] = 0$):

$$\rho^I(t) = \sum_{l,m=1}^{K} \rho^{lm}_{mkt}(t) \otimes |e_l\rangle\langle e_m| \otimes e^{i\gamma(l-m)tH'}\rho_B$$

□

Next we write out the interaction picture operators in $H_I(t)$.

Proposition 5.4.4. *The interaction picture for H_I is given by:*

$$H_I(t) = \sqrt{\kappa\gamma}(A_u \otimes B_u(t) + A_d \otimes B_d(t))$$

Where we have:

$$\begin{aligned} B_u(t) &= \sum_{l=1}^{K-1} |e_{l+1}\rangle\langle e_l| \otimes e^{i\gamma H't} \\ B_d(t) &= \sum_{l=1}^{K-1} |e_l\rangle\langle e_{l+1}| \otimes e^{-i\gamma H't} \end{aligned} \tag{5.18}$$

Proof. The system Hamiltonian, combining equations 5.10 and 5.11 is:

$$H_{sys} = \mathbb{I} \otimes \sum_{l=1}^{K} l|e_l\rangle\langle e_l| \otimes H'$$

Therefore, for A_u, and A_d we get:

$$\begin{aligned} A_u(t) &= e^{iH_{sys}t} A_u e^{-iH_{sys}t} \\ &= e^{i\mathbb{I}t} A_u e^{-i\mathbb{I}t} = A_u \\ A_d(t) &= e^{iH_{sys}t} A_d e^{-iH_{sys}t} \\ &= e^{i\mathbb{I}t} A_d e^{-i\mathbb{I}t} = A_d \end{aligned}$$

For the operators B_u, and B_d, defined in equation 5.13 we note that, as after equation 5.17,

$$e^{iH_{env}t} = \sum_{l=1}^{K} |e_l\rangle\langle e_l| \otimes e^{il\gamma H't}$$

Applying this to (5.13), we get:

$$\begin{aligned} B_u(t) &= \sum_{l=1}^{K-1} |e_{l+1}\rangle\langle e_l| \otimes e^{i(l+1)\gamma H't} e^{-il\gamma H't} \\ &= \sum_{l=1}^{K-1} |e_{l+1}\rangle\langle e_l| \otimes e^{i\gamma H't} \end{aligned}$$

Similarly, we get:

$$\begin{aligned} B_d(t) &= \sum_{l=1}^{K-1} |e_l\rangle\langle e_{l+1}| \otimes e^{il\gamma H't} e^{-i(l+1)\gamma H't} \\ &= \sum_{l=1}^{K-1} |e_l\rangle\langle e_{l+1}| \otimes e^{-i\gamma H't} \end{aligned}$$

□

The next result concerns taking the trace over the Hilbert space $L^2[\mathcal{K}]$.

Proposition 5.4.5. *We assume that the Hamiltonian H' in equation 5.14 is self-adjoint, and that we have (for some orthonormal basis): $\rho_B = \sum_{i=1}^{\infty} q_i |e_i\rangle\langle e_i|$. Then H' has the spectral resolution:*

$$H' = \int_{\mathbb{R}} \omega P(d\omega) \tag{5.19}$$

P is a projection valued measure in the sense of [26] definition 7.10. That is, P maps Borel subsets of $\mathbb{R}$ to projection operators acting on the Hilbert space: $L^2[\mathcal{K}]$. Then we have:

$$Tr[f(H')\rho_B] = \int_{\mathbb{R}} f(\omega) d\mu^{(H',\rho_B)}(\omega) \tag{5.20}$$

Where $\mu^{(H',\rho_B)}$ is a probability measure on $\mathbb{R}$. Furthermore, since H' is a bounded operator we have:

$$\mu^{(H',\rho_B)}(E) < \infty, \text{ for all } E \subset \mathbb{R}$$

Proof. The existence of the spectral resolution 5.19 and the projection valued measure P follows from the assumption that H' is bounded and self adjoint, and from [26] Theorem 7.12. From [26] definition 7.13, it follows that we can define:

$$f(H') = \int_{\sigma(H')} f(\omega) P(d\omega)$$

Where $\sigma(H')$ is the spectrum of H'. Since $L^2[\mathcal{K}]$ is separable, we can write:

$$\begin{aligned}
Tr[f(H')\rho_B] &= \sum_{i=1}^{\infty} q_i \langle e_i || \int_{\sigma(H')} f(\omega) P(d\omega) |e_i\rangle, \text{ for some } q_i, \text{ and orthonormal basis vectors: } |e_i\rangle \\
&= \int_{\sigma(H')} f(\omega) \sum_{i=1}^{\infty} q_i \langle e_i | P(d\omega) | e_i \rangle \\
&= \int_{\sigma(H')} f(\omega) d\mu^{(H'\rho_B)}(\omega) \\
\mu^{(H',\rho_B)}(E) &= \sum_{i=1}^{\infty} q_i \langle e_i | P(E) | e_i \rangle, \text{ for } E \subset \mathbb{R}
\end{aligned}$$

We have that $\langle \psi | P(E) | \psi \rangle < \infty$ for all $E \subset \mathbb{R}$, and all $\psi \in L^2[\mathcal{K}]$, and we can extend the integral to $\mathbb{R}$ by defining $P(E) = 0$ for $E \cup \sigma(H') = \emptyset$. □

Finally, in the Markov approximation of the financial market dynamics, we apply the following.

Proposition 5.4.6 (Strong Coupling Limit)**.** *Assume H' is bounded and self-adjoint, and is represented by the spectral resolution: 5.19. Furthermore, we assume that the measure: $\mu^{(H',\rho_B)}$ given by proposition 5.4.5 is absolutely continuous, $d\mu^{(H'\rho_B)}(\omega) = p(\omega)d\omega$, for some density function $p(\omega)$. Then we have, as $\gamma \to \infty$:*

$$\kappa\gamma Tr[B_\alpha(t)B_\beta(s)\rho^I_{env}] \to \kappa Tr[B_\alpha B_\beta \rho^I_{env}]\delta(t-s) \tag{5.21}$$

$$\kappa\gamma Tr[B_\alpha(s)B_\beta(t)\rho^I_{env}] \to \kappa Tr[B_\alpha B_\beta \rho^I_{env}]\delta(t-s) \tag{5.22}$$

$$\alpha, \beta \in \{u, d\}$$

Here $\rho^I_{env}(s)$ represents an interaction picture state acting on the environment Hilbert space, with the Hamiltonian 5.14, and the stationary state ρ_B acting on $L^2[\mathcal{K}]$.

Proof. The general state acting on $\mathcal{H}_{env}$ (with ρ_B acting on $L^2[\mathcal{K}]$) can be written:

$$\rho_{env} = \sum_{l,m=1}^{K} r_{lm}|e_l\rangle\langle e_m| \otimes \rho_B$$

First note that from the proof of 5.4.3 it follows that under the Hamiltonian 5.14, the interaction picture state is given by:

$$\rho^I_{env}(s) = \sum_{l,m=1}^{K} r_{lm}|e_l\rangle\langle e_m| \otimes e^{i\gamma(l-m)\omega s H'}\rho_B$$

Inserting from proposition 5.4.4, we get, where $S(\alpha) = +1$ for $\alpha = u$, and $S(\alpha) = -1$ for $\alpha = d$:

$$\kappa\gamma Tr[B_\alpha(t)B_\beta(s)\rho^I_{env}(s)] = \kappa\gamma Tr\Big[B_\alpha B_\beta \sum_{l,m=1}^{K} r_{lm}|e_l\rangle\langle e_m|\Big] Tr\Big[\Big(e^{i\gamma(l-m)sH'}e^{i\gamma S(\alpha)tH'}e^{i\gamma S(\beta)sH'}\Big)\rho_B\Big] \tag{5.23}$$

For $\alpha = \beta = u$, 5.23 becomes:

$$\begin{aligned}\kappa\gamma Tr[B_u(t)B_u(s)\rho^I_{env}(s)] &= \kappa \sum_{l=3}^{K} r_{(l-2)l}\Big(\gamma Tr\Big[e^{-2i\gamma s H'}e^{i\gamma s H'}e^{i\gamma t H'}\rho_B\Big]\Big) \\ &= \kappa \sum_{l=1}^{K-2} r_{l(l+2)}\Big(\gamma Tr\Big[e^{i\gamma(t-s)H'}\rho_B\Big]\Big)\end{aligned} \tag{5.24}$$

Where the second line follows since only terms where $l - m = -2$ contribute to the trace, and $S(u) = 1$.

For $\alpha = \beta = d$, we have:

$$\begin{aligned}\kappa\gamma Tr[B_d(t)B_d(s)\rho^I_{env}(s)] &= \kappa \sum_{l=1}^{K-2} r_{(l+2)l}\Big(\gamma Tr\Big[e^{2i\gamma sH'}e^{-i\gamma sH'}e^{-i\gamma tH'}\rho_B\Big]\Big) \\ &= \kappa \sum_{l=1}^{K-2} r_{(l+2)l}\Big(\gamma Tr\Big[e^{i\gamma(s-t)H'}\rho_B\Big]\Big) \end{aligned} \tag{5.25}$$

Finally, for $\alpha = u, \beta = d$ and $\alpha = d, \beta = d$:

$$\begin{aligned}\kappa\gamma Tr[B_u(t)B_d(s)\rho^I_{env}(s)] &= \kappa \sum_{l=1}^{K-1} r_{ll}\Big(\gamma Tr\Big[e^{i\gamma(t-s)H'}\rho_B\Big]\Big) \\ \kappa\gamma Tr[B_d(t)B_u(s)\rho^I_{env}(s)] &= \kappa \sum_{l=1}^{K-1} r_{ll}\Big(\gamma Tr\Big[e^{i\gamma(s-t)H'}\rho_B\Big]\Big)\end{aligned} \tag{5.26}$$

From proposition 5.4.5 we have:

$$\begin{aligned}\gamma Tr\big[e^{i\gamma(t-s)H'}\rho_B\big] &= \int_{\mathbb{R}} e^{i\gamma(t-s)\omega} d\mu^{(H'\rho_B)}(\omega) \\ &= \gamma \int_{\mathbb{R}} e^{i\gamma(t-s)\omega} p(\omega) d\omega\end{aligned}$$

Where the second line follows from the assumption that $\mu^{(H',\rho_B)}$ is absolutely continuous, and can therefore be written using a probability density function $p(\omega)$. Taking the limit $\gamma \to \infty$ we get:

$$\begin{aligned}\gamma Tr\big[e^{i\gamma(t-s)H'}\rho_B\big] &= \gamma \int_{\mathbb{R}} e^{i\gamma\omega(t-s)} p(\omega) d\omega \\ &= \int_{\mathbb{R}} e^{i(t-s)u} p(u/\gamma) du \\ &\to \int_{-\infty}^{\infty} e^{i(t-s)u} du, \text{ as } \gamma \to \infty \\ &= \lim_{u\to\infty} \frac{-i}{(t-s)} \sin((t-s)u) \\ &= \delta(t-s)\end{aligned} \tag{5.27}$$

The result then follows by applying this to equations: 5.24, 5.25 and 5.26. □

5.4.2 Financial Interpretation of the Hilbert Space Structure:

5.4.2.1 Open Quantum Systems Approach:

The operators we are interested in, act on the market Hilbert space: $\mathcal{H}_{mkt}$. For now, we take this Hilbert space to represent the various potential buyers & sellers that make up the market for a particular tradeable security. For example, it might represent a stock exchange if the traded security was a listed equity price. If we used: $\mathcal{H}_{mkt} = \mathbb{C}^N$, we could write (where $|f_i\rangle$, $i = 1 \dots N$ represent the eigenvectors in $\mathbb{C}^N$):

$$X = \sum_{i=1}^{N} x_i |f_i\rangle\langle f_i| \otimes \mathbb{I} \otimes \mathbb{I}$$

The space $\mathcal{H}_{env}$ represents the general environment in which the trading activity occurs. There are two components of this space. The first is a finite dimensional Hilbert space $\mathbb{C}^K$. We interpret the K different eigenstates for this space as K different levels of market risk appetite. For example the eigenstate $|e_1\rangle\langle e_1|$ would represent the most *bearish* state for the market, whereby participants are looking to reduce risk exposure, with a view to protecting the values of their investment portfolios. Similarly $|e_K\rangle\langle e_K|$ would represent the most *bullish* state for the market, with investors looking to build up their risk exposure with a view to maximising returns on their investment portfolios.

The operator $A_u \otimes B_u \otimes \mathbb{I}$ can then be interpreted as an operator which increases the level of market bullishness. The operator B_u shifts the background environment to a higher level of risk appetite, and the operator A_u controls the impact on the operators we measure. Since we have no way of measuring the state of the environment, when we take a measurement of the price, we gain no information regarding the state of $\mathcal{H}_{env}$. This operation is carried out using the partial trace:

$$E^{\rho}[X] = Tr[\rho_{mkt}(t)X]$$

$$\rho_{mkt}(t) = Tr_{env}[\rho(t)]$$

Finally, we consider the second component of the environment space: $L^2[\mathcal{K}]$. Partly the introduction of a space with a continuous spectrum is pragmatic. When we go on to discuss the time evolution, we will need to calculate expressions like:

$$f_{ud}(t,s) = Tr[B_u(t)B_d(s)\rho^I_{env}(s)]$$

As shown in [56], these will not generally converge unless one integrates over a continuous spectrum, and it will certainly not be possible to apply the strong coupling limit. From a financial perspective, this space, and the operator H' ensure that the system Hamiltonian returns a continuous energy spectrum, and will control the energy gained/lost when the environment shifts to a higher/lower level of risk appetite.

5.4.2.2 Choice of the Market Hamiltonian

If we label the system Hamiltonian: $H = H_{mkt} \otimes H_{env}$, then in this section, we discuss the choice of the Hamiltonian: H_{mkt}, that drives the internal dynamics of the market space: $\mathcal{H}_{mkt}$. For any classical model for the dynamics of a traded asset price, for example one based on Ito calculus, there are 2 components of the time evolution:

- The deterministic component of the time evolution, usually termed the *drift* component.
- The random component of the time evolution. This is often modelled classically using a Wiener process, and is often termed the *diffusion* component.

As discussed in chapter 3, the classical approach to risk neutral pricing pricing requires the definition of Martingale measure. I.e, a probability measure Q such that for a derivative payout (with value at time t of V_t) referencing a traded underlying asset (with price at time t of X_t), we have:

$$E^Q[V_t] = V_0$$

The deterministic component of the classical time evolution is determined by the choice of the Martingale measure. For example, assume we enter into a forward contract to sell the underlying at a fixed price: F_T, at time T, and that we can borrow at a risk free interest rate of r. To eliminate risk, we borrow: X_0 to purchase the asset at $t = 0$, which we then hand over at time T. Since we already hold the asset, we are no longer exposed to the random fluctuations in the price of the asset between $t = 0$, and $t = T$. At time T, we must pay back the cash we borrowed with compounded interest: $X_0 e^{rT}$. In return get the amount that we agreed in the original contract: F_T, whilst handing over the asset, which at time T, has the value X_T. Thus, the forward price: F_T, is given in this case by:

$$\begin{aligned} F_T &= E^Q[X_T|X_0] \\ &= X_0 e^{rT} \end{aligned}$$

If we were to set: $\mathcal{H}_{mkt} = L^2[\mathbb{R}]$, we could represent the deterministic drift at the risk free rate: r, using a market Hamiltonian given by:

$$H_{mkt} = -ir\frac{\partial}{\partial x} \tag{5.28}$$

Where r is a constant that represents the risk free interest rate. Applying 5.28, we have:

$$\begin{aligned} |\psi(x,t)\rangle &= e^{iH_{mkt}t}|\psi(x)\rangle \\ &= |\psi(x-rt)\rangle \end{aligned}$$

Note that since:

$$\begin{aligned} [A_u, H_{mkt}] &= [A_d, H_{mkt}] \\ &= 0 \end{aligned}$$

we find that equation 5.30, is translation invariant under this Hamiltonian choice. In other words, using $r \neq 0$ will not impact the dynamics beyond the translation: $|\psi(x)\rangle \rightarrow |\psi(x-rt)\rangle$, and we are free

to choose $r = 0$ without loss of generality. From a financial perspective, setting $r = 0$ means we are modelling the dynamics of forward prices, rather than the current market price (also called the spot price). For many traded underlyings, the market liquidity for forward contracts is sufficient such that it is standard practice to hedge using forward prices, with a maturity that matches the maturity of the derivative, rather than using spot prices.

In theory, we could include kinetic energy and potential energy terms in the market Hamiltonian:

$$H_{mkt} = -ir\frac{\partial}{\partial x} - \frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x) \tag{5.29}$$

This is an important avenue for research, and is discussed further in [6], and [31]. After setting $r = 0$, and hence $H_{mkt} = 0$ in equation 5.28, the dynamics of the market are driven by the interaction with the environment space, which provides the random noise element. By applying 5.29, we are assuming that the market state has its' own internal energy, which will also drive the dynamics. This represents a non-deterministic component to the time evolution, which has no classical counterpart. However, the focus for the current research project is to apply quantum models for the random component of the classical models for the dynamics of traded asset prices. Therefore, for the time being, we choose: $H_{mkt} = \mathbb{I}$, thus ensuring that the only time evolution in the operators we are interested in, comes from the interaction with the environment, which we discuss in 5.4.1.

5.4.3 Time Evolution Mechanism:

In the next proposition we derive the general form for the time evolution, before showing how, under the strong coupling limit: (Proposition 5.4.6), this leads to Markovian dynamics in section 5.4.4. Finally, in sections 5.5 and 5.6, we discuss non-Gaussian extensions.

Proposition 5.4.7. *We let the environment Hilbert space be given by 5.11, the full system Hamiltonian by: 5.12, with B_u, and B_d given by 5.13, and where H_{env} has the product form 5.14. Finally, we assume the*

state starts with the form given by 5.15, and that ρ_B remains in a stationary state (Born approximation), and is therefore independent of time. Then the dynamics of the reduced density matrix are given by:

$$\frac{d\rho_{mkt}(t)}{dt} = -i[H_I(t), \rho^I(0)] - \int_0^t ds \Bigg(\sum_{l,m=1}^{K} \sum_{\alpha,\beta \in \{u,d\}} f_{\alpha\beta}^{lm}(t,s) \Big(A_\alpha A_\beta \rho_{mkt}^{lm}(s) - A_\beta \rho_{mkt}^{lm}(s) A_\alpha \Big) + g_{\alpha\beta}^{lm}(t,s) \Big(\rho_{mkt}^{lm}(s) A_\alpha A_\beta - A_\alpha \rho_{mkt}^{lm}(s) A_\beta \Big) \Bigg) \tag{5.30}$$

Where we denote:

$$f_{\alpha\beta}^{lm}(t,s) = \kappa\gamma Tr[B_\alpha(t) B_\beta(s) r_{lm}(s) |e_l\rangle\langle e_m| \otimes e^{i\gamma(l-m)sH'} \rho_B],\ \alpha,\ \beta \in \{u,d\}$$

$$g_{\alpha\beta}^{lm}(t,s) = \kappa\gamma Tr[B_\alpha(s) B_\beta(t) r_{lm}(s) |e_l\rangle\langle e_m| \otimes e^{i\gamma(l-m)sH'} \rho_B],\ \alpha,\ \beta \in \{u,d\}$$

Proof. We work in the interaction picture, as defined in definition 5.4.2. From proposition 5.4.3, we have:

$$\rho^I(t) = \sum_{l,m=1}^{K} \rho_{mkt}^{lm} \otimes r_{lm}(t) |e_l\rangle\langle e_m| \otimes e^{i\gamma(l-m)tH'} \rho_B$$

Also, from proposition 5.4.4 we have:

$$\begin{aligned} H_I(t) &= \sqrt{\kappa\gamma}(A_u \otimes B_u(t) + A_d \otimes B_d(t)) \\ B_u(t) &= \sum_{l=1}^{K-1} |e_{l+1}\rangle\langle e_l| \otimes e^{i\gamma H' t} \\ B_d(t) &= \sum_{l=1}^{K-1} |e_l\rangle\langle e_{l+1}| \otimes e^{-i\gamma H' t} \end{aligned}$$

Next we feed the interaction picture state: $\rho^I(t)$ into the Von Neumann equation to get:

$$\begin{aligned} \frac{\partial}{\partial t}\rho^I(t) &= -i[H_I(t), \rho^I(t)] \\ \rho^I(t) &= \rho^I(0) - i\int_0^t ds[H_I(s), \rho^I(s)] \end{aligned} \tag{5.31}$$

Note that if we define:

$$\rho_{mkt}^I(t) = Tr_{env}[\rho^I(t)]$$

Then, from definition 5.4.2, we have:

$$\begin{aligned}
\rho^I_{mkt}(t) &= Tr_{env}\Big[e^{i(\mathbb{I}\otimes H_{env})t)}\Big(\sum_{l,m=1}^{K}\rho^{lm}_{mkt}(t)\otimes r_{lm}|e_l\rangle\langle e_m|\otimes\rho_B\Big)e^{-i(\mathbb{I}\otimes H_{env})t)}\Big] \\
&= \sum_{l,m=1}^{K}\rho^{lm}_{mkt}(t)Tr\Big[e^{iH_{env}t}\Big(r_{lm}|e_l\rangle\langle e_m|\otimes\rho_B\Big)e^{-iH_{env}t)}\Big] \\
&= \sum_{l,m=1}^{K}\rho^{lm}_{mkt}(t)Tr[r_{lm}|e_l\rangle\langle e_m|\otimes\rho_B] \\
&= Tr_{env}[\rho(t)] \\
&= \rho_{mkt}(t)
\end{aligned}$$

Now, inserting 5.31 back into the interaction picture Von-Neumann equation, before taking the partial trace over the environment, gives:

$$\begin{aligned}
\frac{\partial}{\partial t}\rho_{mkt}(t) &= -iTr_{env}[H_I(t),\rho^I(0)] - \int_0^t ds Tr_{env}\Big[H_I(t),[H_I(s),\rho^I(s)]\Big] \\
&= -iTr_{env}[H_I(t),\rho^I(0)] - \int_0^t ds Tr_{env}\Big[H_I(t),H_I(s)\rho^I(s)-\rho^I(s)H_I(s)\Big] \\
&= -iTr_{env}[H_I(t),\rho^I(0)] - \int_0^t ds Tr_{env}\Big(H_I(t)H_I(s)\rho^I(s) \\
&- H_I(t)\rho^I(s)H_I(s) - H_I(s)\rho^I(s)H_I(t) + \rho^I(s)H_I(s)H_I(t)\Big) \qquad (5.32)
\end{aligned}$$

Note also that by the cyclicity of the trace we have (where $\rho^I_{env}(s)$ is a state acting on the Hilbert space $\mathcal{H}_{env} = \mathbb{C}^K \otimes L^2[\mathcal{K}]$):

$$Tr[B_u(t)B_d(s)\rho^I_{env}(s)] = Tr[B_d(s)\rho^I_{env}(s)B_u(t)] = Tr[\rho^I_{env}(s)B_u(t)B_d(s)]$$

In the $\rho^I(s)H_I(s)H_I(t)$ term in 5.32, we can therefore write:

$$\sum_{l,m=1}^{K} \rho^{lm}_{mkt}(s)A_uA_dTr[\rho^{I,lm}_{env}(s)B_u(s)B_d(t)] = \sum_{l,m=1}^{K} \rho^{lm}_{mkt}(s)A_uA_dTr[B_u(s)B_d(t)\rho^{I,lm}_{env}(s)] \tag{5.33}$$

$$\sum_{l,m=1}^{K} \rho^{lm}_{mkt}(s)A_dA_uTr[\rho^{I,lm}_{env}(s)B_d(s)B_u(t)] = \sum_{l,m=1}^{K} \rho^{lm}_{mkt}(s)A_dA_uTr[B_d(s)B_u(t)\rho^{I,lm}_{env}(s)]$$

$$\sum_{l,m=1}^{K} \rho^{lm}_{mkt}(s)A_uA_uTr[\rho^{I,lm}_{env}(s)B_u(s)B_u(t)] = \sum_{l,m=1}^{K} \rho^{lm}_{mkt}(s)A_uA_uTr[B_u(s)B_u(t)\rho^{I,lm}_{env}(s)]$$

$$\sum_{l,m=1}^{K} \rho^{lm}_{mkt}(s)A_dA_dTr[\rho^{I,lm}_{env}(s)B_d(s)B_d(t)] = \sum_{l,m=1}^{K} \rho^{lm}_{mkt}(s)A_dA_dTr[B_d(s)B_d(t)\rho^{I,lm}_{env}(s)]$$

$$\rho^{I,lm}(s) = r_{lm}(s)|e_l\rangle\langle e_m| \otimes \rho_B$$

We write:

$$f^{lm}_{\alpha\beta}(t,s) = \kappa\gamma Tr[B_\alpha(t)B_\beta(s)\rho^{I,lm}_{env}(s)] \tag{5.34}$$

$$g^{lm}_{\alpha\beta}(t,s) = \kappa\gamma Tr[B_\alpha(s)B_\beta(t)\rho^{I,lm}_{env}(s)]$$

Therefore 5.33 becomes:

$$\sum_{l,m=1}^{K} \rho^{lm}_{mkt}(s)A_uA_dg^{lm}_{ud}(t,s) + \sum_{l,m=1}^{K} \rho^{lm}_{mkt}(s)A_dA_ug^{lm}_{du}(t,s) + \sum_{l,m=1}^{K} \rho^{lm}_{mkt}(s)A_uA_ug^{lm}_{uu}(t,s) + \sum_{l,m=1}^{K} \rho^{lm}_{mkt}(s)A_dA_dg^{lm}_{dd}(t,s) \tag{5.35}$$

Then in the $H_I(t)\rho^I(s)H_I(s)$ term we get:

$$\sum_{l,m=1}^{K} A_u\rho^{lm}_{mkt}(s)A_dg^{lm}_{ud}(t,s) + \sum_{l,m=1}^{K} A_d\rho^{lm}_{mkt}(s)A_ug^{lm}_{du}(t,s) + \sum_{l,m=1}^{K} A_u\rho^{lm}_{mkt}(s)A_ug^{lm}_{uu}(t,s) + \sum_{l,m=1}^{K} A_d\rho^{lm}_{mkt}(s)A_dg^{lm}_{dd}(t,s) \tag{5.36}$$

Then in the $H_I(s)\rho^I(s)H_I(t)$ term:

$$\sum_{l,m=1}^{K} A_u\rho^{lm}_{mkt}(s)A_d f^{lm}_{ud}(t,s) + \sum_{l,m=1}^{K} A_d\rho^{lm}_{mkt}(s)A_u f^{lm}_{du}(t,s) + \sum_{l,m=1}^{K} A_u\rho^{lm}_{mkt}(s)A_u f^{lm}_{uu}(t,s) + \sum_{l,m=1}^{K} A_d\rho^{lm}_{mkt}(s)A_d f^{lm}_{dd}(t,s) \tag{5.37}$$

and finally in the $H_I(t)H_I(s)\rho^I(s)$ term:

$$\sum_{l,m=1}^{K} A_u A_d\rho^{lm}_{mkt}(s) f^{lm}_{ud}(t,s) + \sum_{l,m=1}^{K} A_d A_u\rho^{lm}_{mkt}(s) f^{lm}_{du}(t,s) + \sum_{l,m=1}^{K} A_u A_u\rho^{lm}_{mkt}(s) f^{lm}_{uu}(t,s) + \sum_{l,m=1}^{K} A_d A_d\rho^{lm}_{mkt}(s) f^{lm}_{dd}(t,s) \tag{5.38}$$

Collecting together 5.35, 5.36, 5.37, and 5.38, leads to 5.30 as required. □

5.4.4 Markovian Approximation:

Remark 5.4.8. *The majority of models of the financial market, applied by practioners, assume Markovian dymamics, partly as a result of tractability, and partly due to the fact that it can be shown that non-Markovian models are (according to many reasonable definitions) arbitrageable. The discussion of whether non-Markovian models of the financial market are reasonable, and investigations into their properties, is an active area of research (see for example [11], [25], and [60]). However, this is not the focus of the current research, and we would therefore like to apply a Markovian approximation to proposition 5.4.7. With this in mind, in this section, we indicate how this can be achieved, by sketching out the mechanisms by which near Markovian dynamics can arise. We use the Singular Coupling Limit, outlined for example in [17], section 3.3, and [56] section 6. We then go on to apply the Markovian approximation for the remainder of the chapter.*

In order to derive the Markovian approximation for proposition 5.4.7, we assume that the environment

remains in a time independent maximum entropy state given by:

$$\rho_{env} = \frac{1}{K}\sum_{i=1}^{K} |e_i\rangle\langle e_i| \otimes \rho_B \tag{5.39}$$

Under 5.39 we have:

$$\begin{aligned}
H_{env}\rho_{env} &= \sum_{l=1}^{K} l|e_l\rangle\langle e_l| \frac{1}{K}\sum_{i=1}^{K} |e_i\rangle\langle e_i| \otimes H'\rho_B \\
&= \sum_{l=1}^{K} l|e_l\rangle\langle e_l| \otimes H'\rho_B \\
&= \sum_{l=1}^{K} l|e_l\rangle\langle e_l| \otimes \rho_B H', \text{ since } [H', \rho_B] = 0 \\
&= \sum_{i=1}^{K} |e_i\rangle\langle e_i| \sum_{l=1}^{K} l|e_l\rangle\langle e_l| \otimes \rho_B H' \\
&= \rho_{env} H_{env}
\end{aligned}$$

Thus we have that: $[H_{env}, \rho_{env}] = 0$, and thus:

$$\begin{aligned}
\rho^I_{env} &= e^{iH_{env}t}\rho_{env}e^{-iH_{env}t} \\
&= \rho_{env}
\end{aligned}$$

We now apply propositions 5.4.4 and 5.4.6 to proposition 5.4.7 to derive the Markovian dynamics.

Proposition 5.4.9 (Born-Markov Approximation)**.** *After applying the strong coupling limit 5.4.6, and assuming the environment state is given by equation 5.39, proposition 5.4.7 becomes:*

$$\frac{d\rho_{mkt}(t)}{dt} = -Tr_{env}[H_I(t), \rho^I(0)] + \sigma^2\Big(A_u\rho_{mkt}(t)A_d + A_d\rho_{mkt}(t)A_u - \frac{1}{2}\{A_uA_d + A_dA_u, \rho_{mkt}(t)\}\Big) \tag{5.40}$$

Where we denote:

$$\sigma^2 = \frac{\kappa(K-1)}{K} \tag{5.41}$$

Proof. First note that, given equation 5.39, we have:

$$\begin{aligned} f_{ud}^{ll}(t,s) = f_{du}^{ll}(t,s) = g_{ud}^{ll}(t,s) = g_{du}^{ll}(t,s) \\ = \frac{\kappa\delta(t-s)}{K},\ l = \{2,\ldots,K-1\} \\ f_{ud}^{KK}(t,s) = g_{ud}^{KK}(t,s) = f_{du}^{11}(t,s) = g_{du}^{11}(t,s) \\ = \frac{\kappa\delta(t-s)}{K} \end{aligned}$$

With all other terms: $f_{\alpha\beta}^{lm}(t,s)$, $g_{\alpha\beta}^{lm}(t,s)$ equal to zero. The result then follows by feeding this into equation 5.30, and integrating from 0 to t. □

Remark 5.4.10 (K=1 case)**.** *Note from equation 5.41, we have that for $K = 1$, $\sigma^2 = 0$, and there is essentially no random diffusion in the market, which then follows unitary evolution, as determined by the system Hamiltonian. This is the equivalent of deterministic drift, with no Brownian motion, in a classical stochastic process.*

The random evolution in the market reduced density matrix arises from the interaction arising from the environment jumping to a higher or lower risk appetite. When $K = 1$, there is only a single level, meaning there is no possibility of the environment making this jump.

Remark 5.4.11 (Alternative Approach: The Weak Coupling Limit)**.** *Before moving on, we briefly consider the main alternative means by which open quantum systems can be approximated using Markovian dynamics. That is the weak coupling limit (see for example [17], [56]). In the weak coupling limit, one assumes that the interaction between the system and the environment is weak in comparison to the market Hamiltonian & the environment Hamiltonian. This means that the typical variation time for the market space: τ_{mkt}, becomes very large as the strength of the interaction with the environment becomes smaller, as does the ratio τ_{mkt}/τ_{env}, where τ_{env} is the typical evolution time for the environment. In other words, the random change, that arises from the interaction Hamiltonian: H_I, causes only slow evolution of the*

market price. We have not pursued this approach for two reasons:

- *Firstly, it requires the expansion of the A_u and A_d operators in terms of the eigenvectors for the market Hamiltonian: H_{mkt}. We would like the model to work in the event that we have $H_{mkt} = 0$.*
- *Secondly, the assumption that the interaction with the environment leads to a slow evolution is not consistent with the fractal nature of the market price. That is, plotting a time-series of price movements over arbitrarily small time intervals (eg price changes every few seconds) leads to a qualitatively similar result as plotting the time-series showing price changes over longer time intervals (eg daily price changes). See for example [40] for further discussion.*

In the singular coupling limit, we assume that the environment Hamiltonian scales with a constant α, and the interaction Hamiltonian by $\sqrt{\alpha}$. This ensures that the typical variation time from the environment: τ_{env} becomes very small. In the limit of $\alpha \to \infty$, the environment settles back to equilibrium effectively instantaneously, and the market price undergoes Markovian evolution, where the memory of prior price movements is essentially forgotten and has no impact of future price changes.

5.4.5 Example 1: Gaussian Case in Finite Dimensions

In this section we assume: $\mathcal{H}_{mkt} = \mathbb{C}^N$, and seek to derive classical diffusion dynamics from the approach in section 5.4.4, before discussing how non-classical diffusion can occur in sections 5.5 and 5.6.

First assume that $\rho_{mkt}(0)$ is a diagonal matrix, so that:

$$\rho_{mkt}(0) = \sum_{i=1}^{N} p_i(0)|f_i\rangle\langle f_i| \tag{5.42}$$

Note, that the state 5.42 can be considered a classical state, as we now explain. The price operator X in this case given by:

$$X = \sum_{i=1}^{N} x_i|f_i\rangle\langle f_i|$$

Where $x_i \in \mathbb{R}$ is the real valued price eigenvalue that is returned for the eigenvector $|f_i\rangle$, and that $x_i > x_j$ for $i > j$. This leads to:

$$\begin{aligned} E[X] &= Tr[X\rho_{mkt}(0)] \\ &= \sum_{i=1}^{N} p_i(0)x_i \end{aligned}$$

So $p_i(0)$ represents the probability of finding the traded market price x_i (at time $t = 0$).

In this setup, A_u represents the operator that shifts the market price higher by one notch, and A_d shifts the market price down by one notch:

Definition 5.4.12.

$$\begin{aligned} A_u &= \sum_{i=1}^{N-1} |f_{i+1}\rangle\langle f_i| \\ A_d &= \sum_{i=1}^{N-1} |f_i\rangle\langle f_{i+1}| \end{aligned}$$

The following proposition shows that under equation 5.30, with A_u, A_d given by definition 5.4.12, the state remains in a similar, essentially classical, state.

Proposition 5.4.13. *Under the assumptions of proposition 5.4.7, with $\mathcal{H}_{mkt} = \mathbb{C}^N$. Let the initial reduced density matrix be given by equation 5.42. Finally, we apply the Born-Markov approximation as in proposition 5.4.9. Then the market state at time t: $\rho_{mkt}(t)$ remains diagonal. In other words we have:*

$$\rho_{mkt}(t) = \sum_{i=1}^{N} p_i(t)|f_i\rangle\langle f_i|$$

where the $\rho_{mkt}(t)$ evolves according to:

$$\begin{aligned} \frac{d\rho_{mkt}(t)}{dt} &= \kappa \sum_{i=2}^{N-1} \big(p_{i+1}(t) + p_{i-1}(t) - 2p_i(t)\big)|f_i\rangle\langle f_i| + \kappa p_1\big(|f_2\rangle\langle f_2| - |f_1\rangle\langle f_1|\big) \\ &\quad + \kappa p_N(t)\big(|f_{N-1}\rangle\langle f_{N-1}| - |f_N\rangle\langle f_N|\big) \end{aligned}$$

Proof. We have:

$$A_u\rho_{mkt}(t)A_d = \sum_{i=1}^{N-1} p_i(t)|f_{i+1}\rangle\langle f_{i+1}|$$

$$A_d\rho_{mkt}(t)A_u = \sum_{i=1}^{N-1} p_i(t)|f_i\rangle\langle f_i|$$

$$\frac{1}{2}\{(A_uA_d + A_dA_u), \rho_{mkt}(t)\} = 2\sum_{i=2}^{N-1} p_i(t)|f_i\rangle\langle f_i| + p_1(t)|f_1\rangle\langle f_1| - p_N(t)|f_N\rangle\langle f_N|$$

Inserting this into equation 5.40 gives the result. □

5.4.6 Example 2: Time Evolution in Infinite Dimensions

In this section assume: $\mathcal{H}_{mkt} = L^2(\mathbb{R})$, with the price operator now given by:

$$X|\psi(x,t)\rangle = |x\psi(x,t)\rangle$$

As before the operators A_u and A_d represent the operators that shift the price up & down by a small amount h:

Definition 5.4.14.

$$A_u|\psi(x,t)\rangle = |\psi(x+h,t)\rangle$$

Proposition 5.4.15. *Under the assumptions of proposition 5.4.7, with $\mathcal{H}_{mkt} = L^2(\mathbb{R})$. Let the reduced density matrix, at time t, be given by:*

$$\rho_{mkt}(t) = \sum_{j\geq 1} p_j|\psi_j(x,t)\rangle\langle\psi_j(x,t)|$$

$$\sum_{j\geq 1} p_j = 1$$

Where $|\psi_j(x,t)\rangle \in L^2(\mathbb{R})$ are a set of orthonormal basis vectors. Finally, we apply the Born-Markov approximation as in proposition 5.4.9. Then we have (for E a compact subset of $\mathbb{R}$):

$$Pr[X \in E] = \int_E p(x,t)dx$$

Where the probability density, given by: $p(x,t) = \sum_{j\geq 1} p_j |\psi_j(x,t)|^2$, satisfies the equation:

$$\frac{dp(x,t)}{dt} = \frac{\sigma^2}{2}\Delta_h p(x,t) \tag{5.43}$$
$$\Delta_h p(x,t) = p(x+h,t) + p(x-h,t) - 2p(x,t)$$

Proof. We have, for some orthonormal basis $\{|e_i(x)\rangle\}$ of $L^2(\mathbb{R})$:

$$\begin{aligned} Pr[X \in E] &= Tr[\rho_{mkt}(t)\mathbb{I}_E(x)] \\ &= \sum_{i\geq 1}\langle e_i| \sum_{j\geq 1} p_j |\psi_j(x,t)\rangle\langle\psi_j(x,t)|\mathbb{I}_E(x)|e_i\rangle \\ &= \sum_{j\geq 1} p_j \sum_{i\geq 1}\langle e_i|\psi_j(x,t)\rangle\langle\psi_j(x,t)|\mathbb{I}_E(x)|e_i\rangle \end{aligned} \tag{5.44}$$

We also have (modulo a phase factor):

$$|\psi_j(x,t)\rangle = \sum_{i\geq 1} a_{ij}(x,t)|e_i\rangle$$
$$a_{ij}(x,t) = \langle e_i|\psi_j(x,t)\rangle$$

Therefore, equation 5.44 becomes:

$$Pr[X \in E] = Tr[\rho_{mkt}(t)\mathbb{I}_E(x)] \tag{5.45}$$

$$= \sum_{j\geq 1} p_j \sum_{i\geq 1} \langle e_i|\psi_j(x,t)\rangle\langle\psi_j(x,t)|\mathbb{I}_E(x)|e_i\rangle \tag{5.46}$$

$$= \sum_{j\geq 1} p_j \sum_{i\geq 1} a_{ij}(x,t)\langle\psi_j(x,t)|\mathbb{I}_E(x)|e_i\rangle \tag{5.47}$$

$$= \sum_{j\geq 1} p_j \langle\psi_j(x,t)|\mathbb{I}_E(x)|\psi_j(x,t)\rangle$$

$$= \sum_{j\geq 1} p_j \int_E |\psi_j(x,t)|^2 dx$$

From which it follows by definition that:

$$p(x,t) = \sum_{j\geq 1} p_j |\psi_j(x,t)|^2 \tag{5.48}$$

We now note that:

$$A_u A_d = A_d A_u = \mathbb{I}$$

Also note that:

$$A_u|\psi(x,t)\rangle\langle\psi(x,t)|A_d = |\psi(x+h,t)\rangle\langle\psi(x+h,t)| \tag{5.49}$$

$$A_d|\psi(x,t)\rangle\langle\psi(x,t)|A_u = |\psi(x-h,t)\rangle\langle\psi(x-h,t)|$$

We differentiate equation 5.45, by applying equation 5.49, together with the Born-Markovian approximation (proposition 5.4.9), to get:

$$\frac{d\rho_{mkt}(t)}{dt} = \frac{\sigma^2}{2}\left(\sum_{j\geq 1} p_j\Big(|\psi_j(x+h,t)\rangle\langle\psi_j(x+h,t)| + |\psi_j(x-h,t)\rangle\langle\psi_j(x-h,t)| - 2|\psi_j(x,t)\rangle\langle\psi_j(x,t)|\Big)\right)$$

Since this is still a linear sum of operators of the form: $|\phi\rangle\langle\phi|$ we can follow the same steps that lead to equation 5.48 to give:

$$\frac{dp(x,t)}{dt} = \frac{\sigma^2}{2}\left(\sum_{j\geq 1} p_j\Big(|\psi_j(x+h,t)|^2 + |\psi_j(x-h,t)|^2 - 2|\psi_j(x,t)|^2\Big)\right)$$

Which, finally leads to:

$$\frac{dp(x,t)}{dt} = \frac{\sigma^2}{2}\Delta_h p(x,t)$$

□

Remark 5.4.16. *It follows from propositions 5.4.13 and 5.4.15, that the probability distribution for the traded price observable, in both the finite & infinite dimensional cases, evolves like a Gaussian process, in the same way as a classical approach. The difference in the quantum case is that where the reduced density matrix has zero Von-Neumann entropy (ie we have full information regarding the market state) there may well still be uncertainty in the final price. This uncertainty relates to the fact that due to imperfections in the operation of the market, there is still uncertainty in the price, despite the fact that we have full information.*

5.5 Non-Gaussian Extension I: Non-Local Operators

5.5.1 Example 3: Non-local Interaction, Finite Dimensions

In section 5.4.5, we have studied the case whereby the market response to a change in the environment to a higher level of risk appetite, is that the price jumps by a fixed amount. Ie, if $\rho_{mkt}(0) = |e_i\rangle\langle e_i|$, then the initial price is given by the eigenvalue: x_i. If the environment were to jump a risk appetite level, the market response (as defined by the interaction Hamiltonian in equation 5.12) would be given by:

$$A_u|e_i\rangle = |f_{i+1}\rangle$$

$$X|e_{i+1}\rangle = x_{i+1}|f_{i+1}\rangle$$

So that the price jumps from x_i to x_{i+1}. We have shown in proposition 5.4.13, that the resulting behaviour can be described by the Gaussian evolution in classical probabilities.

Now, consider the case whereby the response when the environment jumps a risk appetite level (represented by the operator: A_u) is uncertain. The price may jump by 1 level, or more (or the price may not jump at all). In order to introduce this effect, we apply a discrete convolution with a probability distribution labelled P_H:

$$P_H = \sum_i h_i |f_i\rangle \tag{5.50}$$

We start with the operator defined by equation 5.4.12:

$$A_u|f_i\rangle = |f_{i+1}\rangle,\ i < N$$

In order to apply the convolution between $|\psi\rangle$ and H, we can use the following operator:

Definition 5.5.1.

$$H = \sum_{j=1}^{N} \sum_{k=1-j}^{N-j} h_k |f_{j+k}\rangle\langle f_j|$$

The operators A_u^H, A_d^H, together with the new interaction Hamiltonian: H_I can now be defined by:

$$\begin{aligned} A_u^H &= A_u H \\ &= \sum_{i=1}^{N-1} \sum_{j=1}^{N} h_{i-j} |f_{i+1}\rangle\langle f_j| \\ A_d^H &= A_u^{H\dagger} \\ &= H^\dagger A_d \end{aligned}$$

Using this definition for the operators A_u^H, and A_d^H, we can now restate proposition: 5.4.13, with definition 5.5.1 in place of definition 5.4.12.

Proposition 5.5.2. *Let the market Hilbert space be given by: $\mathcal{H}_{mkt} = \mathbb{C}^N$. Furthermore, assume definition 5.5.1 applies with regards to the interaction Hamiltonian. Then, after applying the Born-Markov*

approximation given in proposition 5.4.9, the reduced density matrix for the market evolves according to:

$$\frac{d\rho_{mkt}(t)}{dt} = \sigma^2\Big(A_u^H \rho_{mkt}(t) A_d^H + A_d^H \rho_{mkt}(t) A_u^H - \frac{1}{2}\big\{A_u^H A_d^H + A_d^H A_u^H, \rho_{mkt}(t)\big\}\Big) \tag{5.51}$$

Proof. The result follows from inserting the operators given in definition 5.5.1 into proposition 5.4.9. □

5.5.2 Example 4: Non-local Interaction, Infinite Dimensions

In this section we extend section 5.5, to the infinite dimensional case, where $\mathcal{H}_{mkt} = L^2(\mathbb{R})$. We start with the following:

Definition 5.5.3. *For $P_H \in L^2(\mathbb{R})$ we define the operator: $\hat{H}$ acting on $L^2(\mathbb{R})$, as follows:*

$$\begin{aligned}\hat{H}|\psi(x)\rangle &= |(P_H \star \psi)(x)\rangle \\ &= \int_{\mathbb{R}} P_H(x-y)\psi(y)dy\end{aligned}$$

Proposition 5.5.4. *The adjoint for $\hat{H}$ is given by:*

$$\hat{H}^\dagger|\psi(x)\rangle = \int_{\mathbb{R}} P_H(y-x)\psi(y)dy$$

If $P_H(x)$ is a real valued even function ($P_H(x) = P_H(-x)$), then $\hat{H}$ is self-adjoint.

Proof. We have:

$$\begin{aligned}\langle\phi(x)|\hat{H}\psi(x)\rangle &= \int_{\mathbb{R}} \overline{\phi(x)} \int_{\mathbb{R}} P_H(x-y)\psi(y)dydx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\phi(x)} P_H(y-x)dx\psi(y)dy, \text{ by Fubini's Theorem} \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\phi(x)P_H(y-x)}dx\psi(y)dy, \text{ since } P_H(x) \text{ is real}\end{aligned}$$

Therefore, if $P_H(x) = P_H(-x)$ we have:

$$\begin{aligned}\langle\phi(x)|\hat{H}\psi(x)\rangle &= \int_{\mathbb{R}} \overline{(\phi \star P_H)(x)}\psi(x)dx \\ &= \langle\hat{H}\phi(x)|\psi(x)\rangle\end{aligned}$$

□

We now extend definition 5.5.1 to infinite dimensions:

Definition 5.5.5. *Let $\hat{H}$ be given by definition 5.5.3, and the market Hilbert space be: $\mathcal{H}_{mkt} = L^2(\mathbb{R})$. The operators A_u^H, A_d^H, together with the new interaction Hamiltonian: H_I can now be defined by:*

$$\begin{aligned} A_u^H|\psi(x)\rangle &= A_u\hat{H}|\psi(x)\rangle \\ &= \int_{\mathbb{R}} P_H(x-h-y)\psi(y)dy \\ A_d^{\hat{H}}|\psi(x)\rangle &= A_u^{H\dagger}|\psi(x)\rangle \\ &= \hat{H}^\dagger A_d \\ &= \int_{\mathbb{R}} H(x+h-y)\psi(y)dy \end{aligned}$$

Where $h > 0$ is a fixed parameter.

As per section 5.5, using this definition for the Hamiltonian: H_I, we can now restate proposition: 5.4.15, with definition 5.5.5 in place of 5.4.14.

Proposition 5.5.6. *Let the market Hilbert space be given by $\mathcal{H}_{mkt} = L^2(\mathbb{R})$. Furthermore, assume definition 5.5.5 applies with regards to the interaction Hamiltonian. Then after applying the Born-Markov approximation given in 5.4.9, the reduced density matrix for the market evolves according to:*

$$\begin{aligned} \frac{d\rho_{mkt}(t)}{dt} = \sigma^2\Big(&\hat{H}|\psi(x+h,t)\rangle\langle\psi(x+h,t)|\hat{H}^\dagger + \hat{H}^\dagger|\psi(x-h,t)\rangle\langle\psi(x-h,t)|\hat{H} \\ &- \frac{1}{2}\big\{\{\hat{H},\hat{H}^\dagger\},|\psi(x,t)\rangle\langle\psi(x,t)|\big\}\Big) \end{aligned} \tag{5.52}$$

Proof. First note that:

$$\begin{aligned} A_u^H|\psi(x,t)\rangle\langle\psi(x,t)|A_d^H|\phi(x)\rangle &= A_u\hat{H}|\psi(x,t)\rangle\langle\psi(x,t)|\hat{H}^\dagger A_d\phi(x)\rangle \\ &= A_u|(\psi\star P_H)(x,t)\rangle\langle\hat{H}\psi(x,t)|A_d\phi(x)\rangle \\ &= A_u|(\psi\star P_H)(x,t)\rangle\langle(\psi\star P_H)(x,t)|A_d|\phi(x)\rangle \end{aligned}$$

So we see that:

$$\begin{aligned} A_u^H|\psi(x,t)\rangle\langle\psi(x,t)|A_d^H &= A_u|(\psi \star P_H)(x,t)\rangle\langle(\psi \star P_H)(x,t)|A_d \\ &= |(\psi \star P_H)(x+h,t)\rangle\langle(\psi \star P_H)(x+h,t)| \\ &= \hat{H}|\psi(x+h,t)\rangle\langle\psi(x+h,t)|\hat{H}^\dagger \end{aligned} \tag{5.53}$$

Similarly, we have:

$$\begin{aligned} A_d^H|\psi(x,t)\rangle\langle\psi(x,t)|A_u^H &= |(\psi \star T_x P_H)(x-h,t)\rangle\langle(\psi \star T_x P_H)(x-h,t)| \\ \text{Where, } T_x P_H(x) &= P_H(-x) \\ &= \hat{H}^\dagger|\psi(x-h,t)\rangle\langle\psi(x-h,t)|\hat{H} \end{aligned} \tag{5.54}$$

We also have:

$$\begin{aligned} A_u^H A_d^H|\psi(x,t)\rangle\langle\psi(x,t)| &= A_u\hat{H}\hat{H}^\dagger A_d|\psi(x,t)\rangle\langle\psi(x,t)| \\ &= A_u\hat{H}|(T_x P_H \star \psi(x-h)\rangle\langle\psi(x,t)| \\ &= |\big(P_H \star (T_x P_H \star \psi)\big)(x,t)\rangle\langle\psi(x,t)| \\ &= \hat{H}\hat{H}^\dagger|\psi(x.t)\rangle\langle\psi(x,t)| \end{aligned} \tag{5.55}$$

Finally, we have:

$$A_d^H A_u^H|\psi(x,t)\rangle\langle\psi(x,t)| = \hat{H}^\dagger\hat{H}|\psi(x,t)\rangle\langle\psi(x,t)| \tag{5.56}$$

$$|\psi(x,t)\rangle\langle\psi(x,t)|A_d^H A_u^H = |\psi(x,t)\rangle\langle\psi(x,t)|\hat{H}^\dagger\hat{H}$$

$$|\psi(x,t)\rangle\langle\psi(x,t)|A_u^H A_d^H = |\psi(x,t)\rangle\langle\psi(x,t)|\hat{H}\hat{H}^\dagger$$

Inserting equations 5.53, 5.54, 5.55, 5.56 into proposition 5.4.9, gives the required result. □

Remark 5.5.7. *As is the case for proposition 5.5.2, there is no longer any classical representation for the time evolution of the reduced density matrix, which can no longer in general be easily described in terms of a classical probability density function.*

5.6 Non-Gaussian Extension II: Non-Commutative State

With the operators A_u and A_d given by definition 5.4.12, and $\mathcal{H}_{mkt} = \mathbb{C}^N$, we end up with proposition 5.4.13, that states that where the initial quantum state is represented by a diagonal density matrix, it will remain as a diagonal density matrix. In other words $\rho_{mkt}(t)$ remains in the commutative algebra of diagonal matrices. If one uses definition 5.5.1, for A_u and A_d, then, under proposition 5.5.2, even where $\rho_{mkt}(0)$ starts as a diagonal matrix, this will generally not be the case for $\rho_{mkt}(t)$.

Crucially in both sections 5.4 and 5.5, we have assumed that the environment state remains in the stationary state:

$$\rho_{env} = \frac{1}{K}\sum_{i=1}^{K} |e_i\rangle\langle e_i| \otimes \rho_B$$

Instead, in this section we look at the more general case, and consider the case for non-diagonal ρ_{env}:

$$\rho_{env} = \sum_{l,m=1}^{K} r_{lm}|e_l\rangle\langle e_m| \otimes \rho_B \tag{5.57}$$

Proposition 5.6.1. *We let the environment Hilbert space be given by 5.11, the full system Hamiltonian by: 5.12, with B_u, and B_d given by 5.13, and where H_{env} has the product form 5.14.*

After applying the strong coupling limit 5.4.6 with the environment state given by 5.57, we have:

$$\begin{aligned}\frac{d\rho_{mkt}(t)}{dt} &= -Tr_{env}[H_I(t), \rho^I(0)] + \sigma^2\Big(A_u\rho_{mkt}(t)A_d + A_d\rho_{mkt}(t)A_u - \frac{1}{2}\{A_uA_d + A_dA_u, \rho_{mkt}(t)\}\Big) \\ &+ \nu_u^2\Big(A_u\rho_{mkt}(t)A_u - \frac{1}{2}\{A_uA_u, \rho_{mkt}(t)\}\Big) + \nu_d^2\Big(A_d\rho_{mkt}(t)A_d - \frac{1}{2}\{A_dA_d, \rho_{mkt}(t)\}\Big)\end{aligned} \tag{5.58}$$

Where we denote:

$$\sigma^2 = \kappa\sum_{l=1}^{K-1} r_{ll},\ \nu_u^2 = 2\kappa\sum_{l=1}^{K-2} r_{l(l+2)},\ \nu_d^2 = 2\kappa\sum_{l=1}^{K-2} r_{(l+2)l}$$

Proof. The result follows from applying the strong coupling limit, and feeding equations 5.24, 5.25, and 5.26 into proposition 5.4.7. □

5.7 Numerical Simulations:

In this subsection we examine some numerical simulations. First in section 5.7.1 we briefly consider the setup of the finite difference method used, and specifically the number of dimensions in the market space required for numerical convergence. Then in section 5.7.2, we describe the basic numerical setup used for the remaining sections. In sections 5.7.3 and 5.7.4, we consider what can be learnt from the results on the variance, entropy and kurtosis.

5.7.1 Convergence:

In this section 5.7 we set $\mathcal{H}_{mkt}$ to $\mathbb{C}^N$, and would like to determine an appropriate value for N, such that we model sufficient granularity in the object of interest (in this case the financial market) whilst ensuring convergence. Simultaneously, we do not want to set N too high, for performance reasons.

We have shown in proposition 5.4.13 that in the Gaussian & Markovian case, if $\rho_{mkt}(0)$ is a diagonal matrix, then $\rho_{mkt}(t)$ remains diagonal, and we can write the numerical problem as:

$$\frac{\delta p(x,t)}{\delta t} \approx Lp(x,t)$$

Where $p(x,t)$ is a vector in $\mathbb{C}^N$, and L is a matrix. This enables us to write alternative schemes for simulating $p(x,t)$. For example an explicit scheme:

$$\begin{aligned} \delta p(x,t) &= p(x,t+\delta t) - p(x,t) \\ &\approx \delta t L p(x,t) \\ p(x,t+\delta t) &\approx (1+\delta t L)p(x,t) \end{aligned} \tag{5.59}$$

Or alternatively an implicit scheme:

$$\begin{aligned} \delta p(x,t) &= p(x,t+\delta t) - p(x,t) \\ &\approx \delta t L p(x,t+\delta t) \\ p(x,t+\delta t) &\approx (1-\delta t L)^{-1} p(x,t) \end{aligned} \tag{5.60}$$

It is well known (see for example [62] chapter 19), that, whilst it is faster to run, the explicit scheme will not generally converge unless the spacing in the x variable is sufficiently small, relative to the time step δt. One can resolve this by choosing an implicit scheme such as 5.60. However, this requires the inversion of a large matrix, and will in general be considerably slower. In fact, there are a number of schemes that can be used that combine elements of both 5.59 and 5.60 to ensure both unconditional convergence and speed.

In the general noncommutative case, for example proposition 5.4.9, we have:

$$\frac{\delta \rho_{mkt}(t)}{\delta t} \approx L(\rho_{mkt}(t))$$

where $L(*)$ is now a super-operator, and we cannot generally write it as a matrix acting on a vector. For this reason the simple implicit scheme will not work in this case. For now we stick to an explicit scheme, where L acts on $\rho_{mkt}(t)$ rather than $\rho_{mkt}(t+\delta t)$.

Using the setup described in 5.7.2, we looked at the variance and excess kurtosis (Kurtosis - (3*Variance^2)) after 50 time-steps of $\delta t = 0.001$, versus the dimensionality of the market space. These are shown in figures 5.1 and 5.2 below. From the charts, it is clear that the results are stable for $N \gtrapprox 1000$. With this in mind, and to ensure stability & convergence throughout, we use $N = 1001$.

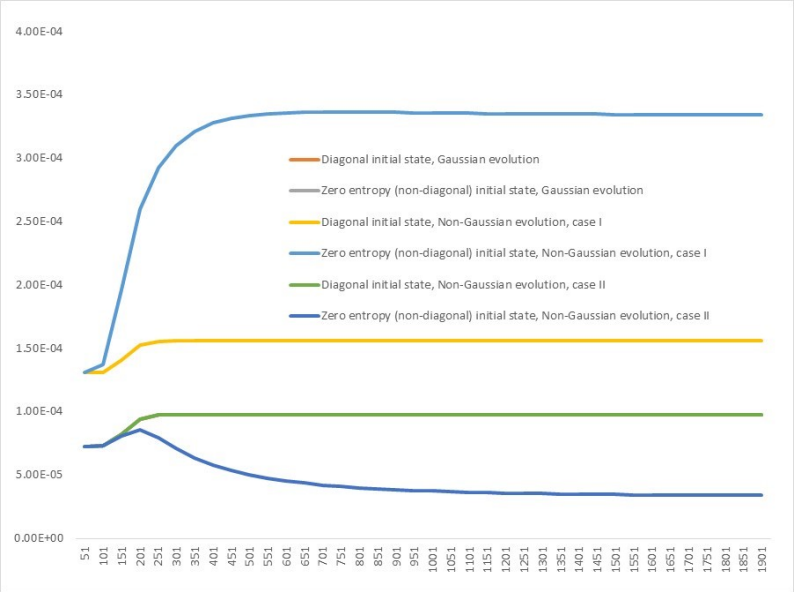

Figure 5.1: Total variance after 50 timesteps versus the dimension for the market space.

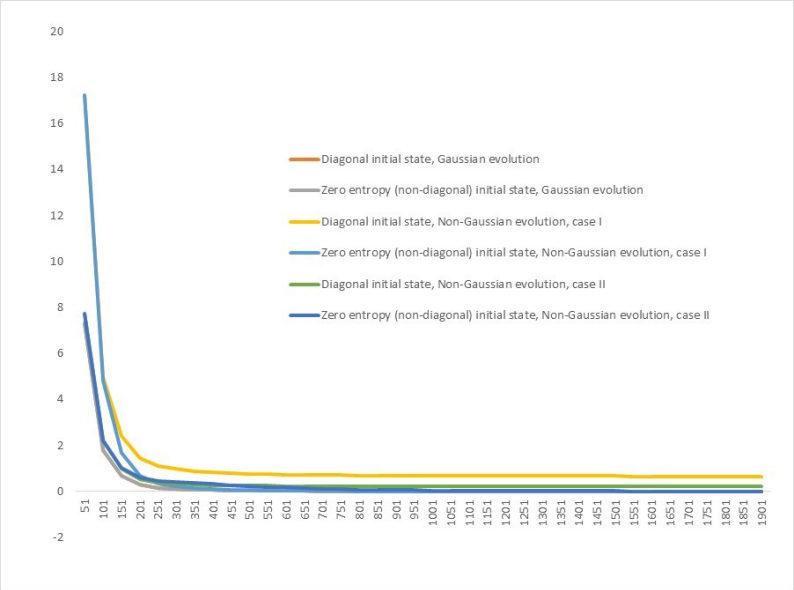

Figure 5.2: Total excess kurtosis after 50 timesteps versus the dimension for the market space.

5.7.2 Basic Setup:

5.7.2.1 Grid Spacing and Initial Market State:

The market Hilbert space is now set to $\mathcal{H}_{mkt} = \mathbb{C}^{1001}$. In order to interpret concepts like variance & kurtosis, we must first define the X operator, which assigns a value to each of the states. In this case we set X to:

$$X = \sum_{i=1}^{1001} x_i |f_i\rangle\langle f_i|$$
$$x_i = -2 + \frac{i-1}{250}$$

We define the initial market state as follows, where $N(x, \mu, \sigma)$ is the normal distribution density function with mean μ, and variance σ^2:

$$\begin{aligned} \rho_0(\theta) &= \theta\rho_c + (1-\theta)\rho_q \\ \rho_c &= \sum_{i=1}^{1001} p_i |f_i\rangle\langle f_i|,\ p_i = N(x_i, 0, 0.005) \\ \rho_q &= (\overline{P} \cdot \overline{P}^T),\ \overline{P}^T = (\sqrt{p_1},\ \sqrt{p_2},\ \ldots,\ \sqrt{p_{1001}}) \end{aligned} \tag{5.61}$$

This ensures that $\rho_0(0)$ is a pure state, and $\rho_0(1)$ is a maximum entropy diagonal state, and also that the initial probability distribution for X is unaffected by the choice of θ.

In order to look at the impact of phase on the time evolution, we also apply the following initial state:

$$\begin{aligned} \rho_0(\phi) &= \phi\rho_c + (1-\phi)\rho_q \\ \rho_c &= \sum_{i=1}^{1001} p_i |f_i\rangle\langle f_i|,\ p_i = N(x_i, 0, 0.005)sgn(x_i) \\ \rho_q &= (\overline{P} \cdot \overline{P}^T),\ \overline{P}^T = (\sqrt{p_1},\ \sqrt{p_2},\ \ldots,\ \sqrt{p_{1001}}) \end{aligned} \tag{5.62}$$

5.7.2.2 Environment State:

We set the environment dimension to $K = 11$, and for both sections 5.7.2.3 and 5.7.2.4, the assumption is that the environment is at thermal equilibrium:

$$\rho_{env} = \frac{1}{11} \sum_{i=1}^{11} |e_i\rangle\langle e_i|$$

Note that the role of the bath state: ρ_B is to allow the strong coupling limit. In other words, the impact is to ensure the evolution is Markovian, and ρ_B does not need to be incorporated into the numerical simulation in any other way.

For section 5.7.2.5, we apply the following state for illustration:

$$\rho_{env} = \frac{1}{11} \sum_{i,j=1}^{11} |e_i\rangle\langle e_j| \tag{5.63}$$

In practice, the off-diagonal terms control the non-Gaussian evolution in this case. By assuming equation 5.63, we reduce the dimensionality of configuring the time evolution to the choice of a single parameter (equation 5.64).

5.7.2.3 Gaussian Evolution:

We apply proposition 5.4.9, with A_u and A_d given by definition 5.4.12, and $\sigma^2/2$ given by:

$$\sigma^2 = \frac{0.4 * \delta t}{\delta x^2}$$

We apply 1000 time-steps. Note, that the objective in this section is to investigate the qualitative behaviour of the models discussed in this chapter, and we scale σ^2 and δt to ensure an appropriate amount of diffusion has occurred after 1000 time-steps.

Finally, we note that under the assumptions in proposition 5.4.9, $Tr[B_u\rho_{env}] = Tr[B_d\rho_{env}] = 0$. Therefore, the term: $-Tr_{env}[H_I(t), \rho^I(0)]$ does not make a contribution.

5.7.2.4 Non-Gaussian Evolution Case I:

We also simulate the first non-Gaussian case explored above in section 5.5.1, with A_u^H, and A_d^H given by 5.5.1. If we have $\overline{V} \in \mathbb{C}^{1001}$, then we have (see definition 5.5.1 and equation 5.50):

$$A_u^H \overline{V}^T \overline{V} A_d^H = A_u (H \overline{V}^T \overline{V} H^\dagger) A_d$$

$$H\overline{V} = \overline{V} \star P_H$$

$$P_H = \sum_{i=1}^{M} h_i |f_i\rangle, \text{ where } M << 1001$$

We would therefore like the vector: $H\overline{V}$ to be normalised. This in turn requires:

$$\sum_k |h_k|^2 = 1$$

In the numerical simulations, we use the following:

$$P_H = (\sqrt{0.05},\ \sqrt{0.125},\ \sqrt{0.7},\ \sqrt{0.125},\ \sqrt{0.05})^T$$

5.7.2.5 Non-Gaussian Evolution Case II:

Finally, we simulate the second non-Gaussian case explored in section 5.6. We apply the basic setup described in sections 5.7.2 and 5.7.2.3, but include the additional terms from proposition 5.6.1. Note:

- The environment state is set to equation 5.63.
- In this case, for illustration purposes, we assume:

$$\sigma^2 = \nu_u^2 = \nu_d^2 = \frac{0.4 * \delta t}{\delta x^2} \tag{5.64}$$

5.7.3 Variance vs Entropy:

Figure 5.3 shows the growth in the variance & the entropy for the Gaussian evolution with $\theta = 0$, and $\theta = 1$. In figure 5.4, we show the first 2 eigenvectors (ranked by eigenvalue), for the $\theta = 0$ Gaussian case.

From proposition 5.4.9, we have the following components in the evolution of $\rho_{mkt}(t)$ at each step:

- $A_u\rho_{mkt}(t)A_d$ and $A_d\rho_{mkt}(t)A_u$. These terms act diagonally, in the sense that:

$$A_u|f_i\rangle\langle f_i|A_d = |f_{i+1}\rangle\langle f_{i+1}| \tag{5.65}$$
$$A_d|f_i\rangle\langle f_i|A_u = |f_{i-1}\rangle\langle f_{i-1}|$$

- $\{A_uA_d + A_dA_u, \rho_{mkt}(t)\}$. These terms also act diagonally, in the sense that:

$$(A_uA_d + A_dA_u)|f_i\rangle\langle f_i| = 2|f_i\rangle\langle f_i|, \text{ for } i = \{2, \ldots, 1000\} \tag{5.66}$$

Now, the probability of finding the value for X as x_i is given by:

$$P(X = x_i) = Tr[\rho_{mkt}(t)|f_i\rangle\langle f_i|] \tag{5.67}$$

So from 5.65, 5.66 and 5.67, we find that since the 2 initial states, with $\theta = 0$, and $\theta = 1$ start with the same probability distribution for X, under Gaussian evolution this remains the case. This can be seen for example in figure 5.3, where the variance in both cases is the same. The entropy difference arises, since despite the fact that the entropy growth over the period is the same, for $\theta = 0$ the market starts in a pure state with entropy of zero.

5.7.3.1 Variance vs Entropy Non-Gaussian Case 1:

In figure 5.5, we show the results for the non-Gaussian time-evolution described in section 5.7.2.4. We first construct initial states using equation 5.61 so that with $\theta = 0$, the state starts in a pure state, whilst with $\theta = 1$, the state is diagonal with respect to the X operator.

We note the following:

- Unlike the Gaussian case, the total variance now depends on the value for θ. In the Gaussian case shown in figure 5.3, the total variance, after the simulation, is identical regardless of the initial entropy. We consider this further in remark 5.7.1.

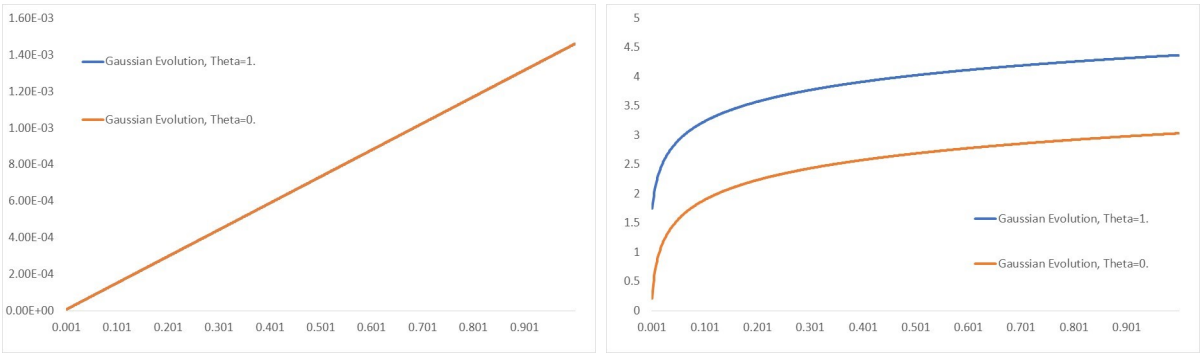

Figure 5.3: The chart shows the growth in the variance & entropy vs time for the Gaussian evolution, $\theta = 0$, and $\theta = 1$.

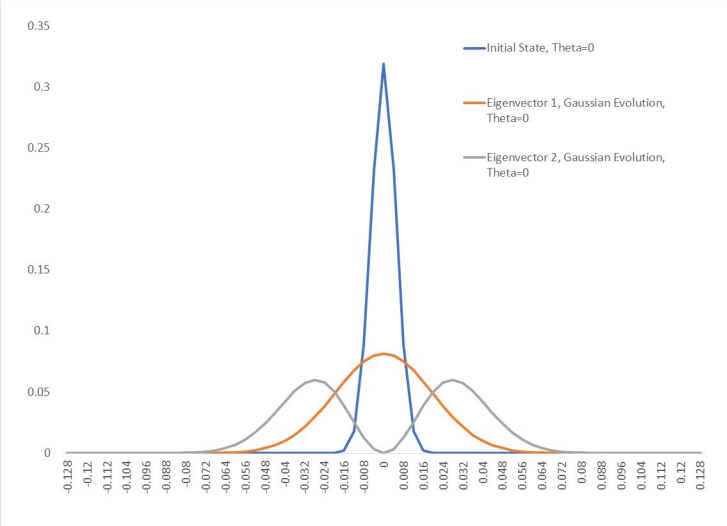

Figure 5.4: The chart shows the initial probability density function, together with the first 2 eigenvectors (by eigenvalue) after 1000 time-steps of Gaussian time evolution.

- Both the entropy gain, and the total variance is highest for the pure quantum initial state: $\theta = 0$. Importantly, initial states which have higher total variance after the simulation, also have higher entropy gain.

- However, due to the higher starting value for the entropy, the maximum final entropy is observed for $\theta = 0.7$.

Remark 5.7.1 (Comments on the Total Variance, for Non-Gaussian Case I)**.** *In order to gain a qualitative understanding of figure 5.5, we consider the simplified setup whereby:*

$$P_H = (h_1, h_0, h_1) \tag{5.68}$$

$$h_0, h_1 \in \mathbb{R}$$

$$h_0^2 + 2h_1^2 = 1$$

We set the initial state to

$$\rho_0 = \sum_{i,j=1}^{N} a_{ij} |f_i\rangle\langle f_j| \tag{5.69}$$

$$\sum_i |a_{ii}|^2 = 1$$

$$a_{ij} \to 0 \text{ for } i, j \to 1, N$$

$$N \to \infty$$

The purpose of proposition 5.7.2, is to highlight why (in this simplified case) the variance depends not just on the diagonal elements of the density matrix, but also on the non-diagonal elements. In other words, we show why in the quantum case, 2 density matrices with the same initial probability distribution for X, have different variance growth rates.

Proposition 5.7.2. *Let the price operator X be given by equation 5.6, the vector P_H by equation 5.68, and the initial state: ρ_0, by equation 5.69, then under the time evolution given by proposition 5.5.2, the rate of change in the total variance is given by:*

$$\frac{\partial(E^{\rho_0}[X^2])}{\partial t} = \sigma^2 \sum_{i=2}^{N-1} x_i^2 h_0^2 (a_{(i+1)(i+1)} + a_{(i-1)(i-1)} - 2a_{ii}) + \sigma^2 \sum_{i=3}^{N-2} x_i^2 h_1^2 (a_{(i+2)(i+2)} + a_{(i-2)(i-2)} - 2a_{ii}) \tag{5.70}$$

$$+ \sigma^2 \sum_{i=3}^{N-2} h_0 h_1 x_i^2 (a_{(i+1)(i+2)} + a_{(i+2)(i+1)} + a_{(i-1)(i-2)} + a_{(i-2)(i-1)}$$

$$- a_{i(i+1)} + a_{(i+1)i} + a_{i(i-1)} + a_{(i-1)i})$$

Proof. See appendix A, section A.3. □

Note that in proposition 5.7.2, the terms:

$$h_0^2 (a_{(i+1)(i+1)} + a_{(i-1)(i-1)} - 2a_{ii})$$

represent the Gaussian terms. If $h_0 = 1$, and $h_1 = 0$, these terms are all that remain. The terms in:

$$h_1^2 (a_{i+2}^2 + a_{i-2}^2 - 2a_i^2)$$

represent additional non-Gaussian terms that act on the diagonal of the density matrix: $\rho(t)$. These do not depend on the relative phase of the various state eigenvectors. The remaining terms:

$$\sigma^2 \sum_{i \geq 3} h_0 h_1 x_i^2 (a_{(i+1)(i+2)} + a_{(i+2)(i+1)} + a_{(i-1)(i-2)} + a_{(i-2)(i-1)}$$

$$- a_{i(i+1)} + a_{(i+1)i} + a_{i(i-1)} + a_{(i-1)i})$$

act on the off-diagonal elements of the density matrix. These do depend on the state. This is highlighted in figure 5.6, where 2 different states with the same initial probability density function & the same initial entropy, have differing total variance.

Overall it is clear that in the Non-Gaussian case, the variance depends on:

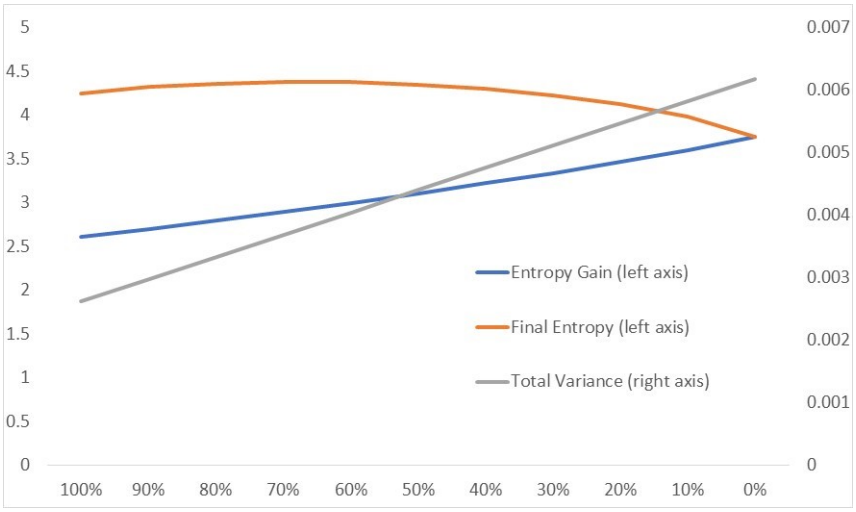

Figure 5.5: The variance for non-Gaussian case I (left vertical axis), and entropy (right vertical axis), against the value for θ used in the initial state.

- *The classical model parameters (for example κ or σ^2) that configure the classical Gaussian contribution to the time-evolution.*
- *The degree of the initial entropy, whereby on the whole states with lower initial entropy will generate more volatility.*
- *The relative phase of the eigenvectors/the off diagonal terms in the initial market density matrix.*

5.7.3.2 Variance vs Entropy Non-Gaussian Case 2:

In figure 5.7, we show the results for the non-Gaussian time-evolution described in section 5.7.2.5. We first construct initial states using equation 5.61 so that with $\theta = 0$, the state starts in a pure state, whilst with $\theta = 1$, the state is diagonal with respect to the X operator.

We note the following:

- As for case I, discussed above, the total variance now depends on the the value chosen for θ in the

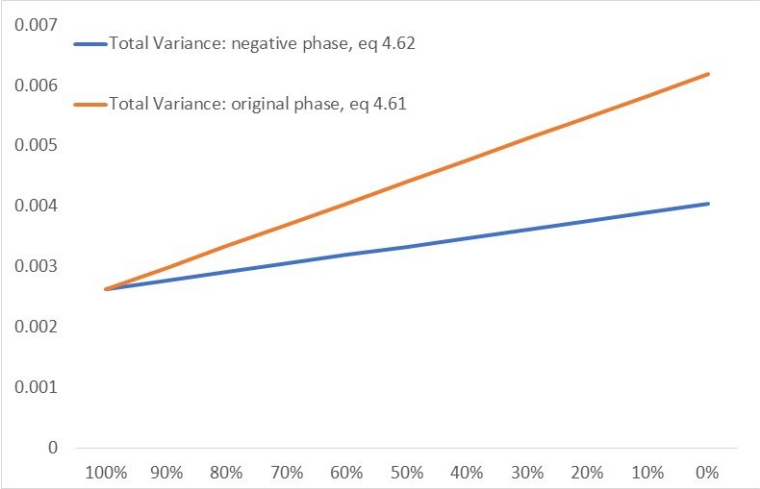

Figure 5.6: The non-Gaussian case I variance versus θ/ϕ, for the original phase (equation 5.61) and the negative phase (equation 5.62).

initial state. However, the largest variance is now seen for the diagonal initial state, and reduces with reducing initial entropy.

- Again, the phase of the initial state is important, as shown in figure 5.8.
- As before, in each case states with higher total variance after the simulation, also have a higher entropy gain.

Remark 5.7.3 (Comments on the Total Variance, for Non-Gaussian Case II)**.** *As above, in order to gain a qualitative understanding of the evolution of the variance, we repeat proposition 5.7.2 for the second variant of non-Gaussian evolution, in proposition 5.7.4 below. For this purpose, we first assume the*

initial state is given by equation 5.71:

$$\rho_0 = \sum_{i,j=1}^{N} a_{ij}|f_i\rangle\langle f_j| \tag{5.71}$$

$$\sum_{i=1}^{N} |a_{ii}|^2 = 1$$

$$a_{ij} \to 0 \text{ for } i, j \to 1, N$$

$$N \to \infty$$

Then we have:

Proposition 5.7.4. *Let the price operator X be given by equation 5.6, and the initial state: ρ_0, by equation 5.71, then under the time evolution given by proposition 5.6.1, the rate of change in the total variance is given by:*

$$\frac{\partial(E^{\rho_0}[X^2])}{\partial t} = \sigma^2 \sum_{i=2}^{N-1} x_i^2(a_{(i+1)(i+1)} + a_{(i-1)(i-1)} - 2a_{ii}) \tag{5.72}$$
$$+ \nu_u^2 \sum_{i=3}^{N-2} x_i^2\Big(a_{(i-1)(i+1)} - \frac{1}{2}\big(a_{(i-2)i} + a_{i(i+2)}\big)\Big) + \nu_d^2 \sum_{i=3}^{N-2} x_i^2\Big(a_{(i+1)(i-1)} - \frac{1}{2}\big(a_{i(i-2)} + a_{(i+2)i}\big)\Big)$$

Proof. See appendix A, section A.3. □

As is the case for the non-Gaussian evolution case I, we have both Gaussian contributions to the evolution of the variance, that depend solely on the diagonal terms in the density matrix:

$$\sigma^2 \sum_{i=2}^{N-1} x_i^2(a_{(i+1)(i+1)} + a_{(i-1)(i-1)} - 2a_{ii})$$

There are also non-Gaussian terms that depend on the off-diagonal components:

$$+\nu_u^2 \sum_{i=3}^{N-2} x_i^2\Big(a_{(i-1)(i+1)} - \frac{1}{2}\big(a_{(i-2)i} + a_{i(i+2)}\big)\Big) + \nu_d^2 \sum_{i=3}^{N-2} x_i^2\Big(a_{(i+1)(i-1)} - \frac{1}{2}\big(a_{i(i-2)} + a_{(i+2)i}\big)\Big)$$

If the terms ν_u, ν_d are zero, then the evolution will be Gaussian, and a diagonal market density matrix will remain diagonal. If $\nu_u \neq 0$ or $\nu_d \neq 0$, then even if the density matrix starts in a diagonal (classical) state, the non-Gaussian evolution will evolve non-zero off-diagonal terms as the simulation progresses.

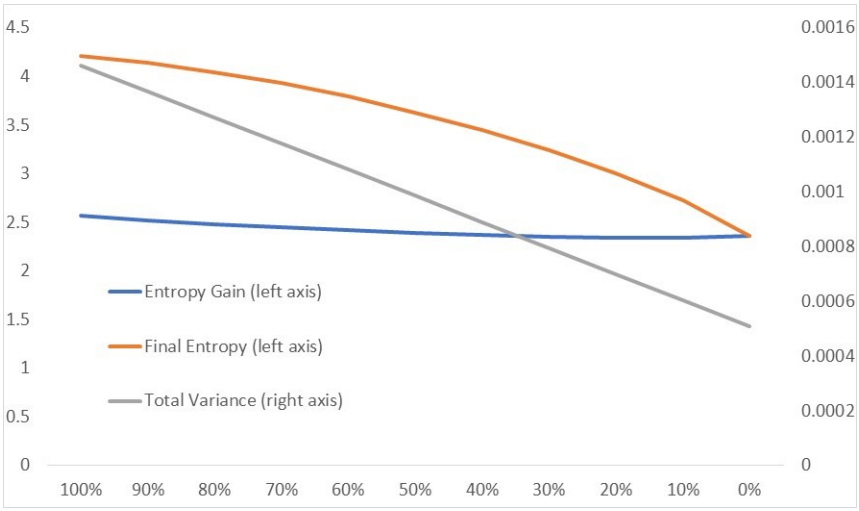

Figure 5.7: The variance for non-Gaussian case II (left vertical axis), and entropy (right vertical axis), against the value for θ used in the initial state.

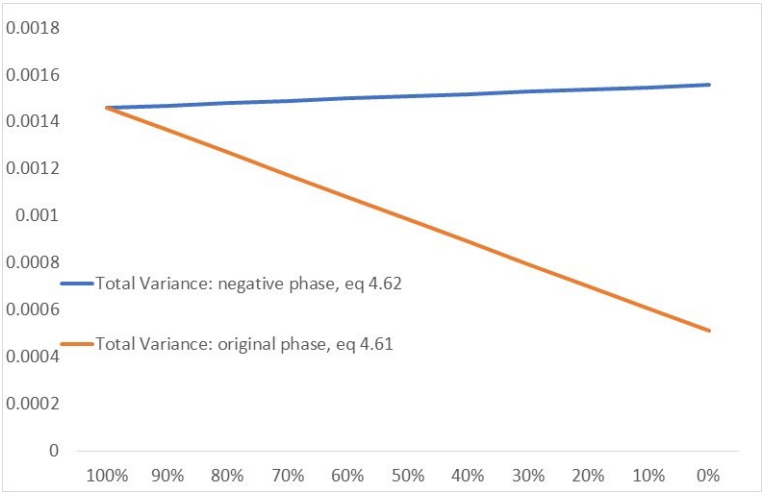

Figure 5.8: The non-Gaussian case II variance versus θ/ϕ, for the original phase (equation 5.61) and the negative phase (equation 5.62).

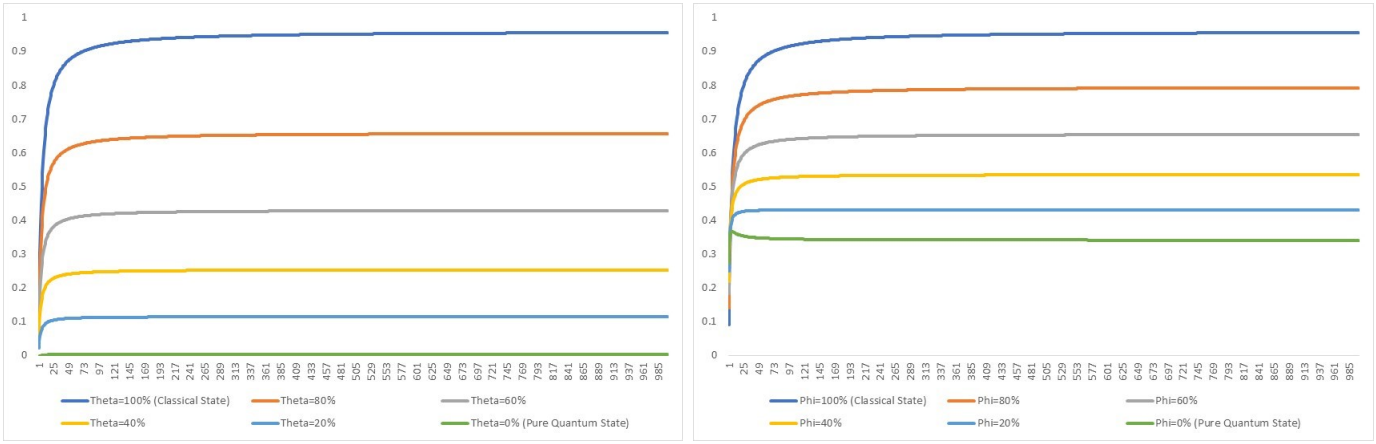

Figure 5.9: The chart shows the excess kurtosis ($Kurtosis - 3 * Variance^2$) for the original phase (left chart) and the negative phase (right chart), and case I non-Gaussian evolution

5.7.4 Excess Kurtosis:

Figure 5.9 shows the excess kurtosis for the 2 examples (original phase and negative phase), for non-Gaussian case I. Similarly figure 5.10 show the same for non-Gaussian evolution case II. These charts also illustrate the degree to which both the entropy and phase of the initial market density matrix impacts the results.

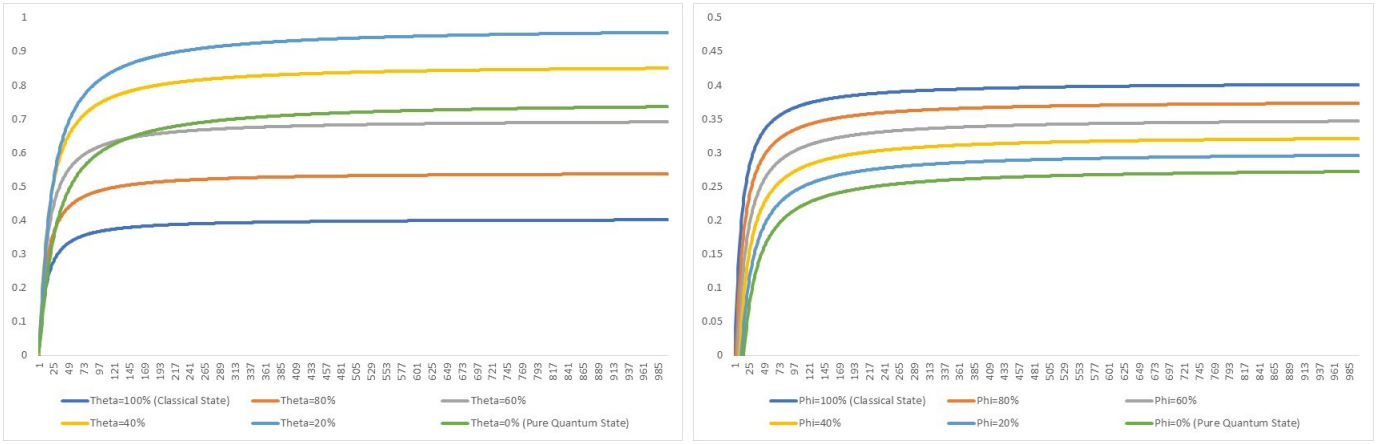

Figure 5.10: The chart shows the excess kurtosis ($Kurtosis - 3 * Variance^2$) for the original phase (left chart) and the negative phase (right chart), and case II non-Gaussian evolution

Chapter 6

Conclusion

In some respects, what we have investigated in this thesis are classically inspired quantum approaches, in the sense that the starting point for much of the research has been looking at ways of modelling the random noise and random diffusion used for much of classical financial modelling, using quantum probability. This allows us to derive, in different ways, the type of parabolic partial differential equations such as the standard heat equation:

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\Delta u$$

that crop up in finance. We can then identify the assumptions that are required for the embedding of the classical approach into a quantum framework. This is important, because we can then look at the impact of relaxing those assumptions, both in terms of the financial interpretation of doing so, and the non-classical behaviour that results. It is these effects that go beyond those observed in a classical model, and the methods used to derive them, that are the key outcomes of this research.

6.1 Quantum Stochastic Calculus:

In [16], Accardi and Boukas have shown how to use quantum stochastic calculus to develop a general quantum Black-Scholes equation. This equation can be thought of as a means of extending the rationale behind non-arbitrage pricing from the world of classical probability spaces, to the world of quantum probability and operators acting on a Hilbert space.

In this thesis, one key contribution we have made is to illustrate how one can tailor both the Hilbert space structure, and the nature of the operators acting on this Hilbert space, to capture noncommutative effects in finance. For example, we have shown how to use the direct sum of two Hilbert spaces to model both the bid & offer prices, and have illustrated how this additional structure allows for noncommutativity to be build into the model.

We have also shown how this setup can be used to tailor different order types (eg market orders vs limit orders), and have demonstrated how to introduce the trade size & market risk appetite into the trade price operators. By doing this, we have captured noncommutativity in the sense that a large market order, where the participant is willing to pay a premium for quick execution, can impact the price obtained from smaller orders. In other words, the noncommutativity arises due to different order types and sizes, in addition to the expected noncommutativity between the price & the instantaneous rate of change in the price.

The analysis culminates in section 3.7, where we have developed a 2 factor model to capture what could happen to the dynamics of a stock price, as it starts to show symptoms of illiquidity indicated by a widening of the bid-offer spread. Similarly, where bid-offer spread is small, the model resembles the geometric Brownian motion implied by the Black-Scholes model more closely. We have also shown how one can introduce a potential function for the bid-offer spread, and discussed how one can tailor the form of the bid-offer spread potential to the economics of price formation.

Another key contribution of this thesis is the consideration of what is meant by a non-arbitrage solution in the quantum context. In particular, we consider two different solution types. The first represents what we call the 'strong non-arbitrage solution'. In this case, by solving a partial differential equation, we construct a zero operator, which means that the solution applies regardless of the quantum state.
What we have found by introducing noncommutative observables, is that a solution may depend on the quantum state. In other words, one cannot necessarily identify a general solution, but instead a solution that is a result of the particular state of the market. This is important since it has enabled us to investigate cases whereby factors such as an imbalance of buyers and sellers, or a heightened market fear factor, impacts the expected price variance.
In general, where a strong non-arbitrage solution exists, it will often take the form of a function of the system coordinates, and time (being the solution to a partial differential equation). This means that the value of a particular derivative price operator acts on the Hilbert space by multiplication. For example, we could have the price operator: $\hat{V}$ acting on the Hilbert space: $L^2(\mathbb{R})$:

$$(\hat{V}\psi)(x,t) = V(x,t)\psi(x,t)$$

Thus in cases like this, different price operators $\hat{V}_i$, form a commutative algebra: $\hat{V}_i\hat{V}_j = \hat{V}_j\hat{V}_i$. In the more general case, it is the weak non-arbitrage case, where we generate noncommutative algebras, situations whereby the order in which different trades are executed, is important.

6.1.1 Further Development:

6.1.1.1 Numerical Methods:

As a general rule, the introduction of noncommutative behaviour will often lead to models that are less tractable to apply. In this thesis, we have attempted to derive results that can easily be interpreted and applied. For example deriving formulae that link the resulting variance to input financial parameters,

also proposition 3.6.2 which derives a formula for the skew & kurtosis in the noncommutative model, and the moment formula in proposition 3.5.7. We have also explored asymptotic methods in chapter 4. However, for general weak non-arbitrage solutions, the challenge is likely to be deriving appropriate mathematical methods for investigating and applying solutions in practice.

6.1.1.2 Configuring the Quantum Stochastic Process:

A quantum stochastic process is in general partly configured by the choice of an operator: L, that acts on the market Hilbert space. For example, we might choose: $L = -i\partial_x$, which one can think of as being akin to a momentum operator. When scaled by the complex valued functions in the Boson Fock space, one is essentially scaling the rate of price change by a random amount, leading to the random diffusion behaviour we expect. One key area for further research therefore, is to explore different choices for the operator L. We have briefly looked at one example in proposition 3.5.3. However, depending on the chosen Hilbert space structure, and different types of noncommutative operators, there are many other possibilities.

6.1.1.3 Non-Canonical Quantum Processes:

The quantum stochastic processes introduced by Hudson and Parthasarathy in [34], and used by Accardi and Boukas in [16] are based on the canonical commutation relations, and the Heisenberg Lie algebra. Indeed the particular representation of this algebra leads to the form of the quantum Ito formula, and the resulting natural quantum representation of the classical Brownian motion:

$$dX = idA_t - idA_t^\dagger$$

In fact, the use of the Heisenberg Lie algebra is a model assumption that must ideally be justified, and other non-canonical examples are investigated by Vilela-Mendes in [46] and by Boukas in [13]. Whilst

the canonical commutation relations are a natural place to start, exploring the use of other Lie algebras is an important avenue for future research.

6.2 Open Quantum Systems:

The importance of the open quantum systems approach to financial modelling is firstly its' ability to model the interaction between a market and its' external environment, and secondly its' ability to quantify the loss of information about what a traded price may be, as one looks further into the future.

For most practical purposes, it is sufficient to model the financial market using a finite dimensional Hilbert space: $\mathbb{C}^N$, whereby there is an orthonormal set of price eigenvectors, each of which represents a fixed price eigenvalue. The market state is then represented by a density matrix, and we consider the 'classical' states to be those diagonal matrices that ensure the system will be found in an eigenstate, even if we don't know which price eigenvalue will be the result of a trade.

In other words, a 'classical' model is one where the market state commutes with the default price operator.

A crucial observation that also motivates the use of the open quantum systems technique is that, given a known probability distribution for a price observable (acting on $\mathbb{C}^N$), it is the diagonal (or 'classical') state that maximises the Von-Neumann entropy. In other words, 'classical' models can be considered as those about which we retain the least information as we use probability models to look further and further into the future. In fact, in chapter 5, we give some simple 'toy' examples of situations whereby we might have varying levels of information regarind a market, whilst the overall uncertainty regarding a particular observable price remains the same.

6.2.1 Non Classical Extension:

In chapter 5, after introducing the basic interaction between the market and the environment, we show how to derive a Lindblad master equation that describes a 'classical' Markovian process, in the sense that a diagonal state remains diagonal. From this, we have gone on to look at two possible 'quantum' extensions, that lead to noncommutative behaviour.

Throughout chapter 5, we focus on the evolution of the quantum state (ie the Schrödinger interpretation). Ie, rather than focusing on the dynamics of an observable as one would when considering a classical stochastic process (for example assuming a traded price follows geometric Brownian motion or something similar) or when applying quantum stochastic calculus (for example using a unitary operator to evolve a traded price) we instead look at the evolution of the quantum state. In other words, rather than deriving the probability measure from our assumptions about the dynamics of a random variable, we derive the dynamics of the random variable by considering how the probability measure (or in this case the quantum state) will evolve.

This different approach enables us to consider directly factors such as how the information entropy changes, as the market evolves, and use this as a crucial measure of non-classical behaviour. In propositions 5.7.2 and 5.7.4 we look at the impact of different quantum extensions on the resulting variance, before going on to quantify the change in the Von-Neumann entropy through numerical simulations. Here an increase in the system entropy relative to the total variance, generally suggests an evolution where we retain less information regarding the future state of the market, and therefore an evolution that is closer in nature to the classical (maximum entropy) evolution.

6.2.2 Further Development:

As is the case with the quantum stochastic calculus, the central objective of the research in chapter 5, is to demonstrate techniques that can be used to extend existing classical modelling of the financial market by incorporating non-classical behaviour. Therefore, one key area for future development, is to take the framework of chapter 5, and extend this to different real world models. For example the stochastic volatility models outlined in [9], or the interest rate modelling outlined in [18].

This can be via building a model from scratch, or potentially by first representing the underlying diffusion in an open quantum systems framework, before applying the kind of extensions investigated here. The majority of stochastic processes, that are used by practitioners in the finance industry, can be mapped via an appropriate change of coordinates, to the standard heat equation (see for example [30]). Therefore, it is reasonable to expect that a good starting point for future research is to build representations of the standard 'classical' models, using the methods of chapter 5, before looking at how they react to the noncommutative extensions discussed here.

It should alsobe noted that in chapter 5, regarding numerical simulations, we have applied explicit finite difference methods to evolve the market density matrices. However, as we have noted, explicit methods can show instability in the event that the number of grid points simulated is not high enough. Therefore, the development of more advanced finite differencing techniques, together with the development of analytic or semi-analytic solutions to the underlying partial differential equations, would enable faster and more accurate results to be obtained.

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Appendix A

Detailed Derivations

A.1 Quantum Stochastic Process for The Price Operator:

A.1.1 The General Price Operator:

X is a self-adjoint operator, defined on a dense subspace of the Hilbert space: $\mathcal{H}$, and we denote the identity operator $\mathbb{I}$.

Then, $X \otimes \mathbb{I}$ is a self adjoint operator densely defined on $\mathcal{H} \otimes \Gamma(L^2(\mathbb{R}^+; \mathbb{C}))$, where $\Gamma(L^2(\mathbb{R}^+; \mathbb{C}))$ represents the symmetric Fock space of square integrable complex valued functions of time.

The time evolution of the general price operator: $X \otimes \mathbb{I}$, is determined by the unitary operator: 3.1:

$$j_t(X) = U_t^*(X \otimes \mathbb{I})U_t$$

Under this time evolution the incremental change in $j_t(X)$ is given by:

Lemma A.1.1.

$$dj_t(X) = j_t(\alpha^\dagger)dA_t^\dagger + j_t(\alpha)dA_t + j_t(\lambda)d\Lambda_t + j_t(\theta)dt \tag{A.1}$$

$$\theta = i[H,X] - \frac{1}{2}\Big(L^*LX + XL^*L - 2L^*XL\Big)$$

$$\alpha = [L^*, X]S$$

$$\alpha^\dagger = S^*[X, L]$$

$$\lambda = S^*XS - X$$

Where again, we have applied the notation convention described in 3.3.1.

Proof. We have:

$$\begin{aligned} dj_t(X) &= d(U_t^*(X \otimes \mathbb{I})U_t) \\ &= dU_t^*(X \otimes \mathbb{I})U_t + U_t^*(X \otimes \mathbb{I})dU_t + dU_t^*(X \otimes \mathbb{I})dU_t \end{aligned}$$

We apply table 3.1 to each of the three terms. First however, note that by taking the adjoint of equation 3.1:

$$dU_t^* = -U_t^*\bigg(\Big(-iH + \frac{LL^*}{2}\Big) \otimes dt + S^*L \otimes dA_t^\dagger - L^* \otimes dA_t + (\mathbb{I} - S^*) \otimes d\Lambda_t\bigg)$$

For the first term we have:

$$dU_t^*(X \otimes \mathbb{I})U_t = U_t^*\bigg(\Big(iHX - \frac{LL^*X}{2}\Big) \otimes dt - S^*LX \otimes dA_t^\dagger + L^*X \otimes dA_t - (\mathbb{I} - S^*)X \otimes d\Lambda_t\bigg)U_t$$

Similarly, for the second term, we have:

$$U_t^*(X \otimes \mathbb{I})dU_t = U_t^*\bigg(\Big(-iXH - \frac{XL^*L}{2}\Big) \otimes dt + XL \otimes dA_t^\dagger - XL^*S \otimes dA_t - X(\mathbb{I} - S) \otimes d\Lambda_t\bigg)U_t$$

For the third term, we must apply the Ito multiplication relationships (see table 3.1).

$$dU_t^*(X \otimes \mathbb{I})dU_t = -(\mathbb{I} - S^*)XL \otimes dA_t^\dagger - L^*X(\mathbb{I} - S) \otimes dA_t + (\mathbb{I} - S^*)X(\mathbb{I} - S) \otimes d\Lambda_t + L^*XL \otimes dt$$

Now collecting together the terms in dt, we have:

$$U_t^*\Big(\Big((iHX - iXH - \frac{L^*LX}{2} - \frac{XLL^*}{2} + L^*XL\Big) \otimes dt\Big)U_t$$
$$= j_t(\theta)dt$$
$$\theta = i[H, X] - \frac{1}{2}\Big(L^*LX + XL^*L - 2L^*XL\Big)$$

Similarly for the $d\Lambda_t$ terms, we have:

$$U_t^*\Big(\Big(S^*XS - X + (\mathbb{I} - S^*)X + X(\mathbb{I} - S)) - (\mathbb{I} - S^*)X - X(\mathbb{I} - S))\Big) \otimes d\Lambda_t\Big)U_t$$
$$= j_t(\lambda)d\Lambda_t$$
$$\lambda = S^*XS - X$$

For the dA_t terms, we have:

$$U_t^*\Big(\Big(L^*X - XL^*S - L^*X(\mathbb{I} - S)\Big) \otimes dA_t\Big)U_t = j_t\Big([L^*, X]S\Big)dA_t$$

Finally, for the $dA_t^\dagger$ terms, we have:

$$U_t^*\Big(\Big(- S^*LX + XL - (\mathbb{I} - S^*)XL\Big) \otimes dA_t\Big)U_t = j_t\Big(S^*[X, L]\Big)dA_t$$

□

Lemma A.1.2. *For: $k \geq 2$, we have:*

$$dj_t(X)^k = j_t(\lambda^{k-1}\alpha^\dagger)dA_t^\dagger + j_t(\alpha\lambda^{k-1})dA_t + j_t(\lambda^k)d\Lambda_t + j_t(\alpha\lambda^{k-2}\alpha^\dagger)dt \tag{A.2}$$

Proof. We proceed by induction. First, we establish the result for $k = 2$, using the Ito multiplication table and A.1.1.

$$dj_t(X)d_j(X) = j_t(\alpha)j_t(\lambda)dA_td\Lambda_t + j_t(\lambda)j_t(\alpha^\dagger)d\Lambda_tdA_t^\dagger + j_t(\lambda)j_t(\lambda)d\Lambda_td\Lambda_t + j_t(\alpha)j_t(\alpha^\dagger)dA_tdA_t^\dagger$$
$$= j_t(\alpha\lambda)dA_t + j_t(\lambda\alpha^\dagger)dA_t^\dagger + j_t(\lambda^2)d\Lambda_t + j_t(\alpha\alpha^\dagger)dt$$

So we have established the result for $k = 2$. Now assume the result holds for $k \leq N$. We have:

$$\begin{aligned} dj_t(X)d_j(X)^N &= j_t(\alpha)j_t(\lambda^N)dA_t d\Lambda_t + j_t(\lambda)j_t(\lambda^{N-1}\alpha^\dagger)d\Lambda_t dA_t^\dagger + j_t(\lambda)j_t(\lambda^N)d\Lambda_t d\Lambda_t \\ &\quad + j_t(\alpha)j_t(\lambda^{N-1}\alpha^\dagger)dA_t dA_t^\dagger \\ &= j_t(\alpha\lambda^N)dA_t + j_t(\lambda^N\alpha^\dagger)dA_t^\dagger + j_t(\lambda^{N+1})d\Lambda_t + j_t(\alpha\lambda^{N-1}\alpha^\dagger)dt \\ &= j_t(X)^{N+1} \end{aligned}$$

So, we have established the result by induction. □

Before proceeding, the following result will be useful.

Lemma A.1.3. *Let L be a linear and symmetric operator on $L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$, whose constituents consist of a linear combination of 1st order partial derivatives. Furthermore, let X be an operator such that:*

$$\begin{aligned} LX &= C \\ C &= \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \\ c &\in \mathbb{C} \end{aligned}$$

Then we have that: $XL^*L + L^*LX - 2L^*XL$ is the zero operator.

Proof. First note that by the product rule we have:

$$\begin{aligned} L^*LX|\psi\rangle &= CL^*|\psi\rangle + L^*XL|\psi\rangle \\ &= 2CL^*|\psi\rangle + XL^*L|\psi\rangle \end{aligned}$$

So that, since L is symmetric, we have:

$$L^*LX + XL^*L = 2CL|\psi\rangle + XL^*L|\psi\rangle \tag{A.3}$$

Similarly, by the product rule we have:

$$L^*XL|\psi\rangle = CL|\psi\rangle + XL^*L|\psi\rangle \tag{A.4}$$

So we see from A.3 and A.4, that $XL^*L + L^*LX = 2L^*XL$ as required. □

A.1.2 The Gaussian Price Operator:

In this section, we assume that we fix the Hilbert space to that of the direct sum: $\mathcal{H} = L^2(\mathbb{R};\mathbb{C})\oplus L^2(\mathbb{R};\mathbb{C})$, as described in section 2.4.2. We also set the Hamiltonian $H = 0$, $S = \mathbb{I}$, and:

$$L = \begin{pmatrix} -i\sigma\partial_x & 0 \\ 0 & -i\sigma\partial_x \end{pmatrix} \tag{A.5}$$
$$X = \begin{pmatrix} x+\epsilon/2 & 0 \\ 0 & x-\epsilon/2 \end{pmatrix}$$

Where, for now, ϵ is assumed to be a real valued constant. This leads to the following process for $j_t(X)$:

Lemma A.1.4. *For $\mathcal{H} = L^2(\mathbb{R};\mathbb{C}) \oplus L^2(\mathbb{R};\mathbb{C})$, $H = 0$, $S = \mathbb{I}$, and X & L given by A.5, then we have:*

$$dj_t(X) = \begin{pmatrix} i\sigma & 0 \\ 0 & i\sigma \end{pmatrix} dA_t^\dagger - \begin{pmatrix} i\sigma & 0 \\ 0 & i\sigma \end{pmatrix} dA_t \tag{A.6}$$
$$dj_t(X)^2 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} dt$$
$$dj_t(X)^k = 0,\ k \geq 3$$

Proof. Since $S = I$, the $d\Lambda_t$ terms in A.1.1 disappear. We are left with:

$$dj_t(X) = j_t([X,L])dA_t^\dagger + j_t([L^*X])dA_t + j_t(\theta)dt$$
$$\theta = -\frac{1}{2}\Big(L^*LX + XL^*L - 2L^*XL\Big)$$

We therefore have from A.1.3 that $\theta = 0$. Further, note that:

$$\begin{aligned}
[X, L] &= \begin{pmatrix} x + \epsilon/2 & 0 \\ 0 & x - \epsilon/2 \end{pmatrix} \begin{pmatrix} -i\sigma\partial_x & 0 \\ 0 & -i\sigma\partial_x \end{pmatrix} - \begin{pmatrix} -i\sigma\partial_x & 0 \\ 0 & -i\sigma\partial_x \end{pmatrix} \begin{pmatrix} x + \epsilon/2 & 0 \\ 0 & x - \epsilon/2 \end{pmatrix} \\
&= \begin{pmatrix} -i\sigma[x, \partial_x] & 0 \\ 0 & -i\sigma[x, \partial_x] \end{pmatrix} \\
&= \begin{pmatrix} i\sigma & 0 \\ 0 & i\sigma \end{pmatrix}
\end{aligned}$$

Thus we have:

$$\alpha = -\begin{pmatrix} i\sigma & 0 \\ 0 & i\sigma \end{pmatrix}, \alpha^\dagger = \begin{pmatrix} i\sigma & 0 \\ 0 & i\sigma \end{pmatrix}, \alpha\alpha^\dagger = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$$

Inserting this into lemmas A.1.1, and A.1.2, along with $\lambda = 0$, gives the result. □

In order to give the Gaussian process a nonzero drift, we can set the H operator, such that $i[H, X] \neq 0$. For example, we have:

Lemma A.1.5. *Let L, X, S, and the Hilbert space: $\mathcal{H}$ be as in lemma A.6. Further we set:*

$$H = \begin{pmatrix} i(\sigma^2/2)\partial_x & 0 \\ 0 & i(\sigma^2/2)\partial_x \end{pmatrix}$$

Then, the Gaussian price process is given by:

$$dj_t(X) = \begin{pmatrix} i\sigma & 0 \\ 0 & i\sigma \end{pmatrix} dA_t^\dagger - \begin{pmatrix} i\sigma & 0 \\ 0 & i\sigma \end{pmatrix} dA_t - \begin{pmatrix} \sigma^2/2 & 0 \\ 0 & \sigma^2/2 \end{pmatrix} dt$$

Proof. The operators α, and $\alpha^\dagger$ depend only on the operator: L, and are therefore unchanged from lemma A.6. Furthermore, since we have:

$$XL^*L + L^*LX - 2L^*XL = 0$$

Then:

$$\begin{aligned}
\theta &= i[H, X] \\
&= i \begin{pmatrix} i(\sigma^2/2)\partial_x & 0 \\ 0 & i(\sigma^2/2)\partial_x \end{pmatrix} \begin{pmatrix} x+\epsilon/2 & 0 \\ 0 & x-\epsilon/2 \end{pmatrix} - i \begin{pmatrix} x+\epsilon/2 & 0 \\ 0 & x-\epsilon/2 \end{pmatrix} \begin{pmatrix} i(\sigma^2/2)\partial_x & 0 \\ 0 & i(\sigma^2/2)\partial_x \end{pmatrix} \\
&= -(\sigma^2/2) \begin{pmatrix} [\partial_x, x] & 0 \\ 0 & [\partial_x, x] \end{pmatrix} \\
&= - \begin{pmatrix} \sigma^2/2 & 0 \\ 0 & \sigma^2/2 \end{pmatrix}
\end{aligned}$$

as required. □

A.1.3 First Extended Quantum Approach:

In this section, we derive the Gaussian price process under the risk-off market sentiment discussed in section 2.5.2. Under this approach, the Hilbert space is given by:

$$\mathcal{H} = L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$$

The bid-offer spread, given by ϵ is now an additional degree of freedom, and the derivative operator becomes:

$$L = \begin{pmatrix} -i\sigma_x\partial_x - i\sigma_\epsilon\partial_\epsilon & 0 \\ 0 & -i\sigma_x\partial_x - i\sigma_\epsilon\partial_\epsilon \end{pmatrix} \tag{A.7}$$

Finally, in section 2.5.2, it is shown that the price operator is then given by:

$$X = \begin{pmatrix} x + \cos(2\theta)\epsilon/2 & -\sin(2\theta)\epsilon/2 \\ -\sin(2\theta)\epsilon/2 & x - \cos(2\theta)\epsilon/2 \end{pmatrix} \tag{A.8}$$

Lemma A.1.6. *Assume that $H = 0$, $S = \mathbb{I}$, L is given by A.7 and X by A.8. Then Gaussian price process becomes:*

$$
\begin{aligned}
dj_t(X) &= \begin{pmatrix} i\sigma_x + i\cos(2\theta)\sigma_\epsilon/2 & -i\sin(2\theta)\sigma_\epsilon/2 \\ -i\sin(2\theta)\sigma_\epsilon/2 & i\sigma_x - i\cos(2\theta)\sigma_\epsilon/2 \end{pmatrix} dA_t^\dagger \\
&- \begin{pmatrix} i\sigma_x + i\cos(2\theta)\sigma_\epsilon/2 & -i\sin(2\theta)\sigma_\epsilon/2 \\ -i\sin(2\theta)\sigma_\epsilon/2 & i\sigma_x - i\cos(2\theta)\sigma_\epsilon/2 \end{pmatrix} dA_t \\
dj_t(X)^2 &= \begin{pmatrix} \sigma_x^2 + \sigma_\epsilon^2/4 + \cos(2\theta)\sigma_x\sigma_\epsilon & \sin(2\theta)\sigma_x\sigma_\epsilon \\ \sin(2\theta)\sigma_x\sigma_\epsilon & \sigma_x^2 + \sigma_\epsilon^2/4 - \cos(2\theta)\sigma_x\sigma_\epsilon \end{pmatrix} dt
\end{aligned}
$$

Proof.

$$
\begin{aligned}
[X, L] &= \begin{pmatrix} x + \cos(2\theta)\epsilon/2 & -\sin(2\theta)\epsilon/2 \\ -\sin(2\theta)\epsilon/2 & x - \cos(2\theta)\epsilon/2 \end{pmatrix} \begin{pmatrix} -i\sigma_x\partial_x - i\sigma_\epsilon\partial_\epsilon & 0 \\ 0 & -i\sigma_x\partial_x - i\sigma_\epsilon\partial_\epsilon \end{pmatrix} \\
&- \begin{pmatrix} -i\sigma_x\partial_x - i\sigma_\epsilon\partial_\epsilon & 0 \\ 0 & -i\sigma_x\partial_x - i\sigma_\epsilon\partial_\epsilon \end{pmatrix} \begin{pmatrix} x + \cos(2\theta)\epsilon/2 & -\sin(2\theta)\epsilon/2 \\ -\sin(2\theta)\epsilon/2 & x - \cos(2\theta)\epsilon/2 \end{pmatrix} \\
&= \begin{pmatrix} -i\sigma_x[x, \partial_x] - i\cos(2\theta)(\sigma_\epsilon/2)[\epsilon, \partial_\epsilon] & -i\sin(2\theta)(\sigma_\epsilon/2)[\epsilon, \partial_\epsilon] \\ -i\sin(2\theta)(\sigma_\epsilon/2)[\epsilon, \partial_\epsilon] & -i\sigma_x[x, \partial_x] + i\cos(2\theta)(\sigma_\epsilon/2)[\epsilon, \partial_\epsilon] \end{pmatrix} \\
&= \begin{pmatrix} i\sigma_x + i\cos(2\theta)(\sigma_\epsilon/2) & i\sin(2\theta)(\sigma_\epsilon/2) \\ i\sin(2\theta)(\sigma_\epsilon/2) & i\sigma_x - i\cos(2\theta)(\sigma_\epsilon/2) \end{pmatrix}
\end{aligned}
$$

Therefore, we have:

$$
\begin{aligned}
\alpha &= -\begin{pmatrix} i\sigma_x + i\cos(2\theta)(\sigma_\epsilon/2) & i\sin(2\theta)(\sigma_\epsilon/2) \\ i\sin(2\theta)(\sigma_\epsilon/2) & i\sigma_x - i\cos(2\theta)(\sigma_\epsilon/2) \end{pmatrix} \\
\alpha^\dagger &= \begin{pmatrix} i\sigma_x + i\cos(2\theta)(\sigma_\epsilon/2) & i\sin(2\theta)(\sigma_\epsilon/2) \\ i\sin(2\theta)(\sigma_\epsilon/2) & i\sigma_x - i\cos(2\theta)(\sigma_\epsilon/2) \end{pmatrix}
\end{aligned} \tag{A.9}
$$

We also have:

$$\alpha\alpha^\dagger = -\begin{pmatrix} i\sigma_x + i\cos(2\theta)(\sigma_\epsilon/2) & i\sin(2\theta)(\sigma_\epsilon/2) \\ i\sin(2\theta)(\sigma_\epsilon/2) & i\sigma_x - i\cos(2\theta)(\sigma_\epsilon/2) \end{pmatrix}\begin{pmatrix} i\sigma_x + i\cos(2\theta)(\sigma_\epsilon/2) & i\sin(2\theta)(\sigma_\epsilon/2) \\ i\sin(2\theta)(\sigma_\epsilon/2) & i\sigma_x - i\cos(2\theta)(\sigma_\epsilon/2) \end{pmatrix}$$
$$= \begin{pmatrix} \sigma_x^2 + \sigma_\epsilon^2/4 + \cos(2\theta)\sigma_x\sigma_\epsilon & \sin(2\theta)\sigma_x\sigma_\epsilon \\ \sin(2\theta)\sigma_x\sigma_\epsilon & \sigma_x^2 + \sigma_\epsilon^2/4 - \cos(2\theta)\sigma_x\sigma_\epsilon \end{pmatrix} \tag{A.10}$$

Since $H = 0$, if we can show $XL^*L + L^*LX - 2L^*XL = 0$, then we have $\theta = 0$. In fact, since we have:

$$LX = \begin{pmatrix} i\sigma_x + (i/2)\cos(2\theta)\sigma_\epsilon & (-i/2)\sin(2\theta)\sigma_\epsilon \\ (-i/2)\sin(2\theta)\sigma_\epsilon & i\sigma_x - (i/2)\cos(2\theta)\sigma_\epsilon \end{pmatrix}$$

we can apply lemma A.1.3, and we have $\theta = 0$. Feeding this, together with A.9 and A.10, into lemmas A.1.1 and A.1.2 gives the required result. □

A.1.4 Quantum State Dependent Volatility:

In this section we use the Hilbert space: $\mathcal{H} = L^2(\mathbb{R})$, and apply the unitary time evolution given by:

$$U_t \in \mathcal{L}\big(\mathcal{H} \otimes \Gamma\big(L^2(\mathbb{R}^+;\mathbb{C})\big)\big) \tag{A.11}$$
$$dU_t = -\Big(\frac{L^*L}{2} \otimes dt + L^* \otimes dA_t - L \otimes dA_t^\dagger\Big)U_t$$
$$L = -i\sigma\frac{\partial}{\partial x} - \nu\frac{\partial^2}{\partial x^2}$$

For $\psi \in L^2(\mathbb{R})$, we set: $(X\psi)(x) = x\psi(x)$, and write:

$$j_t(X) = U_t^*(X \times \mathbb{I})U_t$$

From proposition: 3.4.6, we have:

$$E^\psi\Big[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}j_t(\theta) + \frac{\partial^2 V}{\partial x^2}\frac{j_t(\alpha\alpha^\dagger)}{2}\Big] = 0 \tag{A.12}$$

Where:

$$j_t(\theta) = -\frac{1}{2}\Big(L^*LX + XL^*L - 2L^*XL\Big)$$
$$j_t(\alpha) = [L^*, X]$$

Proposition A.1.7. *With the unitary time evolution defined by A.11, equation A.12 becomes:*

$$E^{\psi}\left[\frac{\partial V}{\partial t} + \frac{\partial^2 V}{\partial x^2}A_{\sigma\nu}\right] = 0 \tag{A.13}$$
$$A_{\sigma\nu} = \frac{\sigma^2}{2} + 2\nu^2\frac{\partial^2}{\partial x^2} + 2i\sigma\nu\frac{\partial}{\partial x}$$

Proof. First note that:

$$XL^*L = -\sigma^2 x\partial_x^2 - 2i\sigma\nu x\partial_x^3 + \nu^2 x\partial_x^4 \tag{A.14}$$

$$L^*LX = -\sigma^2 x\partial_x^2 - 2i\sigma\nu x\partial_x^3 + \nu^2 x\partial_x^4 - 2\sigma^2\partial_x - 6i\sigma\nu\partial_x^2 + 4\nu^2\partial_x^3 \tag{A.15}$$

$$2L^*XL = -2\sigma^2 x\partial_x^2 - 4i\sigma\nu x\partial_x^3 + 2\nu^2 x\partial_x^4 - 2\sigma^2\partial_x - 4i\sigma\nu\partial_x^2 - 2i\sigma\nu\partial_x^2 + 4\nu^2\partial_x^3 \tag{A.16}$$

We find that taking equation A.14 and adding equation A.15, we end up with equation A.16. Therefore, we have:

$$j_t(\theta) = 0$$

Next note that:

$$\begin{aligned}\alpha &= [L^*, X] \\ &= -i\sigma[\partial_x, x] - \nu[\partial_x^2, x] \\ &= -i\sigma - 2\nu\partial_x\end{aligned}$$

Therefore, we have:

$$\alpha\alpha^\dagger = \sigma^2 + 4\nu^2\partial_x^2 + 4i\sigma\nu\partial_x$$

Finally note, that since σ^2, $4\nu^2\partial_x^2$, and $4i\sigma\nu\partial_x$ commute with L, we have:

$$j_t(\alpha\alpha^\dagger) = \sigma^2 + 4\nu^2\partial_x^2 + 4i\sigma\nu\partial_x$$

Inserting this into equation A.12 gives the required result. □

A.1.5 Second Extended Quantum Approach:

In this section we derive a non-Gaussian price process, where X and L are defined according to equation A.5, from section: A.1.2. However, instead of $S = \mathbb{I}$, we use:

$$S = R(\pi/2) \tag{A.17}$$

$$= \begin{pmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{A.18}$$

Lemma A.1.8. *For $\mathcal{H} = L^2(\mathbb{R};\mathbb{C}) \oplus L^2(\mathbb{R};\mathbb{C})$, $H = 0$, X & L given by A.5, and S given by A.17, then we have:*

$$dj_t(X) = \begin{pmatrix} 0 & i\sigma \\ -i\sigma & 0 \end{pmatrix} dA_t + \begin{pmatrix} 0 & i\sigma \\ -i\sigma & 0 \end{pmatrix} dA_t^\dagger + j_t \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} d\Lambda_t \tag{A.19}$$

$$\begin{aligned} dj_t(X)^k = &\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} j_t \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}^{k-2} dt + \begin{pmatrix} 0 & i\sigma \\ -i\sigma & 0 \end{pmatrix} j_t \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}^{k-1} dA_t \\ &+ j_t \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}^{k-1} \begin{pmatrix} 0 & i\sigma \\ -i\sigma & 0 \end{pmatrix} dA_t^\dagger + j_t \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}^{k} d\Lambda_t \end{aligned} \tag{A.20}$$

Proof. We have from lemma A.1.2 that:

$$[X, L] = -\begin{pmatrix} i\sigma & 0 \\ 0 & i\sigma \end{pmatrix}$$

Applying S, we get:

$$\begin{aligned} \alpha &= S * [X, L] \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -i\sigma & 0 \\ 0 & -i\sigma \end{pmatrix} \\ &= \begin{pmatrix} 0 & i\sigma \\ -i\sigma & 0 \end{pmatrix} \end{aligned} \tag{A.21}$$

Similarly, we have:

$$\begin{aligned} \alpha^\dagger &= [L^*, X]S \\ &= \begin{pmatrix} -i\sigma & 0 \\ 0 & -i\sigma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & i\sigma \\ -i\sigma & 0 \end{pmatrix} \end{aligned} \tag{A.22}$$

For λ we get:

$$\begin{aligned} S^*XS - X &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x + \epsilon/2 & 0 \\ 0 & x - \epsilon/2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} x + \epsilon/2 & 0 \\ 0 & x - \epsilon/2 \end{pmatrix} \\ &= \begin{pmatrix} x - \epsilon/2 & 0 \\ 0 & x + \epsilon/2 \end{pmatrix} - \begin{pmatrix} x + \epsilon/2 & 0 \\ 0 & x - \epsilon/2 \end{pmatrix} \\ &= \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \end{aligned} \tag{A.23}$$

Finally, note that:

$$\begin{aligned}
\alpha\lambda^{k-2}\alpha^{\dagger} &= \begin{pmatrix} 0 & i\sigma \\ -i\sigma & 0 \end{pmatrix} \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}^{k-2} \begin{pmatrix} 0 & i\sigma \\ -i\sigma & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & i\sigma \\ -i\sigma & 0 \end{pmatrix} \begin{pmatrix} (-\epsilon)^{k-2} & 0 \\ 0 & \epsilon^{k-2} \end{pmatrix} \begin{pmatrix} 0 & i\sigma \\ -i\sigma & 0 \end{pmatrix} \\
&= \begin{pmatrix} \sigma^2 & - \\ 0 & \sigma^2 \end{pmatrix} \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}^{k-2}
\end{aligned} \tag{A.24}$$

Feeding A.21, A.22, A.23 and A.24 into A.1.1, and A.1.2, gives the results for $dj_t(X)$ and $dj_t(X)^k$. □

A.2 Introduction & Financial Applications: Detailed Derivations

Proposition A.2.1. *Assume:* $\mathcal{H}_{mkt} = L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$*, and that the market initial state is defined by:*

$$|\psi\rangle = \begin{pmatrix} \psi_o(x,\epsilon) \\ \psi_b(x,\epsilon) \end{pmatrix}$$

Furthermore, in the time evolution operator given by equation: 3.1, we set:

$$L = \begin{pmatrix} -i\sigma_x\partial_x - i\sigma_\epsilon\partial_\epsilon & 0 \\ 0 & -i\sigma_x\partial_x - i\sigma_\epsilon\partial_\epsilon \end{pmatrix},\ S = R(\pi/2),\ H = 0$$

Then we have the following:

$$E^{\psi,\varepsilon}[dj_t(X_\theta)^3] = \left(\left(\sigma_x^2\cos(2\theta) + \frac{\sigma_\epsilon^2}{4}\cos(6\theta)\right)\eta_1 + \sigma_x\sigma_\epsilon\cos(4\theta)\epsilon_1 + \left(\frac{\sigma_\epsilon^2}{4}\sin(6\theta) - \sigma_x^2\sin(2\theta)\right)\gamma_1\right)dt$$

$$E^{\psi,\varepsilon}[dj_t(X_\theta)^4] = \left(\left(\sigma_x^2 + \frac{\sigma_\epsilon^2}{4}\right)\epsilon_2 + \eta_2\cos(2\theta)\sigma_x\sigma_\epsilon + \gamma_2\sin(2\theta)\sigma_x\sigma_\epsilon\right)dt$$

$$\begin{aligned}
\eta_k &= \int_{\mathbb{R}^2} \epsilon^k\big(|\psi_o(x,\epsilon)|^2 - |\psi_b(x,\epsilon)|^2\big)dxd\epsilon \\
\gamma_k &= \int_{\mathbb{R}^2} \epsilon^k\big(\overline{\psi_o(x,\epsilon)}\psi_b(x,\epsilon) + \overline{\psi_b(x,\epsilon)}\psi_o(x,\epsilon)\big)dxd\epsilon \\
\epsilon_k &= \int_{\mathbb{R}^2} \epsilon^k\big(|\psi_o(x,\epsilon)|^2 + |\psi_b(x,\epsilon)|^2\big)dxd\epsilon
\end{aligned} \tag{A.25}$$

Proof. We have from lemma A.1.2, that for: $k \geq 2$, we have:

$$dj_t(X_\theta)^k = j_t(\lambda^{k-1}\alpha^\dagger)dA_t^\dagger + j_t(\alpha\lambda^{k-1})dA_t + j_t(\lambda^k)d\Lambda_t + j_t(\alpha\lambda^{k-2}\alpha^\dagger)dt \tag{A.26}$$

We first calculate λ:

$$\begin{aligned}
\lambda &= S^*X_\theta S - X_\theta \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x + \cos(2\theta)\epsilon/2 & -\sin(2\theta)\epsilon/2 \\ -\sin(2\theta)\epsilon/2 & x - \cos(2\theta)\epsilon/2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\cos(2\theta)\epsilon & \sin(2\theta)\epsilon \\ \sin(2\theta)\epsilon & \cos(2\theta)\epsilon \end{pmatrix}
\end{aligned}$$

Next from equation A.9 we have:

$$\begin{aligned}
\alpha &= [L^*, X_\theta]S \\
&= -\begin{pmatrix} i\sigma_x + i\cos(2\theta)(\sigma_\epsilon/2) & i\sin(2\theta)(\sigma_\epsilon/2) \\ i\sin(2\theta)(\sigma_\epsilon/2) & i\sigma_x - i\cos(2\theta)(\sigma_\epsilon/2) \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\alpha^\dagger &= S^*[X_\theta, L] \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i\sigma_x + i\cos(2\theta)(\sigma_\epsilon/2) & i\sin(2\theta)(\sigma_\epsilon/2) \\ i\sin(2\theta)(\sigma_\epsilon/2) & i\sigma_x - i\cos(2\theta)(\sigma_\epsilon/2) \end{pmatrix}
\end{aligned}$$

Taking expectations of equation A.26, we have:

$$E^{\psi,\varepsilon}[dj_t(X_\theta^k] = E^{\psi,\varepsilon}[j_t(\alpha\lambda^{k-2}\alpha^\dagger)]dt$$

For $k = 3$, we have:

$$\alpha\lambda\alpha^\dagger = -\begin{pmatrix} i\sigma_x + i\cos(2\theta)\frac{\sigma_\epsilon}{2} & i\sin(2\theta)\frac{\sigma_\epsilon}{2} \\ i\sin(2\theta)\frac{\sigma_\epsilon}{2} & i\sigma_x - i\cos(2\theta)\frac{\sigma_\epsilon}{2} \end{pmatrix} M(\theta) \begin{pmatrix} i\sigma_x + i\cos(2\theta)\frac{\sigma_\epsilon}{2} & i\sin(2\theta)\frac{\sigma_\epsilon}{2} \\ i\sin(2\theta)\frac{\sigma_\epsilon}{2} & i\sigma_x - i\cos(2\theta)\frac{\sigma_\epsilon}{2} \end{pmatrix}$$

$$M(\theta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\cos(2\theta)\epsilon & \sin(2\theta)\epsilon \\ \sin(2\theta)\epsilon & \cos(2\theta)\epsilon \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\theta)\epsilon & -\sin(2\theta)\epsilon \\ -\sin(2\theta)\epsilon & -\cos(2\theta)\epsilon \end{pmatrix}$$

Therefore, we have:

$$\alpha\lambda\alpha^\dagger = \begin{pmatrix} \sigma_x^2\epsilon\cos(2\theta) + \frac{\sigma_\epsilon^2}{4}\epsilon\cos(6\theta) + \sigma_x\sigma_\epsilon\epsilon\cos(4\theta) & \frac{\sigma_\epsilon^2}{4}\epsilon\sin(6\theta) - \sigma_x^2\sin(2\theta)\epsilon \\ \frac{\sigma_\epsilon^2}{4}\epsilon\sin(6\theta) - \sigma_x^2\sin(2\theta)\epsilon & -\sigma_x^2\epsilon\cos(2\theta) - \frac{\sigma_\epsilon^2}{4}\epsilon\cos(6\theta) + \sigma_x\sigma_\epsilon\epsilon\cos(4\theta) \end{pmatrix}$$

Taking expectations we get:

$$E^{\psi,\varepsilon}[\alpha\lambda\alpha^\dagger] = \begin{pmatrix} \sigma_x^2\epsilon\cos(2\theta) + \frac{\sigma_\epsilon^2}{4}\epsilon\cos(6\theta) + \sigma_x\sigma_\epsilon\epsilon\cos(4\theta) & \frac{\sigma_\epsilon^2}{4}\epsilon\sin(6\theta) - \sigma_x^2\sin(2\theta)\epsilon \\ \frac{\sigma_\epsilon^2}{4}\epsilon\sin(6\theta) - \sigma_x^2\sin(2\theta)\epsilon & -\sigma_x^2\epsilon\cos(2\theta) - \frac{\sigma_\epsilon^2}{4}\epsilon\cos(6\theta) + \sigma_x\sigma_\epsilon\epsilon\cos(4\theta) \end{pmatrix}$$

$$= \left(\left(\sigma_x^2\cos(2\theta) + \frac{\sigma_\epsilon^2}{4}\cos(6\theta)\right)\eta_1 + \sigma_x\sigma_\epsilon\cos(4\theta)\epsilon_1 + \left(\frac{\sigma_\epsilon^2}{4}\sin(6\theta) - \sigma_x^2\sin(2\theta)\right)\gamma_1\right) \qquad \text{(A.27)}$$

The result follows from inserting equation A.27 into equation A.26. For $k = 4$: $\lambda^2 = \begin{pmatrix} \epsilon^2 & 0 \\ 0 & \epsilon^2 \end{pmatrix}$. Therefore:

$$\alpha\lambda^2\alpha^\dagger = -\begin{pmatrix} i\sigma_x + i\cos(2\theta)\frac{\sigma_\epsilon}{2} & i\sin(2\theta)\frac{\sigma_\epsilon}{2} \\ i\sin(2\theta)\frac{\sigma_\epsilon}{2} & i\sigma_x - i\cos(2\theta)\frac{\sigma_\epsilon}{2} \end{pmatrix} \begin{pmatrix} \epsilon^2 & 0 \\ 0 & \epsilon^2 \end{pmatrix} \begin{pmatrix} i\sigma_x + i\cos(2\theta)\frac{\sigma_\epsilon}{2} & i\sin(2\theta)\frac{\sigma_\epsilon}{2} \\ i\sin(2\theta)\frac{\sigma_\epsilon}{2} & i\sigma_x - i\cos(2\theta)\frac{\sigma_\epsilon}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_x^2\epsilon^2 + \frac{\sigma_\epsilon^2\epsilon^2}{4} + \cos(2\theta)\sigma_x\sigma_\epsilon\epsilon^2 & \sin(2\theta)\sigma_x\sigma_\epsilon\epsilon^2 \\ \sin(2\theta)\sigma_x\sigma_\epsilon\epsilon^2 & \sigma_x^2\epsilon^2 + \frac{\sigma_\epsilon^2\epsilon^2}{4} - \cos(2\theta)\sigma_x\sigma_\epsilon\epsilon^2 \end{pmatrix}$$

Taking expectations, we get:

$$\begin{aligned}E^{\psi,\varepsilon}[\alpha\lambda^2\alpha^\dagger] &= \int_{\mathbb{R}^2}\epsilon^2\big(\sigma_x^2+(\sigma_\epsilon^2/4)\big)\big(|\psi_o(x,\epsilon)|^2+|\psi_b(x,\epsilon)|^2\big)dxd\epsilon\\ &+\int_{\mathbb{R}^2}\epsilon^2\cos(2\theta)\sigma_x\sigma_\epsilon\big(|\psi_o(x,\epsilon)|^2-|\psi_b(x,\epsilon)|^2\big)dxd\epsilon\\ &+\int_{\mathbb{R}^2}\epsilon^2\big(\overline{\psi_o(x,\epsilon)}\psi_b(x,\epsilon)+\overline{\psi_o(x,\epsilon)}\psi_b(x,\epsilon)\big)dxd\epsilon\end{aligned}$$

□

A.3 Open Quantum Systems: Detailed Derivations

Proposition A.3.1. *Let the price operator X be given by equation 5.6, the vector P_H by equation 5.68, and the initial state: ρ_0, by equation 5.69, then under the time evolution given by proposition 5.5.2, the rate of change in the total variance is given by:*

$$\begin{aligned}\frac{\partial(E^{\rho_0}[X^2])}{\partial t} &= \sigma^2\sum_{i=2}^{N-1}x_i^2h_0^2(a_{(i+1)(i+1)}+a_{(i-1)(i-1)}-2a_{ii})+\sum_{i=3}^{N-2}x_i^2h_1^2(a_{(i+2)(i+2)}+a_{(i-2)(i-2)}-2a_{ii})\\ &+\sigma^2\sum_{i=3}^{N-2}h_0h_1x_i^2(a_{(i+1)(i+2)}+a_{(i+2)(i+1)}+a_{(i-1)(i-2)}+a_{(i-2)(i-1)}\\ &-a_{i(i+1)}+a_{(i+1)i}+a_{i(i-1)}+a_{(i-1)i})\end{aligned}\tag{A.28}$$

Proof. First note that:

$$\begin{aligned}\frac{\partial(E^{\rho_0}[X^2])}{\partial t} &= \frac{\partial(Tr[X^2\rho_0(t)])}{\partial t}\\ &= \frac{\partial}{\partial t}\Big(\sum_{i=1}^{N}x_i^2a_{ii}(t)\Big)\\ &= \sum_{i=1}^{N}x_i^2\frac{\partial a_{ii}(t)}{\partial t}\\ &= Tr\Big[X^2\frac{\partial\rho_0(t)}{\partial t}\Big]\end{aligned}\tag{A.29}$$

Where the third line of A.29 follows from the fact that under the system Hamiltonian 5.10 we have:

$$e^{-iH_{syst}t}Xe^{iH_{syst}t} = X$$

Applying proposition 5.5.2 to equation A.29, we get:

$$\frac{\partial(E^{\rho_0}[X^2])}{\partial t} = \sigma^2 Tr\Big[X^2(A_u^H\rho_0 A_d^H + A_d^H\rho_0 A_u^H - \frac{1}{2}\{A_u^H A_d^H + A_d^H A_u^H, \rho_0\}\Big] \tag{A.30}$$

We now calculate the terms in A.29 using definition 5.5.1, and proposition 5.5.2. Under 5.68 we have:

$$\begin{aligned} A_d^H &= \sum_{i=1}^{N-1}\big(h_0|f_i\rangle\langle f_{i+1}| + h_{-1}|f_{i+1}\rangle\langle f_{i+1}|\big) + \sum_{i=2}^{N-1} h_1|f_{i-1}\rangle\langle f_{i+1}| \\ A_u^H &= \sum_{i=1}^{N-1}\big(h_0|f_{i+1}\rangle\langle f_i| + h_{-1}|f_{i+1}\rangle\langle f_{i+1}|\big) + \sum_{i=2}^{N-1} h_1|f_{i+1}\rangle\langle f_{i-1}| \end{aligned} \tag{A.31}$$

Since $\rho_0 = \sum_{i,j=1}^{N} a_{ij}|f_i\rangle\langle f_j|$, first we consider an individual $A_u^H a_{ij}|f_i\rangle\langle e_j|A_d^H$ term. We get:

$$\begin{aligned} A_u^H a_{ij}|f_i\rangle\langle f_j|A_d^H = a_{ij}\Big(&h_0^2|f_{i+1}\rangle\langle f_{j+1}| + h_1^2|f_{i+2}\rangle\langle f_{j+2}| + h_{-1}^2|f_i\rangle\langle f_i| \\ &+ h_0h_1(|f_{i+2}\rangle\langle f_{j+1}| + |f_{i+1}\rangle\langle f_{j+2}|) + h_0h_{-1}(|f_i\rangle\langle f_{j+1}| + |f_{i+1}\rangle\langle f_j|) \\ &+ h_{-1}h_1(|f_i\rangle\langle f_{j+2}| + |f_{i+2}\rangle\langle f_j|)\Big) \end{aligned} \tag{A.32}$$

Collecting the diagonal terms from A.32, we get:

$$\begin{aligned} A_u^H\rho_0 A_d^H = &\sum_{i=1}^{N-1} a_{ii}h_0^2|f_{i+1}\rangle\langle f_{i+1}| + \sum_{i=1}^{N-2} a_{ii}h_1^2|f_{i+2}\rangle\langle f_{i+2}| + \sum_{i=1}^{N} a_{ii}h_{-1}^2|f_i\rangle\langle f_i| \\ &+ \sum_{i=1}^{N-2}(a_{i(i+1)} + a_{(i+1)i})h_0h_1|f_{i+2}\rangle\langle f_{i+2}| + \sum_{i=1}^{N-1}(a_{i(i+1)} + a_{(i+1)i})h_0h_{-1}|f_{i+1}\rangle\langle f_{i+1}| \\ &+ \sum_{i=1}^{N-2}(a_{i(i+2)} + a_{(i+2)i})h_{-1}h_1|f_{i+2}\rangle\langle f_{i+2}| \end{aligned} \tag{A.33}$$

Similarly, collecting together the diagonal terms from $A_d^H a_{ij}|e_i\rangle\langle e_j|A_u^H$, we get:

$$\begin{aligned} A_d^H \rho_0 A_u^H = &\sum_{i=2}^{N} a_{ii}h_0^2|f_{i-1}\rangle\langle f_{i-1}| + \sum_{i=3}^{N} a_{ii}h_1^2|f_{i-2}\rangle\langle f_{i-2}| + \sum_{i\geq 1} a_{ii}h_{-1}^2|f_i\rangle\langle f_i| \\ &+ \sum_{i=2}^{N-1}(a_{i(i+1)} + a_{(i+1)i})h_0h_1|f_{i-1}\rangle\langle f_{i-1}| + \sum_{i=1}^{N-1}(a_{i(i+1)} + a_{(i+1)i})h_0h_{-1}|f_i\rangle\langle f_i| \\ &+ \sum_{i=1}^{N-2}(a_{i(i+2)} + a_{(i+2)i})h_{-1}h_1|f_i\rangle\langle f_i| \end{aligned} \tag{A.34}$$

We now consider the individual $A_u^H A_d^H a_{ij}|f_i\rangle\langle f_j|$, and $A_u^H A_d^H a_{ij}|f_i\rangle\langle f_j|$ terms.

$$\begin{aligned} A_u^H A_d^H a_{ij}|f_i\rangle\langle f_j| &= A_d^H A_u^H a_{ij}|f_i\rangle\langle f_j| \\ &= a_{ij}\Big((h_{-1}^2 + h_0^2 + h_1^2)|f_i\rangle\langle f_j| + (h_0h_1 + h_{-1}h_0)(|f_{i+1}\rangle\langle f_j| + |f_{i-1}\rangle\langle f_j|) \\ &+ h_{-1}h_1(|f_{i+2}\rangle\langle f_j| + |f_{i-2}\rangle\langle f_j|)\Big) \end{aligned}$$

Again, collecting together the diagonal terms, we get:

$$\begin{aligned} \frac{1}{2}(A_d^H A_u^H + A_u^H A_d^H)\rho_0 = &\sum_{i=1}^{N} a_{ii}(h_{-1}^2 + h_0^2 + h_1^2)|f_i\rangle\langle f_i| \\ &+ \sum_{i=1}^{N-1}(h_0h_1 + h_{-1}h_0)(a_{i(i+1)}|f_{i+1}\rangle\langle f_{i+1}| + a_{(i+1)i}|f_i\rangle\langle f_i|) \\ &+ \sum_{i=1}^{N-2} h_{-1}h_1(a_{(i+2)i}|f_i\rangle\langle f_i| + a_{i(i+2)}|f_{i+2}\rangle\langle f_{i+2}|) \end{aligned} \tag{A.35}$$

Finally, we consider the individual $a_{ij}|f_i\rangle\langle f_j|A_u^H A_d^H$ terms.

$$\begin{aligned} a_{ij}|f_i\rangle\langle f_j|A_u^H A_d^H &= a_{ij}|f_i\rangle\langle f_j|A_d^H A_u^H \\ &= a_{ij}\Big((h_{-1}^2 + h_0^2 + h_1^2)|f_i\rangle\langle f_j| + (h_0h_1 + h_{-1}h_0)(|f_i\rangle\langle f_{j+1}| + |f_i\rangle\langle f_{j-1}|) \\ &+ h_{-1}h_1(|f_i\rangle\langle f_{j+2}| + |f_i\rangle\langle f_{j-2}|)\Big) \end{aligned}$$

So that for the diagonal terms we get:

$$\begin{aligned}\frac{1}{2}\rho_0(A_d^H A_u^H + A_u^H A_d^H) = &\sum_{i=1}^{N} a_{ii}(h_{-1}^2 + h_0^2 + h_1^2)|f_i\rangle\langle f_i| \\ &+ \sum_{i=1}^{N-1}(h_0h_1 + h_{-1}h_0)(a_{(i+1)i}|f_{i+1}\rangle\langle f_{i+1}| + a_{i(i+1)}|f_i\rangle\langle f_i|) \\ &+ \sum_{i=1}^{N-2} h_{-1}h_1(a_{(i+2)i}|f_{i+2}\rangle\langle f_{i+2}| + a_{i(i+2)}|f_i\rangle\langle f_i|\end{aligned} \tag{A.36}$$

We now feed equations A.33, A.34, A.35 and A.36 into equation A.30. We group the terms together by the coefficients of h_ih_j. First note, that the terms in h_{-1}^2 cancel to zero, and for h_0^2 and h_1^2, we get:

$$\sigma^2 \sum_{i=3}^{N-2} h_0^2 x_i^2(a_{(i+1)(i+1)} + a_{(i-1)(i-1)} - 2a_{ii}) + h_1^2 x_i^2(a_{(i+2)(i+2)} + a_{(i-2)(i-2)} - 2a_{ii}) \tag{A.37}$$

The terms in $h_{-1}h_0$ and $h_{-1}h_1$ also cancel out, leaving the terms in h_0h_1:

$$\begin{aligned}\sigma^2 \sum_{i=3}^{N-2} h_0h_1x_i^2(&a_{(i+1)(i+2)} + a_{(i+2)(i+1)} + a_{(i-1)(i-2)} + a_{(i-2)(i-1)} \\ &- a_{i(i+1)} + a_{(i+1)i} + a_{i(i-1)} + a_{(i-1)i})\end{aligned} \tag{A.38}$$

So that finally we have:

$$\frac{\partial(E^{\rho_0}[X^2])}{\partial t} = \sigma^2 \sum_{i=2}^{N-1} x_i^2 h_0^2(a_{(i+1)(i+1)} + a_{(i-1)(i-1)} - 2a_{ii}) + \sum_{i=3}^{N-2} x_i^2 h_1^2(a_{(i+2)(i+2)} + a_{(i-2)(i-2)} - 2a_{ii}) \tag{A.39}$$

$$\begin{aligned}&+ \sigma^2 \sum_{i=3}^{N-2} h_0h_1x_i^2(a_{(i+1)(i+2)} + a_{(i+2)(i+1)} + a_{(i-1)(i-2)} + a_{(i-2)(i-1)} \\ &- a_{i(i+1)} + a_{(i+1)i} + a_{i(i-1)} + a_{(i-1)i})\end{aligned}$$

□

Proposition A.3.2. *Let the price operator X be given by equation 5.6, and the initial state: ρ_0, by equation 5.71, then under the time evolution given by proposition 5.6.1, the rate of change in the total*

variance is given by:

$$\frac{\partial(E^{\rho_0}[X^2])}{\partial t} = \sigma^2 \sum_{i=2}^{N-1} x_i^2 (a_{(i+1)(i+1)} + a_{(i-1)(i-1)} - 2a_{ii}) \tag{A.40}$$
$$+ \nu_u^2 \sum_{i=3}^{N-2} x_i^2 \Big(a_{(i-1)(i+1)} - \frac{1}{2}\big(a_{(i-2)i} + a_{i(i+2)}\big)\Big) + \nu_d^2 \sum_{i=3}^{N-2} x_i^2 \Big(a_{(i+1)(i-1)} - \frac{1}{2}\big(a_{i(i-2)} + a_{(i+2)i}\big)\Big)$$

Proof. From equation A.29, we have that:

$$\frac{\partial(E^{\rho_0}[X^2])}{\partial t} = Tr\Big[X^2 \frac{\partial \rho_0(t)}{\partial t}\Big] \tag{A.41}$$

Applying proposition 5.6.1 to equation A.41, we get:

$$\begin{aligned}\frac{\partial(E^{\rho_0}[X^2])}{\partial t} &= \sigma^2 Tr\Big[X^2\Big(A_u \rho_{mkt}(t) A_d + A_d \rho_{mkt}(t) A_u \\ &\quad - \frac{1}{2}\{A_u A_d + A_d A_u, \rho_{mkt}(t)\}\Big)\Big] \\ &+ \nu_u^2 Tr\Big[X^2\Big(A_u \rho_{mkt}(t) A_u - \frac{1}{2}\{A_u A_u, \rho_{mkt}(t)\}\Big)\Big] \\ &+ \nu_d^2 Tr\Big[X^2\Big(A_d \rho_{mkt}(t) A_d - \frac{1}{2}\{A_d A_d, \rho_{mkt}(t)\}\Big)\Big]\end{aligned} \tag{A.42}$$

Under definition 5.4.12 and equation 5.71, we have:

$$\begin{aligned}&Tr\Big[X^2\Big(A_u \rho_{mkt}(t) A_d + A_d \rho_{mkt}(t) A_u - \frac{1}{2}\{A_u A_d + A_d A_u, \rho_{mkt}(t)\}\Big)\Big] \\ &\quad = \sum_{i=2}^{N-1} x_i^2 (a_{(i+1)(i+1)} + a_{(i-1)(i-1)} - 2a_{ii}) \\ &Tr\Big[X^2\Big(A_u \rho_{mkt}(t) A_u - \frac{1}{2}\{A_u A_u, \rho_{mkt}(t)\}\Big)\Big] \\ &\quad = \sum_{i=3}^{N-2} x_i^2 \Big(a_{(i-1)(i+1)} - \frac{1}{2}\big(a_{(i-2)i} + a_{i(i+2)}\big)\Big) \\ &Tr\Big[X^2\Big(A_d \rho_{mkt}(t) A_d - \frac{1}{2}\{A_d A_d, \rho_{mkt}(t)\}\Big)\Big] \\ &\quad = \sum_{i=3}^{N-2} x_i^2 \Big(a_{(i+1)(i-1)} - \frac{1}{2}\big(a_{i(i-2)} + a_{(i+2)i}\big)\Big)\end{aligned}$$

The result follows by feeding this into proposition 5.6.1. □