# Hypergraphs, Existential Closure, and Related Problems

by

© **Robert D. Luther**

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# Abstract


In this thesis, we present results from multiple projects with the theme of extending results from graphs to hypergraphs. We first discuss the existential closure property in graphs, a property that is known to hold for most graphs but in practice, examples of these graphs are hard to find. Specifically, we focus on finding necessary conditions for the existence of existentially closed line graphs and line graphs of hypergraphs. We then present constructions for generating infinite families of existentially closed line graphs. Interestingly, when restricting ourselves to existentially closed planar line graphs, we find that there are only finitely many such graphs.

Next, we consider the notion of an existentially closed hypergraph, a novel concept that retains many of the necessary properties of an existentially closed graph. Again, we present constructions for generating infinitely many existentially closed hypergraphs. These constructions use combinatorial designs as the key ingredients, adding to the expansive list of applications of combinatorial designs.

Finally, we extend a classical result of Mader concerning the edge-connectivity of vertex-transitive graphs to linear uniform vertex-transitive hypergraphs. Additionally, we show that if either the linear or uniform properties are absent, then we can generate infinite families of vertex-transitive hypergraphs that do not satisfy the conclusion of the generalised theorem.

Dedicated to my family and friends for being a constant source of positivity in my life and to Melanie for always believing in me.

# Lay summary

In mathematics, a graph is a mathematical structure that can be used to model a relationship between pairs of distinct objects. These objects are referred to as vertices and the relationships between pairs are represented by edges. For example, if we consider the intersections at street corners as our vertices and the streets that connect them as the edges, then we can identify and study the structure of a city map as a mathematical graph.

Much of the focus of this thesis is on hypergraphs, which are a natural generalisation of graphs. Unlike a graph in which edges join pairs of vertices together, in a hypergraph the edges can join together sets of vertices of arbitrary size. We initially focus on a property known as existential closure, a property that has been proven to be present within most graphs. However, in practice, actually finding examples of graphs with this property is historically a difficult problem.

We begin our investigation by examining the existential closure property in some specific families of graphs, notably, line graphs and line graphs of planar graphs. In the general case, we provide constructions for generating infinitely many examples whereas, for line graphs of planar graphs, we prove that there are precisely five such graphs. We then define the notion of an existentially closed hypergraph and provide constructions to generate these graphs as well. Finally, we shift our focus to another well-studied property of hypergraphs, that of edge-connectivity. In particular, we extend a classic result from graphs to hypergraphs.

# Acknowledgements

First and foremost I would like to thank my supervisors, Dr. David Pike and Dr. Andrea Burgess, for their continuous support throughout the entirety of my doctoral program. Their guidance academically (and occasionally emotionally) is something that can never be repaid and will always be appreciated. With your support I have become not just a better mathematician but also a better writer and communicator.

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Finally, I would like to thank my thesis examination members as well as my supervisory committee members.

# Statement of contribution

Unless specifically cited, all results in Chapters 2 to 4 are original contributions based on collaborative work with Dr. Andrea Burgess and Dr. David Pike. Chapter 2 is based on a paper accepted for publication in Graphs and Combinatorics [9]. Chapter 3 is based on recent work which will be submitted for publication once appropriate. Chapter 4 is based on a paper published in the Journal of Graph Theory [8].

Supervision and assistance while editing this thesis were provided by Dr. Andrea Burgess and Dr. David Pike.

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# Chapter 1

# Introduction

A *graph* is a pair of sets $(V, E)$ such that $V$ is a set of distinct elements called *vertices* and $E$ is a set of unordered pairs of vertices called *edges*. If $G$ is a graph, we often write $V(G)$ to denote the vertex set and $E(G)$ to denote the edge set. When a pair of vertices occur as an edge, we say that the two vertices are *adjacent*. We use the same term to describe two edges that share a common vertex. When a vertex $v$ is a member of an edge $e$, we say that $v$ and $e$ are *incident* with each other. Formally, a graph is defined as a pair of sets; however, graphs are often best expressed by a drawing. In a drawing of a graph, the vertices are often depicted as small circles, and the edges are represented by lines drawn between the vertices. For example, the graph in Figure 1.1 is the complete graph on four vertices, typically denoted $K_4$. We say that this graph is *complete* since there is an edge between every pair of vertices.

Two specific families of graphs that we will discuss are line graphs and planar graphs. If $G$ is a graph, the *line graph* of $G$, denoted $L(G)$, is the graph whose vertices are the edges of $G$ and two vertices in $L(G)$ are adjacent if and only if they correspond to adjacent edges in $G$. Line graphs often provide a way to describe relationships between the edges of a graph

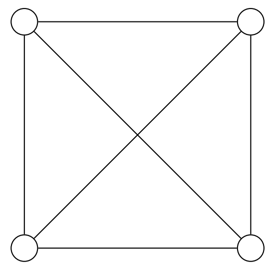

Figure 1.1: The complete graph $K_4$

in a language that graph theorists are familiar with. A *planar* graph is a graph that can be embedded in the plane, that is, it can be drawn in the plane in such a way that no two edges cross each other. Such a drawing is called a *planar embedding* of the graph. For example, Figure 1.2 is a planar embedding of $K_4$ whereas Figure 1.1 is not a planar embedding of $K_4$. Also, Figure 1.3 is a depiction of the line graph of $K_4$, which happens to be planar. Note that the vertices of $L(K_4)$ are precisely the edges of $K_4$.

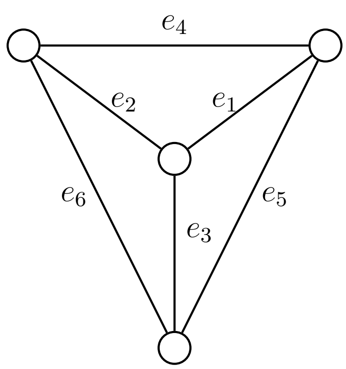


Figure 1.2: A planar embedding of $K_4$

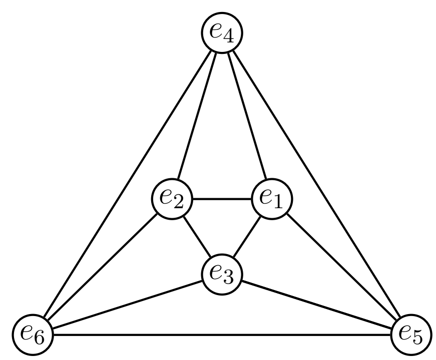


Figure 1.3: A planar embedding of $L(K_4)$

A *hypergraph* is a generalisation of the concept of a graph in which the edge set $E$ is permitted to contain any subset of the vertices in $V$, not just pairs. Formally, a hypergraph $H$ is a pair $(V, E)$ such that $V$ is a set of distinct elements called vertices and $E$ is a collection of subsets of $V$ called *hyperedges* or simply edges. Note that these edges may not all be of the same size in general. However, when every edge in a hypergraph $H$ is of the same cardinality $k$ we say that $H$ is a *uniform* hypergraph or more specifically a *k-uniform* hypergraph.

Additionally, if a hypergraph $H$ has the property that any pair of vertices is contained in at most one edge of $H$ then we say that $H$ is a *linear* hypergraph. For example, Figure 1.4 is a hypergraph on five vertices, having two edges of size three in which any pair of vertices appears in at most one edge. So this hypergraph is a linear 3-uniform hypergraph.

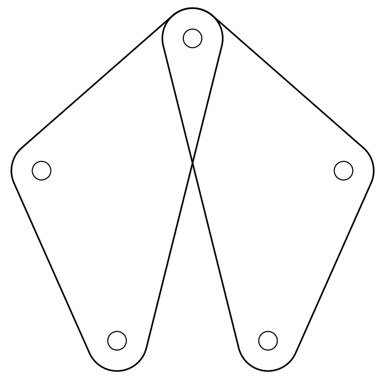

Figure 1.4: A linear 3-uniform hypergraph

Our primary focus in Chapters 2 and 3 is on the existential closure property in certain families of graphs and how to extend this property to hypergraphs. In Chapter 4 we extend a particular connectivity property in certain graphs to similarly structured hypergraphs.

In the next section of the present chapter, we state some standard definitions within graph theory that will be useful throughout this thesis; any additional definitions or terminology will be introduced as needed. For a more comprehensive background on graph theory including standard notation, see [43].

## 1.1 Background Terminology

Let $G = (V, E)$ be a graph and let $u$ and $v$ be two vertices of $V(G)$. The size of the vertex set, $|V(G)|$ is called the *order* of $G$. We say that $u$ is a *neighbour* of $v$ if $u$ and $v$ are adjacent. The set of all neighbours of $v$, denoted $N(v)$, is often called the *neighbourhood* of $v$. If $e = \{u, v\}$ is an edge of $G$, we say that $u$ and $v$ are the *endpoints* of the edge $e$. If

$e = \{u, v\}$ and $f = \{u, v\}$ are two copies of the same edge of $G$, then $e$ and $f$ are said to be *duplicates* of each other. Additionally, if $e = \{v, v\}$ is an edge such that both endpoints are the same vertex, we call $e$ a *loop*. A graph which contains neither duplicate edges nor loops is called *simple*. Two graphs $G$ and $H$ are *isomorphic* if there exists a bijective function $\phi : V(G) \to V(H)$ such that $u$ and $v$ are adjacent in $G$ if and only if $\phi(u)$ and $\phi(v)$ are adjacent in $H$. We say that a graph $H = (V', E')$ is a *subgraph* of $G$, if $V' \subseteq V$ and $E' \subseteq E$; that is, every vertex of $H$ is also a vertex of $G$ and every edge of $H$ is also an edge of $G$. Let $X \subseteq V(G)$ be a subset of the vertices of $G$. The subgraph *induced* by $X$ in $G$, denoted $G[X]$, is the subgraph whose vertex set is $X$ and whose edge set consists of every edge of $G$ which has both endpoints in $X$. The complement graph of $G$, denoted $G^{\mathsf{c}}$, is the graph on the same set of vertices as $G$ such that two vertices are adjacent in the complement precisely when they are not adjacent in $G$. The subgraph $G - v$ of $G$ is the subgraph with vertex set $V(G) \setminus \{v\}$ and edge set equal to the edge set of $G$ with all edges incident with $v$ removed. A *trail* in a graph is an alternating sequence of incident vertices and edges that begins and ends with vertices. If all the vertices in a trail are distinct then we call the trail a *path* and if only the first and last vertices are equal then we call the trail a *cycle*. The path on $n$ vertices is often denoted $P_n$ and the cycle on $n$ vertices is denoted $C_n$.

## 1.2 The Existential Closure Property

For a positive integer $n$, a graph with at least $n$ vertices is *$n$-existentially closed* or simply *$n$-e.c.* if for any set of vertices $S$ of size $n$ and any set $T \subseteq S$, there is a vertex $x \notin S$ adjacent to each vertex of $T$ and no vertex of $S \setminus T$. We say that $x$ is *correctly joined* to $T$ and $S \setminus T$. Hence, for each $n$-subset $S$ of vertices, there exist $2^n$ vertices joined to $S$ in all possible ways. For example, a 1-e.c. graph is one with neither isolated nor universal vertices.

If a graph has the $n$-e.c. property, then it possesses other structural properties such as the following.

**Theorem 1.1.** *[5] Let $G$ be an $n$-e.c. graph where $n$ is a positive integer.*

1. *The graph $G$ is $m$-e.c. for all $1 \leqslant m \leqslant n-1$.*
2. *The graph $G$ has order at least $n + 2^n$, and has at least $n2^{n-1}$ edges.*
3. *The complement of $G$ is $n$-e.c.*
4. *Each graph of order at most $n+1$ occurs as an induced subgraph in $G$.*
5. *If $n > 1$, then for each vertex $x$ of $G$, each of the graphs $G-x$, the subgraph induced by the neighbourhood $N(x)$, and the subgraph induced by $(V(G)\backslash N(x))\backslash\{x\}$ are $(n-1)$-e.c.*

Some examples of $n$-e.c. graphs include the three non-isomorphic 1-e.c. graphs of minimum order 4, depicted in Figure 1.5, and the 2-e.c. graph of order 9 depicted in Figure 1.6. In [10] it was shown that the minimum order of a 2-e.c. graph is nine and in [6] it was established that $K_3 \square K_3$ is the unique 2-e.c. graph on nine vertices; the graph $G \square H$ is called the *cartesian product* of $G$ and $H$ and is the graph on the vertex set $V(G) \times V(H)$ such that two vertices $(u, v)$ and $(u', v')$ are adjacent if and only if $u = u'$ and $v$ is adjacent to $v'$ in $H$ or $v = v'$ and $u$ is adjacent to $u'$ in $G$.

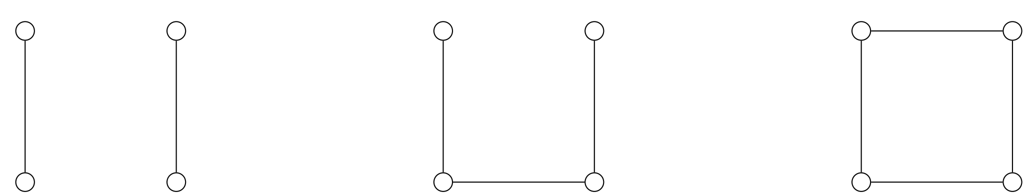

Figure 1.5: The non-isomorphic 1-e.c. graphs of minimum order 4: $2K_2$, $P_4$, and $C_4$

In Chapter 2 we investigate the existential closure property in line graphs. When studying graph properties that represent structure between vertices, it is often interesting to consider

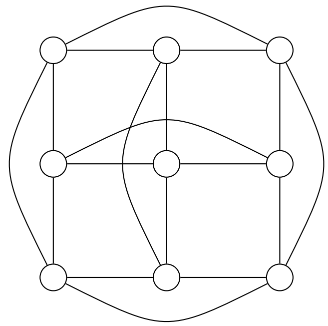

Figure 1.6: $K_3 \square K_3$ - the unique 2-e.c. graph on 9 vertices

how a similar structure can be described in terms of the edges of a graph. Furthermore, when investigating the structure between the edges of a graph, it is often convenient to examine the corresponding line graph.

One such graph property that can be considered on both the vertices and edges is graph colouring. For example, the *chromatic number* of a graph $G$, denoted $\chi(G)$, is the smallest number of colours required to colour the vertices of $G$ such that no two vertices of the same colour are adjacent, whereas the *chromatic index* of $G$, denoted $\chi'(G)$, is the smallest number of colours required to colour the edges of $G$ such that no two edges of the same colour are incident with the same vertex. Note that by definition, for a graph $G$, the chromatic index $\chi'(G)$ is equal to the chromatic number of its line graph, $\chi(L(G))$.

Another structural property with similar behaviour is that of independence. Let $\alpha(G)$ denote the maximum number of pairwise non-adjacent vertices in a graph $G$, and similarly let $\alpha'(G)$ denote the maximum number of pairwise non-adjacent edges. It is easy to observe that $\alpha'(G) = \alpha(L(G))$ for any graph $G$.

In a similar spirit, we define a version of existential closure expressed in terms of the edges of a graph. For a positive integer $n$, we say that a graph is *$n$-line existentially closed* or simply *$n$-line e.c.* if, for any set of edges $S$ of size $n$ and any set $T \subseteq S$, there is an edge $e \notin S$ adjacent to each edge of $T$ and no edge of $S \setminus T$. By definition, a graph is $n$-line e.c. if and only if its line graph is $n$-e.c. and so, our main focus in Chapter 2 will be investigating the $n$-e.c. property in line graphs. In [26], the notation $\Xi(G)$ was first introduced to represent the largest integer $n$ for which the graph $G$ is $n$-e.c. We may similarly define $\Xi'(G)$ to be the largest integer $n$ for which the graph $G$ is $n$-line e.c. and observe that $\Xi'(G) = \Xi(L(G))$.

We continue by finding necessary conditions and providing constructions to generate infinite families of $n$-line e.c. graphs. We then look more specifically at $n$-line e.c. planar graphs and conclude that there are precisely five graphs that are simultaneously 2-line e.c. and planar. Next, we discuss the line graphs of uniform hypergraphs, once again finding some necessary conditions for their existence and providing constructions to generate infinite families of 2-line e.c. hypergraphs.

In Chapter 3 we extend the notion of an $n$-existentially closed graph to uniform hypergraphs as follows. For a $k$-uniform hypergraph $H$, we say that $H$ is *$n$-e.c.* if, for any set of vertices $S$ of size $n$ and any set $T \subseteq S$, there is a set of vertices $X \subseteq V(H) \setminus S$ of size $k-1$ such that for all $z \in T$, $X \cup \{z\}$ is an edge of $H$ and for all $s \in S \setminus T$, $X \cup \{s\}$ is not an edge of $H$. Note that for $k = 2$, this definition agrees with the usual notion of an existentially closed graph.

For a $k$-uniform hypergraph $H$ to be 1-e.c., for each vertex $x$, the hypergraph must have at least one edge containing $x$ and there must exist at least one set of vertices of size $k-1$ which does not form an edge with $x$. The smallest example of a 1-e.c. $k$-uniform hypergraph can be observed by considering the hypergraph on $k+1$ vertices, with two edges that share exactly $k-1$ vertices in common. For example, Figure 1.7 depicts the smallest 1-e.c. 3-uniform hypergraph both in the number of edges and vertices.

We continue in Chapter 3 by identifying multiple necessary conditions for the existence of existentially closed graphs which extend naturally to existentially closed hypergraphs. We also prove that random uniform hypergraphs are asymptotically existentially closed and we

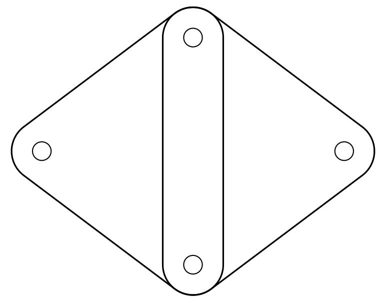

Figure 1.7: The smallest 1-e.c. 3-uniform hypergraph

present constructions for building $n$-e.c. hypergraphs given the existence of certain combinatorial designs.

## 1.3 Edge-Connectivity in Hypergraphs

A graph or hypergraph is *connected* if there exists a path between each pair of vertices. A *cut set* of edges in a graph or hypergraph is a set of edges whose deletion renders the graph or hypergraph disconnected. The *edge-connectivity* of a graph or hypergraph $H$ is the size of a minimum cut set of edges and is denoted $\kappa'(H)$. For a graph or hypergraph $H$, $\delta(H)$ is the minimum degree among the vertices and $\Delta(H)$ is the maximum degree among the vertices, where the degree of a vertex is the number of edges incident with it. For example, the graph $G$ in Figure 1.8 has $\delta(G) = 2$, $\Delta(G) = 3$, and $\kappa'(G) = 1$ since deleting edge $e$ would render the resulting graph disconnected.

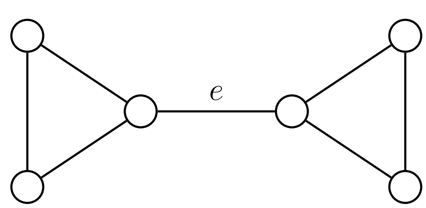


Figure 1.8: A graph $G$ with $\kappa'(G) = 1$

In [44], Whitney observes that, for a graph $G$, $\kappa'(G)$ never exceeds $\delta(G)$, a result that extends naturally to hypergraphs. This bound is in fact tight and a graph or hypergraph $H$ which satisfies $\kappa'(H) = \delta(H)$ is said to be *maximally edge-connected*. For example the graph in Figure 1.8 is not maximally edge-connected. Hellwig and Volkmann list several sufficient conditions for graphs to be maximally edge-connected in their 2008 survey [24].

The subject of connectivity in hypergraphs has been developing recently with results like those in [1, 14, 27]. In [1], Bahmanian and Šajna study various connectivity properties in hypergraphs with an emphasis on cut sets of edges and vertices. In [14], Dewar, Pike and Proos consider both vertex and edge-connectivity in hypergraphs with additional details on the computational complexity of these problems. In [27], Jami and Szigeti investigate the edge-connectivity of permutation hypergraphs. Dankelmann and Meierling extend several well-known sufficient conditions for graphs to be maximally edge-connected to the realm of hypergraphs in [12]. Tong and Shan continue this work with more extensions from graphs to hypergraphs in [41]. Zhao and Meng present sufficient conditions for linear uniform hypergraphs to be maximally edge-connected that generalise results from graphs in [48]. The latter three papers were primarily focused on the properties of distance and girth.

In Chapter 4, we continue our theme of extending results from graphs to hypergraphs by investigating the edge-connectivity of vertex-transitive hypergraphs. A graph or hypergraph $H$ is said to be *vertex-transitive* if, for any two vertices $u$ and $v$ of $V(H)$, there exists some automorphism $\phi$ of $H$ such that $\phi(u) = v$. A graph *automorphism* is an isomorphism (a bijective function between vertex sets that preserves adjacencies) from the graph (or hypergraph) to itself. A classic result of Mader establishes that vertex-transitive graphs are maximally edge-connected [34]. We generalise this result and prove that any linear uniform vertex-transitive hypergraph is maximally edge-connected. We also demonstrate the existence of vertex-transitive hypergraphs which fail to be maximally edge-connected when we relax either the uniformity or linearity conditions of our hypothesis.

# Chapter 2

# Existential Closure in Line Graphs

We begin with a recollection of some definitions and terminology. For a positive integer $n$, a graph with at least $n$ vertices is *$n$-existentially closed* or simply *$n$-e.c.* if for any set of vertices $S$ of size $n$ and any set $T \subseteq S$, there is a vertex $x \notin S$ adjacent to each vertex of $T$ and no vertex of $S \setminus T$. In this case, we say that $x$ is correctly joined to $T$ and $S \setminus T$. The line graph of a graph $G$ is the graph $L(G)$ whose vertices are the edges of $G$ and two vertices in $L(G)$ are adjacent if and only if they correspond to adjacent edges in $G$. Two edges are said to be adjacent when they share at least one end-vertex. We say that a graph is $n$-line e.c. if its corresponding line graph is an $n$-e.c. graph.

The only graphs that fail to be 1-line e.c. are those containing an edge adjacent to every other edge and disconnected graphs that contain a connected component consisting of a single edge. For this reason, we will only consider connected graphs in this chapter. Note that if a graph $G$ has a duplicate edge, then it would be impossible to find a third edge adjacent to one but not the other, and therefore $G$ would not satisfy the 2-line e.c. property. Similarly, if $G$ has a loop at vertex $v$, consider any other edge incident with $v$ and note that no third edge would be adjacent to the loop but not the other edge and so $G$ would not satisfy the 2-line e.c. property. For these reasons, as we continue our exploration of $n$-line e.c. graphs with values of $n$ greater than 1, all graphs we discuss are assumed to be simple.

For small values of $n$, examples of $n$-line e.c. graphs are easy to find. For instance, the graphs $C_4$ and $K_4$ are the only 1-line e.c. graphs of minimum order 4. The graph $K_{3,3}$ is the unique 2-line e.c. graph with 9 edges since it is the only graph with corresponding line graph $K_3 \square K_3$, which is the unique 2-e.c. graph on 9 vertices [6]. A particularly useful result that is easily verified is the following.

**Lemma 2.1.** *The complete graph $K_\ell$ is 2-line e.c. when $\ell \geqslant 6$ and the complete bipartite graph $K_{m,n}$ is 2-line e.c. when $m, n \geqslant 3$.*

In Section 2.1.1, we focus on finding necessary conditions for the existence of $n$-line e.c. graphs. In particular, we prove that if $G$ is $n$-line e.c. then $n$ is at most 2, narrowing our focus to finding examples of 2-line e.c. graphs. In Section 2.1.2, we present constructions that generate infinite families of 2-line e.c. graphs, and in Section 2.2, we prove that there are precisely five graphs which are both 2-line e.c. and planar. Lastly, in Section 2.3, we introduce the problem of existential closure in the line graphs of hypergraphs and present constructions for 2-line e.c. hypergraphs.

## 2.1 $n$-Line e.c. Graphs

When studying a combinatorial object it is often natural to ask what the necessary and sufficient conditions are for such an object to exist. In this section, we look closely at these conditions to ultimately construct more examples of $n$-line e.c. graphs.

### 2.1.1 Necessary Conditions

When studying $n$-line e.c. graphs, some immediate necessary conditions can be observed from parts 1 and 2 of Theorem 1.1.

**Theorem 2.2.** *If $G$ is an $n$-line e.c. graph, then $G$ is $m$-line e.c. for all $1 \leqslant m \leqslant n-1$ and $G$ has at least $n+2^n$ edges.*

The following theorem poses a heavy restriction on the existence of $n$-line e.c. graphs.

**Theorem 2.3.** *Let $G$ be a graph. If $G$ is $n$-line e.c. then $n \leqslant 2$. Alternatively, if $G$ is a graph, then $\Xi'(G) \leqslant 2$.*

**Proof.** By Theorem 2.2, we know that an $n$-line e.c. graph is also $m$-line e.c. for $1 \leqslant m \leqslant n$. So it suffices to show that $G$ cannot be 3-line e.c.

Suppose $G$ is 3-line e.c. and let $e_0$ be an edge of $G$. Since $G$ is also 1-line e.c., there exists an edge $e_1$ not adjacent to $e_0$. Also, since $G$ is 2-line e.c., there exists an edge $e_2$ not adjacent to $e_0$ or $e_1$. Finally, since $G$ is 3-line e.c., there must exist a fourth edge, adjacent to each of the previous three distinct edges, no two of which are adjacent. This is impossible, so $G$ cannot be 3-line e.c. $\square$

As an alternative proof of Theorem 2.3 consider the following: suppose $G$ is a 3-line e.c. graph, that is, $L(G)$ is a 3-e.c. graph. Note that each graph of order four must occur as an induced subgraph in $L(G)$ by Theorem 1 part 4. In particular, $L(G)$ must contain $K_{1,3}$ as an induced subgraph, but this is impossible since all line graphs are necessarily claw-free [2].

Due to the implication of Theorem 2.3, our attention will now focus specifically on 2-line e.c. graphs. A graph $G$ is 2-line e.c. if and only if for each pair of distinct edges $e, f \in E(G)$, the following hold:

($i$) there is another edge adjacent to both $e$ and $f$,

($ii$) there is another edge adjacent to neither $e$ nor $f$, and

($iii$) there is another edge adjacent to $e$ but not to $f$ and vice versa.

Furthermore, for graphs with minimum degree at least three, condition ($iii$) is implied by conditions ($i$) and ($ii$). Suppose $e = \{u, x\}$ and $f = \{v, x\}$ are two adjacent edges in a 2-line e.c. graph $G$. Since $\delta(G) \geqslant 3$, vertex $u$ has at least three neighbours, at most two of which could be $x$ and $v$. So there must exist an edge adjacent to $e$ but not to $f$. Otherwise, letting $e$ and $f$ be two disjoint edges, condition ($ii$) implies there exists a third edge $g$ adjacent to neither $e$ nor $f$. Now apply condition ($i$) to $e$ and $g$ and observe that this edge is adjacent to $e$ but not to $f$. Therefore, when checking a graph for the 2-line e.c. property, it is sufficient to verify only conditions ($i$) and ($ii$).

This simplification of the definition leads to the observation of several consequential properties, one of which involves the concept of diameter in a graph. The *diameter* of a graph is the maximum value of the lengths of shortest paths between all pairs of vertices.

**Lemma 2.4.** *Let $G$ be a 2-line e.c. graph. Then the minimum degree of $G$ is at least three and the diameter of $G$ is at most three.*

**Proof.** Suppose $x$ is a vertex of degree one in $G$, let $e$ be the edge incident with $x$ and let $f$ be any other edge in the graph. Applying condition ($i$) to $e$ and $f$ gives a third edge $g$ adjacent to $e$ and applying ($iii$) to $e$ and $g$ forces a second edge to be incident with $x$, so $x$ must have degree at least two.

Now suppose $x$ is a vertex of degree two in $G$ and let $e$ and $f$ be distinct edges incident with $x$. By applying condition ($i$) to $e$ and $f$, there must exist a third edge $g$ which forms a triangle with $e$ and $f$. Now apply condition ($iii$) to $e$ and $g$ and observe that $x$ must have degree at least three.

In addition, note that any pair of disjoint edges must have a common neighbouring edge, so the length of a shortest path between any two vertices is at most three. □

A particularly useful observation is that if $G$ is a 2-line e.c. graph, then every matching of size two in $G$ is a subgraph of a matching of size three (where a *matching* is a set of disjoint edges), as well as a path of length three.

**Lemma 2.5.** *Let $G$ be a 2-line e.c. graph. Then every matching of size two in $G$ is contained in a matching of size three, and in a path of length three.*

Consequently, $G$ has no induced matching of size two. Such graphs are often referred to as $2K_2$-free graphs; for characterisations of these graphs see [15] and [36]. Therefore, the class of 2-line e.c. graphs is contained within the class of $2K_2$-free graphs.

### 2.1.2 Constructing 2-Line e.c. Graphs

With a specific focus on 2-line e.c. graphs, we are able to develop some constructions for producing an infinite collection of such graphs. These constructions use the notion of the join of two graphs. For two distinct graphs $G$ and $H$, the *join* of $G$ and $H$, denoted $G \vee H$, is the graph on the vertex set $V(G) \cup V(H)$ with edges consisting of $E(G) \cup E(H)$ along with an edge between every vertex of $G$ and every vertex of $H$. Note that $H$ may consist of a single vertex, say $x$, in this case, we write $G \vee x$ to denote the join of $G$ and $H$.

**Theorem 2.6.** *Let $G$ be a 2-line e.c. graph and let $x \notin V(G)$ be a new vertex. Then the join $G \vee x$ is a 2-line e.c. graph.*

**Proof.** We must verify that each pair of edges of $G \vee x$ satisfies the 2-line e.c. property. Any two edges of $G$ retain the 2-line e.c. property. For any two edges incident with $x$, say $e = \{u, x\}$ and $f = \{v, x\}$, any third edge incident with $x$ is adjacent to both $e$ and $f$, and any matching of size at least three in $G$ contains at least one edge which is adjacent to neither $e$ nor $f$.

Now let $e = \{u, x\}$ and $f$ be an edge of $G$. There is an edge adjacent to both since $x$ is adjacent to each vertex of $G$ and there is an edge adjacent to neither since $f$ is a member of a matching in $G$ of size at least three. $\square$

We can extend this result further to allow the addition of any number of new vertices to a 2-line e.c graph.

**Theorem 2.7.** *Let $G$ be a 2-line e.c. graph. Join to $G$ a set $S$ of independent vertices of size $|S| \geqslant 2$ such that each vertex $x \in S$ is adjacent to every vertex of $G$. The resulting graph $G'$ is a 2-line e.c. graph.*

**Proof.** We must verify that each pair of edges of $G'$ satisfies the 2-line e.c. property. By the proof of Theorem 2.6, the only pairs of edges left to check are pairs of the form $e = \{x_1, u\}, f = \{x_2, v\}$ where $x_1, x_2 \in S$ and $u, v \in V(G)$.

To find an edge that is adjacent to neither $e$ nor $f$, simply identify a matching of size three in $G$ and observe that $e$ and $f$ can be adjacent to at most two of the edges of the matching. To find an edge that is adjacent to both, take the edge $\{x_1, v\}$ if $u \neq v$ and any edge incident with $u$ in $G$ if $u = v$. $\square$

As an immediate corollary of Theorem 2.7 and by our previous observation that $K_{m,n}$ is 2-line e.c. for $m, n \geqslant 3$ (Lemma 2.1), we can establish the following.

**Corollary 2.8.** *Any complete multipartite graph with minimum part size at least three is a 2-line e.c. graph.*

We can generalise Theorem 2.7 even further to show that the join of two 2-line e.c. graphs is itself a 2-line e.c. graph.

**Theorem 2.9.** *Let $G_1$ and $G_2$ be two 2-line e.c. graphs each with at least three vertices. Then the resulting graph $G' = G_1 \vee G_2$ is a 2-line e.c. graph.*

**Proof.** We must verify that each pair of edges of $G'$ satisfies the 2-line e.c. property. Any two edges of $G_1$ preserve the 2-line e.c. property, likewise for $G_2$. By Corollary 2.8, any two edges between vertices of $G_1$ and vertices of $G_2$ have the 2-line e.c. property.

Let $e = \{u, v\}$ be an edge of $G_1$ and $f = \{w, x\}$ be an edge with $w \in V(G_1)$ and $x \in V(G_2)$. Either $\{u, x\}$ or $\{v, x\}$ serves as an edge which is adjacent to both $e$ and $f$. Since $e$ is part of a matching of size three in $G_1$, we can find at least one edge that is adjacent to neither $e$ nor $f$. Similar arguments verify that the 2-line e.c. property holds between an edge of $G_2$ and an edge between $G_1$ and $G_2$.

Now let $e = \{u, v\}$ be an edge of $G_1$ and let $f = \{x, y\}$ be an edge of $G_2$. Let $w \in V(G_1)$ and $z \in V(G_2)$ be additional distinct vertices. Note that $\{u, x\}$ is adjacent to both $e$ and $f$, and $\{w, z\}$ is adjacent to neither $e$ nor $f$. $\square$

## 2.2 2-Line e.c. Planar Graphs

Recall that a planar graph is a graph that can be drawn in the plane such that no two edges cross each other. Lemma 2.4 establishes that a 2-line e.c. graph necessarily has diameter at most three. Small diameter in the context of planar graphs imposes upper bounds on the graph orders in terms of the maximum degree $\Delta$.

**Theorem 2.10.** *[23] If $G$ is planar, has diameter 2, and $\Delta \geqslant 8$ then $|V(G)| \leqslant \lfloor \frac{3\Delta}{2} \rfloor + 1$.*

**Theorem 2.11.** *[18] If $G$ is planar and has diameter 3, then $|V(G)| \leqslant 8\Delta + 12$.*

At the same time, planar graphs cannot have a high average degree (a classical application of Euler's Formula shows that the average degree must be below six) and hence they must be relatively sparse. In terms of diameter, the size of a planar graph is also bounded.

**Theorem 2.12.** *[20] If $G$ is a connected planar graph with diameter $D$ then $|E(G)| \leqslant 4|V(G)| - 4 - 3D$.*

Despite these restrictions, infinite families of planar graphs with small diameter are known to exist, for instance, wheel graphs and windmill graphs. This inspired us to ask whether families of planar 2-line e.c. graphs exist as well. However, as we shall see shortly, there are only finitely many such graphs as Theorem 2.15 establishes an upper bound on the order of a planar 2-line e.c. graph.

A classic theorem of Kuratowski characterises planar graphs in terms of forbidden homeomorphic subgraphs. Two graphs are said to be *homeomorphic* if one can be obtained from the other by the subdivision of edges (adding a new vertex to the middle of an existing edge).

**Theorem 2.13.** *[33] A graph $G$ is planar if and only if $G$ does not contain a subgraph homeomorphic to $K_5$ or $K_{3,3}$.*

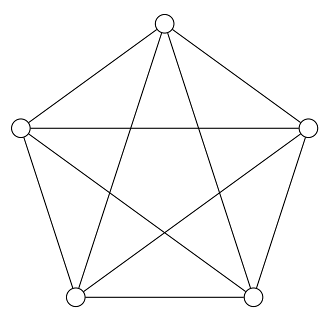

Figure 2.1: The graph $K_5$

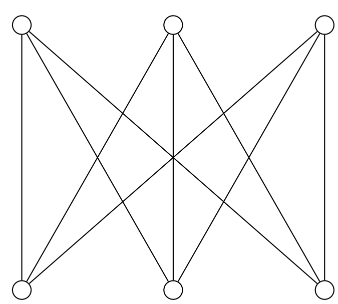

Figure 2.2: The graph $K_{3,3}$

The graphs depicted in Figures 2.1 and 2.2 are the complete graph on five vertices $K_5$ and the complete bipartite graph $K_{3,3}$ respectively. A classic application of Euler's Formula shows that these two graphs are non-planar. An equivalent statement to Kuratowski's due to

Wagner uses the language of graph minors, where a graph $G$ is a *minor* of some other graph $G'$ if $G$ can be obtained from $G'$ using any combination of vertex deletion, edge deletion, or contracting two adjacent vertices to a single vertex.

**Theorem 2.14.** *[42] A graph $G$ is planar if and only if $G$ contains neither $K_5$ nor $K_{3,3}$ as a graph minor.*

The proof of Theorem 2.15 uses the language of graph minors and so will rely on Wagner's Theorem.

**Theorem 2.15.** *If $G$ is a 2-line e.c. planar graph then $|V(G)| \leqslant 12$.*

**Proof.** We proceed by examining the possible sizes of matchings in such a graph $G$. It will be useful to recall from Lemma 2.4 that in a 2-line e.c. graph, the minimum degree, $\delta$, is at least three. Also, by Lemma 2.5, the size of a maximum matching is at least three.

Let $M$ be a maximum matching in $G$. Since $G$ is 2-line e.c., each pair of edges of $M$ must have a common neighbouring edge. So by contracting each edge of $M$, we can observe that $G$ contains $K_{|M|}$ as a minor. If $|M| \geqslant 5$, then $G$ contains $K_5$ as a minor and is therefore not a planar graph. Hence $|M| \leqslant 4$.

Since $M$ is maximum, every edge of $E(G) \setminus M$ must share at least one end-vertex with an edge of $M$. Let $V_M$ be the set of end-vertices of the edges of $M$. Now consider the set of vertices $V(G) \setminus V_M$. We will partition this set into two disjoint sets of vertices, called Type 1 and Type 2 vertices. Precisely, Type 1 vertices are vertices of $V(G) \setminus V_M$ which are each adjacent to both end-vertices of at least one edge of $M$ and Type 2 vertices are vertices of $V(G) \setminus V_M$ which are adjacent to at most one end-vertex of each edge of $M$. Note that since $M$ is maximum, every neighbour of each vertex of $V(G) \setminus V_M$ is a member of $V_M$.

To prove that $|V(G)| \leqslant 12$, we count the total number of possible Type 1 and Type 2 vertices that $G$ could contain. First, suppose $M = \{e_1, e_2, e_3, e_4\}$, and $u$ is a Type 1 vertex.

Without loss of generality, $G$ contains a subgraph with a structure represented by Figure 2.3. Note that, by definition, the Type 1 vertex $u$ is adjacent to both end-vertices of some edge of $M$ and since $\delta(G) \geqslant 3$, $u$ must be adjacent to at least one other vertex of $V_M$.

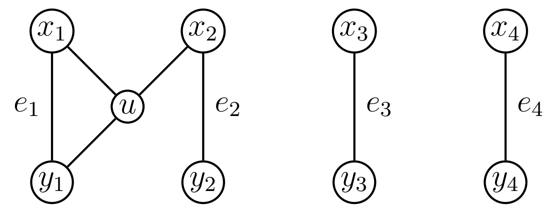


Figure 2.3: The matching $M$ and a Type 1 vertex $u$

Now suppose $v$ is a second Type 1 vertex. If $v$ is adjacent to both end-vertices of $e_1$, then we may form a larger matching $M^* = \{\{v, x_1\}, \{y_1, u\}, e_2, e_3, e_4\}$. If $v$ is adjacent to both end-vertices of $e_2$, then we may form a larger matching $M^* = \{e_1, \{u, x_2\}, \{y_2, v\}, e_3, e_4\}$. Now if $v$ is adjacent to both end-vertices of $e_3$ (or $e_4$), then we first observe that there must be an edge adjacent to both $e_1$ and $e_3$ (or $e_4$). Without loss of generality, let this edge be $\{x_1, x_3\}$ (or $\{x_1, x_4\}$). Then we may form a larger matching $M^* = \{\{y_1, u\}, \{x_1, x_3\}, e_2, \{y_3, v\}, e_4\}$ (or $M^* = \{\{y_1, u\}, \{x_1, x_4\}, e_2, e_3, \{y_4, v\}\}$). So when $|M| = 4$, there can be at most one Type 1 vertex. A similar argument shows that when $|M| = 3$, there is at most one Type 1 vertex as well.

Now suppose for an edge $e = \{x, y\}$ of $M$, $G$ has two Type 2 vertices $u$ and $v$ such that $\{u, x\}$ and $\{v, y\}$ are edges. Then we can augment $M$ by replacing $e$ with the pair of edges $\{u, x\}$ and $\{v, y\}$. So for each edge $e$ of $M$, there is at most one end-vertex of $e$ adjacent to Type 2 vertices. Consequently, if $|M| = 3$, $G$ can have at most two Type 2 vertices as three or more Type 2 vertices would force a $K_{3,3}$ subgraph between the vertices of $V_M$ and the Type 2 vertices. So if $|M| = 3$, $G$ may contain at most nine vertices: six vertices of $V_M$, at most one Type 1 vertex, and at most two Type 2 vertices.

Suppose $|M| = 4$ and $u$ is a Type 1 vertex; the general structure can be observed in

Figure 2.3. If $v$ is a Type 2 vertex adjacent to an end-vertex of $e_1$ (say $x_1$) then we may form a larger matching $M^* = \{\{x_1, v\}, \{y_1, u\}, e_2, e_3, e_4\}$. So no Type 2 vertex can be adjacent to any end-vertex of $e_1$. Therefore, any Type 2 vertices may only be adjacent to end-vertices of the edges $e_2$, $e_3$, $e_4$. Since for each edge of $M$, there is at most one end-vertex adjacent to Type 2 vertices, there are precisely three vertices that are possible neighbours for a Type 2 vertex, one for each edge of $M$ other than $e_1$. In this case, there can be at most two Type 2 vertices, since three or more would force a $K_{3,3}$ subgraph between the vertices of $V_M$ and the Type 2 vertices. So if $|M| = 4$ and $G$ contains a Type 1 vertex, $G$ may contain at most eleven vertices: eight vertices of $V_M$, one Type 1 vertex, and at most two Type 2 vertices.

Now suppose $|M| = 4$ and $G$ contains no Type 1 vertex. By our previous observation, for each edge of $M$, there is at most one end-vertex adjacent to Type 2 vertices. So a Type 2 vertex has only four possible neighbours, namely one end-vertex from each edge of $M$. Note that at most two Type 2 vertices may share three common neighbours among the vertices of $V_M$ since otherwise, $G$ would contain a $K_{3,3}$ subgraph between the Type 2 vertices and the three common neighbours of $V_M$. Now suppose that there are two such Type 2 vertices having three common neighbours, say $x, y, z \in V_M$. Then any additional Type 2 vertex must share exactly two of these three common neighbours, say $x$ and $y$. This vertex's third neighbour lies on a path with $z$ of length three consisting of two edges of $M$ and an adjacent edge shared between them since, by Lemma 2.5, every pair of disjoint edges is contained in a path of length 3. By contracting this path, we observe a $K_{3,3}$ minor between the Type 2 vertices and the vertices of $V_M$. Finally, assuming no pair of Type 2 vertices have three common neighbours, we conclude that there can be at most four Type 2 vertices since this is the number of distinct 3-subsets of a 4-set. So if $|M| = 4$ and $G$ does not contain any Type 1 vertices, $G$ may contain at most twelve vertices: eight vertices of $V_M$ and at most four Type 2 vertices. $\square$

By computer search of all planar graphs up to order 12, we established that there are precisely five 2-line e.c. planar graphs. These are the graphs named Tc20, Tc30, Tc39, Tc43, and Tc44, on page 246 of [39]. For a planar representation of each graph, see Figures 2.4 to 2.8. Table 2.1 lists the rows of the lower triangle of the adjacency matrix for each graph. This search was aided by the program `plantri`, authored by Brinkmann and McKay [7]. The code used to test these graphs can be found in Appendix A. The graphs Tc20, Tc30, and Tc39 have faces of sizes three and four whereas the graphs Tc43 and Tc44 are triangulations. Tc43 and Tc44 are also named the heptahedral graph 34 and the Johnson solid skeleton 13 respectively. The heptahedral graphs were first enumerated by Kirkman [31] and Hermes [25] and the Johnson solid skeleton is the planar embedding of the pentagonal bipyramid $J_{1,3}$ [28].

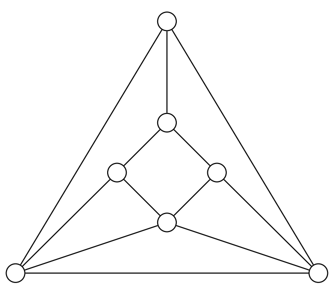

Figure 2.4: Tc20

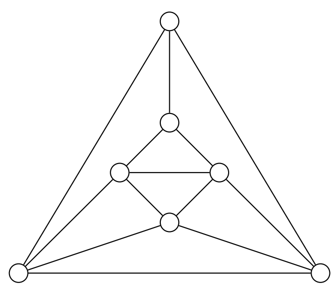

Figure 2.5: Tc30

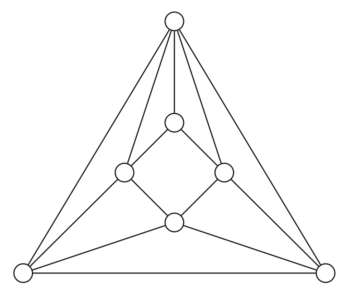

Figure 2.6: Tc39

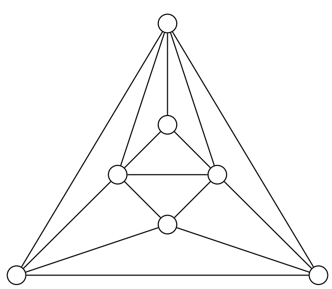

Figure 2.7: Tc43

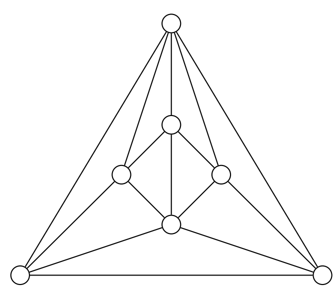

Figure 2.8: Tc44

| | |
|---|---|
| Tc20 | 1 10 100 0101 01101 001111 |
| Tc30 | 1 11 101 1001 01001 011101 |
| Tc39 | 1 11 101 1001 01001 011111 |
| Tc43 | 1 11 101 1101 01001 011111 |
| Tc44 | 1 11 101 1001 11001 011111 |

Table 2.1: All planar 2-line e.c. graphs represented by the rows of the lower triangle of their adjacency matrices

## 2.3 $n$-Line e.c. Hypergraphs

Recall that a hypergraph $H$ is a pair $(V, E)$ such that $V$ is a set of distinct vertices and $E$ is a collection of subsets of $V$ called hyperedges or simply edges. A hypergraph in which all edges have the same cardinality $k$ is called a $k$-uniform hypergraph. The *line graph* of a hypergraph $H$, denoted $L(H)$, is the graph with vertex set $E(H)$ such that adjacency of vertices in $L(H)$ corresponds with adjacency of edges in $H$, where two edges in $H$ are adjacent if and only if they share at least one vertex. We say that a hypergraph is *$n$-line existentially closed* or *$n$-line e.c.* if its corresponding line graph is an $n$-e.c. graph. A *matching* in a hypergraph is a set of edges in which no two edges contain a common vertex.

We know from Theorem 2.3 that a graph cannot be $n$-line e.c. for $n \geqslant 3$. To find examples of $n$-e.c. line graphs for $n \geqslant 3$ we instead consider the line graphs of hypergraphs. The idea of existential closure properties in the line graphs of hypergraphs already has a history in the literature, although under a more specific set of parameters. In particular, the block-intersection graphs of designs (which may be viewed as the line graphs of certain hypergraphs), have been studied with the $n$-existential closure property in mind. In a 2005 paper by Forbes, Grannell, and Griggs [19], the block-intersection graphs of Steiner triple systems are investigated and in a 2007 paper by McKay and Pike [35], the block-intersection graphs of more general balanced incomplete block designs were considered. Existential closure was also examined in the block intersection graphs of infinite designs in [26] and [37].

**Theorem 2.16.** *Let $H$ be a hypergraph with edges of size at most $k$. If $H$ is $n$-line e.c. then $n \leqslant k$. Consequently, if $H$ is a $k$-uniform $n$-line e.c. hypergraph, then $n \leqslant k$.*

**Proof.** Suppose $H$ is $(k+1)$-line e.c. and let $e$ be an edge of $H$. By the same method detailed in the proof of Theorem 2.3, we can build a matching of size $k+1$ in $H$ which contains $e$.

Now, since $H$ is $(k+1)$-line e.c., there must exist a $(k+2)^{\text{nd}}$ edge, adjacent to each of the previous $k+1$ edges, no two of which are adjacent. This is impossible, so $H$ cannot be $(k+1)$-line e.c. In particular, this result holds when $H$ is a $k$-uniform $n$-line e.c. hypergraph as well. $\square$

Note that we can use existing examples of $n$-e.c. graphs to construct $n$-line e.c. hypergraphs for any $n$ as follows. Let $G$ be an $n$-e.c. graph and form a set of size $\deg(v)$ for each vertex $v \in V(G)$ consisting of the edges with which it is incident. From this, we can build a hypergraph $H$ with $V(H) = E(G)$ and $E(H)$ consisting of the sets we have just formed. Note that the line graph of $H$ is isomorphic to $G$, so $H$ is an $n$-line e.c. hypergraph. Also, if $G$ happens to be $k$-regular, then the resulting hypergraph $H$ would be $k$-uniform.

We can apply this construction to any given set of $n$-e.c. graphs to produce additional examples of $n$-line e.c. hypergraphs. One explicit family of $n$-e.c. graphs is the set of Paley graphs. Paley graphs are constructed from finite fields by taking the set of elements of a field as the vertex set and making two vertices adjacent if those elements differ by a quadratic residue in the field. For more information on Paley graphs see [21]. In [3] and [4] it was shown that for any $n$, every sufficiently large Paley graph is $n$-e.c. Since Paley graphs are necessarily regular and form an infinite family of graphs, we conclude that there exist infinitely many sufficiently large uniform $n$-line e.c. hypergraphs for any $n$.

**Theorem 2.17.** *There exist infinitely many uniform $n$-line e.c. hypergraphs for any $n$.*

If we once again focus on $n = 2$, then we can find analogous results for hypergraphs to those presented in Section 2.1.2.

**Theorem 2.18.** *Let $X$ and $Y$ be disjoint sets of vertices. Let $H$ be the hypergraph with vertex set $X \cup Y$ along with all possible edges of size $k \geqslant 3$ such that each edge contains at least one vertex of $X$ and at least one vertex of $Y$. If $|X| \geqslant |Y| \geqslant 2k - 1$, then $H$ is a $k$-uniform 2-line e.c. hypergraph.*

**Proof.** We must verify that each pair of edges in $H$ satisfies the 2-line e.c. property. Let $e = \{u_1, u_2, \ldots, u_k\}$ and $f = \{v_1, v_2, \ldots, v_k\}$ be two distinct edges of $H$. Without loss of generality, we may assume that $u_1, v_1 \in X$ and $u_k, v_k \in Y$.

Any third edge containing $u_1$ and $v_1$ is adjacent to both $e$ and $f$. Similarly, any edge $g = \{x_1, x_2, \ldots, x_k\}$ where each $x_i$ is distinct from each $u_i$ and each $v_i$ is adjacent to neither $e$ nor $f$. This is possible since any edge of $H$ can contain at most $k-1$ vertices of $X$ (likewise for $Y$), so since $|X| \geqslant |Y| \geqslant 2k - 1 = 2(k-1) + 1$, such an edge $g$ exists. Finally, since $e \neq f$, there exists at least one vertex of $e$ which is not contained in $f$ and vice versa. So pick this vertex and observe that at least one of its incident edges will serve as an edge that is adjacent to $e$ and not $f$ and vice versa. $\square$

Using Theorem 2.18 and similar arguments as used in the proof of Theorem 2.9 we can establish the following corollary.

**Corollary 2.19.** *Let $H_1$ and $H_2$ be $k$-uniform 2-line e.c. hypergraphs on distinct sets of vertices. Let $H$ be the hypergraph $H_1 \cup H_2$ along with all possible edges of size $k$ such that each edge has a non-empty intersection with both $V(H_1)$ and $V(H_2)$. If $|V(H_1)| \geqslant |V(H_2)| \geqslant 2k - 1$, then $H$ is a $k$-uniform 2-line e.c. hypergraph.*

# Chapter 3

# Existential Closure in Uniform Hypergraphs

We again recall some useful definitions and terminology. A hypergraph in which all edges have the same cardinality $h$ is called an $h$-uniform hypergraph. For an $h$-uniform hypergraph $H$, we say that $H$ is $n$-e.c. if, for any set of vertices $S$ of size $n$ and any set $T \subseteq S$, there is a set of vertices $X \subseteq V(H) \setminus S$ of size $h-1$ such that for all $z \in T$, $X \cup \{z\}$ is an edge of $H$ and for all $s \in S \setminus T$, $X \cup \{s\}$ is not an edge of $H$. We again say that the set $X$ is correctly joined to $T$ and $S \setminus T$. Note that for $h = 2$, this definition agrees with the usual notion of an existentially closed graph.

In Section 3.1, we identify multiple necessary conditions for the existence of existentially closed graphs which extend naturally to existentially closed hypergraphs. Many of these results mirror those listed in Theorem 1.1.

In Section 3.2, we prove that random uniform hypergraphs are asymptotically existentially closed. In particular, for a large enough number of vertices and for any $n$, random uniform hypergraphs are $n$-existentially closed. In a sense, this implies that most uniform hypergraphs are existentially closed. However, as this result is non-constructive, we are still left without examples of such hypergraphs. We remedy this by presenting constructions for building existentially closed hypergraphs from combinatorial designs. In particular, we construct infinitely many $n$-e.c. uniform hypergraphs for any $n$ given an appropriate combinatorial design, which is known to exist whenever the obvious necessary conditions are met and the order is sufficiently large.

## 3.1 Necessary Conditions

As is the case for graphs, some immediate structural properties of $n$-e.c. hypergraphs are easily observed. Note that the following results mirror those listed in Theorem 1.1.

**Theorem 3.1.** *Let $H$ be an $h$-uniform hypergraph. If $H$ is $n$-e.c. then $H$ is $m$-e.c. for each $1 \leqslant m \leqslant n$.*

**Proof.** Let $S = \{v_1, v_2, \ldots, v_m\}$ be a set of vertices of $H$ where $1 \leqslant m \leqslant n$ and choose $T \subseteq S$. Pick vertices $v_{m+1}, v_{m+2}, \ldots, v_n \in V(H) \setminus S$. Since $H$ is $n$-e.c. there exists an $(h-1)$-set $X$ correctly joined to $T$ and $(S \setminus T) \cup \{v_{m+1}, v_{m+2}, \ldots, v_n\}$. In particular, this set $X$ is also correctly joined to $T$ and $S \setminus T$ and so $H$ is $m$-e.c. $\square$

We also identify some lower bounds on the number of vertices and edges in an $n$-e.c. hypergraph.

**Theorem 3.2.** *Let $H$ be an $h$-uniform hypergraph. If $H$ is $n$-e.c. then $H$ has at least $n2^{n-1}$ edges and at least $n + \ell$ vertices where $\ell$ is the smallest positive integer such that $\binom{\ell}{h-1} \geqslant 2^n$.*

**Proof.** Let $S$ be an $n$-set in $V(H)$. For each $x \in S$, $x$ is contained in $2^{n-1}$ sets $T \subseteq S$, each of which is correctly joined to at least one appropriate set $X$. Each set $X$ forms an edge with vertex $x$ and thus $\deg(x) \geqslant 2^{n-1}$. Thus, since each set $X$ is disjoint from $S$, $H$ has at least $n2^{n-1}$ edges.

Note that $H$ must have at least $n$ vertices, plus enough other vertices to form at least $2^n$ sets of size $h-1$. Let $\ell$ be the smallest positive integer such that $\binom{\ell}{h-1} \geqslant 2^n$. Then $H$ has at least $n+\ell$ vertices. $\square$

Observe that when $h=2$, $\ell = 2^n$ which agrees with part 2 of Theorem 1.

For an $h$-uniform hypergraph $H$, define $H^{\mathsf{c}}$ as the hypergraph on the vertex set $V(H)$ where an $h$-set $e$ of vertices is an edge of $H^{\mathsf{c}}$ if and only if $e$ is not an edge of $H$. We call $H^{\mathsf{c}}$ the *$h$-uniform complement* of $H$ or simply, the *complement* of $H$.

**Theorem 3.3.** *Let $H$ be an $h$-uniform hypergraph. If $H$ is $n$-e.c. then the complement $H^{\mathsf{c}}$ is also $n$-e.c.*

**Proof.** Let $S$ be an $n$-set of vertices in $V(H^{\mathsf{c}})$ and $T \subseteq S$. Since $H$ is $n$-e.c. there is an $(h-1)$-set $X$ which is correctly joined to $S \setminus T$ and $T$, meaning $X$ forms an edge in $H$ with each vertex of $S \setminus T$ and with no vertex of $T$. But this means that $X$ forms an edge in $H^{\mathsf{c}}$ with each vertex of $T$ and with no vertex of $S \setminus T$. Thus, $H^{\mathsf{c}}$ is $n$-e.c. $\square$

For a hypergraph $H = (V, E)$, we say that the hypergraph $H' = (V', E')$ is a *subgraph* of $H$ if $V' \subseteq V$ and $E' \subseteq E$. In other words, $H'$ is a subgraph of $H$ if and only if every vertex of $V'$ is also a vertex of $V$ and every edge of $E'$ is also an edge of $E$. Note that this definition of a subgraph of a hypergraph coincides with the definition of a "strong subhypergraph" in [14] and a "hypersubgraph" in [1].

Now let $H$ be a hypergraph and let $Y \subseteq V(H)$ be a subset of vertices of $H$. We denote the subgraph induced by $Y$ in $H$ by $H[Y]$. That is, $H[Y]$ is the hypergraph on the vertex set $Y$ whose edges are precisely the edges of $H$ in which each vertex is a member of $Y$. Also, for $v \in V(H)$ the neighbourhood of $v$, denoted $N(v)$, is the set of all vertices which occur together with $v$ in at least one edge of $H$.

**Theorem 3.4.** *Let $H$ be an $h$-uniform hypergraph. If $H$ is $n$-e.c. then for each vertex $v \in V(H)$, the hypergraphs $H - v$ and $H[N(v)]$ are $(n-1)$-e.c.*

**Proof.** Let $S$ be an $(n-1)$-set of vertices in $V(H-v)$ and $T \subseteq S$. Since $H$ is $n$-e.c. there is an $(h-1)$-set $X$ which is correctly joined to $T$ and $(S \cup \{v\}) \setminus T$ in $H$. Note that by definition, $v \notin X$. So $X$ forms an edge in $H - v$ with each vertex of $T$ and with no vertex of $S \setminus T$. Thus, $H - v$ is $(n-1)$-e.c.

Now let $S$ be an $(n-1)$-set of vertices in $H[N(v)]$ and $T \subseteq S$. Since $H$ is $n$-e.c. there is an $(h-1)$-set $X$ which is correctly joined to $T \cup \{v\}$ and $(S \cup \{v\}) \setminus (T \cup \{v\})$ in $H$. Note that since $X$ forms an edge with each vertex of $T \cup \{v\}$, then $X \cup \{v\}$ is an edge of $H$ and so each vertex of $X$ is contained in $N(v)$. Therefore $X$ forms an edge in $H[N(v)]$ with each vertex of $T$ and with no vertex of $S \setminus T$. Thus, $H[N(v)]$ is $(n-1)$-e.c. $\square$

In Section 3.3 we will see examples of $h$-uniform $n$-e.c. hypergraphs $H$ constructed from combinatorial designs. These hypergraphs have the additional property that each pair of vertices appears together in at least one edge of $H$. This means that for any $v \in V(H)$, the neighbourhood of $v$ is $N(v) = V(H) \setminus \{v\}$ and so the set of non-neighbours of $v$ is $(V(H) \setminus N(v)) \setminus \{v\}$ which is an empty set of vertices. So the subgraph induced by this set is the empty graph and is therefore not $n$-e.c. for any $n$.

To continue extending results listed in Theorem 1.1 to $h$-uniform hypergraphs, we define a slightly altered notion of a set of non-neighbours of a vertex in a hypergraph. For $v \in V(H)$, let $A(v)$ be the set of all vertices that occur together with $v$ in at least one edge of $H^{\mathsf{c}}$. Note that for a graph, $A(v) = (V(H) \setminus N(v)) \setminus \{v\}$. With this distinction, we may establish the following result.

**Theorem 3.5.** *Let $H$ be an $h$-uniform hypergraph. If $H$ is $n$-e.c. then for each vertex $v \in V(H)$, the hypergraph $H[A(v)]$ is $(n-1)$-e.c.*

**Proof.** Let $S$ be an $(n-1)$-set of vertices in $H[A(v)]$ and $T \subseteq S$. Since $H$ is $n$-e.c. there is an $(h-1)$-set $X$ which is correctly joined to $T$ and $(S \cup \{v\}) \setminus T$ in $H$. Note that $X$ does not form an edge with any vertex of $(S \cup \{v\}) \setminus T$. In particular, $X$ does not form an edge with $v$, so $X \subseteq A(v)$. Now note that $X$ forms an edge in $H[A(v)]$ with each vertex of $T$ and with no vertex of $S \setminus T$. Thus, $H[A(v)]$ is $(n-1)$-e.c. $\square$

In the next section, we prove that random uniform hypergraphs are asymptotically existentially closed, a result that mirrors that of random graphs.

## 3.2 Random $n$-e.c. Hypergraphs

One of the earliest results on existentially closed graphs was that random finite graphs are asymptotically $n$-e.c. [17]. We show that this result also extends to $n$-e.c. hypergraphs.

A *random $h$-uniform hypergraph*, denoted $H_h(m,p)$, is an $h$-uniform hypergraph on $m$ vertices in which each set of vertices $e \subseteq V(H)$ of size $h$ is chosen to be an edge of $H$ randomly and independently with probability $p$, where $p$ may depend on $m$. Thus, for $h = 2$ this model reduces to the well-known Erdős-Rényi model $G(m,p)$ [16]. For some early results on random hypergraphs, see [29].

**Theorem 3.6.** *With probability 1 as $m \to \infty$, $H_h(m,p)$ satisfies the $n$-e.c. property where $p$ is a fixed real number such that $0 < p < 1$ and $n > 1$.*

**Proof.** Fix an $n$-set $S$ of vertices and fix $T \subseteq S$. For a given $(h-1)$-set $X \subseteq V \setminus S$, the probability that $X$ is not correctly joined to $T$ and $S \setminus T$ is $1 - p^n$. The probability that no set of size $h-1$ is correctly joined to $T$ and $S \setminus T$ is therefore

$$(1-p^n)^{\binom{m-n}{h-1}}.$$

As there are $\binom{m}{n}$ choices for $S$ and $2^n$ choices for $T \subseteq S$, the probability that $H_h(m,p)$ is not $n$-e.c. is at most

$$\binom{m}{n} 2^n \left(1-p^n\right)^{\binom{m-n}{h-1}}.$$

Since $n$ and $p$ are fixed, the probability that $H_h(m,p)$ is not $n$-e.c. tends to 0 as $m \to \infty$. $\Box$

Since random $h$-uniform hypergraphs asymptotically satisfy the $n$-e.c. property, we should expect to see many examples of these hypergraphs. However, as is the case for graphs, it is not immediately clear how to find examples of these hypergraphs. In the next section, we present constructions for building existentially closed hypergraphs from combinatorial designs, most notably, balanced incomplete block designs and $t$-designs.

## 3.3 $n$-e.c. Hypergraphs from Designs

The initial construction we present for producing an infinite family of 2-e.c. $k$-uniform hypergraphs makes use of a well-known topic within combinatorics, Latin squares. A *Latin square of order* $n$ is an $n \times n$ array consisting of $n$ symbols such that each symbol occurs in each row and each column precisely once. A pair of Latin squares of the same order is said to be *orthogonal* if, when superimposed, the entries viewed as ordered pairs are all unique. A set of Latin squares of the same order in which any two form an orthogonal pair is said to be a set of *mutually orthogonal Latin squares* or *MOLS* for short. It is well-known that the maximum possible number of mutually orthogonal Latin squares of order $n$ is $(n-1)$. Such a set of MOLS is referred to as a *complete set* of MOLS. Complete sets of MOLS of order $n$ are known to exist when $n$ is a prime or power of a prime. For more information on Latin squares including applications, see [13]. For an example of a complete set of MOLS of order 4, see Figure 3.1. In this example, if we pick any two squares and superimpose them together, each ordered pair of the symbols $\{0, 1, 2, 3\}$ appears precisely once. So this is indeed a complete set of mutually orthogonal Latin squares.

| | | | |
|---|---|---|---|
| 0 | 1 | 2 | 3 |
| 2 | 3 | 0 | 1 |
| 1 | 0 | 3 | 2 |
| 3 | 2 | 1 | 0 |

| | | | |
|---|---|---|---|
| 0 | 1 | 2 | 3 |
| 3 | 2 | 1 | 0 |
| 2 | 3 | 0 | 1 |
| 1 | 0 | 3 | 2 |

| | | | |
|---|---|---|---|
| 0 | 1 | 2 | 3 |
| 1 | 0 | 3 | 2 |
| 3 | 2 | 1 | 0 |
| 2 | 3 | 0 | 1 |

Figure 3.1: A complete set of MOLS of order 4

Now suppose $L$ is a set of $\ell$ mutually orthogonal Latin squares of order $k+1$. We will form a $k$-uniform hypergraph $H_L$ in the following way. Let $V$ be a set of $(k+1)^2$ vertices organised into a $(k+1) \times (k+1)$ array $A$. To form the edge set $E$, we will collect edges in two ways. First, for each row (respectively each column) of $A$, take all $(k+1)$ $k$-sets of vertices within the row (respectively column) to be edges of $E$. Second, for each Latin square in $L$ and for each symbol within the squares of $L$, take note of the position of each occurrence of that symbol and then take the corresponding $(k+1)$-set of vertices within $A$. Now take all $k$-subsets of vertices within this set as edges of $E$. The resulting hypergraph $H_L = (V, E)$ is a $k$-uniform hypergraph on $(k+1)^2$ vertices and $(\ell+2)(k+1)^2$ edges.

| | | | |
|---|---|---|---|
| 0 | 1 | 2 | 3 |
| 4 | 5 | 6 | 7 |
| 8 | 9 | a | b |
| c | d | e | f |

Figure 3.2: The $4 \times 4$ array $A$

For example, consider the complete set of MOLS in Figure 3.1 as our ingredient set $L$. Then the constructed hypergraph $H_L = (V, E)$ is a 3-uniform hypergraph on 16 vertices.

Organise the vertex set $V$ into a $4 \times 4$ array $A$ (see Figure 3.2). Then form the edge set $E$ according to the construction above. For instance, the 4-sets acquired from the symbols in the first square in $L$ are the following:

$$\{0, 6, 9, f\}, \{1, 7, 8, e\}, \{2, 4, b, d\}, \{3, 5, a, c\}.$$

We then take all 3-subsets of these 4-sets as edges of $E$. The resulting hypergraph $H_L$ will have 80 edges according to our construction. For the full hypergraph, see Example B.1 in Appendix B.

Our next theorem asserts that if $L$ is a complete set of MOLS, then the resulting hypergraph $H_L$ is 2-existentially closed.

**Theorem 3.7.** *If $L$ is a complete set of MOLS of order $k+1$ and $k \geqslant 3$, then the hypergraph $H_L$ is 2-existentially closed.*

**Proof.** We must verify that for any 2-set of vertices $S = \{u, v\}$ and any $T \subseteq S$, there is a $(k-1)$-set $X \subseteq V(H_L) \setminus S$ such that $X$ forms an edge with each vertex of $T$ and with no vertex of $S \setminus T$.

When $|T| = 0$, we take $X$ to be any $(k-1)$-set of vertices all of which occur in the same row together but a different row than that of $u$ and $v$. Here, $X$ forms an edge with neither $u$ nor $v$. When $|T| = 1$, say $T = \{u\}$, if $u$ and $v$ are in distinct columns then we take $X$ to be a $(k-1)$-set of vertices which each occur in the same column as $u$ and note that $X$ forms an edge with $u$ but not with $v$. Otherwise, if $u$ and $v$ are in the same column, we take $X$ to be a $(k-1)$-set of vertices which each occur in the same row as $u$ and note that $X$ forms an edge with $u$ but not with $v$.

Finally, when $|T| = 2$, if $u$ and $v$ happen to be in the same row (respectively column), we take $X$ to be the set containing the other $k-1$ vertices in that row (respectively column) and note that $X$ forms an edge with each of $u$ and $v$. Otherwise, when $u$ and $v$ are in distinct rows and columns, we find the Latin square among $L$ in which the positions corresponding to $u$ and $v$ share a common symbol; this is guaranteed to exist since $L$ is a complete set of MOLS. We then take $X$ to be the set containing the other $k-1$ vertices corresponding to the positions of the other occurrences of the shared symbol in that Latin square and note that $X$ forms an edge with each of $u$ and $v$. □

Since complete sets of MOLS are known to exist whenever the order of the Latin squares is a prime or prime power, Theorem 3.7 implies that there are infinitely many 2-e.c. uniform hypergraphs.

Using similar techniques, we may construct 2-e.c. uniform hypergraphs given the existence of affine planes. An affine plane is another well-known structure within discrete mathematics and can be constructed given a complete set of MOLS. From the point of view of design theory, an *affine plane* of order $n$ is a set of $n^2$ elements along with a collection of sets of size $n$ such that each pair of elements is contained in exactly one set. In design theory, the set of all affine planes forms a particular family of balanced incomplete block designs. In general, a *balanced incomplete block design* or *BIBD* with parameters $(v, k, \lambda)$ is a pair $\mathcal{D} = (V, \mathcal{B})$ such that $V$ is a set of $v$ distinct elements called *points* and $\mathcal{B}$ is a collection of $k$-subsets of $V$ called *blocks* such that each pair of points of $V$ occurs in exactly $\lambda$ blocks of $\mathcal{B}$.

The total number of blocks in a BIBD is denoted $b$ and the number of blocks which contain any given point is called the *replication number* and is denoted $r$. In particular, $b = \frac{\lambda v(v-1)}{k(k-1)}$ and $r = \frac{\lambda(v-1)}{k-1}$. Balanced incomplete block designs are known to exist asymptotically whenever the necessary divisibility conditions are met [45, 46, 47]. For more information on BIBDs including constructions and known examples, see [11].

For example, Figure 3.3 is a graphical representation of the unique (7,3,1)-BIBD. This

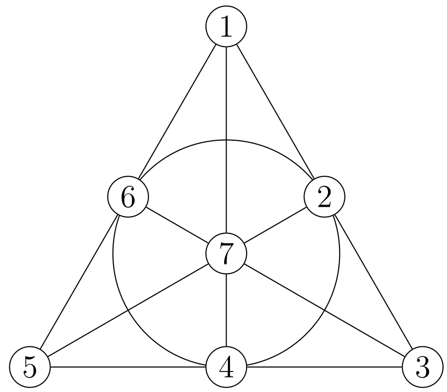


Figure 3.3: The Fano plane

design is known as the Fano plane and is significant within multiple branches of mathematics including design theory, projective geometry, and group theory. The Fano plane is a member of a family of designs similar to affine planes called projective planes. Here, each line (including the curved one) passes through exactly three points and determines a block of size 3 consisting of those points. The corresponding blocks of this design are then $\{1,2,3\},\{3,4,5\},\{1,5,6\},\{1,4,7\},\{2,5,7\},\{3,6,7\},$ and $\{2,4,6\}$. Note that each pair of points occurs in precisely one block. For more information on the Fano plane see [11].

Any design with $v > k$ other than the design in which the block set $\mathcal{B}$ is the set of all $k$-subsets of $V$, is actually 1-e.c. when viewed as a hypergraph. Indeed, each point occurs in exactly $r$ blocks, so each point occurs at least once and no point occurs $\binom{v}{k}$ times.

Now suppose $\mathcal{D}$ is a $(v,k,1)$-BIBD with $k \geqslant 3$. For each $h$ such that $3 \leqslant h \leqslant k$, we will form an $h$-uniform hypergraph $H_{\mathcal{D},h}$ in the following way. Let the vertex set $V$ be the point set of $\mathcal{D}$. For each block $B$ of $\mathcal{D}$, take all $\binom{k}{h}$ $h$-subsets of $B$ as edges of the edge set $E$. The resulting hypergraph $H_{\mathcal{D},h} = (V,E)$ is an $h$-uniform hypergraph with $v$ vertices and $b\binom{k}{h}$ edges where $b$ is the total number of blocks in $\mathcal{D}$.

**Theorem 3.8.** *Let $\mathcal{D}$ be a $(v,k,1)$-BIBD with $k \geqslant 4$. If $v \geqslant k+2$ and $3 \leqslant h \leqslant k-1$, then the hypergraph $H_{\mathcal{D},h}$ is 2-existentially closed.*

**Proof.** We must verify that for any 2-set of vertices $S = \{u, v\}$ and any $T \subseteq S$, there is an $(h-1)$-set $X \subseteq V(H_{\mathcal{D},h}) \setminus S$ such that $X$ forms an edge with each vertex of $T$ and with no vertex of $S \setminus T$. Note that since $h \geqslant 3$ and $\lambda = 1$, the number of times any $(h-1)$-set $X$ occurs within a block of $\mathcal{D}$ is at most once; otherwise, the block containing $X$ would contain a pair which occurs more than $\lambda$ times among the blocks of $\mathcal{D}$. So any $(h-1)$-set $X$ chosen directly from a block of $\mathcal{D}$ is unique.

When $|T| = 2$, let $B$ be the unique block containing both $u$ and $v$. Since $k \geqslant h+1$, $B$ contains $u$, $v$ and at least $h-1$ other points. So take $X$ to be an $(h-1)$-set of points in $B \setminus \{u, v\}$ and note that $X$ forms an edge with each of $u$ and $v$ in $H_{\mathcal{D},h}$.

When $|T| = 1$, say $T = \{u\}$, let $B$ be a block among the $r$ blocks that contain $u$ other than the unique block which contains both $u$ and $v$. Such a block exists since the replication number $r = \frac{v-1}{k-1}$ is greater than 1 whenever $v > k$. Now take $X$ to be an $(h-1)$-set of points other than $u$ in this block and note that $X$ forms an edge with $u$ but not with $v$.

Finally, when $|T| = 0$, if there exists a block $B$ which contains neither $u$ nor $v$, then we can choose an $(h-1)$-set $X$ within $B$ and note that $X$ forms an edge with neither $u$ nor $v$. There are $b$ blocks in total, $r$ blocks containing $u$, $r$ blocks containing $v$, and exactly one block containing both $u$ and $v$ (which is counted twice among the blocks containing $u$ and $v$). So if $b > 2r - 1$ then there exists an appropriate block $B$ from which to choose an $(h-1)$-set $X$. Recall that $b = \frac{\lambda v(v-1)}{k(k-1)}$, $r = \frac{\lambda(v-1)}{k-1}$ and $\lambda = 1$, so

$$\begin{array}{rrcl} & \frac{v(v-1)}{k(k-1)} & > & \frac{2(v-1)}{k-1} - 1 \\ \Leftrightarrow & v(v-1) & > & 2k(v-1) - k(k-1) \\ \Leftrightarrow & (v-2k)(v-1) & > & k(1-k). \end{array}$$

Now when $v \geqslant k+2$,

$$(v-2k)(v-1) \geqslant (2-k)(k+1) > k(1-k)$$

and thus $b > 2r-1$ holds. $\square$

Note that if $h = 2$ this construction would yield a graph (i.e., a 2-uniform hypergraph) but such a graph would not even be 1-existentially closed. Indeed, since each pair of points in a design occurs precisely $\lambda$ times, the resulting graph would be complete and trivially not 1-e.c. Also, if $h = k$ then the hypergraph $H_{\mathcal{D},h}$ is simply the design $\mathcal{D}$ itself. Note that any design with $\lambda = 1$ cannot be 2-e.c. since by definition, there would need to exist a set $X$ of size $k-1$ which forms an edge (or block) with at least two distinct points, violating $\lambda = 1$. However, finding designs with higher values of $\lambda$ that are $n$-e.c. for $n \geqslant 2$ is an open problem.

To find examples of $n$-e.c. hypergraphs for values of $n \geqslant 3$, we make use of a natural generalisation of balanced incomplete block designs. A $t$-$(v, k, \lambda)$ *block design*, or $t$-*design* for short, is a pair $\mathcal{D} = (V, \mathcal{B})$ such that $V$ is a set of $v$ distinct points and $\mathcal{B}$ is a collection of blocks of size $k$ such that each $t$-subset of points of $V$ occurs in exactly $\lambda$ blocks of $\mathcal{B}$. Note that a $t$-design with $t = 2$ is exactly a balanced incomplete block design. Infinitely many nontrivial $t$-designs without repeated blocks are known to exist for all $t$ [40]. Asymptotically, $t$-designs are known to exist whenever the necessary divisibility conditions are met [30]. For more information on $t$-designs including constructions and known examples, see [11].

Now suppose $\mathcal{D}$ is a $t$-$(v, k, 1)$-design with $k \geqslant 3$. For each $h$ such that $3 \leqslant h \leqslant k$, we will form an $h$-uniform hypergraph $H_{\mathcal{D},h}$ in the following way. Let the vertex set $V$ be the point set of $\mathcal{D}$. For each block $B$ of $\mathcal{D}$, take all $\binom{k}{h}$ $h$-subsets of $B$ as edges of the edge set $E$. The resulting hypergraph $H_{\mathcal{D},h} = (V, E)$ is an $h$-uniform hypergraph with $v$ vertices and $b\binom{k}{h}$ edges where $b$ is the number of blocks in $\mathcal{D}$.

To show that $H_{\mathcal{D},h}$ is an existentially closed hypergraph for certain values of $v$, $k$, and $h$, we make use of a result that can be found as a remark in Chapter II, Section 4.2 of [11] that allows us to count the number of blocks in a design that contain certain points while avoiding other certain points.

**Lemma 3.9.** *[11, §II.4.2] Let $\mathcal{D} = (V, \mathcal{B})$ be a $t$-$(v, k, \lambda)$ block design and let $I$ and $J$ be disjoint subsets of $V$ with $|I| = i$, $|J| = j$ and $i + j \leqslant t$. If $\lambda_{i,j}$ is the number of blocks that contain each point of $I$ and no point of $J$, then $\lambda_{i,j} = \lambda\binom{v-i-j}{k-i}/\binom{v-t}{k-t}$.*

**Theorem 3.10.** *Let $\mathcal{D}$ be a $t$-$(v, k, 1)$-design with $k \geqslant 2t$. If $v \geqslant k+t$ and $t+1 \leqslant h \leqslant k-t+1$, then the hypergraph $H_{\mathcal{D},h}$ is $t$-existentially closed.*

**Proof.** We must verify that for any $t$-set of vertices $S$ and any $T \subseteq S$, there is an $(h-1)$-set $X \subseteq V(H_{\mathcal{D},h}) \setminus S$ such that $X$ forms an edge with each vertex of $T$ and with no vertex of $S \setminus T$. Note that since $h \geqslant t + 1$ and $\lambda = 1$, the number of times any $(h - 1)$-set $X$ occurs within a block of $\mathcal{D}$ is at most once; otherwise, the block containing $X$ would contain a $t$-set which occurs more than $\lambda$ times among the blocks of $\mathcal{D}$. So any $(h-1)$-set $X$ chosen directly from a block of $\mathcal{D}$ is unique.

When $|T| = t$, let $B$ be the unique block containing all $t$ points of $T$. Since $k \geqslant t+h-1$, $B$ contains the $t$ points of $T$ and at least $h-1$ other points. So take $X$ to be an $(h-1)$-set of points in $B \setminus T$ and note that $X$ forms an edge with each vertex of $T$ in $H_{\mathcal{D},h}$.

Now suppose $0 \leqslant |T| \leqslant t - 1$. For notational simplicity, let $|T| = i$. If there exists a block $B$ which contains the $i$ points of $T$ but none of the $t-i$ points of $S \setminus T$, then we can choose an $(h-1)$-set $X$ consisting of points of $B$ other than the $i$ points of $T$ and note that $X$ forms an edge with each vertex of $T$ but with no vertex of $S \setminus T$. By Lemma 3.9, the number of such blocks is precisely $\lambda_{i,t-i} = \binom{v-t}{k-i}/\binom{v-t}{k-t}$. Note that $\lambda_{i,t-i}$ is a positive integer so long as $v - t \geqslant k - i$ and $v - t \geqslant k - t$. Since $0 \leqslant i \leqslant t - 1$, these inequalities hold whenever $v \geqslant k + t$. □

Since infinitely many $t$-designs are known to exist for all $t$ [40], Theorem 3.10 implies the following.

**Corollary 3.11.** *There exist infinitely many $n$-e.c. hypergraphs for any $n$.*

## 3.4 A Brief Note

Although hypergraphs are a natural generalisation of the concept of a graph, we often encounter multiple ways to generalise properties from graphs to hypergraphs. For example, the concept of connectivity in graphs can be extended in multiple ways when looking at hypergraphs. In particular, there are distinct definitions of *weak connectivity* and *strong connectivity* in hypergraphs [14]. In Chapter 4, our definition of connectivity in a hypergraph is equivalent to the notion of weak connectivity. These nuances require extra care on the researcher's part to ensure definitions and terminology are translated appropriately from graphs to hypergraphs.

During the early stages of this project, we discussed an initial notion of an $n$-e.c. hypergraph which appeared to extend more naturally from graphs. Specifically, we considered defining an $n$-existentially closed hypergraph as a hypergraph in which for any set of vertices $S$ of size $n$ and for any set $T \subseteq S$, there is a vertex $x \notin S$ which occurs in at least one edge with each vertex of $T$ and occurs in no edges with any vertex of $S \setminus T$. This idea was not restricted to uniform hypergraphs; however, we noticed that this is equivalent to asking whether or not the 2-section of a hypergraph is existentially closed. The *2-section* of a hypergraph $H$ is the graph $G_H$ that has the same vertex set as $H$ and each edge of size $k$ within $H$ is replaced by a subgraph in $G_H$ isomorphic to the complete graph $K_k$ on the same set of $k$ vertices. This means that instead of studying existential closure in hypergraphs, we would have been considering existentially closed graphs that are isomorphic to the 2-section of certain hypergraphs.

Additionally, many necessary properties were lost in this scenario. For example, we found examples of hypergraphs whose complement hypergraph failed to retain the $n$-e.c. property in this scenario. Specifically, under this notion, a 1-e.c. hypergraph would be a hypergraph in which its 2-section does not contain any *isolated* or *universal* vertices (vertices of degree 0 or degree one less than the total number of vertices, respectively). Here, the smallest connected 1-e.c. 3-uniform hypergraph is the hypergraph depicted in Figure 3.4. However, the complement of this hypergraph is not 1-e.c. since its 2-section contains universal vertices. Note that the complement of the 2-section of a hypergraph $H$ is not necessarily isomorphic to the 2-section of the complement of $H$.

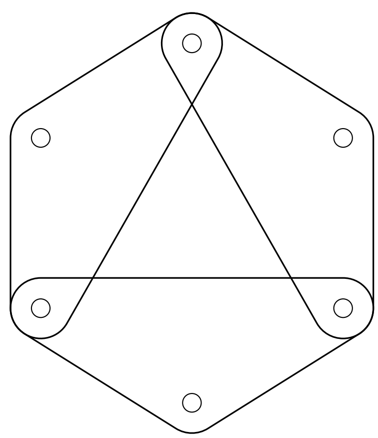

Figure 3.4: A 1-e.c. hypergraph under the initial notion

Ultimately, we settled on the definition of an existentially closed hypergraph previously discussed that allows many of the necessary properties of existentially closed graphs to be extended naturally to hypergraphs.

# Chapter 4

# The Edge-Connectivity of Vertex-Transitive Hypergraphs

Recall that a graph or hypergraph is connected if there is a path connecting each pair of vertices, where a path is a sequence of alternating incident vertices and edges without repetition. A cut set of edges in a graph or hypergraph is a set of edges whose deletion renders the graph or hypergraph disconnected. The edge-connectivity of a graph or hypergraph $H$ is the size of a minimum cut set of edges and is denoted $\kappa'(H)$. For a graph or hypergraph $H$, $\delta(H)$ is the minimum degree among the vertices and $\Delta(H)$ is the maximum degree among the vertices, where the degree of a vertex is the number of edges incident with it. A graph (or hypergraph) isomorphism $\phi$ from the graph (or hypergraph) to itself is called an automorphism. A graph or hypergraph $H$ is said to be vertex-transitive if, for any two vertices $u$ and $v$ of $V(H)$, there exists some automorphism $\phi$ of $H$ such that $\phi(u) = v$. Note that any vertex-transitive graph or hypergraph must also be *regular*, meaning all vertices have the same degree. A linear hypergraph is one in which any pair of vertices is contained in at most one edge. A uniform hypergraph is one in which each edge has the same cardinality;

moreover, if each edge has cardinality $k$, then we say that the hypergraph is $k$-uniform.

Recall that a graph or hypergraph $H$ is said to be maximally edge-connected if and only if $\kappa'(H) = \delta(H)$. A classic result of Mader asserts the edge-connectivity of vertex-transitive graphs.

**Theorem 4.1.** *[34] Let $G$ be a vertex-transitive and connected graph. Then $G$ is maximally edge-connected.*

Our main result in this chapter is a generalisation of Mader's Theorem to linear uniform hypergraphs. In particular, we show the following:

**Theorem 4.2.** *Let $H$ be a linear $k$-uniform hypergraph with $k \geqslant 3$. If $H$ is vertex-transitive and connected, then $H$ is maximally edge-connected.*

In Section 4.1 we demonstrate the existence of vertex-transitive hypergraphs that fail to be maximally edge-connected when we relax either the uniformity or linearity conditions of Theorem 4.2 and in Section 4.2 we present the proof of Theorem 4.2. The contents of this chapter are based on a paper published in the *Journal of Graph Theory* [8].

## 4.1 Non-Uniform and Non-Linear Hypergraphs

In this section, we present two examples of vertex-transitive hypergraphs that are not maximally edge-connected. Both examples meet all of the criteria of the hypothesis of Theorem 4.2 except for linearity in the first case and uniformity in the second.

### 4.1.1 Uniform but Non-Linear Hypergraphs

For $k \geqslant 3$, let $H$ be the complete $k$-uniform hypergraph on $n \geqslant k + 2$ vertices, i.e. $V(H)$ consists of $n$ vertices and $E(H)$ is equal to the set of all $k$-subsets of $V(H)$. Then $H$ is a connected $k$-uniform hypergraph which is simple but non-linear, where a *simple* hypergraph is one with no repeated edges and no loops. For any two vertices $u$ and $v$, there exists an automorphism $\phi$ such that $\phi(u) = v$, $\phi(v) = u$ and $\phi(w) = w$ for any other vertex $w$. Therefore $H$ is also vertex-transitive.

Now let $H_1, H_2, \ldots, H_k$ be distinct copies of $H$, each with its own vertex set $V(H_i) = V(H) \times \{i\}$. Take $H^*$ to be the union of these copies along with $n$ edges of the form $E_v = \{(v, 1), (v, 2), \ldots, (v, k)\}$ (one for each vertex $v \in V(H)$). Then $H^*$ is a connected $k$-uniform hypergraph which is simple but non-linear.

Now we must verify that $H^*$ is vertex-transitive. For any two vertices within the same copy of $H$, we can find an automorphism $\phi$ of $H^*$ similar to the ones described for $H$; for example, to map $(u, 1)$ to $(v, 1)$, use the map $\phi : H^* \to H^*$ defined by

$$\text{for all } i, \phi(u, i) = (v, i), \phi(v, i) = (u, i) \text{ and } \phi(w, i) = (w, i) \text{ when } w \notin \{u, v\}.$$

For any two vertices within an edge of the form $E_v$, simply take an automorphism $\psi$ of $H^*$ which exchanges the two corresponding copies of $H$ and fixes the rest; for example, to map $(v, 1)$ to $(v, 2)$, use the map $\psi : H^* \to H^*$ defined by

$$\psi(u, 1) = (u, 2), \psi(u, 2) = (u, 1) \text{ and } \psi(u, i) = (u, i) \text{ when } i \notin \{1, 2\}.$$

Finally, for any two vertices in general, we may take a composition (if needed) of the two types of automorphisms we have just described. Therefore, $H^*$ is a vertex-transitive hypergraph. However, so long as $n \geqslant k + 2$ and $k \geqslant 3$,

$$\kappa'(H^*) \leqslant n < \binom{n-1}{k-1} + 1 = \Delta(H) + 1 = \Delta(H^*)$$

and so $H^*$ is not maximally edge-connected.

As an example of this construction with $k = 3$ and $n = 5$, for $1 \leqslant i \leqslant 3$, let $H_i = (V_i, E_i)$ where $V_i = \{a, b, c, d, e\} \times \{i\}$ and $E_i$ is the set of all 3-subsets of $V_i$. Then $V(H^*)$ consists of the following 15 ordered pairs:

$$(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3), (d, 1), (d, 2), (d, 3), (e, 1), (e, 2), (e, 3).$$

According to our construction, the edge set $E(H^*) = E_1 \cup E_2 \cup E_3$ along with the five edges listed below:

{(a,1), (a,2), (a,3)}
{(b,1), (b,2), (b,3)}
{(c,1), (c,2), (c,3)}
{(d,1), (d,2), (d,3)}
{(e,1), (e,2), (e,3)}

Then $H^* = (V, E)$ is a 3-uniform vertex-transitive hypergraph on 15 vertices and 35 edges with edge-connectivity $\kappa'(H^*) \leqslant 5$ and degree $\Delta(H^*) = \binom{5-1}{3-1} + 1 = 7$.

### 4.1.2 Linear but Non-Uniform Hypergraphs

To construct an example of a vertex-transitive hypergraph that is linear but non-uniform, we rely on a well-known example from combinatorial designs, a finite affine plane. A finite *affine plane* of order $n$ is a set of $n^2 + n$ lines on $n^2$ points such that each line contains $n$ points and each point lies on $n + 1$ lines. Additionally, each pair of points lie on a unique line and the lines of an affine plane can be partitioned into $n + 1$ equivalence classes under the equivalence relation of parallelism; we will refer to these classes as parallel classes. We give a direct construction of a finite affine plane of prime order as follows.

Let $k$ be an odd prime and form a $k \times k$ array $A$ such that the entry in row $i$ and column

$j$ is $a_{i,j} = (i-1)k + j$, where $i, j \in \{1, 2, \ldots, k\}$. Let the first parallel class $\Pi_0$ be the set of all rows of $A$, that is,

$$\Pi_0 = \big\{\{1, \ldots, k\}, \{k+1, \ldots, 2k\}, \ldots, \{(k-1)k+1, \ldots, k^2\}\big\}.$$

For each $i = 1, \ldots, k$, form the lines of parallel class $\Pi_i$ by selecting a point from row 1 and $k-1$ other points, one from each subsequent row, such that each subsequent point is located $(i-1)$ cells to the right of the last point (wrapping around if necessary). Repeat this process for each point in row 1 to form all $k$ lines of parallel class $\Pi_i$. Precisely, $\Pi_i$ is the collection of lines $\{B_{i,j}\}$ with $j = 1, 2, \ldots, k$ such that each line is a set of points $B_{i,j} = \{tk + s \mid t \in \{0, 1, \ldots, k-1\}\}$ where $s$ is the unique integer between 1 and $k$ inclusive for which $s \equiv (i-1)t + j \pmod{k}$.

Now let $H$ be the $k$-uniform hypergraph with vertex set $V(H) = \{1, 2, \ldots, k^2\}$ and edge set $E(H) = \bigcup_{i=1}^{k} \Pi_i$. Note that we have intentionally left out the class $\Pi_0$. To verify that $H$ is vertex-transitive, let $x$ and $y$ be two vertices of $H$. Find the parallel class among $\Pi_0, \Pi_1, \ldots, \Pi_k$ which contains the pair $\{x, y\}$ in a line together and write the lines of this class in order as a permutation $\sigma$. For example, if $x$ and $y$ are both contained in the line $B_{i,j}$, then $\sigma = (L_{i,1})(L_{i,2}) \cdots (L_{i,k})$, where $(L_{i,\ell})$ is a list of the points of $B_{i,\ell}$ written in a fixed order as a permutation. Note that one of $\sigma$, $\sigma^2$,...,$\sigma^{k-1}$ is an automorphism in $H$ which maps $x$ to $y$ (and preserves the parallel classes).

Now take a copy of $H$ (denoted $H'$) on the vertex set $\{1', 2', \ldots, (k^2)'\}$ with edges corresponding to those of $H$. Using the parallel class $\Pi_0$, form $k$ additional edges of size $2k$ as follows. For each $i \in \{1, 2, \ldots, k\}$, let $e_i$ be the edge containing the $k$ vertices of the $i^{\text{th}}$ row of $A$ along with the corresponding vertices in $H'$. In particular, for each $i \in \{1, 2, \ldots, k\}$,

$$e_i = \{(i-1)k+1, (i-1)k+2, \ldots, (i-1)k+k, ((i-1)k+1)', ((i-1)k+2)', \ldots, ((i-1)k+k)'\}.$$

Let $H^*$ be the hypergraph on the vertex set $V(H) \cup V(H')$ with edge set $E(H) \cup E(H')$ along with the $k$ edges of the form $e_i$, each of size $2k$. Note that $H^*$ is a connected linear non-uniform hypergraph with edges of sizes $k$ and $2k$. By composing the automorphisms described for $H$ with the automorphism which maps each vertex of $H$ to its copy in $H'$, we can verify that $H^*$ is also vertex-transitive. However the edge-connectivity $\kappa'(H^*) = k$ whereas the degree $\Delta(H^*) = k + 1$, and so $H^*$ is not maximally edge-connected.

As an example of this construction, let $P$ be the affine plane of order 3 on the point set $\{1, 2, \ldots, 9\}$ with lines shown in Table 4.1.

| | | | |
|---|---|---|---|
| $\Pi_0$ : | $\{1, 2, 3\}$, | $\{4, 5, 6\}$, | $\{7, 8, 9\}$ |
| $\Pi_1$ : | $\{1, 4, 7\}$, | $\{2, 5, 8\}$, | $\{3, 6, 9\}$ |
| $\Pi_2$ : | $\{1, 5, 9\}$, | $\{3, 4, 8\}$, | $\{2, 6, 7\}$ |
| $\Pi_3$ : | $\{1, 6, 8\}$, | $\{2, 4, 9\}$, | $\{3, 5, 7\}$ |

Table 4.1: Lines of the affine plane $P$

Note that the parallel classes $\Pi_0, \ldots, \Pi_3$ of $P$ are indicated horizontally. Now let $H$ denote $P$ with $\Pi_0$ removed and observe that $H$ may be viewed as a connected linear 3-uniform hypergraph. By our argument above, $H$ is vertex-transitive.

Now let $H'$ be the hypergraph on the vertex set $\{1', 2', \ldots, 9'\}$ with edges corresponding to those of $H$. Form three additional edges of size 6 as follows:

$$\{1, 2, 3, 1', 2', 3'\}, \{4, 5, 6, 4', 5', 6'\}, \{7, 8, 9, 7', 8', 9'\}.$$

Let $H^*$ be the hypergraph on the vertex set $\{1, 2, \ldots, 9, 1', 2', \ldots, 9'\}$ with the edge set consisting of $E(H) \cup E(H')$ along with the three edges of size 6. Note that $H^*$ is a connected linear non-uniform hypergraph with edges of sizes 3 and 6 and, by our argument above, $H^*$ is also vertex-transitive. However the edge-connectivity $\kappa'(H^*) = 3$ whereas the degree $\Delta(H^*) = 4$, and so $H^*$ is not maximally edge-connected.

## 4.2 A Generalisation of Mader's Theorem

Let $H$ be a hypergraph with vertex set $V(H)$. For $Y \subseteq V(H)$, we let $\partial(Y)$ denote the set of edges in $H$ in which each edge has at least one vertex in $Y$ and at least one vertex in $V \setminus Y$. A key part of the proof of our main theorem is the following lemma.

**Lemma 4.3.** *Let $H$ be a $k$-uniform hypergraph and $X, Y \subseteq V(H)$. Then*

$$|\partial(X \cup Y)| + |\partial(X \cap Y)| \leqslant |\partial(X)| + |\partial(Y)|.$$

**Proof.** In a Venn diagram of two (possibly intersecting) sets, there are four distinct regions. For our subsets $X$ and $Y$, these are $X \setminus Y$, $Y \setminus X$, $X \cap Y$ and $(X \cup Y)^C$. Any edges that contain vertices in more than one of these regions will contribute to the values of $|\partial(X \cup Y)| + |\partial(X \cap Y)|$ and $|\partial(X)| + |\partial(Y)|$.

When $k = 2$, we have $\binom{4}{2} = 6$ pairs of regions and hence, six types of relevant edges that may exist. By checking each pair of regions, we see that $|\partial(X)| + |\partial(Y)|$ accounts for all of the edges of $\partial(X \cup Y) \cup \partial(X \cap Y)$ but counts any edges with vertices in both $X \setminus Y$ and $Y \setminus X$ twice, whereas $|\partial(X \cup Y)| + |\partial(X \cap Y)|$ does not count these edges at all.

When $k = 3$, we have $\binom{4}{3} = 4$ additional types of possible edges. Then $|\partial(X)| + |\partial(Y)|$ accounts for all of the edges of $\partial(X \cup Y) \cup \partial(X \cap Y)$ but counts any edges with vertices in both $X \setminus Y$ and $Y \setminus X$ twice, whereas $|\partial(X \cup Y)| + |\partial(X \cap Y)|$ counts these edges at most once.

When $k \geqslant 4$, there is only one additional type of possible edge, one that contains vertices from all four regions. Such edges are contained in each of $\partial(X)$, $\partial(Y)$, $\partial(X \cup Y)$, and $\partial(X \cap Y)$, and so they are counted twice by both $|\partial(X)| + |\partial(Y)|$ and $|\partial(X \cup Y)| + |\partial(X \cap Y)|$. $\square$

We now proceed with the proof of our main result in this chapter. Note that the examples detailed in Section 4.1 imply the necessity of the linear and uniform conditions in the statement of this result.

**Theorem 4.2.** *Let $H$ be a linear $k$-uniform hypergraph with $k \geqslant 3$. If $H$ is vertex-transitive and connected, then $H$ is maximally edge-connected.*

**Proof.** Since $\kappa'(H) \leqslant \Delta(H)$, it suffices to show that $\kappa'(H) \geqslant \Delta(H)$. Choose a proper non-empty subset $X \subset V(H)$ such that

$(i)$ $|\partial(X)|$ is minimum and

$(ii)$ $|X|$ is minimum (subject to $(i)$).

Note that by condition $(i)$, $|\partial(X)| = \kappa'(H)$, so it suffices to show that $|\partial(X)| \geqslant \Delta(H)$. By definition $\partial(X) = \partial(V(H) \setminus X)$, so condition $(ii)$ implies that $|X| \leqslant \frac{1}{2}|V(H)|$. In [21] such a set $X$ is referred to as an *edge atom*.

Now suppose there exists $\phi \in \mathrm{Aut}(H)$ such that $\emptyset \neq X \cap \phi(X) \neq X$. Then by Lemma 4.3,

$$|\partial(X \cup \phi(X))| + |\partial(X \cap \phi(X))| \leqslant |\partial(X)| + |\partial(\phi(X))| = 2|\partial(X)|.$$

If $|\partial(X \cup \phi(X))| < |\partial(X)|$ then the set $X \cup \phi(X)$ contradicts our choice of $X$ by condition $(i)$. Otherwise, $|\partial(X \cap \phi(X))| \leqslant |\partial(X)|$, but then $X \cap \phi(X)$ contradicts our choice of $X$ by condition $(i)$ or $(ii)$. Therefore, for every $\phi \in \mathrm{Aut}(H)$, either $X \cap \phi(X) = X$ or $X \cap \phi(X) = \emptyset$. For this reason, we say that $X$ is a *block of imprimitivity* (for more information on this terminology, see [21]). This proof so far has loosely followed the proof of Mader's Theorem found in [21]; however, to proceed from here we must make use of original techniques.

Now, for $Y \subseteq V(H)$, we let $\partial_i(Y)$ denote the set of edges in $H$ in which each edge has exactly $i$ vertices in $Y$ and $k - i$ vertices in $V \setminus Y$. Note that $\partial(Y) = \bigcup_{i=1}^{k-1} \partial_i(Y)$. For any

$x \in X$ and $1 \leqslant i \leqslant k$, let $a_i$ be the number of neighbours of $x$ in $X$ which occur in edges of $\partial_i(X)$. Similarly, let $b_i$ be the number of neighbours of $x$ in $V \setminus X$ which occur in edges of $\partial_i(X)$. Since $X$ is a block of imprimitivity, the values of $a_i$ and $b_i$ for $1 \leqslant i \leqslant k$ do not depend on the choice of $x \in X$. Indeed, if $x$ and $y$ are two vertices in $X$ then since $X$ is a block of imprimitivity, any automorphism $\phi$ such that $\phi(x) = y$ must satisfy $X \cap \phi(X) = X$. In particular, $|\partial_i(X)| = |\partial_i(\phi(X))|$ for any $i$, so any neighbour of $x$ counted by $a_i$ has an image that is a neighbour of $y$. Repeating this argument with an automorphism that maps $y$ to $x$ verifies that $a_i$ is independent of our choice of vertex. A similar argument verifies that $b_i$ is independent of our choice of vertex.

If $|X| = 1$, then $\partial(X) = \Delta(H)$, so from now on we assume $|X| \geqslant 2$. Let $x, y \in X$ and note that by definition, $a_k = |\partial_k(X)|(k-1)$ and $b_1 = |\partial_1(X)|(k-1)$, and thus

$$|\partial(X \setminus \{y\})| = |\partial(X)| + \frac{a_k}{k-1} - \frac{b_1}{k-1}.$$

So, if $a_k \leqslant b_1$ then $X \setminus \{y\}$ contradicts our choice of $X$. Otherwise we assume $a_k > b_1$ which implies $\partial_k(X)$ is nonempty.

For the remainder of the proof, we will refer to an edge contained in the set $\partial_i(X)$ as a $\partial_i$-edge. If $|X| = k$ then $X$ is simply a single $\partial_k$-edge. Then by linearity, the only edges of $\partial(X)$ are $\partial_1$-edges and by vertex transitivity, $|\partial(X)| = k(\Delta(H) - 1)$. Now $|\partial(X)| = k(\Delta(H) - 1)$ is strictly greater than $|\partial(\{x\})| = \Delta(H)$ as long as $k \geqslant 3$ and $\Delta(H) \geqslant 2$. But this is easy to confirm as a connected hypergraph $H'$ with $\Delta(H') = 1$ would be a single edge of $k$ vertices. So $\{x\}$ contradicts our choice of $X$ by condition $(ii)$. Hence $|X|$ must be strictly greater than $k$.

Now since $a_k \neq 0$ and $X$ is a block of imprimitivity, every vertex of $X$ must be incident with at least one $\partial_k$-edge. Then the collection of $\partial_k$-edges is either a collection of non-intersecting edges or a collection of edges in which each vertex of $X$ lies at the intersection of at least two of these edges. In the first case, there must be paths in $H$ connecting the disjoint $\partial_k$-edges. But then any one of the $\partial_k$-edges would be a better choice for our set $X$ by condition $(ii)$.

Therefore, we know that each vertex of $X$ lies at the intersection of at least two $\partial_k$-edges. For $x \in X$, let $r_x$ be the number of $\partial_k$-edges within $X$ which contain $x$. Observe that $r_x = \frac{a_k}{k-1}$ and so $r_x$ does not depend on our choice of $x$. So we will simply use $r$ to denote the number of $\partial_k$-edges within $X$ which contain any given vertex of $X$. Observe that the degree of $H$, $\Delta(H)$, must be strictly greater than $r$, since otherwise, every neighbour of any vertex in $X$ must also be a vertex of $X$ and therefore either $H$ is disconnected or $X = V(H)$.

In addition, we note that $\Delta(H)$ must be strictly greater than $|\partial(X)|$, since otherwise $\kappa'(H) = |\partial(X)| = \Delta(H)$. Also $|\partial(X)| \geqslant \frac{|X|(\Delta(H)-r)}{k-1}$, since the edges of $\partial(X)$ can be shared by at most $k-1$ vertices of $X$. Therefore,

$$\Delta(H) > \frac{|X|(\Delta(H) - r)}{k-1};$$

rearranging for $|X|$ gives a strict upper bound

$$|X| < \frac{\Delta(H)(k-1)}{\Delta(H) - r}.$$

Observe that $X$ contains the vertex $x$ and at least $r(k-1)$ other vertices. So

$$\frac{\Delta(H)(k-1)}{\Delta(H) - r} > |X| \geqslant 1 + r(k-1).$$

This implies $\Delta(H)(k-1) > (\Delta(H) - r) + (\Delta(H) - r)r(k-1)$ and since $\Delta(H) - r > 0$, we have $\Delta(H)(k-1) > (\Delta(H) - r)r(k-1)$. Dividing both sides by $k - 1 \neq 0$ we have $\Delta(H) > (\Delta(H) - r)r$.

Now $\Delta(H) > (\Delta(H) - r)r$ rearranges to $r^2 > \Delta(H)(r-1)$. To make the arithmetic easier, let $d$ be the difference $\Delta(H) - r$, and note that $d$ is a positive integer. Substitute $d + r$ for $\Delta(H)$ and continue:

$$\begin{aligned} & r^2 > (d+r)(r-1) \\ \Rightarrow\ & r^2 > r^2 + dr - d - r \\ \Rightarrow\ & 0 > dr - d - r \\ \Rightarrow\ & d > r(d-1). \end{aligned}$$

If $d > 1$ then $r < \frac{d}{d-1}$, a ratio of two consecutive positive integers, so $1 \leqslant r < \frac{d}{d-1} \leqslant 2$ which implies $r = 1$. This means that each vertex of $X$ is incident with a single $\partial_k$-edge of $X$. But we previously established that each vertex of $X$ lies at the intersection of at least two $\partial_k$-edges, a contradiction.

Finally, if $d = 1$, then each vertex is incident with a single boundary edge. Recall the lower bound $|X| \geqslant 1 + r(k-1)$. Replacing $r$ with $\Delta(H) - d = \Delta(H) - 1$, we get $|X| \geqslant 1 + (\Delta(H) - 1)(k-1)$, which is strictly greater than $(\Delta(H) - 1)(k-1)$. So we have $\frac{|X|}{k-1} > \Delta(H) - 1$. Observe that $|\partial(X)| \geqslant \frac{|X|}{k-1}$ since boundary edges take up vertices of $X$ at most $k-1$ at a time. Therefore $\frac{|X|}{k-1} > \Delta(H) - 1$ implies $|\partial(X)| > \Delta(H) - 1$ and so $\kappa'(H) = |\partial(X)| \geqslant \Delta(H)$. $\square$

# Chapter 5

# Summary and Open Problems

In summary, this thesis discussed various novel concepts within graph theory, specifically, the notion of an $n$-line e.c. graph and an $n$-e.c. hypergraph. In addition, we extended a classic result of Mader from graphs to hypergraphs.

In Chapter 2, we focused on finding necessary conditions for the existence of existentially closed line graphs and then presented constructions for generating infinitely many such graphs. Next, we considered existentially closed line graphs of planar graphs and in comparison, proved that there are only finitely many such graphs. Open problems include finding more families of 2-line e.c. graphs and potentially finding conditions that are both necessary and sufficient for their existence. We then considered the line graphs of hypergraphs and presented constructions to build infinitely many uniform 2-line e.c. hypergraphs. By Theorem 2.17, there exist sufficiently large uniform $n$-line e.c. hypergraphs for any value of $n$. However, as this is an asymptotic result, there is room for the discovery of families of $n$-line e.c. hypergraphs for values of $n \geqslant 3$.

In Chapter 3, we extended the notion of an existentially closed graph to an existentially closed hypergraph. We then developed constructions for generating infinite families of these hypergraphs. Interestingly, our constructions make use of structures from design theory such as Latin squares and block designs. This adds to the ever-growing list of applications of design theory to other areas of mathematics. In Section 3.3, we noted that most balanced incomplete block designs are 1-e.c. when viewed as hypergraphs. It remains to be seen if designs viewed as hypergraphs could be $n$-e.c. for larger values of $n$.

One possible area of future work involves testing *Paley hypergraphs* for the existential closure property. As we mentioned in Section 2.3, sufficiently large Paley graphs are known to be $n$-e.c. for any $n$ [3, 4]. Fortunately, the notion of a Paley hypergraph already exists in the literature. This extension of Paley graphs was first introduced by Kocay in [32] and was later refined by Potočnik and Šajna in [38] and again by Dueck (Gosselin) in [22]. Additionally, as our definition of existentially closed hypergraphs is specific to uniform hypergraphs, it remains to see if there is a similar concept for non-uniform hypergraphs.

In Chapter 4, we extended Mader's connectivity result on vertex-transitive graphs to an analogous result on vertex-transitive hypergraphs. In particular, we proved that if $H$ is a linear $k$-uniform hypergraph with $k \geqslant 3$ and $H$ is vertex-transitive, then $H$ is maximally edge-connected. Our generalisation of Mader's Theorem introduced two hypergraph-specific terms to the hypothesis, namely that we require a linear uniform hypergraph. We demonstrated that if we drop either of these restrictions in the hypothesis then we can generate an infinite family of hypergraphs that are vertex-transitive but not maximally edge-connected. So in general, these additions to the hypothesis are required. Open problems include finding families of vertex-transitive hypergraphs that may not be uniform or linear but are still maximally edge-connected or, more generally, finding other families of hypergraphs that are maximally edge-connected.

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# Appendix A

# Code used to test for 2-line e.c. graphs

Original code `matrix.c` takes input from `plantri` and outputs the rows of the lower triangle of the adjacency matrix for each graph.

```
//matrix.c
#include<stdio.h>
#include<stdlib.h>

int convert(int);

int m=3;   //the ceiling of the number of possible edges divided by 6,
           //the number of ascii chars per graph w/o first char
char G[3]; //G[m]
int F[3];  //F[m]
int tot=9; //total number of graphs in input file

int main()
{
 int i,j;
 char var;
 FILE * fp;
 FILE * fp2;

 fp=fopen("planar.6","r");
```

```
 fp2=fopen("mat.6","w");

for(j=0;j<tot;j++)
   {
   for(i=0;i<m+2;i++)
      {
      fscanf(fp,"%c",&var);
      if(var!=32 && i!=0){G[i-1]=var;}
      }

   for (i=0; i<m; i++)
      {
      F[i]=convert(G[i]-63);
      fprintf(fp2, "%06d", F[i]);
      }
   fprintf(fp2, "\n");
   }
 fclose(fp);
 fclose(fp2);
 return 0;
}

int convert(int dec) //converts binary strings into decimal numbers
{
    if (dec == 0)
    {
        return 0;
    }
    else
    {
        return (dec % 2 + 10 * convert(dec / 2));
    }
}
```

Original code `2ec.c` takes input from `matrix.c` and checks for the 2-line e.c. property.

```
//2ec.c
/*

Compiled using: gcc <fname.c>
Executed by:  a.out n

*/

#include<stdio.h>
#include<stdlib.h>
#include<time.h>

/* Global variables. */
int n=6;      // number of vertices
int A[6][6];  // A[n][n]
int m=3;      // the ceiling of the number of possible edges divided by 6
int str[3*6]; // str[m*6]
int tot=9;    // total number of graphs

int card(int a, int b, int c, int d);

int main()
{
 int r = 1; // assume 2-line ec until we learn otherwise
 int p0 = 1;
 int p1 = 1;
 int p2 = 1;
 int p3 = 1;
 int i,j,k,l;
 int P[4]={0,0,0,0};
 int x,y,z,u,v,w;
 int card_a, card_b;
 FILE * fp;
```

```
 FILE * fp2;
 char var;

 fp=fopen("mat.6","r");
 fp2=fopen("results.6","w");

for(l=0;l<tot;l++)
{
for(i=0;i<m*6+1;i++)
{
fscanf(fp,"%c",&var);
if(var!=10){str[i]=var-48;}

}

 k=0;
 for (j=1;j<n;j++)
    {
    for (i=0;i<j;i++)
       {
       A[i][j]=str[k];
       k++;
       }
    }

 for (x = 0; x < n; x++)
    {
    for (y = 0; y < n; y++)
       {
       if (A[x][y] == 1)
         {
         for (u = x; u < n; u++)
            {
            if (u == x){k = y+1;}
```

```
if (u != x){k = 0;}
for (v = k; v < n; v++)
   {
   if (A[u][v] == 1)
     {
     P[0]=0;
     P[1]=0;
     P[2]=0;
     P[3]=0;

     for (w = 0; w < n; w++)
        {
         for (z = 0; z < n; z++)
            {
            if (A[w][z] == 1)
              {
              card_a = card(x,y,w,z);
              card_b = card(u,v,w,z);
              if ((card_a == 2)||(card_b == 2)) continue;
              P[2 * card_b + card_a] = 1;
              }
            }
        }

     if (P[0]==0 || P[1]==0 || P[2]==0 || P[3]==0)
       {
       r = 0;
       if(P[0]==0){p0=0;}
       if(P[1]==0){p1=0;}
       if(P[2]==0){p2=0;}
       if(P[3]==0){p3=0;}
       }

     }
```

```
              }
            }
          }
        }
     }

 if (r == 1)
   {fprintf(fp2, "Graph %d is 2-line e.c. \n", l);}
r=1;
p0=1;
p1=1;
p2=1;
p3=1;
}
 fclose(fp);
 fclose(fp2);
 return 0;
}

int card (int a, int b, int c, int d)
 {
 if ((a==c && b==d)||(a==d && b==c)) return 2;
 if (a!=c && a!=d && b!=c && b!=d) return 0;
 if ((a==c && b!=d)||(a==d && b!=c)||(b==c && a!=d)||(b==d && a!=c))
    {return 1;}
 }
```

# Appendix B

# Example Hypergraph

**Example B.1.** *A 3-uniform hypergraph constructed from a complete set of MOLS of order 4 as described in Section 3.3:*

$H = (V, E)$ where $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, a, b, c, d, e, f\}$ and the 80 edges of $E$ are represented by the unordered triples listed below:

| | | | |
|---|---|---|---|
| 012 | 013 | 023 | 123 |
| 456 | 457 | 467 | 567 |
| 89a | 89b | 8ab | 9ab |
| cde | cdf | cef | def |
| 048 | 04c | 08c | 48c |
| 159 | 15d | 19d | 59d |
| 26a | 26e | 2ae | 6ae |
| 37b | 37f | 3bf | 7bf |
| 069 | 06f | 09f | 69f |
| 178 | 17e | 18e | 78e |
| 24b | 24d | 2bd | 4bd |
| 35a | 35c | 3ac | 5ac |
| 07a | 07b | 0ab | 7ab |
| 16b | 16c | 1bc | 6bc |
| 258 | 25f | 28f | 58f |
| 349 | 34e | 39e | 49e |
| 05b | 05e | 0be | 5be |
| 14a | 14f | 1af | 4af |
| 279 | 27c | 29c | 79c |
| 368 | 36d | 38d | 68d |