Coends and categorical Hopf algebras

by

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Abstract

K. Shimizu has proved that, in a braided finite tensor category over an algebraically closed field, the triviality of the Müger centre implies that a certain Hopf pairing is non-degenerate. It is an open question whether the hypothesis that the base field is algebraically closed is necessary. In this thesis, we show, following some unpublished notes of Y. Sommerhäuser and his coauthors, that this hypothesis is indeed not necessary in the case of the category of finite-dimensional modules over a finite-dimensional quasitriangular ribbon Hopf algebra H. In this category, the coend can be constructed as the dual space of H.

We first review some basics of category theory, the construction of a coend as a categorical Hopf algebra, and duals and homomorphic images of categorical Hopf algebras. We then prove the result mentioned above. We conclude by constructing an example of a similar category where the dual space fails to be a coend.

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Statement of contribution

The material in Chapter 1 is standard and can be found in many textbooks, for example in [5], [9], and [13]. The material in Chapter 2 is more recent, but can now also be found in textbooks, for example in [6]. In Chapter 3, we provide a detailed proof of a frequently needed fact for which we are not aware of a reference in the literature, namely Theorem 3.2.3. This result was obtained during joint sessions with my supervisor. The main result of Chapter 4 is Theorem 4.6.1, which is taken from unpublished notes of the authors of [7]. In Chapter 5, I supplied the proof of Proposition 5.2.2. This result plays a role in the construction of the non-example in 5.3, which was again obtained during joint sessions with my supervisor.

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Chapter 1

Category theory

This chapter provides a review of the basics of category theory, along with some related preliminary concepts.

1.1 Categories, functors, and natural transformations

We begin with the formal definition of a category.

Definition 1.1.1. A *category* $\mathcal{C}$ consists of a class $\mathrm{Ob}(\mathcal{C})$, whose elements are called the *objects* of $\mathcal{C}$; a class $\mathrm{Hom}(\mathcal{C})$, whose elements are called the *morphisms* of $\mathcal{C}$; and maps

$$\begin{aligned} \mathrm{id}\colon & \mathrm{Ob}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{C}) \\ \mathrm{dom}\colon & \mathrm{Hom}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}) \\ \mathrm{cod}\colon & \mathrm{Hom}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}) \\ \circ\colon & \mathrm{Hom}(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \mathrm{Hom}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{C}), \end{aligned}$$

where

$$\mathrm{Hom}(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \mathrm{Hom}(\mathcal{C}) = \{(f,g) \in \mathrm{Hom}(\mathcal{C}) \times \mathrm{Hom}(\mathcal{C}) : \mathrm{dom}\, f = \mathrm{cod}\, g\},$$

satisfying the following axioms:

1. For all objects $X \in \mathrm{Ob}(\mathcal{C})$,

$$\mathrm{dom}(\mathrm{id}_X) = \mathrm{cod}(\mathrm{id}_X) = X.$$

2. For all morphisms $f \in \mathrm{Hom}(\mathcal{C})$ with $\operatorname{dom} f = X$ and $\operatorname{cod} f = Y$,

$$\mathrm{id}_Y \circ f = f \circ \mathrm{id}_X = f.$$

3. For all morphisms $f, g, h \in \mathrm{Hom}(\mathcal{C})$ with $\operatorname{dom} f = \operatorname{cod} g$ and $\operatorname{dom} g = \operatorname{cod} h$,

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Note that $\mathrm{id}(X)$ is denoted by id_X. For any objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, we denote by $\mathrm{Hom}(X, Y)$, or $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ if the category is to be emphasized, the class of all morphisms $f \in \mathrm{Hom}(\mathcal{C})$ with $\operatorname{dom} f = X$ and $\operatorname{cod} f = Y$. We also write $f \colon X \to Y$ to indicate that $f \in \mathrm{Hom}(X, Y)$. We call these classes *hom-sets*, despite the existence of categories for which the hom-sets are proper classes. See, for instance, functor categories in [9, Ch. II.4, p. 41]. We will refer to the objects in $\mathrm{Ob}(\mathcal{C})$ simply as *objects in* $\mathcal{C}$, and the morphisms in $\mathrm{Hom}(\mathcal{C})$ as *morphisms in* $\mathcal{C}$.

One immediate example of a category is the category **Set** of sets together with functions between sets. Categories are particularly useful for describing sets with a mathematical structure, together with morphisms that preserve the structure. Some examples include the category of groups together with group homomorphisms; the category of vector spaces over a field K together with K-linear maps; the category of left A-modules, where A is an algebra, together with A-linear maps; the category of topological spaces together with continuous functions; and so on. The composition in each of the above categories is simply composition of functions.

Every category $\mathcal{C}$ gives rise to a second category, known as the *opposite category* $\mathcal{C}^{\mathrm{op}}$. Its objects are the objects of $\mathcal{C}$, and its morphisms are obtained by "reversing the arrows." In other words, it is defined by taking $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and, for all objects X and Y in $\mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X).$$

The composition $\circ_{\mathcal{C}^{\mathrm{op}}}$ in this category is defined by

$$g \circ_{\mathcal{C}^{\mathrm{op}}} f = f \circ g,$$

for all pairs $(f, g) \in \mathrm{Hom}(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \mathrm{Hom}(\mathcal{C})$.

Another construction of a category from a given category $\mathcal{C}$ is the *product category* $\mathcal{C} \times \mathcal{C}$, whose objects are pairs (X, Y) of objects in $\mathcal{C}$ and whose morphisms are pairs (f, g) of morphisms in $\mathcal{C}$. Its composition is defined by

$$(h, k) \circ (f, g) = (h \circ f, k \circ g)$$

and $\mathrm{id}_{(X,Y)} = (\mathrm{id}_X, \mathrm{id}_Y)$.

We call a morphism $f\colon X \to Y$ in a category $\mathcal{C}$ an *isomorphism* if there exists a morphism $g\colon Y \to X$ in $\mathcal{C}$ such that

$$g \circ f = \mathrm{id}_X \quad \text{and} \quad f \circ g = \mathrm{id}_Y.$$

Observe that this notion generalizes, and unifies, the usual notions of isomorphism for various mathematical structures.

We now define the notion of a functor, which can be viewed as a morphism of categories.

Definition 1.1.2. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of a map $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and a map $F\colon \mathrm{Hom}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{D})$, both denoted by F, satisfying the following axioms:

1. For any object $X \in \mathrm{Ob}(\mathcal{C})$,
$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}.$$

2. For any morphism $f \in \mathrm{Hom}(\mathcal{C})$,
$$\begin{aligned} \mathrm{dom}(F(f)) &= F(\mathrm{dom}(f)) \\ \mathrm{cod}(F(f)) &= F(\mathrm{cod}(f)). \end{aligned}$$

3. For any pair $(f, g) \in \mathrm{Hom}(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \mathrm{Hom}(\mathcal{C})$,
$$F(f \circ g) = F(f) \circ F(g).$$

The second axiom in the above definition can be restated by saying that for any morphism $f\colon X \to Y$ in $\mathcal{C}$, the morphism $F(f)$ is a morphism $F(X) \to F(Y)$ in $\mathcal{D}$.

A related concept is that of a *contravariant functor* $F\colon \mathcal{C} \to \mathcal{D}$, which assigns to each morphism $f\colon X \to Y$ in $\mathcal{C}$ a morphism $F(f)\colon F(Y) \to F(X)$ in $\mathcal{D}$ and satisfies

$$F(f \circ g) = F(g) \circ F(f)$$

for all pairs $(f, g) \in \mathrm{Hom}(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \mathrm{Hom}(\mathcal{C})$. Note the reversal of order. A functor as in Definition 1.1.2 is then called a *covariant functor*.

An important example of covariant and contravariant functors comes from hom-sets.

Example 1.1.1. For each object X in a category $\mathcal{C}$, we have the *covariant hom-functor*

$$\begin{aligned} \operatorname{Hom}(X,-)\colon \mathcal{C} &\to \mathbf{Set} \\ Y &\mapsto \operatorname{Hom}(X,Y) \\ f &\mapsto \operatorname{Hom}(X,f), \end{aligned}$$

where $\operatorname{Hom}(X,f) \in \operatorname{Hom}(\mathbf{Set})$ is the map

$$\begin{aligned} \operatorname{Hom}(X,f)\colon \operatorname{Hom}(X,\operatorname{dom} f) &\to \operatorname{Hom}(X,\operatorname{cod} f) \\ g &\mapsto f \circ g. \end{aligned}$$

We call this map *post-composition with* f.

We also have, for each object Y in $\mathcal{C}$, the *contravariant hom-functor*

$$\begin{aligned} \operatorname{Hom}(-,Y)\colon \mathcal{C} &\to \mathbf{Set} \\ X &\mapsto \operatorname{Hom}(X,Y) \\ f &\mapsto \operatorname{Hom}(f,Y), \end{aligned}$$

where $\operatorname{Hom}(f,Y) \in \operatorname{Hom}(\mathbf{Set})$ is the map

$$\begin{aligned} \operatorname{Hom}(f,Y)\colon \operatorname{Hom}(\operatorname{cod} f,Y) &\to \operatorname{Hom}(\operatorname{dom} f,Y) \\ g &\mapsto g \circ f. \end{aligned}$$

We call this map *pre-composition with* f. Observe that for composable morphisms f and g in $\mathcal{C}$, the map

$$\operatorname{Hom}(f \circ g,Y)\colon \operatorname{Hom}(\operatorname{cod} f,Y) \to \operatorname{Hom}(\operatorname{dom} g,Y)$$

is defined by

$$h \mapsto h \circ (f \circ g),$$

and the map

$$\operatorname{Hom}(g,Y) \circ \operatorname{Hom}(f,Y)\colon \operatorname{Hom}(\operatorname{cod} f,Y) \to \operatorname{Hom}(\operatorname{dom} g,Y)$$

is defined by

$$h \mapsto h \circ f \mapsto (h \circ f) \circ g = h \circ (f \circ g).$$

Hence

$$\mathrm{Hom}(f \circ g, Y) = \mathrm{Hom}(g, Y) \circ \mathrm{Hom}(f, Y)$$

so that $\mathrm{Hom}(-, Y)$ is indeed contravariant.

The following concept can be viewed as a morphism of functors.

Definition 1.1.3. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* η from F to G is a function that assigns to each object X in $\mathcal{C}$ a morphism $\eta_X\colon F(X) \to G(X)$ in $\mathcal{D}$ such that for any morphism $f\colon X \to Y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

If η_X is an isomorphism in $\mathcal{D}$ for every object X in $\mathcal{C}$, then η is called a *natural isomorphism*; two functors are said to be isomorphic if there exists a natural isomorphism between them.

The concept of a natural isomorphism allows us to define the notions of equivalent and isomorphic categories.

Definition 1.1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then $\mathcal{C}$ and $\mathcal{D}$ are said to be *equivalent* (respectively, *isomorphic*) if there exists functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ such that the functor $G \circ F\colon \mathcal{C} \to \mathcal{C}$ is isomorphic (respectively, equal) to the identity functor $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ and the functor $F \circ G\colon \mathcal{D} \to \mathcal{D}$ is isomorphic (respectively, equal) to the identity functor $\mathrm{id}_{\mathcal{D}}\colon \mathcal{D} \to \mathcal{D}$.

1.2 Tensor categories and braidings

A tensor category is a category equipped with a tensor product:

Definition 1.2.1. A *tensor category* is a category $\mathcal{C}$ together with a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (for which we denote $\otimes(X, Y) = X \otimes Y$ and $\otimes(f, g) = f \otimes g$), a *unit object* I, and

i) a natural isomorphism λ, called the *left unit constraint*, from the functor defined by

$$X \mapsto I \otimes X$$
$$f \mapsto \mathrm{id}_I \otimes f,$$

for all objects X in $\mathcal{C}$ and morphisms f in $\mathcal{C}$, to the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$;

ii) a natural isomorphism ρ, called the *right unit constraint*, from the functor defined by

$$\begin{aligned} X &\mapsto X \otimes I \\ f &\mapsto f \otimes \mathrm{id}_I, \end{aligned}$$

for all objects X in $\mathcal{C}$ and morphisms f in $\mathcal{C}$, to the identity functor $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$; and

iii) a natural isomorphism α, called the *associativity constraint*, from $\otimes \circ (\otimes \times \mathrm{id}_{\mathcal{C}})$ to $\otimes \circ (\mathrm{id}_{\mathcal{C}} \times \otimes)$,

satisfying the following axioms:

1. *Pentagon Axiom*: For all objects X, Y, Z, W in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccccc} & & (X \otimes Y) \otimes (Z \otimes W) & & \\ & \overset{\alpha_{X \otimes Y, Z, W}}{\nearrow} & & \overset{\alpha_{X, Y, Z \otimes W}}{\searrow} & \\ ((X \otimes Y) \otimes Z) \otimes W & & & & X \otimes (Y \otimes (Z \otimes W)) \\ \downarrow{\scriptstyle \alpha_{X,Y,Z} \otimes \mathrm{id}_W} & & & & \uparrow{\scriptstyle \mathrm{id}_X \otimes \alpha_{Y,Z,W}} \\ (X \otimes (Y \otimes Z)) \otimes W & & \xrightarrow{\alpha_{X, Y \otimes Z, W}} & & X \otimes ((Y \otimes Z) \otimes W) \end{array} \tag{1.1}$$

2. *Triangle Axiom*: For all objects X, Y in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} (X \otimes I) \otimes Y & \xrightarrow{\alpha_{X,I,Y}} & X \otimes (I \otimes Y) \\ & {\scriptstyle \rho_X \otimes \mathrm{id}_Y} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_X \otimes \lambda_Y} & \\ & X \otimes Y & \end{array} \tag{1.2}$$

The naturality of the left unit constraint λ means that for all morphisms $f \colon X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} I \otimes X & \xrightarrow{\lambda_X} & X \\ \downarrow{\scriptstyle \mathrm{id}_I \otimes f} & & \downarrow{\scriptstyle f} \\ I \otimes Y & \xrightarrow{\lambda_Y} & Y \end{array}$$

commutes; the naturality of the right unit constraint ρ is analogous. The naturality of the associativity constraint α means that the diagram

$$\begin{array}{ccc}
(X \otimes Y) \otimes Z & \xrightarrow{\alpha_{X,Y,Z}} & X \otimes (Y \otimes Z) \\
\Big\downarrow{\scriptstyle (f\otimes g)\otimes h} & & \Big\downarrow{\scriptstyle f\otimes(g\otimes h)} \\
(X' \otimes Y') \otimes Z' & \xrightarrow{\alpha_{X',Y',Z'}} & X' \otimes (Y' \otimes Z')
\end{array}$$

commutes for all morphisms $f\colon X \to X'$, $g\colon Y \to Y'$, and $h\colon Z \to Z'$ in $\mathcal{C}$.

As a consequence of the axioms in Definition 1.2.1, the diagrams

$$\begin{array}{ccc}
(I \otimes X) \otimes Y & \xrightarrow{\alpha_{I,X,Y}} & I \otimes (X \otimes Y) \\
{\scriptstyle \lambda_X \otimes \mathrm{id}_Y} \searrow & & \swarrow {\scriptstyle \lambda_{X\otimes Y}} \\
& X \otimes Y &
\end{array}$$

and

$$\begin{array}{ccc}
(X \otimes Y) \otimes I & \xrightarrow{\alpha_{X,Y,I}} & X \otimes (Y \otimes I) \\
{\scriptstyle \rho_{X\otimes Y}} \searrow & & \swarrow {\scriptstyle \mathrm{id}_X \otimes \rho_Y} \\
& X \otimes Y &
\end{array}$$

commute, and we have

$$\lambda_I = \rho_I.$$

For a proof, see [5, Lem. XI.2.2, p. 283] and [5, Lem. XI.2.3, p. 284].

Note also that, since $\otimes$ is a functor, we have

$$(f \circ f') \otimes (g \circ g') = (f \otimes g) \circ (f' \otimes g')$$

whenever this composition is defined. We will use this interchange property extensively.

Tensor categories are also known as *monoidal categories* (cf. [9, Ch. XI, p. 252]). Our terminology follows [5, Def. XI.2.1, p. 282]. In [2], these terms have different meanings (cf. [2, Def. 2.1.1, p. 21] and [2, Def. 4.1.1, p. 65]).

The category of vector spaces over a field is the prototypical example:

Example 1.2.1. For any two vector spaces V and W over a field K, there exists a vector space $V \otimes W$, called the *tensor product* of V and W, with a bilinear map $\otimes\colon V \times W \to V \otimes W$ that is universal in the sense that, for any vector space U and bilinear map $\varphi\colon V \times W \to U$, there is a unique linear map $f\colon V \otimes W \to U$ such that $f \circ \otimes = \varphi$. The category of vector spaces over K, equipped with this tensor product, is a tensor category. The unit object is the base field K. The left unit constraint λ is defined on V as the isomorphism

$$\begin{aligned}\lambda_V\colon K \otimes V &\to V\\ \lambda \otimes v &\mapsto \lambda v,\end{aligned}$$

and the right unit constraint ρ is the defined on V as the isomorphism

$$\begin{aligned}\rho_V\colon V \otimes K &\to V\\ v \otimes \lambda &\mapsto \lambda v.\end{aligned}$$

The associativity constraint α is defined on spaces U, V, and W as the isomorphism

$$\begin{aligned}\alpha_{U,V,W}\colon (U \otimes V) \otimes W &\to U \otimes (V \otimes W)\\ (u \otimes v) \otimes w &\mapsto u \otimes (v \otimes w).\end{aligned}$$

We also have the following basic example.

Example 1.2.2. Let $\mathcal{C} = \mathbf{Set}$ and let $\otimes = \times$ be the Cartesian product. This means that $X \otimes Y = X \times Y$ and $f \otimes g = f \times g$ for all sets X and Y and functions f and g. Then $\mathcal{C}$ is a tensor category with the unit object being any set with exactly one element, which we denote by

$$I = \{*\}.$$

The left and right unit constraints are defined on each set X as

$$\begin{aligned}\lambda_X\colon \{*\} \times X &\to X\\ (*, x) &\mapsto x\end{aligned}$$

and

$$\begin{aligned}\rho_X\colon X \times \{*\} &\to X\\ (x, *) &\mapsto x,\end{aligned}$$

respectively, and the associativity constraint is defined on sets X, Y, and Z by

$$\begin{aligned}\alpha_{X,Y,Z}\colon (X \times Y) \times Z &\to X \times (Y \times Z)\\ ((x, y), z) &\mapsto (x, (y, z)).\end{aligned}$$

We say that a tensor category is *strict* if each λ_X, ρ_X, and $\alpha_{X,Y,Z}$ is an identity morphism. Every tensor category is tensor equivalent to a strict tensor category by [2, Thm. 2.8.5, p. 36] and [5, Prop. XI.5.1, p. 289]. Therefore, we will usually assume that the category is strict. Strictness requires in particular that $X \otimes I = X = I \otimes X$ and $f \otimes \mathrm{id}_I = f = \mathrm{id}_I \otimes f$ for all objects X and morphisms f, and that parentheses can be ignored in tensor products of several objects.

Example 1.2.3. Let G be a group (or a monoid). Let $\mathcal{C}$ be the category whose objects are the elements of G and whose morphisms are defined by $\mathrm{Hom}(g,h) = \{*\}$ for all $g, h \in G$. Then composition is the unique map

$$\begin{aligned} \mathrm{Hom}(g,h) \times \mathrm{Hom}(h,k) &\to \mathrm{Hom}(g,k) \\ (*,*) &\mapsto *. \end{aligned}$$

Now define the functor

$$\begin{aligned} \otimes\colon \mathcal{C} \times \mathcal{C} &\to \mathcal{C} \\ (g,h) &\mapsto gh \\ (*,*) &\mapsto *. \end{aligned}$$

Observe that for all $g, h, k \in G$,

$$(g \otimes h) \otimes k = (gh)k = g(hk) = g \otimes (h \otimes k)$$

and hence we can define an associativity constraint

$$\alpha_{g,h,k} = \mathrm{id}_{ghk} = *.$$

The corresponding unit object must be $I = e$, the identity element of G, and if we define the left and right unit constraints as $\lambda_g\colon e \otimes g \to g$ and $\rho_g\colon g \otimes e \to g$, respectively, then this makes $\mathcal{C}$ a tensor category. Notice also that

$$\lambda_g = \rho_g = \mathrm{id}_g = *$$

and hence $\mathcal{C}$ is an example of a strict tensor category.

A tensor category can also carry the structure of a braiding. We define a braided tensor category as follows.

Definition 1.2.2. Let $\mathcal{C}$ be a tensor category with tensor product $\otimes$ and associativity constraint α. A *braiding* on $\mathcal{C}$ is a natural isomorphism σ from the functor $\otimes$ to the functor $\otimes_{\text{rev}}\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$, defined by $(X,Y)\mapsto Y\otimes X$, such that the diagrams

$$\begin{array}{ccc}
(X\otimes Y)\otimes Z & \xrightarrow{\alpha_{X,Y,Z}} & X\otimes(Y\otimes Z) \\
{\scriptstyle \sigma_{X\otimes Y,Z}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_X\otimes\sigma_{Y,Z}} \\
Z\otimes(X\otimes Y) & & X\otimes(Z\otimes Y) \\
{\scriptstyle \alpha^{-1}_{Z,X,Y}}\big\downarrow & & \big\downarrow{\scriptstyle \alpha^{-1}_{X,Z,Y}} \\
(Z\otimes X)\otimes Y & \xleftarrow{\sigma_{X,Z}\otimes\mathrm{id}_Y} & (X\otimes Z)\otimes Y
\end{array} \tag{1.3}$$

and

$$\begin{array}{ccc}
X\otimes(Y\otimes Z) & \xrightarrow{\alpha^{-1}_{X,Y,Z}} & (X\otimes Y)\otimes Z \\
{\scriptstyle \sigma_{X,Y\otimes Z}}\big\downarrow & & \big\downarrow{\scriptstyle \sigma_{X,Y}\otimes\mathrm{id}_Z} \\
(Y\otimes Z)\otimes X & & (Y\otimes X)\otimes Z \\
{\scriptstyle \alpha_{Y,Z,X}}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_{Y,X,Z}} \\
Y\otimes(Z\otimes X) & \xleftarrow{\mathrm{id}_Y\otimes\sigma_{X,Z}} & Y\otimes(X\otimes Z)
\end{array} \tag{1.4}$$

commute.

Every braiding σ on $\mathcal{C}$ gives rise to a second braiding $\tilde{\sigma}$ on $\mathcal{C}$, defined as $\tilde{\sigma}_{X,Y}=\sigma^{-1}_{Y,X}$. The braiding axioms (1.3) and (1.4) are simply interchanged with respect to $\tilde{\sigma}$. From these axioms, one can also deduce the so-called *Yang-Baxter equation*:

$$\begin{aligned}
&(\mathrm{id}_Z\otimes\sigma_{X,Y})\circ(\sigma_{X,Z}\otimes\mathrm{id}_Y)\circ(\mathrm{id}_X\otimes\sigma_{Y,Z}) \\
&=(\sigma_{Y,Z}\otimes\mathrm{id}_X)\circ(\mathrm{id}_Y\otimes\sigma_{X,Z})\circ(\sigma_{X,Y}\otimes\mathrm{id}_Z)
\end{aligned} \tag{1.5}$$

The following example is known as the *trivial braiding*; for a non-trivial braiding, see (4.24).

Example 1.2.4. For vector spaces V and W over a field K, consider the map

$$\begin{aligned}
\varphi\colon V\times W &\to W\otimes V \\
(v,w) &\mapsto w\otimes v,
\end{aligned}$$

which can be expressed as $\otimes \circ \tau$, where

$$\begin{aligned} \tau \colon V \times W &\to W \times V \\ (v, w) &\mapsto (w, v). \end{aligned}$$

This map is bilinear, so by the universal property of the tensor product there is a linear map $\sigma_{V,W} \colon V \otimes W \to W \otimes V$ with

$$\sigma_{V,W}(v \otimes w) = \varphi(w, v) = w \otimes v.$$

We call $\sigma_{V,W}$ the *flip map*. The collection of these linear maps $\sigma_{V,W}$, indexed by vector spaces V and W, defines a braiding on the category of vector spaces over K.

We will frequently make use of the result proved in [5, Prop. XIII.1.2, p. 316], which, when interpreted in a strict category, states that every braiding σ on a tensor category $\mathcal{C}$ satisfies

$$\sigma_{I,X} = \mathrm{id}_X = \sigma_{X,I} \tag{1.6}$$

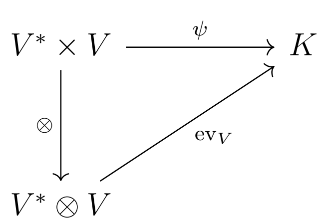
for all objects X in $\mathcal{C}$.

1.3 Dual objects

Let V be a finite-dimensional vector space over a field K and denote by $V^* = \mathrm{Hom}_K(V, K)$ its dual space. The map

$$\begin{aligned} \psi \colon V^* \times V &\to K \\ (\varphi, v) &\mapsto \varphi(v) \end{aligned}$$

is bilinear, so by the universal property there exists a linear map $\mathrm{ev}_V \colon V^* \otimes V \to K$, called the *evaluation map*, such that the diagram



commutes, i.e., $\mathrm{ev}_V(\varphi \otimes v) = \psi(\varphi, v) = \varphi(v)$.

Since V is finite-dimensional, it also has the *coevaluation map*

$$\begin{aligned} \mathrm{coev}_V \colon K &\to V \otimes V^* \\ \lambda &\mapsto \lambda \sum_{i=1}^{n} v_i \otimes v_i^*, \end{aligned}$$

where $\{v_1, \ldots, v_n\}$ is a basis for V and $\{v_1^*, \ldots, v_n^*\}$ is the corresponding dual basis. This is often denoted by db_V and called the *dual basis map*. In Appendix A, we prove that $\mathrm{coev}_V = \mathrm{db}_V$ does not depend on the choice of basis.

Proposition 1.3.1. *The diagram*

$$\begin{array}{ccccccc} V & & \xrightarrow{\quad\quad\quad\quad\mathrm{id}_V\quad\quad\quad\quad} & & & & V \\ \Big\downarrow{\scriptstyle \lambda_V^{-1}} & & & & & & \Big\uparrow{\scriptstyle \rho_V} \\ K \otimes V & \xrightarrow{\mathrm{coev}_V \otimes \mathrm{id}_V} & (V \otimes V^*) \otimes V & \xrightarrow{\alpha_{V,V^*,V}} & V \otimes (V^* \otimes V) & \xrightarrow{\mathrm{id}_V \otimes \mathrm{ev}_V} & V \otimes K \end{array}$$

commutes.

Proof. Let $v \in V$. Under the composition, we have

$$\begin{aligned} v \mapsto 1_K \otimes v \mapsto \Big(\sum_{i=1}^{n} v_i \otimes v_i^*\Big) \otimes v &= \sum_{i=1}^{n} (v_i \otimes v_i^*) \otimes v \\ \mapsto \sum_{i=1}^{n} v_i \otimes (v_i^* \otimes v) \mapsto \sum_{i=1}^{n} v_i \otimes v_i^*(v) &\mapsto \sum_{i=1}^{n} v_i^*(v) v_i \\ = v \end{aligned}$$

and hence the composition is equal to id_V. $\square$

Proposition 1.3.2. *The diagram*

$$\begin{array}{ccccccc} V^* & & \xrightarrow{\quad\quad\quad\quad\mathrm{id}_{V^*}\quad\quad\quad\quad} & & & & V^* \\ \Big\downarrow{\scriptstyle \rho_{V^*}^{-1}} & & & & & & \Big\uparrow{\scriptstyle \lambda_{V^*}} \\ V^* \otimes K & \xrightarrow{\mathrm{id}_{V^*} \otimes \mathrm{coev}_V} & V^* \otimes (V \otimes V^*) & \xrightarrow{\alpha_{V^*,V,V^*}^{-1}} & (V^* \otimes V) \otimes V^* & \xrightarrow{\mathrm{ev}_V \otimes \mathrm{id}_{V^*}} & K \otimes V^* \end{array}$$

commutes.

Proof. Let $\varphi \in V^*$. Under the composition, we have

$$\begin{aligned} \varphi &\mapsto \varphi \otimes 1_K \mapsto \varphi \otimes \left(\sum_{i=1}^{n} v_i \otimes v_i^* \right) = \sum_{i=1}^{n} \varphi \otimes (v_i \otimes v_i^*) \\ &\mapsto \sum_{i=1}^{n} (\varphi \otimes v_i) \otimes v_i^* \mapsto \sum_{i=1}^{n} \varphi(v_i) \otimes v_i^* \mapsto \sum_{i=1}^{n} \varphi(v_i) v_i^* \\ &= \varphi \end{aligned}$$

and hence the composition is equal to id_{V^*}. □

The above properties for the evaluation and coevaluation maps for a finite-dimensional vector space and its dual space are taken to be the defining properties of the notion of a left dual object in a tensor category. We also have an analogous notion of a right dual object.

Definition 1.3.1. Let $\mathcal{C}$ be a tensor category and let X be an object in $\mathcal{C}$. A *left dual* of X is an object X^* in $\mathcal{C}$ together with morphisms $\mathrm{ev}_X \colon X^* \otimes X \to I$ and $\mathrm{coev}_X \colon I \to X \otimes X^*$ in $\mathcal{C}$ such that the diagrams

$$\begin{array}{ccccccc} X & & & \xrightarrow{\mathrm{id}_X} & & & X \\ \downarrow{\scriptstyle \lambda_X^{-1}} & & & & & & \uparrow{\scriptstyle \rho_X} \\ I \otimes X & \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} & (X \otimes X^*) \otimes X & \xrightarrow{\alpha_{X,X^*,X}} & X \otimes (X^* \otimes X) & \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} & X \otimes I \end{array}$$

and

$$\begin{array}{ccccccc} X^* & & & \xrightarrow{\mathrm{id}_{X^*}} & & & X^* \\ \downarrow{\scriptstyle \rho_{X^*}^{-1}} & & & & & & \uparrow{\scriptstyle \lambda_{X^*}} \\ X^* \otimes I & \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} & X^* \otimes (X \otimes X^*) & \xrightarrow{\alpha^{-1}_{X^*,X,X^*}} & (X^* \otimes X) \otimes X^* & \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} & I \otimes X^* \end{array}$$

commute. A *right dual* of X is an object ${}^*X$ in $\mathcal{C}$ together with morphisms $\mathrm{ev}'_X \colon X \otimes {}^*X \to I$ and $\mathrm{coev}'_X \colon I \to {}^*X \otimes X$ in $\mathcal{C}$ such that the diagrams

$$\begin{array}{ccccccc} X & & & \xrightarrow{\mathrm{id}_X} & & & X \\ \downarrow{\scriptstyle \rho_X^{-1}} & & & & & & \uparrow{\scriptstyle \lambda_X} \\ X \otimes I & \xrightarrow{\mathrm{id}_X \otimes \mathrm{coev}'_X} & X \otimes ({}^*X \otimes X) & \xrightarrow{\alpha^{-1}_{X,X^*,X}} & (X \otimes {}^*X) \otimes X & \xrightarrow{\mathrm{ev}'_X \otimes \mathrm{id}_X} & I \otimes X \end{array}$$

and

$$
\begin{array}{ccccccc}
{}^*X & & & \xrightarrow{\mathrm{id}_{{}^*X}} & & & {}^*X \\
\downarrow{\scriptstyle \lambda^{-1}_{{}^*X}} & & & & & & \uparrow{\scriptstyle \rho_{{}^*X}} \\
I \otimes {}^*X & \xrightarrow{\mathrm{coev}'_X \otimes \mathrm{id}_{{}^*X}} & ({}^*X \otimes X) \otimes {}^*X & \xrightarrow{\alpha^{-1}_{{}^*X, X, {}^*X}} & {}^*X \otimes (X \otimes X^*) & \xrightarrow{\mathrm{id}_{{}^*X} \otimes \mathrm{ev}'_X} & {}^*X \otimes I
\end{array}
$$

commute.

Unlike dual vector spaces, dual objects need not be unique. We will see, however, that left and right duals are unique up to isomorphism. It follows immediately from the definitions that if X^* is a left dual of X, then X is a right dual of X^*; and if ${}^*X$ is a right dual of X, then X is a left dual of ${}^*X$. Thus, we have

$$
{}^*(X^*) \cong X \cong ({}^*X)^*.
$$

If $\mathcal{C}$ is a tensor category in which every object has a left (respectively, right) dual, we say that $\mathcal{C}$ has *left duality* (respectively, *right duality*).

In a braided category $\mathcal{C}$ with braiding σ and left duality, we have the following relations:

$$
(\mathrm{id}_X \otimes \sigma^{-1}_{Y,X^*}) \circ (\mathrm{coev}_X \otimes \mathrm{id}_Y) = (\sigma_{Y,X} \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_Y \otimes \mathrm{coev}_X) \tag{1.7}
$$

$$
(\mathrm{id}_Y \otimes \mathrm{ev}_X) \circ (\sigma^{-1}_{Y,X^*} \otimes \mathrm{id}_X) = (\mathrm{ev}_X \otimes \mathrm{id}_Y) \circ (\mathrm{id}_{X^*} \otimes \sigma_{Y,X}) \tag{1.8}
$$

To verify (1.7), for example, we apply (1.6), the naturality of σ, and the axiom (1.4) of a braiding to obtain

$$
\begin{aligned}
(\mathrm{coev}_X \otimes \mathrm{id}_Y) &= (\mathrm{coev}_X \otimes \mathrm{id}_Y) \circ \sigma_{Y,I} \\
&= \sigma_{Y, X \otimes X^*} \circ (\mathrm{id}_Y \otimes \mathrm{coev}_X) \\
&= (\mathrm{id}_X \otimes \sigma_{Y,X^*}) \circ (\sigma_{Y,X} \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_Y \otimes \mathrm{coev}_X),
\end{aligned}
$$

which is equivalent to (1.7). These relations are true for any braiding σ, so in particular they are true for the braiding defined by $\tilde{\sigma}_{X,Y} = \sigma^{-1}_{Y,X}$. Thus we have the equivalent relations:

$$
(\mathrm{id}_X \otimes \sigma_{X^*,Y}) \circ (\mathrm{coev}_X \otimes \mathrm{id}_Y) = (\sigma^{-1}_{X,Y} \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_Y \otimes \mathrm{coev}_X) \tag{1.9}
$$

$$
(\mathrm{id}_Y \otimes \mathrm{ev}_X) \circ (\sigma_{X^*,Y} \otimes \mathrm{id}_X) = (\mathrm{ev}_X \otimes \mathrm{id}_Y) \circ (\mathrm{id}_{X^*} \otimes \sigma^{-1}_{X,Y}) \tag{1.10}
$$

1.4 Dual morphisms

Let $\mathcal{C}$ be a tensor category with left duality, and let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. For respective left dual objects X^* and Y^* of X and Y, we define a morphism $f^*\colon Y^* \to X^*$ by the following diagram:

$$\begin{array}{ccccc}
Y^* & & \xrightarrow{\quad f^* \quad} & & X^* \\
\downarrow{\scriptstyle \rho_{Y^*}} & & & & \uparrow{\scriptstyle \lambda_{X^*}} \\
Y^* \otimes I & & & & I \otimes X^* \\
\downarrow{\scriptstyle \mathrm{id}_{Y^*} \otimes \mathrm{coev}_X} & & & & \uparrow{\scriptstyle \mathrm{ev}_Y \otimes \mathrm{id}_{X^*}} \\
Y^* \otimes (X \otimes X^*) & \xrightarrow{\alpha^{-1}_{Y^*,X,X^*}} & (Y^* \otimes X) \otimes X^* & \xrightarrow{(\mathrm{id}_{Y^*} \otimes f) \otimes \mathrm{id}_{X^*}} & (Y^* \otimes Y) \otimes X^*
\end{array} \tag{1.11}$$

Note that, since

$$((\mathrm{id}_{Y^*} \otimes f) \otimes \mathrm{id}_{X^*}) \circ \alpha^{-1}_{Y^*,X,X^*} = \alpha^{-1}_{Y^*,Y,X^*} \circ (\mathrm{id}_{Y^*} \otimes (f \otimes \mathrm{id}_{X^*}))$$

by the naturality of α, we can also define f^* by the diagram

$$\begin{array}{ccccc}
Y^* & & \xrightarrow{\quad f^* \quad} & & X^* \\
\downarrow{\scriptstyle \rho_{Y^*}} & & & & \uparrow{\scriptstyle \lambda_{X^*}} \\
Y^* \otimes I & & & & I \otimes X^* \\
\downarrow{\scriptstyle \mathrm{id}_{Y^*} \otimes \mathrm{coev}_X} & & & & \uparrow{\scriptstyle \mathrm{ev}_Y \otimes \mathrm{id}_{X^*}} \\
Y^* \otimes (X \otimes X^*) & \xrightarrow{\mathrm{id}_{Y^*} \otimes (f \otimes \mathrm{id}_{X^*})} & Y^* \otimes (Y \otimes X^*) & \xrightarrow{\alpha^{-1}_{Y^*,Y,X^*}} & (Y^* \otimes Y) \otimes X^*
\end{array}$$

The morphism f^* is called the *left dual morphism* of f.

The following theorem gives equivalent characterizations of the dual morphism f^*.

Theorem 1.4.1. *The following are equivalent for morphisms $f\colon X \to Y$ and $g\colon Y^* \to X^*$ in a tensor category $\mathcal{C}$ with left duality.*

1. $g = f^*$.

2. $\mathrm{ev}_X \circ (g \otimes \mathrm{id}_X) = \mathrm{ev}_Y \circ (\mathrm{id}_{Y^*} \otimes f)$*, i.e., the diagram*

$$\begin{array}{ccc} Y^* \otimes X & \xrightarrow{\mathrm{id}_{Y^*}\otimes f} & Y^* \otimes Y \\ \downarrow{\scriptstyle g\otimes \mathrm{id}_X} & & \downarrow{\scriptstyle \mathrm{ev}_Y} \\ X^* \otimes X & \xrightarrow{\mathrm{ev}_X} & I \end{array} \tag{1.12}$$

commutes.

3. $(f \otimes \mathrm{id}_{X^*}) \circ \mathrm{coev}_X = (\mathrm{id}_Y \otimes g) \circ \mathrm{coev}_Y$*, i.e., the diagram*

$$\begin{array}{ccc} I & \xrightarrow{\mathrm{coev}_Y} & Y \otimes Y^* \\ \downarrow{\scriptstyle \mathrm{coev}_X} & & \downarrow{\scriptstyle \mathrm{id}_Y\otimes g} \\ X \otimes X^* & \xrightarrow{f\otimes \mathrm{id}_{X^*}} & Y \otimes X^* \end{array} \tag{1.13}$$

commutes.

Proof. We use the defining properties in Definition 1.3.1 of the left duals X^* and Y^*. First, we prove that $g = f^*$ is equivalent to (1.12). If $g = f^*$, then

$$\begin{aligned} &\mathrm{ev}_X \circ (g \otimes \mathrm{id}_X) \\ &= \mathrm{ev}_X \circ (f^* \otimes \mathrm{id}_X) \\ &= \mathrm{ev}_X \circ (\mathrm{ev}_Y \otimes \mathrm{id}_{X^*} \otimes \mathrm{id}_X) \circ (\mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{X^*} \otimes \mathrm{id}_X) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X) \\ &= \mathrm{ev}_Y \circ (\mathrm{id}_{Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{ev}_X) \circ (\mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{X^*} \otimes \mathrm{id}_X) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X) \\ &= \mathrm{ev}_Y \circ (\mathrm{id}_{Y^*} \otimes f) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{id}_X \otimes \mathrm{ev}_X) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X) \\ &= \mathrm{ev}_Y \circ (\mathrm{id}_{Y^*} \otimes f). \end{aligned}$$

Conversely, if

$$\mathrm{ev}_X \circ (g \otimes \mathrm{id}_X) = \mathrm{ev}_Y \circ (\mathrm{id}_{Y^*} \otimes f),$$

then

$$\begin{aligned} f^* &= (\mathrm{ev}_Y \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X) \\ &= (\mathrm{ev}_X \otimes \mathrm{id}_{X^*}) \circ (g \otimes \mathrm{id}_X \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X) \\ &= (\mathrm{ev}_X \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X) \circ g \\ &= g. \end{aligned}$$

Next, we prove that $g = f^*$ is equivalent to (1.13). If $g = f^*$, then

$$\begin{aligned}
&(\mathrm{id}_Y \otimes g) \circ \mathrm{coev}_Y \\
&= (\mathrm{id}_Y \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_Y \otimes \mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_Y \otimes \mathrm{id}_{Y^*} \otimes \mathrm{coev}_X) \circ \mathrm{coev}_Y \\
&= (\mathrm{id}_Y \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_Y \otimes \mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{X^*}) \circ (\mathrm{coev}_Y \otimes \mathrm{id}_X \otimes \mathrm{id}_{X^*}) \circ \mathrm{coev}_X \\
&= (\mathrm{id}_Y \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{X^*}) \circ (\mathrm{coev}_Y \otimes \mathrm{id}_Y \otimes \mathrm{id}_{X^*}) \circ (f \otimes \mathrm{id}_{X^*}) \circ \mathrm{coev}_X \\
&= (f \otimes \mathrm{id}_{X^*}) \circ \mathrm{coev}_X.
\end{aligned}$$

Conversely, if

$$(f \otimes \mathrm{id}_{X^*}) \circ \mathrm{coev}_X = (\mathrm{id}_Y \otimes g) \circ \mathrm{coev}_Y,$$

then

$$\begin{aligned}
f^* &= (\mathrm{ev}_Y \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X) \\
&= (\mathrm{ev}_Y \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{id}_Y \otimes g) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_Y) \\
&= g \circ (\mathrm{ev}_Y \otimes \mathrm{id}_{Y^*}) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_Y) \\
&= g.
\end{aligned}$$

This proves the theorem. □

Although dual objects are not unique in general, any two left duals or right duals of an object X are isomorphic. We prove this in the case of left duals.

Lemma 1.4.1. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms in a tensor category $\mathcal{C}$ with left duality, then*

$$(g \circ f)^* = f^* \circ g^*.$$

Proof. By applying Theorem 1.4.1 first to f and then to g, we have

$$\begin{aligned}
\mathrm{ev}_X \circ ((f^* \circ g^*) \otimes \mathrm{id}_X) &= \mathrm{ev}_X \circ (f^* \otimes \mathrm{id}_X) \circ (g^* \otimes \mathrm{id}_X) \\
&= \mathrm{ev}_Y \circ (\mathrm{id}_{Y^*} \otimes f) \circ (g^* \otimes \mathrm{id}_X) \\
&= \mathrm{ev}_Y \circ (g^* \otimes \mathrm{id}_Y) \circ (\mathrm{id}_{Z^*} \otimes f) \\
&= \mathrm{ev}_Z \circ (\mathrm{id}_{Z^*} \otimes g) \circ (\mathrm{id}_{Z^*} \otimes f) \\
&= \mathrm{ev}_Z \circ (\mathrm{id}_{Z^*} \otimes (g \circ f)).
\end{aligned}$$

Thus, applying Theorem 1.4.1 to $g \circ f$, we have $(g \circ f)^* = f^* \circ g^*$. □

Proposition 1.4.1. *If X is an object in a tensor category $\mathcal{C}$ with left duality, then any two left duals of X are isomorphic.*

Proof. Let X^* be a left dual of X, and let Y^* be another left dual of $Y = X$. Then $f = \mathrm{id}_X \colon X \to Y$ has a dual $f^* \colon Y^* \to X^*$ and $g = \mathrm{id}_X \colon Y \to X$ has a dual $g^* \colon X^* \to Y^*$. By Lemma 1.4.1,

$$g^* \circ f^* = (f \circ g)^* = (\mathrm{id}_Y)^* = \mathrm{id}_{Y^*}$$

and

$$f^* \circ g^* = (g \circ f)^* = (\mathrm{id}_X)^* = \mathrm{id}_{X^*}.$$

This means that f^* and g^* are inverse morphisms, and hence X^* and Y^* are isomorphic. $\square$

Note that, by Theorem 1.4.1, the isomorphism g^* in the above proof is characterized by the property

$$\mathrm{ev}_Y \circ (g^* \otimes \mathrm{id}_X) = \mathrm{ev}_X.$$

For each X and Y in $\mathcal{C}$, the object $X \otimes Y$ has a left dual object $(X \otimes Y)^*$ with evaluation $\mathrm{ev}_{X\otimes Y}$ and coevaluation $\mathrm{coev}_{X\otimes Y}$, but it also has the left dual $Y^* \otimes X^*$ with evaluation

$$\overline{\mathrm{ev}}_{X\otimes Y} = \mathrm{ev}_Y \circ (\mathrm{id}_{Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_Y) \tag{1.14}$$

and coevaluation

$$\overline{\mathrm{coev}}_{X\otimes Y} = (\mathrm{id}_X \otimes \mathrm{coev}_Y \otimes \mathrm{id}_{X^*}) \circ \mathrm{coev}_X. \tag{1.15}$$

Proposition 1.4.1 then implies the existence of an isomorphism $\gamma_{X,Y} \colon Y^* \otimes X^* \to (X \otimes Y)^*$, which is dual to $\mathrm{id}_{X\otimes Y}$. By Theorem 1.4.1, the morphism $\gamma_{X,Y}$ is characterized by the properties

$$\mathrm{ev}_{X\otimes Y} \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y}) = \mathrm{ev}_Y \circ (\mathrm{id}_{Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_Y) \tag{1.16}$$

and

$$\mathrm{coev}_{X\otimes Y} = (\mathrm{id}_{X\otimes Y} \otimes \gamma_{X,Y}) \circ (\mathrm{id}_X \otimes \mathrm{coev}_Y \otimes \mathrm{id}_{X^*}) \circ \mathrm{coev}_X. \tag{1.17}$$

Furthermore, the collection of morphisms $\gamma_{X,Y}$ defines a natural isomorphism, as shown in the following proposition.

Proposition 1.4.2. *In a tensor category $\mathcal{C}$ with left duality, the isomorphisms $\gamma_{X,Y}$ dual to $\mathrm{id}_{X\otimes Y}$, and characterized by* (1.16) *and* (1.17)*, define a natural isomorphism from the functor $(X, Y) \mapsto Y^* \otimes X^*$, $(f, g) \mapsto g^* \otimes f^*$ to the functor $(X, Y) \mapsto (X \otimes Y)^*$, $(f, g) \mapsto (f \otimes g)^*$.*

Proof. Let $f\colon X \to X'$ and $g\colon Y \to Y'$ be morphisms in $\mathcal{C}$. Naturality of γ means that

$$\gamma_{X,Y} \circ (g^* \otimes f^*) = (f \otimes g)^* \circ \gamma_{X',Y'}. \tag{1.18}$$

With respect to $\mathrm{ev}_{X\otimes Y}$, the morphism $(f \otimes g)^*$ is the dual morphism of $f \otimes g$, and with respect to the evaluation $\overline{\mathrm{ev}}_{X\otimes Y}$ defined in (1.14), the morphism $g^* \otimes f^*$ is the dual morphism of $f \otimes g$. By Theorem 1.4.1, this means that

$$\mathrm{ev}_{X\otimes Y} \circ ((f \otimes g)^* \otimes \mathrm{id}_{X\otimes Y}) = \mathrm{ev}_{X'\otimes Y'} \circ (\mathrm{id}_{(X'\otimes Y')^*} \otimes f \otimes g) \tag{1.19}$$

and

$$\overline{\mathrm{ev}}_{X\otimes Y} \circ (g^* \otimes f^* \otimes \mathrm{id}_{X\otimes Y}) = \overline{\mathrm{ev}}_{X'\otimes Y'} \circ (\mathrm{id}_{Y'^*\otimes X'^*} \otimes f \otimes g). \tag{1.20}$$

By (1.16), we can rewrite (1.20) as

$$\begin{aligned} &\mathrm{ev}_{X\otimes Y} \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (g^* \otimes f^* \otimes \mathrm{id}_{X\otimes Y}) \\ &= \mathrm{ev}_{X'\otimes Y'} \circ (\gamma_{X',Y'} \otimes \mathrm{id}_{X'\otimes Y'}) \circ (\mathrm{id}_{Y'^*\otimes X'^*} \otimes f \otimes g) \\ &= \mathrm{ev}_{X'\otimes Y'} \circ (\mathrm{id}_{(X'\otimes Y')^*} \otimes f \otimes g) \circ (\gamma_{X',Y'} \otimes \mathrm{id}_{X\otimes Y}) \end{aligned}$$

and, by (1.19), this further equals

$$\mathrm{ev}_{X\otimes Y} \circ ((f \otimes g)^* \otimes \mathrm{id}_{X\otimes Y}) \circ (\gamma_{X',Y'} \otimes \mathrm{id}_{X\otimes Y}).$$

In other words, we have

$$\mathrm{ev}_{X\otimes Y} \circ ((\gamma_{X,Y} \circ (g^* \otimes f^*)) \otimes \mathrm{id}_{X\otimes Y}) = \mathrm{ev}_{X\otimes Y} \circ (((f \otimes g)^* \circ \gamma_{X',Y'}) \otimes \mathrm{id}_{X\otimes Y}).$$

Composing both sides with $\gamma^{-1}_{X',Y'} \otimes \mathrm{id}_{X\otimes Y}$ yields

$$\mathrm{ev}_{X\otimes Y} \circ ((\gamma_{X,Y} \circ (g^* \otimes f^*) \circ \gamma^{-1}_{X',Y'}) \otimes \mathrm{id}_{X\otimes Y}) = \mathrm{ev}_{X\otimes Y} \circ ((f \otimes g)^* \otimes \mathrm{id}_{X\otimes Y}).$$

Applying Theorem 1.4.1 again, this shows that both $\gamma_{X,Y} \circ (g^* \otimes f^*) \circ \gamma^{-1}_{X',Y'}$ and $(f \otimes g)^*$ are the dual morphism of $f \otimes g$ with respect to $\mathrm{ev}_{X\otimes Y}$ and, in particular, they are equal. This proves (1.18). □

In a tensor category with right duality, there is also the notion of a *right dual morphism* ${}^*f\colon {}^*Y \to {}^*X$, defined as

$${}^*f = \rho_{{}^*X} \circ (\mathrm{id}_{{}^*X} \otimes \mathrm{ev}'_Y) \circ (\mathrm{id}_{{}^*X} \otimes f \otimes \mathrm{id}_{{}^*Y}) \circ (\mathrm{coev}'_X \otimes \mathrm{id}_{{}^*Y}) \circ \lambda_{{}^*Y}.$$

A similar treatment to that given for left dual morphisms shows that ${}^*f$ is characterized by the properties

$$\mathrm{ev}'_X \circ (\mathrm{id}_X \otimes {}^*f) = \mathrm{ev}'_Y \circ (f \otimes \mathrm{id}_Y) \tag{1.21}$$

and

$$(\mathrm{id}_{{}^*X} \otimes f) \circ \mathrm{coev}'_X = ({}^*f \otimes \mathrm{id}_Y) \circ \mathrm{coev}'_Y, \tag{1.22}$$

analogously to Theorem 1.4.1, and that any two right duals of an object X are isomorphic. Furthermore, we have a natural isomorphism $\gamma'_{X,Y} \colon {}^*Y \otimes {}^*X \to {}^*(X \otimes Y)$ that is right dual to $\mathrm{id}_{X\otimes Y}$. It is characterized by the property

$$\mathrm{ev}'_{X\otimes Y} \circ (\mathrm{id}_{X\otimes Y} \otimes \gamma'_{X,Y}) = \mathrm{ev}'_X \circ (\mathrm{id}_X \otimes \mathrm{ev}'_Y \otimes \mathrm{id}_{{}^*X}). \tag{1.23}$$

1.5 Abelian and linear categories

We now discuss abelian and linear categories. First, we need several definitions, which can also be found in [9, Ch. VIII] and [3, Ch. 2].

We define a *monomorphism* as a morphism f that is *left-cancellable*, which means that

$$f \circ g = f \circ h \implies g = h,$$

and we define an *epimorphism* as a morphism f that is *right-cancellable*, which means that

$$g \circ f = h \circ f \implies g = h.$$

A *product* of a set $\{X_i\}_{i\in I}$ of objects in a category $\mathcal{C}$, indexed by a set I, is defined as an object X in $\mathcal{C}$ together with morphisms $\{p_i \colon X \to X_i\}_{i\in I}$, called *projections*, with the following universal property: For any object Y in $\mathcal{C}$ and morphisms $\{f_i \colon Y \to X_i\}_{i\in I}$ in $\mathcal{C}$, there exists a unique morphism $f \colon Y \to X$ such that $f_i = p_i \circ f$ for all $i \in I$. A *coproduct* of a set $\{X_i\}_{i\in I}$ of objects in $\mathcal{C}$, indexed by a set I, is defined as an object X in $\mathcal{C}$ together with morphisms $\{q_i \colon X_i \to X\}_{i\in I}$, called *injections*, with the following universal property: For any object Y in $\mathcal{C}$ and morphisms $\{f_i \colon X_i \to Y\}_{i\in I}$ in $\mathcal{C}$, there exists a unique morphism $f \colon X \to Y$ such that $f_i = f \circ q_i$ for all $i \in I$.

A *zero object* in a category $\mathcal{C}$ is defined as an object, denoted by 0, such that for all objects X in $\mathcal{C}$, the sets $\mathrm{Hom}(0, X)$ and $\mathrm{Hom}(X, 0)$ each contain exactly one morphism. It can be shown that any two zero objects in a category are isomorphic, and therefore each $\mathrm{Hom}(X, Y)$

contains a distinguished morphism $X \to 0 \to Y$, which we call the *zero morphism* and denote by $0_{X,Y}$, or simply 0.

If $\mathcal{C}$ is a category with a zero object, then we define a *kernel* of a morphism $f\colon X \to Y$ in $\mathcal{C}$ as an object $\ker(f)$ in $\mathcal{C}$ together with a morphism $i\colon \ker(f) \to X$ such that $f \circ i = 0$, and i is universal in the sense that for any other morphism $j\colon W \to X$ such that $f \circ j = 0$, there is a unique morphism $j_0\colon W \to \ker(f)$ such that $j = i \circ j_0$. We also have the notion of a *cokernel* of f. This is an object $\operatorname{coker}(f)$ together with a morphism $\pi\colon Y \to \operatorname{coker}(f)$ such that $\pi \circ f = 0$, and π has the analogous universal property.

If $\mathcal{C}$ is a category for which every morphism has a kernel and a cokernel, we define an *image* of f, denoted by $\operatorname{im}(f)$, as a kernel of a cokernel of f. That is, if $\pi\colon Y \to \operatorname{coker}(f)$ is a cokernel of f, then an image of f is defined as a kernel $i\colon \operatorname{im}(f) = \ker(\pi) \to Y$. Note that, since $\pi \circ f = 0$, there exists a unique morphism $g\colon A \to \operatorname{im}(f)$ such that $f = i \circ g$, by the universal property of the kernel of π. Thus we can view i as a generalized inclusion map.

The notions of kernel and cokernel can be generalized to the notions of *equalizer* and *coequalizer*, respectively. An equalizer of two morphisms $f, f'\colon X \to Y$ in $\mathcal{C}$ is a morphism $g\colon Z \to X$ in $\mathcal{C}$ such that $f \circ g = f' \circ g$, and g is universal in the sense that for any morphism $g'\colon Z' \to X$ with $f \circ g' = f' \circ g'$, there exists a unique morphism $h\colon Z \to Z'$ such that $g = g' \circ h$. A coequalizer of f and f' is a morphism g such that $g \circ f = g \circ f'$ that is universal in the analogous way. Notice that the kernel and cokernel of a morphism f are, respectively, the equalizer and coequalizer of f and the zero morphism 0.

If $\mathcal{C}$ is a category with a zero object and a product defined for any two objects, then we say that $\mathcal{C}$ is an *additive category* if each of its hom-sets $\operatorname{Hom}(X, Y)$ has an abelian group structure and the composition map $\circ\colon \operatorname{Hom}(X, Y) \times \operatorname{Hom}(Y, Z) \to \operatorname{Hom}(X, Z)$ is additive in each component, i.e., it is $\mathbb{Z}$-bilinear. If K is a field, then we say that $\mathcal{C}$ is a *K-linear category* if each of its hom-sets is a K-vector space and the composition map is K-bilinear (cf. [21, Ch. 4, p. 65]).

We are now ready to define an abelian category.

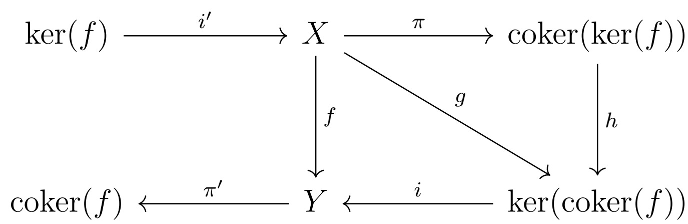
Definition 1.5.1. An *abelian category* $\mathcal{C}$ is an additive category such that

1. Every morphism has a kernel and a cokernel.
2. Every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.
3. Every morphism is the composition of an epimorphism and a monomorphism.

We have the following lemma about the unique morphism g induced by an image.

Lemma 1.5.1. *If $f\colon X \to Y$ is a morphism with image $i\colon \mathrm{im}(f) \to Y$ in an abelian category, then the unique morphism g such that $f = i \circ g$ is an epimorphism.*

Proof. We first establish the existence of a morphism $h\colon \mathrm{coker}(\ker(f)) \to \ker(\mathrm{coker}(f))$ that makes the following diagram commutative:



We have $f \circ i' = 0$, since i' is the kernel of f, and therefore $i \circ g \circ i' = 0$. This implies that $g \circ i' = 0$, because i is a monomorphism by [13, Lem. 1, 1.9]. Therefore, by the universal property of π, there exists a unique morphism $h\colon \mathrm{coker}(\ker(f)) \to \ker(\mathrm{coker}(f))$ such that $g = h \circ \pi$, as required. Abelian categories can be characterized by the property that this uniquely determined h is an isomorphism, as in [13, Ch. 4, p. 164]. It is also a fact that π is an epimorphism, and since isomorphisms are in particular epimorphisms and the composition of epimorphisms is an epimorphism, this implies that g is an epimorphism. □

We also have the notions of additive and K-linear functors. If $\mathcal{C}$ and $\mathcal{D}$ are additive (respectively, K-linear) categories, then a functor $F\colon \mathcal{C} \to \mathcal{D}$ is called an *additive functor* (respectively, a *K-linear functor*) if for all objects X and Y in $\mathcal{C}$, the map

$$F\colon \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

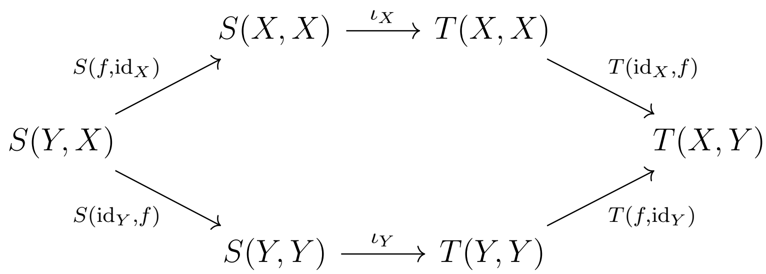
is a homomorphism of groups (respectively, of K-vector spaces). See [3, Ch. II, Prop. 9.5] and [21, Ch. 4, p. 65]. Two K-linear categories are said to be equivalent (respectively, isomorphic) if there exists a K-linear equivalence (respectively, a K-linear isomorphism) between them. We then say that a category is *finite* if it is K-linearly equivalent to a category of modules over a finite-dimensional K-algebra.

When we speak of an *abelian tensor category*, we mean an abelian category with a tensor product that is $\mathbb{Z}$-bilinear; when we speak of a *K-linear tensor category*, we mean a K-linear category with a tensor product that is K-bilinear.

1.6 Dinatural transformations and coends

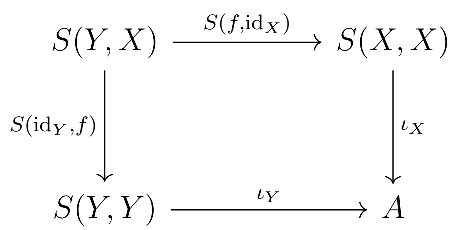
The notion of a dinatural transformation is defined as follows.

Definition 1.6.1. Let $S, T\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be functors. A *dinatural transformation* ι from S to T is a function that assigns to each object X in $\mathcal{C}$ a morphism $\iota_X\colon S(X,X) \to T(X,X)$ in $\mathcal{D}$ such that for any morphism $f\colon X \to Y$ in $\mathcal{C}$, the following diagram commutes:



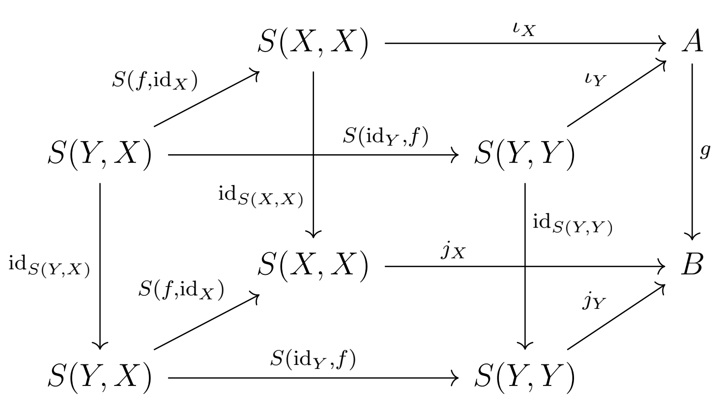
(1.24)

We are interested in the case where $T\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is a constant functor, defined by $T(X,Y) = A$ and $T(f,g) = \mathrm{id}_A$ for some object A in $\mathcal{D}$. In this case, (1.24) reduces to



We will often denote the dinatural transformation ι from $S\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ to a constant functor $T\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$, defined by $T(X,Y) \to L$, as $\iota_X\colon S(X,X) \to L$, where it is understood that X varies over all objects in $\mathcal{C}$.

If $g\colon A \to B$ is a morphism in $\mathcal{D}$, then $j_X = g \circ \iota_X\colon S(X,X) \to B$ is again a dinatural transformation. This follows from the commutativity of the following diagram:

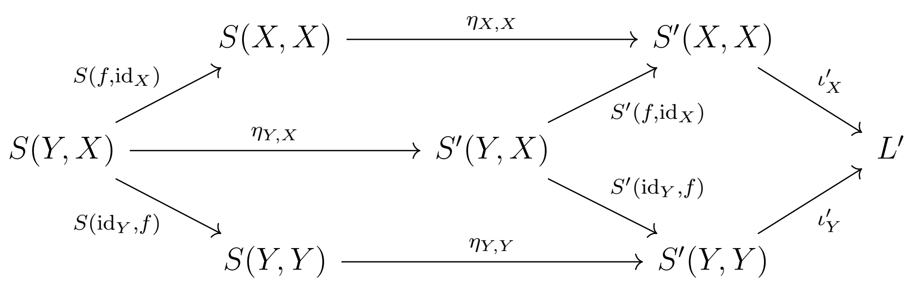


This leads us to the following definition.

Definition 1.6.2. A *coend* for a functor $S\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is an object L in $\mathcal{D}$ together with a dinatural transformation $\iota_X\colon S(X,X) \to L$ that is universal in the sense that for every dinatural transformation $j_X\colon S(X,X) \to M$, there is a unique morphism $g\colon L \to M$ in $\mathcal{C}$ such that $j_X = g \circ \iota_X$ for all objects X in $\mathcal{C}$.

As with other universal properties, the above definition implies that any two coends are isomorphic, and that the isomorphism is unique.

If we have another functor $S'\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ with a universal dinatural transformation $\iota'_X\colon S'(X,X) \to L'$, and we have a natural transformation $\eta_{X,Y}\colon S(X,Y) \to S'(X,Y)$, then $j_X = \iota'_X \circ \eta_{X,X}\colon S(X,X) \to L'$ is also a dinatural transformation because the diagram



commutes as a consequence of [7, Lem. 2.2, p. 39], which is a version of [9, Prop. 1, p. 228] for coends.

Now let $\mathcal{C}$ be a K-linear category with left duality and consider the functor $S\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ defined by

$$\begin{aligned} (X,Y) &\mapsto X^* \otimes Y \\ (f,g) &\mapsto f^* \otimes g. \end{aligned} \tag{1.25}$$

If $\mathcal{C}$ is finite, then it follows from [6, Cor. 5.1.8, p. 267] that there exists a coend L for this functor with universal dinatural transformation $\iota_X\colon X^* \otimes X \to L$. Henceforth, when we speak of a coend without further qualification, we will mean the coend for this functor.

We now show that $\iota_X \otimes \iota_Y$ is a dinatural transformation. Let $\mathcal{E} = \mathcal{C} \times \mathcal{C}$ and consider the functor $S'\colon \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{C}$ defined by

$$\begin{aligned} (X',Y',X,Y) &\mapsto X'^* \otimes X \otimes Y'^* \otimes Y \\ (f',g',f,g) &\mapsto f'^* \otimes f \otimes g'^* \otimes g. \end{aligned} \tag{1.26}$$

Note that if we have a morphism

$$(f', g', f, g)\colon (X', Y', X, Y) \to (\tilde{X}', \tilde{Y}', \tilde{X}, \tilde{Y})$$

in $\mathcal{E}^{\mathrm{op}} \times \mathcal{E}$, then f' is a morphism $\tilde{X}' \to X'$ and g' is a morphism $\tilde{Y}' \to Y$, and therefore

$$S'(f', g', f, g) = f'^{*} \otimes f \otimes g'^{*} \otimes g$$

is a morphism

$$X'^{*} \otimes X \otimes Y'^{*} \otimes Y \to \tilde{X}'^{*} \otimes \tilde{X} \otimes \tilde{Y}'^{*} \otimes \tilde{Y},$$

as required. Then $\iota_X \otimes \iota_Y \colon S'(X, Y, X, Y) \to L \otimes L$ is dinatural, i.e., the diagram

$$\begin{array}{ccc}
\tilde{X}^{*} \otimes X \otimes \tilde{Y}^{*} \otimes Y & \xrightarrow{f^{*} \otimes \mathrm{id}_X \otimes g^{*} \otimes \mathrm{id}_Y} & X^{*} \otimes X \otimes Y^{*} \otimes Y \\
{\scriptstyle \mathrm{id}_{\tilde{X}^{*}} \otimes f \otimes \mathrm{id}_{\tilde{Y}^{*}} \otimes g} \downarrow & & \downarrow {\scriptstyle \iota_X \otimes \iota_Y} \\
\tilde{X}^{*} \otimes \tilde{X} \otimes \tilde{Y}^{*} \otimes \tilde{Y} & \xrightarrow{\iota_{\tilde{X}} \otimes \iota_{\tilde{Y}}} & L \otimes L
\end{array}$$

commutes, because for all objects (X, Y) in $\mathcal{E}$ and morphisms $(f, g)\colon (X, Y) \to (\tilde{X}, \tilde{Y})$ in $\mathcal{E}$,

$$\begin{aligned}
(\iota_X \otimes \iota_Y) \circ (f^{*} \otimes \mathrm{id}_X \otimes g^{*} \otimes \mathrm{id}_Y) &= (\iota_X \circ (f^{*} \otimes \mathrm{id}_X)) \otimes (\iota_Y \circ (g^{*} \otimes \mathrm{id}_Y)) \\
&= (\iota_{\tilde{X}} \circ (\mathrm{id}_{\tilde{X}} \otimes f)) \otimes (\iota_{\tilde{Y}} \circ (\mathrm{id}_{\tilde{Y}} \otimes g)) \\
&= (\iota_{\tilde{X}} \otimes \iota_{\tilde{Y}}) \circ (\mathrm{id}_{\tilde{X}} \otimes f \otimes \mathrm{id}_{\tilde{Y}} \otimes g)
\end{aligned}$$

by the dinaturality of ι_X and ι_Y.

It is explained in [7, p. 39] that $\iota_X \otimes \iota_Y$ is in fact universally dinatural:

Theorem 1.6.1. *Let* $\iota_X \colon X^{*} \otimes X \to L$ *be the universal dinatural transformation for the functor defined by* (1.25)*. Then the dinatural transformation*

$$\iota_X \otimes \iota_Y \colon X^{*} \otimes X \otimes Y^{*} \otimes Y \to L \otimes L$$

is universal, and hence $L \otimes L$ *is a coend for the functor* (1.26).

1.7 Hopf algebras in categories

The notions of an algebra, a coalgebra, a bialgebra, and a Hopf algebra can be generalized to the categorical setting. We begin with the definition of an algebra in a tensor category $\mathcal{C}$.

Definition 1.7.1. Let $\mathcal{C}$ be a tensor category. An *algebra* in $\mathcal{C}$ is an object A in $\mathcal{C}$; a morphism $m_A\colon A \otimes A \to A$, called the *product* of A; and a morphism $u_A\colon I \to A$, called the *unit* of A, such that the diagrams

$$\begin{array}{ccccc} (A \otimes A) \otimes A & \xrightarrow{\alpha_{A,A,A}} & A \otimes (A \otimes A) & \xrightarrow{\mathrm{id}_A \otimes m_A} & A \otimes A \\ \downarrow{\scriptstyle m_A \otimes \mathrm{id}_A} & & & & \downarrow{\scriptstyle m_A} \\ A \otimes A & & \xrightarrow{\qquad m_A \qquad} & & A \end{array} \tag{1.27}$$

and

$$\begin{array}{ccc} I \otimes A & \xrightarrow{u_A \otimes \mathrm{id}_A} & A \otimes A \\ & {\scriptstyle \lambda_A} \searrow & \downarrow{\scriptstyle m_A} \\ & & A \end{array} \qquad \begin{array}{ccc} A \otimes I & \xrightarrow{\mathrm{id}_A \otimes u_A} & A \otimes A \\ & {\scriptstyle \rho_A} \searrow & \downarrow{\scriptstyle m_A} \\ & & A \end{array} \tag{1.28}$$

commute. The property expressed in (1.27) is called the *associativity* of m_A, and the property expressed in (1.28) is called the *unitality* of m_A.

A homomorphism of algebras in $\mathcal{C}$ is defined as follows.

Definition 1.7.2. Let A and B be algebras in a tensor category $\mathcal{C}$. A morphism $f\colon A \to B$ is called an *algebra homomorphism* if

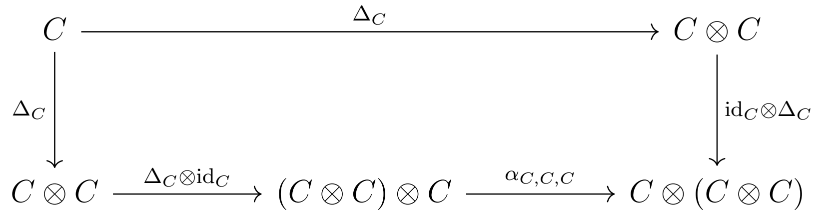
$$f \circ m_A = m_B \circ (f \otimes f)$$

and

$$f \circ u_A = u_B.$$

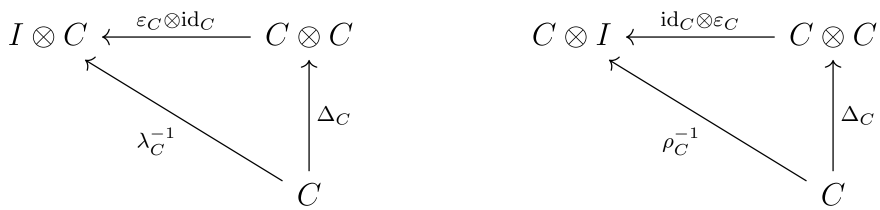
The definition of a coalgebra in a tensor category $\mathcal{C}$ is obtained by reversing the arrows in the definition of an algebra in $\mathcal{C}$. In other words, a coalgebra in $\mathcal{C}$ is an algebra in the opposite category $\mathcal{C}^{\mathrm{op}}$:

Definition 1.7.3. Let $\mathcal{C}$ be a tensor category. A *coalgebra* in $\mathcal{C}$ is an object C in $\mathcal{C}$; a morphism $\Delta_C\colon C \to C \otimes C$, called the *coproduct* of C; and a morphism $\varepsilon_C\colon C \to I$, called the *counit* of C, such that the diagrams



(1.29)

and



(1.30)

commute. The property expressed in (1.29) is called the *coassociativity* of Δ_C, and the property expressed in (1.30) is called the *counitality* of ε_C.

The definition of a coalgebra homomorphism in a tensor category $\mathcal{C}$ is then obtained by reversing the arrows in the definition of an algebra homomorphism in $\mathcal{C}$; that is, a coalgebra homomorphism in $\mathcal{C}$ is an algebra homomorphism in $\mathcal{C}^{\mathrm{op}}$:

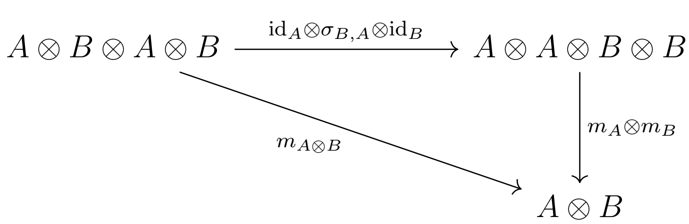
Definition 1.7.4. Let C and D be coalgebras in a tensor category $\mathcal{C}$. A morphism $f\colon C \to D$ is called a *coalgebra homomorphism* if

$$\Delta_D \circ f = (f \otimes f) \circ \Delta_C$$

and

$$\varepsilon_D \circ f = \varepsilon_C.$$

If A and B are algebras in a braided category $\mathcal{C}$ with braiding σ, then we can define an algebra structure on the tensor product $A \otimes B$. Suppressing α, we define the product $m_{A \otimes B}$ by the diagram



(1.31)

and we define the unit $u_{A\otimes B}$ by the diagram

$$\begin{array}{ccc} I & \xrightarrow{\lambda_I^{-1}=\rho_I^{-1}} & I\otimes I \\ & \searrow^{u_{A\otimes B}} & \downarrow{\scriptstyle u_A\otimes u_B} \\ & & A\otimes B \end{array} \tag{1.32}$$

With an algebra structure on the tensor product of algebras, we can consider whether a coproduct is an algebra homomorphism. The notions of an algebra and a coalgebra can then be combined.

Definition 1.7.5. Let $\mathcal{C}$ be a braided tensor category. A *bialgebra* in $\mathcal{C}$ is an object B in $\mathcal{C}$ that is both an algebra and a coalgebra in $\mathcal{C}$, such that the coproduct Δ_B and and counit ε_B are algebra homomorphisms. If B and B' are bialgebras in $\mathcal{C}$, then a morphism $f\colon B \to B'$ is called a *bialgebra homomorphism* if it is both an algebra and coalgebra homomorphism.

By Definition 1.7.2, the coproduct Δ_B being an algebra homomorphism means that

$$\Delta_B \circ m_B = m_{B\otimes B} \circ (\Delta_B \otimes \Delta_B) = (m_B \otimes m_B) \circ (\mathrm{id}_B \otimes \sigma_{B,B} \otimes \mathrm{id}_B) \circ (\Delta_B \otimes \Delta_B)$$

and

$$\Delta_B \circ u_B = u_{B\otimes B} = (u_B \otimes u_B) \circ \rho_I^{-1}.$$

Note that, for $\varepsilon_B\colon B \to I$ to be an algebra homomorphism, we need to have an algebra structure on I. Assuming strictness (so that $I \otimes I = I$), we take $\lambda_I = \rho_I = \mathrm{id}_I$ to be both the multiplication $m_I\colon I \otimes I \to I$ and the unit $u_I\colon I \to I$.

Exactly as in the vector space case, Δ_B and ε_B are algebra homomorphisms if and only if m_B and u_B are coalgebra homomorphisms [5, Thm. III.2.1, p. 45], so a bialgebra can be equivalently defined in terms of the latter requirement.

We can now define the notion of a Hopf algebra in a braided category $\mathcal{C}$.

Definition 1.7.6. Let $\mathcal{C}$ be a braided tensor category. A *Hopf algebra* in $\mathcal{C}$ is a bialgebra H in $\mathcal{C}$ with a morphism $S_H\colon H \to H$, called the *antipode*, such that the diagrams

$$\begin{array}{ccccc} H & \xrightarrow{\varepsilon_H} & I & \xrightarrow{u_H} & H \\ \downarrow{\scriptstyle \Delta_H} & & & & \uparrow{\scriptstyle m_H} \\ H\otimes H & & \xrightarrow{S_H\otimes \mathrm{id}_H} & & H\otimes H \end{array} \tag{1.33}$$

and

$$\begin{array}{ccccc} H & \xrightarrow{\varepsilon_H} & I & \xrightarrow{u_H} & H \\ \Big\downarrow{\scriptstyle \Delta_H} & & & & \Big\uparrow{\scriptstyle m_H} \\ H \otimes H & & \xrightarrow{\mathrm{id}_H \otimes S_H} & & H \otimes H \end{array} \tag{1.34}$$

commute. If H and H' are Hopf algebras in $\mathcal{C}$, then a bialgebra homomorphism $f\colon H \to H'$ is called a *Hopf algebra homomorphism* if

$$S_{H'} \circ f = f \circ S_H.$$

The properties expressed in (1.33) and (1.34) are called the *antipode equations* for H. Note that the ordinary notion of a Hopf algebra over a field K is a Hopf algebra in the category of vector spaces over K, braided with the flip map. It can be shown, exactly as in the vector space case (cf. [20, Lem. 4.0.4, p. 81f], [5, Exerc. III.8.9, p. 69]), that every bialgebra homomorphism between Hopf algebras is automatically a Hopf algebra homomorphism.

Chapter 2

The coend as a Hopf algebra

In this chapter, we show that the coend for the functor (1.25) in a braided finite tensor category $\mathcal{C}$ is a Hopf algebra in $\mathcal{C}$. This was first shown in [8], and also summarized in [6].

2.1 Product

Let $\mathcal{C}$ be a braided finite tensor category with braiding σ, and let L be the coend of the functor (1.25), with universal dinatural transformation ι. We mentioned in Theorem 1.6.1 that

$$\iota_X \otimes \iota_Y \colon X^* \otimes X \otimes Y^* \otimes Y \to L \otimes L$$

is a universal dinatural transformation. We can define another dinatural transformation $\xi_{X,Y} \colon X^* \otimes X \otimes Y^* \otimes Y \to L$ by

$$\xi_{X,Y} = \iota_{X\otimes Y} \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (\sigma_{X^*\otimes X, Y^*} \otimes \mathrm{id}_Y) \tag{2.1}$$

where $\gamma_{X,Y} \colon Y^* \otimes X^* \to (X \otimes Y)^*$ is the natural isomorphism characterized by the property (1.16). The dinaturality of ξ means that the diagram

$$\begin{array}{ccc}
X'^* \otimes X \otimes Y'^* \otimes Y & \xrightarrow{f^*\otimes \mathrm{id}_X \otimes g^* \otimes \mathrm{id}_Y} & X^* \otimes X \otimes Y^* \otimes Y \\
\Big\downarrow {\scriptstyle \mathrm{id}_{X'^*}\otimes f \otimes \mathrm{id}_{Y'^*} \otimes g} & & \Big\downarrow {\scriptstyle \xi_{X,Y}} \\
X'^* \otimes X' \otimes Y'^* \otimes Y' & \xrightarrow{\quad \xi_{X',Y'} \quad} & L
\end{array}$$

commutes for all morphisms $f\colon X \to X'$ and $g\colon Y \to Y'$ in $\mathcal{C}$. By the naturality of σ, we have

$$\begin{aligned} &\xi_{X,Y} \circ (f^* \otimes \mathrm{id}_X \otimes g^* \otimes \mathrm{id}_Y) \\ &= \iota_{X\otimes Y} \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (\sigma_{X^*\otimes X, Y^*} \otimes \mathrm{id}_Y) \circ (f^* \otimes \mathrm{id}_X \otimes g^* \otimes \mathrm{id}_Y) \\ &= \iota_{X\otimes Y} \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (g^* \otimes f^* \otimes \mathrm{id}_X \otimes \mathrm{id}_Y) \circ (\sigma_{X'^*\otimes X, Y'^*} \otimes \mathrm{id}_Y), \end{aligned}$$

and, by the naturality of γ proved in Proposition 1.4.2, this equals

$$\iota_{X\otimes Y} \circ ((f\otimes g)^* \otimes \mathrm{id}_X \otimes \mathrm{id}_Y) \circ (\gamma_{X',Y'} \otimes \mathrm{id}_X \otimes \mathrm{id}_Y) \circ (\sigma_{X'^*\otimes X, Y'^*} \otimes \mathrm{id}_Y).$$

By the dinaturality of ι, we have

$$\iota_{X\otimes Y} \circ ((f\otimes g)^* \otimes \mathrm{id}_{X\otimes Y}) = \iota_{X'\otimes Y'} \circ (\mathrm{id}_{(X'\otimes Y')^*} \otimes f \otimes g),$$

and therefore the above expression is equal to

$$\begin{aligned} &\iota_{X'\otimes Y'} \circ (\mathrm{id}_{(X'\otimes Y')^*} \otimes f \otimes g) \circ (\gamma_{X',Y'} \otimes \mathrm{id}_X \otimes \mathrm{id}_Y) \circ (\sigma_{X'^*\otimes X, Y'^*} \otimes \mathrm{id}_Y) \\ &= \iota_{X'\otimes Y'} \circ (\gamma_{X',Y'} \otimes \mathrm{id}_{X'} \otimes \mathrm{id}_{Y'}) \circ (\mathrm{id}_{Y'^*} \otimes \mathrm{id}_{X'^*} \otimes f \otimes g) \circ (\sigma_{X'^*\otimes X, Y'^*} \otimes \mathrm{id}_Y) \\ &= \iota_{X'\otimes Y'} \circ (\gamma_{X',Y'} \otimes \mathrm{id}_{X'} \otimes \mathrm{id}_{Y'}) \circ (\sigma_{X'^*\otimes X', Y'^*} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_{X'^*} \otimes f \otimes \mathrm{id}_{Y'^*} \otimes g) \\ &= \xi_{X',Y'} \circ (\mathrm{id}_{X'^*} \otimes f \otimes \mathrm{id}_{Y'^*} \otimes g). \end{aligned}$$

This proves that ξ is dinatural. By the universality of $\iota_X \otimes \iota_Y$ stated in Theorem 1.6.1, there exists a unique morphism $m_L\colon L\otimes L \to L$ such that $m_L \circ (\iota_X \otimes \iota_Y) = \xi_{X,Y}$ for all objects X and Y in $\mathcal{C}$.

We now prove that this m_L is associative, i.e., that

$$m_L \circ (m_L \otimes \mathrm{id}_L) = m_L \circ (\mathrm{id}_L \otimes m_L).$$

We will use a more general version of Theorem 1.6.1, which states that

$$\iota_X \otimes \iota_Y \otimes \iota_Z \colon X^* \otimes X \otimes Y^* \otimes Y \otimes Z^* \otimes Z \to L \otimes L \otimes L$$

is a universal dinatural transformation. Since $\iota_X \otimes \iota_Y \otimes \iota_Z$ is dinatural, each of

$$m_L \circ (m_L \otimes \mathrm{id}_L) \circ (\iota_X \otimes \iota_Y \otimes \iota_Z)$$

and

$$m_L \circ (\mathrm{id}_L \otimes m_L) \circ (\iota_X \otimes \iota_Y \otimes \iota_Z)$$

also define a dinatural transformation. Therefore, by the uniqueness in the universal property of $\iota_X \otimes \iota_Y \otimes \iota_Z$, it is sufficient to prove that

$$m_L \circ (m_L \otimes \mathrm{id}_L) \circ (\iota_X \otimes \iota_Y \otimes \iota_Z) = m_L \circ (\mathrm{id}_L \otimes m_L) \circ (\iota_X \otimes \iota_Y \otimes \iota_Z). \tag{2.2}$$

On the one hand,

$$\begin{aligned}
&m_L \circ (m_L \otimes \mathrm{id}_L) \circ (\iota_X \otimes \iota_Y \otimes \iota_Z) \\
&= m_L \circ (\xi_{X,Y} \otimes \iota_Z) \\
&= m_L \circ (\iota_{X\otimes Y} \otimes \iota_Z) \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y} \otimes \mathrm{id}_{Z^*\otimes Z}) \circ (\sigma_{X^*\otimes X, Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \\
&= \xi_{X\otimes Y, Z} \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y} \otimes \mathrm{id}_{Z^*\otimes Z}) \circ (\sigma_{X^*\otimes X, Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \\
&= \iota_{X\otimes Y\otimes Z} \circ (\gamma_{X\otimes Y, Z} \otimes \mathrm{id}_{X\otimes Y\otimes Z}) \circ (\sigma_{(X\otimes Y)^*\otimes X\otimes Y, Z^*} \otimes \mathrm{id}_Z) \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y} \otimes \mathrm{id}_{Z^*\otimes Z}) \\
&\qquad \circ (\sigma_{X^*\otimes X, Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \\
&= \iota_{X\otimes Y\otimes Z} \circ (\gamma_{X\otimes Y, Z} \otimes \mathrm{id}_{X\otimes Y\otimes Z}) \circ (\mathrm{id}_{Z^*} \otimes \gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y} \otimes \mathrm{id}_Z) \\
&\qquad \circ (\sigma_{Y^*\otimes X^*\otimes X\otimes Y, Z^*} \otimes \mathrm{id}_Z) \circ (\sigma_{X^*\otimes X, Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}),
\end{aligned}$$

where we have used the naturality of σ in the last step, and, on the other hand,

$$\begin{aligned}
&m_L \circ (\mathrm{id}_L \otimes m_L) \circ (\iota_X \otimes \iota_Y \otimes \iota_Z) \\
&= m_L \circ (\iota_X \otimes \xi_{Y,Z}) \\
&= m_L \circ (\iota_X \otimes \iota_{Y\otimes Z}) \circ (\mathrm{id}_{X^*\otimes X} \otimes \gamma_{Y,Z} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*\otimes X} \otimes \sigma_{Y^*\otimes Y, Z^*} \otimes \mathrm{id}_Z) \\
&= \xi_{X, Y\otimes Z} \circ (\mathrm{id}_{X^*\otimes X} \otimes \gamma_{Y,Z} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*\otimes X} \otimes \sigma_{Y^*\otimes Y, Z^*} \otimes \mathrm{id}_Z) \\
&= \iota_{X\otimes Y\otimes Z} \circ (\gamma_{X, Y\otimes Z} \otimes \mathrm{id}_{X\otimes Y\otimes Z}) \circ (\sigma_{X^*\otimes X, (Y\otimes Z)^*} \otimes \mathrm{id}_{Y\otimes Z}) \\
&\qquad \circ (\mathrm{id}_{X^*\otimes X} \otimes \gamma_{Y,Z} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*\otimes X} \otimes \sigma_{Y^*\otimes Y, Z^*} \otimes \mathrm{id}_Z) \\
&= \iota_{X\otimes Y\otimes Z} \circ (\gamma_{X, Y\otimes Z} \otimes \mathrm{id}_{X\otimes Y\otimes Z}) \circ (\gamma_{Y,Z} \otimes \mathrm{id}_{X^*\otimes X} \otimes \mathrm{id}_{Y\otimes Z}) \\
&\qquad \circ (\sigma_{X^*\otimes X, Z^*\otimes Y^*} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*\otimes X} \otimes \sigma_{Y^*\otimes Y, Z^*} \otimes \mathrm{id}_Z).
\end{aligned}$$

Now, by the axioms (1.3) and (1.4) of a braiding, we have

$$\begin{aligned}
&(\sigma_{X^*\otimes X, Z^*\otimes Y^*} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*\otimes X} \otimes \sigma_{Y^*\otimes Y, Z^*} \otimes \mathrm{id}_Z) \\
&= (\mathrm{id}_{Z^*} \otimes \sigma_{X^*\otimes X, Y^*} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\sigma_{X^*\otimes X, Z^*} \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_{Y\otimes Z}) \\
&\qquad \circ (\mathrm{id}_{X^*\otimes X} \otimes \sigma_{Y^*, Z^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*\otimes X} \otimes \mathrm{id}_{Y^*} \otimes \sigma_{Y, Z^*} \otimes \mathrm{id}_Z),
\end{aligned}$$

and, by the Yang-Baxter equation (1.5), this equals

$$\begin{aligned}
&(\sigma_{Y^*,Z^*} \otimes \mathrm{id}_{X^*\otimes X} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{Y^*} \otimes \sigma_{X^*\otimes X,Z^*} \otimes \mathrm{id}_{Y\otimes Z}) \\
&\quad \circ (\sigma_{X^*\otimes X,Y^*} \otimes \mathrm{id}_{Z^*} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*\otimes X} \otimes \mathrm{id}_{Y^*} \otimes \sigma_{Y,Z^*} \otimes \mathrm{id}_Z) \\
&= (\sigma_{Y^*,Z^*} \otimes \mathrm{id}_{X^*\otimes X} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{Y^*} \otimes \sigma_{X^*\otimes X,Z^*} \otimes \mathrm{id}_{Y\otimes Z}) \\
&\qquad \circ (\mathrm{id}_{Y^*\otimes X^*\otimes X} \otimes \sigma_{Y,Z^*} \otimes \mathrm{id}_Z) \circ (\sigma_{X^*\otimes X,Y^*} \otimes \mathrm{id}_{Y\otimes Z^*\otimes Z}) \\
&= (\sigma_{Y^*,Z^*} \otimes \mathrm{id}_{X^*\otimes X} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{Y^*} \otimes \sigma_{X^*\otimes X\otimes Y,Z^*} \otimes \mathrm{id}_Z) \\
&\qquad \circ (\sigma_{X^*\otimes X,Y^*} \otimes \mathrm{id}_{Y\otimes Z^*\otimes Z}) \\
&= (\sigma_{Y^*\otimes X^*\otimes X\otimes Y,Z^*} \otimes \mathrm{id}_Z) \circ (\sigma_{X^*\otimes X,Y^*} \otimes \mathrm{id}_{Y\otimes Z^*\otimes Z}),
\end{aligned}$$

where we have applied axiom (1.3) twice at the end. Thus, if we can show that

$$(\gamma_{X\otimes Y,Z} \otimes \mathrm{id}_{X\otimes Y\otimes Z}) \circ (\mathrm{id}_{Z^*} \otimes \gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y\otimes Z}) = (\gamma_{X,Y\otimes Z} \otimes \mathrm{id}_{X\otimes Y\otimes Z}) \circ (\gamma_{Y,Z} \otimes \mathrm{id}_{X^*\otimes X\otimes Y\otimes Z}),$$

then (2.2) will follow. It is sufficient to prove that

$$\gamma_{X\otimes Y,Z} \circ (\mathrm{id}_{Z^*} \otimes \gamma_{X,Y}) = \gamma_{X,Y\otimes Z} \circ (\gamma_{Y,Z} \otimes \mathrm{id}_{X^*}),$$

and, letting $f = \gamma_{X\otimes Y,Z} \circ (\mathrm{id}_{Z^*} \otimes \gamma_{X,Y})$ and $g = \gamma_{X,Y\otimes Z} \circ (\gamma_{Y,Z} \otimes \mathrm{id}_{X^*})$, it is further sufficient to prove that

$$\mathrm{ev}_{X\otimes Y\otimes Z} \circ (f \otimes \mathrm{id}_{X\otimes Y\otimes Z}) = \mathrm{ev}_{X\otimes Y\otimes Z} \circ (g \otimes \mathrm{id}_{X\otimes Y\otimes Z}).$$

This holds because, on the one hand,

$$\begin{aligned}
\mathrm{ev}_{X\otimes Y\otimes Z} \circ (f \otimes \mathrm{id}_{X\otimes Y\otimes Z}) &= \mathrm{ev}_{X\otimes Y\otimes Z} \circ (\gamma_{X\otimes Y,Z} \otimes \mathrm{id}_{X\otimes Y\otimes Z}) \circ (\mathrm{id}_{Z^*} \otimes \gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y\otimes Z}) \\
&= \mathrm{ev}_Z \circ (\mathrm{id}_{Z^*} \otimes \mathrm{ev}_{X\otimes Y} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{Z^*} \otimes \gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y\otimes Z}) \\
&= \mathrm{ev}_Z \circ (\mathrm{id}_{Z^*} \otimes \mathrm{ev}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{Z^*} \otimes \mathrm{id}_{Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z),
\end{aligned}$$

and, on the other hand,

$$\begin{aligned}
\mathrm{ev}_{X\otimes Y\otimes Z} \circ (g \otimes \mathrm{id}_{X\otimes Y\otimes Z}) &= \mathrm{ev}_{X\otimes Y\otimes Z} \circ (\gamma_{X,Y\otimes Z} \otimes \mathrm{id}_{X\otimes Y\otimes Z}) \circ (\gamma_{Y,Z} \otimes \mathrm{id}_{X^*} \otimes \mathrm{id}_{X\otimes Y\otimes Z}) \\
&= \mathrm{ev}_{Y\otimes Z} \circ (\mathrm{id}_{(X\otimes Y)^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_{Y\otimes Z}) \circ (\gamma_{Y,Z} \otimes \mathrm{id}_{X^*} \otimes \mathrm{id}_{X\otimes Y\otimes Z}) \\
&= \mathrm{ev}_{Y\otimes Z} \circ (\gamma_{Y,Z} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{Z^*\otimes Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_{Y\otimes Z}) \\
&= \mathrm{ev}_Z \circ (\mathrm{id}_{Z^*} \otimes \mathrm{ev}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{Z^*} \otimes \mathrm{id}_{Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z).
\end{aligned}$$

This establishes the associativity of m_L.

We remark here that there is an alternative way to express this multiplication. We arrived at the multiplication by means of the dinatural transformation

$$\xi_{X,Y} = \iota_{X\otimes Y} \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (\sigma_{X^*,Y^*} \otimes \mathrm{id}_{X\otimes Y}) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,Y^*} \otimes \mathrm{id}_Y).$$

We could instead begin by defining

$$\eta_{X,Y} = \iota_{Y\otimes X} \circ (\gamma_{Y,X} \otimes \mathrm{id}_{Y\otimes X}) \circ (\mathrm{id}_{X^*\otimes Y^*} \otimes \sigma_{X,Y}) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,Y^*} \otimes \mathrm{id}_Y). \tag{2.3}$$

We show that this is in fact equal to $\xi_{X,Y}$. This will follow if

$$\iota_{Y\otimes X} \circ (\gamma_{Y,X} \otimes \mathrm{id}_{Y\otimes X}) \circ (\mathrm{id}_{X^*\otimes Y^*} \otimes \sigma_{X,Y}) = \iota_{X\otimes Y} \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (\sigma_{X^*,Y^*} \otimes \mathrm{id}_{X\otimes Y}).$$

Observe that

$$\begin{aligned}\iota_{Y\otimes X} \circ (\gamma_{Y,X} \otimes \mathrm{id}_{Y\otimes X}) \circ (\mathrm{id}_{X^*\otimes Y^*} \otimes \sigma_{X,Y}) &= \iota_{Y\otimes X} \circ (\mathrm{id}_{(Y\otimes X)^*} \otimes \sigma_{X,Y}) \circ (\gamma_{Y,X} \otimes \mathrm{id}_{X\otimes Y}) \\ &= \iota_{X\otimes Y} \circ (\sigma^*_{X,Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (\gamma_{Y,X} \otimes \mathrm{id}_{X\otimes Y})\end{aligned}$$

by the dinaturality of ι, so it is sufficient to prove that

$$\sigma^*_{X,Y} \circ \gamma_{Y,X} = \gamma_{X,Y} \circ \sigma_{X^*,Y^*}. \tag{2.4}$$

Using the dual objects $X^*\otimes Y^*$ and $Y^*\otimes X^*$ of $Y\otimes X$ and $X\otimes Y$, respectively, one can show that $\sigma_{X^*,Y^*} = \sigma^*_{X,Y}$ by Theorem 1.4.1. Then, via $\gamma_{X,Y}\colon Y^*\otimes X^* \to (X\otimes Y)^*$, this can be transported to other versions of the dual object so that we arrive at (2.4). Thus $\eta_{X,Y} = \xi_{X,Y}$ and, in particular, η is a dinatural transformation, so by the universality of $\iota_X \otimes \iota_Y$ there is a unique morphism m_L such that $m_L \circ (\iota_X \otimes \iota_Y) = \eta_{X,Y}$. The uniqueness implies that this is the same morphism m_L obtained from $\xi_{X,Y}$.

2.2 Unit

We now discuss the unit of L. In this section, we will not assume that the unit constraints are identities. First, we show that the unit object I is both a left and right dual of itself. Recall that $\lambda_I = \rho_I$, and observe that

$$\mathrm{id}_I \otimes \lambda_I = \lambda_{I\otimes I} = \lambda_I \otimes \mathrm{id}_I$$

by naturality. Therefore

$$\begin{aligned}\lambda_I \circ (\mathrm{id}_I \otimes \lambda_I) \circ (\lambda_I^{-1} \otimes \mathrm{id}_I) \circ \lambda_I^{-1} &= \lambda_I \circ (\mathrm{id}_I \otimes \lambda_I) \circ (\lambda_I \circ (\lambda_I \otimes \mathrm{id}_I))^{-1} \\ &= \lambda_I \circ \lambda_{I \otimes I} \circ (\lambda_I \circ \lambda_{I\otimes I})^{-1} \\ &= \mathrm{id}_I,\end{aligned}$$

and

$$\begin{aligned}\lambda_I \circ (\lambda_I \otimes \mathrm{id}_I) \circ (\mathrm{id}_I \otimes \lambda_I^{-1}) \circ \lambda_I^{-1} &= \lambda_I \circ (\lambda_I \otimes \mathrm{id}_I) \circ (\lambda_I \circ (\mathrm{id}_I \otimes \lambda_I))^{-1} \\ &= \lambda_I \circ \lambda_{I \otimes I} \circ (\lambda_I \circ \lambda_{I\otimes I})^{-1} \\ &= \mathrm{id}_I.\end{aligned}$$

These relations show that the object I, together with the morphisms

$$\mathrm{ev}_I = \lambda_I = \rho_I$$

and

$$\mathrm{coev}_I = \lambda_I^{-1} = \rho_I^{-1},$$

satisfies the conditions of both a left dual and a right dual in Definition 1.3.1.

Now, using the left dual $I^* = I$, we can define the unit of L as

$$u_L = \iota_I \circ \lambda_I^{-1} \colon I \to L, \tag{2.5}$$

where ι is the universal dinatural transformation associated to L. We prove the left unitality of u_L, i.e.,

$$m_L \circ (u_L \otimes \mathrm{id}_L) = \lambda_L. \tag{2.6}$$

By the universality of ι, it is sufficient to prove that

$$m_L \circ (u_L \otimes \mathrm{id}_L) \circ \lambda_L^{-1} \circ \iota_X = \iota_X.$$

By the naturality of λ, we have $\lambda_L^{-1} \circ \iota_X = (\mathrm{id}_I \otimes \iota_X) \circ \lambda_{X^* \otimes X}^{-1}$, and hence

$$\begin{aligned}m_L \circ (u_L \otimes \mathrm{id}_L) \circ \lambda_L^{-1} \circ \iota_X &= m_L \circ (\iota_I \otimes \mathrm{id}_L) \circ (\lambda_I^{-1} \otimes \mathrm{id}_L) \circ (\mathrm{id}_I \otimes \iota_X) \circ \lambda_{X^* \otimes X}^{-1} \\ &= m_L \circ (\iota_I \otimes \iota_X) \circ (\lambda_I^{-1} \otimes \mathrm{id}_{X^* \otimes X}) \circ \lambda_{X^* \otimes X}^{-1} \\ &= \xi_{I,X} \circ (\lambda_I^{-1} \otimes \mathrm{id}_{X^* \otimes X}) \circ \lambda_{X^* \otimes X}^{-1},\end{aligned}$$

where ξ is the dinatural transformation defined as in (2.1). Inserting the definition of $\xi_{I,X}$, and applying the naturality of σ, this equals

$$\begin{aligned}&\iota_{I\otimes X}\circ(\gamma_{I,X}\otimes \mathrm{id}_{I\otimes X})\circ(\sigma_{I^*\otimes I,X^*}\otimes \mathrm{id}_X)\circ(\lambda_I^{-1}\otimes \mathrm{id}_{X^*\otimes X})\circ\lambda^{-1}_{X^*\otimes X}\\ &=\iota_{I\otimes X}\circ(\gamma_{I,X}\otimes \mathrm{id}_{I\otimes X})\circ(\mathrm{id}_{X^*}\otimes\lambda_I^{-1}\otimes \mathrm{id}_X)\circ(\sigma_{I,X^*}\otimes \mathrm{id}_X)\circ\lambda^{-1}_{X^*\otimes X}.\end{aligned}$$

Now by [5, Lem. XI.2.2, p. 283], we have

$$\lambda^{-1}_{X^*\otimes X}=(\lambda^{-1}_{X^*}\otimes \mathrm{id}_X),$$

and by [5, Prop. XIII.1.2, p. 316], we have

$$\sigma_{I,X^*}=\rho^{-1}_{X^*}\circ\lambda_{X^*},$$

so the above expression reduces to

$$\begin{aligned}&\iota_{I\otimes X}\circ(\gamma_{I,X}\otimes \mathrm{id}_{I\otimes X})\circ(\mathrm{id}_{X^*}\otimes\lambda_I^{-1}\otimes \mathrm{id}_X)\circ(\rho^{-1}_{X^*}\otimes \mathrm{id}_X)\\ &=\iota_{I\otimes X}\circ(\gamma_{I,X}\otimes \mathrm{id}_{I\otimes X})\circ(\mathrm{id}_{X^*}\otimes\rho_I^{-1}\otimes \mathrm{id}_X)\circ(\rho^{-1}_{X^*}\otimes \mathrm{id}_X).\end{aligned}$$

By another application of [5, Lem. XI.2.2, p. 283] and the naturality of ρ, this equals

$$\begin{aligned}&\iota_{I\otimes X}\circ(\gamma_{I,X}\otimes \mathrm{id}_{I\otimes X})\circ(\rho^{-1}_{X^*\otimes I}\otimes \mathrm{id}_X)\circ(\rho^{-1}_{X^*}\otimes \mathrm{id}_X)\\ &=\iota_{I\otimes X}\circ(\gamma_{I,X}\otimes \mathrm{id}_{I\otimes X})\circ(\rho^{-1}_{X^*}\otimes \mathrm{id}_I\otimes \mathrm{id}_X)\circ(\rho^{-1}_{X^*}\otimes \mathrm{id}_X).\end{aligned}$$

To complete the proof, we need the following relation:

$$\gamma_{I,X}\circ\rho^{-1}_{X^*}=\lambda^*_X. \tag{2.7}$$

By Theorem 1.4.1, this is equivalent to

$$\mathrm{ev}_X\circ(\mathrm{id}_{X^*}\otimes\lambda_X)=\mathrm{ev}_{I\otimes X}\circ(\gamma_{I,X}\otimes \mathrm{id}_{I\otimes X})\circ(\rho^{-1}_{X^*}\otimes \mathrm{id}_{I\otimes X}).$$

When the unit constraints are not assumed to be identities, the characterization (1.16) of γ says that

$$\begin{aligned}\mathrm{ev}_{I\otimes X}\circ(\gamma_{I,X}\otimes \mathrm{id}_{I\otimes X})&=\mathrm{ev}_X\circ(\mathrm{id}_{X^*}\otimes\lambda_X)\circ(\mathrm{id}_{X^*}\otimes \mathrm{ev}_I\otimes \mathrm{id}_X)\\ &=\mathrm{ev}_X\circ(\mathrm{id}_{X^*}\otimes\lambda_X)\circ(\mathrm{id}_{X^*}\otimes\lambda_I\otimes \mathrm{id}_X),\end{aligned}$$

and, as a consequence of the Triangle Axiom (1.2), we have

$$\rho_{X^*}^{-1} \otimes \mathrm{id}_I = \mathrm{id}_{X^*} \otimes \lambda_I^{-1}.$$

Therefore

$$\begin{aligned}
&\mathrm{ev}_{I\otimes X} \circ (\gamma_{I,X} \otimes \mathrm{id}_{I\otimes X}) \circ (\rho_{X^*}^{-1} \otimes \mathrm{id}_{I\otimes X}) \\
&= \mathrm{ev}_X \circ (\mathrm{id}_{X^*} \otimes \lambda_X) \circ (\mathrm{id}_{X^*} \otimes \lambda_I \otimes \mathrm{id}_X) \circ (\rho_{X^*}^{-1} \otimes \mathrm{id}_{I\otimes X}) \\
&= \mathrm{ev}_X \circ (\mathrm{id}_{X^*} \otimes \lambda_X) \circ (\mathrm{id}_{X^*} \otimes \lambda_I \otimes \mathrm{id}_X) \circ (\mathrm{id}_{X^*} \otimes \lambda_I^{-1} \otimes \mathrm{id}_X) \\
&= \mathrm{ev}_X \circ (\mathrm{id}_{X^*} \otimes \lambda_X),
\end{aligned}$$

and this establishes (2.7).

Thus, continuing the calculation,

$$\begin{aligned}
&\iota_{I\otimes X} \circ (\gamma_{I,X} \otimes \mathrm{id}_{I\otimes X}) \circ (\rho_{X^*}^{-1} \otimes \mathrm{id}_I \otimes \mathrm{id}_X) \circ (\rho_{X^*}^{-1} \otimes \mathrm{id}_X) \\
&= \iota_{I\otimes X} \circ (\lambda_X^* \otimes \mathrm{id}_{I\otimes X}) \circ (\rho_{X^*}^{-1} \otimes \mathrm{id}_X) \\
&= \iota_X \circ (\mathrm{id}_{X^*} \otimes \lambda_X) \circ (\rho_{X^*}^{-1} \otimes \mathrm{id}_X) \\
&= \iota_X \circ (\rho_{X^*} \otimes \mathrm{id}_X) \circ (\rho_{X^*}^{-1} \otimes \mathrm{id}_X) \\
&= \iota_X,
\end{aligned}$$

where we have used the dinaturality of ι and another application of (1.2). This establishes the left unitality of u_L; the right unitality is proved similarly.

2.3 Coproduct

To define the coproduct of L, we begin with a dinatural transformation $\zeta_X \colon X^* \otimes X \to L \otimes L$ defined by

$$\zeta_X = (\iota_X \otimes \iota_X) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X). \tag{2.8}$$

The dinaturality of ζ means that the diagram

$$\begin{array}{ccc}
Y^* \otimes X & \xrightarrow{f^* \otimes \mathrm{id}_X} & X^* \otimes X \\
\Big\downarrow{\scriptstyle \mathrm{id}_{Y^*} \otimes f} & & \Big\downarrow{\scriptstyle \zeta_X} \\
Y^* \otimes Y & \xrightarrow{\zeta_Y} & L \otimes L
\end{array}$$

commutes for all morphisms $f\colon X \to Y$ in $\mathcal{C}$. By the dinaturality of ι and Theorem 1.4.1, we have

$$\begin{aligned}\zeta_X \circ (f^* \otimes \mathrm{id}_X) &= (\iota_X \otimes \iota_X) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X) \circ (f^* \otimes \mathrm{id}_X)\\ &= (\iota_X \otimes \iota_X) \circ (f^* \otimes \mathrm{id}_X \otimes \mathrm{id}_{X^*} \otimes \mathrm{id}_X) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X)\\ &= (\iota_Y \otimes \iota_X) \circ (\mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{X^*} \otimes \mathrm{id}_X) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X)\\ &= (\iota_Y \otimes \iota_X) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{id}_Y \otimes f^* \otimes \mathrm{id}_X) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_Y \otimes \mathrm{id}_X)\\ &= (\iota_Y \otimes \iota_Y) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Y^*} \otimes f) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_Y \otimes \mathrm{id}_X)\\ &= (\iota_Y \otimes \iota_Y) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_Y \otimes \mathrm{id}_Y) \circ (\mathrm{id}_{Y^*} \otimes f)\\ &= \zeta_Y \circ (\mathrm{id}_{Y^*} \otimes f),\end{aligned}$$

and this proves that ζ is dinatural. By the universality of ι, there is a unique morphism $\Delta_L\colon L \to L \otimes L$ such that $\Delta_L \circ \iota_X = \zeta_X$ for all objects X in $\mathcal{C}$.

We now prove the coassociativity of Δ_L, which means that

$$(\Delta_L \otimes \mathrm{id}_L) \circ \Delta_L = (\mathrm{id}_L \otimes \Delta_L) \circ \Delta_L.$$

Since ι is dinatural, each of $(\Delta_L \otimes \mathrm{id}_L) \circ \Delta_L \circ \iota_X$ and $(\mathrm{id}_L \otimes \Delta_L) \circ \Delta_L \circ \iota_X$ also define a dinatural transformation. Therefore, by the universality of ι, it is sufficient to prove that

$$(\Delta_L \otimes \mathrm{id}_L) \circ \Delta_L \circ \iota_X = (\mathrm{id}_L \otimes \Delta_L) \circ \Delta_L \circ \iota_X.$$

We have

$$\begin{aligned}&(\Delta_L \otimes \mathrm{id}_L) \circ \Delta_L \circ \iota_X\\ &= (\Delta_L \otimes \mathrm{id}_L) \circ \zeta_X\\ &= (\Delta_L \otimes \mathrm{id}_L) \circ (\iota_X \otimes \iota_X) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X)\\ &= (\zeta_X \otimes \iota_X) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X)\\ &= (\iota_X \otimes \iota_X \otimes \iota_X) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X \otimes \mathrm{id}_{X^* \otimes X}) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X)\\ &= (\iota_X \otimes \iota_X \otimes \iota_X) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{coev}_X \otimes \mathrm{id}_X),\end{aligned}$$

and a similar calculation shows that

$$(\mathrm{id}_L \otimes \Delta_L) \circ \Delta_L \circ \iota_X = (\iota_X \otimes \iota_X \otimes \iota_X) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{coev}_X \otimes \mathrm{id}_X).$$

This establishes the coassociativity of Δ_L.

We now prove that Δ_L is an algebra homomorphism, which means that

$$\Delta_L \circ m_L = m_{L\otimes L} \circ (\Delta_L \otimes \Delta_L)$$

and

$$\Delta_L \circ u_L = u_{L\otimes L}.$$

For the multiplicativity of Δ_L, it is sufficient by the universality of $\iota_X \otimes \iota_Y$ to prove that

$$\Delta_L \circ m_L \circ (\iota_X \otimes \iota_Y) = m_{L\otimes L} \circ (\Delta_L \otimes \Delta_L) \circ (\iota_X \otimes \iota_Y),$$

i.e.,

$$\Delta_L \circ \xi_{X,Y} = (m_L \otimes m_L) \circ (\mathrm{id}_L \otimes \sigma_{L,L} \otimes \mathrm{id}_L) \circ (\zeta_X \otimes \zeta_Y),$$

where ξ is the dinatural transformation defined as in (2.1). For the left-hand side of this equation, we have

$$\begin{aligned}
&\Delta_L \circ \xi_{X,Y} \\
&= \Delta_L \circ \iota_{X\otimes Y} \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (\sigma_{X^*\otimes X, Y^*} \otimes \mathrm{id}_Y) \\
&= \zeta_{X\otimes Y} \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (\sigma_{X^*\otimes X, Y^*} \otimes \mathrm{id}_Y) \\
&= (\iota_{X\otimes Y} \otimes \iota_{X\otimes Y}) \circ (\mathrm{id}_{(X\otimes Y)^*} \otimes \mathrm{coev}_{X\otimes Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (\sigma_{X^*\otimes X, Y^*} \otimes \mathrm{id}_Y) \\
&= (\iota_{X\otimes Y} \otimes \iota_{X\otimes Y}) \circ (\gamma_{X,Y} \otimes \mathrm{coev}_{X\otimes Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (\sigma_{X^*\otimes X, Y^*} \otimes \mathrm{id}_Y).
\end{aligned}$$

For the right-hand side, we have

$$\begin{aligned}
&(m_L \otimes m_L) \circ (\mathrm{id}_L \otimes \sigma_{L,L} \otimes \mathrm{id}_L) \circ (\zeta_X \otimes \zeta_Y) \\
&= (m_L \otimes m_L) \circ (\mathrm{id}_L \otimes \sigma_{L,L} \otimes \mathrm{id}_L) \circ (\iota_X \otimes \iota_X \otimes \iota_Y \otimes \iota_Y) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X \otimes \mathrm{id}_{Y^*} \otimes \mathrm{coev}_Y \otimes \mathrm{id}_Y),
\end{aligned}$$

which, by the naturality of σ, equals

$$\begin{aligned}
&(m_L \otimes m_L) \circ (\iota_X \otimes \iota_Y \otimes \iota_X \otimes \iota_Y) \circ (\mathrm{id}_{X^*\otimes X} \otimes \sigma_{X^*\otimes X, Y^*\otimes Y} \otimes \mathrm{id}_{Y^*\otimes Y}) \\
&\quad \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X \otimes \mathrm{id}_{Y^*} \otimes \mathrm{coev}_Y \otimes \mathrm{id}_Y) \\
&= (\xi_{X,Y} \otimes \xi_{X,Y}) \circ (\mathrm{id}_{X^*\otimes X} \otimes \sigma_{X^*\otimes X, Y^*\otimes Y} \otimes \mathrm{id}_{Y^*\otimes Y}) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X \otimes \mathrm{id}_{Y^*} \otimes \mathrm{coev}_Y \otimes \mathrm{id}_Y) \\
&= (\iota_{X\otimes Y} \otimes \iota_{X\otimes Y}) \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y} \otimes \gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (\sigma_{X^*\otimes X, Y^*} \otimes \mathrm{id}_Y \otimes \sigma_{X^*\otimes X, Y^*} \otimes \mathrm{id}_Y) \\
&\qquad \circ (\mathrm{id}_{X^*\otimes X} \otimes \sigma_{X^*\otimes X, Y^*\otimes Y} \otimes \mathrm{id}_{Y^*\otimes Y}) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X \otimes \mathrm{id}_{Y^*} \otimes \mathrm{coev}_Y \otimes \mathrm{id}_Y).
\end{aligned}$$

By the braiding axioms (1.3) and (1.4), we have

$$\sigma_{X^*\otimes X, Y^*\otimes Y} = (\mathrm{id}_{Y^*}\otimes\sigma_{X^*,Y}\otimes\mathrm{id}_X)\circ(\sigma_{X^*,Y^*}\otimes\mathrm{id}_Y\otimes\mathrm{id}_X)$$
$$\circ(\mathrm{id}_{X^*}\otimes\mathrm{id}_{Y^*}\otimes\sigma_{X,Y})\circ(\mathrm{id}_{X^*}\otimes\sigma_{X,Y^*}\otimes\mathrm{id}_Y),$$

and hence the above expression becomes

$$\begin{aligned}
&(\iota_{X\otimes Y}\otimes\iota_{X\otimes Y})\circ(\gamma_{X,Y}\otimes\mathrm{id}_{X\otimes Y}\otimes\gamma_{X,Y}\otimes\mathrm{id}_{X\otimes Y})\circ(\sigma_{X^*\otimes X,Y^*}\otimes\mathrm{id}_Y\otimes\sigma_{X^*\otimes X,Y^*}\otimes\mathrm{id}_Y)\\
&\quad\circ(\mathrm{id}_{X^*\otimes X\otimes Y^*}\otimes\sigma_{X^*,Y}\otimes\mathrm{id}_{X\otimes Y^*\otimes Y})\circ(\mathrm{id}_{X^*\otimes X}\otimes\sigma_{X^*,Y^*}\otimes\mathrm{id}_{Y\otimes X\otimes Y^*\otimes Y})\\
&\quad\circ(\mathrm{id}_{X^*\otimes X\otimes X^*\otimes Y^*}\otimes\sigma_{X,Y}\otimes\mathrm{id}_{Y^*\otimes Y})\circ(\mathrm{id}_{X^*\otimes X\otimes X^*}\otimes\sigma_{X,Y^*}\otimes\mathrm{id}_{Y\otimes Y^*\otimes Y})\\
&\quad\circ(\mathrm{id}_{X^*}\otimes\mathrm{coev}_X\otimes\mathrm{id}_{X\otimes Y^*}\otimes\mathrm{coev}_Y\otimes\mathrm{id}_Y)\\
&=(\iota_{X\otimes Y}\otimes\iota_{X\otimes Y})\circ(\gamma_{X,Y}\otimes\mathrm{id}_{X\otimes Y}\otimes\gamma_{X,Y}\otimes\mathrm{id}_{X\otimes Y})\circ(\sigma_{X^*\otimes X,Y^*}\otimes\mathrm{id}_Y\otimes\sigma_{X^*\otimes X,Y^*}\otimes\mathrm{id}_Y)\\
&\qquad\circ(\mathrm{id}_{X^*\otimes X\otimes Y^*}\otimes\sigma_{X^*,Y}\otimes\mathrm{id}_{X\otimes Y^*\otimes Y})\circ(\mathrm{id}_{X^*\otimes X}\otimes\sigma_{X^*,Y^*}\otimes\mathrm{id}_{Y\otimes X\otimes Y^*\otimes Y})\\
&\qquad\circ(\mathrm{id}_{X^*}\otimes\mathrm{coev}_X\otimes\mathrm{id}_{Y^*}\otimes\sigma_{X,Y}\otimes\mathrm{id}_{Y^*\otimes Y})\circ(\mathrm{id}_{X^*}\otimes\sigma_{X,Y^*}\otimes\mathrm{coev}_Y\otimes\mathrm{id}_Y).
\end{aligned}$$

Applying (1.9) to the morphism $(\mathrm{id}_X\otimes\sigma_{X^*,Y^*})\circ(\mathrm{coev}_X\otimes\mathrm{id}_{Y^*})$ and the braiding axiom (1.3) to $\sigma_{X^*\otimes X,Y^*}$, this equals

$$\begin{aligned}
&(\iota_{X\otimes Y}\otimes\iota_{X\otimes Y})\circ(\gamma_{X,Y}\otimes\mathrm{id}_{X\otimes Y}\otimes\gamma_{X,Y}\otimes\mathrm{id}_{X\otimes Y})\circ(\sigma_{X^*,Y^*}\otimes\mathrm{id}_{X\otimes Y}\otimes\sigma_{X^*,Y^*}\otimes\mathrm{id}_{X\otimes Y})\\
&\quad\circ(\mathrm{id}_{X^*}\otimes\sigma_{X,Y^*}\otimes\mathrm{id}_{Y\otimes X^*}\otimes\sigma_{X,Y^*}\otimes\mathrm{id}_Y)\circ(\mathrm{id}_{X^*\otimes X\otimes Y^*}\otimes\sigma_{X^*,Y}\otimes\mathrm{id}_{X\otimes Y^*\otimes Y})\\
&\quad\circ(\mathrm{id}_{X^*}\otimes\sigma^{-1}_{X,Y^*}\otimes\mathrm{id}_{X^*\otimes Y\otimes X\otimes Y^*\otimes Y})\circ(\mathrm{id}_{X^*\otimes Y^*}\otimes\mathrm{coev}_X\otimes\sigma_{X,Y}\otimes\mathrm{id}_{Y^*\otimes Y})\\
&\quad\circ(\mathrm{id}_{X^*}\otimes\sigma_{X,Y^*}\otimes\mathrm{coev}_Y\otimes\mathrm{id}_Y)\\
&=(\iota_{X\otimes Y}\otimes\iota_{X\otimes Y})\circ(\gamma_{X,Y}\otimes\mathrm{id}_{X\otimes Y}\otimes\gamma_{X,Y}\otimes\mathrm{id}_{X\otimes Y})\circ(\sigma_{X^*,Y^*}\otimes\mathrm{id}_{X\otimes Y}\otimes\sigma_{X^*,Y^*}\otimes\mathrm{id}_{X\otimes Y})\\
&\qquad\circ(\mathrm{id}_{X^*\otimes Y^*\otimes X}\otimes\sigma_{X^*,Y}\otimes\sigma_{X,Y^*}\otimes\mathrm{id}_Y)\circ(\mathrm{id}_{X^*\otimes Y^*\otimes X\otimes X^*}\otimes\sigma_{X,Y}\otimes\mathrm{id}_{Y^*\otimes Y})\\
&\qquad\circ(\mathrm{id}_{X^*\otimes Y^*}\otimes\mathrm{coev}_X\otimes\mathrm{id}_X\otimes\mathrm{coev}_Y\otimes\mathrm{id}_Y)\circ(\mathrm{id}_{X^*}\otimes\sigma_{X,Y^*}\otimes\mathrm{id}_Y).
\end{aligned}$$

Then, applying (1.7) to the morphism $(\sigma_{X,Y}\otimes\mathrm{id}_{Y^*})\circ(\mathrm{id}_X\otimes\mathrm{coev}_Y)$, we obtain

$$\begin{aligned}
&(\iota_{X\otimes Y}\otimes\iota_{X\otimes Y})\circ(\gamma_{X,Y}\otimes\mathrm{id}_{X\otimes Y}\otimes\gamma_{X,Y}\otimes\mathrm{id}_{X\otimes Y})\circ(\sigma_{X^*,Y^*}\otimes\mathrm{id}_{X\otimes Y}\otimes\sigma_{X^*,Y^*}\otimes\mathrm{id}_{X\otimes Y})\\
&\quad\circ(\mathrm{id}_{X^*\otimes Y^*\otimes X}\otimes\sigma_{X^*,Y}\otimes\sigma_{X,Y^*}\otimes\mathrm{id}_Y)\circ(\mathrm{id}_{X^*\otimes Y^*\otimes X\otimes X^*\otimes Y}\otimes\sigma^{-1}_{X,Y^*}\otimes\mathrm{id}_Y)\\
&\quad\circ(\mathrm{id}_{X^*\otimes Y^*}\otimes\mathrm{coev}_X\otimes\mathrm{coev}_Y\otimes\mathrm{id}_{X\otimes Y})\circ(\mathrm{id}_{X^*}\otimes\sigma_{X,Y^*}\otimes\mathrm{id}_Y)\\
&=(\iota_{X\otimes Y}\otimes\iota_{X\otimes Y})\circ(\gamma_{X,Y}\otimes\mathrm{id}_{X\otimes Y}\otimes\gamma_{X,Y}\otimes\mathrm{id}_{X\otimes Y})\circ(\sigma_{X^*,Y^*}\otimes\mathrm{id}_{X\otimes Y}\otimes\sigma_{X^*,Y^*}\otimes\mathrm{id}_{X\otimes Y})\\
&\qquad\circ(\mathrm{id}_{X^*\otimes Y^*\otimes X}\otimes\sigma_{X^*,Y}\otimes\mathrm{id}_{Y^*\otimes X\otimes Y})\circ(\mathrm{id}_{X^*\otimes Y^*\otimes X\otimes X^*}\otimes\mathrm{coev}_Y\otimes\mathrm{id}_{X\otimes Y})\\
&\qquad\circ(\mathrm{id}_{X^*\otimes Y^*}\otimes\mathrm{coev}_X\otimes\mathrm{id}_{X\otimes Y})\circ(\mathrm{id}_{X^*}\otimes\sigma_{X,Y^*}\otimes\mathrm{id}_Y).
\end{aligned}$$

Applying (1.7) again, to the morphism $(\sigma_{X^*,Y} \otimes \mathrm{id}_{Y^*}) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_Y)$, we obtain

$$\begin{aligned}&(\iota_{X\otimes Y} \otimes \iota_{X\otimes Y}) \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y} \otimes \gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (\sigma_{X^*,Y^*} \otimes \mathrm{id}_{X\otimes Y} \otimes \sigma_{X^*,Y^*} \otimes \mathrm{id}_{X\otimes Y})\\ &\circ (\mathrm{id}_{X^*\otimes Y^*\otimes X\otimes Y} \otimes \sigma^{-1}_{X^*,Y^*} \otimes \mathrm{id}_{X\otimes Y}) \circ (\mathrm{id}_{X^*\otimes Y^*\otimes X} \otimes \mathrm{coev}_Y \otimes \mathrm{id}_{X^*\otimes X\otimes Y})\\ &\circ (\mathrm{id}_{X^*\otimes Y^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_{X\otimes Y}) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,Y^*} \otimes \mathrm{id}_Y),\end{aligned}$$

which simplifies, using the braiding axiom (1.3), to

$$\begin{aligned}&(\iota_{X\otimes Y} \otimes \iota_{X\otimes Y}) \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y} \otimes \gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (\mathrm{id}_{Y^*\otimes X^*\otimes X} \otimes \mathrm{coev}_Y \otimes \mathrm{id}_{X^*\otimes X\otimes Y})\\ &\circ (\mathrm{id}_{Y^*\otimes X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_{X\otimes Y}) \circ (\sigma_{X^*\otimes X,Y^*} \otimes \mathrm{id}_Y).\end{aligned}$$

Finally, by the characterization (1.17) of γ, this equals

$$(\iota_{X\otimes Y} \otimes \iota_{X\otimes Y}) \circ (\gamma_{X,Y} \otimes \mathrm{coev}_{X\otimes Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (\sigma_{X^*\otimes X,Y^*} \otimes \mathrm{id}_Y).$$

This establishes the multiplicativity of Δ_L. That Δ_L preserves the unit follows from the definitions of Δ_L and u_L, and the definition of $u_{L\otimes L}$ as in (1.32):

$$\Delta_L \circ u_L = \Delta_L \circ \iota_I = \zeta_I = \iota_I \otimes \iota_I = u_{L\otimes L}.$$

2.4 Counit

We now obtain the counit of L. Observe that $\mathrm{ev}_X \colon X^* \otimes X \to I$ from Definition 1.3.1 defines a dinatural transformation, since the dual morphism f^* of any morphism $f \colon X \to Y$ in $\mathcal{C}$ is characterized by the commutativity of the diagram

$$\begin{array}{ccc} Y^* \otimes X & \xrightarrow{f^*\otimes \mathrm{id}_X} & X^* \otimes X \\ {\scriptstyle \mathrm{id}_{Y^*}\otimes f}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_X} \\ Y^* \otimes Y & \xrightarrow{\mathrm{ev}_Y} & I \end{array} \tag{2.9}$$

by Theorem 1.4.1. Therefore, by the universality of ι, there is a unique morphism $\varepsilon_L \colon L \to I$ such that $\varepsilon_L \circ \iota_X = \mathrm{ev}_X$ for all objects X in $\mathcal{C}$.

The counitality of ε_L means that

$$(\varepsilon_L \otimes \mathrm{id}_L) \circ \Delta_L = \mathrm{id}_L = (\mathrm{id}_L \otimes \varepsilon_L) \circ \Delta_L.$$

For the left equation, it is sufficient by the universality of ι to prove that

$$(\varepsilon_L \otimes \mathrm{id}_L) \circ \Delta_L \circ \iota_X = \iota_X,$$

i.e.,

$$(\varepsilon_L \otimes \mathrm{id}_L) \circ \zeta_X = \iota_X,$$

where ζ is the dinatural transformation defined as in (2.8). By Definition 1.3.1, we have

$$\begin{aligned}(\varepsilon_L \otimes \mathrm{id}_L) \circ \zeta_X &= (\varepsilon_L \otimes \mathrm{id}_L) \circ (\iota_X \otimes \iota_X) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X) \\ &= (\mathrm{ev}_X \otimes \iota_X) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X) \\ &= \iota_X \circ (\mathrm{ev}_X \otimes \mathrm{id}_{X^*} \otimes \mathrm{id}_X) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X) \\ &= \iota_X,\end{aligned}$$

as required. The right equation is proved similarly.

Next, we prove that ε_L is an algebra homomorphism. Recalling that $m_I = \lambda_I = \mathrm{id}_I$, the multiplicativity of ε_L means that

$$\varepsilon_L \circ m_L = m_I \circ (\varepsilon_L \otimes \varepsilon_L) = \varepsilon_L \otimes \varepsilon_L.$$

It is sufficient by the universality of $\iota_X \otimes \iota_Y$ to prove that

$$\varepsilon_L \circ m_L \circ (\iota_X \otimes \iota_Y) = (\varepsilon_L \otimes \varepsilon_L) \circ (\iota_X \otimes \iota_Y),$$

i.e.,

$$\varepsilon_L \circ \xi_{X,Y} = \mathrm{ev}_X \otimes \mathrm{ev}_Y,$$

where ξ is the dinatural transformation defined as in (2.1). By the characterization (1.16) of γ, the naturality of σ, and the fact that $\sigma_{I,Y^*} = \mathrm{id}_{Y^*}$ by (1.6), we have

$$\begin{aligned}\varepsilon_L \circ \xi_{X,Y} &= \varepsilon_L \circ \iota_{X\otimes Y} \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (\sigma_{X^*\otimes X,Y^*} \otimes \mathrm{id}_Y) \\ &= \mathrm{ev}_{X\otimes Y} \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y}) \circ (\sigma_{X^*\otimes X,Y^*} \otimes \mathrm{id}_Y) \\ &= \mathrm{ev}_Y \circ (\mathrm{id}_{Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_Y) \circ (\sigma_{X^*\otimes X,Y^*} \otimes \mathrm{id}_Y) \\ &= \mathrm{ev}_Y \circ (\sigma_{I,Y^*} \otimes \mathrm{id}_Y) \circ (\mathrm{ev}_X \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_Y) \\ &= \mathrm{ev}_X \otimes \mathrm{ev}_Y,\end{aligned}$$

as required.

That ε_L preserves the unit follows from the definitions of ε_L and u_L, and the definitions $u_I = \lambda_I = \mathrm{id}_I$ and $\mathrm{ev}_I = \lambda_I = \mathrm{id}_I$:

$$\varepsilon_L \circ u_L = \varepsilon_L \circ \iota_I = \mathrm{ev}_I = u_I.$$

2.5 Antipode

So far we have shown that L is a bialgebra in $\mathcal{C}$. We now describe its antipode. We begin once again by defining a dinatural transformation. Let

$$\chi_X = (\mathrm{ev}_X \otimes \iota_{X^*}) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X^{**} \otimes X^*, X}) \circ (\mathrm{coev}_{X^*} \otimes \mathrm{id}_{X^* \otimes X}). \tag{2.10}$$

The dinaturality of χ means that the diagram

$$\begin{array}{ccc} Y^* \otimes X & \xrightarrow{f^* \otimes \mathrm{id}_X} & X^* \otimes X \\ {\scriptstyle \mathrm{id}_{Y^*} \otimes f} \downarrow & & \downarrow {\scriptstyle \chi_X} \\ Y^* \otimes Y & \xrightarrow{\chi_Y} & I \end{array}$$

commutes for all morphisms $f\colon X \to Y$ in $\mathcal{C}$. By the naturality of σ, the characterizations of dual morphisms given in Theorem 1.4.1, and the dinaturality of ι, we have

$$\begin{aligned}
&\chi_Y \circ (\mathrm{id}_{Y^*} \otimes f) \\
&= (\mathrm{ev}_Y \otimes \iota_{Y^*}) \circ (\mathrm{id}_{Y^*} \otimes \sigma_{Y^{**} \otimes Y^*, Y}) \circ (\mathrm{coev}_{Y^*} \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_{Y^*} \otimes f) \\
&= (\mathrm{ev}_Y \otimes \iota_{Y^*}) \circ (\mathrm{id}_{Y^*} \otimes \sigma_{Y^{**} \otimes Y^*, Y}) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{id}_{Y^{**}} \otimes \mathrm{id}_{Y^*} \otimes f) \circ (\mathrm{coev}_{Y^*} \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_X) \\
&= (\mathrm{ev}_Y \otimes \iota_{Y^*}) \circ (\mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_{Y^{**}} \otimes \mathrm{id}_{Y^*}) \circ (\mathrm{id}_{Y^*} \otimes \sigma_{Y^{**} \otimes Y^*, X}) \circ (\mathrm{coev}_{Y^*} \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_X) \\
&= (\mathrm{ev}_X \otimes \iota_{Y^*}) \circ (f^* \otimes \mathrm{id}_X \otimes \mathrm{id}_{Y^{**}} \otimes \mathrm{id}_{Y^*}) \circ (\mathrm{id}_{Y^*} \otimes \sigma_{Y^{**} \otimes Y^*, X}) \circ (\mathrm{coev}_{Y^*} \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_X) \\
&= (\mathrm{ev}_X \otimes \iota_{Y^*}) \circ (\mathrm{id}_{X^*} \otimes \sigma_{Y^{**} \otimes Y^*, X}) \circ (f^* \otimes \mathrm{id}_{Y^{**}} \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_X) \circ (\mathrm{coev}_{Y^*} \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_X) \\
&= (\mathrm{ev}_X \otimes \iota_{Y^*}) \circ (\mathrm{id}_{X^*} \otimes \sigma_{Y^{**} \otimes Y^*, X}) \circ (\mathrm{id}_{X^*} \otimes f^{**} \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_X) \circ (\mathrm{coev}_{X^*} \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_X) \\
&= (\mathrm{ev}_X \otimes \iota_{Y^*}) \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_X \otimes f^{**} \otimes \mathrm{id}_{Y^*}) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X^{**} \otimes Y^*, X}) \circ (\mathrm{coev}_{X^*} \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_X) \\
&= (\mathrm{ev}_X \otimes \iota_{X^*}) \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_X \otimes \mathrm{id}_{X^{**}} \otimes f^*) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X^{**} \otimes Y^*, X}) \circ (\mathrm{coev}_{X^*} \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_X) \\
&= (\mathrm{ev}_X \otimes \iota_{X^*}) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X^{**} \otimes X^*, X}) \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_{X^{**}} \otimes f^* \otimes \mathrm{id}_X) \circ (\mathrm{coev}_{X^*} \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_X) \\
&= (\mathrm{ev}_X \otimes \iota_{X^*}) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X^{**} \otimes X^*, X}) \circ (\mathrm{coev}_{X^*} \otimes \mathrm{id}_{X^*} \otimes \mathrm{id}_X) \circ (f^* \otimes \mathrm{id}_X) \\
&= \chi_X \circ (f^* \otimes \mathrm{id}_X),
\end{aligned}$$

as required. Thus, by the universality of ι, there exists a unique morphism S_L such that $S_L \circ \iota_X = \chi_X$ for all objects X in $\mathcal{C}$.

We now show that S_L satisfies the antipode equations. Recall that this means that

$$m_L \circ (S_L \otimes \mathrm{id}_L) \circ \Delta_L = u_L \circ \varepsilon_L = m_L \circ (\mathrm{id}_L \otimes S_L) \circ \Delta_L.$$

For the left antipode equation, it is sufficient by the universality of ι to prove that

$$m_L \circ (S_L \otimes \mathrm{id}_L) \circ \Delta_L \circ \iota_X = u_L \circ \varepsilon_L \circ \iota_X,$$

i.e.,

$$m_L \circ (S_L \otimes \mathrm{id}_L) \circ \zeta_X = u_L \circ \mathrm{ev}_X,$$

where ζ is the dinatural transformation (2.8). By the braiding axiom (1.4), the naturality of σ, the relation $\sigma_{X^{**}\otimes X^*,I} = \mathrm{id}_{X^{**}\otimes X^*}$ by (1.6), and Definition 1.3.1 of a left dual, we have

$$\begin{aligned}
&m_L \circ (S_L \otimes \mathrm{id}_L) \circ \zeta_X \\
&= m_L \circ (S_L \otimes \mathrm{id}_L) \circ (\iota_X \otimes \iota_X) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X) \\
&= m_L \circ (\chi_X \otimes \iota_X) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X) \\
&= m_L \circ (\mathrm{ev}_X \otimes \iota_{X^*} \otimes \iota_X) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X^{**}\otimes X^*,X} \otimes \mathrm{id}_{X^*\otimes X}) \\
&\qquad \circ (\mathrm{coev}_{X^*} \otimes \mathrm{id}_{X^*\otimes X} \otimes \mathrm{id}_{X^*\otimes X}) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X) \\
&= \xi_{X^*,X} \circ (\mathrm{ev}_X \otimes \mathrm{id}_{X^{**}\otimes X^*} \otimes \mathrm{id}_{X^*\otimes X}) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X^{**}\otimes X^*,X} \otimes \mathrm{id}_{X^*\otimes X}) \\
&\qquad \circ (\mathrm{coev}_{X^*} \otimes \mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X) \\
&= \iota_{X^*\otimes X} \circ (\gamma_{X^*,X} \otimes \mathrm{id}_{X^*\otimes X}) \circ (\sigma_{X^{**}\otimes X^*,X^*} \otimes \mathrm{id}_X) \circ (\mathrm{ev}_X \otimes \mathrm{id}_{X^{**}\otimes X^*} \otimes \mathrm{id}_{X^*\otimes X}) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \sigma_{X^{**}\otimes X^*,X} \otimes \mathrm{id}_{X^*\otimes X}) \circ (\mathrm{coev}_{X^*} \otimes \mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X) \\
&= \iota_{X^*\otimes X} \circ (\gamma_{X^*,X} \otimes \mathrm{id}_{X^*\otimes X}) \circ (\mathrm{ev}_X \otimes \mathrm{id}_{X^*\otimes X^{**}\otimes X^*} \otimes \mathrm{id}_X) \circ (\mathrm{id}_{X^*\otimes X} \otimes \sigma_{X^{**}\otimes X^*,X^*} \otimes \mathrm{id}_X) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \sigma_{X^{**}\otimes X^*,X} \otimes \mathrm{id}_{X^*\otimes X}) \circ (\mathrm{coev}_{X^*} \otimes \mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X) \\
&= \iota_{X^*\otimes X} \circ (\gamma_{X^*,X} \otimes \mathrm{id}_{X^*\otimes X}) \circ (\mathrm{ev}_X \otimes \mathrm{id}_{X^*\otimes X^{**}\otimes X^*} \otimes \mathrm{id}_X) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X^{**}\otimes X^*,X\otimes X^*} \otimes \mathrm{id}_X) \\
&\qquad \circ (\mathrm{coev}_{X^*} \otimes \mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X) \\
&= \iota_{X^*\otimes X} \circ (\gamma_{X^*,X} \otimes \mathrm{id}_{X^*\otimes X}) \circ (\mathrm{ev}_X \otimes \mathrm{id}_{X^*\otimes X^{**}\otimes X^*} \otimes \mathrm{id}_X) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X^{**}\otimes X^*,X\otimes X^*} \otimes \mathrm{id}_X) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_{X^{**}} \otimes \mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X) \circ (\mathrm{coev}_{X^*} \otimes \mathrm{id}_{X^*\otimes X}) \\
&= \iota_{X^*\otimes X} \circ (\gamma_{X^*,X} \otimes \mathrm{id}_{X^*\otimes X}) \circ (\mathrm{ev}_X \otimes \mathrm{id}_{X^*\otimes X^{**}\otimes X^*} \otimes \mathrm{id}_X) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_{X^{**}\otimes X^*} \otimes \mathrm{id}_X) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X^{**}\otimes X^*,I} \otimes \mathrm{id}_X) \circ (\mathrm{coev}_{X^*} \otimes \mathrm{id}_{X^*\otimes X}) \\
&= \iota_{X^*\otimes X} \circ (\gamma_{X^*,X} \otimes \mathrm{id}_{X^*\otimes X}) \circ (\mathrm{coev}_{X^*} \otimes \mathrm{id}_{X^*\otimes X}).
\end{aligned}$$

Now by the dinaturality of ι,

$$\iota_{X^*\otimes X} \circ (\mathrm{ev}_X^* \otimes \mathrm{id}_{X^*\otimes X}) = \iota_I \circ (\mathrm{id}_{I^*} \otimes \mathrm{ev}_X) = u_L \circ \mathrm{ev}_X,$$

so if $\mathrm{ev}_X^* = \gamma_{X^*,X} \circ \mathrm{coev}_{X^*}$, then the result will follow. By Theorem 1.4.1, it is sufficient to prove that

$$\mathrm{ev}_I \circ (\mathrm{id}_{I^*} \otimes \mathrm{ev}_X) = \mathrm{ev}_{X^*\otimes X} \circ ((\gamma_{X^*,X} \circ \mathrm{coev}_{X^*}) \otimes \mathrm{id}_{X^*\otimes X}),$$

and this follows from the characterization (1.16) of γ. If we express the multiplication m_L in the alternative form obtained from (2.3), then the proof that S_L satisfies the right antipode equation is analogous. This establishes that the coend L is a Hopf algebra in $\mathcal{C}$.

2.6 The Hopf pairing

In this section, we consider the morphism $\omega\colon L \otimes L \to I$ that is induced by the dinatural transformation defined by

$$\omega_{X,Y} = (\mathrm{ev}_X \otimes \mathrm{ev}_Y) \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Y^*,X} \circ \sigma_{X,Y^*}) \otimes \mathrm{id}_Y).$$

By the universality of $\iota_X \otimes \iota_Y$, there exists a unique morphism $\omega\colon L \otimes L \to I$ such that $\omega_{X,Y} = \omega \circ (\iota_X \otimes \iota_Y)$. We will prove that ω is a *Hopf pairing*, which means that it satisfies the following identities:

$$\omega \circ (m_L \otimes \mathrm{id}_L) = \omega \circ (\mathrm{id}_L \otimes \omega \otimes \mathrm{id}_L) \circ (\mathrm{id}_L \otimes \mathrm{id}_L \otimes \Delta_L) \tag{2.11}$$

$$\omega \circ (\mathrm{id}_L \otimes m_L) = \omega \circ (\mathrm{id}_L \otimes \omega \otimes \mathrm{id}_L) \circ (\Delta_L \otimes \mathrm{id}_L \otimes \mathrm{id}_L) \tag{2.12}$$

$$\omega \circ (u_L \otimes \mathrm{id}_L) = \varepsilon_L = \omega \circ (\mathrm{id}_L \otimes u_L) \tag{2.13}$$

See also [17, p. 10].

The dinaturality of $\omega_{X,Y}$ means that for all morphisms $f\colon X \to \tilde{X}$ and $g\colon Y \to \tilde{Y}$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc}
\tilde{X}^* \otimes X \otimes \tilde{Y}^* \otimes Y & \xrightarrow{f^*\otimes \mathrm{id}_X \otimes g^* \otimes \mathrm{id}_Y} & X^* \otimes X \otimes Y^* \otimes Y \\
\Big\downarrow{\scriptstyle \mathrm{id}_{\tilde{X}^*}\otimes f \otimes \mathrm{id}_{\tilde{Y}^*} \otimes g} & & \Big\downarrow{\scriptstyle \omega_{X,Y}} \\
\tilde{X}^* \otimes \tilde{X} \otimes \tilde{Y}^* \otimes \tilde{Y} & \xrightarrow{\omega_{\tilde{X},\tilde{Y}}} & L \otimes L
\end{array}$$

By the naturality of σ and Theorem 1.4.1, we have

$$
\begin{aligned}
&\omega_{X,Y} \circ (f^* \otimes \mathrm{id}_X \otimes g^* \otimes \mathrm{id}_Y) \\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_Y) \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Y^*,X} \circ \sigma_{X,Y^*}) \otimes \mathrm{id}_Y) \circ (f^* \otimes \mathrm{id}_X \otimes g^* \otimes \mathrm{id}_Y) \\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_Y) \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_X \otimes g^* \otimes \mathrm{id}_Y) \circ (f^* \otimes (\sigma_{\tilde{Y}^*,X} \circ \sigma_{X,\tilde{Y}^*}) \otimes \mathrm{id}_Y) \\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_Y) \circ (f^* \otimes \mathrm{id}_X \otimes g^* \otimes \mathrm{id}_Y) \circ (\mathrm{id}_{\tilde{X}^*} \otimes (\sigma_{\tilde{Y}^*,X} \circ \sigma_{X,\tilde{Y}^*}) \otimes \mathrm{id}_Y) \\
&= (\mathrm{ev}_{\tilde{X}} \otimes \mathrm{ev}_{\tilde{Y}}) \circ (\mathrm{id}_{\tilde{X}^*} \otimes f \otimes \mathrm{id}_{\tilde{Y}^*} \otimes g) \circ (\mathrm{id}_{\tilde{X}^*} \otimes (\sigma_{\tilde{Y}^*,X} \circ \sigma_{X,\tilde{Y}^*}) \otimes \mathrm{id}_Y) \\
&= (\mathrm{ev}_{\tilde{X}} \otimes \mathrm{ev}_{\tilde{Y}}) \circ (\mathrm{id}_{\tilde{X}^*} \otimes (\sigma_{\tilde{Y}^*,\tilde{X}} \circ \sigma_{\tilde{X},\tilde{Y}^*}) \otimes g) \circ (\mathrm{id}_{\tilde{X}^*} \otimes f \otimes \mathrm{id}_{\tilde{Y}^*} \otimes \mathrm{id}_Y) \\
&= (\mathrm{ev}_{\tilde{X}} \otimes \mathrm{ev}_{\tilde{Y}}) \circ (\mathrm{id}_{\tilde{X}^*} \otimes (\sigma_{\tilde{Y}^*,\tilde{X}} \circ \sigma_{\tilde{X},\tilde{Y}^*}) \otimes \mathrm{id}_{\tilde{Y}}) \circ (\mathrm{id}_{\tilde{X}^*} \otimes f \otimes \mathrm{id}_{\tilde{Y}^*} \otimes g) \\
&= \omega_{\tilde{X},\tilde{Y}} \circ (\mathrm{id}_{\tilde{X}^*} \otimes f \otimes \mathrm{id}_{\tilde{Y}^*} \otimes g)
\end{aligned}
$$

as required.

To prove (2.11), it is sufficient by the universality of $\iota_X \otimes \iota_Y \otimes \iota_Z$ to prove that

$$\omega \circ (m_L \otimes \mathrm{id}_L) \circ (\iota_X \otimes \iota_Y \otimes \iota_Z) = \omega \circ (\mathrm{id}_L \otimes \omega \otimes \mathrm{id}_L) \circ (\mathrm{id}_L \otimes \mathrm{id}_L \otimes \Delta_L) \circ (\iota_X \otimes \iota_Y \otimes \iota_Z),$$

i.e.,

$$\omega \circ (\xi_{X,Y} \otimes \iota_Z) = \omega \circ (\mathrm{id}_L \otimes \omega \otimes \mathrm{id}_L) \circ (\iota_X \otimes \iota_Y \otimes \zeta_Z),$$

where ξ is defined as in (2.1) and ζ is defined as in (2.8). For the right-hand side of this equation, we have

$$
\begin{aligned}
&\omega \circ (\mathrm{id}_L \otimes \omega \otimes \mathrm{id}_L) \circ (\iota_X \otimes \iota_Y \otimes \zeta_Z) \\
&= \omega \circ (\mathrm{id}_L \otimes \omega \otimes \mathrm{id}_L) \circ (\iota_X \otimes \iota_Y \otimes \iota_Z \otimes \iota_Z) \circ (\mathrm{id}_{X^* \otimes X} \otimes \mathrm{id}_{Y^* \otimes Y} \otimes \mathrm{id}_{Z^*} \otimes \mathrm{coev}_Z \otimes \mathrm{id}_Z) \\
&= \omega \circ (\iota_X \otimes \iota_Z) \circ (\mathrm{id}_{X^* \otimes X} \otimes \omega_{Y,Z} \otimes \mathrm{id}_{Z^* \otimes Z}) \circ (\mathrm{id}_{X^* \otimes X} \otimes \mathrm{id}_{Y^* \otimes Y} \otimes \mathrm{id}_{Z^*} \otimes \mathrm{coev}_Z \otimes \mathrm{id}_Z) \\
&= \omega_{X,Z} \circ (\mathrm{id}_{X^* \otimes X} \otimes \mathrm{ev}_Y \otimes \mathrm{ev}_Z \otimes \mathrm{id}_{Z^* \otimes Z}) \circ (\mathrm{id}_{X^* \otimes X \otimes Y^*} \otimes (\sigma_{Z^*,Y} \circ \sigma_{Y,Z^*}) \otimes \mathrm{id}_{Z \otimes Z^* \otimes Z}) \\
&\qquad \circ (\mathrm{id}_{X^* \otimes X} \otimes \mathrm{id}_{Y^* \otimes Y} \otimes \mathrm{id}_{Z^*} \otimes \mathrm{coev}_Z \otimes \mathrm{id}_Z) \\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Z^*,X} \circ \sigma_{X,Z^*}) \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^* \otimes X} \otimes \mathrm{ev}_Y \otimes \mathrm{ev}_Z \otimes \mathrm{id}_{Z^* \otimes Z}) \\
&\qquad \circ (\mathrm{id}_{X^* \otimes X \otimes Y^*} \otimes \mathrm{id}_{Y \otimes Z^*} \otimes \mathrm{coev}_Z \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^* \otimes X \otimes Y^*} \otimes (\sigma_{Z^*,Y} \circ \sigma_{Y,Z^*}) \otimes \mathrm{id}_Z) \\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Z^*,X} \circ \sigma_{X,Z^*}) \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^* \otimes X} \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{Z^* \otimes Z}) \\
&\qquad \circ (\mathrm{id}_{X^* \otimes X \otimes Y^*} \otimes (\sigma_{Z^*,Y} \circ \sigma_{Y,Z^*}) \otimes \mathrm{id}_Z),
\end{aligned}
$$

where we have used Definition 1.3.1 of a left dual Z^*. For the left-hand side, we have by the characterization (1.16) of γ and the braiding axioms (1.3) and (1.4) that

$$\begin{aligned}
&\omega \circ (\xi_{X,Y} \otimes \iota_Z) \\
&= \omega \circ (\iota_{X\otimes Y} \otimes \iota_Z) \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y} \otimes \mathrm{id}_{Z^*\otimes Z}) \circ (\sigma_{X^*\otimes X,Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \\
&= \omega_{X\otimes Y,Z} \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y} \otimes \mathrm{id}_{Z^*\otimes Z}) \circ (\sigma_{X^*\otimes X,Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \\
&= (\mathrm{ev}_{X\otimes Y} \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{(X\otimes Y)^*} \otimes (\sigma_{Z^*,X\otimes Y} \circ \sigma_{X\otimes Y,Z^*}) \otimes \mathrm{id}_Z) \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y} \otimes \mathrm{id}_{Z^*\otimes Z}) \\
&\qquad \circ (\sigma_{X^*\otimes X,Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \\
&= (\mathrm{ev}_{X\otimes Y} \otimes \mathrm{ev}_Z) \circ (\gamma_{X,Y} \otimes \mathrm{id}_{X\otimes Y\otimes Z^*} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{Y^*\otimes X^*} \otimes (\sigma_{Z^*,X\otimes Y} \circ \sigma_{X\otimes Y,Z^*}) \otimes \mathrm{id}_Z) \\
&\qquad \circ (\sigma_{X^*\otimes X,Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \\
&= (\mathrm{ev}_Y \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \circ (\mathrm{id}_{Y^*\otimes X^*} \otimes (\sigma_{Z^*,X\otimes Y} \circ \sigma_{X\otimes Y,Z^*}) \otimes \mathrm{id}_Z) \\
&\qquad \circ (\sigma_{X^*\otimes X,Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \\
&= (\mathrm{ev}_Y \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \circ (\mathrm{id}_{Y^*\otimes X^*} \otimes \sigma_{Z^*,X\otimes Y} \otimes \mathrm{id}_Z) \\
&\qquad \circ (\mathrm{id}_{Y^*\otimes X^*} \otimes \sigma_{X\otimes Y,Z^*} \otimes \mathrm{id}_Z) \circ (\sigma_{X^*\otimes X,Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \\
&= (\mathrm{ev}_Y \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \circ (\mathrm{id}_{Y^*\otimes X^*} \otimes \mathrm{id}_X \otimes \sigma_{Z^*,Y} \otimes \mathrm{id}_Z) \\
&\qquad \circ (\mathrm{id}_{Y^*\otimes X^*} \otimes \sigma_{Z^*,X} \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{Y^*\otimes X^*} \otimes \sigma_{X,Z^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z) \\
&\qquad \circ (\mathrm{id}_{Y^*\otimes X^*} \otimes \mathrm{id}_X \otimes \sigma_{Y,Z^*} \otimes \mathrm{id}_Z) \circ (\sigma_{X^*,Y^*} \otimes \mathrm{id}_X \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \\
&= (\mathrm{ev}_Y \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \circ (\sigma_{X^*,Y^*} \otimes \mathrm{id}_X \otimes \sigma_{Z^*,Y} \otimes \mathrm{id}_Z) \\
&\qquad \circ (\mathrm{id}_{X^*\otimes Y^*} \otimes (\sigma_{Z^*,X} \circ \sigma_{X,Z^*}) \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,Y^*} \otimes \sigma_{Y,Z^*} \otimes \mathrm{id}_Z).
\end{aligned}$$

An application of (1.10) to σ_{X^*,Y^*}, followed by an application of (1.8) to $\sigma_{Z^*,Y}$, yields

$$\begin{aligned}
&(\mathrm{ev}_Y \otimes \mathrm{ev}_Z) \circ (\mathrm{ev}_X \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \circ (\mathrm{id}_{X^*} \otimes \sigma^{-1}_{X,Y^*} \otimes \sigma_{Z^*,Y} \otimes \mathrm{id}_Z) \\
&\quad \circ (\mathrm{id}_{X^*\otimes Y^*} \otimes (\sigma_{Z^*,X} \circ \sigma_{X,Z^*}) \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,Y^*} \otimes \sigma_{Y,Z^*} \otimes \mathrm{id}_Z) \\
&= (\mathrm{ev}_Y \otimes \mathrm{ev}_Z) \circ (\mathrm{ev}_X \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \circ (\mathrm{id}_{X^*} \otimes \sigma^{-1}_{X,Y^*} \otimes \sigma_{Z^*,Y} \otimes \mathrm{id}_Z) \\
&\qquad \circ (\mathrm{id}_{X^*\otimes Y^*} \otimes (\sigma_{Z^*,X} \circ \sigma_{X,Z^*}) \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,Y^*} \otimes \sigma_{Y,Z^*} \otimes \mathrm{id}_Z) \\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{X^*\otimes X} \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_{X\otimes Y^*} \otimes \sigma_{Z^*,Y} \otimes \mathrm{id}_Z) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \sigma^{-1}_{X,Y^*} \otimes \mathrm{id}_{Z^*\otimes Y} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*\otimes Y^*} \otimes (\sigma_{Z^*,X} \circ \sigma_{X,Z^*}) \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,Y^*} \otimes \sigma_{Y,Z^*} \otimes \mathrm{id}_Z) \\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{X^*\otimes X} \otimes \mathrm{id}_{Z^*} \otimes \mathrm{ev}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_X \otimes \sigma^{-1}_{Z^*,Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \sigma^{-1}_{X,Y^*} \otimes \mathrm{id}_{Z^*\otimes Y} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*\otimes Y^*} \otimes (\sigma_{Z^*,X} \circ \sigma_{X,Z^*}) \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,Y^*} \otimes \sigma_{Y,Z^*} \otimes \mathrm{id}_Z).
\end{aligned}$$

Applying (1.3) to $\sigma^{-1}_{X\otimes Z^*,Y^*}$, the naturality of σ^{-1}, and then (1.3) again, this equals

$$\begin{aligned}
&(\mathrm{ev}_X \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{X^*\otimes X} \otimes \mathrm{id}_{Z^*} \otimes \mathrm{ev}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \sigma^{-1}_{X\otimes Z^*,Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z)\\
&\quad \circ (\mathrm{id}_{X^*\otimes Y^*} \otimes (\sigma_{Z^*,X} \circ \sigma_{X,Z^*}) \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,Y^*} \otimes \sigma_{Y,Z^*} \otimes \mathrm{id}_Z)\\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{X^*\otimes X} \otimes \mathrm{id}_{Z^*} \otimes \mathrm{ev}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Z^*,X} \circ \sigma_{X,Z^*}) \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z)\\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \sigma^{-1}_{X\otimes Z^*,Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,Y^*} \otimes \sigma_{Y,Z^*} \otimes \mathrm{id}_Z)\\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{X^*\otimes X} \otimes \mathrm{id}_{Z^*} \otimes \mathrm{ev}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Z^*,X} \circ \sigma_{X,Z^*}) \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z)\\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_X \otimes \sigma^{-1}_{Z^*,Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \sigma^{-1}_{X,Y^*} \otimes \mathrm{id}_{Z^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z)\\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,Y^*} \otimes \sigma_{Y,Z^*} \otimes \mathrm{id}_Z)\\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{X^*\otimes X} \otimes \mathrm{id}_{Z^*} \otimes \mathrm{ev}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Z^*,X} \circ \sigma_{X,Z^*}) \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z)\\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_X \otimes \sigma^{-1}_{Z^*,Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_{X\otimes Y^*} \otimes \sigma_{Y,Z^*} \otimes \mathrm{id}_Z)\\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Z^*,X} \circ \sigma_{X,Z^*}) \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_{X\otimes Z^*} \otimes \mathrm{ev}_Y \otimes \mathrm{id}_Z)\\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_X \otimes \sigma^{-1}_{Z^*,Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_{X\otimes Y^*} \otimes \sigma_{Y,Z^*} \otimes \mathrm{id}_Z).
\end{aligned}$$

Finally, we apply (1.8) to $\sigma^{-1}_{Z^*,Y^*}$ to obtain

$$\begin{aligned}
&(\mathrm{ev}_X \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Z^*,X} \circ \sigma_{X,Z^*}) \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_X \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{Z^*} \otimes \mathrm{id}_Z)\\
&\quad \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_X \otimes \mathrm{id}_{Y^*} \otimes \sigma_{Z^*,Y} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_{X\otimes Y^*} \otimes \sigma_{Y,Z^*} \otimes \mathrm{id}_Z),
\end{aligned}$$

which is equal to the right-hand side.

To prove (2.12), it is sufficient by the universality of $\iota_X \otimes \iota_Y \otimes \iota_Z$ to prove that

$$\omega \circ (\iota_X \otimes \xi_{Y,Z}) = \omega \circ (\mathrm{id}_L \otimes \omega \otimes \mathrm{id}_L) \circ (\zeta_X \otimes \iota_Y \otimes \iota_Z).$$

For the right-hand side, we have

$$\begin{aligned}
&\omega \circ (\mathrm{id}_L \otimes \omega \otimes \mathrm{id}_L) \circ (\zeta_X \otimes \iota_Y \otimes \iota_Z)\\
&= \omega \circ (\mathrm{id}_L \otimes \omega \otimes \mathrm{id}_L) \circ (\iota_X \otimes \iota_X \otimes \iota_Y \otimes \iota_Z) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X \otimes \mathrm{id}_{Y^*\otimes Y} \otimes \mathrm{id}_{Z^*\otimes Z})\\
&= \omega \circ (\iota_X \otimes \iota_Z) \circ (\mathrm{id}_{X^*\otimes X} \otimes \omega_{X,Y} \otimes \mathrm{id}_{Z^*\otimes Z}) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X \otimes \mathrm{id}_{Y^*\otimes Y} \otimes \mathrm{id}_{Z^*\otimes Z})\\
&= \omega_{X,Z} \circ (\mathrm{id}_{X^*\otimes X} \otimes \mathrm{ev}_X \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \circ (\mathrm{id}_{X^*\otimes X\otimes X^*} \otimes (\sigma_{Y^*,X} \circ \sigma_{X,Y^*}) \otimes \mathrm{id}_{Y\otimes Z^*\otimes Z})\\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X \otimes \mathrm{id}_{Y^*\otimes Y} \otimes \mathrm{id}_{Z^*\otimes Z})\\
&= \omega_{X,Z} \circ (\mathrm{id}_{X^*\otimes X} \otimes \mathrm{ev}_X \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_{X\otimes Y^*} \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z})\\
&\qquad \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Y^*,X} \circ \sigma_{X,Y^*}) \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z})\\
&= \omega_{X,Z} \circ (\mathrm{id}_{X^*} \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Y^*,X} \circ \sigma_{X,Y^*}) \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}),
\end{aligned}$$

where we have applied Definition 1.3.1. For the left-hand side, we have

$$
\begin{aligned}
&\omega \circ (\iota_X \otimes \xi_{Y,Z}) \\
&= \omega \circ (\iota_X \otimes \iota_{Y\otimes Z}) \circ (\mathrm{id}_{X^*\otimes X} \otimes \gamma_{Y,Z} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*\otimes X} \otimes \sigma_{Y^*\otimes Y,Z^*} \otimes \mathrm{id}_Z) \\
&= \omega_{X,Y\otimes Z} \circ (\mathrm{id}_{X^*\otimes X} \otimes \gamma_{Y,Z} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*\otimes X} \otimes \sigma_{Y^*\otimes Y,Z^*} \otimes \mathrm{id}_Z) \\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*} \otimes \sigma_{(Y\otimes Z)^*,X} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,(Y\otimes Z)^*} \otimes \mathrm{id}_{Y\otimes Z}) \\
&\qquad \circ (\mathrm{id}_{X^*\otimes X} \otimes \gamma_{Y,Z} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*\otimes X} \otimes \sigma_{Y^*\otimes Y,Z^*} \otimes \mathrm{id}_Z) \\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*} \otimes \sigma_{(Y\otimes Z)^*,X} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*} \otimes \gamma_{Y,Z} \otimes \mathrm{id}_X \otimes \mathrm{id}_{Y\otimes Z}) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,Z^*\otimes Y^*} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*\otimes X} \otimes \sigma_{Y^*\otimes Y,Z^*} \otimes \mathrm{id}_Z) \\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_X \otimes \gamma_{Y,Z} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*} \otimes \sigma_{Z^*\otimes Y^*,X} \otimes \mathrm{id}_{Y\otimes Z}) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,Z^*\otimes Y^*} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*\otimes X} \otimes \sigma_{Y^*\otimes Y,Z^*} \otimes \mathrm{id}_Z) \\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{X^*\otimes X} \otimes \mathrm{id}_{Z^*} \otimes \mathrm{ev}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \sigma_{Z^*,X} \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_{Y\otimes Z}) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_{Z^*} \otimes \sigma_{Y^*,X} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_{Z^*} \otimes \sigma_{X,Y^*} \otimes \mathrm{id}_{Y\otimes Z}) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,Z^*} \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*\otimes X} \otimes \sigma_{Y^*\otimes Y,Z^*} \otimes \mathrm{id}_Z) \\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{X^*\otimes X} \otimes \mathrm{id}_{Z^*} \otimes \mathrm{ev}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \sigma_{Z^*,X} \otimes \mathrm{id}_{Y^*} \otimes \mathrm{id}_{Y\otimes Z}) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_{Z^*} \otimes (\sigma_{Y^*,X} \circ \sigma_{X,Y^*}) \otimes \mathrm{id}_{Y\otimes Z}) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X\otimes Y^*\otimes Y,Z^*} \otimes \mathrm{id}_Z) \\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{X^*} \otimes \sigma_{Z^*,X} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_{Z^*\otimes X} \otimes \mathrm{ev}_Y \otimes \mathrm{id}_Z) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \sigma_{X\otimes Y^*\otimes Y,Z^*} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Y^*,X} \circ \sigma_{X,Y^*}) \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*} \otimes \mathrm{id}_Z) \\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{X^*} \otimes \sigma_{Z^*,X} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,Z^*} \otimes \mathrm{id}_Z) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_X \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{Z^*} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Y^*,X} \circ \sigma_{X,Y^*}) \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*} \otimes \mathrm{id}_Z) \\
&= (\mathrm{ev}_X \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Z^*,X} \circ \sigma_{X,Z^*}) \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_X \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{Z^*} \otimes \mathrm{id}_Z) \\
&\qquad \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Y^*,X} \circ \sigma_{X,Y^*}) \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*} \otimes \mathrm{id}_Z) \\
&= \omega_{X,Z} \circ (\mathrm{id}_{X^*\otimes X} \otimes \mathrm{ev}_Y \otimes \mathrm{id}_{Z^*\otimes Z}) \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Y^*,X} \circ \sigma_{X,Y^*}) \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Z^*\otimes Z}),
\end{aligned}
$$

as required, where we have used the characterization (1.16) of γ and several applications of the naturality of σ and the braiding axioms (1.3) and (1.4).

Finally, we prove the left equality in (2.13); the right equality is proved similarly. For this calculation, we do not assume that the unit constraint λ is the identity transformation. It is sufficient by the universality of ι to prove that

$$\omega \circ (u_L \otimes \mathrm{id}_L) \circ \lambda_L^{-1} \circ \iota_X = \varepsilon_L \circ \iota_X = \mathrm{ev}_X.$$

We have

$$\begin{aligned}
&\omega \circ (u_L \otimes \mathrm{id}_L) \circ \lambda_L^{-1} \circ \iota_X \\
&= \omega \circ ((\iota_I \circ \lambda_I^{-1}) \otimes \mathrm{id}_L) \circ \lambda_L^{-1} \circ \iota_X \\
&= \omega \circ ((\iota_I \circ \lambda_I^{-1}) \otimes \mathrm{id}_L) \circ (\mathrm{id}_I \otimes \iota_X) \circ \lambda_{X^*\otimes X}^{-1} \\
&= \omega \circ (\iota_I \otimes \iota_X) \circ (\lambda_I^{-1} \otimes \mathrm{id}_{X^*\otimes X}) \circ \lambda_{X^*\otimes X}^{-1} \\
&= \omega_{I,X} \circ (\lambda_I^{-1} \otimes \mathrm{id}_{X^*\otimes X}) \circ \lambda_{X^*\otimes X}^{-1} \\
&= \lambda_I \circ (\mathrm{ev}_I \otimes \mathrm{ev}_X) \circ (\mathrm{id}_{I^*} \otimes (\sigma_{X^*,I} \circ \sigma_{I,X^*}) \otimes \mathrm{id}_X) \circ (\lambda_I^{-1} \otimes \mathrm{id}_{X^*\otimes X}) \circ \lambda_{X^*\otimes X}^{-1} \\
&= \lambda_I \circ (\lambda_I \otimes \mathrm{ev}_X) \circ (\lambda_I^{-1} \otimes \mathrm{id}_{X^*\otimes X}) \circ \lambda_{X^*\otimes X}^{-1} \\
&= \lambda_I \circ (\mathrm{id}_I \otimes \mathrm{ev}_X) \circ \lambda_{X^*\otimes X}^{-1} \\
&= \lambda_I \circ \lambda_I^{-1} \circ \mathrm{ev}_X \\
&= \mathrm{ev}_X,
\end{aligned}$$

where we have used the naturality of λ, and the fact that $\sigma_{X^*,I} \circ \sigma_{I,X^*} = \mathrm{id}_{I\otimes X^*}$ as a consequence of [5, Prop. XIII.1.2, p. 316].

Chapter 3

Duals and homomorphic images of categorical Hopf algebras

In this chapter, we show that a dual object of a Hopf algebra in a braided category $\mathcal{C}$ with duality is again a Hopf algebra in $\mathcal{C}$, and that, if $\mathcal{C}$ is an abelian tensor category, an image of a Hopf algebra homomorphism in $\mathcal{C}$ is a Hopf algebra in $\mathcal{C}$. We then consider morphisms $\omega' \colon H \to H^*$ and $\omega'' \colon H \to {}^*H$ induced by the Hopf pairing. We see that these morphisms are in fact Hopf algebra homomorphisms, so that their images are Hopf subalgebras.

3.1 Duals of categorical Hopf algebras

Let H be a Hopf algebra in a braided category $\mathcal{C}$ with left duality. We now discuss how the left dual object H^* is again a Hopf algebra in $\mathcal{C}$. The structure morphisms on the dual object are defined slightly differently than in the case of the dual vector space of an ordinary Hopf algebra.

Recall that if H is a finite-dimensional Hopf algebra in the ordinary sense, then the dual space H^* is also a Hopf algebra, whose product can be expressed in Sweedler notation as

$$(\varphi\psi)(h) = \varphi(h_{(1)})\psi(h_{(2)}) \tag{3.1}$$

and whose coproduct can be expressed as

$$\varphi_{(1)}(h)\varphi_{(2)}(h') = \varphi(hh'), \tag{3.2}$$

for $\varphi, \psi \in H^*$ and $h, h' \in H$. Equation (3.1) means that

$$m_{H^*}(\varphi \otimes \psi) = \Delta_H^*(\varphi \otimes \psi),$$

where $\varphi \otimes \psi \in H^* \otimes H^*$ is viewed as the element in $(H \otimes H)^*$ defined by

$$(\varphi \otimes \psi)(h \otimes h') = \varphi(h)\psi(h')$$

for $h, h' \in H$; and (3.2) means that

$$\Delta_{H^*}(\varphi) = m_H^*(\varphi),$$

where $m_H^*(\varphi)$ is viewed as an element of $H^* \otimes H^*$. These identifications are valid because, in the finite-dimensional case, the vector space $H^* \otimes H^*$ is isomorphic to $(H \otimes H)^*$.

In the categorical context, however, the objects $X^* \otimes Y^*$ and $(X \otimes Y)^*$ are not isomorphic in general. Instead, $Y^* \otimes X^*$ is isomorphic to $(X \otimes Y)^*$, by the natural isomorphism γ characterized by (1.16). Thus, for a left dual object H^* of a Hopf algebra H in $\mathcal{C}$, we define the product as

$$m_{H^*} = \Delta_H^* \circ \gamma_{H,H} \tag{3.3}$$

and the coproduct as

$$\Delta_{H^*} = \gamma_{H,H}^{-1} \circ m_H^*. \tag{3.4}$$

The unit of the dual H^* is defined as the dual of the counit of H:

$$u_{H^*} = \varepsilon_H^* \colon I \cong I^* \to H^*.$$

The counit of the dual H^* is defined as the dual of the unit of H:

$$\varepsilon_{H^*} = u_H^* \colon H^* \to I^* \cong I.$$

The antipode of H^* is the dual of the antipode of H:

$$S_{H^*} = S_H^* \colon H^* \to H^*.$$

We prove only that Δ_{H^*} is multiplicative, i.e., that

$$\Delta_{H^*} \circ m_{H^*} = (m_{H^*} \otimes m_{H^*}) \circ (\mathrm{id}_{H^*} \otimes \sigma_{H^*,H^*} \otimes \mathrm{id}_{H^*}) \circ (\Delta_{H^*} \otimes \Delta_{H^*}).$$

Using the definition (1.11) of a dual morphism and Definition 1.3.1 of a left dual object,

$$\begin{aligned}
&\Delta_{H^*} \circ m_{H^*} \\
&= \gamma_{H,H}^{-1} \circ m_H^* \circ \Delta_H^* \circ \gamma_{H,H} \\
&= \gamma_{H,H}^{-1} \circ (\mathrm{ev}_H \otimes \mathrm{id}_{(H\otimes H)^*}) \circ (\mathrm{id}_{H^*} \otimes m_H \otimes \mathrm{id}_{(H\otimes H)^*}) \circ (\mathrm{id}_{H^*} \otimes \mathrm{coev}_{H\otimes H}) \\
&\qquad \circ (\mathrm{ev}_{H\otimes H} \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{(H\otimes H)^*} \otimes \Delta_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{(H\otimes H)^*} \otimes \mathrm{coev}_H) \circ \gamma_{H,H} \\
&= \gamma_{H,H}^{-1} \circ (\mathrm{ev}_{H\otimes H} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{(H\otimes H)^*}) \circ (\mathrm{id}_{(H\otimes H)^*\otimes H\otimes H} \otimes \mathrm{id}_{H^*} \otimes m_H \otimes \mathrm{id}_{(H\otimes H)^*}) \\
&\qquad \circ (\mathrm{id}_{(H\otimes H)^*} \otimes \Delta_H \otimes \mathrm{id}_{H^*} \otimes \mathrm{id}_{H\otimes H\otimes (H\otimes H)^*}) \circ (\mathrm{id}_{(H\otimes H)^*} \otimes \mathrm{coev}_H \otimes \mathrm{coev}_{H\otimes H}) \circ \gamma_{H,H} \\
&= \gamma_{H,H}^{-1} \circ (\mathrm{ev}_{H\otimes H} \otimes \mathrm{id}_{(H\otimes H)^*}) \circ (\mathrm{id}_{(H\otimes H)^*} \otimes \Delta_H \otimes \mathrm{ev}_H \otimes \mathrm{id}_{(H\otimes H)^*}) \\
&\qquad \circ (\mathrm{id}_{(H\otimes H)^*} \otimes \mathrm{coev}_H \otimes m_H \otimes \mathrm{id}_{(H\otimes H)^*}) \circ (\mathrm{id}_{(H\otimes H)^*} \otimes \mathrm{coev}_{H\otimes H}) \circ \gamma_{H,H} \\
&= \gamma_{H,H}^{-1} \circ (\mathrm{ev}_{H\otimes H} \otimes \mathrm{id}_{(H\otimes H)^*}) \circ (\mathrm{id}_{(H\otimes H)^*} \otimes \Delta_H \otimes \mathrm{id}_{(H\otimes H)^*}) \\
&\qquad \circ (\mathrm{id}_{(H\otimes H)^*} \otimes m_H \otimes \mathrm{id}_{(H\otimes H)^*}) \circ (\mathrm{id}_{(H\otimes H)^*} \otimes \mathrm{coev}_{H\otimes H}) \circ \gamma_{H,H} \\
&= (\mathrm{ev}_{H\otimes H} \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\gamma_{H,H} \otimes \mathrm{id}_{H\otimes H} \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H^*} \otimes (\Delta_H \circ m_H) \otimes \mathrm{id}_{H^*\otimes H^*}) \\
&\qquad \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \mathrm{id}_{H\otimes H} \otimes \gamma_{H,H}^{-1}) \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \mathrm{coev}_{H\otimes H}).
\end{aligned}$$

Since Δ_H is an algebra homomorphism, we have

$$\Delta_H \circ m_H = (m_H \otimes m_H) \circ (\mathrm{id}_H \otimes \sigma_{H,H} \otimes \mathrm{id}_H) \circ (\Delta_H \otimes \Delta_H),$$

and, by the characterizations (1.16) and (1.17) of γ, we have

$$\mathrm{ev}_{H\otimes H} \circ (\gamma_{H,H} \otimes \mathrm{id}_{H\otimes H}) = \mathrm{ev}_H \circ (\mathrm{id}_{H^*} \otimes \mathrm{ev}_H \otimes \mathrm{id}_H) \tag{3.5}$$

and

$$(\mathrm{id}_{H\otimes H} \otimes \gamma_{H,H}^{-1}) \circ \mathrm{coev}_{H\otimes H} = (\mathrm{id}_H \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*}) \circ \mathrm{coev}_H. \tag{3.6}$$

Thus, the above expression is equal to

$$\begin{aligned}
&(\mathrm{ev}_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*} \otimes \mathrm{ev}_H \otimes \mathrm{id}_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H^*} \otimes m_H \otimes m_H \otimes \mathrm{id}_{H^*\otimes H^*}) \\
&\quad \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \mathrm{id}_H \otimes \sigma_{H,H} \otimes \mathrm{id}_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \Delta_H \otimes \Delta_H \otimes \mathrm{id}_{H^*\otimes H^*}) \\
&\quad \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \mathrm{id}_H \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \mathrm{coev}_H) \\
&= (\mathrm{ev}_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*} \otimes m_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H\otimes H} \otimes \mathrm{id}_{H^*\otimes H^*}) \\
&\qquad \circ (\mathrm{id}_{H^*\otimes H^*} \otimes m_H \otimes \mathrm{id}_{H\otimes H} \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \mathrm{id}_H \otimes \sigma_{H,H} \otimes \mathrm{id}_H \otimes \mathrm{id}_{H^*\otimes H^*}) \\
&\qquad \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \mathrm{id}_{H\otimes H} \otimes \Delta_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \mathrm{id}_{H\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*}) \\
&\qquad \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \Delta_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \mathrm{coev}_H).
\end{aligned}$$

Applying Theorem 1.4.1 to m_H and Δ_H, this equals

$$\begin{aligned}
&(\mathrm{ev}_{H\otimes H}\otimes \mathrm{id}_{H^*\otimes H^*})\circ(m_H^*\otimes \mathrm{id}_{H\otimes H}\otimes \mathrm{id}_{H^*\otimes H^*})\circ(\mathrm{id}_{H^*}\otimes \mathrm{ev}_{H\otimes H}\otimes \mathrm{id}_{H\otimes H}\otimes \mathrm{id}_{H^*\otimes H^*})\\
&\quad\circ(\mathrm{id}_{H^*}\otimes m_H^*\otimes \mathrm{id}_{H\otimes H}\otimes \mathrm{id}_{H\otimes H}\otimes \mathrm{id}_{H^*\otimes H^*})\circ(\mathrm{id}_{H^*\otimes H^*}\otimes \mathrm{id}_H\otimes \sigma_{H,H}\otimes \mathrm{id}_H\otimes \mathrm{id}_{H^*\otimes H^*})\\
&\quad\circ(\mathrm{id}_{H^*\otimes H^*}\otimes \mathrm{id}_{H\otimes H}\otimes \mathrm{id}_{H\otimes H}\otimes \Delta_H^*\otimes \mathrm{id}_{H^*})\circ(\mathrm{id}_{H^*\otimes H^*}\otimes \mathrm{id}_{H\otimes H}\otimes \mathrm{coev}_{H\otimes H}\otimes \mathrm{id}_{H^*})\\
&\quad\circ(\mathrm{id}_{H^*\otimes H^*}\otimes \mathrm{id}_{H\otimes H}\otimes \Delta_H^*)\circ(\mathrm{id}_{H^*\otimes H^*}\otimes \mathrm{coev}_{H\otimes H})\\
&=(\mathrm{ev}_{H\otimes H}\otimes \mathrm{id}_{H^*\otimes H^*})\circ(\mathrm{id}_{(H\otimes H)^*}\otimes \mathrm{ev}_{H\otimes H}\otimes \mathrm{id}_{H\otimes H}\otimes \mathrm{id}_{H^*\otimes H^*})\\
&\qquad\circ(\mathrm{id}_{(H\otimes H)^*}\otimes \mathrm{id}_{(H\otimes H)^*}\otimes \mathrm{id}_H\otimes \sigma_{H,H}\otimes \mathrm{id}_H\otimes \mathrm{id}_{H^*\otimes H^*})\\
&\qquad\circ(m_H^*\otimes m_H^*\otimes \mathrm{id}_{H\otimes H}\otimes \mathrm{id}_{H\otimes H}\otimes \Delta_H^*\otimes \Delta_H^*)\\
&\qquad\circ(\mathrm{id}_{H^*\otimes H^*}\otimes \mathrm{id}_{H\otimes H}\otimes \mathrm{coev}_{H\otimes H}\otimes \mathrm{id}_{(H\otimes H)^*})\circ(\mathrm{id}_{H^*\otimes H^*}\otimes \mathrm{coev}_{H\otimes H}).
\end{aligned}$$

From (3.5) and (3.6), we have

$$\mathrm{ev}_{H\otimes H}=\mathrm{ev}_H\circ(\mathrm{id}_{H^*}\otimes \mathrm{ev}_H\otimes \mathrm{id}_H)\circ(\gamma_{H,H}^{-1}\otimes \mathrm{id}_{H\otimes H})$$

and

$$\mathrm{coev}_{H\otimes H}=(\mathrm{id}_{H\otimes H}\otimes \gamma_{H,H})\circ(\mathrm{id}_H\otimes \mathrm{coev}_H\otimes \mathrm{id}_{H^*})\circ \mathrm{coev}_H,$$

and hence the above expression equals

$$\begin{aligned}
&(\mathrm{ev}_H\otimes \mathrm{id}_{H^*\otimes H^*})\circ(\mathrm{id}_{H^*}\otimes \mathrm{ev}_H\otimes \mathrm{id}_{H\otimes H^*\otimes H^*})\circ(\gamma_{H,H}^{-1}\otimes \mathrm{id}_{H\otimes H\otimes H^*\otimes H^*})\\
&\circ(\mathrm{id}_{(H\otimes H)^*}\otimes \mathrm{ev}_H\otimes \mathrm{id}_{H\otimes H\otimes H^*\otimes H^*})\circ(\mathrm{id}_{(H\otimes H)^*\otimes H^*}\otimes \mathrm{ev}_H\otimes \mathrm{id}_{H\otimes H\otimes H\otimes H^*\otimes H^*})\\
&\circ(\mathrm{id}_{(H\otimes H)^*}\otimes \gamma_{H,H}^{-1}\otimes \mathrm{id}_{H\otimes H\otimes H\otimes H\otimes H^*\otimes H^*})\circ(\mathrm{id}_{(H\otimes H)^*\otimes(H\otimes H)^*\otimes H}\otimes \sigma_{H,H}\otimes \mathrm{id}_{H\otimes H^*\otimes H^*})\\
&\circ(m_H^*\otimes m_H^*\otimes \mathrm{id}_{H\otimes H\otimes H\otimes H}\otimes \Delta_H^*\otimes \Delta_H^*)\circ(\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H\otimes H\otimes H}\otimes \gamma_{H,H}\otimes \mathrm{id}_{(H\otimes H)^*})\\
&\circ(\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H\otimes H}\otimes \mathrm{coev}_H\otimes \mathrm{id}_{H^*\otimes(H\otimes H)^*})\circ(\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H}\otimes \mathrm{coev}_H\otimes \mathrm{id}_{(H\otimes H)^*})\\
&\circ(\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H}\otimes \gamma_{H,H})\circ(\mathrm{id}_{H^*\otimes H^*\otimes H}\otimes \mathrm{coev}_H\otimes \mathrm{id}_{H^*})\circ(\mathrm{id}_{H^*\otimes H^*}\otimes \mathrm{coev}_H),
\end{aligned}$$

which further equals

$$\begin{aligned}
&(\mathrm{ev}_H\otimes \mathrm{id}_{H^*\otimes H^*})\circ(\mathrm{id}_{H^*}\otimes \mathrm{ev}_H\otimes \mathrm{id}_{H\otimes H^*\otimes H^*})\circ(\gamma_{H,H}^{-1}\otimes \mathrm{id}_{H\otimes H\otimes H^*\otimes H^*})\\
&\circ(\mathrm{id}_{(H\otimes H)^*}\otimes \mathrm{ev}_H\otimes \mathrm{id}_{H\otimes H\otimes H^*\otimes H^*})\circ(\mathrm{id}_{(H\otimes H)^*\otimes H^*}\otimes \sigma_{H,H}\otimes \mathrm{id}_{H\otimes H^*\otimes H^*})\\
&\circ(\mathrm{id}_{(H\otimes H)^*\otimes H^*}\otimes \mathrm{ev}_H\otimes \mathrm{id}_{H\otimes H\otimes H\otimes H^*\otimes H^*})\circ(\mathrm{id}_{(H\otimes H)^*}\otimes \gamma_{H,H}^{-1}\otimes \mathrm{id}_{H\otimes H\otimes H\otimes H\otimes H^*\otimes H^*})\\
&\circ(m_H^*\otimes m_H^*\otimes \mathrm{id}_{H\otimes H\otimes H\otimes H}\otimes \Delta_H^*\otimes \Delta_H^*)\circ(\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H\otimes H\otimes H}\otimes \gamma_{H,H}\otimes \mathrm{id}_{(H\otimes H)^*})\\
&\circ(\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H\otimes H}\otimes \mathrm{coev}_H\otimes \mathrm{id}_{H^*\otimes(H\otimes H)^*})\circ(\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H}\otimes \mathrm{coev}_H\otimes \mathrm{id}_{(H\otimes H)^*})\\
&\circ(\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H}\otimes \gamma_{H,H})\circ(\mathrm{id}_{H^*\otimes H^*\otimes H}\otimes \mathrm{coev}_H\otimes \mathrm{id}_{H^*})\circ(\mathrm{id}_{H^*\otimes H^*}\otimes \mathrm{coev}_H).
\end{aligned}$$

Applying (1.8) to the morphism $(\mathrm{ev}_H \otimes \mathrm{id}_H) \circ (\mathrm{id}_{H^*} \otimes \sigma_{H,H})$, we have

$$
\begin{aligned}
&(\mathrm{ev}_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H\otimes H^*\otimes H^*}) \circ (\gamma_{H,H}^{-1} \otimes \mathrm{id}_{H\otimes H\otimes H^*\otimes H^*}) \\
&\circ (\mathrm{id}_{(H\otimes H)^*\otimes H} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H\otimes H^*\otimes H^*}) \circ (\mathrm{id}_{(H\otimes H)^*} \otimes \sigma_{H,H^*}^{-1} \otimes \mathrm{id}_{H\otimes H\otimes H^*\otimes H^*}) \\
&\circ (\mathrm{id}_{(H\otimes H)^*\otimes H^*} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H\otimes H\otimes H\otimes H^*\otimes H^*}) \circ (\mathrm{id}_{(H\otimes H)^*} \otimes \gamma_{H,H}^{-1} \otimes \mathrm{id}_{H\otimes H\otimes H\otimes H\otimes H^*\otimes H^*}) \\
&\circ (m_H^* \otimes m_H^* \otimes \mathrm{id}_{H\otimes H\otimes H\otimes H} \otimes \Delta_H^* \otimes \Delta_H^*) \circ (\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H\otimes H\otimes H} \otimes \gamma_{H,H} \otimes \mathrm{id}_{(H\otimes H)^*}) \\
&\circ (\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*\otimes (H\otimes H)^*}) \circ (\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{(H\otimes H)^*}) \\
&\circ (\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H} \otimes \gamma_{H,H}) \circ (\mathrm{id}_{H^*\otimes H^*\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \mathrm{coev}_H) \\
&= (\mathrm{ev}_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H\otimes H^*\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H^*\otimes H} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H\otimes H^*\otimes H^*}) \\
&\quad \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \sigma_{H,H^*}^{-1} \otimes \mathrm{id}_{H\otimes H\otimes H^*\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H^*\otimes H^*} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H\otimes H\otimes H\otimes H^*\otimes H^*}) \\
&\quad \circ (\gamma_{H,H}^{-1} \otimes \gamma_{H,H}^{-1} \otimes \mathrm{id}_{H\otimes H\otimes H\otimes H\otimes H^*\otimes H^*}) \circ (m_H^* \otimes m_H^* \otimes \mathrm{id}_{H\otimes H\otimes H\otimes H} \otimes \Delta_H^* \otimes \Delta_H^*) \\
&\quad \circ (\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H\otimes H\otimes H} \otimes \gamma_{H,H} \otimes \gamma_{H,H}) \circ (\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*\otimes H^*\otimes H^*}) \\
&\quad \circ (\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H^*\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*}) \\
&\quad \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \mathrm{coev}_H) \\
&= (\mathrm{ev}_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H\otimes H^*\otimes H^*}) \circ (\mathrm{id}_{H^*} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H^*\otimes H\otimes H\otimes H^*\otimes H^*}) \\
&\quad \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \sigma_{H,H^*}^{-1} \otimes \mathrm{id}_{H\otimes H\otimes H^*\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H^*\otimes H^*} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H\otimes H\otimes H\otimes H^*\otimes H^*}) \\
&\quad \circ (\Delta_{H^*} \otimes \Delta_{H^*} \otimes \mathrm{id}_{H\otimes H\otimes H\otimes H} \otimes m_{H^*} \otimes m_{H^*}) \circ (\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*\otimes H^*\otimes H^*}) \\
&\quad \circ (\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H^*\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*}) \\
&\quad \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \mathrm{coev}_H),
\end{aligned}
$$

and, applying (1.10) to the morphism $(\mathrm{ev}_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H^*} \otimes \sigma_{H,H^*}^{-1})$, we obtain

$$
\begin{aligned}
&(\mathrm{ev}_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H\otimes H^*\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H\otimes H\otimes H^*\otimes H^*}) \\
&\circ (\mathrm{id}_{H^*} \otimes \sigma_{H^*,H^*} \otimes \mathrm{id}_{H\otimes H\otimes H\otimes H^*\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H^*\otimes H^*} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H\otimes H\otimes H\otimes H^*\otimes H^*}) \\
&\circ (\Delta_{H^*} \otimes \Delta_{H^*} \otimes \mathrm{id}_{H\otimes H\otimes H\otimes H} \otimes m_{H^*} \otimes m_{H^*}) \circ (\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*\otimes H^*\otimes H^*}) \\
&\circ (\mathrm{id}_{H^*\otimes H^*\otimes H\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H^*\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \mathrm{coev}_H) \\
&= (m_{H^*} \otimes m_{H^*}) \circ (\mathrm{ev}_H \otimes \mathrm{id}_{H^*\otimes H^*\otimes H^*\otimes H^*}) \circ (\mathrm{id}_{H^*} \otimes \mathrm{ev}_H \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*\otimes H^*\otimes H^*}) \\
&\quad \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \mathrm{ev}_H \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H^*\otimes H^*} \otimes \mathrm{ev}_H \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*}) \\
&\quad \circ (\mathrm{id}_{H^*\otimes H^*\otimes H^*\otimes H^*} \otimes \mathrm{coev}_H) \circ (\mathrm{id}_{H^*} \otimes \sigma_{H^*,H^*} \otimes \mathrm{id}_{H^*}) \circ (\Delta_{H^*} \otimes \Delta_{H^*}).
\end{aligned}
$$

Finally, several applications of Definition 1.3.1 yields

$$\Delta_{H^*} \circ m_{H^*} = (m_{H^*} \otimes m_{H^*}) \circ (\mathrm{id}_{H^*} \otimes \sigma_{H^*,H^*} \otimes \mathrm{id}_{H^*}) \circ (\Delta_{H^*} \otimes \Delta_{H^*}).$$

By similar calculations, one can show that the right dual ${}^*H$ of a Hopf algebra H in a braided category $\mathcal{C}$ with right duality is again a Hopf algebra in $\mathcal{C}$. Its product and coproduct are defined similarly, by taking

$$m_{{}^*H} = {}^*\Delta_H \circ \gamma'_{H,H}$$

and

$$\Delta_{{}^*H} = (\gamma'_{H,H})^{-1} \circ {}^*m_H,$$

where γ' is the natural isomorphism characterized by (1.23). The unit and counit of ${}^*H$ are the right dual morphisms of the counit and unit of H, respectively.

3.2 Images of Hopf algebra homomorphisms

In this section, we prove that the image of a Hopf algebra homomorphism $f\colon A \to B$ in an abelian tensor category $\mathcal{C}$ with left duality is a Hopf subalgebra of B. First, we need the following lemma.

Lemma 3.2.1. *If $f\colon X \to Y$ is an epimorphism in a tensor category $\mathcal{C}$ with left duality, then both $f \otimes \mathrm{id}_Z$ and $\mathrm{id}_Z \otimes f$ are also epimorphisms, where Z is any object in $\mathcal{C}$.*

Proof. We prove that $f \otimes \mathrm{id}_Z$ is an epimorphism, the other assertion is proved similarly. Suppose that

$$g \circ (f \otimes \mathrm{id}_Z) = h \circ (f \otimes \mathrm{id}_Z)$$

for some morphisms $g\colon Y \otimes Z \to W$ and $h\colon Y \otimes Z \to W$. Then

$$(g \otimes \mathrm{id}_{Z^*}) \circ (f \otimes \mathrm{id}_{Z \otimes Z^*}) \circ (\mathrm{id}_X \otimes \mathrm{coev}_Z) = (h \otimes \mathrm{id}_{Z^*}) \circ (f \otimes \mathrm{id}_{Z \otimes Z^*}) \circ (\mathrm{id}_X \otimes \mathrm{coev}_Z),$$

that is,

$$(g \otimes \mathrm{id}_{Z^*}) \circ (\mathrm{id}_Y \otimes \mathrm{coev}_Z) \circ f = (h \otimes \mathrm{id}_{Z^*}) \circ (\mathrm{id}_Y \otimes \mathrm{coev}_Z) \circ f.$$

Since f is an epimorphism, this implies that

$$(g \otimes \mathrm{id}_{Z^*}) \circ (\mathrm{id}_Y \otimes \mathrm{coev}_Z) = (h \otimes \mathrm{id}_{Z^*}) \circ (\mathrm{id}_Y \otimes \mathrm{coev}_Z).$$

Then

$$\begin{aligned}&(\mathrm{id}_W \otimes \mathrm{ev}_Z) \circ (g \otimes \mathrm{id}_{Z^* \otimes Z}) \circ (\mathrm{id}_Y \otimes \mathrm{coev}_Z \otimes \mathrm{id}_Z)\\ &= (\mathrm{id}_W \otimes \mathrm{ev}_Z) \circ (h \otimes \mathrm{id}_{Z^* \otimes Z}) \circ (\mathrm{id}_Y \otimes \mathrm{coev}_Z \otimes \mathrm{id}_Z),\end{aligned}$$

that is,

$$g \circ (\mathrm{id}_{Y\otimes Z} \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_Y \otimes \mathrm{coev}_Z \otimes \mathrm{id}_Z)$$
$$= h \circ (\mathrm{id}_{Y\otimes Z} \otimes \mathrm{ev}_Z) \circ (\mathrm{id}_Y \otimes \mathrm{coev}_Z \otimes \mathrm{id}_Z).$$

This implies that $g = h$, by applying Definition 1.3.1 of a left dual Z^*. Thus $f \otimes \mathrm{id}_Z$ is an epimorphism. □

It can be shown similarly that if f is a monomorphism, then both $f \otimes \mathrm{id}_Z$ and $\mathrm{id}_Z \otimes f$ are also monomorphisms. This and the above lemma are actually parts of a more general theorem [3, Theorem 7.7, Ch. 2, p. 68].

Next, we prove that if f is an algebra homomorphism, then $C = \mathrm{im}(f)$ is a subalgebra, which means that there exists a multiplication $m_C\colon C \otimes C \to C$ for C such that the diagram

$$\begin{array}{ccc} C \otimes C & \xrightarrow{i\otimes i} & B \otimes B \\ \Big\downarrow{\scriptstyle m_C} & & \Big\downarrow{\scriptstyle m_B} \\ C & \xrightarrow{i} & B \end{array}$$

commutes, where $i\colon \mathrm{im}(f) \to B$ is an image of f.

Theorem 3.2.1. *Let $\mathcal{C}$ be an abelian tensor category with left duality. If $f\colon A \to B$ is an algebra homomorphism in $\mathcal{C}$, then $C = \mathrm{im}(f)$ is a subalgebra of B.*

Proof. Since f is an algebra homomorphism, we have

$$m_B \circ (f \otimes f) = f \circ m_A.$$

Recall that there exists a unique morphism $g\colon A \to \mathrm{im}(f)$ such that $f = i \circ g$ by the universal property of the kernel of the cokernel. Hence,

$$m_B \circ (i \otimes i) \circ (g \otimes g) = i \circ g \circ m_A.$$

Now since $\pi \circ i = 0$, where $\pi\colon B \to \mathrm{coker}(f)$ is the cokernel of f, we have

$$\pi \circ m_B \circ (i \otimes i) \circ (g \otimes g) = \pi \circ i \circ g \circ m_A$$
$$= 0.$$

But $g \otimes g$ is an epimorphism by Lemmas 1.5.1 and 3.2.1, so this implies that

$$\pi \circ m_B \circ (i \otimes i) = 0.$$

Now, by the universal property of π, there exists a unique morphism $m_C \colon C \otimes C \to C$ such that

$$m_B \circ (i \otimes i) = i \circ m_C.$$

Furthermore, it can be shown that m_C satisfies the definition of a product on C, and hence C is a subalgebra of B. Its unit satisfies $i \circ u_C = u_B$. □

The next task is to prove that $C = \operatorname{im}(f)$ is a subcoalgebra, which means that there exists a coproduct $\Delta_C \colon C \to C \otimes C$ such that the diagram

$$\begin{array}{ccc} C \otimes C & \xrightarrow{i \otimes i} & B \otimes B \\ \uparrow{\scriptstyle \Delta_C} & & \uparrow{\scriptstyle \Delta_B} \\ C & \xrightarrow{i} & B \end{array} \tag{3.7}$$

commutes, where $i \colon \operatorname{im}(f) \to B$ is an image of f.

Theorem 3.2.2. *Let $\mathcal{C}$ be an abelian tensor category with left duality. If $f \colon A \to B$ is a coalgebra homomorphism in $\mathcal{C}$, then $C = \operatorname{im}(f)$ is a subcoalgebra of B.*

Proof. We first prove that there exists a morphism $\delta_C \colon C \to C \otimes B$ such that

$$(i \otimes \mathrm{id}_B) \circ \delta_C = \Delta_B \circ i,$$

where $i \colon \operatorname{im}(f) \to B$ is an image of f. Let $\pi \colon B \to \operatorname{coker}(f)$ be a cokernel of f, and write $f = i \circ g$. Then, using the fact that f is a coalgebra homomorphism, and that $\pi \circ f = 0$, we have

$$\begin{aligned} (\pi \otimes \mathrm{id}_B) \circ \Delta_B \circ i \circ g &= (\pi \otimes \mathrm{id}_B) \circ \Delta_B \circ f \\ &= (\pi \otimes \mathrm{id}_B) \circ (f \otimes f) \circ \Delta_A \\ &= (0 \otimes f) \circ \Delta_A \\ &= 0 \circ \Delta_A \\ &= 0. \end{aligned}$$

Since g is an epimorphism by Lemma 1.5.1, this implies that $(\pi \otimes \mathrm{id}_B) \circ \Delta_B \circ i = 0$. It can be shown by similar reasoning as in Lemma 3.2.1 that $i \otimes \mathrm{id}_B$ is the kernel of $\pi \otimes \mathrm{id}_B$. Thus, by the universal property of the kernel, there exists a unique morphism δ_C such that

$$\Delta_B \circ i = (i \otimes \mathrm{id}_B) \circ \delta_C,$$

as asserted.

Next, we show that $(\mathrm{id}_C \otimes \pi) \circ \delta_C = 0$. Letting $D = \mathrm{coker}(f)$, and again using the fact that f is a coalgebra homomorphism, we have

$$\begin{aligned}(i \otimes \mathrm{id}_D) \circ (\mathrm{id}_C \otimes \pi) \circ \delta_C \circ g &= (\mathrm{id}_B \otimes \pi) \circ (i \otimes \mathrm{id}_B) \circ \delta_C \circ g \\ &= (\mathrm{id}_B \otimes \pi) \circ \Delta_B \circ i \circ g \\ &= (\mathrm{id}_B \otimes \pi) \circ \Delta_B \circ f \\ &= (\mathrm{id}_B \otimes \pi) \circ (f \otimes f) \circ \Delta_A \\ &= (f \otimes 0) \circ \Delta_A \\ &= 0 \circ \Delta_A \\ &= 0.\end{aligned}$$

Since g is an epimorphism, this implies that

$$(i \otimes \mathrm{id}_D) \circ (\mathrm{id}_C \otimes \pi) \circ \delta_C = 0,$$

and since $i \otimes \mathrm{id}_D$ is a monomorphism, this further implies that

$$(\mathrm{id}_C \otimes \pi) \circ \delta_C = 0.$$

Now, by the universal property of the kernel of $\mathrm{id}_C \otimes \pi$, which is $\mathrm{id}_C \otimes i$, there exists a unique morphism $\Delta_C \colon C \to C \otimes C$ such that

$$\delta_C = (\mathrm{id}_C \otimes i) \circ \Delta_C,$$

and hence

$$(i \otimes i) \circ \Delta_C = (i \otimes \mathrm{id}_B) \circ \delta_C = \Delta_B \circ i.$$

Furthermore, it can be shown that Δ_C satisfies the definition of a coproduct on C, and therefore C is a subcoalgebra of B. Its counit is $\varepsilon_C = \varepsilon_B \circ i$. $\square$

Finally, we prove that $\mathrm{im}(f)$ has an antipode, which makes it a Hopf subalgebra of B.

Theorem 3.2.3. *Let $\mathcal{C}$ be an abelian tensor category with left duality. If $f\colon A \to B$ is a Hopf algebra homomorphism in $\mathcal{C}$, then $C = \mathrm{im}(f)$ is a Hopf subalgebra of B.*

Proof. Let $\pi\colon B \to \mathrm{coker}(f)$ be a cokernel of f and let $i\colon \mathrm{im}(f) \to B$ be an image of f, and write $f = i \circ g$. First, we show that there exists a morphism $S_C\colon C \to C$ such that $i \circ S_C = S_B \circ i$. Since f is a Hopf algebra homomorphism, we have

$$\begin{aligned} \pi \circ S_B \circ i \circ g &= \pi \circ S_B \circ f \\ &= \pi \circ f \circ S_A \\ &= 0 \circ S_A \\ &= 0. \end{aligned}$$

Since g is an epimorphism by Lemma 1.5.1, this implies that $\pi \circ S_B \circ i = 0$. Therefore, by the universal property of the kernel of π, there exists a unique morphism S_C such that $i \circ S_C = S_B \circ i$, as required. We prove that S_C satisfies the left antipode equation

$$m_C \circ (S_C \otimes \mathrm{id}_C) \circ \Delta_C = u_C \circ \varepsilon_C.$$

The right antipode equation is proved similarly. We have

$$\begin{aligned} i \circ m_C \circ (S_C \otimes \mathrm{id}_C) \circ \Delta_C &= m_B \circ (i \otimes i) \circ (S_C \otimes \mathrm{id}_C) \circ \Delta_C \\ &= m_B \circ (S_B \otimes \mathrm{id}_B) \circ (i \otimes i) \circ \Delta_C \\ &= m_B \circ (S_B \otimes \mathrm{id}_B) \circ \Delta_B \circ i \\ &= u_B \circ \varepsilon_B \circ i \\ &= i \circ u_C \circ \varepsilon_C, \end{aligned}$$

and since i is left-cancellable, the result follows. □

3.3 The homomorphisms induced by the Hopf pairing

Let H be a Hopf algebra in an abelian tensor category $\mathcal{C}$ with left duality. For a Hopf pairing $\omega\colon H \otimes H \to I$, we define the morphism

$$\omega' = (\omega \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \mathrm{coev}_H). \tag{3.8}$$

The pairing ω is said to be *non-degenerate* when ω' is an isomorphism. In this section, we show that $\omega'\colon H \to H^*$ is a Hopf algebra homomorphism. It then follows by Theorem 3.2.3 that $\mathrm{im}(\omega')$ is a Hopf subalgebra of H^*. First, we prove that ω' is an algebra homomorphism.

Theorem 3.3.1. *The morphism $\omega'\colon H \to H^*$ defined in* (3.8) *is an algebra homomorphism.*

Proof. We prove that

$$\omega' \circ m_H = m_{H^*} \circ (\omega' \otimes \omega').$$

We have

$$\begin{aligned}
&m_{H^*} \circ (\omega' \otimes \omega') \\
&= \Delta_H^* \circ \gamma_{H,H} \circ (\omega \otimes \mathrm{id}_{H^*} \otimes \omega \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \mathrm{coev}_H \otimes \mathrm{id}_H \otimes \mathrm{coev}_H) \\
&= (\mathrm{ev}_{H\otimes H} \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{(H\otimes H)^*} \otimes \Delta_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{(H\otimes H)^*} \otimes \mathrm{coev}_H) \circ \gamma_{H,H} \\
&\qquad \circ (\omega \otimes \mathrm{id}_{H^*} \otimes \omega \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \mathrm{coev}_H \otimes \mathrm{id}_H \otimes \mathrm{coev}_H) \\
&= (\mathrm{ev}_{H\otimes H} \otimes \mathrm{id}_{H^*}) \circ (\gamma_{H,H} \otimes \mathrm{id}_{H\otimes H\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \Delta_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \mathrm{coev}_H) \\
&\qquad \circ (\omega \otimes \mathrm{id}_{H^*} \otimes \omega \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \mathrm{coev}_H \otimes \mathrm{id}_H \otimes \mathrm{coev}_H) \\
&= (\mathrm{ev}_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H^*} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \Delta_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H^*\otimes H^*} \otimes \mathrm{coev}_H) \\
&\qquad \circ (\omega \otimes \mathrm{id}_{H^*} \otimes \omega \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \mathrm{coev}_H \otimes \mathrm{id}_H \otimes \mathrm{coev}_H) \\
&= (\omega \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H\otimes H} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H\otimes H^*}) \\
&\qquad \circ (\mathrm{id}_H \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H\otimes H^*}) \circ (\mathrm{id}_{H\otimes H^*} \otimes \Delta_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H\otimes H^*} \otimes \mathrm{coev}_H) \\
&\qquad \circ (\mathrm{id}_H \otimes \omega \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H\otimes H} \otimes \mathrm{coev}_H) \\
&= (\omega \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H\otimes H^*}) \circ (\mathrm{id}_{H\otimes H^*} \otimes \Delta_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H\otimes H^*} \otimes \mathrm{coev}_H) \\
&\qquad \circ (\mathrm{id}_H \otimes \omega \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H\otimes H} \otimes \mathrm{coev}_H) \\
&= (\omega \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \omega \otimes \mathrm{id}_{H\otimes H^*}) \circ (\mathrm{id}_{H\otimes H\otimes H} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H\otimes H^*}) \\
&\qquad \circ (\mathrm{id}_{H\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H\otimes H\otimes H^*}) \circ (\mathrm{id}_{H\otimes H} \otimes \Delta_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H\otimes H} \otimes \mathrm{coev}_H) \\
&= (\omega \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \omega \otimes \mathrm{id}_{H\otimes H^*}) \circ (\mathrm{id}_{H\otimes H} \otimes \Delta_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H\otimes H} \otimes \mathrm{coev}_H),
\end{aligned}$$

where we have used the characterization (1.16) of γ, and two applications of Definition 1.3.1 of a left dual H^*. Now, by applying property (2.11) of the Hopf pairing ω, this equals

$$\begin{aligned}
(\omega \otimes \mathrm{id}_{H^*}) \circ (m_H \otimes \mathrm{id}_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H\otimes H} \otimes \mathrm{coev}_H) &= (\omega \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \mathrm{coev}_H) \circ m_H \\
&= \omega' \circ m_H,
\end{aligned}$$

as asserted. ∎

Next, we prove that ω' is also a coalgebra homomorphism.

Theorem 3.3.2. *The morphism* $\omega' \colon H \to H^*$ *defined in* (3.8) *is a coalgebra homomorphism.*

Proof. We prove that
$$\Delta_{H^*} \circ \omega' = (\omega' \otimes \omega') \circ \Delta_H.$$

Inserting the definitions and applying (3.6),

$$\begin{aligned}
&\Delta_{H^*} \circ \omega' \\
&= \gamma_{H,H}^{-1} \circ m_H^* \circ (\omega \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \mathrm{coev}_H) \\
&= \gamma_{H,H}^{-1} \circ (\mathrm{ev}_H \otimes \mathrm{id}_{(H\otimes H)^*}) \circ (\mathrm{id}_{H^*} \otimes m_H \otimes \mathrm{id}_{(H\otimes H)^*}) \circ (\mathrm{id}_{H^*} \otimes \mathrm{coev}_{H\otimes H}) \circ (\omega \otimes \mathrm{id}_{H^*}) \\
&\qquad \circ (\mathrm{id}_H \otimes \mathrm{coev}_H) \\
&= (\mathrm{ev}_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*} \otimes m_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H\otimes H} \otimes \gamma_{H,H}^{-1}) \circ (\mathrm{id}_{H^*} \otimes \mathrm{coev}_{H\otimes H}) \\
&\qquad \circ (\omega \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \mathrm{coev}_H) \\
&= (\mathrm{ev}_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*} \otimes m_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H^*\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*}) \\
&\qquad \circ (\mathrm{id}_{H^*} \otimes \mathrm{coev}_H) \circ (\omega \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \mathrm{coev}_H) \\
&= (\omega \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H\otimes H} \otimes \mathrm{ev}_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_H \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H\otimes H^*\otimes H^*}) \\
&\qquad \circ (\mathrm{id}_H \otimes m_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \mathrm{coev}_H) \\
&= (\omega \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_H \otimes m_H \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_{H\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \mathrm{coev}_H),
\end{aligned}$$

where we have also used Definition 1.3.1 of a left dual H^*. Now, applying the property (2.12) of the Hopf pairing ω, this equals

$$\begin{aligned}
&(\omega \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_H \otimes \omega \otimes \mathrm{id}_{H\otimes H^*\otimes H^*}) \circ (\Delta_H \otimes \mathrm{id}_{H\otimes H\otimes H^*\otimes H^*}) \\
&\quad \circ (\mathrm{id}_{H\otimes H} \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \mathrm{coev}_H),
\end{aligned}$$

which further equals

$$\begin{aligned}
&(\omega \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_H \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \omega \otimes \mathrm{id}_{H^*}) \circ (\Delta_H \otimes \mathrm{id}_{H\otimes H^*}) \\
&\quad \circ (\mathrm{id}_H \otimes \mathrm{coev}_H) \\
&= (\omega \otimes \mathrm{id}_{H^*\otimes H^*}) \circ (\mathrm{id}_H \otimes \mathrm{coev}_H \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \omega \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H\otimes H} \otimes \mathrm{coev}_H) \circ \Delta_H \\
&= (\omega' \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_H \otimes \omega') \circ \Delta_H \\
&= (\omega' \otimes \omega') \circ \Delta_H,
\end{aligned}$$

as asserted. □

This establishes that ω' is a bialgebra homomorphism. As mentioned in 1.7, it follows that ω' is a Hopf algebra homomorphism, i.e., that

$$\omega' \circ S_H = S_{H^*} \circ \omega'.$$

We also have the morphism $\omega'' \colon H \to {}^*H$ defined in [17, p. 10, (3.4)] as

$$\omega'' = (\mathrm{id}_{{}^*H} \otimes \omega) \circ (\mathrm{coev}'_H \otimes \mathrm{id}_H), \tag{3.9}$$

where coev' is the right dual coevaluation as defined in Definition 1.3.1. This morphism is related to ω' by

$$(\omega'')^* = \omega'$$

and non-degeneracy of ω can be equivalently defined in terms of ω''. Furthermore, ω'' is also a Hopf algebra homomorphism.

Chapter 4

Category of modules over a quasitriangular Hopf algebra

Following the conventions of [17, p. 3], we define the *Müger centre* of a braided abelian tensor category $\mathcal{C}$ as the subcategory of $\mathcal{C}$ consisting of all objects X in $\mathcal{C}$ such that

$$\sigma_{Y,X} \circ \sigma_{X,Y} = \mathrm{id}_{X \otimes Y}$$

for all objects Y in $\mathcal{C}$. We say that the Müger centre is *trivial* if all of its objects are isomorphic to the direct sum of finitely many copies of the unit object. In our context, a direct sum is the same thing as a coproduct, as defined in 1.5. It is proved in [17, Theorem 1.1, p. 3] that if $\mathcal{C}$ is a braided finite tensor category over an algebraically closed field, then triviality of the Müger centre of $\mathcal{C}$ implies that the Hopf pairing ω in $\mathcal{C}$ is non-degenerate. In this chapter, we prove this implication in the case where $\mathcal{C}$ is the category of finite-dimensional modules over a finite-dimensional quasitriangular ribbon Hopf algebra H, without the hypothesis that the base field is algebraically closed.

4.1 Quasitriangular Hopf algebras

Before we recall the definition of a quasitriangular Hopf algebra, we will need the following notation. Let A be an algebra and let $T = \sum_{i=1}^{n} a_i \otimes b_i \in A \otimes A$. For a given integer $k \geq 2$, we denote by T_{pq} the tensor in $A^{\otimes k}$ that has, for each $i = 1, \ldots, n$, the element a_i in the p-th tensor factor and the element b_i in the q-th tensor factor, and 1_A in all remaining tensor factors. For example, for $k = 2$ we have $T_{21} = \sum_{i=1}^{n} b_i \otimes a_i$.

Definition 4.1.1. A *quasitriangular Hopf algebra* is a Hopf algebra H with a bijective antipode S_H and an invertible element

$$R = \sum_{i=1}^{n} a_i \otimes b_i \in H \otimes H,$$

called a *universal R-matrix*, satisfying the following axioms for all $h \in H$:

1. $\Delta_H^{\mathrm{cop}}(h) = R\Delta_H(h)R^{-1}$
2. $(\Delta_H \otimes \mathrm{id}_H)(R) = R_{13}R_{23}$
3. $(\mathrm{id}_H \otimes \Delta_H)(R) = R_{13}R_{12}$

where Δ_H is the coproduct of H and Δ_H^{cop} is the coopposite coproduct, defined by

$$\Delta_H^{\mathrm{cop}}(h) = h_{(2)} \otimes h_{(1)}$$

in Sweedler notation.

By [5, Theorem VIII.2.4, p. 175], we have the following lemma.

Lemma 4.1.1. *The universal R-matrix of a quasitriangular Hopf algebra H satisfies the following properties:*

1. $(\varepsilon_H \otimes \mathrm{id}_H)(R) = (\mathrm{id} \otimes \varepsilon_H)(R) = 1$
2. $(S_H \otimes \mathrm{id}_H)(R) = (\mathrm{id}_H \otimes S_H^{-1})(R) = R^{-1}$
3. $(S_H \otimes S_H)(R) = R$

where ε_H is the counit of H.

If H is a quasitriangular Hopf algebra with R-matrix $R = \sum_{i=1}^{n} a_i \otimes b_i$, then there is a so-called *quasisymmetry* $\sigma_{X,Y} \colon X \otimes Y \to Y \otimes X$ defined for all H-modules X and Y by

$$\sigma_{X,Y}(x \otimes y) = \sum_{i=1}^{n} (b_i \cdot y) \otimes (a_i \cdot x), \tag{4.1}$$

where $\cdot$ denotes the action by H. The first axiom in Definition 4.1.1 is equivalent to $\sigma_{X,Y}$ being H-linear for each pair X and Y, and the second and third axioms in Definition 4.1.1

are equivalent to $\sigma_{X,Y}$ satisfying the braiding axioms (1.3) and (1.4) for each X and Y. The invertibility of R is equivalent to $\sigma_{X,Y}$ being bijective. Thus, σ is a braiding in the category of H-modules.

The notion of a ribbon element is defined as follows.

Definition 4.1.2. Let H be a quasitriangular Hopf algebra, and let R denote its universal R-matrix. A *ribbon element* in H is a nonzero central element $v \in H$ satisfying

$$\Delta_H(v) = (R_{21}R)(v \otimes v) \tag{4.2}$$

and

$$S_H(v) = v. \tag{4.3}$$

With a specified ribbon element, H is called a *ribbon Hopf algebra.*

A version of the above definition can be found in [16, p. 7]. In Proposition 4.1.2, we will see that two of the axioms in that definition are in fact consequences of the above definition, and that a ribbon element in [16, p. 7] is effectively the inverse of a ribbon element in Definition 4.1.2.

In a braided category with duality, we also have the notion of a ribbon twist.

Definition 4.1.3. Let $\mathcal{C}$ be a braided category with braiding σ and left duality. A natural isomorphism θ from the identity functor to itself is called a *ribbon twist* if it satisfies

$$\theta_{X\otimes Y} = \sigma_{Y,X} \circ \sigma_{X,Y} \circ (\theta_X \otimes \theta_Y) \tag{4.4}$$

and

$$\theta_{X^*} = \theta_X^*. \tag{4.5}$$

With a specified ribbon twist, $\mathcal{C}$ is called a *ribbon category.*

The naturality of θ means that for each morphism $f\colon X \to Y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \theta_X}\downarrow & & \downarrow{\scriptstyle \theta_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

Note also that

$$(\theta_X \otimes \theta_Y) \circ \sigma_{Y,X} \circ \sigma_{X,Y} = \sigma_{Y,X} \circ (\theta_Y \otimes \theta_X) \circ \sigma_{X,Y} = \sigma_{Y,X} \circ \sigma_{X,Y} \circ (\theta_X \otimes \theta_Y)$$

by the naturality of σ, so the axiom (4.4) can be expressed in various ways.

If a quasitriangular Hopf algebra H has a ribbon element v, then we can define a ribbon twist θ on the category of finite-dimensional modules over H. By finite-dimensional, we are referring to the dimension of the vector space structure on the module induced by the base field of H.

Proposition 4.1.1. *Let H be a quasitriangular Hopf algebra with a ribbon element v. For each finite-dimensional H-module X, define*

$$\begin{aligned} \theta_X \colon X &\to X \\ x &\mapsto v \cdot x. \end{aligned}$$

Then θ is a ribbon twist in the category of finite-dimensional H-modules.

Proof. First note that each θ_X is H-linear, which means that it is a morphism in the category: For any $h \in H$, we have

$$\theta_X(h \cdot x) = v \cdot (h \cdot x) = vh \cdot x = hv \cdot x = h \cdot (v \cdot x) = h \cdot \theta_X(x)$$

by the associativity of the module action and the centrality of v. Now let X and Y be any finite-dimensional H-modules and let $x \otimes y \in X \otimes Y$. Then

$$\theta_{X \otimes Y}(x \otimes y) = v \cdot (x \otimes y) = \Delta_H(v) \cdot (x \otimes y)$$

by definition, and we have

$$\begin{aligned} (\sigma_{Y,X} \circ \sigma_{X,Y} \circ (\theta_X \otimes \theta_Y))(x \otimes y) &= (\sigma_{Y,X} \circ \sigma_{X,Y})((v \cdot x) \otimes (v \cdot y)) \\ &= \sum_{i=1}^{n} \sigma_{Y,X}(((b_i v) \cdot y) \otimes ((a_i v) \cdot x)) \\ &= \sum_{i,j=1}^{n} ((b_j a_i v) \cdot x) \otimes ((a_j b_i v) \cdot y)) \\ &= ((R_{21} R)(v \otimes v)) \cdot (x \otimes y) \\ &= \Delta_H(v) \cdot (x \otimes y), \end{aligned}$$

where we have used the axiom (4.2) for the ribbon element v. Thus, θ satisfies (4.4). Next, we prove that $\theta_{X^*} = \theta_X^*$, where X^* is the dual space of the vector space structure on X induced by the base field of H. For all $\varphi \in X^*$ and $x \in X$, we have

$$\theta_X^*(\varphi)(x) = \varphi(\theta_X(x))$$

by the definition of the dual morphism of θ_X, and we have

$$\theta_{X^*}(\varphi)(x) = (v \cdot \varphi)(x) = \varphi(S_H(v) \cdot x) = \varphi(v \cdot x) = \varphi(\theta_X(x)),$$

where we have applied the definition of the module action

$$(h \cdot \varphi)(x) = \varphi(S_H(h) \cdot x) \tag{4.6}$$

of H on linear forms, and the axiom (4.3) for v. Thus, θ satisfies axiom (4.5). Note that the naturality of θ simply follows from the fact that the morphisms in this category are H-linear. Hence, θ is a ribbon twist. □

The element

$$u = \sum_{i=1}^{n} S_H(b_i)a_i$$

is known as the *Drinfel'd element*. The Drinfel'd element has many interesting properties [5], one of which is that it is invertible with inverse given by

$$u^{-1} = \sum_{i=1}^{n} S_H^{-2}(b_i)a_i. \tag{4.7}$$

Lemma 4.1.1 allows us to rewrite this as

$$u^{-1} = \sum_{i=1}^{n} b_i S_H^2(a_i). \tag{4.8}$$

Another property of the Drinfel'd element is that it satisfies

$$S_H^2(h) = uhu^{-1} \tag{4.9}$$

for all $h \in H$. The Drinfel'd element of H is related to a ribbon element in H by the following proposition.

Proposition 4.1.2. *If v is a ribbon element in a quasitriangular Hopf algebra H, then*

$$v^2 = S_H(u^{-1})u^{-1}, \tag{4.10}$$

where S_H is the antipode of H and u is its Drinfel'd element.

Proof. The definition of a ribbon element implies that

$$\begin{aligned} v \otimes v &= (R_{21}R)^{-1}\Delta_H(v) \\ &= R^{-1}R_{21}^{-1}\Delta_H(v). \end{aligned}$$

Noting that R_{21} is an R-matrix for H^{op}, which has antipode S_H^{-1}, and applying the second property in Lemma 4.1.1, this equals

$$\sum_{i,j=1}^{n} S_H(a_i)b_j v_{(1)} \otimes b_i S_H(a_j)v_{(2)}.$$

If we apply the antipode to the second tensor factor and then apply the multiplication, this implies

$$\begin{aligned} vS_H(v) &= \sum_{i,j=1}^{n} S_H(a_i)b_j v_{(1)} S_H(b_i S_H(a_j)v_{(2)}) \\ &= \sum_{i,j=1}^{n} S_H(a_i)b_j v_{(1)} S_H(v_{(2)})S_H^2(a_j)S_H(b_i). \end{aligned}$$

Since $v = S_H(v)$, we have $v^2 = vS_H(v)$. Hence, applying the antipode equation, the form (4.8) of u^{-1}, and the third property in Lemma 4.1.1,

$$\begin{aligned} v^2 &= \sum_{i,j=1}^{n} S_H(a_i)b_j v_{(1)} S_H(v_{(2)})S_H^2(a_j)S_H(b_i) \\ &= \varepsilon_H(v)\sum_{i,j} S_H(a_i)b_j S_H^2(a_j)S_H(b_i) \\ &= \varepsilon_H(v)\sum_{i} S_H(a_i)u^{-1}S_H(b_i) \\ &= \varepsilon_H(v)\sum_{i} a_i u^{-1} b_i. \end{aligned}$$

Now, observe that (4.9) implies

$$u^{-1}h = S_H^{-2}(h)u^{-1}$$

for all $h \in H$. Therefore, using the form (4.7) of u^{-1} and the third property in Lemma 4.1.1,

$$\begin{aligned} v^2 &= \varepsilon_H(v) \sum_i a_i S_H^{-2}(b_i) u^{-1} \\ &= \varepsilon_H(v) S_H(u^{-1}) u^{-1}. \end{aligned}$$

Thus, if we can show that $\varepsilon_H(v) = 1$, then the result will follow. By the counit equation, the fact that $\varepsilon_H \otimes \mathrm{id}_H$ is an algebra homomorphism, and Lemma 4.1.1, we have

$$\begin{aligned} v &= (\varepsilon_H \otimes \mathrm{id}_H)(\Delta_H(v)) \\ &= (\varepsilon_H \otimes \mathrm{id}_H)(R_{21} R (v \otimes v)) \\ &= (\varepsilon_H \otimes \mathrm{id}_H)(R_{21})(\varepsilon_H \otimes \mathrm{id}_H)(R) \varepsilon_H(v) v \\ &= \varepsilon_H(v) v. \end{aligned}$$

Now since $v \neq 0$, this implies that $\varepsilon_H(v) = 1$. $\square$

4.2 The dual space as a coend

Following [22, 4.5] (see also [7, 2.4], [10, 7.4]), we now show that the dual space H^* of a finite-dimensional Hopf algebra H is a coend in the category of finite-dimensional H-modules. The dual space H^* is an object in this category (i.e., an H-module) when equipped with the *coadjoint action* of H on H^*, defined for $\varphi \in H^*$ and $h \in H$ by

$$(h \cdot \varphi)(h') = \varphi(S_H(h_{(1)}) h' h_{(2)}) \tag{4.11}$$

for all $h' \in H$, where S_H is the antipode of H. To see that this defines a module action on H^*, we first consider the right module action on H defined by

$$h' \cdot h = S_H(h_{(1)}) h' h_{(2)}.$$

This is a right action because, for all $h, h', h'' \in H$, we have

$$\begin{aligned} h'' \cdot (hh') &= S_H(h_{(1)} h'_{(1)}) h'' h_{(2)} h'_{(2)} \\ &= S_H(h'_{(1)}) S_H(h_{(1)}) h'' h_{(2)} h'_{(2)} \\ &= (S_H(h_{(1)}) h'' h_{(2)}) \cdot h' \\ &= (h'' \cdot h) \cdot h', \end{aligned}$$

and the remaining module axioms are immediate. It then follows that

$$((hh') \cdot \varphi)(h'') = \varphi((h'' \cdot h) \cdot h') = (h \cdot (h' \cdot \varphi))(h''), \tag{4.12}$$

where we have used $\cdot$ for both actions. The remaining module axioms for H^* are again immediate. Thus $h \cdot \varphi$, as defined by (4.11), defines a module action on H^*.

Now for each finite-dimensional H-module X, let X^* denote its dual space and define $\iota_X \colon X^* \otimes X \to H^*$ by

$$\iota_X(\varphi \otimes x)(h) = \varphi(h \cdot x) \tag{4.13}$$

for all $h \in H$. First, we show that ι_X is H-linear, and therefore a morphism in the category. For all $h, h' \in H$, we have

$$\begin{aligned} \iota_X(h \cdot (\varphi \otimes x))(h') &= \iota_X((h_{(1)} \cdot \varphi) \otimes (h_{(2)} \cdot x))(h') \\ &= (h_{(1)} \cdot \varphi)((h' h_{(2)}) \cdot x)) \\ &= \varphi((S_H(h_{(1)}) h' h_{(2)}) \cdot x), \end{aligned}$$

where we have used the action (4.6) of H on X^*. Using the action (4.11) on H^*, we have

$$\begin{aligned} (h \cdot \iota_X(\varphi \otimes x))(h') &= \iota_X(\varphi \otimes x)(S_H(h_{(1)}) h' h_{(2)}) \\ &= \varphi((S_H(h_{(1)}) h' h_{(2)}) \cdot x) \end{aligned}$$

also, and hence ι_X is H-linear. Next, we show that ι is dinatural, i.e., that the diagram

$$\begin{array}{ccc} Y^* \otimes X & \xrightarrow{f^* \otimes \mathrm{id}_X} & X^* \otimes X \\ {\scriptstyle \mathrm{id}_{Y^*} \otimes f} \downarrow & & \downarrow {\scriptstyle \iota_X} \\ Y^* \otimes Y & \xrightarrow{\iota_Y} & H^* \end{array}$$

commutes for all finite-dimensional H-modules X and Y and all H-module homomorphisms $f \colon X \to Y$. For any $\varphi \otimes x \in Y^* \otimes X$ and $h \in H$, we have

$$\begin{aligned} ((\iota_X \circ (f^* \otimes \mathrm{id}_X))(\varphi \otimes x))(h) &= (\iota_X(f^*(\varphi) \otimes x))(h) \\ &= f^*(\varphi)(h \cdot x) \\ &= \varphi(f(h \cdot x)) \\ &= \varphi(h \cdot f(x)) \end{aligned}$$

and

$$((\iota_Y \circ (\mathrm{id}_{Y^*} \otimes f))(\varphi \otimes x))(h) = \iota_Y(\varphi \otimes f(x))(h)$$
$$= \varphi(h \cdot f(x)).$$

Hence

$$\iota_X \circ (f^* \otimes \mathrm{id}_X) = \iota_Y \circ (\mathrm{id}_{Y^*} \otimes f)$$

so that ι is dinatural. Finally, we prove that ι is universally dinatural. Let $\xi_X : X^* \otimes X \to Z$ be another dinatural transformation, and define $r : H^* \to Z$, as in [22, Lem. 4.3, p. 498], by

$$r(\varphi) = \xi_H(\varphi \otimes 1_H).$$

For any $x \in X$, consider the H-module homomorphism $f : H \to X$ defined by $f(h) = h \cdot x$. For any $\varphi \in X^*$, we have

$$f^*(\varphi) = \iota_X(\varphi \otimes x)$$

by the definition (4.13) of ι_X. Since f is a morphism in this category, we have

$$\xi_X \circ (\mathrm{id}_{X^*} \otimes f) = \xi_H \circ (f^* \otimes \mathrm{id}_H)$$

by the dinaturality of ξ. Evaluating on $\varphi \otimes 1_H$ gives

$$\xi_X(\varphi \otimes x) = \xi_H(f^*(\varphi) \otimes 1_H) = r(f^*(\varphi)) = r(\iota_X(\varphi \otimes x)).$$

This is true for any $\varphi \otimes x \in X^* \otimes X$, and hence $\xi_X = r \circ \iota_X$. Thus, H^* is a coend with universal dinatural transformation $\iota_X : X^* \otimes X \to H^*$ defined by (4.13).

4.3 The structure morphisms on the dual space

Let H be a finite-dimensional quasitriangular Hopf algebra with R-matrix $R = \sum_{i=1}^n a_i \otimes b_i$ and ribbon element v. We have seen that H^* is a coend in the category of finite-dimensional H-modules. This category has the braiding (4.1), and therefore H^* is a Hopf algebra in this category with the structure morphisms discussed in Chapter 2. In this section, we show that we can describe these structure morphisms explicitly. We will denote this coend by L, to distinguish it from H^* equipped with the usual dual Hopf algebra structure. We will see that $L = H^*$ as a coalgebra, but L has a different multiplication and antipode.

Recall that the multiplication on H^* is defined for $\varphi, \psi \in H^*$ and $h \in H$ by

$$m_{H^*}(\varphi \otimes \psi)(h) = (\varphi\psi)(h) = \varphi(h_{(1)})\psi(h_{(2)}).$$

We prove that the multiplication $m_L \colon L \otimes L \to L$ and antipode $S_L \colon L \to L$ for L are defined, as in [6, p. 331], by

$$(m_L(\varphi \otimes \psi))(h) = \sum_{i=1}^{n} \varphi(h_{(2)}a_i)\psi(S_H(b_{i(1)})h_{(1)}b_{i(2)}),$$

where S_H is the antipode of H, and

$$(S_L(\varphi))(h) = \sum_{i=1}^{n} \varphi(S_H(a_i)v^2 S_H(h)ub_i) = \sum_{i=1}^{n} \varphi(S_H(a_i)S_H(h)S_H(u^{-1})b_i),$$

where $u = \sum_{i=1}^{n} S_H(b_i)a_i$ is the Drinfel'd element of H. The two expressions for S_L are equal as a consequence of Proposition 4.1.2. We will make use of both forms in what follows.

We must first prove that m_L is a morphism in the category, i.e., that it is H-linear. Let $\varphi, \psi \in L$ and let $h, h' \in H$. Then

$$\begin{aligned}(m_L(h \cdot (\varphi \otimes \psi)))(h') &= (m_L((h_{(1)} \cdot \varphi) \otimes (h_{(2)} \cdot \psi)))(h') \\ &= \sum_{i=1}^{n} ((h_{(1)} \cdot \varphi)(h'_{(2)}a_i))((h_{(2)} \cdot \psi)(S_H(b_{i(1)})h'_{(1)}b_{i(2)})) \\ &= \sum_{i=1}^{n} \varphi(S_H(h_{(1)})h'_{(2)}a_i h_{(2)})\psi(S_H(h_{(3)})S_H(b_{i(1)})h'_{(1)}b_{i(2)}h_{(4)}) \\ &= \sum_{i=1}^{n} \varphi(S_H(h_{(1)})h'_{(2)}a_i h_{(2)})\psi(S_H(b_{i(1)}h_{(3)})h'_{(1)}b_{i(2)}h_{(4)})\end{aligned}$$

and, using the fact that S_H is both an algebra and coalgebra antihomomorphism,

$$\begin{aligned}(h \cdot (m_L(\varphi \otimes \psi))(h') &= (m_L(\varphi \otimes \psi))(S_H(h_{(1)})h'h_{(2)}) \\ &= \sum_{i=1}^{n} \varphi(S_H(h_{(1)})_{(2)}h'_{(2)}h_{(3)}a_i)\psi(S_H(b_{i(1)})S_H(h_{(1)})_{(1)}h'_{(1)}h_{(2)}b_{i(2)}) \\ &= \sum_{i=1}^{n} \varphi(S_H(h_{(1)})h'_{(2)}h_{(4)}a_i)\psi(S_H(b_{i(1)})S_H(h_{(2)})h'_{(1)}h_{(3)}b_{i(2)}) \\ &= \sum_{i=1}^{n} \varphi(S_H(h_{(1)})h'_{(2)}h_{(4)}a_i)\psi(S_H(h_{(2)}b_{i(1)})h'_{(1)}h_{(3)}b_{i(2)}).\end{aligned}$$

These two expressions are equal as a consequence of the axiom of the R-matrix

$$\sum_{i=1}^{n} h_{(2)}a_i \otimes h_{(1)}b_i = \sum_{i=1}^{n} a_i h_{(1)} \otimes b_i h_{(2)},$$

because it implies that

$$\sum_{i=1}^{n} h_{(1)} \otimes h_{(4)}a_i \otimes h_{(2)}b_{i(1)} \otimes h_{(3)}b_{i(2)} = \sum_{i=1}^{n} h_{(1)} \otimes a_i h_{(2)} \otimes b_{i(1)}h_{(3)} \otimes b_{i(2)}h_{(4)}.$$

To prove that m_L is the multiplication on L, it is sufficient by the universality of $\iota_X \otimes \iota_Y$, where $\iota_X \colon X^* \otimes X \to L$ is defined by (4.13), to prove that

$$m_L \circ (\iota_X \otimes \iota_Y) = \eta_{X,Y},$$

where η is the dinatural transformation (2.3). Let $\varphi \otimes x \otimes \psi \otimes y \in X^* \otimes X \otimes Y^* \otimes Y$, and let $h \in H$. On the one hand,

$$\begin{aligned}
&(\eta_{X,Y}(\varphi \otimes x \otimes \psi \otimes y))(h) \\
&= ((\iota_{Y\otimes X} \circ (\gamma_{Y,X} \otimes \mathrm{id}_{Y\otimes X}) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X,Y^*\otimes Y}))(\varphi \otimes x \otimes \psi \otimes y))(h) \\
&= \sum_{i=1}^{n} ((\iota_{Y\otimes X} \circ (\gamma_{Y,X} \otimes \mathrm{id}_{Y\otimes X}))(\varphi \otimes (b_i \cdot (\psi \otimes y)) \otimes (a_i \cdot x)))(h) \\
&= \sum_{i=1}^{n} ((\iota_{Y\otimes X} \circ (\gamma_{Y,X} \otimes \mathrm{id}_{Y\otimes X}))(\varphi \otimes (b_{i(1)} \cdot \psi) \otimes (b_{i(2)} \cdot y) \otimes (a_i \cdot x)))(h) \\
&= \sum_{i=1}^{n} \iota_{Y\otimes X}((b_{i(1)} \cdot \psi) \otimes \varphi \otimes (b_{i(2)} \cdot y) \otimes (a_i \cdot x))(h) \\
&= \sum_{i=1}^{n} ((b_{i(1)} \cdot \psi) \otimes \varphi)(h \cdot ((b_{i(2)} \cdot y) \otimes (a_i \cdot x))) \\
&= \sum_{i=1}^{n} ((b_{i(1)} \cdot \psi) \otimes \varphi)((h_{(1)}b_{i(2)} \cdot y) \otimes (h_{(2)}a_i \cdot x)) \\
&= \sum_{i=1}^{n} (b_{i(1)} \cdot \psi)(h_{(1)}b_{i(2)} \cdot y)\varphi((h_{(2)}a_i \cdot x) \\
&= \sum_{i=1}^{n} \psi(S_H(b_{i(1)})h_{(1)}b_{i(2)} \cdot y)\varphi(h_{(2)}a_i \cdot x),
\end{aligned}$$

where we have regarded $\gamma_{X,Y}$ as a map $Y^* \otimes X^* \to X^* \otimes Y^*$, since $(X \otimes Y)^* \cong X^* \otimes Y^*$

when H is finite-dimensional. On the other hand,

$$\begin{aligned}
&((m_L \circ (\iota_X \otimes \iota_Y))(\varphi \otimes x \otimes \psi \otimes y))(h) \\
&= (m_L(\iota_X(\varphi \otimes x) \otimes \iota_Y(\psi \otimes y))(h) \\
&= \sum_{i=1}^{n} ((\iota_X(\varphi \otimes x))(h_{(2)}a_i)(\iota_Y(\psi \otimes y))(S_H(b_{i(1)})h_{(1)}b_{i(2)}) \\
&= \sum_{i=1}^{n} \varphi(h_{(2)}a_i \cdot x)\psi(S_H(b_{i(1)})h_{(1)}b_{i(2)} \cdot y).
\end{aligned}$$

Thus $m_L \circ (\iota_X \otimes \iota_Y) = \eta_{X,Y}$, so that m_L is the product on L.

Next, we show that

$$\Delta_L = m_H^* = \Delta_{H^*}$$

defines the coproduct on L. Again, we must first establish that $\Delta_L \colon L \to L \otimes L$ is H-linear. Let $\varphi \in L$ and $h, h', h'' \in H$. Then

$$\begin{aligned}
(h \cdot \Delta_L(\varphi))(h' \otimes h'') &= ((h_{(1)} \cdot \varphi_{(1)}) \otimes (h_{(2)} \cdot \varphi_{(2)}))(h' \otimes h'') \\
&= \varphi_{(1)}(S_H(h_{(1)})h'h_{(2)})\varphi_{(2)}(S_H(h_{(3)})h''h_{(4)}) \\
&= \varphi(S_H(h_{(1)})h'h_{(2)}S_H(h_{(3)})h''h_{(4)}) \\
&= \varphi(S_H(h_{(1)})h'h''\varepsilon_H(h_{(2)})h_{(3)}) \\
&= \varphi(S_H(h_{(1)})h'h''h_{(2)}) \\
&= (h \cdot \varphi)(h'h'') \\
&= (m_H^*(h \cdot \varphi))(h' \otimes h'') \\
&= (\Delta_L(h \cdot \varphi))(h' \otimes h''),
\end{aligned}$$

as required.

To prove that Δ_L is the coproduct on L, it is sufficient by the universality of ι to prove that

$$\Delta_L \circ \iota_X = \zeta_X,$$

where ζ is the dinatural transformation defined in (2.8). Let $\varphi \otimes x \in X^* \otimes X$, and let $h, h' \in H$. Then

$$\begin{aligned}
((\Delta_L \circ \iota_X)(\varphi \otimes x))(h \otimes h') &= \iota_X(\varphi \otimes x)(hh') \\
&= \varphi(hh' \cdot x),
\end{aligned}$$

and, noting that the coevaluation in this category is the dual basis map,

$$\begin{aligned}(\zeta_X(\varphi\otimes x))(h\otimes h') &= (((\iota_X\otimes\iota_X)\circ(\mathrm{id}_{X^*}\otimes\mathrm{coev}_X\otimes\mathrm{id}_X))(\varphi\otimes x))(h\otimes h')\\ &= \Big((\iota_X\otimes\iota_X)\Big(\varphi\otimes\sum_i(x_i\otimes x_i^*)\otimes x\Big)\Big)(h\otimes h')\\ &= \sum_i(\iota_X(\varphi\otimes x_i)\otimes\iota_X(x_i^*\otimes x))(h\otimes h')\\ &= \sum_i\varphi(h\cdot x_i)x_i^*(h'\cdot x)\\ &= \varphi\Big(h\cdot\sum_i x_i^*(h'\cdot x)x_i\Big)\\ &= \varphi(hh'\cdot x).\end{aligned}$$

Therefore $\Delta_L\circ\iota_X=\zeta_X$, so that Δ_L is the coproduct on L.

Next, we show that S_L is the antipode of L. To see that S_L is H-linear, observe that

$$\begin{aligned}S_L(h\cdot\varphi)(h') &= \sum_{i=1}^n(h\cdot\varphi)(S_H(a_i)v^2S_H(h')ub_i)\\ &= \sum_{i=1}^n\varphi(S_H(h_{(1)})S_H(a_i)v^2S_H(h')ub_ih_{(2)})\\ &= \sum_{i=1}^n\varphi(v^2S_H(a_ih_{(1)})S_H(h')ub_ih_{(2)})\end{aligned}$$

for all $\varphi\in L$ and $h,h'\in H$, and, using the property $S_H^2(h)=uhu^{-1}$ of the Drinfel'd element,

$$\begin{aligned}(h\cdot S_L(\varphi))(h') &= S_L(\varphi)(S_H(h_{(1)})h'h_{(2)})\\ &= \sum_{i=1}^n\varphi(S_H(a_i)v^2S_H(S_H(h_{(1)})h'h_{(2)})ub_i)\\ &= \sum_{i=1}^n\varphi(v^2S_H(a_i)S_H(h_{(2)})S_H(h')S_H^2(h_{(1)})ub_i)\\ &= \sum_{i=1}^n\varphi(v^2S_H(h_{(2)}a_i)S_H(h')uh_{(1)}b_i).\end{aligned}$$

These two expression are equal as a consequence of the axiom of the R-matrix

$$\sum_{i=1}^n h_{(2)}a_i\otimes h_{(1)}b_i=\sum_{i=1}^n a_ih_{(1)}\otimes b_ih_{(2)}.$$

To prove that S_L is the antipode for L, it is sufficient by universality of ι to prove that

$$S_L \circ \iota_X = \chi_X,$$

where χ is the dinatural transformation (2.10). For all $\varphi \otimes x \in X^* \otimes X$ and $h \in H$, we have

$$\begin{aligned}
((S_L \circ \iota_X)(\varphi \otimes x))(h) &= S_L(\iota_X(\varphi \otimes x))(h) \\
&= \sum_i \iota_X(\varphi \otimes x)(S_H(a_i)S_H(h)S_H(u^{-1})b_i) \\
&= \sum_i \varphi(S_H(a_i)S_H(h)S_H(u^{-1})b_i \cdot x).
\end{aligned}$$

Rewriting χ_X using the braiding axiom (1.3), we also have

$$\begin{aligned}
\chi_X(\varphi \otimes x)(h) &= (((\mathrm{ev}_X \otimes \iota_{X^*}) \circ (\mathrm{id}_{X^*} \otimes \sigma_{X^{**},X} \otimes \mathrm{id}_{X^*}) \circ (\mathrm{coev}_{X^*} \otimes \sigma_{X^*,X}))(\varphi \otimes x))(h) \\
&= \sum_{i,j} ((\mathrm{ev}_X \otimes \iota_{X^*})((\mathrm{id}_{X^*} \otimes \sigma_{X^{**},X} \otimes \mathrm{id}_{X^*})(x_i^* \otimes x_i^{**} \otimes (b_j \cdot x) \otimes (a_j \cdot \varphi))))(h) \\
&= \sum_{i,j,k} ((\mathrm{ev}_X \otimes \iota_{X^*})(x_i^* \otimes (b_k b_j \cdot x) \otimes (a_k \cdot x_i^{**}) \otimes (a_j \cdot \varphi)))(h) \\
&= \sum_{i,j,k} x_i^*(b_k b_j \cdot x)\iota_{X^*}((a_k \cdot x_i^{**}) \otimes (a_j \cdot \varphi))(h) \\
&= \sum_{i,j,k} x_i^*(b_k b_j \cdot x)(a_k \cdot x_i^{**})(h a_j \cdot \varphi) \\
&= \sum_{i,j,k} x_i^*(b_k b_j \cdot x)x_i^{**}(S_H(a_k) h a_j \cdot \varphi) \\
&= \sum_{i,j,k} x_i^*(b_k b_j \cdot x)(S_H(a_k) h a_j \cdot \varphi)(x_i) \\
&= \sum_{j,k} (S_H(a_k) h a_j \cdot \varphi)\Big(\sum_i x_i^*(b_k b_j \cdot x)x_i\Big) \\
&= \sum_{j,k} (S_H(a_k) h a_j \cdot \varphi)(b_k b_j \cdot x) \\
&= \sum_{j,k} \varphi(S_H(S_H(a_k) h a_j) b_k b_j \cdot x) \\
&= \sum_{j,k} \varphi(S_H(a_j)S_H(h)S_H^2(a_k) b_k b_j \cdot x) \\
&= \sum_{j} \varphi(S_H(a_j)S_H(h)S_H(u^{-1}) b_j \cdot x),
\end{aligned}$$

where we have used the property $S_H(u^{-1}) = \sum_{i=1}^n S_H^2(a_i)b_i$. Hence $S_L \circ \iota_X = \chi_X$, as required.

Finally, we show that the unit and counit of L are those of H^*. The unit object in this category is the base field K, equipped with the *trivial action*, defined by

$$h \cdot \lambda = \varepsilon_H(h)\lambda$$

for $h \in H$ and $\lambda \in K$. Recall that the unit element in H^* is the counit ε_H of H, so that the unit map $u_{H^*}\colon K \to H^*$ for H^* is the map $\lambda \mapsto \lambda\varepsilon_H$. We will denote this by u_L. When L is equipped with the coadjoint action (4.11), this map is H-linear, as

$$\begin{aligned}(u_L(h \cdot \lambda))(h') &= (u_L(\varepsilon_H(h)\lambda))(h') \\ &= \varepsilon_H(h)\lambda\varepsilon_H(h') \\ &= \lambda\varepsilon_H(hh')\end{aligned}$$

and

$$\begin{aligned}(h \cdot u_L(\lambda))(h') &= \lambda\varepsilon_H(S_H(h_{(1)})h'h_{(2)}) \\ &= \lambda\varepsilon_H(S_H(h_{(1)})h_{(2)})\varepsilon_H(h') \\ &= \lambda\varepsilon_H(hh')\end{aligned}$$

for all $\lambda \in K$ and $h, h' \in H$.

Now recall that we defined the unit for the coend in (2.5) as $\iota_I \circ \lambda_I^{-1}$, and hence the unit for L in this category is the map $\iota_K\colon K^* \otimes K \cong K \to L$, where ι is defined by (4.13). For all $\lambda \in K$ and $h \in H$, we have

$$\begin{aligned}(u_L(\lambda))(h) &= \lambda\varepsilon_H(h) \\ &= h \cdot \lambda \\ &= (\iota_K(\mathrm{id}_K \otimes \lambda))(h),\end{aligned}$$

and hence $u_L = u_{H^*}$ is the unit of L. The counit $\varepsilon_{H^*}\colon H^* \to K$ for H^* is evaluation at the unit element $\varphi \mapsto \varphi(1_H)$. We will denote this by ε_L. This map is H-linear because

$$\begin{aligned}\varepsilon_L(h \cdot \varphi) &= (h \cdot \varphi)(1_H) \\ &= \varphi(S_H(h_{(1)})1_H h_{(2)}) \\ &= \varepsilon_H(h)\varphi(1_H) \\ &= h \cdot \varepsilon_L(\varphi)\end{aligned}$$

for all $\lambda \in K$ and $h \in H$. Now let $\varphi \otimes x \in X^* \otimes X$ and observe that

$$\begin{aligned}(\varepsilon_L \circ \iota_X)(\varphi \otimes x) &= \varepsilon_L(\iota_X(\varphi \otimes x)) \\ &= \iota_X(\varphi \otimes x)(1_H) \\ &= \varphi(x) \\ &= \mathrm{ev}_X(\varphi \otimes x),\end{aligned}$$

and hence $\varepsilon_L \circ \iota_X = \mathrm{ev}_X$. By universality of ι, this implies that $\varepsilon_L = \varepsilon_{H^*}$ is the counit of L.

4.4 Coaction by the coend

The definition of a comodule over a coalgebra is obtained by reversing the arrows in the commutative diagrams that define a module over an algebra. These diagrams are interpreted in the category of vector spaces over a field K. This notion can be defined in an arbitrary tensor category as follows.

Definition 4.4.1. Let C be a coalgebra in a tensor category $\mathcal{C}$. A *right comodule* over C (or *right C-comodule*) is an object X in $\mathcal{C}$ together with a morphism $\delta_X \colon X \to X \otimes C$, called the *coaction*, such that the diagrams

$$\begin{array}{ccc} X \otimes C \otimes C & \xleftarrow{\mathrm{id}_X \otimes \Delta_C} & X \otimes C \\ \Big\uparrow{\scriptstyle \delta_X \otimes \mathrm{id}_C} & & \Big\uparrow{\scriptstyle \delta_X} \\ X \otimes C & \xleftarrow[\delta_X]{} & X \end{array} \tag{4.14}$$

and

$$\begin{array}{ccc} X \otimes I & \xleftarrow{\mathrm{id}_X \otimes \varepsilon_C} & X \otimes C \\ & \nwarrow{\scriptstyle \cong} & \Big\uparrow{\scriptstyle \delta_X} \\ & & X \end{array} \tag{4.15}$$

commute, where Δ_C and ε_C are the coproduct and counit of C, respectively.

We know that the coend L in a braided finite tensor category $\mathcal{C}$ is a Hopf algebra in $\mathcal{C}$ and, in particular, a coalgebra. The following coaction, found in [17, p. 13, (3.14)], makes every object in $\mathcal{C}$ a right comodule over L.

Proposition 4.4.1. *If L is a coend in a braided finite tensor category $\mathcal{C}$, with universal dinatural transformation ι, then for each object X in $\mathcal{C}$ the morphism*

$$\delta_X = (\mathrm{id}_X \otimes \iota_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X), \tag{4.16}$$

defines a right coaction on X by L.

Proof. We have to verify that

$$(\mathrm{id}_X \otimes \Delta_L) \circ \delta_X = (\delta_X \otimes \mathrm{id}_L) \circ \delta_X \tag{4.17}$$

and

$$(\mathrm{id}_X \otimes \varepsilon_L) \circ \delta_X = \mathrm{id}_X. \tag{4.18}$$

On the one hand, we have

$$\begin{aligned}(\mathrm{id}_X \otimes \Delta_L) \circ \delta_X &= (\mathrm{id}_X \otimes \Delta_L) \circ (\mathrm{id}_X \otimes \iota_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X)\\ &= (\mathrm{id}_X \otimes \iota_X \otimes \iota_X) \circ (\mathrm{id}_X \otimes \mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X)\\ &= (\mathrm{id}_X \otimes \iota_X \otimes \iota_X) \circ (\mathrm{coev}_X \otimes \mathrm{coev}_X \otimes \mathrm{id}_X)\end{aligned}$$

because Δ_L is the unique morphism satisfying $\Delta_L \circ \iota_X = (\iota_X \otimes \iota_X) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X \otimes \mathrm{id}_X)$ for all X in $\mathcal{C}$, as discussed in 2.3. On the other hand, we have

$$\begin{aligned}(\delta_X \otimes \mathrm{id}_L) \circ \delta_X &= (\delta_X \otimes \mathrm{id}_L) \circ (\mathrm{id}_X \otimes \iota_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X)\\ &= (\mathrm{id}_X \otimes \iota_X \otimes \mathrm{id}_L) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X \otimes \mathrm{id}_L) \circ (\mathrm{id}_X \otimes \iota_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X)\\ &= (\mathrm{id}_X \otimes \iota_X \otimes \iota_X) \circ (\mathrm{coev}_X \otimes \mathrm{coev}_X \otimes \mathrm{id}_X),\end{aligned}$$

and this proves (4.17). For (4.18), observe that

$$\begin{aligned}(\mathrm{id}_X \otimes \varepsilon_L) \circ \delta_X &= (\mathrm{id}_X \otimes \varepsilon_L) \circ (\mathrm{id}_X \otimes \iota_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X)\\ &= (\mathrm{id} \otimes \mathrm{ev}_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X)\\ &= \mathrm{id}_X\end{aligned}$$

because ε_L is the unique morphism satisfying $\varepsilon_L \circ \iota_X = \mathrm{ev}_X$, as discussed in 2.4. □

The notion of a module over an algebra can also be defined in an arbitrary tensor category, in the same way that we have for comodules. The arrows in the commutative diagrams that define a module over an algebra in a tensor category are reversed relative to Definition 4.4.1.

The definition of a module homomorphism and that of a comodule homomorphism are also reversed relative to one another: If X and Y are modules in a tensor category $\mathcal{C}$, with actions $\alpha_X \colon X \otimes A \to X$ and $\alpha_Y \colon Y \otimes A \to Y$ by an algebra A in $\mathcal{C}$, then a morphism $f \colon X \to Y$ is a *module homomorphism* if it satisfies

$$f \circ \alpha_X = \alpha_Y \circ (f \otimes \mathrm{id}_A); \tag{4.19}$$

and if X and Y are comodules in $\mathcal{C}$, with coactions $\delta_X \colon X \to X \otimes C$ and $\delta_Y \colon Y \to Y \otimes C$ by a coalgebra C in $\mathcal{C}$, then a morphism $f \colon X \to Y$ is a *comodule homomorphism* if it satisfies

$$\delta_Y \circ f = (f \otimes \mathrm{id}_C) \circ \delta_X. \tag{4.20}$$

Proposition 4.4.2. *With respect to the coaction* (4.16)*, every morphism $f \colon X \to Y$ in $\mathcal{C}$ is a comodule homomorphism, i.e.,*

$$\delta_Y \circ f = (f \otimes \mathrm{id}_L) \circ \delta_X.$$

Proof. By the dinaturality of ι and Theorem 1.4.1,

$$\begin{aligned}
\delta_Y \circ f &= (\mathrm{id}_Y \otimes \iota_Y) \circ (\mathrm{coev}_Y \otimes \mathrm{id}_Y) \circ f \\
&= (\mathrm{id}_Y \otimes \iota_Y) \circ (\mathrm{id}_Y \otimes \mathrm{id}_{Y^*} \otimes f) \circ (\mathrm{coev}_Y \otimes \mathrm{id}_X) \\
&= (\mathrm{id}_Y \otimes \iota_X) \circ (\mathrm{id}_Y \otimes f^* \otimes \mathrm{id}_X) \circ (\mathrm{coev}_Y \otimes \mathrm{id}_X) \\
&= (\mathrm{id}_Y \otimes \iota_X) \circ (f \otimes \mathrm{id}_{X^*} \otimes \mathrm{id}_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X) \\
&= (f \otimes \mathrm{id}_L) \circ (\mathrm{id}_X \otimes \iota_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X) \\
&= (f \otimes \mathrm{id}_L) \circ \delta_X
\end{aligned}$$

as asserted. □

Let $\mathcal{C}$ be a tensor category with duality. It is explained in [17, p. 12] that if C is a coalgebra in $\mathcal{C}$, then ${}^*C$ is an algebra in $\mathcal{C}$; and if X is a right C-comodule with coaction δ_X, then X becomes a right ${}^*C$-module with action $\alpha_X \colon X \otimes {}^*C \to X$ defined by

$$\alpha_X = (\mathrm{id}_X \otimes \mathrm{ev}'_C) \circ (\delta_X \otimes \mathrm{id}_{{}^*C}).$$

Thus, since every object X in a braided finite tensor category $\mathcal{C}$ is a right comodule over the coend L, with coaction (4.16), every X in $\mathcal{C}$ is also a right module over $A = {}^*L$ with action

$$\alpha_X = (\mathrm{id}_X \otimes \mathrm{ev}'_L) \circ (\delta_X \otimes \mathrm{id}_A). \tag{4.21}$$

It should be remarked that a standard result for finite-dimensional vector spaces states that a right C-comodule becomes a *left* ${}^*C$-module (cf. [12, Lem. 1.6.4], [15, Prop. 3.2.2]). The apparent discrepancy arises from the dualization process of the coalgebra C: The usual convention for finite-dimensional vector spaces X and Y is to identify ${}^*(X \otimes Y)$ with ${}^*X \otimes {}^*Y$, and then the product on ${}^*C$ is simply the dual of the coproduct on C. In the categorical context, however, the dualization of the coproduct requires the isomorphism γ' defined by (1.23). Dualizing vector spaces in this way, identifying ${}^*(X \otimes Y)$ with ${}^*Y \otimes {}^*X$, effectively reverses the multiplication on ${}^*C$ and thus gives rise to a *right* ${}^*C$-module.

Just as every morphism in the category becomes a comodule homomorphism with respect to the coaction (4.16), every morphism becomes a module homomorphism with respect to the action (4.21):

Proposition 4.4.3. *With respect to the action* (4.21)*, every morphism* $f\colon X \to Y$ *in* $\mathcal{C}$ *is a module homomorphism, i.e.,*

$$f \circ \alpha_X = \alpha_Y \circ (f \otimes \mathrm{id}_A).$$

Proof. By Proposition 4.4.2,

$$\begin{aligned} f \circ \alpha_X &= f \circ (\mathrm{id}_X \otimes \mathrm{ev}'_L) \circ (\delta_X \otimes \mathrm{id}_A) \\ &= (\mathrm{id}_Y \otimes \mathrm{ev}'_L) \circ (f \otimes \mathrm{id}_L \otimes \mathrm{id}_A) \circ (\delta_X \otimes \mathrm{id}_A) \\ &= (\mathrm{id}_Y \otimes \mathrm{ev}'_L) \circ (\delta_Y \otimes \mathrm{id}_A) \circ (f \otimes \mathrm{id}_A) \\ &= \alpha_Y \circ (f \otimes \mathrm{id}_A) \end{aligned}$$

as asserted. □

In the special case, discussed in 4.2, of the dual space H^* of a finite-dimensional Hopf algebra H as a coend L in the category of finite-dimensional H-modules, we have that every object X (i.e., every finite-dimensional H-module) is a right comodule over L with coaction $\delta_X\colon X \to X \otimes L$ defined by (4.16) and a right module over $A = {}^*L$ with action $\alpha_X\colon X \otimes A \to X$ defined by (4.21). For a right dual of L, we can take $A = H$ equipped with the (left) adjoint action

$$h \cdot h' = h_{(2)} h' S_H^{-1}(h_{(1)}).$$

Then A is a right dual of L with right evaluation

$$\begin{aligned} \mathrm{ev}'_L\colon L \otimes A &\to K \\ \varphi \otimes a &\mapsto \varphi(a). \end{aligned}$$

To see that ev'_L is a morphism in the category, i.e., H-linear, recall that L is equipped with the coadjoint action (4.11) and observe that

$$\begin{aligned}\varphi(h \cdot h') &= \varphi(h_{(2)} h' S_H^{-1}(h_{(1)})) \\ &= \varphi(S_H(S_H^{-1}(h_{(2)}))h' S_H^{-1}(h_{(1)})) \\ &= \varphi(S_H(S_H^{-1}(h)_{(1)})h' S_H^{-1}(h)_{(2)}) \\ &= (S_H^{-1}(h) \cdot \varphi)(h')\end{aligned}$$

for all $\varphi \in L$ and $h, h' \in H$. By the bijectivity of S_H, this can equivalently be expressed as

$$(h \cdot \varphi)(h') = \varphi(S_H(h) \cdot h')$$

(cf. [5, Ch. XIV, p. 347]), from which it follows that

$$\begin{aligned}\mathrm{ev}'_L(h \cdot (\varphi \otimes a)) &= \mathrm{ev}'_L((h_{(1)} \cdot \varphi) \otimes (h_{(2)} \cdot a)) \\ &= (h_{(1)} \cdot \varphi)(h_{(2)} \cdot a) \\ &= \varphi(S_H(h_{(1)})h_{(2)} \cdot a) \\ &= \varepsilon_H(h)\varphi(a) \\ &= h \cdot \mathrm{ev}'_L(\varphi \otimes a),\end{aligned}$$

where we have applied the counit equation for H. In this category, we can express the coaction by L explicitly because ι_X is given by (4.13) and the coevaluation in this category is the dual basis map: Letting $\{x_i\}_i$ be a basis for X and $\{x_i^*\}_i$ be the corresponding dual basis for X^*,

$$\delta_X(x) = \sum_i x_i \otimes \iota_X(x_i^* \otimes x).$$

This further gives an explicit expression for the action $\alpha_X \colon X \otimes A \to X$ induced by this coaction:

$$\begin{aligned}\alpha_X(x \otimes a) &= (\mathrm{id}_X \otimes \mathrm{ev}'_L)(\delta_X(x) \otimes a) \\ &= \sum_i x_i \otimes \iota_X(x_i^* \otimes x)(a) \\ &= \sum_i x_i x_i^*(a \cdot x)\end{aligned}$$

and hence

$$\alpha_X(x \otimes a) = a \cdot x. \tag{4.22}$$

This shows that the right action of $a \in A$ by α_X coincides with the left action of a given by the H-module structure of X. This is consistent with the following lemma, which shows that $A = H^{\mathrm{op}}$ as an algebra.

Lemma 4.4.1. *The multiplication $m_A \colon A \otimes A \to A$ on $A = {}^*L$ is given by*

$$m_A(a \otimes b) = ba$$

for all $a \otimes b \in A \otimes A$.

Proof. The multiplication on the right dual $A = {}^*L$ is given by

$$m_A = {}^*\Delta_L \circ \gamma'_{L,L},$$

where γ' is the natural isomorphism characterized by (1.23). Thus, by the characterization (1.21) of a right dual morphism, and the fact that the coproduct on L is the same as that on H^*, we have

$$\begin{aligned}
\varphi(m_A(a \otimes b)) &= \mathrm{ev}'_L(\varphi \otimes m_A(a \otimes b)) \\
&= (\mathrm{ev}'_L \circ (\mathrm{id}_L \otimes m_A))(\varphi \otimes a \otimes b) \\
&= (\mathrm{ev}'_L \circ (\mathrm{id}_L \otimes {}^*\Delta_L) \circ (\mathrm{id}_L \otimes \gamma'_{L,L}))(\varphi \otimes a \otimes b) \\
&= (\mathrm{ev}'_{L\otimes L} \circ (\Delta_L \otimes \mathrm{id}_{{}^*(L\otimes L)}) \circ (\mathrm{id}_L \otimes \gamma'_{L,L}))(\varphi \otimes a \otimes b) \\
&= (\mathrm{ev}'_{L\otimes L} \circ (\mathrm{id}_{L\otimes L} \otimes \gamma'_{L,L}) \circ (\Delta_L \otimes \mathrm{id}_{A\otimes A}))(\varphi \otimes a \otimes b) \\
&= (\mathrm{ev}'_L \circ (\mathrm{id}_L \otimes \mathrm{ev}'_L \otimes \mathrm{id}_A) \circ (\Delta_L \otimes \mathrm{id}_{A\otimes A}))(\varphi \otimes a \otimes b) \\
&= (\mathrm{ev}'_L \circ (\mathrm{id}_L \otimes \mathrm{ev}'_L \otimes \mathrm{id}_A))(\varphi_{(1)} \otimes \varphi_{(2)} \otimes a \otimes b) \\
&= \mathrm{ev}'_L(\varphi_{(1)} \otimes \varphi_{(2)}(a) b) \\
&= \varphi_{(1)}(b)\varphi_{(2)}(a) \\
&= \varphi(ba).
\end{aligned}$$

Since this is true for an arbitrary $\varphi \in L$, this implies that $m_A(a \otimes b) = ba$. □

In the case $X = L$, the relation (4.22) shows that the right action $\alpha_L \colon L \otimes A \to L$ is given by

$$\alpha_L(\varphi \otimes a) = \varphi_{(1)}(S(a_{(1)}))\varphi_{(3)}(a_{(2)})\varphi_{(2)},$$

because the coadjoint action (4.11) on L can be expressed as

$$a \cdot \varphi = \varphi_{(1)}(S(a_{(1)}))\varphi_{(3)}(a_{(2)})\varphi_{(2)}.$$

Similarly, (4.22) shows that the right action $\alpha_A\colon A \otimes A \to A$ is given by

$$\alpha_A(a' \otimes a) = a_{(2)} a' S^{-1}(a_{(1)}). \tag{4.23}$$

4.5 Yetter-Drinfel'd Hopf algebras

We now study the concept of a Yetter-Drinfel'd Hopf algebra, which will play a role in the proof of the theorem in the next section. Let H be a Hopf algebra over a field K, and let X be a left H-comodule with coaction $\delta_X\colon X \to H \otimes X$. We use the following Sweedler notation for the coaction:

$$\delta_X(x) = x^{(1)} \otimes x^{(2)} \in H \otimes X.$$

As defined in [18, Par. 1.1, p. 7], a left H-comodule X that is also a left H-module and satisfies the condition

$$\delta_X(h \cdot x) = h_{(1)} x^{(1)} S_H(h_{(3)}) \otimes h_{(2)} \cdot x^{(2)}$$

for all $h \in H$ and $x \in X$ is called a *left Yetter-Drinfel'd module*, and a right H-comodule X that is also a right H-module and satisfies the condition

$$\delta_X(x \cdot h) = x^{(1)} \cdot h_{(2)} \otimes S_H(h_{(1)}) x^{(2)} h_{(3)}$$

for all $h \in H$ and $x \in X$ is called a *right Yetter Drinfel'd module*. There also exist notions of left-right and right-left Yetter-Drinfel'd modules, but we will not need them in what follows.

Left Yetter-Drinfel'd modules over H together with morphisms that are both H-linear and H-colinear form a category, which we denote by ${}^H_H\mathcal{YD}$. This category is a tensor category: The tensor product of two left Yetter-Drinfel'd modules X and Y over H is again a left Yetter-Drinfel'd module over H, with the diagonal action $\alpha_{X\otimes Y}\colon H \otimes X \otimes Y \to X \otimes Y$ defined by

$$\alpha_{X\otimes Y}(h \otimes x \otimes y) = h_{(1)} \cdot x \otimes h_{(2)} \cdot y$$

and the codiagonal coaction $\delta_{X\otimes Y}\colon X \otimes Y \to H \otimes X \otimes Y$ defined by

$$\delta_{X\otimes Y}(x \otimes y) = x^{(1)} y^{(1)} \otimes x^{(2)} \otimes y^{(2)}.$$

The unit object is the base field K, which, like every vector space, is a left Yetter-Drinfel'd module when equipped with the trivial module action $h \cdot \lambda = \varepsilon_H(h)\lambda$ and trivial comodule coaction $\delta_K(\lambda) = 1_H \otimes \lambda$. Moreover, the quasisymmetry $\sigma_{X,Y} \colon X \otimes Y \to Y \otimes X$ defined by

$$\sigma_{X,Y}(x \otimes y) = x^{(1)} \cdot y \otimes x^{(2)} \tag{4.24}$$

is bijective if H has a bijective antipode, and therefore defines a braiding on ${}^H_H\mathcal{YD}$.

If H is quasitriangular, with R-matrix $R = \sum_{i=1}^n a_i \otimes b_i$, then any left H-module X is a left Yetter-Drinfel'd module over H when equipped with the left coaction $\delta_X \colon X \to H \otimes X$ defined, as in [4, Par. 1.3, p. 94] and [12, Prop. 10.6.7, p. 211], by

$$\delta_X(x) = \sum_{i=1}^n b_i \otimes a_i \cdot x.$$

See also [11]. If we equip a second H-module Y with this H-comodule structure, then any H-linear map from X to Y is also H-colinear with respect to this coaction. This coaction defined on $X \otimes Y$, viewed as an H-module with the diagonal action, coincides with the codiagonal coaction on $X \otimes Y$. Furthermore, the quasisymmetry (4.24) coincides with the quasisymmetry (4.1) on the category of left H-modules. This assignment from left modules over H to left Yetter-Drinfel'd modules over H thus defines a *strict braided tensor functor*, which is a functor that preserves the tensor product and braiding.

A Hopf algebra in ${}^H_H\mathcal{YD}$ is called a *Yetter-Drinfel'd Hopf algebra*. Since the functor described above preserves the tensor product and braiding, the structure morphisms for a Hopf algebra A in the category of left modules over a quasitriangular Hopf algebra H also satisfy the axioms of a Hopf algebra in the category of left Yetter-Drinfel'd modules, and is therefore a Yetter-Drinfel'd Hopf algebra.

As noted in [18, Lem. 1.2, p. 9], the fact that left Yetter-Drinfel'd modules over H are the same as right Yetter-Drinfel'd modules over $H^{\mathrm{op\,cop}}$ implies that if A is a left Yetter-Drinfel'd Hopf algebra over H, then $A^{\mathrm{op\,cop}}$ is a right Yetter-Drinfel'd Hopf algebra over $H^{\mathrm{op\,cop}}$. Here, the opposite multiplication and coopposite comultiplication of A are defined in the ordinary sense, as

$$m_{A^{\mathrm{op}}}(a \otimes b) = ba$$

and

$$\Delta_{A^{\mathrm{cop}}}(a) = a_{(2)} \otimes a_{(1)}.$$

For the coaction on $A^{\mathrm{op\,cop}}$, we use Sweedler indices with square brackets:

$$\delta_{A^{\mathrm{op\,cop}}}(a) = a^{[1]} \otimes a^{[2]} = a^{(2)} \otimes a^{(1)}$$

Now applying [18, Lem. 1.3, p. 10], we have that if H is finite-dimensional, then $A^{\mathrm{op\,cop}}$ is a left Yetter-Drinfel'd module over $(H^{\mathrm{op\,cop}})^*$ with action

$$\alpha_{A^{\mathrm{op\,cop}}}(\varphi \otimes a) = a^{[1]}\varphi(a^{[2]}) = a^{(2)}\varphi(a^{(1)}), \tag{4.25}$$

and coaction

$$\delta_{A^{\mathrm{op\,cop}}}(a) = a^{\{1\}} \otimes a^{\{2\}} = \sum_i h_i^* \otimes a \cdot h_i, \tag{4.26}$$

where $\{h_i\}_i$ is a basis of $H^{\mathrm{op\,cop}}$. Observe that we can equivalently express this coaction by the equation

$$a^{\{1\}}(h)a^{\{2\}} = a \cdot h,$$

which is dual to the action. Finally, we note that the assignment from right Yetter-Drinfel'd modules over a finite-dimensional Hopf algebra to left Yetter-Drinfel'd modules over its dual given in [18, Lem. 1.3, p. 10] again defines a strict braided tensor functor. This implies that the left Yetter-Drinfel'd module $A^{\mathrm{op\,cop}}$ over $(H^{\mathrm{op\,cop}})^*$ is in fact a left Yetter-Drinfel'd Hopf algebra in the category ${}^{(H^{\mathrm{op\,cop}})^*}_{(H^{\mathrm{op\,cop}})^*}\mathcal{YD}$. In summary, we have the following lemma.

Lemma 4.5.1. *If A is a Hopf algebra in the category of modules over a finite-dimensional quasitriangular Hopf algebra H, then A is a left Yetter-Drinfel'd Hopf algebra over H and $A^{\mathrm{op\,cop}}$ is a left Yetter-Drinfel'd Hopf algebra over $(H^{\mathrm{op\,cop}})^*$.*

We will also need the following definitions: A *left integral* in a Yetter-Drinfel'd Hopf algebra A is an element $\Lambda \in A$ such that

$$a\Lambda = \varepsilon_A(a)\Lambda$$

for all $a \in A$, and a *right integral* in A is an element $\Gamma \in A$ such that

$$\Gamma a = \varepsilon_A(a)\Gamma$$

for all $a \in A$. It is proved in [19, Prop. 2.10, p. 432] that every finite-dimensional Yetter-Drinfel'd Hopf algebra contains non-zero left and right integrals, which are unique up to scalar multiples, and that there exists a character $\iota_A \colon H \to K$ (i.e., an algebra homomorphism to the base field) and a grouplike element $g_A \in H$, known respectively as the *integral character* and *integral group element* of A, satisfying the following properties:

$$h \cdot \Lambda_A = \iota_A(h)\Lambda_A, \qquad \delta_A(\Lambda_A) = g_A \otimes \Lambda_A$$
$$h \cdot \Gamma_A = \iota_A(h)\Gamma_A, \qquad \delta_A(\Gamma_A) = g_A \otimes \Gamma_A$$

4.6 Non-degeneracy and triviality of the Müger centre

We are now ready to prove the main result of this chapter. Let H be a finite-dimensional quasitriangular ribbon Hopf algebra, with R-matrix $R = \sum_{i=1}^{n} a_i \otimes b_i$ and ribbon element v. Let $\mathcal{C}$ be the category of finite-dimensional left H-modules. Recall that the dual space H^* is a coend in $\mathcal{C}$, and hence a Hopf algebra in $\mathcal{C}$ with the structure morphisms discussed in 4.3. We denote this coend by L to distinguish it from the usual dual Hopf algebra H^*. Let $A = {}^*L$, and note that A is again a Hopf algebra in $\mathcal{C}$ with the structure morphisms defined as in 3.1. We know that the morphism $\omega''\colon L \to A$ defined by

$$\omega'' = (\mathrm{id}_A \otimes \omega) \circ (\mathrm{coev}'_L \otimes \mathrm{id}_L) \tag{4.27}$$

as in (3.9) is a Hopf algebra homomorphism, and therefore $B = \mathrm{im}(\omega'')$ is a Hopf subalgebra of A by the discussion in 3.2. We prove that if the Müger center of $\mathcal{C}$ is trivial, then ω'' is an isomorphism, which means by definition that the Hopf pairing ω is non-degenerate.

We first prove several lemmas. The first is a relation between the counits of ${}^*L$ and L, which is true by virtue of ω'' being a Hopf algebra homomorphism, but we give a direct proof.

Lemma 4.6.1. *Let $A = {}^*L$ and let $\omega''\colon L \to A$ be defined by* (4.27)*. Then*

$$\varepsilon_A \circ \omega'' = \varepsilon_L.$$

Proof. The counit ε_A is the right dual of the unit u_L of L. By the the characterization (1.21) of a right dual morphism, this is equivalent to

$$\mathrm{ev}'_L \circ (u_L \otimes \mathrm{id}_A) = \mathrm{ev}'_I \circ (\mathrm{id}_I \otimes \varepsilon_A) = \varepsilon_A,$$

where we have used the fact that ev'_I is the left, and right, unit constraint. Thus

$$\begin{aligned}
\varepsilon_A \circ \omega'' &= \mathrm{ev}'_L \circ (u_L \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \omega) \circ (\mathrm{coev}'_L \otimes \mathrm{id}_L) \\
&= \mathrm{ev}'_L \circ (\mathrm{id}_L \otimes \mathrm{id}_A \otimes \omega) \circ (u_L \otimes \mathrm{coev}'_L \otimes \mathrm{id}_L) \\
&= \omega \circ (\mathrm{ev}'_L \otimes \mathrm{id}_{L\otimes L}) \circ (\mathrm{id}_L \otimes \mathrm{coev}'_L \otimes \mathrm{id}_L) \circ (u_L \otimes \mathrm{id}_L) \\
&= \omega \circ (u_L \otimes \mathrm{id}_L),
\end{aligned}$$

where we have used Definition 1.3.1 of a right dual ${}^*L$. This is equal to ε_L by the property (2.13) of the Hopf pairing ω. □

From Lemma 4.6.1 we also obtain the following relation.

Lemma 4.6.2. *Let* Γ *be a right integral of* $B = \operatorname{im}(\omega'')$*. Then for all* $a \in A$ *and* $\varphi \in L$*,*

$$\Gamma a \omega''(\varphi) = \Gamma a \varepsilon_B(\omega''(\varphi)). \tag{4.28}$$

Proof. For any $a \in A$ and $\varphi \in L$ we have

$$\Gamma a \omega''(\varphi) = \Gamma a_{(2)} \varepsilon_A(a_{(1)}) \omega''(\varphi) = \Gamma a_{(3)} \omega''(\varphi) S^{-1}(a_{(2)}) a_{(1)}$$

by the counit equation and skew-antipode equation. Recall from Proposition 4.4.3 that the action defined by (4.21) makes every morphism in $\mathcal{C}$ an A-module homomorphism, and recall that the action of A on itself is given by (4.23). Therefore, we have

$$\Gamma a \omega''(\varphi) = \Gamma \alpha_A(\omega''(\varphi) \otimes a_{(2)}) a_{(1)} = \Gamma \omega''(\alpha_L(\varphi \otimes a_{(2)})) a_{(1)}.$$

But for any $\varphi \in L$, we have by Lemma 4.6.1 that

$$\Gamma \omega''(\varphi) = \Gamma \varepsilon_B(\omega''(\varphi)) = \Gamma \varepsilon_A(\omega''(\varphi)) = \Gamma \varepsilon_L(\varphi).$$

Therefore, using the fact that Γ is a right integral and the A-linearity of ω'' and ε_B,

$$\begin{aligned}
\Gamma \omega''(\alpha_L(\varphi \otimes a_{(2)})) a_{(1)} &= \Gamma \varepsilon_B(\omega''(\alpha_L(\varphi \otimes a_{(2)}))) a_{(1)} \\
&= \Gamma \alpha_K(\varepsilon_B(\omega''(\varphi)) \otimes a_{(2)}) a_{(1)} \\
&= \Gamma (a_{(2)} \cdot \varepsilon_B(\omega''(\varphi)) a_{(1)} \\
&= \Gamma \varepsilon_B(\omega''(\varphi)) \varepsilon_A(a_{(2)}) a_{(1)} \\
&= \Gamma a \varepsilon_B(\omega''(\varphi)),
\end{aligned}$$

where we have also used the relation between the right action by A and the H-module structure given in (4.22). □

In Lemma 4.6.2 the multiplication was carried out in A. Recalling that $A = H^{\mathrm{op}}$ as an algebra by Lemma 4.4.1, the result states that

$$\omega''(\varphi) a \Gamma = \varepsilon_H(\omega''(\varphi)) a \Gamma \tag{4.29}$$

in terms of the multiplication in H.

In the category that we are currently considering, we have the following expression for ω.

Lemma 4.6.3. *The Hopf pairing ω for the coend L can be expressed by*

$$\omega(\varphi \otimes \psi) = \sum_{i,j} \varphi(b_j a_i)\psi(S_H(a_j b_i)) \tag{4.30}$$

for all $\varphi, \psi \in L$, where S_H is the antipode of H.

Proof. Recall that ω is the unique morphism satisfying

$$\omega \circ (\iota_X \otimes \iota_Y) = (\mathrm{ev}_X \otimes \mathrm{ev}_Y) \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Y^*,X} \circ \sigma_{X,Y^*}) \otimes \mathrm{id}_Y)$$

and that, in this category, σ is the braiding defined by (4.1). The coend L has the universal dinatural transformation ι defined by (4.13). Thus

$$\begin{aligned}
&(\omega \circ (\iota_X \otimes \iota_Y))(\varphi \otimes x \otimes \psi \otimes y) \\
&= ((\mathrm{ev}_X \otimes \mathrm{ev}_Y) \circ (\mathrm{id}_{X^*} \otimes (\sigma_{Y^*,X} \circ \sigma_{X,Y^*}) \otimes \mathrm{id}_Y))(\varphi \otimes x \otimes \psi \otimes y) \\
&= \sum_{i,j} (\mathrm{ev}_X \otimes \mathrm{ev}_Y)(\varphi \otimes (b_j a_i \cdot x) \otimes (a_j b_i \cdot \psi) \otimes y) \\
&= \sum_{i,j} \varphi(b_j a_i \cdot x)(a_j b_i \cdot \psi)(y) \\
&= \sum_{i,j} \varphi(b_j a_i \cdot x)\psi(S_H(a_j b_i) \cdot y),
\end{aligned}$$

where we have applied the action (4.6). But the map $f \colon L \otimes L \to K$ defined by

$$f(\varphi \otimes \psi) = \sum_{i,j} \varphi(b_j a_i)\psi(S_H(a_j b_i))$$

also satisfies

$$\begin{aligned}
(f \circ (\iota_X \otimes \iota_Y))(\varphi \otimes x \otimes \psi \otimes y) &= f(\iota_X(\varphi \otimes x) \otimes \iota_Y(\psi \otimes y)) \\
&= \sum_{i,j} (\iota_X(\varphi \otimes x))(b_j a_i)(\iota_Y(\psi \otimes y))(S_H(a_j b_i)) \\
&= \sum_{i,j} \varphi(b_j a_i \cdot x)\psi(S_H(a_j b_i) \cdot y).
\end{aligned}$$

Since f and ω are in particular linear maps over K and every $\varphi \in L$ can be expressed as $\iota_H(\varphi \otimes 1_H)$, where H has the left regular representation, this implies that $f = \omega$ by universality of $\iota_X \otimes \iota_Y$. $\square$

We now prove the main result.

Theorem 4.6.1. *If the Müger centre of $\mathcal{C}$ is trivial, then ω is non-degenerate.*

Proof. Let Γ be a right integral of $B = \mathrm{im}(\omega'')$. Then, by (4.29), we have

$$\omega''(\varphi)a\Gamma = \varepsilon_H(\omega''(\varphi))a\Gamma \tag{4.31}$$

for all $a \in A$ and $\varphi \in L$. Making the identification $L = ({}^*L)^* = A^*$, the right coevaluation map $\mathrm{coev}'_L\colon K \to A \otimes L$ can be expressed as

$$\mathrm{coev}'_L(1) = \sum_i x_i \otimes x_i^*$$

where $\{x_i\}_i$ is a basis of A and $\{x_i^*\}_i$ is the corresponding dual basis of L. By Lemma 4.6.3, we therefore have

$$\begin{aligned}\omega''(\varphi) &= \sum_k x_k \omega(x_k^* \otimes \varphi)\\ &= \sum_{i,j,k} x_k x_k^*(b_j a_i)\varphi(S_H(a_j b_i))\\ &= \sum_{i,j} b_j a_i \varphi(S_H(a_j b_i))\end{aligned}$$

for all $\varphi \in L$. Now substituting this expression for $\omega''(\varphi)$ into (4.31), and using the fact that ε_H is an algebra homomorphism and has the property

$$(\varepsilon_H \otimes \mathrm{id}_H)(R) = (\mathrm{id}_H \otimes \varepsilon_H)(R) = 1$$

by Lemma 4.1.1, we have

$$\begin{aligned}\sum_{i,j} b_j a_i \varphi(S_H(a_j b_i))a\Gamma &= \sum_{i,j} \varepsilon_H(b_j a_i)\varphi(S_H(a_j b_i))a\Gamma\\ &= \varphi\Big(S_H\Big(\sum_{i,j} a_j \varepsilon_H(b_j)\varepsilon_H(a_i)b_i\Big)\Big)a\Gamma\\ &= \varphi(S_H(1))a\Gamma\\ &= \varphi(1)a\Gamma.\end{aligned}$$

This implies that

$$\sum_{i,j} b_j a_i a\Gamma \otimes S_H(a_j b_i) = a\Gamma \otimes 1,$$

which further implies, by the bijectivity of S_H, that

$$\sum_{i,j} b_j a_i a\Gamma \otimes a_j b_i = a\Gamma \otimes 1. \tag{4.32}$$

Now consider the left H-module $X = H\Gamma$, and let Y be a finite-dimensional left H-module. Then (4.32) implies that for any $a\Gamma \in X$ and $y \in Y$,

$$\begin{aligned}(\sigma_{Y,X} \circ \sigma_{X,Y})(a\Gamma \otimes y) &= \sum_{ij} b_j a_i \cdot a\Gamma \otimes a_j b_i \cdot y \\ &= a\Gamma \otimes y\end{aligned}$$

and hence $\sigma_{Y,X} \circ \sigma_{X,Y} = \mathrm{id}_{X \otimes Y}$ for all Y in $\mathcal{C}$, which means that X is in the Müger centre of $\mathcal{C}$.

Now assume that the Müger centre is trivial. Then X is a direct sum of finitely many copies of the unit object, which is the base field K equipped with the trivial action. By linearity of the action, this means that for all $a \in A$ and $a'\Gamma \in X$,

$$a \cdot (a'\Gamma) = \varepsilon_H(a) a'\Gamma.$$

In particular,

$$a\Gamma = \varepsilon_H(a)\Gamma$$

for all $a \in A$, which means that Γ is a left integral of H. It follows that Γ is a right integral of A since $A = H^{\mathrm{op}}$ as an algebra and

$$\varepsilon_A = {}^*u_L = {}^*(\varepsilon_H^*) = \varepsilon_H.$$

Thus, every right integral of B is a right integral of A. Now recall that $A^{\mathrm{op\,cop}}$ is a left Yetter-Drinfel'd Hopf algebra over $(H^{\mathrm{op\,cop}})^*$ by Lemma 4.5.1. Furthermore, every right integral of A is a left integral of $A^{\mathrm{op\,cop}}$, so by [19, Prop. 2.10, p. 432] any $a \in A$ can be expressed as

$$a = \rho_{A^{\mathrm{op\,cop}}}(m^{\mathrm{op}}(a \otimes \Gamma_{[1]}))\iota_{A^{\mathrm{op\,cop}}}(S_{(H^{\mathrm{op\,cop}})^*}({\Gamma_{[2]}}^{\{1\}}))S_{A^{\mathrm{op\,cop}}}({\Gamma_{[2]}}^{\{2\}}),$$

where $\rho_{A^{\mathrm{op\,cop}}}$ is a right integral of $(A^{\mathrm{op\,cop}})^*$ and $\iota_{A^{\mathrm{op\,cop}}}$ is the integral character of $A^{\mathrm{op\,cop}}$, and we have denoted the Sweedler indices for the coproduct with square brackets to distinguish them from those for A. Now since $S_A = S_{A^{\mathrm{op\,cop}}}$ and $S_H = S_{H^{\mathrm{op\,cop}}}$, and $S_{H^*} = S_H^*$, we have

$$a = \rho_{A^{\mathrm{op\,cop}}}(\Gamma_{(2)}a)\iota_{A^{\mathrm{op\,cop}}}(S_{H^*}({\Gamma_{(1)}}^{\{1\}}))S_A({\Gamma_{(1)}}^{\{2\}}).$$

Next, recall that $H^{\mathrm{op\,cop}}$ acts on $A^{\mathrm{op\,cop}}$ by (4.25). Thus, noting that Γ is a right integral for A, we have as a consequence of [19, Prop. 2.10(2), p. 432], applied to both $A^{\mathrm{op\,cop}}$ and A, that

$$\iota_{A^{\mathrm{op\,cop}}}(\varphi)\Gamma = \varphi \cdot \Gamma = \Gamma^{[1]}\varphi(\Gamma^{[2]}) = \Gamma^{(2)}\varphi(\Gamma^{(1)}) = \Gamma\varphi(g_A),$$

where g_A is the integral group element of A. This implies that

$$\iota_{A^{\mathrm{op\,cop}}}(\varphi) = \varphi(g_A).$$

Therefore, recalling (4.26), and the fact that S_A is H-linear, we have

$$\begin{aligned}
a &= \rho_{A^{\mathrm{op\,cop}}}(\Gamma_{(2)}a)(S_{H^*}({\Gamma_{(1)}}^{\{1\}})(g_A))S_A({\Gamma_{(1)}}^{\{2\}}) \\
&= \rho_{A^{\mathrm{op\,cop}}}(\Gamma_{(2)}a)\sum_i (S_{H^*}(h_i^*)(g_A))S_A(\Gamma_{(1)} \cdot h_i) \\
&= \rho_{A^{\mathrm{op\,cop}}}(\Gamma_{(2)}a)\sum_i h_i^*(S_H(g_A))S_A(\Gamma_{(1)} \cdot h_i) \\
&= \rho_{A^{\mathrm{op\,cop}}}(\Gamma_{(2)}a)S_A\Big(\Gamma_{(1)} \cdot \sum_i h_i^*(g_A^{-1})h_i\Big) \\
&= \rho_{A^{\mathrm{op\,cop}}}(\Gamma_{(2)}a)S_A(\Gamma_{(1)} \cdot g_A^{-1}) \\
&= \rho_{A^{\mathrm{op\,cop}}}(\Gamma_{(2)}a)S_A(g_A^{-1} \cdot \Gamma_{(1)})
\end{aligned}$$

where $\{h_i\}_i$ is a basis of H, and we have used the fact that the right action by $H^{\mathrm{op\,cop}}$ is equal to the left action by H (using $\cdot$ for both actions). Since Γ is an element of B, and since the antipode of B is S_A restricted to B, the right hand side of this equation is an element of B. This shows that $A \subseteq B$, and hence $A = B$. Since ω'' is in particular a linear map between finite-dimensional vector spaces, it follows that ω'' is an isomorphism. $\square$

Chapter 5

A non-universal dual space

In this chapter, we construct an example of a braided finite tensor category containing a Hopf algebra A whose dual A^* with the coadjoint action fails to be a coend in the category of A-modules. This non-example shows that there are certain hidden properties that are automatically satisfied in the category discussed in 4.2. This construction involves the notion of a $\mathcal{C}$-category, which can be thought of as a category with an action by a tensor category.

5.1 Categories over a tensor category

Let $\mathcal{C}$ be a tensor category. We define, as in [14, 2.1, p. 94-96], a $\mathcal{C}$*-category* as a category $\mathcal{D}$ together with a functor

$$\otimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{D}$$

and natural isomorphisms

$$\beta_{X,Y,Z}\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

and

$$\pi_Z\colon I \otimes Z \to Z,$$

for objects X and Y in $\mathcal{C}$ and Z in $\mathcal{D}$, where $\otimes$ also denotes the tensor product of $\mathcal{C}$. For two $\mathcal{C}$-categories $(\mathcal{D}, \otimes)$ and $(\mathcal{D}', \otimes)$, a functor $\omega\colon \mathcal{D} \to \mathcal{D}'$ together with a natural isomorphism

$$\nu_{X,Z}\colon \omega(X \otimes Z) \to X \otimes \omega(Z),$$

for objects X in $\mathcal{C}$ and Z in $\mathcal{D}$, is called a *$\mathcal{C}$-functor*. For two $\mathcal{C}$-functors (ω, ν) and (ω', ν'), a natural transformation $\varphi\colon \omega \to \omega'$ is called a *$\mathcal{C}$-transformation* (or a *$\mathcal{C}$-morphism*) if the diagram

$$\begin{array}{ccc} \omega(X \otimes Z) & \xrightarrow{\varphi(X\otimes Z)} & \omega'(X \otimes Z) \\ \downarrow{\scriptstyle \nu_{X,Z}} & & \downarrow{\scriptstyle \nu'_{X,Z}} \\ X \otimes \omega(Z) & \xrightarrow{\mathrm{id}_X \otimes \varphi(Z)} & X \otimes \omega'(Z) \end{array}$$

commutes.

Any tensor category $\mathcal{C}$ is a $\mathcal{C}$-category. Another example is the category $\mathcal{C}^C$ of right comodules over a coalgebra C in $\mathcal{C}$. As explained in [14, 2.1.6, p. 95], the category $\mathcal{C}^C$ is a $\mathcal{C}$-category because if X is an object in $\mathcal{C}$ and Y in an object in $\mathcal{C}^C$ with coaction δ_Y, then $X \otimes Y$ is a right comodule over C with coaction $\delta_{X\otimes Y} = \mathrm{id}_X \otimes \delta_Y$.

For two $\mathcal{C}$-functors ω and ω', we denote the set of natural transformations from ω to ω' by $\mathrm{Nat}(\omega, \omega')$, and the subset of $\mathcal{C}$-transformations by $\mathrm{Nat}_{\mathcal{C}}(\omega, \omega')$, as in [14, Def. 2.3, p. 96]. Let $\mathcal{C}$ be a tensor category and let C be a coalgebra in $\mathcal{C}$. Let

$$\omega\colon \mathcal{C}^C \to \mathcal{C}$$

denote the *forgetful functor*, which sends each right comodule over C to its underlying object. We have the following proposition.

Proposition 5.1.1. *Let $Z \in \mathcal{C}$, and suppose that $g\colon C \to Z$ is a morphism in $\mathcal{C}$. For each object X in $\mathcal{C}^C$, let $\delta_X\colon X \to X \otimes C$ denote its coaction. Then the collection of morphisms $\tilde{\delta}_X\colon \omega(X) \to \omega(X) \otimes Z$ defined by*

$$\tilde{\delta}_X = (\mathrm{id}_X \otimes g) \circ \delta_X \tag{5.1}$$

is a natural transformation $\tilde{\delta}\colon \omega \to \omega \otimes Z$, where $\omega \otimes Z\colon \mathcal{C}^C \to \mathcal{C}$ is the functor defined on objects by $X \mapsto \omega(X) \otimes Z$.

Proof. Naturality of $\tilde{\delta}$ means that the diagram

$$\begin{array}{ccc} \omega(X) & \xrightarrow{\tilde{\delta}_X} & \omega(X) \otimes Z \\ \downarrow{\scriptstyle \omega(f)} & & \downarrow{\scriptstyle \omega(f)\otimes \mathrm{id}_Z} \\ \omega(Y) & \xrightarrow{\tilde{\delta}_Y} & \omega(Y) \otimes Z \end{array}$$

commutes for any morphism f in $\mathcal{C}^C$. Since f is a comodule homomorphism, and $\omega(f)$ is simply the morphism f viewed as a morphism in $\mathcal{C}$, we have

$$\begin{aligned}
\tilde{\delta}_Y \circ \omega(f) &= (\mathrm{id}_Y \otimes g) \circ \delta_Y \circ \omega(f) \\
&= (\mathrm{id}_Y \otimes g) \circ (\omega(f) \otimes \mathrm{id}_C) \circ \delta_X \\
&= (\omega(f) \otimes \mathrm{id}_Z) \circ (\mathrm{id}_X \otimes g) \circ \delta_X \\
&= (\omega(f) \otimes \mathrm{id}_Z) \circ \tilde{\delta}_X
\end{aligned}$$

as required. □

Both ω and $\omega \otimes Z$ are $\mathcal{C}$-functors, as observed in [14, 2.1.8, p. 95] and [14, 2.1.9, p. 96]. The natural transformation $\tilde{\delta}\colon \omega \to \omega \otimes Z$, defined as in (5.1) for some morphism $g\colon C \to Z$ in $\mathcal{C}$, is a $\mathcal{C}$-transformation because

$$\tilde{\delta}_{X\otimes Y} = (\mathrm{id}_{X\otimes Y} \otimes g) \circ \delta_{X\otimes Y} = (\mathrm{id}_{X\otimes Y} \otimes g) \circ (\mathrm{id}_X \otimes \delta_Y) = \mathrm{id}_X \otimes \tilde{\delta}_Y$$

for all $X \in \mathcal{C}$ and $Y \in \mathcal{C}^C$. The next proposition shows that the converse is also true.

Proposition 5.1.2. *If $\tilde{\delta}\colon \omega \to \omega \otimes Z$ is a $\mathcal{C}$-transformation, then there exists a morphism $g\colon C \to Z$ in $\mathcal{C}$ such that*

$$\tilde{\delta}_X = (\mathrm{id}_X \otimes g) \circ \delta_X \tag{5.2}$$

for all X in $\mathcal{C}^C$.

Proof. Suppose that $\tilde{\delta}\colon \omega \to \omega \otimes Z$ is a $\mathcal{C}$-transformation. Note that C is in $\mathcal{C}^C$ with coaction $\delta_C = \Delta_C$, the coproduct on C. Then $X \otimes C$ is in $\mathcal{C}^C$ for any X in $\mathcal{C}$, with coaction $\delta_{X\otimes C} = \mathrm{id}_X \otimes \delta_C = \mathrm{id}_X \otimes \Delta_C$. With these coactions, $\delta_X\colon X \to X \otimes C$ is a comodule homomorphism, because

$$\delta_{X\otimes C} \circ \delta_X = (\mathrm{id}_X \otimes \delta_C) \circ \delta_X = (\mathrm{id}_X \otimes \Delta_C) \circ \delta_X = (\delta_X \otimes \mathrm{id}_C) \circ \delta_X$$

by the comodule axiom (4.14) for X, and hence δ_X is a morphism in $\mathcal{C}^C$. Therefore, by the naturality of $\tilde{\delta}$, we have that the diagram

$$\begin{array}{ccc}
X & \xrightarrow{\tilde{\delta}_X} & X \otimes Z \\
\Big\downarrow{\scriptstyle \delta_X} & & \Big\downarrow{\scriptstyle \delta_X \otimes \mathrm{id}_Z} \\
X \otimes C & \xrightarrow{\tilde{\delta}_{X\otimes C}} & X \otimes C \otimes Z
\end{array}$$

commutes. Since $\tilde{\delta}$ is a $\mathcal{C}$-transformation, we also have $\tilde{\delta}_{X\otimes C} = \mathrm{id}_X \otimes \tilde{\delta}_C$. Therefore

$$(\delta_X \otimes \mathrm{id}_Z) \circ \tilde{\delta}_X = (\mathrm{id}_X \otimes \tilde{\delta}_C) \circ \delta_X,$$

which implies

$$(\mathrm{id}_X \otimes \varepsilon_C \otimes \mathrm{id}_Z) \circ (\delta_X \otimes \mathrm{id}_Z) \circ \tilde{\delta}_X = (\mathrm{id}_X \otimes \varepsilon_C \otimes \mathrm{id}_Z) \circ (\mathrm{id}_X \otimes \tilde{\delta}_C) \circ \delta_X.$$

Thus, letting

$$g = (\varepsilon_C \otimes \mathrm{id}_Z) \circ \tilde{\delta}_C,$$

and applying the comodule axiom (4.15) for X, we have

$$\tilde{\delta}_X = (\mathrm{id}_X \otimes g) \circ \delta_X$$

as asserted. □

We now consider a braided category $\mathcal{C}$, with braiding σ, and a Hopf algebra H in $\mathcal{C}$.

Lemma 5.1.1. *If X and Y are objects in $\mathcal{C}^H$ with respective coactions δ_X and δ_Y, then*

$$\delta_{X\otimes Y} = (\mathrm{id}_{X\otimes Y} \otimes m_H) \circ (\mathrm{id}_X \otimes \sigma_{H,Y} \otimes \mathrm{id}_H) \circ (\delta_X \otimes \delta_Y)$$

defines a right coaction on $X \otimes Y$ by H, and hence $X \otimes Y$ is an object in $\mathcal{C}^H$. Furthermore, if f and g are morphisms in $\mathcal{C}^H$, then $f \otimes g$ is a morphism in $\mathcal{C}^H$, and hence $\mathcal{C}^H$ is a tensor category.

Proof. By the naturality of σ and the coassociativity comodule axiom (4.14) for δ_X and δ_Y,

$$\begin{aligned}
&(\delta_{X\otimes Y} \otimes \mathrm{id}_H) \circ \delta_{X\otimes Y} \\
&= (\mathrm{id}_{X\otimes Y} \otimes m_H \otimes \mathrm{id}_H) \circ (\mathrm{id}_X \otimes \sigma_{H,Y} \otimes \mathrm{id}_H \otimes \mathrm{id}_H) \circ (\delta_X \otimes \delta_Y \otimes \mathrm{id}_H) \circ (\mathrm{id}_{X\otimes Y} \otimes m_H) \\
&\qquad \circ (\mathrm{id}_X \otimes \sigma_{H,Y} \otimes \mathrm{id}_H) \circ (\delta_X \otimes \delta_Y) \\
&= (\mathrm{id}_{X\otimes Y} \otimes m_H \otimes m_H) \circ (\mathrm{id}_X \otimes \sigma_{H,Y} \otimes \mathrm{id}_H \otimes \mathrm{id}_{H\otimes H}) \circ (\delta_X \otimes \delta_Y \otimes \mathrm{id}_{H\otimes H}) \\
&\qquad \circ (\mathrm{id}_X \otimes \sigma_{H,Y} \otimes \mathrm{id}_H) \circ (\delta_X \otimes \delta_Y) \\
&= (\mathrm{id}_{X\otimes Y} \otimes m_H \otimes m_H) \circ (\mathrm{id}_X \otimes \sigma_{H,Y} \otimes \mathrm{id}_H \otimes \mathrm{id}_{H\otimes H}) \circ (\mathrm{id}_X \otimes \mathrm{id}_H \otimes \sigma_{H,Y\otimes H} \otimes \mathrm{id}_H) \\
&\qquad \circ (\delta_X \otimes \mathrm{id}_H \otimes \delta_Y \otimes \mathrm{id}_H) \circ (\delta_X \otimes \delta_Y) \\
&= (\mathrm{id}_{X\otimes Y} \otimes m_H \otimes m_H) \circ (\mathrm{id}_X \otimes \sigma_{H,Y} \otimes \mathrm{id}_H \otimes \mathrm{id}_{H\otimes H}) \circ (\mathrm{id}_X \otimes \mathrm{id}_H \otimes \sigma_{H,Y\otimes H} \otimes \mathrm{id}_H) \\
&\qquad \circ (\mathrm{id}_X \otimes \Delta_H \otimes \mathrm{id}_Y \otimes \Delta_H) \circ (\delta_X \otimes \delta_Y).
\end{aligned}$$

By the braiding axiom (1.4), this equals

$$
\begin{aligned}
&(\mathrm{id}_{X\otimes Y}\otimes m_H\otimes m_H)\circ(\mathrm{id}_X\otimes\sigma_{H,Y}\otimes\mathrm{id}_H\otimes\mathrm{id}_{H\otimes H})\circ(\mathrm{id}_X\otimes\mathrm{id}_H\otimes\mathrm{id}_Y\otimes\sigma_{H,H}\otimes\mathrm{id}_H)\\
&\quad\circ(\mathrm{id}_X\otimes\mathrm{id}_H\otimes\sigma_{H,Y}\otimes\mathrm{id}_H\otimes\mathrm{id}_H)\circ(\mathrm{id}_X\otimes\Delta_H\otimes\mathrm{id}_Y\otimes\Delta_H)\circ(\delta_X\otimes\delta_Y)\\
&=(\mathrm{id}_{X\otimes Y}\otimes m_H\otimes m_H)\circ(\mathrm{id}_X\otimes\mathrm{id}_{Y\otimes H}\otimes\sigma_{H,H}\otimes\mathrm{id}_H)\circ(\mathrm{id}_X\otimes\sigma_{H,Y}\otimes\mathrm{id}_{H\otimes H}\otimes\mathrm{id}_H)\\
&\qquad\circ(\mathrm{id}_X\otimes\mathrm{id}_H\otimes\sigma_{H,Y}\otimes\mathrm{id}_{H\otimes H})\circ(\mathrm{id}_X\otimes\Delta_H\otimes\mathrm{id}_Y\otimes\Delta_H)\circ(\delta_X\otimes\delta_Y)\\
&=(\mathrm{id}_{X\otimes Y}\otimes m_H\otimes m_H)\circ(\mathrm{id}_X\otimes\mathrm{id}_{Y\otimes H}\otimes\sigma_{H,H}\otimes\mathrm{id}_H)\circ(\mathrm{id}_X\otimes\sigma_{H\otimes H,Y}\otimes\mathrm{id}_H\otimes\mathrm{id}_H)\\
&\qquad\circ(\mathrm{id}_X\otimes\Delta_H\otimes\mathrm{id}_Y\otimes\Delta_H)\circ(\delta_X\otimes\delta_Y)\\
&=(\mathrm{id}_{X\otimes Y}\otimes m_H\otimes m_H)\circ(\mathrm{id}_X\otimes\mathrm{id}_{Y\otimes H}\otimes\sigma_{H,H}\otimes\mathrm{id}_H)\circ(\mathrm{id}_X\otimes\mathrm{id}_Y\otimes\Delta_H\otimes\Delta_H)\\
&\qquad\circ(\mathrm{id}_X\otimes\sigma_{H,Y}\otimes\mathrm{id}_H)\circ(\delta_X\otimes\delta_Y)\\
&=(\mathrm{id}_{X\otimes Y}\otimes\Delta_H)\circ(\mathrm{id}_X\otimes\mathrm{id}_Y\otimes m_H)\circ(\mathrm{id}_X\otimes\sigma_{H,Y}\otimes\mathrm{id}_H)\circ(\delta_X\otimes\delta_Y)\\
&=(\mathrm{id}_{X\otimes Y}\otimes\Delta_H)\circ\delta_{X\otimes Y},
\end{aligned}
$$

where we have also used braiding axiom (1.3), another application of the naturality of σ, and the fact that Δ_H is an algebra homomorphism. This shows that $\delta_{X\otimes Y}$ satisfies the coassociativity comodule axiom (4.14). Using the fact that ε_H is an algebra homomorphism, the naturality of σ, and the counital comodule axiom (4.15) for δ_X and δ_Y, we have

$$
\begin{aligned}
&(\mathrm{id}_{X\otimes Y}\otimes\varepsilon_H)\circ\delta_{X\otimes Y}\\
&=(\mathrm{id}_{X\otimes Y}\otimes\varepsilon_H)\circ(\mathrm{id}_{X\otimes Y}\otimes m_H)\circ(\mathrm{id}_X\otimes\sigma_{H,Y}\otimes\mathrm{id}_H)\circ(\delta_X\otimes\delta_Y)\\
&=(\mathrm{id}_{X\otimes Y}\otimes m_I)\circ(\mathrm{id}_{X\otimes Y}\otimes\varepsilon_H\otimes\varepsilon_H)\circ(\mathrm{id}_X\otimes\sigma_{H,Y}\otimes\mathrm{id}_H)\circ(\delta_X\otimes\delta_Y)\\
&=(\mathrm{id}_{X\otimes Y}\otimes m_I)\circ(\mathrm{id}_X\otimes\sigma_{I,Y})\circ(\mathrm{id}_X\otimes\varepsilon_H\otimes\mathrm{id}_Y\otimes\varepsilon_H)\circ(\delta_X\otimes\delta_Y)\\
&=\mathrm{id}_{X\otimes Y},
\end{aligned}
$$

where we have recalled that $\sigma_{I,Y}=\mathrm{id}_Y$ by (1.6). This shows that $\delta_{X\otimes Y}$ also satisfies the counital comodule axiom (4.15) and is therefore a right coaction on $X\otimes Y$ by H. Now let $f\colon X\to Y$ and $g\colon X'\to Y'$ be morphisms in $\mathcal{C}^H$. Then,

$$
\begin{aligned}
&\delta_{Y\otimes Y'}\circ(f\otimes g)\\
&=(\mathrm{id}_{Y\otimes Y'}\otimes m_H)\circ(\mathrm{id}_Y\otimes\sigma_{H,Y'}\otimes\mathrm{id}_H)\circ(\delta_Y\otimes\delta_{Y'})\circ(f\otimes g)\\
&=(\mathrm{id}_{Y\otimes Y'}\otimes m_H)\circ(\mathrm{id}_Y\otimes\sigma_{H,Y'}\otimes\mathrm{id}_H)\circ(f\otimes\mathrm{id}_H\otimes g\otimes\mathrm{id}_H)\circ(\delta_X\otimes\delta_{X'})\\
&=(\mathrm{id}_{Y\otimes Y'}\otimes m_H)\circ(f\otimes g\otimes\mathrm{id}_H\otimes\mathrm{id}_H)\circ(\mathrm{id}_X\otimes\sigma_{H,X'}\otimes\mathrm{id}_H)\circ(\delta_X\otimes\delta_{X'})\\
&=(f\otimes g\otimes\mathrm{id}_H)\circ(\mathrm{id}_X\otimes\mathrm{id}_{X'}\otimes m_H)\circ(\mathrm{id}_X\otimes\sigma_{H,X'}\otimes\mathrm{id}_H)\circ(\delta_X\otimes\delta_{X'})\\
&=(f\otimes g\otimes\mathrm{id}_H)\circ\delta_{X\otimes X'}
\end{aligned}
$$

by the defining property (4.20) of a comodule homomorphism and the naturality of σ, and hence $f \otimes g$ is a morphism in $\mathcal{C}^H$. $\square$

We may regard H as a right comodule over itself via the *coadjoint coaction*

$$\delta_H = (\mathrm{id}_H \otimes m_H) \circ (\mathrm{id}_H \otimes S_H \otimes \mathrm{id}_H) \circ (\sigma_{H,H} \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes \Delta_H) \circ \Delta_H, \qquad (5.3)$$

as defined in [14, 2.5.1, p. 106]. Thus, we may define the natural transformation $\tilde{\delta}$ in (5.1) in terms of a comodule homomorphism $g\colon H \to Z$. The next lemma shows that each morphism in this natural transformation is then a comodule homomorphism.

Lemma 5.1.2. *Let $Z \in \mathcal{C}^H$ and let $g\colon H \to Z$ be a morphism in $\mathcal{C}^H$. With the coaction on $X \otimes Z$ defined as in Lemma 5.1.1, the morphism $\tilde{\delta}_X = (\mathrm{id}_X \otimes g) \circ \delta_X$ is a comodule homomorphism for each X in $\mathcal{C}^H$, i.e., the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\ \tilde{\delta}_X\ } & X \otimes Z \\ \downarrow{\scriptstyle \delta_X} & & \downarrow{\scriptstyle \delta_{X \otimes Z}} \\ X \otimes H & \xrightarrow{\ \tilde{\delta}_X \otimes \mathrm{id}_H\ } & X \otimes Z \otimes H \end{array}$$

commutes.

Proof. Using the fact that $g\colon H \to Z$ is a comodule homomorphism, the naturality of σ, and the coassociativity comodule axiom (4.14) for X, we have

$$\begin{aligned}
&\delta_{X\otimes Z} \circ \tilde{\delta}_X \\
&= (\mathrm{id}_{X\otimes Z} \otimes m_H) \circ (\mathrm{id}_X \otimes \sigma_{H,Z} \otimes \mathrm{id}_H) \circ (\delta_X \otimes \delta_Z) \circ (\mathrm{id}_X \otimes g) \circ \delta_X \\
&= (\mathrm{id}_{X\otimes Z} \otimes m_H) \circ (\mathrm{id}_X \otimes \sigma_{H,Z} \otimes \mathrm{id}_H) \circ (\delta_X \otimes \mathrm{id}_{Z\otimes H}) \circ (\mathrm{id}_X \otimes g \otimes \mathrm{id}_H) \\
&\qquad \circ (\mathrm{id}_X \otimes \delta_H) \circ \delta_X \\
&= (\mathrm{id}_{X\otimes Z} \otimes m_H) \circ (\mathrm{id}_X \otimes \sigma_{H,Z} \otimes \mathrm{id}_H) \circ (\mathrm{id}_{X\otimes H} \otimes g \otimes \mathrm{id}_H) \circ (\mathrm{id}_{X\otimes H} \otimes \delta_H) \\
&\qquad \circ (\delta_X \otimes \mathrm{id}_H) \circ \delta_X \\
&= (\mathrm{id}_{X\otimes Z} \otimes m_H) \circ (\mathrm{id}_X \otimes g \otimes \mathrm{id}_H \otimes \mathrm{id}_H) \circ (\mathrm{id}_X \otimes \sigma_{H,H} \otimes \mathrm{id}_H) \circ (\mathrm{id}_{X\otimes H} \otimes \delta_H) \\
&\qquad \circ (\mathrm{id}_X \otimes \Delta_H) \circ \delta_X \\
&= (\mathrm{id}_X \otimes g \otimes \mathrm{id}_H) \circ (\mathrm{id}_X \otimes \mathrm{id}_H \otimes m_H) \circ (\mathrm{id}_X \otimes \sigma_{H,H} \otimes \mathrm{id}_H) \circ (\mathrm{id}_{X\otimes H} \otimes \delta_H) \\
&\qquad \circ (\mathrm{id}_X \otimes \Delta_H) \circ \delta_X.
\end{aligned}$$

As a consequence of [14, Lem. 2.13, p. 108], we have

$$(\mathrm{id}_H \otimes m_H) \circ (\sigma_{H,H} \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes \delta_H) \circ \Delta_H = \Delta_H,$$

and hence

$$\begin{aligned} \delta_{X\otimes Z} \circ \tilde{\delta}_X &= (\mathrm{id}_X \otimes g \otimes \mathrm{id}_H) \circ (\mathrm{id}_X \otimes \Delta_H) \circ \delta_X \\ &= (\mathrm{id}_X \otimes g \otimes \mathrm{id}_H) \circ (\delta_X \otimes \mathrm{id}_H) \circ \delta_X \\ &= (\tilde{\delta}_X \otimes \mathrm{id}_H) \circ \delta_X \end{aligned}$$

as required. □

Putting everything together, we have the following theorem.

Theorem 5.1.1. *If H is a Hopf algebra in a braided category $\mathcal{C}$, viewed as a comodule over itself via the coadjoint coaction defined in* (5.3)*, and Z is in $\mathcal{C}^H$, then the map*

$$\mathrm{Hom}_{\mathcal{C}^H}(H, Z) \to \mathrm{Nat}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}^H}, \mathrm{id}_{\mathcal{C}^H} \otimes Z)$$

sending each morphism $g\colon H \to Z$ in $\mathcal{C}^H$ to the natural transformation $\tilde{\delta}\colon \mathrm{id}_{\mathcal{C}^H} \to \mathrm{id}_{\mathcal{C}^H} \otimes Z$ defined by

$$\tilde{\delta}_X = (\mathrm{id}_X \otimes g) \circ \delta_X \tag{5.4}$$

is a bijection.

Proof. By Lemma 5.1.2, we have that $\tilde{\delta}_X$ is a comodule homomorphism for each X in $\mathcal{C}^H$. By the same argument as in Proposition 5.1.1, we have that $\tilde{\delta}_X$ defines a natural transformation, and we have observed that it is in fact a $\mathcal{C}$-transformation. Conversely, if $\tilde{\delta}\colon \mathrm{id}_{\mathcal{C}^H} \to \mathrm{id}_{\mathcal{C}^H} \otimes Z$ is a $\mathcal{C}$-transformation, then we know from Proposition 5.1.2 that

$$\tilde{\delta}_X = (\mathrm{id}_X \otimes g) \circ \delta_X$$

for

$$g = (\varepsilon_H \otimes \mathrm{id}_Z) \circ \tilde{\delta}_H.$$

Thus, it remains to prove that g is a comodule homomorphism. We first observe that the counit $\varepsilon_H\colon H \to I$ is a comodule homomorphism with respect to the coadjoint coaction (5.3) on H and the trivial coaction $\delta_I\colon I \to I \otimes H$ on I, defined as the unit $u_H\colon I \to H = I \otimes H$.

We have

$$\begin{aligned}
&(\varepsilon_H \otimes \mathrm{id}_H) \circ \delta_H \\
&= (\varepsilon_H \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes m_H) \circ (\mathrm{id}_H \otimes S_H \otimes \mathrm{id}_H) \circ (\sigma_{H,H} \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes \Delta_H) \circ \Delta_H \\
&= m_H \circ (S_H \otimes \mathrm{id}_H) \circ (\varepsilon_H \otimes \mathrm{id}_H \otimes \mathrm{id}_H) \circ (\sigma_{H,H} \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes \Delta_H) \circ \Delta_H \\
&= m_H \circ (S_H \otimes \mathrm{id}_H) \circ (\sigma_{H,I} \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes \varepsilon_H \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes \Delta_H) \circ \Delta_H \\
&= m_H \circ (S_H \otimes \mathrm{id}_H) \circ \Delta_H \\
&= u_H \circ \varepsilon_H \\
&= \delta_I \circ \varepsilon_H,
\end{aligned}$$

where we have used the naturality of σ, the fact that $\sigma_{H,I} = \mathrm{id}_H$, and the counit and antipode equations for H. This implies that $\varepsilon_H \otimes \mathrm{id}_Z$ is a comodule homomorphism by Lemma 5.1.1. Since $\tilde{\delta}_H$ is a comodule homomorphism, it follows that g is a comodule homomorphism. □

5.2 Coends over a tensor category

We now let $\mathcal{C}$ be a braided category with left duality, and show that the $\mathcal{C}$-transformations $\tilde{\delta}\colon \mathrm{id}_{\mathcal{C}^H} \to \mathrm{id}_{\mathcal{C}^H} \otimes Z$ discussed in 5.1 are in bijection with certain dinatural transformations $\iota_X\colon X^* \otimes X \to Z$. This leads us to the notion of a $\mathcal{C}$-coend (cf. [14, Def. 3.2, p. 111], [1, Def. 3.1, p. 160]). We first prove the following.

Proposition 5.2.1. *The set* $\mathrm{Nat}(\mathrm{id}_{\mathcal{C}^H}, \mathrm{id}_{\mathcal{C}^H} \otimes Z)$ *is in bijection with the set of dinatural transformations* $\iota_X\colon X^* \otimes X \to Z$*, for* X *in* $\mathcal{C}^H$*.*

Proof. For each $\tilde{\delta} \in \mathrm{Nat}(\mathrm{id}_{\mathcal{C}^H}, \mathrm{id}_{\mathcal{C}^H} \otimes Z)$ and X in $\mathcal{C}^H$, define

$$\iota_X = (\mathrm{ev}_X \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \tilde{\delta}_X). \tag{5.5}$$

Let $f\colon X \to Y$ be a morphism in $\mathcal{C}^H$. Then

$$\begin{aligned}
\iota_X \circ (f^* \otimes \mathrm{id}_X) &= (\mathrm{ev}_X \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \tilde{\delta}_X) \circ (f^* \otimes \mathrm{id}_X) \\
&= (\mathrm{ev}_X \otimes \mathrm{id}_Z) \circ (f^* \otimes \mathrm{id}_{X\otimes Z}) \circ (\mathrm{id}_{Y^*} \otimes \tilde{\delta}_X) \\
&= (\mathrm{ev}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{Y^*} \otimes f \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{Y^*} \otimes \tilde{\delta}_X) \\
&= (\mathrm{ev}_Y \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{Y^*} \otimes \tilde{\delta}_Y) \circ (\mathrm{id}_{Y^*} \otimes f) \\
&= \iota_Y \circ (\mathrm{id}_{Y^*} \otimes f),
\end{aligned}$$

where we have used Theorem 1.4.1 and the naturality of $\tilde{\delta}$. This shows that the collection of morphisms $\iota_X \colon X^* \otimes X \to Z$, which is the image of $\tilde{\delta}$ under this mapping, is a dinatural transformation. The inverse map sends a dinatural transformation $\iota_X \colon X^* \otimes X \to Z$ to the collection of morphisms defined by

$$\tilde{\delta}_X = (\mathrm{id}_X \otimes \iota_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X) \tag{5.6}$$

(cf. [5, Prop. XIV.2.2, p. 343]). It remains to prove that $\tilde{\delta}$ is a natural transformation. Naturality of $\tilde{\delta}$ means that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\delta}_X} & X \otimes Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes \mathrm{id}_Z} \\ Y & \xrightarrow{\tilde{\delta}_Y} & Y \otimes Z \end{array} \tag{5.7}$$

commutes for each morphism $f \colon X \to Y$ in $\mathcal{C}^H$. We have

$$\begin{aligned} (f \otimes \mathrm{id}_Z) \circ \tilde{\delta}_X &= (f \otimes \mathrm{id}_Z) \circ (\mathrm{id}_X \otimes \iota_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X) \\ &= (\mathrm{id}_Y \otimes \iota_X) \circ (f \otimes \mathrm{id}_{X^*} \otimes \mathrm{id}_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_X) \\ &= (\mathrm{id}_Y \otimes \iota_X) \circ (\mathrm{id}_Y \otimes f^* \otimes \mathrm{id}_X) \circ (\mathrm{coev}_Y \otimes \mathrm{id}_X) \\ &= (\mathrm{id}_Y \otimes \iota_Y) \circ (\mathrm{id}_Y \otimes \mathrm{id}_{Y^*} \otimes f) \circ (\mathrm{coev}_Y \otimes \mathrm{id}_X) \\ &= (\mathrm{id}_Y \otimes \iota_Y) \circ (\mathrm{coev}_Y \otimes \mathrm{id}_Y) \circ f \\ &= \tilde{\delta}_Y \circ f, \end{aligned}$$

where we have used Theorem 1.4.1 and the dinaturality of ι. This establishes the desired bijection. □

If we restrict this bijection to the set $\mathrm{Nat}_{\mathcal{C}}(\mathrm{id}_{\mathcal{C}^H}, \mathrm{id}_{\mathcal{C}^H} \otimes Z)$ of $\mathcal{C}$-transformations, then we have a bijection with some subset of the dinatural transformations $\iota_X \colon X^* \otimes X \to Z$. The next proposition characterizes these dinatural transformations.

Proposition 5.2.2. *Let $\tilde{\delta} \in \mathrm{Nat}(\mathrm{id}_{\mathcal{C}^H}, \mathrm{id}_{\mathcal{C}^H} \otimes Z)$, and let ι be the corresponding dinatural transformation under the bijection in Proposition 5.2.1. Then $\tilde{\delta}$ is a $\mathcal{C}$-transformation if and only if*

$$\iota_{X \otimes Y} = \iota_Y \circ (\mathrm{id}_{Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_Y) \circ (\gamma^{-1}_{X,Y} \otimes \mathrm{id}_{X \otimes Y}). \tag{5.8}$$

Proof. We first show that $\tilde{\delta} \in \mathrm{Nat}(\mathrm{id}_{\mathcal{C}^H}, \mathrm{id}_{\mathcal{C}^H} \otimes Z)$ is a $\mathcal{C}$-transformation if and only if

$$\iota_{X\otimes Y} = \iota_Y \circ (\mathrm{ev}_{X\otimes Y} \otimes \mathrm{id}_{Y^*\otimes Y}) \circ (\mathrm{id}_{(X\otimes Y)^*} \otimes \mathrm{id}_X \otimes \mathrm{coev}_Y \otimes \mathrm{id}_Y). \tag{5.9}$$

Suppose that $\tilde{\delta}\colon \mathrm{id}_{\mathcal{C}^H} \to \mathrm{id}_{\mathcal{C}^H} \otimes Z$ is a $\mathcal{C}$-transformation, and recall that this means that

$$\tilde{\delta}_{X\otimes Y} = \mathrm{id}_X \otimes \tilde{\delta}_Y.$$

Applying the bijection (5.5), we have

$$\begin{aligned}
\iota_{X\otimes Y} &= (\mathrm{ev}_{X\otimes Y} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{(X\otimes Y)^*} \otimes \tilde{\delta}_{X\otimes Y}) \\
&= (\mathrm{ev}_{X\otimes Y} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{(X\otimes Y)^*} \otimes \mathrm{id}_X \otimes \tilde{\delta}_Y) \\
&= (\mathrm{ev}_{X\otimes Y} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{(X\otimes Y)^*} \otimes \mathrm{id}_X \otimes \mathrm{id}_Y \otimes \iota_Y) \circ (\mathrm{id}_{(X\otimes Y)^*} \otimes \mathrm{id}_X \otimes \mathrm{coev}_Y \otimes \mathrm{id}_Y) \\
&= \iota_Y \circ (\mathrm{ev}_{X\otimes Y} \otimes \mathrm{id}_{Y^*\otimes Y}) \circ (\mathrm{id}_{(X\otimes Y)^*} \otimes \mathrm{id}_X \otimes \mathrm{coev}_Y \otimes \mathrm{id}_Y).
\end{aligned}$$

Conversely, suppose ι satisfies (5.9). Then, applying the bijection (5.6), we have

$$\begin{aligned}
\tilde{\delta}_{X\otimes Y} &= (\mathrm{id}_{X\otimes Y} \otimes \iota_{X\otimes Y}) \circ (\mathrm{coev}_{X\otimes Y} \otimes \mathrm{id}_{X\otimes Y}) \\
&= (\mathrm{id}_{X\otimes Y} \otimes \iota_Y) \circ (\mathrm{id}_{X\otimes Y} \otimes \mathrm{ev}_{X\otimes Y} \otimes \mathrm{id}_{Y^*\otimes Y}) \\
&\qquad \circ (\mathrm{id}_{X\otimes Y} \otimes \mathrm{id}_{(X\otimes Y)^*} \otimes \mathrm{id}_X \otimes \mathrm{coev}_Y \otimes \mathrm{id}_Y) \circ (\mathrm{coev}_{X\otimes Y} \otimes \mathrm{id}_{X\otimes Y}) \\
&= (\mathrm{id}_{X\otimes Y} \otimes \iota_Y) \circ (\mathrm{id}_{X\otimes Y} \otimes \mathrm{ev}_{X\otimes Y} \otimes \mathrm{id}_{Y^*\otimes Y}) \\
&\qquad \circ (\mathrm{coev}_{X\otimes Y} \otimes \mathrm{id}_X \otimes \mathrm{id}_{Y\otimes Y^*} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes \mathrm{coev}_Y \otimes \mathrm{id}_Y) \\
&= (\mathrm{id}_{X\otimes Y} \otimes \iota_Y) \circ (\mathrm{id}_X \otimes \mathrm{coev}_Y \otimes \mathrm{id}_Y) \\
&= \mathrm{id}_X \otimes \tilde{\delta}_Y,
\end{aligned}$$

which means that $\tilde{\delta}$ is $\mathcal{C}$-transformation.

By applying (3.1), the condition (5.9) can equivalently be expressed as

$$\begin{aligned}
\iota_{X\otimes Y} &= \iota_Y \circ (\mathrm{ev}_Y \otimes \mathrm{id}_{Y^*\otimes Y}) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_Y \otimes \mathrm{id}_{Y^*\otimes Y}) \circ (\gamma_{X,Y}^{-1} \otimes \mathrm{id}_{X\otimes Y} \otimes \mathrm{id}_{Y^*\otimes Y}) \\
&\qquad \circ (\mathrm{id}_{(X\otimes Y)^*} \otimes \mathrm{id}_X \otimes \mathrm{coev}_Y \otimes \mathrm{id}_Y) \\
&= \iota_Y \circ (\mathrm{ev}_Y \otimes \mathrm{id}_{Y^*\otimes Y}) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{coev}_Y \otimes \mathrm{id}_Y) \circ (\mathrm{id}_{Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_Y) \\
&\qquad \circ (\gamma_{X,Y}^{-1} \otimes \mathrm{id}_X \otimes \mathrm{id}_Y) \\
&= \iota_Y \circ (\mathrm{id}_{Y^*} \otimes \mathrm{ev}_X \otimes \mathrm{id}_Y) \circ (\gamma_{X,Y}^{-1} \otimes \mathrm{id}_{X\otimes Y}).
\end{aligned}$$

Thus, $\tilde{\delta}$ is a $\mathcal{C}$-transformation if and only if (5.8) holds. $\square$

In the case $Z = H$ with the coadjoint coaction and $g = \mathrm{id}_H$, the $\mathcal{C}$-transformation $\tilde{\delta}$ defined in (5.4) takes the form $\tilde{\delta}_X = \delta_X$ for X in $\mathcal{C}^H$, and $\tilde{\delta}$ corresponds to a dinatural transformation $\iota_X \colon X^* \otimes X \to H$ satisfying (5.8) and (5.9) by Proposition 5.2.2. Explicitly,

$$\iota_X = (\mathrm{ev}_X \otimes \mathrm{id}_H) \circ (\mathrm{id}_{X^*} \otimes \delta_X). \tag{5.10}$$

The next proposition shows that ι is universal among dinatural transformations satisfying (5.8).

Proposition 5.2.3. *Let $\iota_X \colon X^* \otimes X \to H$ be the dinatural transformation defined by* (5.10)*. For any dinatural transformation $j_X \colon X^* \otimes X \to Z$ satisfying* (5.8)*, there exists a unique comodule homomorphism $g \colon H \to Z$ such that*

$$j_X = g \circ \iota_X. \tag{5.11}$$

Proof. Under the bijection in Proposition 5.2.1, the dinatural transformation j corresponds to a natural transformation $\tilde{\delta} \in \mathrm{Nat}(\mathrm{id}_{\mathcal{C}^H}, \mathrm{id}_{\mathcal{C}^H} \otimes Z)$. Since j satisfies (5.8), this $\tilde{\delta}$ is a $\mathcal{C}$-transformation by Proposition 5.2.2. By Theorem 5.1.1,

$$\tilde{\delta}_X = (\mathrm{id}_X \otimes g) \circ \delta_X$$

for a unique comodule homomorphism $g \colon H \to Z$. Hence, applying (5.5),

$$\begin{aligned} j_X &= (\mathrm{ev}_X \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \tilde{\delta}_X) \\ &= (\mathrm{ev}_X \otimes \mathrm{id}_Z) \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_X \otimes g) \circ (\mathrm{id}_{X^*} \otimes \delta_X) \\ &= g \circ (\mathrm{ev}_X \otimes \mathrm{id}_H) \circ (\mathrm{id}_{X^*} \otimes \delta_X) \\ &= g \circ \iota_X \end{aligned}$$

as required. □

In light of this result, we will refer to H as a $\mathcal{C}$*-coend*. Thus H, together with the dinatural transformation

$$\iota_X = (\mathrm{ev}_X \otimes \mathrm{id}_H) \circ (\mathrm{id}_{X^*} \otimes \delta_X),$$

is a $\mathcal{C}$-coend in $\mathcal{C}^H$.

5.3 The non-example

We are now ready to construct our example of a braided finite tensor category containing a Hopf algebra A whose dual A^* with the coadjoint action is not a coend in the category of A-modules. We begin by constructing a Hopf algebra, in a category $\mathcal{C}^C$ of comodules over a coalgebra in a category $\mathcal{C}$, that is a $\mathcal{C}$-coend but not a coend. This can then be dualized.

Let G be the cyclic group of order 2 and $H = K[G]$ be the group algebra over a field K. We let $\mathcal{C}$ be the category of finite-dimensional H-modules. Note that H is quasitriangular with R-matrix $R = 1_H \otimes 1_H$, so that $\mathcal{C}$ is a braided category in which the braiding is the flip map. The Hopf algebra H, which is a Hopf algebra in the category of vector spaces, is in fact a Hopf algebra in $\mathcal{C}$ when endowed with the trivial action. We will denote H, viewed as a Hopf algebra in $\mathcal{C}$, by C. We now verify that the structure morphisms of C are indeed H-linear and, hence, morphisms in $\mathcal{C}$. Let $h \in H$ and $c \in C$. For the coproduct $\Delta_C = \Delta_H$, we have

$$\begin{aligned}
h \cdot \Delta_C(c) &= h_{(1)} \cdot c_{(1)} \otimes h_{(2)} \cdot c_{(2)} \\
&= \varepsilon_H(h_{(1)})c_{(1)} \otimes \varepsilon_H(h_{(2)})c_{(2)} \\
&= \varepsilon_H(h_{(1)}\varepsilon_H(h_{(2)}))c_{(1)} \otimes c_{(2)} \\
&= \varepsilon_H(h)\Delta_C(c) \\
&= \Delta_C(\varepsilon_H(h)c) \\
&= \Delta_C(h \cdot c),
\end{aligned}$$

and, recalling that the unit object in the category of H-modules is the base field K with the trivial H-action, we have for the counit $\varepsilon_C = \varepsilon_H$ that

$$\begin{aligned}
h \cdot \varepsilon_C(c) &= \varepsilon_H(h)\varepsilon_C(c) \\
&= \varepsilon_C(\varepsilon_H(h)c) \\
&= \varepsilon_C(h \cdot c).
\end{aligned}$$

This shows that $\Delta_C = \Delta_H$ and $\varepsilon_C = \varepsilon_H$ are H-linear. For the multiplication $m_C = m_H$, we have for $h \in H$ and $c, c' \in C$ that

$$\begin{aligned}
m_C(h \cdot (c \otimes c')) &= m_C(\varepsilon_H(h_{(1)})c \otimes \varepsilon_H(h_{(2)})c') \\
&= \varepsilon_H(h_{(1)}\varepsilon_H(h_{(2)}))m_C(c \otimes c') \\
&= \varepsilon_H(h)m_C(c \otimes c'),
\end{aligned}$$

and therefore m_C is H-linear. Finally, $S_C = S_H$ and $u_C = u_H$ are H-linear as a consequence of their K-linearity. Note that the coproduct $\Delta_C = \Delta_H$ remains an algebra homomorphism when viewed in $\mathcal{C}$, since the braiding coincides with the flip map.

We now consider the category $\mathcal{C}^C$ of comodules over C in the category $\mathcal{C}$. Let a be the non-identity element of G and, for each X in $\mathcal{C}^C$, define

$$\tilde{\delta}_X(x) = a \cdot x.$$

Each $\tilde{\delta}_X$ is an H-module homomorphism because, for all $h \in H$ and $x \in X$,

$$\begin{aligned}
\tilde{\delta}_X(h \cdot x) &= a \cdot (h \cdot x) \\
&= ah \cdot x \\
&= ha \cdot x \\
&= h \cdot (a \cdot x) \\
&= h \cdot \tilde{\delta}_X(x)
\end{aligned}$$

since a is central in G and, therefore, in H. Each $\tilde{\delta}_X$ is a C-comodule homomorphism because for all $x \in X$,

$$\begin{aligned}
(\delta_X \circ \tilde{\delta}_X)(x) &= \delta_X(a \cdot x) \\
&= a \cdot \delta_X(x) \\
&= a_{(1)} \cdot x^{(1)} \otimes a_{(2)} \cdot x^{(2)} \\
&= a_{(1)} \cdot x^{(1)} \otimes \varepsilon_H(a_{(2)}) x^{(2)} \\
&= a_{(1)} \varepsilon_H(a_{(2)}) \cdot x^{(1)} \otimes x^{(2)} \\
&= a \cdot x^{(1)} \otimes x^{(2)} \\
&= ((\tilde{\delta}_X \otimes \mathrm{id}_H) \circ \delta_X)(x),
\end{aligned}$$

where we have used the fact that δ_X is H-linear, being a morphism in $\mathcal{C}$. Next observe that for any morphism $f \colon X \to Y$ in $\mathcal{C}^C$,

$$\begin{aligned}
(f \circ \tilde{\delta}_X)(x) &= f(a \cdot x) \\
&= a \cdot f(x) \\
&= \tilde{\delta}_Y(f(x)) \\
&= (\tilde{\delta}_Y \circ f)(x)
\end{aligned}$$

since f is, in particular, H-linear. Thus, we have constructed a natural transformation $\tilde{\delta}\colon \mathrm{id}_{\mathcal{C}^C} \to \mathrm{id}_{\mathcal{C}^C}$.

Now if $\tilde{\delta}$ were a $\mathcal{C}$-transformation, then by applying Theorem 5.1.1 with $Z = K$, there would exist a unique morphism $g\colon C \to K$ in $\mathcal{C}$ such $\tilde{\delta}_X = (\mathrm{id}_X \otimes g) \circ \delta_X$ for all X in $\mathcal{C}^C$. Let $X = H$ and equip it with the regular H-action and the trivial C-coaction $\delta_X(x) = x \otimes 1_C$, and observe that X is in $\mathcal{C}^C$ since the trivial C-coaction on X is H-linear:

$$\begin{aligned} h \cdot (x \otimes 1_C) &= h_{(1)} \cdot x \otimes h_{(2)} \cdot 1_C \\ &= h_{(1)} \cdot x \otimes \varepsilon_H(h_{(2)}) 1_C \\ &= h_{(1)} \varepsilon_H(h_{(2)}) \cdot x \otimes 1_C \\ &= h \cdot x \otimes 1_C. \end{aligned}$$

But

$$\tilde{\delta}_X(a) = a \cdot a = a^2 = 1_G$$

while

$$((\mathrm{id}_X \otimes g) \circ \delta_X)(a) = (\mathrm{id}_X \otimes g)(a \otimes 1_C) = g(1_C)a,$$

and these are not equal since 1_G and a are linearly independent. Therefore, $\tilde{\delta}$ is not a $\mathcal{C}$-transformation.

We know from 5.2 that C, together with the coadjoint coaction (5.3), is a $\mathcal{C}$-coend in $\mathcal{C}^C$, with the dinatural transformation

$$\iota_X = (\mathrm{ev}_X \otimes \mathrm{id}_C) \circ (\mathrm{id}_{X^*} \otimes \delta_X),$$

which is universal among dinatural transformations satisfying (5.8). If ι is universal among all dinatural transformations $i\colon X^* \otimes X \to Z$ in $\mathcal{C}^C$, then for the dinatural transformation

$$j_X = \mathrm{ev}_X \circ (\mathrm{id}_{X^*} \otimes \tilde{\delta}_X)$$

corresponding to $\tilde{\delta}$ under the bijection in Proposition 5.2.1, there exists a unique comodule homomorphism $g\colon C \to K$ such that $j_X = g \circ \iota_X$. But then

$$\begin{aligned} j_X &= g \circ \iota_X \\ &= g \circ (\mathrm{ev}_X \otimes \mathrm{id}_C) \circ (\mathrm{id}_{X^*} \otimes \delta_X) \\ &= \mathrm{ev}_X \circ (\mathrm{id}_{X^*} \otimes \mathrm{id}_X \otimes g) \circ (\mathrm{id}_{X^*} \otimes \delta_X) \end{aligned}$$

implies by the general adjunction from [5, Prop. XIV.2.2, p. 343] that

$$\tilde{\delta}_X = (\mathrm{id}_X \otimes g) \circ \delta_X,$$

which we have shown is a contradiction. This shows that ι is not universal among all dinatural transformations in $\mathcal{C}^C$, and hence C together with ι is a $\mathcal{C}$-coend in $\mathcal{C}^C$ but not a coend.

Now recall that left and right duals of a Hopf algebra in a category are again Hopf algebras in that category, and that the left and right duals of the structure morphisms are the structure morphisms for the left and right dual Hopf algebras, respectively. We can thus view C as the dual Hopf algebra A^* in $\mathcal{C}$, where $A = {}^*C$ is a right dual of C. Hence A^*, which has the trivial action by H and coadjoint coaction by $A^* = C$, is a $\mathcal{C}$-coend in the category $\mathcal{C}^C$ that is not a coend.

We now dualize this case. Recall that if C is a coalgebra in $\mathcal{C}$ then the category $\mathcal{C}^C$ of right C-comodules can be identified with the category $\mathcal{C}_{^*C}$ of right ${}^*C$-modules, by the discussion in [17, p. 12]. The correspondence between the C-coaction and ${}^*C$-action is given by the adjunction in [5, Prop. XIV.2.2, p. 343]. If X is in $\mathcal{C}_{^*C}$ with action $\alpha_X \colon X \otimes {}^*C \to X$, then the corresponding coaction $\delta_X \colon X \to X \otimes C$ is given by

$$\delta_X = (\alpha_X \otimes \mathrm{id}_C) \circ (\mathrm{id}_X \otimes \mathrm{coev}_{^*C}). \tag{5.12}$$

Thus, for the case $C = A^*$, we can identify the category $\mathcal{C}^C$ with the category $\mathcal{C}_A$ by this correspondence. Since coends are preserved by an isomorphism of categories, it follows that A^*, equipped with the action corresponding to the coadjoint coaction, is a $\mathcal{C}$-coend in $\mathcal{C}_A$ that is not a coend.

This non-example is particularly interesting because, in the case where $\mathcal{C}$ is the category of finite-dimensional vector spaces and H is a Hopf algebra in $\mathcal{C}$, we have proved that H^* together with the coadjoint action is a coend in the category of finite-dimensional H-modules, and we now prove that the the coadjoint action of H on H^* is the action corresponding to the coadjoint coaction of H^* on H^*.

Proposition 5.3.1. *Let H be a finite-dimensional Hopf algebra. Then the coadjoint action of H on H^* corresponds to the coadjoint coaction of H^* on H^* under the correspondence given in* (5.12)*.*

Proof. The coadjoint coaction (5.3) of H^* on H^* is given by

$$\delta_{H^*} = (\mathrm{id}_{H^*} \otimes m_{H^*}) \circ (\mathrm{id}_{H^*} \otimes S_{H^*} \otimes \mathrm{id}_{H^*}) \circ (\sigma_{H^*,H^*} \otimes \mathrm{id}_{H^*}) \circ (\mathrm{id}_{H^*} \otimes \Delta_{H^*}) \circ \Delta_{H^*}.$$

Evaluating this on $\varphi \in H^*$, we obtain

$$\delta_{H^*}(\varphi) = \varphi_{(2)} \otimes S_{H^*}(\varphi_{(1)})\varphi_{(3)}.$$

Recall that the coend L from 4.3 is equal to H^* as a coalgebra. By Lemma 4.4.1, we have $A = {}^*L = H^{\mathrm{op}}$ as an algebra. Therefore, by the correspondence in [17, p. 12], the right coaction δ_{H^*} corresponds to the right action $\alpha_{H^*} \colon H^* \otimes H^{\mathrm{op}} \to H^*$ defined by

$$\alpha_{H^*} = (\mathrm{id}_{H^*} \otimes \mathrm{ev}_A) \circ (\delta_{H^*} \otimes \mathrm{id}_A).$$

Evaluating this on $\varphi \otimes h \in H^* \otimes H^{\mathrm{op}}$ gives

$$\begin{aligned} \alpha_{H^*}(\varphi \otimes h) &= (\mathrm{id}_{H^*} \otimes \mathrm{ev}_H)(\varphi_{(2)} \otimes S_{H^*}(\varphi_{(1)})\varphi_{(3)} \otimes h) \\ &= \varphi_{(2)} \otimes S_{H^*}(\varphi_{(1)}(h_{(1)}))\varphi_{(3)}(h_{(2)}) \\ &= \varphi_{(1)}(S_H(h_{(1)}))\varphi_{(3)}(h_{(2)})\varphi_{(2)}, \end{aligned}$$

and evaluating on $h' \in H$ gives

$$\begin{aligned} \alpha_{H^*}(\varphi \otimes h)(h') &= \varphi_{(1)}(S_H(h_{(1)})\varphi_{(2)}(h')\varphi_{(3)}(h_{(2)}) \\ &= \varphi(S_H(h_{(1)})h'h_{(2)}). \end{aligned}$$

Since a right action by H^{op} is the same as a left action by H, this shows that the right coadjoint coaction of H^* on H^* corresponds to the left coadjoint action of H on H^*. $\square$

Thus, since we know that H^* with the coadjoint action is a coend in the category of finite-dimensional H-modules, it must be the case that every dinatural transformation in this category satisfies the property (5.8).

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Appendix A

Dual basis map

Here, we establish that the coevaluation map $\mathrm{coev}_V = \mathrm{db}_V \colon V^* \otimes V \to K$ in the category of vector spaces over a field K, also known as the dual basis map, is independent of the choice of basis. Recall from 1.3 that it is defined by

$$\begin{aligned} \mathrm{coev}_V \colon K &\to V \otimes V^* \\ \lambda &\mapsto \lambda \sum_{i=1}^{n} v_i \otimes v_i^*, \end{aligned}$$

where $\{v_1, \ldots, v_n\}$ is a basis for V and $\{v_1^*, \ldots, v_n^*\}$ is the corresponding dual basis.

Define $\psi \colon V \times V^* \to \mathrm{End}(V)$ by

$$\psi(v, \varphi)(v') = \varphi(v')v$$

for all $(v, \varphi) \in V \times V^*$ and $v' \in V$. Then ψ is bilinear, so by the universal property of the tensor product there exists a linear map $T \colon V \otimes V^* \to \mathrm{End}(V)$ such that the diagram

$$\begin{array}{ccc} V \times V^* & \xrightarrow{\ \psi\ } & \mathrm{End}(V) \\ \big\downarrow \otimes & \nearrow_{T} & \\ V \otimes V^* & & \end{array}$$

commutes. In other words, for all $(v, \varphi) \in V \times V^*$ and $v' \in V$

$$T(v \otimes \varphi)(v') = \varphi(v')v.$$

Now let $v' = \sum_{i=1}^n \lambda_i v_i$ be any vector in V, and observe that

$$\begin{aligned} T\Big(\sum_{i=1}^n v_i \otimes v_i^*\Big)(v') &= \Big(\sum_{i=1}^n T(v_i \otimes v_i^*)\Big)(v') = \sum_{i=1}^n T(v_i \otimes v_i^*)(v') \\ &= \sum_{i=1}^n v_i^*(v')v_i = \sum_{i,j=1}^n \lambda_j v_i^*(v_j)v_i = \sum_{i=1}^n \lambda_i v_i \\ &= v', \end{aligned}$$

which means that

$$T\Big(\sum_{i=1}^n v_i \otimes v_i^*\Big) = \mathrm{id}_V. \tag{A.1}$$

Note that, in particular,

$$v' = \sum_{i=1}^n v_i^*(v')v_i.$$

By writing $\varphi \in V^*$ as a linear combination of the v_i^*, a similar calculation also shows that

$$\varphi = \sum_{i=1}^n \varphi(v_i)v_i^*.$$

Now let $T'\colon \mathrm{End}(V) \to V \otimes V^*$ be defined by $T'(f) = \sum_{i=1}^n f(v_i) \otimes v_i^*$ for all $f \in \mathrm{End}(V)$. Observe that

$$\begin{aligned} (T' \circ T)(v \otimes \varphi) &= T'(T(v \otimes \varphi)) = \sum_{i=1}^n T(v \otimes \varphi)(v_i) \otimes v_i^* \\ &= \sum_{i=1}^n \varphi(v_i)v \otimes v_i^* = \sum_{i=1}^n v \otimes \varphi(v_i)v_i^* = v \otimes \sum_{i=1}^n \varphi(v_i)v_i^* \\ &= v \otimes \varphi \end{aligned}$$

for all $v \otimes \varphi \in V \otimes V^*$, and hence $T' \circ T = \mathrm{id}_{V \otimes V^*}$. Furthermore,

$$\begin{aligned} (T \circ T')(f)(v) &= T(T'(f))(v) = T\Big(\sum_{i=1}^n f(v_i) \otimes v_i^*\Big)(v) \\ &= \sum_{i=1}^n T(f(v_i) \otimes v_i^*)(v) = \sum_{i=1}^n v_i^*(v)f(v_i) = f\Big(\sum_{i=1}^n v_i^*(v)v_i\Big) \\ &= f(v) \end{aligned}$$

for all $f \in \mathrm{End}(V)$ and $v \in V$, and hence $T \circ T' = \mathrm{id}_{\mathrm{End}(V)}$.

This shows that T' is the inverse of T, and therefore T is a bijection. Now since coev_V is determined by $\mathrm{coev}_V(1) = \sum_{i=1}^{n} v_i \otimes v_i^*$, the bijectivity of T and relation (A.1) show that coev_V is independent of the choice of basis.