# Entropy and entanglement in a bipartite quasi-Hermitian system and its Hermitian counterparts

by

© **Abedalsalam Abumoise**

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# Abstract

We provide a brief introduction to the mathematical aspects of Hermitian and non-Hermitian quantum theory. Subsequently, we derive the general form of a two-dimensional Hamiltonian possessing $PT$ symmetry, where $T$ is the complex conjugation operator. We then investigate the diagonalizability of such Hamiltonians at the transition point.

Furthermore, we analyze the time-independent solutions of the Dyson map for a non-Hermitian, time-independent Hamiltonian. During this exploration, we uncover certain features of these solutions at the transition point.

As a central finding of this thesis, we consider a quantum oscillator coupled to a bath of $N$ other oscillators. The total system evolves with a quasi-Hermitian Hamiltonian. Associated to it is a family of Hermitian systems, parameterized by a unitary map $W$. Our main goal is to find the influence of $W$ on the entropy and the entanglement in the Hermitian systems. We calculate explicitly the reduced density matrix of the single oscillator for all Hermitian systems and show that, regardless of $W$, their von Neumann entropy oscillates with a common period which is twice that of the non-Hermitian system. We show that generically, the oscillator and the bath are entangled for almost all times. While the amount of entanglement depends on the choice of $W$, it is independent of $W$ when averaged over a period. These results describe some universality in the physical properties of all Hermitian systems associated to a given non-Hermitian one.

To my loving parents and sweet wife.

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# Statement of contribution

In this thesis, Chapter 6 presents collaborative research work by Abedalsalam Abumoise, Marco Merkli, and Graham Cox. All results included in this chapter were developed by all parties. Note that the content of Chapter 6 has been previously disseminated in a publication [26]. While the writing of the work was primarily conducted by Abedalsalam, the supervision, and editing of this work were undertaken by Marco and Graham.

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# List of symbols

| | |
|---|---|
| $\mathbb{R}$ | Real numbers |
| $\mathbb{C}$ | Complex numbers |
| $\mathcal{H}$ | Hilbert space |
| $\mathcal{H}^*$ | Dual space of $\mathcal{H}$ |
| $\mathcal{L}(\mathcal{H})$ | Space of linear operators acting on $\mathcal{H}$ |
| $\mathcal{B}(\mathcal{H})$ | Space of bounded linear operator acting on $\mathcal{H}$ |
| $\|\cdot\rangle$ | Ket vector in Hilbert space $\mathcal{H}$ |
| $\langle\cdot\|$ | Bra vector in dual space $\mathcal{H}^*$ |
| $\langle\cdot\|\cdot\rangle$ | Inner product |
| $\|\cdot\|$ | Modulus of complex number |
| $\|\cdot\|$ | Norm function |
| $H$ | Hamiltonian (energy) operator acting on $\mathcal{H}$ |
| $\langle O\rangle$ | Expectation value of the operator $O$ |
| $\otimes$ | Tensor product |
| $\rho$ | Density operator acting on $\mathcal{H}$ |
| $tr(O)$ | Trace of an operator $O$ |
| $\mathcal{E}(\rho)$ | Entropy of the density matrix |
| $P$ | Linear involution operator $P^2 = \mathbb{1}$ |
| $T$ | Anti-linear involution operator $T^2 = \mathbb{1}$ |
| $\hat{x}$ | Momentum operator |
| $\hat{p}$ | Position operator |
| $O^*$ | The Hilbert space adjoint of the operator $O$ |
| $O^t$ | The transpose of the operator $O$ |
| spec$(O)$ or $\sigma(O)$ | The spectrum of the operator $O$ |
| res$(O)$ | The resolvent of the operator $O$ |
| $[A, B] = AB - BA$ | The commutation relationship between operators $A$ and $B$ |

# Chapter 1

# Introduction

In recent years, there has been a growing interest in the exploration of extensions to quantum mechanics that permit the use of non-Hermitian Hamiltonians. A notable example of this is found in $PT$ symmetric quantum theory, where non-Hermitian Hamiltonians yielding purely real spectra are frequently encountered. Many of these Hamiltonians are, in fact, quasi-Hermitian, as has been extensively discussed in the literature (e.g., [4, 5, 7, 12, 25, 29, 27, 28, 34, 38]). In the case of quasi-Hermitian Hamiltonians, it is possible to construct corresponding Hermitian Hamiltonians, which can then be examined using the conventional methods of quantum theory.

The growing interest in non-Hermitian quantum mechanics, or non-Hermitian Hamiltonians, can be attributed to several compelling observations. Despite the long-standing prevalence of Hermitian operators (Hamiltonians) in traditional quantum mechanics, some of the reasons behind the increasing popularity of non-Hermitian quantum mechanics are the following:

1. Complex Energy Eigenvalues: Hermitian operators in quantum mechanics have real eigenvalues, which correspond to observable quantities like energy. However, in certain complex and open quantum systems, non-Hermitian operators can lead to complex energy eigenvalues. These systems can describe processes with decay, amplification, or non-unitary dynamics, which are relevant in fields like quantum optics and nuclear physics.

2. $PT$ symmetry: Some non-Hermitian Hamiltonians exhibit $PT$ symmetry, which means that they are invariant under the combined operations of the parity ($P$)

and time reversal ($T$) operators,

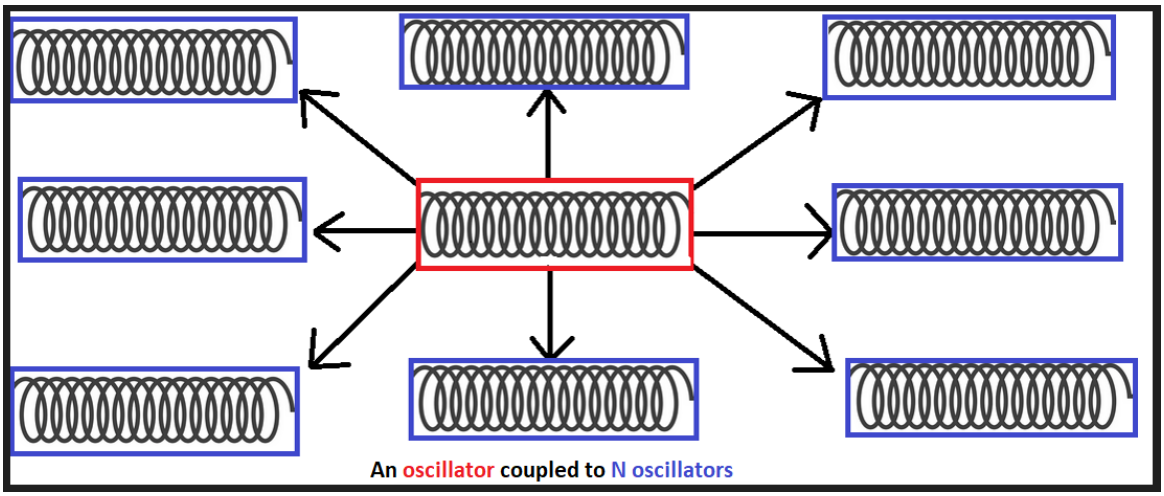
$$H = PTHPT.$$

$PT$ symmetric systems have been studied extensively and exhibit interesting properties. In particular, a $PT$ symmetric $H$ can have real spectrum, despite being non-Hermitian. This concept has been applied to various physical systems, leading to the study of exceptional points and non-Hermitian physics. More details are given in Subsection 2.2.1

3. Topological Phases: Non-Hermitian systems can exhibit topological phases and associated phenomena, including topological phase transitions. These phases have been explored in the context of non-Hermitian systems, leading to a deeper understanding of the role of non-Hermiticity in quantum mechanics and its connection to topology. See subsection 2.2.1

4. Quantum Mechanics with Gain and Loss: Non-Hermitian Hamiltonians can model open quantum systems that interact with their environment. This is particularly relevant in quantum optics, where gain and loss are common phenomena. Non-Hermitian operators are used to describe the dynamics of quantum systems coupled to external baths or reservoirs. See example 2 in Subsection 2.2.2.

An illustration of a non-Hermitian system can be seen in the case of an oscillator interacting with a 'bath' consisting of $N$ independent oscillators.



The Hamiltonian of the above system is given by

$$H = \nu N_{\text{tot}} + (g+\kappa)\sqrt{N}\, a^* Q + (g-\kappa)\sqrt{N}\; a Q^*, \tag{1.1}$$

where $\nu > 0$ and $g, \kappa \in \mathbb{R}$ are parameters and

$$N_{\text{tot}} = a^* a + \sum_{n=1}^{N} q_n^* q_n, \qquad Q = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} q_n. \tag{1.2}$$

Here $a^*$ and $a$ are respectively the creation and annihilation operators for the oscillator. Similarly, $q_i^*$ and $q_i$ represent the creation and annihilation operators for the $i^{th}$ oscillator associated in the bath.

The Hamiltonian in (1.1) is Hermitian only when $\kappa = 0$. Consequently, the application of Hermitian quantum theory is not a valid option for dealing with such Hamiltonians. Hence, there arose a need to formulate a new theory capable of handling non-Hermitian operators. Notably, a key feature of the Hamiltonian described in (1.1) is its similarity to its conjugate transpose through a metric operator, a characteristic subsequently labeled as quasi-Hermiticity.

## 1.1 Quasi-Hermitian systems

In this thesis, we employ the notation $\mathcal{H}$ to denote the Hilbert space, with the inner product $\langle\cdot|\cdot\rangle$ exhibiting anti-linearity in the first argument. In other words, we have

$$\langle c_1\psi_1 + c_2\psi_2|\psi_3\rangle = \bar{c}_1\langle\psi_1|\psi_3\rangle + \bar{c}_2\langle\psi_2|\psi_3\rangle,$$

where $c_j \in \mathbb{C}$ and $|\psi_j\rangle \in \mathcal{H}$.

Let $\mathcal{H}$ be a finite-dimensional Hilbert space. An operator $\eta$ is said to be positive, denoted as $\eta > 0$, if $\langle\psi|\eta\psi\rangle > 0$ for all nonzero $|\psi\rangle \in \mathcal{H}$. This is equivalent with saying that $\eta^* = \eta$ and all eigenvalues of $\eta$ are strictly positive. Here $A^*$ is the adjoint of the operator $A$, defined by $\langle\psi|A\phi\rangle = \langle A^*\psi|\phi\rangle$ for all $|\phi\rangle, |\psi\rangle \in \mathcal{H}$. An operator $H$ on $\mathcal{H}$ is called ($\eta$-)*quasi-Hermitian* if there exists a positive operator $\eta > 0$ such that

$$H^* = \eta H \eta^{-1}. \tag{1.3}$$

Quasi-hermiticity is a special case of *pseudo-Hermiticity*, where (1.3) holds with an invertible (but not necessarily positive) Hermitian operator $\eta$. Pseudo- and quasi-Hermitian Hamiltonians arise in $PT$ symmetric quantum theory, see for instance [42] and references therein.

## 1.2 Hermitian counterparts

Let $H$ be a non-Hermitian operator on a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot|\cdot\rangle$, a candidate for the Hamiltonian of a physical system. In order to obtain a Hermitian quantum theory, one could either:

(1) Modify the inner product of $\mathcal{H}$ to $\langle\cdot|\eta\,\cdot\rangle$ for some metric operator $\eta > 0$, such that $H$ becomes Hermitian in the Hilbert space $\mathcal{H}_\eta$ with this new inner product; or

(2) Take a similarity transformation (invertible map) $S$ such that the transformed $h = SHS^{-1}$ is Hermitian in the original Hilbert space $\mathcal{H}$.

If $H$ is quasi-Hermitian, then both options (1) and (2) are possible, but neither the metric nor the similarity transform in options (1) and (2) are unique. To explore this non-uniqueness, we first notice that any quasi-Hermitian $H$ is diagonalizable [13, 16, 29]. More precisely,

$$H = \sum_{n=1}^{N} E_n |\psi_n\rangle\langle\phi_n|, \tag{1.4}$$

where the $E_n \in \mathbb{R}$ are the eigenvalues and the $\{|\psi_n\rangle, |\phi_n\rangle\}_{n=1}^N$ form a complete bi-orthonormal family, meaning that $\langle\psi_k|\phi_l\rangle = \delta_{kl}$ and $\sum |\psi_n\rangle\langle\phi_n| = \mathbb{1}$. Let us consider the case where all eigenvalues $E_n$ are distinct for simplicity (a discussion including degenerate eigenvalues can be done similarly, but this is not our focus here). Then the decomposition (1.4) is unique, it is the spectral representation of the operator $H$, and the $P_n \equiv |\psi_n\rangle\langle\phi_n|$ are the uniquely defined (generally not orthogonal) spectral projections. The vectors $|\psi_n\rangle$ and $|\phi_n\rangle$, however, are determined only up to a joint scaling $|\phi_n\rangle \mapsto z_n|\phi_n\rangle$ and $|\psi_n\rangle \mapsto \frac{1}{z_n}|\psi_n\rangle$, with $0 \neq z_n \in \mathbb{C}$ arbitrary.

• First let us explore the option (1). A metric $\eta$ is called a *metric for* $H$ if $H$ is $\eta$-quasi-Hermitian. Let $A$ be a linear operator on $\mathcal{H}$, with adjoint $A^*$ as defined

above. If $A$ is viewed as an operator on $\mathcal{H}_\eta$, then $\langle\phi|\eta A\psi\rangle = \langle\eta^{-1}A^*\eta\phi|\eta\psi\rangle$, so the adjoint of $A$ in $\mathcal{H}_\eta$ is

$$A^\ddagger = \eta^{-1}A^*\eta. \tag{1.5}$$

It follows that a given $\eta > 0$ is a metric for $H$ if and only if $H^\ddagger = H$, that is, if and only if $H$ is Hermitian acting on $\mathcal{H}_\eta$. It is well known (see e.g. [34, 16]) that

$$\eta \text{ is a metric for } H \iff \eta = \sum_{n=1}^{N} x_n|\phi_n\rangle\langle\phi_n| \text{ for some } x_1,\dots,x_N > 0, \tag{1.6}$$

where the $|\phi_n\rangle$ are the vectors appearing in (1.4). The multitude of metrics obtained by varying the $x_j$ in (1.6) naturally appears due to the fact that $|\phi_n\rangle$ is only determined up to an arbitrary nonzero scaling factor $z_n$ (as explained after (1.4)), which results in the scaling $x_n \mapsto x_n|z_n|^2$. Given this non-uniqueness of the metric, which one should be chosen to define the physical Hilbert space $\mathcal{H}_\eta$?

One answer is that the metric is fixed provided that instead of just $H$, one chooses a whole irreducible family of operators to be Hermitian observables. Namely, it is shown in [38] (see also [35] for the two-dimensional case) that if there is a family of operators $\{A_i\}_i$ on $\mathcal{H}$, and positive operators $\eta$, $\eta'$ such that $A_i^* = \eta A_i \eta^{-1}$ and $A_i^* = \eta' A_i (\eta')^{-1}$ for all $i$, then

$$\eta' \text{ is a scalar multiple of } \eta \iff \{A_i\}_i \text{ is an irreducible family of operators on } \mathcal{H}.$$

Recall a set $\mathcal{O}(\mathcal{V})$ of operators on a vector space $\mathcal{V}$ is **irreducible** if there are no proper subspaces of $\mathcal{V}$ (i.e. except $\mathcal{V}$ and $\{0\}$) that are invariant under all $\hat{\mathcal{O}} \in \mathcal{O}(\mathcal{V})$.

This means that for an irreducible family of quasi-Hermitian operators, there is exactly one metric (up to a scalar multiple) that makes those operators Hermitian. The chosen family[1] can then be viewed as the physical observables of the theory and the space of pure states is $\mathcal{H}$ with inner product $\langle\cdot|\cdot\rangle_\eta$.

On the other hand, if interested only in the single observable $H$ (the Hamiltonian), one should keep the $x_n$ in (1.6) general.

- Next, let us investigate option (2) for $H$ of the form (1.4). Let $\eta$ be a metric

[1]Examples of irreducible families are the Pauli matrices for a spin, with the Euclidean inner product on $\mathbb{C}^2$, or the position $\hat{x}$ and momentum $\hat{p} = -i\hbar\nabla_x$ for a quantum particle (rather, the bounded Weyl operators generated by them) with the inner product $\langle\psi|\phi\rangle = \int_{\mathbb{R}^3} \bar{\psi}(x)\phi(x)\,d^3x$.

for $H$, so it is of the form (1.6). We find all invertible $S$ such that the transformed $h \equiv SHS^{-1}$ is Hermitian,

$$h = SHS^{-1} = \big(SHS^{-1}\big)^* = h^*. \tag{1.7}$$

One readily sees that (1.7) is equivalent to $TH = HT$, where $T = \eta^{-1}S^*S$. That $T$ commutes with $H$, as in (1.4), is equivalent to $T$ being diagonal in the same bi-orthonormal system as $H$, that is, $T = \sum_{n=1}^N t_n|\psi_n\rangle\langle\phi_n|$ for some $t_n \in \mathbb{C}$. Now

$$S^*S = \eta T = \Big(\sum_{n=1}^N x_n|\phi_n\rangle\langle\phi_n|\Big)\Big(\sum_{k=1}^N t_k|\psi_k\rangle\langle\phi_k|\Big) = \sum_{n=1}^N x_n t_n|\phi_n\rangle\langle\phi_n|, \tag{1.8}$$

and as $S^*S > 0$ and $x_n > 0$, we have $t_n > 0$ as well. It follows from (1.6) and (1.8) that $\eta T$ is also a metric for $H$. In fact, (1.6) and (1.8) show that given a fixed metric $\eta$ for $H$ and varying $\eta T$ over all operators $T > 0$ that commute with $H$, we obtain all of the metrics for $H$. We conclude that given $\eta$, the $S$ we are looking for are the solutions of $S^*S = \eta T$, where $T > 0$ is an operator that commutes with $H$ (equivalently, is diagonal in the same bi-orthonormal system as $H$). The general solution is

$$S = W\sqrt{\eta T}, \tag{1.9}$$

where $W$ is any unitary and where for a positive operator $A$, $\sqrt{A}$ is the unique positive operator whose square equals $A$.

Once $W$ and $T$ are chosen, the associated Hermitian $h$ in (1.7) becomes

$$h_{W,T} = W\sqrt{\eta T}\, H \frac{1}{\sqrt{\eta T}} W^*. \tag{1.10}$$

We stress with this notation that $h$ depends on the choice of $W$ and $T$. The $h$ obtained from two different choices of unitaries, say $V$ and $W$, are unitarily equivalent, with $h_{V,T} = U h_{W,T} U^*$ and $U = VW^*$. In this sense, the choice of $W$ is globally immaterial. However, if the Hilbert space has a local structure, say is of bipartite nature $\mathcal{H} = \mathcal{H}_{\rm S} \otimes \mathcal{H}_{\rm B}$, then the global unitary $U$ may well change the local properties of the two local subsystems, in which case the choice of $W$ will play a physically relevant role. We also point out that the spectrum of $h_{W,T}$ does not depend on either $W$ or $T$ (or $\eta$, for that matter).

In Chapter 6, we start with a given quasi-Hermitian Hamiltonian $H$ and an arbitrary metric $\eta$ for $H$ and we view $\mathcal{H}_\eta$ as the physical Hilbert space. We analyze the class of all associated *Hermitian systems* $h_{W,T}$, where $W$ and $T$ vary over all unitaries and all positive operators commuting with $H$, respectively. As explained above, considering all metrics $\eta$ is the same as considering all metrics $\eta T$, so varying over $T$ is redundant if $\eta$ is kept arbitrary. We may then set $T = \mathbb{1}$ and only consider

$$S = W\sqrt{\eta}, \qquad h_W = W\sqrt{\eta}\, H \frac{1}{\sqrt{\eta}} W^* \tag{1.11}$$

for all $W$ and $\eta$.

## 1.3 States, reduced states, von Neumann entropy

Consider now a fixed metric $\eta$, so that the physical Hilbert space is $\mathcal{H}_\eta$ and $H$ is Hermitian on $\mathcal{H}_\eta$, $H^\ddagger = H$, where $H^\ddagger$ be defined as stated in equation (1.5). Then $e^{-itH}$ is the unitary Schrödinger dynamics on $\mathcal{H}_\eta$. The average of an observable $A$ on $\mathcal{H}_\eta$ in the state $|\psi\rangle \in \mathcal{H}_\eta$ is given by

$$\langle\psi|A\psi\rangle_\eta = \langle\psi|\eta A\psi\rangle = \mathrm{tr}\big(|\psi\rangle\langle\psi|\eta A\big) = \mathrm{tr}\big(\widetilde{\rho}A\big), \tag{1.12}$$

where

$$\widetilde{\rho} = |\psi\rangle\langle\psi|\eta \tag{1.13}$$

is a density matrix on $\mathcal{H}_\eta$ (a positive, trace-one operator). This $\widetilde{\rho}$ is called the 'generalized density matrix' in [39]. It is important to point out that the trace in (1.12) is a purely algebraic quantity: it is the sum of the eigenvalues of the operator, and therefore does not depend on the choice of the metric.

To arrive at a Hermitian Hamiltonian, it is necessary to make a choice for the unitary $W$ in (1.11). The associated Hermitian Hamiltonian $h_W$ is then given by (1.11). Let

$$|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle, \quad |\phi(t)\rangle = e^{-ith_W}|\phi(0)\rangle \tag{1.14}$$

be the evolution of the initial states $|\psi(0)\rangle$, $|\phi(0)\rangle$ with respect to $H$ and $h_W$, respectively. The states are related by

$$|\phi(t)\rangle = S|\psi(t)\rangle, \quad S = W\sqrt{\eta}, \tag{1.15}$$

and the density matrices associated to these vector states for the non-Hermitian (see (1.13)) and the Hermitian systems are

$$\rho_H(t) = |\psi(t)\rangle\langle\psi(t)|\eta \quad \text{and} \quad \rho_{h_W}(t) = |\phi(t)\rangle\langle\phi(t)|, \tag{1.16}$$

respectively. (We adopt the notation $\rho_{h_W}$ and $\rho_H$ for the density matrices on the Hermitian and non-Hermitian sides of the problem from [18].) It is clear from (1.15) that

$$\rho_{h_W}(t) = S|\psi(t)\rangle\langle\psi(t)|S^* = S|\psi(t)\rangle\langle\psi(t)|(S^*S)S^{-1} = S\rho_H(t)S^{-1}. \tag{1.17}$$

It follows that $\rho_{h_W}(t)$ and $\rho_H(t)$ have the same eigenvalues, and hence the same von Neumann entropy, $\mathcal{E}(\rho_{h_W}(t)) = \mathcal{E}(\rho_H(t))$, where

$$\mathcal{E}(\rho) = -\mathrm{tr}\big(\rho\ln\rho\big) = -\sum_i \lambda_i \ln\lambda_i \tag{1.18}$$

and $\{\lambda_i\}$ are the eigenvalues of $\rho$.

Consider now a bipartite system with $\mathcal{H} = \mathcal{H}_{\rm S} \otimes \mathcal{H}_{\rm B}$ ('system' and 'bath'). We consider the *reduced states* (denoted by an overbar) defined by tracing out the degrees of freedom of the subsystem $\mathcal{H}_{\rm B}$,

$$\bar\rho_H(t) = \mathrm{tr}_{\mathcal{H}_{\rm B}}\big(\rho_H(t)\big), \qquad \bar\rho_{h_W}(t) = \mathrm{tr}_{\mathcal{H}_{\rm B}}\big(\rho_{h_W}(t)\big). \tag{1.19}$$

In some recent works [18, 22, 21, 11], the dynamics of a bipartite system generated by a non-Hermitian Hamiltonian $H$ is studied, with particular focus on the von Neumann entropy of the reduced density matrix $\bar\rho_H(t)$. The strategy proposed in those works is to examine the entropy of $\bar\rho_{h_W}(t)$ as a proxy for that of $\bar\rho_H(t)$. In this respect, however, one should observe the following facts:

1. The operator $\bar\rho_H(t)$ always satisfies $\mathrm{tr}_{\mathcal{H}_{\rm S}}(\bar\rho_H(t)) = 1$, but for some choices of $\eta$

the eigenvalues of $\bar{\rho}_H(t)$ can be complex, in which case it is not a valid density matrix.

2. Even if the metric $\eta$ is chosen such that $\bar{\rho}_H(t)$ is a density matrix, for generic choices of $W$ the von Neumann entropies $\mathcal{E}(\bar{\rho}_H(t))$ and $\mathcal{E}(\bar{\rho}_{h_W}(t))$ are not the same. The latter in fact depends on the choice of $W$.

To understand the normalization of the trace mentioned in fact 1. above, we observe (using $\mathbb{1}_{\mathrm{S}}$ as the system observable) that

$$\mathrm{tr}_{\mathcal{H}_{\mathrm{S}}}(\bar{\rho}_H(t)) = \mathrm{tr}_{\mathcal{H}_{\mathrm{S}}}(\bar{\rho}_H(t)\mathbb{1}_{\mathrm{S}}) = \mathrm{tr}_{\mathcal{H}_{\mathrm{S}}\otimes\mathcal{H}_{\mathrm{B}}}\big(\rho_H(t)(\mathbb{1}_{\mathrm{S}}\otimes\mathbb{1}_{\mathrm{B}})\big) = \mathrm{tr}_{\mathcal{H}_{\mathrm{S}}\otimes\mathcal{H}_{\mathrm{B}}}(\rho_H(t)) = 1.$$

If $S = S_{\mathrm{S}} \otimes S_{\mathrm{B}}$, then $\bar{\rho}_h = S_{\mathrm{S}}\bar{\rho}_H S_{\mathrm{S}}^{-1}$ and so the spectra and thus the von Neumann entropies of $\bar{\rho}_h$ and $\bar{\rho}_H$ coincide. However, if $S$ is entangling (not of product form $S_{\mathrm{S}} \otimes S_{\mathrm{B}}$), then the eigenvalues of the two reduced density matrices are not the same in general, and neither are their entropies.

These difficulties are resolved in Chapter 6, where we study the concrete model used in [18]. In particular, we determine for which choices of $\eta$ the reduced operator $\bar{\rho}_H(t)$ is indeed a density matrix, and then we find the von Neumann entropy of $\bar{\rho}_{h_W}(t)$ *for all possible choices of the unitary* $W$.

## 1.4 Main results and organization of the thesis

Our main findings are summed up as follows:

1. **Diagonalizability at transition point.** We successfully compute all $2 \times 2$ Hamiltonians possessing $PT$ symmetry, where $P$ is an involutive $2 \times 2$ matrix, meaning $P^2 = \mathbb{1}$, and $T$ is the complex conjugate operator with respect to a fixed basis acting on the complex space $\mathbb{C}^2$. We then proceed to delineate the regions where these Hamiltonians exhibit unbroken and broken symmetry, pinpointing the specific transition points. Subsequently, our investigation focuses on the diagonalizability of these Hamiltonians precisely at these transition points. See Chapter 3.

2. **Existence of metrics and their properties.** In Chapters 5, we consider a general setup of a $PT$ symmetric system. In Chapter 5 we introduce a generic

model of a family of Hamiltonians $H(\varkappa)$ indexed by a parameter $\varkappa$ varying around a symmetry transition point $\varkappa_0$, with $\varkappa < \varkappa_0$ corresponding to the symmetry unbroken regime. Our main result is Theorem 5.2.2. It shows that at the transition point $\varkappa_0$, the Hamiltonian $H(\varkappa_0)$ is diagonalizable if and only if there exists a time-independent metric $\eta$, that is, if and only if $H^*(\varkappa_0)\eta = \eta H(\varkappa_0)$ for some $\eta > 0$.

We show in Section 5.2.2 that in the $PT$ symmetry broken regime, there are *no* time-independent metrics (Proposition 5.2.5). Furthermore, we show in Theorem 5.2.3 that in the $PT$ symmetry unbroken regime, any possibly time-dependent metric is a bounded function of time, while in the $PT$ symmetry broken regime, one can always find a time-dependent metric which grows exponentially in time.

3. **Analysis of an explicit $PT$ symmetric open system.** In Chapter 6 we introduce an $N+1$ oscillator model with a $PT$ symmetric Hamiltonian - it is the same model that has been considered in [18]. In Sections 6.1.2 and 6.1.3 we obtain the explicit formulas for the states $|\psi(t)\rangle$ and $|\phi(t)\rangle$ as well as the reduced states $\bar{\rho}_H(t)$ and $\bar{\rho}_{h_W}(t)$.

   – **Metrics.** We find all possible metrics for the given Hamiltonian and show that the operator $\bar{\rho}_H(t)$ is a density matrix exactly when $\eta$ is of product form $\Lambda_{\mathrm{S}} \otimes \Lambda_{\mathrm{B}}$ – otherwise $\bar{\rho}_H(t)$ has complex eigenvalues. See Section 6.1.1. Consequently, for the further analysis we take $\eta$ of product form.

   – **Subsystem entropy.** The reduced states $\bar{\rho}_H(t)$ and $\bar{\rho}_{h_W}(t)$ are periodic[2] in time, both having the same period regardless of the choice of $W$. The von Neumann entropy of $\bar{\rho}_H(t)$ and $\bar{\rho}_{h_W}(t)$ is periodic in time as well, but for *generic initial conditions and generic* $W$, the period of the entropy of the Hermitian system is *double* that of the non-Hermitian system. See Section 6.2.

   – SB **entanglement.** We show that the non-Hermitian and the Hermitian SB (system-bath) states $|\psi(t)\rangle$, $|\phi(t)\rangle$ are entangled for all times except at periodically reoccurring single instants. Given any entangled state $|\psi\rangle$, one can find $W$ such that the associated $|\phi\rangle$ is disentangled, and for any disentangled $|\psi\rangle$ there

[2]The whole SB complex consists of $N+1$ oscillators, so the energy spectrum of all the Hamiltonians involved consists of discrete eigenvalues only, without continuous spectrum. This explains the periodicity.

are $W$ such that $|\phi\rangle$ is entangled. However, in an averaged sense, the choice of $W$ does not influence the entanglement at all. Namely, the *concurrence* of the time-averaged density matrix, $\langle\rho\rangle = \frac{1}{T}\int_0^T |\phi(t)\rangle\langle\phi(t)|\, dt$, where $T$ is the period of $|\phi(t)\rangle\langle\phi(t)|$, is *independent* of $W$. Its value is determined entirely by the initial condition and the choice of the metric. We identify the initial states for which $\langle\rho\rangle$ is separable and for which it is maximally entangled. See Section 6.3.

The structure of this thesis is as follows:

Chapter 2: Concise introduction to key concepts in quantum mechanics, such as $PT$ symmetry, and pseudo-Hermiticity, which are pertinent to the research presented.

Chapter 3: Investigation of the diagonalizability of $2 \times 2$ matrices that exhibit $PT$ symmetry.

Chapter 4: Introductory discussion of the Dyson map.

Chapter 5: Analysis of time-independent solutions of the Dyson map. Exploration of the properties of these solutions, focusing on their positivity and invertibility.

Chapter 6: Analysis of the concrete $N+1$-oscillator $PT$ symmetric system. Based on the published work [26].

# Chapter 2

# Basics of Hermitian and Non-Hermitian Quantum Mechanics

## 2.1 The basic postulates of quantum mechanics

Quantum mechanics serves as a mathematical framework for the formulation of physical theories. In this chapter, we provide a concise overview of the fundamental postulates that underpin quantum mechanics. These postulates establish a vital link between physical reality and the mathematical structure of quantum mechanics [37].

### 2.1.1 Postulate 1: Space of pure states

Any physical system is described by a complex Hilbert space $\mathcal{H}$, known as the (pure) state space of the system. The system is completely described by its state vector, which is a unit vector in the system state space,

$$|\varphi\rangle \in \mathcal{H}, \qquad \|\varphi\| = 1.$$

Such a state (vector) is also called a "**ket**" (or a wave function).

To any ket $|\varphi\rangle$ is associated the "**bra**", denoted by $\langle\varphi|$, defined to be the element

in the dual space $\mathcal{H}^*$ of $\mathcal{H}$ acting as

$$\langle\varphi|\Big(|\psi\rangle\Big) = \langle\varphi|\psi\rangle, \tag{2.1}$$

where the right hand side is the inner product of $|\varphi\rangle$ and $|\psi\rangle$ in $\mathcal{H}$.

**Examples.**

1. A single spin has a Hilbert space $\mathcal{H} = \mathbb{C}^2$. It has basis $B = \{|\uparrow\rangle, |\downarrow\rangle\}$, where

$$|\uparrow\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \quad |\downarrow\rangle = \begin{bmatrix}0\\1\end{bmatrix}.$$

   Any state can be written as a linear combination of the basis elements,

$$|\psi\rangle \in \mathbb{C}^2 \qquad |\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle, \qquad \alpha, \beta \in \mathbb{C}. \tag{2.2}$$

   The interpretation of the complex numbers $\alpha$, $\beta$ is that $|\alpha|^2$, $|\beta|^2$ are probabilities of finding the spin in the state up or down, respectively (upon measurement, see Postulate 4 below). The normalization $\|\psi\|^2 = |\alpha|^2 + |\beta|^2 = 1$ is consistent with this probability interpretation of the coordinates.

2. A single particle in three-dimensional space is described by the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, d^3x)$ of complex-valued square-integrable functions. A (pure) state is given by a square-integrable normalized function $\psi(x)$. The physical interpretation of the 'component' $\psi(x)$ is this: $|\psi(x)|^2 d^3x$ is the probability density of finding the particle at location $x \in \mathbb{R}^3$.

### 2.1.2 Postulate 2: Dynamics (Schrödinger equation)

The state of a quantum system evolves in time according to an evolution equation, the Schrödinger equation. Namely, the orbit $t \mapsto |\psi(t)\rangle$ satisfies the first-order linear differential equation

$$i\hbar\frac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle. \tag{2.3}$$

Here $\hbar$ is the *Planck constant* and $H$ is a self-adjoint operator acting on the pure state Hilbert space $\mathcal{H}$, called the *Hamiltonian*. Equation (2.3) is solved by

$$|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle, \tag{2.4}$$

where we have chosen units so that the Planck constant $\hbar = 1$ . The unitary group

$$t \mapsto U(t) = e^{-itH} \tag{2.5}$$

is often called the *propagator*, as it pushes the initial condition to the state at time $t$.

**Example.** If a spin is initially in the state $|\psi(0)\rangle = \alpha_0|\uparrow\rangle + \beta_0|\downarrow\rangle$, then according to the Schrödinger equation (2.4), with Hamiltonian

$$H = \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2.6}$$

the state at time $t$ is

$$|\psi(t)\rangle = \alpha_0 e^{-it/2}|\uparrow\rangle + \beta_0 e^{it/2}|\downarrow\rangle. \tag{2.7}$$

### 2.1.3 Postulate 3: Composition of systems

If two systems have Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ then the composite system is described by the tensor product,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2. \tag{2.8}$$

**Examples.**

1. The Hilbert space describing $N$ particles is given by

$$\bigotimes_{i=1}^{N} \mathcal{H}_i = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N,$$

   where for each $1 \leq i \leq N$, $\mathcal{H}_i = L^2(\mathbb{R}^3, d^3x)$.

2. The composite space $\mathcal{H}$ of a spin and a single particle is

$$\mathcal{H} = \mathbb{C}^2 \otimes L^2(\mathbb{R}^3, d^3x).$$

If a state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ can be written as $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ with $|\psi_j\rangle \in \mathcal{H}_j$ , then it is called a (tensor) **product state** or **a separable state**, or a **disentangled state**. A state that is not of product form is called a non-separable state, or an **entangled state**.

### 2.1.4 Postulate 4: Measurements

Every physical observable (energy, position, momentum, etc) is associated with a self-adjoint operator $A = A^*$. The Hamiltonian $H$ (see Postulate 2) is the observable of *energy*. Suppose the spectral decomposition of $A$ is given by

$$A = \sum_j \lambda_j P_j, \tag{2.9}$$

where the $P_j$ are the spectral projections and $\lambda_j$ the eigenvalues. When measuring the observable $A$ in any state, the possible measurement outcomes are one of $\{\lambda_1, \lambda_2, \dots\}$. When the measurement is performed on the state $|\psi\rangle$, the outcome $\lambda_i$ will occur with probability

$$p_i = \|P_j|\psi\rangle\|^2 = \langle\psi|P_j\psi\rangle. \tag{2.10}$$

If the measurement reveals the outcome $\lambda_j$, then the state of the system *immediately after measurement* is

$$|\psi_{\text{post}}\rangle = \frac{P_j|\psi\rangle}{\|P_j|\psi\rangle\|}. \tag{2.11}$$

This part of the postulate is called the "wave function collapse" and (2.11) is called the post-measurement state.

**Examples.**

1. Consider the spin with Hamiltonian (2.6),

$$H = \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \tfrac{1}{2}P_+ - \tfrac{1}{2}P_- \tag{2.12}$$

   (with obvious notation for the spectral projections). The measurement outcomes for the energy are $\pm 1/2$ in any state. Upon measurement of the energy in the

state $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$, the measurement value $+1/2$ occurs with probability

$$p_+ = \|P_+|\psi\rangle\|^2 = |\alpha|^2.$$

2. Let $A$ be an observable and $|\psi\rangle$ a state. The expectation value (statistical average) of $A$ with respect to the state $|\psi\rangle$ is denoted by $\langle A\rangle$. From Postulate 4 we know that the possible measurement outcomes of $A$ are its eigenvalues, where each eigenvalue $\lambda_j$ will occur with probability $p_j$. Thus the average of $A$ is

$$\langle A\rangle = \sum_j \lambda_j p_j. \tag{2.13}$$

Using the probability formula in equation (2.10) and the spectral decomposition of $A$ in (2.9) we have

$$\begin{aligned} \langle A\rangle &= \sum_j \lambda_j \langle\psi|P_j\psi\rangle \\ &= \langle\psi|\sum_j \lambda_j P_j\psi\rangle \\ &= \langle\psi|A\psi\rangle \\ &= \mathrm{tr}\big(|\psi\rangle\langle\psi|A\big). \end{aligned} \tag{2.14}$$

The trace of an operator $X$ (if it exists) is given by

$$\mathrm{tr}(X) = \sum_{n\in\mathbb{N}} \langle e_n|Xe_n\rangle, \tag{2.15}$$

for any orthonormal basis $\{|e_n\rangle\}_{n\in\mathbb{N}}$ of $\mathcal{H}$. The definition of trace is independent of the choice of the orthonormal basis.

### 2.1.5 Density matrix

The average of an observable $\mathcal{O}$ in the pure state $|\varphi\rangle$ is $\langle\varphi|\mathcal{O}\varphi\rangle$, see (2.14). Suppose now that our knowledge of the state is not perfect, namely, that we only know that with probabilities $p_j$ our state is $|\varphi_j\rangle$. The collection $\{|\varphi_j\rangle, p_j\}$ is called an *ensemble of pure states*. The average of the observable $\mathcal{O}$ associated to that ensemble is naturally

defined to be

$$\langle \mathcal{O} \rangle = \sum_j p_j \langle \varphi_j | \mathcal{O} \varphi_j \rangle. \tag{2.16}$$

By defining the *density matrix*

$$\rho := \sum_j p_j |\varphi_j\rangle\langle\varphi_j|, \tag{2.17}$$

we see that

$$\langle \mathcal{O} \rangle = \mathrm{tr}\big(\rho\, \mathcal{O}\big). \tag{2.18}$$

The density matrix $\rho$, (2.17), is called a **mixed state** [8] if its rank exceeds one. If $\rho$ has rank one, then it is called a pure state, $\rho = |\varphi\rangle\langle\varphi|$ (see (2.14)). More generally, any operator $\rho$ acting on $\mathcal{H}$ satisfying the following properties is a density matrix:

- $\rho \geq 0$ (positive, in particular self-adjoint),
- $\mathrm{tr}(\rho) = 1$ (normalized).

**Examples.**

1. For any $0 \leq p \leq 1$, the following is a family of density matrices of a spin,

$$\rho = p|\uparrow\rangle\langle\uparrow| + (1-p)|\downarrow\rangle\langle\downarrow| = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}.$$

   Here, $\rho$ is pure if and only if $p \in \{0, 1\}$.

2. Let $|\psi\rangle$ be a general pure state of a spin, (2.2). The associated density matrix (written in the basis $|\uparrow\rangle$, $|\downarrow\rangle$) reads

$$\rho = |\psi\rangle\langle\psi| = \begin{bmatrix} |\alpha|^2 & \bar{\alpha}\beta \\ \alpha\bar{\beta} & |\beta|^2 \end{bmatrix}. \tag{2.19}$$

### 2.1.6 Reduced density matrix and partial trace

Let $X \otimes Y$ be an operator on the composite system $\mathcal{H}_1 \otimes \mathcal{H}_2$, formed by two separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. We define the partial trace over $\mathcal{H}_2$ by

$$\mathrm{tr}_2(X \otimes Y) = X\mathrm{tr}(Y). \tag{2.20}$$

$\mathrm{tr}_2$ extends by linearity and countinuity to a linear map from $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ to $\mathcal{B}(\mathcal{H}_1)$ [40]. The partial trace is important when we study the physical state of a subsystem of the composite system. In other words, if $\rho_{12}$ is a density matrix of the composite system $\mathcal{H}_1 \otimes \mathcal{H}_2$, then the *reduced states* are

$$\rho_1 = \mathrm{tr}_2(\rho_{12}), \qquad \rho_2 = \mathrm{tr}_1(\rho_{12}). \tag{2.21}$$

$\rho_1$ and $\rho_2$ are called the **reduced density operators** for the systems 1 and 2, respectively. The point of this construction is the following.

Suppose $\rho_{12}$ is the density matrix of the composite system $\mathcal{H}_1 \otimes \mathcal{H}_2$, and we want to find the average of an observable $\mathcal{O}_1$ of system 1 only. This average is

$$\mathrm{tr}_{\mathcal{H}_1 \otimes \mathcal{H}_2}\Big(\rho_{12}(\mathcal{O}_1 \otimes \mathbb{1}_2)\Big) = \mathrm{tr}_{\mathcal{H}_1}(\rho_1 \mathcal{O}_1). \tag{2.22}$$

This means we can use the reduced density matrix of a composite system if we are interested in the properties of a subsystem only.

**Example.** The Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ describes the pure states of two spins. Consider the pure state (Bell state)

$$|\psi\rangle = \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}, \tag{2.23}$$

where $|\uparrow\uparrow\rangle = |\uparrow\rangle \otimes |\uparrow\rangle$ , $|\downarrow\downarrow\rangle = |\downarrow\rangle \otimes |\downarrow\rangle$ and call its density matrix $\rho_{12} = |\psi\rangle\langle\psi|$. The reduction to the first spin is

$$\rho_1 = \mathrm{tr}_2\rho_{12} = \tfrac{1}{2}\big(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\big) = \frac{1}{2}\mathbb{1}. \tag{2.24}$$

This example shows that the reduced state of a pure state can actually be a mixed state. (Note, the rank of $\rho_1$ is two.)

### 2.1.7 Evolution of quantum systems

According to Postulate 2, the dynamics of the pure initial state $|\psi(0)\rangle$ is given by $|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle$. Equivalently, the propagator

$$U(t) = e^{-itH}, \tag{2.25}$$

satisfies the evolution equation

$$i\frac{dU(t)}{dt} = HU(t). \tag{2.26}$$

This is the setup of Postulate 2, which implicitly assumes that the system considered is *closed*, meaning that it is not in contact with 'external agents'. (Strictly speaking, thus, the only closed system is the whole universe, since in reality, *any* system is in contact with its surroundings.)

How does the dynamics look for a closed system in a mixed state? Suppose the system is described by the density matrix $\rho(0)$ at time zero. To get the equation of motion for this state, we use the definition of the density matrix in (2.17),

$$\rho(0) = \sum_j p_j |\varphi_j(0)\rangle\langle\varphi_j(0)|. \tag{2.27}$$

Now the evolution of $|\varphi_j\rangle$ is given by $|\varphi_j(t)\rangle = U(t)|\varphi_j(0)\rangle$ and so the density matrix at time $t$ is

$$\begin{aligned} \rho(t) &= \sum_j p_j U(t)|\varphi_j(0)\rangle\langle\varphi_j(0)|U^*(t) \\ &= U(t)\rho(0)U^*(t), \end{aligned} \tag{2.28}$$

where $U^*(t)$ is the adjoint of $U(t)$. With (2.25) this becomes

$$\rho(t) = e^{-itH}\rho(0)e^{itH}. \tag{2.29}$$

Equation (2.29) is called the **Liouville-von Neumann equation** [9]. In differential form, it takes the shape

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)]. \tag{2.30}$$

Let $\mathcal{O}$ be an observable. Its average in the state $\rho(t)$ is

$$\begin{aligned} \mathrm{tr}\Big(\rho(t)\mathcal{O}\Big) &= \mathrm{tr}\big(e^{-itH}\rho(0)e^{itH}\mathcal{O}\big) \\ &= \mathrm{tr}\big(\rho(0)\mathcal{O}(t)\big), \end{aligned} \tag{2.31}$$

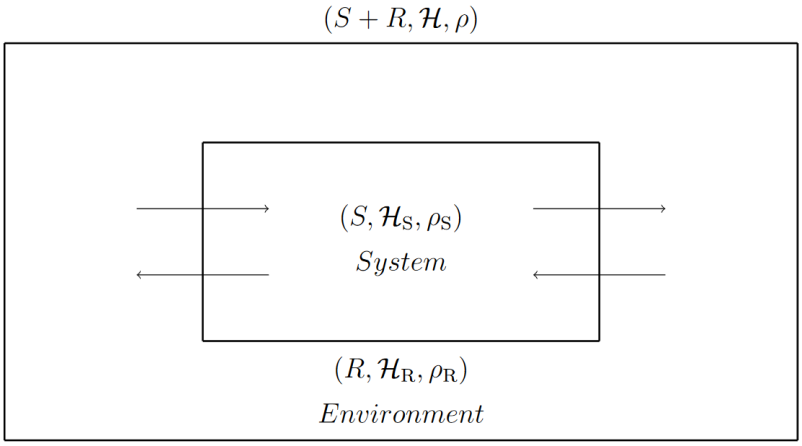
where

$$\mathcal{O}(t) = e^{itH}\mathcal{O}e^{-itH}. \tag{2.32}$$

The map $t \mapsto \mathcal{O}(t)$ called the **Heisenberg evolution** of the observable $\mathcal{O}$ [23].

### 2.1.8 Open systems

An open system is a system in contact with an 'environment', with which the system can exchange energy, matter, information, etc. The following diagram illustrates what we mean by an open quantum system.




In the above figure, the system $S$ is described by a Hilbert space $\mathcal{H}_S$ and a state $\rho_S$. It is coupled with the environment $R$ ("reservoir") which is described by a Hilbert space $\mathcal{H}_R$ and a state $\rho_R$.

Postulate 3 tells us that the total system $S + R$ is given by the tensor product $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_R$. A state of the joint system, in which the system and reservoir parts are not correlated (no entanglement) is given by $\rho = \rho_S \otimes \rho_R$. The total Hamiltonian

$H$ for the composite system has the form

$$H = H_S \otimes \mathbb{1}_R + \mathbb{1}_S \otimes H_R + H_{\mathrm{C}}, \tag{2.33}$$

where $H_S$ and $H_R$ are the Hamiltonians of the system and environment. $H_C$ is the Hamiltonian of the interaction between the system and environment, which acts on the total system $\mathcal{H}$. Even though the evolution of the total complex $S+R$ is given by the Schrödinger equation (unitary propagator), the time evolution of the open subsystem $S$ is not, in general, unitary. The non-unitary dynamics of the open system comes from the interaction between the system and the environment. It reflects the fact that the system can lose energy, matter, etc.

Let $\rho(0)$ be the initial state of the complex $S+R$. The reduced density matrix of $S$ at time $t$ is given by

$$\rho_S(t) = \mathrm{tr}_R\{U(t)\rho(0)U^*(t)\}, \tag{2.34}$$

where we take the partial trace over the reservoir degrees of freedom. In equation (2.34), $U(t)\rho(0)U^*(t)$ is the (closed, unitary) evolution of the *total* complex $S+R$.

The differential form of (2.34) is

$$\frac{d}{dt}\mathrm{tr}_R\rho(t) = \frac{d}{dt}\rho_S(t) = -i\mathrm{tr}_R\big[H, \rho(t)\big], \tag{2.35}$$

where $H$ is as in (2.33).

An observable $\mathcal{O}_S$ of the open system $S$ has the form

$$\mathcal{O}_S = \mathcal{O} \otimes \mathbb{1}_R, \tag{2.36}$$

where operator $\mathcal{O}$ acting on $\mathcal{H}_S$ and $\mathbb{1}_R$ stands for the identity operator of $\mathcal{H}_R$. The average value of $\mathcal{O}_S$ is given by

$$\langle \mathcal{O}_S(t) \rangle = \mathrm{tr}_S(\rho_S(t)\mathcal{O}_S), \tag{2.37}$$

where $\rho_S(t)$ as is in (2.34).

### 2.1.9 The Von Neumann entropy

The von Neumann entropy of the density operator $\rho$ is defined by

$$\mathcal{E}(\rho) = -\mathrm{tr}(\rho \log(\rho)). \tag{2.38}$$

If the trace is infinite, we set $\mathcal{E}(\rho) = +\infty$. If we use the spectral decomposition (2.17) of $\rho$ , we find the representation

$$\mathcal{E}(\rho) = -\sum_j p_j \log(p_j). \tag{2.39}$$

Here are some fundamental properties of entropy. For a more in-depth exploration, please see reference [37].

**Proposition 2.1.1.** *The entropy $\mathcal{E}(\rho)$ is a non-negative function. Moreover,*

*(i) The state $\rho$ is pure if and only if $\mathcal{E}(\rho) = 0$.*

*(ii) In a $N$-dimensional Hilbert space, the entropy is at most $\log N$. The maximum entropy is taken by a completely mixed state. More precisely, $\mathcal{E}(\rho) = \log N$ if and only if the system is in the completely mixed state $\rho = \frac{\mathbb{1}}{N}$, where $\mathbb{1}$ is the identity operator.*

*(iii) For a bipartite pure state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2$, $\mathcal{E}(\rho_1) = \mathcal{E}(\rho_2)$.*

### 2.1.10 Creation and annihilation operators of the harmonic oscillator

We begin with the Hamiltonian operator for the harmonic oscillator $H$ acting on the square-integrable functions Hilbert space $L^2(\mathbb{R}, dx)$ expressed as

$$H = \frac{\hat{p}^2}{2\mu} + \frac{1}{2}\mu\omega^2\hat{x}^2, \tag{2.40}$$

where $\mu$ is the particle's mass, $\omega$ is the angular frequency of the oscillator, $\hat{x}$ is the position operator (the multiplication operator acting as $\hat{x}\psi(x) = x\psi(x)$), and $\hat{p}$ is the momentum operator (given by $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ in the coordinate basis). We next introduce

the operators $Q$ and $P$, related to $\hat{x}$ and $\hat{p}$ by the equations

$$\hat{x} = \left(\frac{\hbar}{\mu\omega}\right)^{\frac{1}{2}} Q, \tag{2.41}$$

and

$$\hat{p} = (\mu\omega\hbar)^{\frac{1}{2}} P. \tag{2.42}$$

Using

$$[\hat{x}, \hat{p}] = i\hbar, \tag{2.43}$$

it can easily be shown that

$$[Q, P] = i. \tag{2.44}$$

Substituting equations (2.41) and (2.42) into equation (2.40) we obtain

$$H = \frac{1}{2}\hbar\omega(P^2 + Q^2). \tag{2.45}$$

We next define

$$H_1 = \frac{H}{\hbar\omega} = \frac{1}{2}(P^2 + Q^2) \tag{2.46}$$

It is easy to see that if $\psi$ is an eigenfunction of $H_1$ with the eigenvalue $\lambda$, then $\psi$ is also an eigenfunction of $H$ with eigenvalue $E = \hbar\omega\lambda$. We next define the **annihilation** operator by

$$\hat{a} = \frac{1}{\sqrt{2}}(Q + iP). \tag{2.47}$$

The adjoint of the annihilation operator

$$\hat{a}^* = \frac{1}{\sqrt{2}}(Q - iP). \tag{2.48}$$

is called a **creation** operator. Clearly, $\hat{a}$ is not Hermitian. Using (2.44), it is easy to show the commutator between creation and annihilation operators is given by

$$[\hat{a}, \hat{a}^*] = 1. \tag{2.49}$$

Using the definition of creation and annihilation operators in (2.47) and (2.48)

respectively, the expression for $H_1$ becomes

$$H_1 = \hat{a}^*\hat{a} + \frac{1}{2} = N + \frac{1}{2}, \tag{2.50}$$

where

$$N = \hat{a}^*\hat{a}, \tag{2.51}$$

is called the **number** operator. We see that if $\psi_n$ is an eigenfunction function of $N$ with eigenvalue $n$; i.e. $N\psi_n = n\psi_n$, then $\psi_n$ is also an eigenfunction of $H_1$ with eigenvalue $n + 1/2$ as well as an eigenfunction of $H$ with eigenvalue $(n + 1/2)\hbar\omega$.

## 2.2 Non-Hermitian Quantum Theory

Traditionally, the Hamiltonian is a Hermitian (self-adjoint) operator, $H^* = H$. Let us therefore first recall the principal reasons why Hermiticity is a very necessary property to have in a physical system.

- One primary justification lies in the fact that Hermiticity ensures the realness of energy values. This is easily seen when starting from the time-independent Schrödinger equation for some state vector $|\psi\rangle$ involving a time-independent Hamiltonian $H$ and its conjugate $H^*$

$$H|\psi\rangle = E|\psi\rangle \qquad \text{and} \qquad \langle\psi|H^* = \bar{E}\langle\psi|. \tag{2.52}$$

  Multiplying the first equation by the bra state $\langle\psi|$ from the left and the second equation by the $|\psi\rangle$ from the right, and subsequently taking the difference we obtain

$$\langle\psi|H|\psi\rangle - \langle\psi|H^*|\psi\rangle = (E - \bar{E})\langle\psi|\psi\rangle. \tag{2.53}$$

  Thus when $H$ is Hermitian, i.e. $H = H^*$, the left-hand side vanishes and since $\langle\psi|\psi\rangle \neq 0$ it follows that the energy $E$ must be real.

- The second justification is rooted in Hermiticity's role in safeguarding the conservation of probability densities, or in other words, ensuring the unitarity of the propagator $e^{-itH}$. This is easily seen by starting from the evolution of a

state at time $t = 0$ to a state at time $t$

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle. \tag{2.54}$$

Taking the conjugate of this equation, multiplying by $\langle\psi(t)|$ from the left and assuming once more that the Hamiltonian is Hermitian we obtain

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|e^{itH^*}e^{-iH}\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle. \tag{2.55}$$

This means the normalization at the time $t = 0$ is the same as at any other arbitrary time $t$, i.e. it is conserved.

However, non-Hermitian operators can also have purely real eigenvalues, and consequently, non-Hermitian Hamiltonians might also be proposed to generate the dynamics according to (2.3). This was first observed in Schrödinger operators $H$ with complex potentials that are "$PT$ symmetric" [4], namely,

$$H(PT) = (PT)H,$$

for a linear (parity) operator $P$ and an anti-linear (time-reversal) operator $T$. Recall that the definition of an anti-linear operator $A$ acting on a Hilbert space $\mathcal{H}$ is

$$A\big(\alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle\big) := \bar{\alpha}_1 A|\psi_1\rangle + \bar{\alpha}_2 A|\psi_2\rangle.$$

Here $\alpha_j \in \mathbb{C}$ and $|\psi_j\rangle \in \mathcal{H}$.

### 2.2.1 $PT$ symmetry and its ramifications

In this section, we review the basics of $PT$ symmetric quantum theory. $PT$ symmetric quantum theory replaces the Hermiticity constraint on observables in standard quantum mechanics by the physically-motivated constraint of invariance under $PT$ symmetry. The operator $PT$ acting on the Hilbert space $\mathcal{H}$ is assumed to be the composition of $P$, a linear operator, and $T$, an antilinear operator, such that their combined action is an antiunitary involution on $\mathcal{H}$. An antiunitary operator, is a

bijective antilinear map $U : \mathcal{H} \to \mathcal{H}$ on Hilbert space $\mathcal{H}$ such that

$$\langle Ux|Uy\rangle = \overline{\langle x|y\rangle},$$

for all $|x\rangle, |y\rangle \in \mathcal{H}$. An antiunitary operator $U$ is called an antiunitary involution if $U^2 = \mathbb{1}$.

### 2.2.2 Definition of $PT$ symmetry

$PT$ symmetry is a notion originating in physics. It can be defined abstractly in mathematical terms as follows.

**Definition 2.1** ($PT$ symmetry)**.** [2] *Let $\mathcal{H}$ be a Hilbert space and let $P \in \mathcal{L}(\mathcal{H})$ and let $T$ be an anti-linear operator on $\mathcal{H}$, satisfying $P^2 = \mathbb{1}$, $T^2 = \mathbb{1}$, and $[P, T] = 0$ (commutator). A linear operator $H \in \mathcal{L}(\mathcal{H})$ is called $PT$* symmetric *if*

$$(PT)H(PT)^{-1} = H. \tag{2.56}$$

The operator $PT$ is invertible due to the invertibility of both $P$ and $T$, with their respective inverses being $P^{-1} = P$ and $T^{-1} = T$. Furthermore, as $[P, T] = 0$, we can establish that

$$(PT)^{-1} = T^{-1}P^{-1} = TP = PT.$$

In general, the operators $H, P, T$ may be unbounded; then one has to consider delicate domain questions (which are never addressed in the physical literature). We are assuming here that $H$, $P$, and $T$ are all *bounded operators*, defined on all of $\mathcal{H}$. Then we have $(PT)^{-1} = PT$ and (2.56) is equivalent to $[PT, H] = 0$. Often, $H$ is called a *Hamiltonian*, even though it is not a Hermitian operator, in general. The operators $P$ and $T$ are also called the *parity and time-reversal operators*.

**Examples.**

1. The founding example comes from basic quantum theory [2]. The Hilbert space is $\mathcal{H} = L^2(\mathbb{R}, dx)$, the state space of a quantum particle (moving on a line). The Hamiltonian is $H = \hat{p}^2 + i\hat{x}^3$, where $\hat{p} = -i\partial_x$ and $\hat{x}$ is the operator of multiplication by $x$. (This $H$ is not a bounded operator.) Elements in $\mathcal{H}$ are

called wave functions and denoted by $\psi(x)$. The operators $P, T$ are defined as

$$P\psi(x) = \psi(-x) \tag{2.57}$$
$$T\psi(x) = \overline{\psi(x)} \tag{2.58}$$

(complex conjugate). Equivalently, the action of $P$ and $T$ can be given on the operators $\hat{p}$, $\hat{x}$: $P\hat{x}P = -\hat{x}$, $P\hat{p}P = -\hat{p}$, $T\hat{x}T = \hat{x}$, $T\hat{p}T = -\hat{p}$. With this, it is evident that $H$ is $PT$ symmetric.

Note: In this setup, the Hilbert space has the direct sum decomposition $\mathcal{H} = \mathcal{H}_{\mathrm{e}}^{\mathrm{Re}} \oplus \mathcal{H}_{\mathrm{o}}^{\mathrm{Re}} \oplus \mathcal{H}_{\mathrm{e}}^{\mathrm{Im}} \oplus \mathcal{H}_{\mathrm{o}}^{\mathrm{Im}}$ into summands of even and odd, real and imaginary functions. Assuming $\psi(x) \in \mathcal{H}$, we can express $\psi(x)$ as a combination of even functions, $E(x)$, and odd function, $O(x)$, as follows

$$\psi(x) = E(x) + O(x),$$

since $E(x)$ and $O(x)$ are complex-valued functions we can represent $\psi(x)$ in the following manner:

$$\psi(x) = Re(E(x)) + iIm(E(x)) + Re(O(x)) + iIm(O(x)). \tag{2.59}$$

By utilizing the definitions of $P$ and $T$ in equations (2.57) and (2.58), we can directly compute the action of the operator $PT$ on $\psi(x)$ defined in equation (2.59) as follows:

$$\begin{aligned} PT\psi(x) &= Re(E(x)) - iIm(E(x)) - Re(O(x)) + iIm(O(x)) \\ &= (\mathbb{1} \oplus -\mathbb{1} \oplus -\mathbb{1} \oplus \mathbb{1})\psi(x) \end{aligned}$$

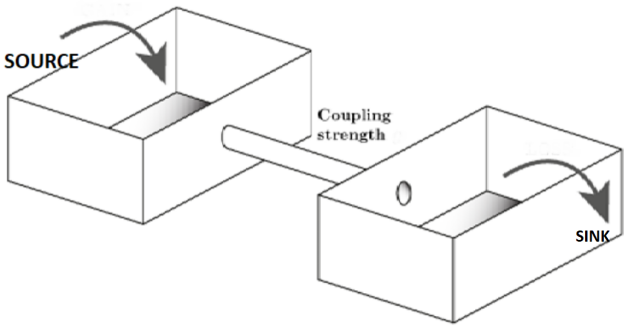
This means $PT = \mathbb{1} \oplus -\mathbb{1} \oplus -\mathbb{1} \oplus \mathbb{1}$. Hence the spectrum of $PT$ is $\{-1, 1\}$. The spectrum of $PT$ generally lies on the complex unit circle (see the proof of Theorem 2.2.2).

2. Consider the case of a two-level system $\mathcal{H} = \mathbb{C}^2$, with a possibly non-Hermitian Hamiltonian $H$ given by

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}, \tag{2.60}$$

where $r$, $s$ and $\theta$ are real parameters, the Hamiltonian in equation (2.60) is used

to describe a basic two-site system with gain and loss [2].




To gain a deeper insight into the system outlined by the Hamiltonian in equation (2.60), let's begin by considering the case where $s = 0$. The Hamiltonian that describes the time evolution of the one-dimensional system in the left box is the $1 \times 1$ matrix $H_1 = re^{i\theta}$, where $r > 0$ and $0 < \theta < \pi$ so that $\mathrm{Im}(H_1) > 0$. The solution to the time-dependent Schrödinger equation for this system

$$i\frac{d}{dt}|\psi(t)\rangle = H_1|\psi(t)\rangle,$$

is $|\psi(t)\rangle = e^{-itH_1}|\psi(0)\rangle$, which grows exponentially with time because there is a source in the left box. Similarly, the Hamiltonian that describes the time evolution of the one-dimensional system in the right box is the $1\times1$ matrix $H_2 = re^{-i\theta}$, so that $\mathrm{Im}(H_2) < 0$. The solution to the time-dependent Schrödinger equation for the right system is $|\psi(t)\rangle = e^{-itH_2}|\psi(0)\rangle$, which decays exponentially with time because there is a sink in the right box. The left and right systems taken together are described by the diagonal matrix Hamiltonian

$$H = \begin{pmatrix} re^{i\theta} & 0 \\ 0 & re^{-i\theta} \end{pmatrix}. \tag{2.61}$$

The system governed by the Hamiltonian $H$ in equation (2.61) is not in equilibrium. This is because the eigenfunction in the left box undergoes exponential decay, while the eigenfunction in the right box experiences exponential growth. To attain equilibrium, it is necessary to establish a strong coupling between the boxes, achieved through the coupling constant $s$.

The Hamiltonian $H$ in (2.60) is $PT$ symmetric if we define the time reversal

and parity operators as

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \quad \text{and} \quad P = \sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{(Pauli } x \text{ matrix)}. \tag{2.62}$$

3. For $\varkappa \in \mathbb{R}$ define $H \in \mathcal{L}(\mathbb{C}^2)$ by

$$H = \begin{pmatrix} 1 & i\varkappa \\ i\varkappa & -1 \end{pmatrix}. \tag{2.63}$$

$H$ is $PT$ symmetric for $T$ as in (2.62) and

$$P = \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{(Pauli } z \text{ matrix)}. \tag{2.64}$$

4. Let $H$ be real $N \times N$ matrix, then $H$ is $PT$ symmetric if we define $P = \mathbb{1}$ and $T$ to be the complex conjugation. An $N \times N$ matrix is real if and only if it is $PT$ symmetric with $P = \mathbb{1}$ and $T$ is the complex conjugation.

### 2.2.3 $PT$ symmetry breaking

A $PT$ symmetric $H$ belongs to one of two regimes: the unbroken or the broken symmetry regime. As we will see, the regimes determine the spectral properties of $H$.

**Definition 2.2** (Broken and unbroken symmetry regimes)**.** [2] *Let $H$ be a $PT$ symmetric Hamiltonian.*

1. *If every eigenvector of $H$ is an eigenvector of $PT$, then we say the $PT$ symmetry of $H$ is unbroken, or, that we are in the unbroken regime.*

2. *If $H$ has an eigenvector that is not an eigenvector of $PT$, then we say the $PT$ symmetry of $H$ is broken, or, that we are in the broken regime.*

3. *When $H = H(\epsilon)$ depends on a parameter $\epsilon \in \mathbb{R}$, then $\epsilon_0$ is called an exceptional (or transition) point if for all $\epsilon$ close enough to $\epsilon_0$, $H(\epsilon)$ is in different regimes for $\epsilon < \epsilon_0$ and $\epsilon > \epsilon_0$.*

**Examples.**

1. The class of quantum mechanical Hamiltonians

$$H(\epsilon) = \hat{p}^2 + \hat{x}^2(i\hat{x})^\epsilon \qquad \epsilon \in \mathbb{R}. \tag{2.65}$$

In [4, 5] was shown numerically that $H$ is in the unbroken symmetry regime whenever $\epsilon \geq 0$, and it is in the broken regime when $\epsilon < 0$, The phase transition occurs at $\epsilon_0 = 0$. Note that if we restrict ourselves to $\epsilon = 1$, the Hamiltonian (2.65) reduces to the Hamiltonian we have discussed earlier, $H = \hat{p}^2 + i\hat{x}^3$. It's worth noting that at $\epsilon = 0$ the Hamiltonian reduces to a harmonic oscillator.

2. The eigenvalues and eigenvectors of the Hamiltonian in (2.60) are

$$\lambda_\pm = r\cos\theta \pm (s^2 - r^2\sin^2\theta)^{\frac{1}{2}}, \tag{2.66}$$

and

$$|\psi_+\rangle = \frac{1}{\sqrt{2\cos\varkappa}}\begin{pmatrix} e^{\frac{i\varkappa}{2}} \\ e^{\frac{-i\varkappa}{2}} \end{pmatrix}, \qquad |\psi_-\rangle = \frac{i}{\sqrt{2\cos\varkappa}}\begin{pmatrix} e^{\frac{-i\varkappa}{2}} \\ -e^{\frac{i\varkappa}{2}} \end{pmatrix} \tag{2.67}$$

respectively. The parameter $\varkappa$ satisfies the equation $\sin\varkappa = \frac{r}{s}\sin\theta$. Note that $\varkappa$ is real if and only if $s^2 \geq r^2\sin^2\theta$ if and only if we are in the region of unbroken $PT$ symmetry. When $s^2 < r^2\sin^2\theta$, $H$ is in the broken region.

3. [2] The eigenvalues and eigenvectors of the Hamiltonian in (2.63) are

$$E_\pm = \pm i\beta(\varkappa) \quad \text{and} \quad |\psi_\pm(\varkappa)\rangle = \frac{1}{\sqrt{\varkappa^2 + |\pm\beta(\varkappa) - i|}}\begin{pmatrix} \pm\beta(\varkappa) - i \\ \varkappa \end{pmatrix}, \tag{2.68}$$

where $\beta(\varkappa) = \sqrt{\varkappa^2 - 1}$. One can check that if $\varkappa \in (-\infty, -1) \cup (1, \infty)$ then the eigenvalues are real and the corresponding eigenvectors of $H$ are eigenvectors of $PT$ corresponding to the eigenvalues $\lambda = \pm 1$, indeed, $PT|\psi_\pm(\varkappa)\rangle = -|\psi_\pm(\varkappa)\rangle$. On the other hand, for $\varkappa \in (-1, 1)$, the eigenvalues are complex conjugate pairs and the eigenstates of $H$ are not eigenstates of $PT$. The exceptional points are $\varkappa_0 = \pm 1$.

### 2.2.4 Symmetry phases and spectrum

The purpose of this section is to solve the eigenvalue problems associated to the $PT$ symmetric operators and explain the features of the spectrum of such operators.

Suppose $H$ is $PT$ symmetric. The resolvent satisfies

$$(H-z)^{-1} = (PTHT^{-1}P^{-1} - z)^{-1} = PT(H-\bar{z})^{-1}T^{-1}P^{-1},$$

where we used that $PTzT^{-1}P^{-1} = \bar{z}$ (complex conjugate). This implies

$$z \in \text{spec}(H) \quad \Longleftrightarrow \quad \bar{z} \in \text{spec}(H). \tag{2.69}$$

We thus have the following result:

**Theorem 2.2.1.** *Suppose $H$ is $PT$ symmetric, with $H$, $P$ and $T$ bounded. Then the spectrum of $H$ is invariant under complex conjugation. In particular, the eigenvalues of $H$ are either real or come in complex conjugate pairs.*

Note: if $P$ and $T$ are bounded, then so are $P^{-1}$ and $T^{-1}$ since $P^{-1} = P$ and $T^{-1} = T$. The next result links the reality of eigenvalues of $H$ and the $PT$ symmetry unbroken regime.

**Theorem 2.2.2.** [1] *Let $H$ be a $PT$ symmetric operator.*

(a) *If an eigenvector of $H$ is also an eigenvector of $PT$, then the associated eigenvalue of $H$ is real.*

(b) *Suppose $E$ is a real,* simple *eigenvalue of $H$. Then the associated eigenvector is also an eigenvector of $PT$.*

**Proof.** We prove (a) first. Note that the spectrum of $PT$ lies on the complex unit circle:

$$PT\psi = \lambda\psi \quad \Rightarrow \quad \psi = (PT)^2\psi = PT\lambda\psi = \bar{\lambda}PT\psi = |\lambda|^2\psi \quad \Rightarrow \quad |\lambda| = 1.$$

Next, consider $H\psi = z\psi$ with $\psi$ satisfying $PT\psi = \lambda\psi$. Multiplying by $PT$ gives

$$PTH\psi = \bar{z}PT\psi = \lambda\bar{z}\psi. \tag{2.70}$$

Since $PTH = HPT$ we have $PTH\psi = H\lambda\psi = \lambda z\psi$. Combining with (2.70) gives $z = \bar{z}$, as $\lambda \neq 0$.

Now we prove (b). Consider $H\psi = E\psi$. Since $PTH = HPT$ we have $PTH\psi = E(PT\psi) = H(PT\psi)$, now the simplicity of $E$ implies $PT\psi \in \text{span}\{\psi\}$. Thus $PT\psi = \lambda\psi$ for some nonzero $\lambda \in \mathbb{C}$, this means $\psi$ is an eigenvector of $PT$ with eigenvalue $\lambda$. □

In part (b) of theorem 2.2.2, the simplicity of the eigenvalue condition is crucial. To illustrate this, consider the scenario where $P = \mathbb{1}$ and $T$ represents complex conjugation. A matrix is then $PT$ symmetric if and only if its entries are real. If we select $H$ to be the identity matrix, every vector in $\mathbb{C}^2$ becomes an eigenvector for the eigenvalue $E = 1$. However, not every vector in $\mathbb{C}^2$ serves as an eigenvector for $PT$. For example, if we take

$$PT\begin{pmatrix}1\\ i\end{pmatrix} = \begin{pmatrix}1\\ -i\end{pmatrix},$$

it demonstrates an eigenvector of $H$ with a real eigenvalue that isn't an eigenvector for $PT$.

### 2.2.5 Pseudo-Hermiticity and Quasi-Hermiticity

Pseudo-Hermiticity and quasi-Hermiticity are concepts related to linear operators in quantum mechanics and mathematical physics. Both terms involve a generalization of the Hermitian property of operators, which plays a crucial role in quantum mechanics.

**Definition 2.3.** [34] *A linear operator $H$ on a Hilbert space $\mathcal{H}$ is called* pseudo-Hermitian *if there is a Hermitian, bounded, invertible linear operator $\eta$ such that*

$$H^* = \eta H \eta^{-1}. \tag{2.71}$$

*An operator $\eta$ satisfying* (2.71) *is called a* pseudo-metric operator *associated with $H$. The set of all pseudo-metric operators associated to $H$ is denoted by $\mathfrak{M}_H$. The set of all* strictly positive *pseudo-metric operators associated with $H$ is denoted by $\mathfrak{M}_H^+$. $H$ is called* quasi-Hermitian *if* (2.71) *holds for an $\eta \in \mathfrak{M}_H^+$.*

Just as for the $PT$ symmetric case, the spectrum of a pseudo-Hermitian operator $H$ is also invariant under complex conjugation – it satisfies (2.69). This follows easily

from (2.71).

The following is a link between $PT$ symmetry and pseudo-Hermiticity.

**Lemma.** *Suppose $H$ is a $PT$ symmetric operator acting on a finite dimensional Hilbert space. Suppose further that $H$ is real symmetric (* i.e.*, $H^t = H$) and $P$ is Hermitian (* i.e.*, $P^* = P$). Then $P \in \mathfrak{M}_H$,* i.e.*, $H$ is pseudo-Hermitian with pseudo-metric $P$.*

**Proof.** Fix a basis such that $THT = \bar{H}$, where $\bar{H}$ is the entry-wise complex conjugate of $H$. From equation (2.56) we have

$$H = PTHPT = PTHTP = P\bar{H}P, \tag{2.72}$$

The second equality holds because $P$ and $T$ commute. Since $P^{-1} = P$ equation (2.72) can be written as

$$PHP = \bar{H} = H^* \tag{2.73}$$

The second equality is true by assumption $H^t = H$, where $^t$ denotes the transpose. □

**Theorem 2.2.3.** *[43] If $H$ finite-dimensional $PT$ symmetric Hamiltonian, then $H$ is pseudo-Hermitian.*

**Proof.** According to the definition of $PT$ symmetry, the Hamiltonian $H$ exhibits similarity to its complex conjugate, denoted as $\bar{H}$. Consequently, in accordance with Theorem 2 presented in [43], which asserts that pseudo-Hermiticity is equivalent to being similar to its complex conjugate, we can conclude that $H$ is pseudo-Hermitian. □

Let $\{|\psi_n\rangle\}_{n\in\mathbb{N}}$, $\{|\phi_n\rangle\}_{n\in\mathbb{N}}$, be two bases of $\mathcal{H}$, satisfying $\langle\phi_m|\psi_n\rangle = \delta_{m,n}$. The collection $\{|\phi_n\rangle, |\psi_n\rangle\}_{n\in\mathbb{N}}$ is called a *biorthonormal basis* [29], or BIOB, of $\mathcal{H}$. We have the following completeness relation

$$\sum_n |\phi_n\rangle\langle\psi_n| = \mathbb{1} = \sum_n |\psi_n\rangle\langle\phi_n|, \tag{2.74}$$

which is easily verified by applying basis vectors $|\psi_n\rangle$ or $|\phi_n\rangle$ to the sums and noting that the action is that of the identity.

An operator $H$ on $\mathcal{H}$ is called *diagonalizable* if it is of the form

$$H = \sum_n E_n |\psi_n\rangle\langle\phi_n| \tag{2.75}$$

for some BIOB $\{|\phi_n\rangle, |\psi_n\rangle\}_{n\in\mathbb{N}}$ and some $E_n \in \mathbb{C}$. This means that the spectrum of $H$ consists exactly of the eigenvalues $E_n$ (with associated eigenvectors $|\psi_n\rangle$). The adjoint of the operator $H$ in the equation (2.75) given by

$$H^* = \sum_n \bar{E}_n |\phi_n\rangle\langle\psi_n|, \tag{2.76}$$

so the $|\phi_n\rangle$ are the eigenvectors of $H^*$ with corresponding eigenvalues $\bar{E}_n$.

**Theorem 2.2.4.** *[29, 27] Let $H : \mathcal{H} \to \mathcal{H}$ be a linear operator. Then*

1. *If $H$ is pseudo-Hermitian, then the spectrum of $H$ is invariant under complex conjugation.*
2. *Suppose $H$ is diagonalizable and the spectrum of $H$ is invariant under complex conjugation. Then $H$ is pseudo-Hermitian.*

In particular, this result shows that a diagonalizable, $PT$ symmetric $H$ is pseudo-Hermitian.

**Proof.**

1. Let $\eta \in \mathfrak{M}_H$. We have

$$(H - z)^{-1} = (\eta^{-1} H^* \eta - z)^{-1} = \eta^{-1}(H^* - z)^{-1}\eta,$$

so $H - z$ is invertible if and only if $H^* - z$ is invertible. This means that $\mathrm{spec}(H) = \mathrm{spec}(H^*)$. The result follows since $\mathrm{spec}(H^*) = \overline{\mathrm{spec}(H)}$.

Now we prove part 2. Let $S_R$ be the collection of real eigenvalues $E$ of $H$ (repeated according to their multiplicity) and let $S_P$ be the collection of pairs $(E, \bar{E})$ of the complex conjugate, non-real eigenvalues of $H$, also repeated according to their multiplicity, and where $\mathrm{Im}E > 0$. We denote the eigenvectors as

$$\begin{aligned} E \in S_R: &\quad H\psi_E = E\psi_E,\ H^*\phi_E = E\phi_E \\ (E, \bar{E}) \in S_P: &\quad H\psi_E^+ = E\psi_E^+,\ H^*\phi_E^+ = \bar{E}\phi_E^+,\quad H\psi_E^- = \bar{E}\psi_E^-,\ H^*\phi_E^- = E\phi_E^-. \end{aligned}$$

They can be normalized as to form a BIOB. We have

$$H = \sum_{E \in S_R} E|\psi_E\rangle\langle\phi_E| + \sum_{(E,\bar{E})\in S_P} E|\psi_E^+\rangle\langle\phi_E^+| + \bar{E}|\psi_E^-\rangle\langle\phi_E^-|.$$

Let $\eta_E$, $\eta_E^{\pm} \in \mathbb{C}$ be any non-zero numbers and set

$$\eta = \sum_{E \in S_R} \eta_E|\phi_E\rangle\langle\phi_E| + \sum_{(E,\bar{E})\in S_P} \eta_E^+|\phi_E^-\rangle\langle\phi_E^+| + \eta_E^-|\phi_E^+\rangle\langle\phi_E^-|.$$

Using the biorthogonality and the completeness relation (2.74), it is easy to verify that $\eta$ is invertible, and

$$\eta^{-1} = \sum_{E \in S_R} \frac{1}{\eta_E}|\psi_E\rangle\langle\psi_E| + \sum_{(E,\bar{E})\in S_P} \frac{1}{\eta_E^+}|\psi_E^+\rangle\langle\psi_E^-| + \frac{1}{\eta_E^-}|\psi_E^-\rangle\langle\psi_E^+|.$$

Furthermore, $\eta H \eta^{-1} = H^*$, as is also easily verified using biorthogonality. Finally, $\eta = \eta^*$ if and only if $\eta_E \in \mathbb{R}$ and $\eta_E^- = (\bar{\eta}_E^+)$ for all $E$. This shows that $H$ is pseudo-Hermitian. This completes the proof. □

Note, if all eigenvalues are real (no complex conjugate non-real pairs), then in the above construction we can take $\eta = \sum_{E\in \mathrm{spec}(H)} \eta_E|\phi_E\rangle\langle\phi_E|$ and we have $\eta > 0 \Leftrightarrow \eta_E > 0 \, \forall E$. This shows that if $H$ is diagonalizable and has a purely real spectrum, then $H$ is quasi-Hermitian, see also Theorem (2.2.5) below.

**Theorem 2.2.5.** *Let $H : \mathcal{H} \to \mathcal{H}$ be a linear operator. Then*

1. *If $H$ is bounded and quasi-Hermitian, then the spectrum of $H$ is real.*
2. *As a partial converse: If $H$ is diagonalizable and has real spectrum, then $H$ is quasi-Hermitian.*

**Proof.**

1. Since $\eta$ is strictly positive we can define $H_0$ as

$$H_0 = \sqrt{\eta} H (\sqrt{\eta})^{-1}, \tag{2.77}$$

equation (2.77) together with equation (2.71) implies $H_0$ is Hermitian i.e, $H_0 = H_0^*$.

So the eigenvalues of $H_0$ are real. Then equation (2.77) implies that the eigenvalues of $H$ are real too.

2. Suppose $H$ is diagonalizable i.e., there exists an invertible matrix $Q$ and a diagonal matrix $D$ such that $H = QDQ^{-1}$. Since the eigenvalues of $H$ are all real $D = D^*$. Now we set $\eta = QQ^*$. We have

$$H^* = (Q^{-1})^* D^* Q^* = (Q^{-1})^* DQ^* = (Q^{-1})^* \big(Q^{-1} H Q\big) Q^* = \eta^{-1} H \eta.$$

□

The following example is given to illustrate why we need the diagonalizability condition in part (2) of Theorem 2.2.5.

**Example.** The operator

$$H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.78}$$

is not diagonalizable. Any $\eta$ belonging to the set $\mathfrak{M}_H$, which is any $2 \times 2$ matrix $\eta$ that satisfies the equation $H^*\eta = \eta H$, can be expressed in the following form:

$$\eta = \begin{pmatrix} 0 & b \\ b & d \end{pmatrix}, \tag{2.79}$$

where $b$, $d$ are real numbers. For $\eta$ to be strictly positive, we need $\mathrm{tr}(\eta) > 0$ and $\det(\eta) > 0$. Here $\det(\eta) = -b^2 < 0$, therefore, $\eta$ can't be positive definite, so $\mathfrak{M}_H^+$ is empty set.

# Chapter 3

# Diagonalizablity and $PT$ symmetric regime of two dimensional $PT$ symmetric Hamiltonian

In the field of quantum mechanics, the study of non-Hermitian Hamiltonians has unveiled interesting phenomena that challenge conventional expectations. One remarkable class of non-Hermitian Hamiltonians is the $PT$ symmetric Hamiltonians, where $PT$ symmetry refers to a combined operation involving parity $P$ and time-reversal $T$ symmetries. In recent years, the diagonalizability of $PT$ symmetric Hamiltonians has attracted significant attention, especially in two-dimensional systems. This exploration is motivated by the remarkable behavior exhibited by such systems, particularly at the transition points where $PT$ symmetry undergoes a critical transformation. This discussion delves into the fascinating world of two-dimensional $PT$ symmetric Hamiltonians, focusing on their diagonalizability properties at these critical transition points.

## 3.1 Diagonalizablity at the transition point

**Definition 3.1.** Let $(x, y, z, w)$ be the complex entry of the matrix $H \in \mathcal{L}(\mathbb{C}^2)$. Define $B_\epsilon(x, y, z, w)$ as a ball centered around $(x, y, z, w)$ with radius $\epsilon$. We designate $(x_0, y_0, z_0, w_0)$ as a transition point if for every $\epsilon > 0$, within some regions of the ball

$B_\epsilon(x_0, y_0, z_0, w_0)$, the eigenvalues $\lambda_\pm(x, y, z, w)$ are real, while in the remaining parts, they form a complex conjugate pair.

**Theorem 3.1.1.** *Let $H$, $P$ and $T$ be operators acting on $\mathcal{H} = \mathbb{C}^2$ such that $H$ is $PT$ symmetric, then $H$ is either nondiagonalizable or proportional to the identity at a transition point.*

**Proof.** Given that $H$ is a $2 \times 2$ matrix, it possesses at most two eigenvalues, denoted as

$$\lambda_\pm = r \pm \sqrt{E}, \tag{3.1}$$

where

$$r = \frac{x + w}{2}, \qquad E = \big(\frac{x - w}{2}\big)^2 + yz. \tag{3.2}$$

Because $H$ is $PT$ symmetric, the eigenvalues $\lambda_\pm$ are either real or form a complex conjugate pair.

1. If $\lambda_\pm$ are real, we have

$$2r = \lambda_+ + \lambda_-, \qquad 2\sqrt{E} = \lambda_+ - \lambda_-. \tag{3.3}$$

   Since the sum and subtraction of real numbers result in real numbers, Equations (3.3) imply that both $r$ and $\sqrt{E}$ are real, thus indicating that $E > 0$.

2. If $\lambda_\pm$ are complex conjugate pair ($\lambda_+ = \bar{\lambda}_-$), we have

$$2r = \lambda_+ + \bar{\lambda}_+ = 2Re(\lambda_+) \qquad 2i\sqrt{-E} = \lambda_+ - \bar{\lambda}_+ = 2iIm(\lambda_+) \tag{3.4}$$

   Since $Re(\lambda_+)$ and $Im(\lambda_+)$ are real, it follows from equation (3.4) that both $r$ and $\sqrt{-E}$ are real, thereby implying that $E < 0$.

If $E > 0$, two distinct real eigenvalues emerge, leading to the diagonalizability of $H$ and placement within the unbroken regime. In the case of $E < 0$, a pair of complex conjugate eigenvalues arises, still resulting in diagonalizability but placing $H$ in the broken regime. Due to calculations in equations (3.3) and (3.4) and the fact the quantities $r$ and $E$ depend continuously on the parameter space $(x, y, z, w)$, then they are both real and $E$ equals zero at the transition point according to definition 3.1. If $E = 0$, the eigenvalues become $\lambda_\pm = \lambda = r$, where $r$ may have geometric multiplicity

2, yielding $H = r\mathbb{1}$ ( real proportional to the identity), or geometric multiplicity 1, causing $H$ to be non-diagonalizable. □

In the subsequent subsection, we investigate the various potential scenarios described in Theorem 3.1.1, taking into account all allowable choices of $H$ and $P$, while assuming that $T$ operates as the complex conjugation operator.

### 3.1.1 Case Analysis: Theorem 3.1.1

Let $H$ and $P$ be a linear operator acting on the space $\mathbb{C}^2$

$$H = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.5}$$

Suppose further $T$ is the complex conjugation operator,

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}. \tag{3.6}$$

We assume the operators $H, P$ and $T$ satisfy the following

1. $P^2 = \mathbb{1}$,
2. $[P, T] = 0$,
3. $[H, PT] = 0$.

The first assumption implies $P = P^{-1}$, while the definition of complex conjugation automatically implies $T = T^{-1}$. From the second assumption we have

$$P = TPT = \bar{P}, \tag{3.7}$$

the notation bar is the matrix entry-wise complex conjugate. Equation (3.7) implies that matrix $P$ is a real involution matrix. Therefore, it can be concluded that:

$$P = \pm\mathbb{1} \quad \text{or} \quad P = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{3.8}$$

where $a^2 + bc = 1$ and $a, b$ and $c$ are real numbers. After utilizing the commutativity property $[H, PT] = 0$, the following result is obtained:

$$H = PTHPT = PTHTP = P\bar{H}P. \tag{3.9}$$

Or equivalently

$$PH = \bar{H}P.$$

Assuming that $P = \pm\mathbb{1}$, equation (3.9) yields $H = \bar{H}$, which implies that any real matrix can be considered as a $PT$ symmetric matrix. It is important to note that if a matrix $H$ is symmetric ($H = H^t$), then $H$ will also be Hermitian. Equation (3.9) shows that $H$ is $P$-pseudo Hermitian. We omit the computation for the case $P = \pm\mathbb{1}$ because in physics, the operator $P$ typically signifies spatial reflection. When $P = \pm\mathbb{1}$, there is no reflection, yet since $H$ remains $PT$ symmetric, the outcomes from theorem 3.1.1 remain valid.

Let

$$P = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{3.10}$$

where $a^2 + bc = 1$, and $a, b$, and $c$ are real numbers. We aim to determine all possible solutions of $H$ which satisfy (3.9), or more specifically, all $H$ that satisfy $PH = \bar{H}P$, that is

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} \bar{x} & \bar{y} \\ \bar{z} & \bar{w} \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix}. \tag{3.11}$$

Simplifying equation (3.11) we have

$$\begin{aligned} bz_1 &= cy_1, \\ 2ay_1 &= b(x_1 - w_1), \\ 2az_1 &= c(x_1 - w_1), \end{aligned}$$

and

$$\begin{aligned} x_2 &= -w_2, \\ 2ax_2 &= -(cy_2 + bz_2), \\ 2aw_2 &= (cy_2 + bz_2). \end{aligned}$$

The variables $x_1, y_1, z_1$ and $w_1$ represent the real parts of $x, y, z$, and $w$, respectively, while $x_2, y_2, z_2$ and $w_2$ represent their corresponding imaginary parts. The solutions for the above systems fall into two distinct cases.

- **Case 1:** If $c \neq 0$, then

$$H = \begin{pmatrix} \bar{w} + \frac{2a}{c} z_1 & \frac{b}{c}\bar{z} + \frac{2ai}{c} w_2 \\ z & w \end{pmatrix}, \tag{3.12}$$

  In this context, $z = z_1 + iz_2$ , while $w = w_1 + iw_2$. The eigenvalues and eigenvectors of $H$ in (3.12) can be expressed as follows:

$$\lambda_\pm = \left(w_1 + \frac{a}{c} z_1\right) \pm E \quad \text{and} \quad v_\pm = \begin{pmatrix} \lambda_\pm - w \\ z \end{pmatrix}. \tag{3.13}$$

  Here, $E = \left[(\frac{a}{c} z_1)^2 + \frac{b}{c}|z|^2 - \left(w_2^2 + \frac{2a}{c} z_2 w_2\right)\right]^{\frac{1}{2}}$. When $E \neq 0$ , the eigenvectors $v_\pm$ are linearly independent, and therefore, $H$ is diagonalizable. However, when $E = 0$ and $z \neq 0$, the matrix $H$ has only one eigenvalue with algebraic multiplicity 2 and geometric multiplicity 1, making $H$ non-diagonalizable. Assume $z = 0$ and since $E = 0$ we have $w_2 = 0$ and therefore $H$ in equation (3.12) reduces to $H = w_1 \mathbb{1}$.

- **Case 2:** If $c$ is assumed to be equal to 0, then $a$ takes the values of $\pm 1$. Let's consider the scenario where $a$ is equal to 1 as the case when $a$ is equal to $-1$ is analogous. In this context, the matrices $P$ and $H$ can be expressed as:

$$P = \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} x - \frac{ib}{2} z & \frac{b}{2}(x - w) + iy \\ iz & w + \frac{ib}{2} z \end{pmatrix}, \tag{3.14}$$

  where $x, y, z$ and $w$ are real numbers. The eigenvalues and eigenvectors of $H$ are:

$$\lambda_\pm = \frac{x + w}{2} \pm E \quad \text{and} \quad v_\pm = \begin{pmatrix} (x - w) - ibz \pm 2E \\ 2iz \end{pmatrix}, \tag{3.15}$$

  where $E = \left((x - w)^2 - (4zy + b^2 z^2)\right)^{\frac{1}{2}}$. If $E \neq 0$, then the eigenvectors $v_\pm$ are linearly independent, and thus $H$ is diagonalizable. However, if $E = 0$ and $z \neq 0$, the matrix $H$ has one eigenvalue with algebraic multiplicity 2 and

geometric multiplicity 1, and hence $H$ is not diagonalizable. However if $E = 0$ and we assume $z = 0$, we get $x = w$ and hence $H$ in equation (3.14) becomes

$$H = \begin{pmatrix} x & iy \\ 0 & x \end{pmatrix}.$$

Therefore, if $y \neq 0$, $H$ is not diagonalizable, and it becomes proportional to the identity ($H = x\mathbb{1}$) when $y = 0$.

## 3.2 $PT$ symmetry regimes

The objective of this section is to determine the $PT$ symmetry regimes of the operators $H$ that were obtained from equations (3.12) and (3.14).

- **Unbroken regime:** Assume the quantity $E$ that appears in equations (3.13) and (3.15) be real. Then the operator $H$ in (3.12) and (3.14) possesses two distinct nonzero real eigenvalues. According to theorem (2.2.2) part (b), the associated eigenvectors are eigenvectors of $PT$, indicating that we are in the unbroken regime.

- **Broken regime:** Assume the quantity $E$ that appears in equations (3.13) and (3.15) be pure imaginary. Then the operator $H$ in (3.12) and (3.14) has a complex conjugate pair of eigenvalues. According to theorem (2.2.2) part (a), the associated eigenvectors are not eigenvectors of $PT$, indicating that we are in the broken regime.

- **Transition point:**

  1. Let $H$ be defined as shown in equation (3.12). Recall that the eigenvalues of $H$ appears in equation (3.13) depends on the quantity

  $$E = \big[\big(\frac{a}{c}z_1\big)^2 + \frac{b}{c}|z|^2\big) - \big(w_2^2 + \frac{2a}{c}z_2w_2\big)\big]^{\frac{1}{2}}.$$

  Suppose $E = 0$ and $z \neq 0$ i.e.,

  $$(\frac{a}{c}z_1)^2 + \frac{b}{c}|z|^2 = w_2^2 + \frac{2a}{c}z_2w_2. \tag{3.16}$$

In this case, $H$ has one eigenvalue $\lambda = w_1 + \frac{2a}{c}z_1$ with algebraic multiplicity 2 and corresponding eigenvector

$$v = \begin{pmatrix} \frac{2a}{c}z_1 - iw_2 \\ z \end{pmatrix}. \tag{3.17}$$

Now we claim that $v$ in equation (3.17) is an eigenvector of $PT$, that is

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} \frac{2a}{c}z_1 + iw_2 \\ \bar{z} \end{pmatrix} = e^{i\theta} \begin{pmatrix} \frac{2a}{c}z_1 - iw_2 \\ z \end{pmatrix}, \tag{3.18}$$

for some $\theta \in [0, 2\pi)$. To prove the claim note that one can easily show the following three qualities

$$\begin{aligned} |\frac{2a^2}{c}z_1 + iaw_2 + b\bar{z}|^2 - |\frac{2a}{c}z_1 - iw_2|^2 &= 0, \\ |2az_1 + icw_2 - a\bar{z}|^2 - |z|^2 &= 0, \\ \frac{\frac{2a^2}{c}z_1 + iaw_2 + b\bar{z}}{\frac{2a}{c}z_1 - iw_2} - \frac{2az_1 + icw_2 - a\bar{z}}{z} &= 0, \end{aligned}$$

hold if $E = 0$ and $z \neq 0$.

2. Let $H$ and $P$ defined as in equation (3.14). Then the eigenvalues of $H$ in equation (3.15) depends on the quantity

$$E = ((x - w)^2 - 4yz + b^2z^2)^{\frac{1}{2}}.$$

Suppose $E = 0$ and $z \neq 0$ i.e.,

$$(x - w)^2 = 4yz + b^2z^2,$$

we have only one eigenvalue $\lambda = \frac{x+w}{2}$ with algebraic multiplicity 2 and corresponding eigenvector

$$v = \begin{pmatrix} (x - w) - ibz \\ 2iz \end{pmatrix}. \tag{3.19}$$

We claim vector $v$ in equation (3.19) is an eigenvector of $PT$ i.e,

$$\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix} \begin{pmatrix} (x-w)+ibz \\ -2iz \end{pmatrix} = e^{i\theta} \begin{pmatrix} (x-w)-ibz \\ 2iz \end{pmatrix}, \tag{3.20}$$

for some $\theta \in [0, 2\pi)$. Note that (3.20) is valid since

$$\begin{aligned} |(x-w)-ibz| &= |e^{i\theta}||(x-w)-ibz|, \\ |2iz| &= |e^{i\theta}||2iz|, \end{aligned}$$

and

$$\frac{(x-w)-ibz}{(x-w)-ibz} = e^{i\theta} = \frac{2iz}{2iz} \tag{3.21}$$

Equation (3.21) implies that $\theta = 2n\pi$ for any integer $n$ solves equation (3.20).

In both scenarios mentioned above, at the transition point assuming $z = 0$ implies that $H$ is real and proportional to the identity. Every vector in $\mathbb{C}^2$ becomes an eigenvector of $H$. Consequently, there exist vectors in $\mathbb{C}^2$ that are not eigenvectors of $PT$, where $P$ is defined as in (3.14) and (3.10). In this case, one can envision the broken region as a closed subset in the parameter space, with its unbroken counterpart being the open complement. However, if $H$ is not diagonalizable at the transition point, it becomes apparent that at these junctures, the only eigenvector of $H$ is also an eigenvector of $PT$. Thus, it is plausible to regard the transition point as an integral part of the unbroken region. Consequently, one may conceptualize the unbroken region as a closed entity and its open complement as the broken region.

# Chapter 4

# Dyson Map

## 4.1 The Dyson map: Setup and motivation

The idea of mapping a non-Hermitian Hamiltonian to a Hermitian one was originally presented by Dyson in the context of the theory of magnetization [14, 15]. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot|\cdot\rangle$ and let $H$ be an operator on $\mathcal{H}$ that is not necessarily Hermitian with respect to $\langle\cdot|\cdot\rangle$. Denote by $|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle$ the solution of the evolution equation

$$i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle, \tag{4.1}$$

with initial state $|\psi(0)\rangle$. Next, let $S(t)$ be a differentiable family of operators on $\mathcal{H}$ such that $S(t)$ is invertible for each $t$, and set

$$|\varphi(t)\rangle = S(t)|\psi(t)\rangle. \tag{4.2}$$

Passing from $|\psi(t)\rangle$ to $|\varphi(t)\rangle$ represents a (possibly time-dependent) change of variables. The evolution equation for $|\varphi(t)\rangle$ is

$$i\partial_t|\varphi(t)\rangle = h(t)|\varphi(t)\rangle, \tag{4.3}$$

with

$$h(t) = S(t)HS(t)^{-1} + i\dot{S}(t)S(t)^{-1}, \tag{4.4}$$

the dot being the time derivative. Conversely, if $|\varphi(t)\rangle$ solves equation (4.3) with $h(t)$ in equation (4.4), then $|\psi(t)\rangle$ solves (4.1). Equation (4.4) is called the *time-dependent Dyson equation* [18]. By means of $S(t)$, one may thus equivalently solve (4.1) or (4.3). If $H$ is not Hermitian, one can look for $S(t)$ such that the resulting $h(t)$ is Hermitian, hence trading a non-Hermitian problem with constant Hamiltonian $H$ for a Hermitian problem with time-dependent Hamiltonian $h(t)$. One readily sees that

$$h(t)^* = h(t) \quad \Longleftrightarrow \quad i\partial_t\big(S(t)^*S(t)\big) = H^*\big(S(t)^*S(t)\big) - \big(S(t)^*S(t)\big)H. \tag{4.5}$$

The operator $\eta(t) = S(t)^*S(t)$ is non-negative. This is because, for any non-zero $|\psi\rangle \in \mathcal{H}$, the following inequality holds:

$$\langle \eta(t)\psi|\psi\rangle = \langle S(t)\psi|S(t)\psi\rangle = \|S(t)\psi\|^2 \geq 0$$

The equation for $\eta(t)$, according to (4.5), is

$$i\partial_t\eta(t) = H^*\eta(t) - \eta(t)H. \tag{4.6}$$

This is called the *quasi-Hermiticity relation* in [17]; note that it simplifies to (1.3) if $\eta$ does not depend on time. It is clear that (4.6) has a unique solution for any initial condition $\eta(0)$, namely

$$\eta(t) = e^{-itH^*}\eta(0)e^{itH}, \tag{4.7}$$

and that $\eta(t)$ is positive for all times if and only if it is positive at some $t_0$.

A strategy to study the dynamics generated by a non-Hermitian $H$ is to find a transformation $S(t)$ such that $h(t)$, as given by (4.4), is Hermitian, and then analyze the dynamics of this Hermitian system using usual quantum theoretical methods. Finding $S(t)$ for a specific Hamiltonian $H$ is not easy, however. It often involves making a judicious ansatz containing parameters that must solve rather complicated differential equations, which are obtained by imposing the self-adjointness of $h(t)$. This can be done explicitly for some models [10, 17, 18, 33, 36, 19, 20].

Given $H$, we seek *all* possible $S(t)$, and the resulting Hermitian Hamiltonians $h(t)$, with the sole requirement that $\eta(t) = S(t)^*S(t)$ is non-negative and satisfies the quasi-Hermiticity relation (4.6). The solution $\eta(t)$ is uniquely determined by the initial condition $\eta(0)$, which we may choose to be any positive, invertible operator.

The most general form of $S(t)$ is thus

$$S(t) = W(t)\sqrt{\eta(0)}e^{itH}, \tag{4.8}$$

where $W(t)$ is any unitary family and $\sqrt{\eta(0)}$ denotes the unique positive operator squaring to $\eta(0)$. The $h(t)$ associated to (4.8) by (4.4) is

$$h(t) = i\dot{W}(t)W(t)^*. \tag{4.9}$$

Note that we are entirely free to choose $W(t)$. For instance, given an arbitrary $A = A^*$, the choice $W(t) = e^{-itA}$ yields $h(t) = A$. This means any time-independent Hermitian $h$ can be obtained from a suitable choice of $S(t)$. A particularly simple choice is $S(t) = e^{itH}$, which results from undoing the dynamics $e^{-itH}$ (going backwards in time) and has $h = 0$.

More generally, suppose $A(t)$ is a continuous family of operators, and let $W(t)$ solve the differential equation

$$i\dot{W}(t) = A(t)W(t). \tag{4.10}$$

It is easily shown that if $A(t) = A(t)^*$ for all $t$ and the initial condition $W(0)$ is unitary, then the solution $W(t)$ is unitary for all $t$. Choosing this $W(t)$, we find from (4.9) the Hermitian Hamiltonian $h(t) = i\dot{W}(t)W(t)^* = i\dot{W}(t)W(t)^{-1} = A(t)$. This means any time-*dependent* Hermitian $h(t)$ can also be obtained from a suitable choice of $S(t)$.

# Chapter 5

# A generic model for $PT$ symmetry breaking

In Section 5.1 we introduce a generic model of a family of Hamiltonians $H(\varkappa)$ indexed by a parameter $\varkappa$ varying around a symmetry transition point $\varkappa_0$, with $\varkappa < \varkappa_0$ corresponding to the symmetry unbroken regime.

We study the existence and properties of metrics for $H(\varkappa)$ in Section 5.2. The main result of this chapter is Theorem 5.2.2. It shows that at the transition point $\varkappa_0$, the Hamiltonian $H(\varkappa_0)$ is diagonalizable if and only if there exists a time-independent metric $\eta$, that is, if and only if $H^*(\varkappa_0)\eta = \eta H(\varkappa_0)$ for some $\eta > 0$.

We show in Section 5.2.2 that in the $PT$ symmetry broken regime, there are *no* time-independent metrics (Proposition 5.2.5). Furthermore, we show in Theorem 5.2.3 that in the $PT$ symmetry unbroken regime, any possibly time-dependent metric is a bounded function of time, while in the $PT$ symmetry broken regime, one can always find a time-dependent metric which grows exponentially in time.

## 5.1 A family of non-Hermitian Hamiltonians

We consider a family of linear operators $H(\varkappa) \in \mathcal{B}(\mathbb{C}^N)$, where $\varkappa \in I$ varies in the open interval

$$I = (\varkappa_0 - \epsilon, \varkappa_0 + \epsilon), \tag{5.1}$$

centered at $\varkappa_0 \in \mathbb{R}$ having radius $\epsilon > 0$. The interval is the disjoint union

$$I = I_- \cup \{\varkappa_0\} \cup I_+, \tag{5.2}$$

where

$$I_- = (\varkappa_0 - \epsilon, \varkappa_0), \quad I_+ = (\varkappa_0, \varkappa_0 + \epsilon). \tag{5.3}$$

We make the following assumptions.

(A1) $\varkappa \mapsto H(\varkappa)$ is a continuous map from $I$ to $\mathcal{B}(\mathbb{C}^N)$.

(A2) For $\varkappa \in I_-$, the spectrum of $H(\varkappa)$ consists of $N$ simple, real eigenvalues $\{E_j^-(\varkappa)\}_{j=1}^N$.

(A3) For $\varkappa \in I_+$, the spectrum of $H(\varkappa)$ consists of $m$ real eigenvalues $\{E_j^+(\varkappa)\}_{j=1}^m$ and of $p$ pairs of complex conjugate eigenvalues $\{E_{m+j}^+(\varkappa), \bar{E}_{m+j}^+(\varkappa)\}_{j=1}^p$, all of which are simple. We allow for $m = 0$ or $p = 0$ (no eigenvalues real or all of them real). We have $N = m + 2p$.

We refer to $\varkappa \in I_-$ as the unbroken symmetry phase (regime) and to $\varkappa \in I_+$ as the broken symmetry phase (regime). The point $\varkappa_0$ is called the symmetry breaking, or transition point. As a consequence of (A2) and (A3), $H(\varkappa)$ is diagonalizable for $\varkappa \neq \varkappa_0$ and we have the spectral representation of $H(\varkappa)$,

$$H(\varkappa) = \sum_{E \in \mathrm{spec}(H(\varkappa))} E\, P(E). \tag{5.4}$$

For each eigenvalue $E \in \mathrm{spec}(H(\varkappa))$, the associated spectral (Riesz) projection is

$$P(E) = \frac{-1}{2\pi i} \int_{\Gamma(E)} (H(\varkappa) - z)^{-1} dz, \tag{5.5}$$

where $\Gamma(E)$ is a circle in the complex plane, centered at $E$ and such that all eigenvalues other than $E$ are outside $\Gamma(E)$. The projections satisfy $\forall E, E' \in \mathrm{spec}(H(\varkappa))$

$$P(E)P(E') = \delta_{E,E'} P(E) \qquad \text{and} \qquad \sum_{E \in \mathrm{spec}(H(\varkappa))} P(E) = \mathbb{1}. \tag{5.6}$$

Due to the simplicity assumption in (A2) and (A3) we have

$$P(E) = |\psi(E)\rangle\langle\phi(E)|, \tag{5.7}$$

with

$$\langle\phi(E)|\psi(E')\rangle = \delta_{E,E'}. \tag{5.8}$$

Moreover, for all $E \in \mathrm{spec}(H(\varkappa))$,

$$H(\varkappa)|\psi(E)\rangle = E|\psi(E)\rangle \quad \text{and} \quad H(\varkappa)^*|\phi(E)\rangle = \bar{E}|\phi(E)\rangle. \tag{5.9}$$

Of course, the eigenvalues $E$ of $H(\varkappa)$ depend on $\varkappa$, and so do the eigenvectors $|\psi(E)\rangle$ and $|\phi(E)\rangle$.

We now collect a few results about the continuity of spectral data of the family $H(\varkappa)$. We start off with a general result about the continuity of the spectrum.

**Proposition 5.1.1.** *Let $A$ and $B$ be bounded operators on a Hilbert space $\mathcal{H}$ and denote their spectra by $\sigma(A)$, $\sigma(B)$. For any $\epsilon > 0$ there exists a $\delta > 0$ such that if $\|A - B\| < \delta$, then* $\mathrm{dist}(\sigma(A), \sigma(B)) < \epsilon$.

**Proof.** Let $z \in \rho(A) = \mathbb{C}\backslash\sigma(A)$, the resolvent set of $A$. From the relation $B - z = [\mathbb{1} - (A - B)(A - z)^{-1}](A - z)$ we see that $B - z$ is invertible provided $\|(A - B)(A - z)^{-1}\| < 1$, because then the right side is the product of two invertible operators. Thus,

$$\{z \ : \ \|A - B\| \, \|(A - z)^{-1}\| < 1\} \subset \rho(B). \tag{5.10}$$

The function $z \mapsto (A-z)^{-1}$ is holomorphic in $\rho(A)$ and $(A-z)^{-1} \to 0$ for $|z| \to \infty$, so it is bounded for all $z$ in the complement of any $\epsilon$-neighbourhood of $\sigma(A)$: $\forall \epsilon > 0 \, \exists b < \infty$ such that $\|(A - z)^{-1}\| \leq b$ whenever $\mathrm{dist}(z, \sigma(A)) \geq \epsilon$. Take $\|A - B\| < \delta := 1/b$. Then according to (5.10), each $z$ with $\mathrm{dist}(z, \sigma(A)) \geq \epsilon$ belongs to $\rho(B)$. Hence $\sigma(B)$ is contained in $\{z : \mathrm{dist}(z, \sigma(A)) < \epsilon\}$. □

**Proposition 5.1.2.** *Suppose $\varkappa \mapsto H(\varkappa)$ is a family of bounded operators on a finite-dimensional Hilbert space, which is continuous in an open interval $I$ around some $\varkappa_0 \in \mathbb{R}$, and such that $H(\varkappa)$ is diagonalizable for all $I \ni \varkappa \neq \varkappa_0$. Then the following holds.*

(1) *The eigenvalues and the Riesz projections of $H(\varkappa)$ are continuous in $\varkappa \in I$.*

(2) *One can find eigenvectors* $|\psi(E)\rangle$ *and* $|\phi(E)\rangle$ *of* $H(\varkappa)$ *and* $H(\varkappa)^*$ *that are continuous vector valued functions in* $\varkappa$ *for* $\varkappa \in I$, $\varkappa \neq \varkappa_0$, *and such that* (5.7) *holds.*

**Proof.** (1) We first show the continuity of the eigenvalues. Let $\mu \in I$ and let $\eta$ be small. We use proposition 5.1.1 with $A = H(\mu)$ and $B = H(\mu + \eta)$. Given any $\varepsilon > 0$ there exists a $\delta_0 > 0$ such that if $\|H(\mu+\eta) - H(\mu)\| < \delta_0$, then the eigenvalues of $H(\mu + \eta)$ lie in the union of the $\varepsilon$-balls centered at the eigenvalues of $H(\mu)$. By the continuity of $H(\varkappa)$ at $\varkappa = \mu$, there exists a $\delta > 0$ such that if $|\eta| < \delta$ then $\|H(\mu + \eta) - H(\mu)\| < \delta_0$. Hence for $|\eta| < \delta$ each eigenvalue of $H(\eta + \mu)$ lies at a distance $< \varepsilon$ to one of the eigenvalues of $H(\mu)$. This shows the continuity of the eigenvalues of $H(\varkappa)$ at $\varkappa = \mu$.

Now we show the continuity of the Riesz projections. Again, let $\mu \in I$ and let $\eta$ be small. We have

$$(H(\mu+\eta) - z)^{-1} = (H(\mu) - z)^{-1} + (H(\mu+\eta) - z)^{-1}[H(\mu) - H(\mu+\eta)](H(\mu) - z)^{-1}.$$

For $\varkappa$ close to $\mu$, denote by $P(\varkappa)$ the Riesz projection given by (5.5) for a fixed contour $\Gamma$ around an eigenvalue $E$ of $H(\mu)$. Then

$$P(\mu+\eta) - P(\mu) = \frac{-1}{2\pi i}\int_\Gamma (H(\mu+\eta) - z)^{-1}[H(\mu) - H(\mu+\eta)](H(\mu) - z)^{-1}dz.$$

It follows that $\|P(\mu+\eta) - P(\mu)\| \leq C\|H(\mu) - H(\mu+\eta)\| \to 0$ as $\eta \to 0$. This shows that $P(\varkappa)$ is continuous at $\mu \in I$.

(2) Let $\varkappa = \varkappa_* \in I\backslash\{\varkappa_0\}$ be fixed. We construct a family of eigenvectors which is continuous at $\varkappa_*$. Pick any nonzero vector $\chi \in \mathrm{Ran}P(\varkappa_*)$ and set $\psi_\varkappa = P(\varkappa)\chi$. The map $\varkappa \mapsto \psi_\varkappa$ is continuous at $\varkappa_*$ due to point (1) of the proposition. Since $P(\varkappa_*)\chi = \chi$, we have by the inverse triangle inequality,

$$\|\psi_\varkappa\| \geq \|\chi\| - \|[P(\varkappa) - P(\varkappa_*)]\chi\| \geq \big(1 - \|P(\varkappa) - P(\varkappa_*)\|\big)\|\chi\|.$$

By part (1) we have $\|\psi_\varkappa\| > \|\chi\|/2$ for all $\varkappa$ in a neighbourhood of $\varkappa_*$. Hence $\varkappa \mapsto \psi_\varkappa$ is a continuous family of nonzero vectors in a neighborhood of $\varkappa_*$. Now $\psi_\varkappa$ is indeed an eigenvector of $H(\varkappa)$ with eigenvalue $E(\varkappa)$: Since $H(\varkappa)$ is diagonalizable, we have

$H(\varkappa)P(\varkappa) = E(\varkappa)P(\varkappa)$ and so

$$H(\varkappa)\psi_\varkappa = H(\varkappa)P(\varkappa)\chi = E(\varkappa)P(\varkappa)\chi = E(\varkappa)\psi_\varkappa.$$

The argument for $H(\varkappa)^*$ is the same. Set $K(\varkappa) = H(\varkappa)^*$. Then $K(\varkappa)$ is continuous in $\varkappa \in I$ and diagonalizable for $\varkappa \neq \varkappa_*$, and one can repeat the above argument.

Note: This procedure is extended easily to the case where the eigenvalues are not simple. Then $\dim P(\varkappa) = d > 1$ is constant in a neighbourhood of $\varkappa_*$ and one chooses $d$ linearly independent vectors $\chi_1, \ldots, \chi_d \in \mathrm{Ran}P(\varkappa_*)$ and proceeds as above. □

## 5.2 Solutions of the Dyson equation

Let $H$ be a diagonalizable operator in $\mathcal{B}(\mathbb{C}^N)$

$$H = \sum_{E \in \sigma(H)} EP(E), \tag{5.11}$$

where $\sigma(H) \subset \mathbb{C}$ is the spectrum and $P(E)$ is the eigenprojection associated to $E$. The eigenvalues are not assumed to be simple here. The set $\sigma(H)$ automatically consists of at most $N$ distinct points. Consider the linear operator $\mathcal{L}$ acting on $\mathcal{B}(\mathbb{C}^N)$ as

$$\mathcal{L}\eta = H^*\eta - \eta H, \qquad \eta \in \mathcal{B}(\mathbb{C}^N). \tag{5.12}$$

We want to solve the Dyson equation (4.6),

$$i\partial_t \eta(t) = \mathcal{L}\eta(t), \quad t \geq 0, \tag{5.13}$$

with a given initial condition

$$\eta(0) = \eta_0 \in \mathcal{B}(\mathbb{C}^N). \tag{5.14}$$

The solution can be written as

$$\eta(t) = e^{-it\mathcal{L}}\eta_0 = e^{-itH^*}\eta_0 e^{itH}. \tag{5.15}$$

The operator $\mathcal{L}$ is not Hermitian w.r.t. the inner product $\langle A, B\rangle = \mathrm{tr}(A^*B)$ on $\mathcal{B}(\mathbb{C}^N)$ unless $H = H^*$ is Hermitian as an operator on $\mathbb{C}^N$ with respect to the Euclidean inner product. In fact, the adjoint of $\mathcal{L}$ is $\mathcal{L}^*\eta = H\eta - \eta H^*$.

For $E, F \in \sigma(H)$, define the operators $\mathcal{P}(E, F)$ acting on $\eta \in \mathcal{B}(\mathbb{C}^N)$ by

$$\mathcal{P}(E, F)\eta = P(E)^*\eta P(F). \tag{5.16}$$

These are the spectral projections of $\mathcal{L}$,

$$\mathcal{L} = \sum_{E,F\in\sigma(H)} (\bar{E} - F)\mathcal{P}(E, F). \tag{5.17}$$

Clearly we have (as it should be)

$$\mathcal{P}(E, F)\mathcal{P}(E', F') = \mathcal{P}(E, F)\delta_{E,E'}\delta_{F,F'} \quad \text{and} \quad \sum_{E,F\in\sigma(H)} \mathcal{P}(E, F) = \mathbb{1}. \tag{5.18}$$

**Proposition 5.2.1.** *For any operator $\eta \in \mathcal{B}(\mathbb{C}^N)$, the following are equivalent*

(i) *$\eta$ is a time-independent solution of the Dyson equation* (5.13)

(ii) $\mathcal{L}\eta = 0$

(iii) $P(E)^*\eta P(F) = 0$ *whenever* $\bar{E} \neq F$

(iv)
$$\eta = \sum_{E\in\sigma(H)} P(E)^*\eta P(\bar{E}) \tag{5.19}$$

**Proof.** (i) $\Longleftrightarrow$ (ii): $\partial_t\eta = 0$ is equivalent to $\mathcal{L}\eta = 0$ by (5.13).
(ii) $\Longrightarrow$ (iii): From (5.17) we get

$$\mathcal{L}\eta = \sum_{E,F\in\sigma(H)} (\bar{E} - F)P(E)^*\eta P(F) = 0. \tag{5.20}$$

Let $E_0, F_0 \in \sigma(H)$. Apply $P(E_0)^*$ and $P(F_0)$ to equation (5.20),

$$P(E_0)^*\mathcal{L}\eta P(F_0) = \sum_{E,F\in\sigma(H)} (\bar{E} - F)P(E_0)^*P(E)^*\eta P(F)P(F_0) = 0. \tag{5.21}$$

Due to the disjointness condition (5.6), the relation (5.21) reduces to

$$(\bar{E}_0 - F_0)P(E_0)^*\eta P(F_0) = 0.$$

This means $P(E_0)^*\eta P(F_0) = 0$ whenever $\bar{E}_0 \neq F_0$. Since $E_0$ and $F_0$ are arbitrary eigenvalues, (iii) holds.

(iii) $\Longrightarrow$ (iv): Using the completeness relation $\mathbb{1} = \sum_{E\in\sigma(H)} P(E) = \sum_{E\in\sigma(H)} P(E)^*$ we obtain

$$\eta = \sum_{E,F\in\sigma(H)} P(E)^*\eta P(F) = \sum_{E\in\sigma(H)} P(E)^*\eta P(\bar{E}).$$

(iv) $\Longrightarrow$ (iii): For $\eta$ of the form (5.19) and $E, F \in \sigma(H)$, we have by the disjointness of the projections, that $P(E)^*\eta P(F) = 0$ unless $E = \bar{F}$.

Finally (iii) $\Longrightarrow$ (i):

$$\begin{aligned} e^{-itH^*}\eta e^{itH} &= \Big(\sum_{E\in\sigma(H)} e^{-it\bar{E}}P(E)^*\Big)\eta\Big(\sum_{F\in\sigma(H)} e^{itF}P(F)\Big) \\ &= \sum_{E,F\in\sigma(H)} e^{-it(\bar{E}-F)}P(E)^*\eta P(F) \\ &= \sum_{E\in\sigma(H)} P(E)^*\eta P(\bar{E}) = \eta. \end{aligned}$$

This completes the proof of Proposition 5.2.1. □

### 5.2.1 Metrics at the transition point $\varkappa_0$

Suppose now $H(\varkappa)$ depends continuously on $\varkappa \in I_- = (\varkappa_0 - \epsilon, \varkappa_0)$ and that the limit

$$\lim_{\varkappa\to(\varkappa_0)_-} H(\varkappa) = H_- \tag{5.22}$$

exists. For each $\varkappa \in I_-$, consider the Dyson equation

$$i\partial_t\eta(t,\varkappa) = \mathcal{L}(\varkappa)\eta(t,\varkappa), \quad t \geq 0, \tag{5.23}$$

with $\mathcal{L}(\varkappa)\eta(t,\varkappa) = H^*(\varkappa)\eta(t,\varkappa) - \eta(t,\varkappa)H(\varkappa)$ and initial condition

$$\eta(0,\varkappa) = \eta_0(\varkappa) \in \mathcal{B}(\mathbb{C}^N). \tag{5.24}$$

Denote the distinct eigenvalues of $H_-$ by $\{E_1^-, \ldots, E_r^-\} \subset \mathbb{C}$, $1 \leq r \leq N$.

**Proposition 5.2.2.** *Each eigenvalue $E(\varkappa)$ of $H(\varkappa)$ is continuous in $\varkappa \in I_-$ and converges to one $E_\ell^-$ as $\varkappa \to (\varkappa_0)_-$. Conversely, each $E_\ell^-$ is the limit of at least one eigenvalue $E(\varkappa)$ of $H(\varkappa)$ as $\varkappa \to (\varkappa_0)_-$.*

**Proof.** Define the operator

$$\bar{H}(\varkappa) = \begin{cases} H(\varkappa), & \varkappa \in I_- \\ H_-, & \varkappa \geq \varkappa_0 \end{cases}.$$

The map $\varkappa \mapsto \bar{H}(\varkappa)$ is continuous in $\varkappa$ in a real, open interval centered at $\varkappa_0$. It follows from Proposition 5.1.2 that the eigenvalues of $\bar{H}(\varkappa)$ are continuous in $\varkappa$ and in particular, that any arbitrarily small neighborhood of the spectrum of $H_-$ contains the spectrum of $H(\varkappa)$, provided that $\varkappa$ close enough to $\varkappa_0$.

To see that every eigenvalue $E_\ell^-$ is the limit of an eigenvalue of $H(\varkappa)$, we let $\Gamma$ be a circle in the complex plane, centered at $E_\ell^-$ of radius small enough so that no other eigenvalues of $H_-$ are contained inside or on it. Then for $\varkappa$ close to $\varkappa_0$, there are no eigenvalues of $H(\varkappa)$ on $\Gamma$, since the spectrum of that operator has to cluster around the spectrum of $H_-$. Thus the projection

$$P(\varkappa) = \frac{-1}{2\pi i}\int_\Gamma (H(\varkappa) - z)^{-1} dz$$

is well defined, and as $\varkappa \to (\varkappa_0)_-$, it converges to the spectral (Riesz) projection of $H_-$ associated to $E_\ell^-$. So their ranks coincide, *i.e.*, $\mathrm{rank}P(\varkappa) \geq 1$ for $\varkappa$ sufficiently close to $(\varkappa_0)_-$. This rank is the sum of the algebraic multiplicities of all eigenvalues of $H(\varkappa)$ inside $\Gamma$. □

The next result shows that if $H_-$ has a real, non-semisimple eigenvalue, then the Dyson equation for $H_-$ does not have a strictly positive, time-independent solution.

Recall that an eigenvalue $\lambda$ is classified as semisimple if its geometric and algebraic multiplicities are equal. In the case where both the geometric and algebraic

multiplicities are one, $\lambda$ is called a simple eigenvalue. An eigenvalue $\lambda$ is said to be non-semisimple if its geometric multiplicity is strictly less than its algebraic multiplicity.

**Theorem 5.2.1.** *Suppose that $H_-$ has a real eigenvalue that is not semisimple. Then the kernel of $\mathcal{L}_{H_-}$ does not contain any strictly positive element.*

**Proof.** The result follows immediately from the following observation. □

**Proposition 5.2.3.** *For $H \in \mathcal{B}(\mathbb{C}^N)$ define the linear operator $\mathcal{L}_H$ acting on $\eta \in \mathcal{B}(\mathbb{C}^N)$ by $\mathcal{L}_H\eta = H^*\eta - \eta H$. If $H$ has a real eigenvalue that is not semisimple, then the kernel of $\mathcal{L}_H$ does not contain any strictly positive element.*

**Proof.** According to the Jordan decomposition, there is an invertible map $Q$ such that $H = Q^{-1}JQ$, where $J$ has the following properties. In a certain (Jordan) basis of $\mathbb{C}^N$, $J$ takes the form of a block diagonal matrix, with blocks $J_k$, called Jordan blocks. Each $J_k$ is either a $1 \times 1$ matrix, or a square matrix of size greater than one, in which case $J_k$ has a constant diagonal (the entry is an eigenvalue of $H$), and the super diagonal consists of the entry 1, and all other matrix elements are zero.

Take an $H$ which is not diagonalizable. Then there is at least one Jordan block, $J_k$, of size $\geq 2$. Splitting off the diagonal gives $J_k = E\mathbb{1}_k + N_k$, where $E$ is a non-semisimple eigenvalue of $H$ and $N_k$ is the nilpotent matrix with zeros everywhere except ones on the super diagonal. Thus there exists a $\psi \in \mathbb{C}^N$ such that $N_k\psi \neq 0$ but $N_k^2\psi = 0$. This means that

$$(J - E\mathbb{1})\psi \neq 0 \quad \text{but} \quad (J - E\mathbb{1})^2\psi = 0. \tag{5.25}$$

Using the decomposition $H = Q^{-1}JQ$ and setting $\xi = (Q^{-1})^*\eta Q^{-1}$, we get for *real* $E$

$$H^*\eta - \eta H = 0 \quad \Longleftrightarrow \quad J^*\xi - \xi J = 0 \quad \Longleftrightarrow \quad (J - E\mathbb{1})^*\xi - \xi(J - E\mathbb{1}) = 0. \tag{5.26}$$

Since strict positivity of $\eta$ and $\xi$ are equivalent, it suffices to show that if the last equality in (5.26) holds, then $\xi$ cannot be strictly positive. Now we show that indeed, if $\xi$ is positive, then $(J - E\mathbb{1})\psi$ is in the kernel of $\xi$: due to (5.25),

$$0 = \langle \psi, \xi(J - E\mathbb{1})^2\psi\rangle = \langle \psi, (J - E\mathbb{1})^*\xi(J - E\mathbb{1})\psi\rangle = \|\sqrt{\xi}(J - E\mathbb{1})\psi\|^2.$$

This completes the proof of Proposition 5.2.3 and hence that of Theorem 5.2.1. □

The following is a consequence of Theorem 5.2.1. Suppose that the Dyson equation for $H(\varkappa)$ has a strictly positive, time-independent solution $\eta(\varkappa)$ for each $\varkappa \in I_-$, and that $\eta(\varkappa) \to \eta_-$ as $\varkappa \to (\varkappa_0)-$. Then of course, $\eta_-$ is a non-negative, time-independent solution of the Dyson equation for $H_-$ (*i.e.*, $0 \leq \eta_- \in \ker\mathcal{L}_{H_-}$), but due to Theorem 5.2.1, $\eta_-$ cannot be invertible. that is, $\eta_-$ is not strictly positive.

**Proposition 5.2.4.** *If $H$ is diagonalizable and has a real spectrum as in* (5.11)*, then the kernel of $\mathcal{L}_H$ contains a strictly positive element $\eta > 0$.*

**Proof.** Suppose $H$ is diagonalizable and has real spectrum. We need to show that there is an $\eta > 0$ such that $\mathcal{L}\eta = 0$. Let $\eta_0 > 0$ and set

$$\eta = \sum_{E\in\sigma(H)} P(E)^*\eta_0 P(E).$$

The disjointness of the projections, (5.6), easily implies that $\sum_{E\in\sigma(H)} P(E)^*\eta P(E) = \eta$ and so Proposition 5.2.1, (5.19), shows that $\eta$ is in the kernel of $\mathcal{L}$. (We use here that the spectrum is real.) We now show that $\eta > 0$. For any $\psi \in \mathbb{C}^N$ we have

$$\langle\psi|\eta\psi\rangle = \sum_{E\in\sigma(H)} \|\sqrt{\eta_0}P(E)\psi\|^2 \geq 0, \tag{5.27}$$

which shows that $\eta \geq 0$. Now suppose $\eta\psi = 0$. Then by (5.27), $P(E)\psi = 0$ for all $E \in \sigma(H)$. Hence (see (5.6)) $0 = \sum_{E\in\sigma(H)} P(E)\psi = \psi$. It follows that $\eta > 0$. □

As an immediate application of Propositions 5.2.3 and 5.2.4 suited to the world of $PT$ symmetry, we have the following result.

**Theorem 5.2.2.** *Suppose that $\varkappa \mapsto H(\varkappa)$ is continuous in a neighborhood of $\varkappa_0$ and has real eigenvalues for $\varkappa < \varkappa_0$. Then $H(\varkappa_0)$ is diagonalizable if and only if the kernel of $\mathcal{L}_{H(\varkappa_0)}$ contains a strictly positive element $\eta > 0$.*

**Proof.** Due to the continuity of $\varkappa \mapsto H(\varkappa)$ at $\varkappa_0$, the spectrum of $H(\varkappa_0)$ is real. Proposition 5.2.4 thus gives the implication ($\Rightarrow$). The implication ($\Leftarrow$ ) is correct since it is just the contrapositive of proposition 5.2.3. □

Theorem 5.2.2 shows that at an exceptional point $\varkappa_0$, there is a time-independent metric $\eta > 0$ satisfying the quasi-Hermiticity equation (4.6) if and only if $H(\varkappa_0)$ is diagonalizable. Based on observations made in concrete examples, it is claimed in the

$PT$ symmetry literature that $H(\varkappa_0)$ is not diagonalizable at exceptional points $\varkappa_0$. Theorem 5.2.2 shows that the statement of this claim is true if and only if there is no time-independent metric for $\varkappa = \varkappa_0$.

### 5.2.2 Metrics in the $PT$ symmetry broken regime $\varkappa \in I_+$

**Proposition 5.2.5.** *Any time-independent solution of the Dyson equation in the interval $I_+$ cannot be strictly positive.*

**Proof.** Recall, $\eta$ is a time-independent solution to the Dyson equation (5.15) if and only if $\eta \in ker(\mathcal{L}(\varkappa))$, that is, if and only if

$$H^*(\varkappa)\eta - \eta H(\varkappa) = 0. \tag{5.28}$$

Due to assumption in (A.3), the formulas of $H(\varkappa)$ and $H^*(\varkappa)$ in equation (5.4) can be written as

$$\begin{aligned} H(\varkappa) &= \sum_{i=1}^{m} E_i P(E_i) + \sum_{j=1}^{p} E_{m+j} P(E_{m+j}) + \bar{E}_{m+j} P(\bar{E}_{m+j}), \\ H^*(\varkappa) &= \sum_{i=1}^{m} E_i P(E_i)^* + \sum_{j=1}^{p} \bar{E}_{m+j} P(E_{m+j})^* + E_{m+j} P(\bar{E}_{m+j})^*. \end{aligned} \tag{5.29}$$

For simplicity of notation, we drop the plus sign at the top of eigenvalues ($E_k \equiv E_k^+$) as we only consider the case $\varkappa \in I_+$ here. Combining (5.28) and (5.29) we have

$$\begin{aligned} &\Big( \sum_{i=1}^{m} E_i P(E_i)^* + \sum_{j=1}^{p} \bar{E}_{m+j} P(E_{m+j})^* + E_{m+j} P(\bar{E}_{m+j})^* \Big) \eta \\ &\qquad = \eta \Big( \sum_{i=1}^{m} E_i P(E_i) + \sum_{j=1}^{p} E_{m+j} P(E_{m+j}) + \bar{E}_{m+j} P(\bar{E}_{m+j}) \Big). \end{aligned} \tag{5.30}$$

Now, we use the ordered basis formed by the eigenvectors of $H(\varkappa)$

$$\{\psi(E_1), \cdots, \psi(E_m), \psi(E_{m+1}), \psi(\bar{E}_{m+1}), \cdots, \psi(E_{m+p}), \psi(\bar{E}_{m+p})\}$$

together with equation (5.30) to calculate the matrix elements of $\eta$. Assume that all

of the eigenvalues are simple, then for all $k, l = 1 \cdots, m$ and $j, n = 1, \cdots, p$ we have

$$\begin{aligned} \langle \psi(E_k) | \eta \psi(E_l) \rangle &= 0, \text{ if } k \neq l, \\ \langle \psi(E_k) | \eta \psi(E_{m+j}) \rangle = \langle \psi(E_k) | \eta \psi(\bar{E}_{m+j}) \rangle &= 0, \ \forall \ \{k, j\}, \\ \langle \psi(E_{m+j}) | \eta \psi(E_{m+n}) \rangle &= 0, \text{ if } \bar{E}_{m+j} \neq E_{m+n}. \end{aligned} \tag{5.31}$$

Then, the matrix representation of $\eta$ is

$$\eta = \left[\begin{array}{ccc|ccc} x_1 & & 0 & & & \\ & \ddots & & & 0 & \\ & & \ddots & & & \\ 0 & & x_m & & & \\ \hline & & & \begin{bmatrix} 0 & y_{m+1} \\ z_{m+1} & 0 \end{bmatrix} & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & \begin{bmatrix} 0 & y_{m+p} \\ z_{m+p} & 0 \end{bmatrix} \end{array}\right]_{N \times N},$$

where

$$\begin{aligned} x_i &= \langle \psi(E_i), \eta \psi(E_i) \rangle, \\ y_{m+j} &= \langle \psi(E_{m+j}), \eta \psi(\bar{E}_{m+j}) \rangle, \\ z_{m+j} &= \langle \psi(\bar{E}_{m+j}), \eta \psi(E_{m+j}) \rangle. \end{aligned}$$

Clearly the matrix $\eta$ is a block diagonal matrix with blocks $\{A_k\}_{k=1}^m$ and $\{A_{m+j}\}_{j=1}^p$ where $A_k = [x_k]$ and $A_{m+j} = \begin{bmatrix} 0 & y_{m+j} \\ z_{m+j} & 0 \end{bmatrix}$. Then, by matrix algebra, we have

$$\begin{aligned} det(\eta - \lambda \mathbb{1}) &= det(A_1 - \lambda \mathbb{1}) \times \cdots \times det(A_{m+p} - \lambda \mathbb{1}), \\ &= (x_1 - \lambda) \times \cdots \times (x_m - \lambda) \times (\lambda^2 - \mu_{m+1}) \times \cdots \times (\lambda^2 - \mu_{m+p}), \end{aligned} \tag{5.32}$$

where $\mu_{m+j} = y_{m+j} z_{m+j}$. Equation (5.32) shows that $\eta$ cannot have only positive eigenvalues. □

Our next result concerns the relation between boundedness properties of metric

operators $\eta(t)$ and the symmetry phases.

**Theorem 5.2.3.** *Let $H$ be a diagonalizable linear operator on $\mathbb{C}^N$.*

1. *If the spectrum of $H$ is real, then any (not necessarily positive definite) solution $\eta(t)$ of (5.15) satisfies $\sup_{t\in\mathbb{R}} \|\eta(t)\| < \infty$.*
2. *If $H$ has a complex conjugate pair of eigenvalues $z = x \pm iy$, then there is an initial condition $\eta(0) > 0$ and a constant $C > 0$ such that the solution $\eta(t)$ of (5.15) satisfies $\|\eta(t)\| \geq Ce^{2|yt|}$ for all $t \in \mathbb{R}$.*

**Proof.** We have

$$H = \sum_k E_k |\psi_k\rangle\langle\phi_k|, \tag{5.33}$$

for a BIOS $\{|\phi_k\rangle, |\psi_k\rangle\}$ and eigenvalues $E_k \in \mathbb{C}$. Then by the functional calculus, we obtain from (5.15)

$$\eta(t) = \sum_{k,l} e^{it(E_l - \bar{E}_k)} \eta_{k,l} |\phi_k\rangle\langle\phi_l|, \qquad \eta_{k,l} = \langle\psi_k|\eta(0)\psi_l\rangle. \tag{5.34}$$

If all $E_j$ are real then clearly $\eta(t)$ is bounded in $t$, as asserted in point 1. To address point 2, take $\eta(0) = \sum_k |\phi_k\rangle\langle\phi_k|$. Since $\{|\phi_k\rangle\}_k$ is a basis of $\mathbb{C}^N$, this $\eta(0)$ has full rank $N$ and hence $\eta(0) > 0$. We obtain from (5.34),

$$\eta(t) = \sum_k e^{-2t\mathrm{Im}E_k} |\phi_k\rangle\langle\phi_k|.$$

Suppose now that there is a pair of complex conjugate eigenvalues, say $E_1 = x + iy$, $E_2 = x - iy$. Then $\eta(t)|\psi_1\rangle = e^{-2ty}|\phi_1\rangle$ and $\eta(t)|\psi_2\rangle = e^{2ty}|\phi_2\rangle$. Thus $\|\eta(t)\| \geq Ce^{2|yt|}$. This concludes the proof. □

# Chapter 6

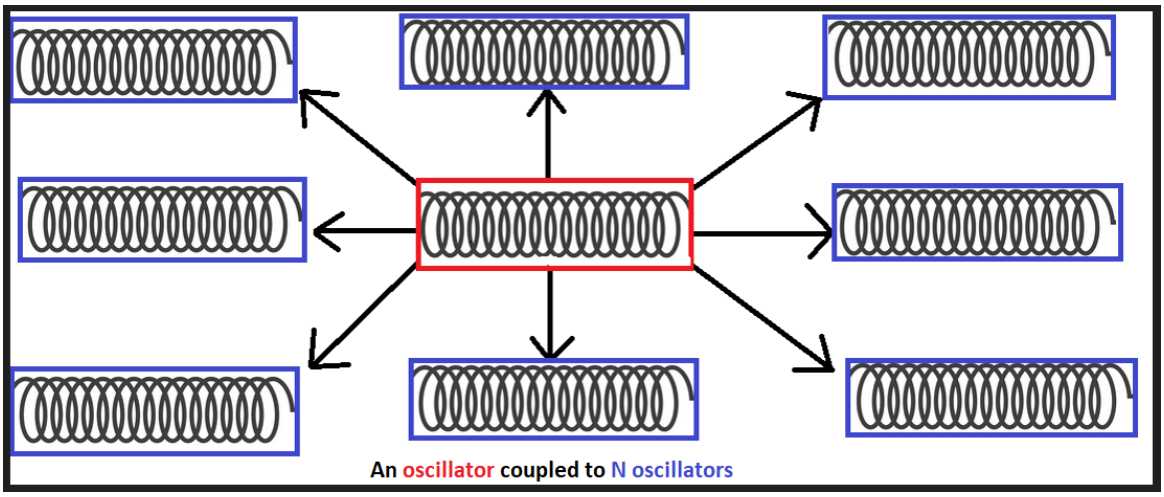
# Entropy and entanglement in a bipartite quasi-Hermitian system and its Hermitian counterparts

In this chapter, we present the content of our published paper [26], having the following

**Abstract.** We consider a quantum oscillator coupled to a bath of $N$ other oscillators. The total system evolves with a quasi-Hermitian Hamiltonian. Associated to it is a family of Hermitian systems, parameterized by a unitary map $W$. Our main goal is to find the influence of $W$ on the entropy and the entanglement in the Hermitian systems. We calculate explicitly the reduced density matrix of the single oscillator for all Hermitian systems and show that, regardless of $W$, their von Neumann entropy oscillates with a common period which is twice that of the non-Hermitian system. We show that, generically, the oscillator and the bath are entangled for almost all times. While the amount of entanglement depends on the choice of $W$, the entanglement of the time-averaged density matrix is entirely independent of $W$. These results describe some universality in the physical properties of all Hermitian systems associated to a given non-Hermitian one.

## 6.1 Model

We consider an oscillator with creation and annihilation operators $a^*$, $a$ is coupled to a 'bath' of $N$ independent oscillators with creation and annihilation operators $q_i^*$, $q_i$, $i = 1, \ldots, N$.




The total Hilbert space of the $N + 1$ oscillators is

$$\mathcal{H} = \mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{B}}, \tag{6.1}$$

where $\mathcal{H}_{\mathrm{S}}$ is the space of a single oscillator and $\mathcal{H}_{\mathrm{B}}$ is that of the other $N$. We denote the inner product by $\langle \cdot | \cdot \rangle$ and let $^*$ denote the adjoint in this inner product. The commutation relations are $[a, a^*] = 1 = [q_i, q_i^*]$, and all operators belonging to different oscillators commute. This open quantum system, non-Hermitian model was used in [18].

### 6.1.1 The quasi-Hermitian system

The coupled total system–bath Hamiltonian is

$$H = \nu N_{\mathrm{tot}} + (g + \kappa)\sqrt{N}\, a^* Q + (g - \kappa)\sqrt{N}\, a Q^*, \tag{6.2}$$

where $\nu > 0$ and $g, \kappa \in \mathbb{R}$ are parameters and

$$N_{\text{tot}} = a^* a + \sum_{n=1}^{N} q_n^* q_n, \qquad Q = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} q_n. \tag{6.3}$$

Due to the different prefactors of $\kappa$ in the interaction term of (6.2), $H$ is $^*$-Hermitian if and only if $\kappa = 0$.

The 'uncoupled' ($g = \kappa = 0$) Hamiltonian is simply $\nu N_{\text{tot}}$, a multiple of the total number operator $N_{\text{tot}}$. As $H$ commutes with $N_{\text{tot}}$, each eigenspace of $N_{\text{tot}}$, with a fixed number of excitations (in the system plus the bath) is left invariant. Denote by $|0_{\text{S}}0_{\text{B}}\rangle$ the 'vacuum' zero excitation state, where all oscillators are in the ground state. The single excitation space is defined as

$$\mathcal{E}_1 = \text{span}\big\{|1_{\text{S}}0_{\text{B}}\rangle, |0_{\text{S}}1_1\rangle, |0_{\text{S}}1_2\rangle, \ldots, |0_{\text{S}}1_N\rangle\big\}, \tag{6.4}$$

where $|1_{\text{S}}0_{\text{B}}\rangle = a^*|0_{\text{S}}0_{\text{B}}\rangle$ and $|0_{\text{S}}1_i\rangle = q_i^*|0_{\text{S}}0_{\text{B}}\rangle$ for $i = 1, \ldots, N$. When $H$ is applied to a vector in $\mathcal{E}_1$ the result is again a vector in $\mathcal{E}_1$. Moreover, due to the collective, symmetric nature of the system-bath interaction in (6.2), $H$ leaves the even smaller space

$$\mathcal{H}_1 = \text{span}\big\{|e_{\text{S}}\rangle,\ |e_{\text{B}}\rangle\big\} \tag{6.5}$$

invariant, where

$$|e_{\text{S}}\rangle = |1_{\text{S}}0_{\text{B}}\rangle, \qquad |e_{\text{B}}\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} |0_{\text{S}}1_n\rangle. \tag{6.6}$$

Those two vectors describe states in which a single excitation is either in S (the state $|e_{\text{S}}\rangle$) or in B, collectively spread over the $N$ bath oscillators (the state $|e_{\text{B}}\rangle$). Therefore, we may view $H$ as an operator on $\mathcal{H}_1$. When we do this we denote it by $H_1$, which has the form

$$H_1 = \nu \mathbb{1} + (g - \kappa)\sqrt{N}\ |e_{\text{B}}\rangle\langle e_{\text{S}}| + (g + \kappa)\sqrt{N}\ |e_{\text{S}}\rangle\langle e_{\text{B}}|. \tag{6.7}$$

The eigenvalues of $H_1$ are

$$\omega_{\pm} = \nu \pm \omega, \qquad \omega = \sqrt{N}\sqrt{g^2 - \kappa^2}, \tag{6.8}$$

which are real for $\kappa^2 \leq g^2$ and (purely imaginary) complex conjugates for $\kappa^2 > g^2$.

See [18] for a discussion of the $PT$ symmetry of $H$. The operator $H_1$ is diagonalizable except at the transition points defined by $\kappa^2 = g^2 \neq 0$, where $H_1$ reduces to a Jordan block. Note that increasing the number $N$ of oscillators in the bath simply amounts to speeding up the dynamics (the frequency $\omega$) by a factor $\sqrt{N}$.

We consider the '$PT$ symmetry unbroken regime' $\kappa^2 < g^2$, so that $\omega_\pm \in \mathbb{R}$. For definiteness we take $g > 0$ (the case $g < 0$ can be dealt with in the same fashion), so

$$0 \leq |\kappa| < g, \tag{6.9}$$

which is equivalent to $g + \kappa > 0$ and $g - \kappa > 0$. Then we have $\omega > 0$ and

$$a_1 = \sqrt{g+\kappa} > 0, \quad a_2 = \sqrt{g-\kappa} > 0, \tag{6.10}$$

where the equalities in (6.10) define the quantities $a_1$, $a_2$. The two linearly independent (not normalized) eigenvectors of $H_1$ and its adjoint $H_1^*$ are

$$|v_\pm\rangle \propto a_1|e_{\rm S}\rangle \pm a_2|e_{\rm B}\rangle \quad \text{and} \quad |v_\pm^*\rangle \propto a_2|e_{\rm S}\rangle \pm a_1|e_{\rm B}\rangle,$$

respectively. They satisfy $H_1|v_\pm\rangle = \omega_\pm|v_\pm\rangle$ and $H_1^*|v_\pm^*\rangle = \omega_\pm|v_\pm^*\rangle$. Note that $|v_\pm^*\rangle$ denote the eigenvectors of $H^*$, not to be confused with the complex conjugates of the eigenvectors $|v_\pm\rangle$ of $H$. We normalize the vectors as

$$|v_\pm\rangle = \tfrac{1}{\sqrt{2}}\Big(\sqrt{\tfrac{a_1}{a_2}}|e_{\rm S}\rangle \pm \sqrt{\tfrac{a_2}{a_1}}|e_{\rm B}\rangle\Big) \quad \text{and} \quad |v_\pm^*\rangle = \tfrac{1}{\sqrt{2}}\Big(\sqrt{\tfrac{a_2}{a_1}}|e_{\rm S}\rangle \pm \sqrt{\tfrac{a_1}{a_2}}|e_{\rm B}\rangle\Big). \tag{6.11}$$

Then $\{|v_\pm\rangle, |v_\pm^*\rangle\}$ is a bi-orthonormal basis, satisfying $\langle v_\pm^*|v_\mp\rangle = 0$ and $\langle v_\pm^*|v_\pm\rangle = 1$, and the operator $H_1$ can be written as

$$H_1 = \omega_+|v_+\rangle\langle v_+^*| + \omega_-|v_-\rangle\langle v_-^*|. \tag{6.12}$$

Using this, one easily finds

$$\begin{aligned} e^{-itH_1} &= e^{-it\omega_+}|v_+\rangle\langle v_+^*| + e^{-it\omega_-}|v_-\rangle\langle v_-^*| \\ &= e^{-it\nu}\cos(\omega t)\mathbb{1} - ie^{-it\nu}\sin(\omega t)\Big(\frac{a_1}{a_2}|e_{\rm S}\rangle\langle e_{\rm B}| + \frac{a_2}{a_1}|e_{\rm B}\rangle\langle e_{\rm S}|\Big). \end{aligned} \tag{6.13}$$

We consider initial states which are vectors in $\mathcal{H}_1$, as defined in (6.5), so the dynamics generated by $H$ is entirely given by the operator $H_1$ from (6.7). We still

consider the regime (6.9), so that the spectrum of $H_1$ consists of two distinct real eigenvalues. Comparing (1.4), (1.6) and (6.12), we see that $H_1$ is quasi-Hermitian and the set of all associated metrics is

$$\mathcal{M}_+ = \big\{\eta = x_1|v_+^*\rangle\langle v_+^*| + x_2|v_-^*\rangle\langle v_-^*| \ : \ x_1, x_2 > 0\big\}. \tag{6.14}$$

Written as a matrix in the basis $\{|e_{\rm S}\rangle, |e_{\rm B}\rangle\}$, we obtain from (6.11)

$$\eta = \frac{1}{2}\begin{pmatrix} (x_1+x_2)\, a_2/a_1 & x_1 - x_2 \\ x_1 - x_2 & (x_1+x_2)\, a_1/a_2 \end{pmatrix}. \tag{6.15}$$

This is diagonal exactly when $x_1 = x_2$. As we will see in Appendix A, this is equivalent to $\eta$ being the *restriction to* $\mathcal{H}_1$ *of a product metric* $\Lambda_{\rm S} \otimes \Lambda_{\rm B}$ *on* $\mathcal{H}$. And as we discuss below after (6.23), this is also equivalent to the reduced system state $\bar{\rho}_H(t)$ in (6.22) being a positive operator.

### 6.1.2 Reduced non-Hermitian system dynamics

Fix an $\eta \in \mathcal{M}_+$ and take an initial state of the form

$$|\psi(0)\rangle = A|e_{\rm S}\rangle + B|e_{\rm B}\rangle \tag{6.16}$$

for some $A, B \in \mathbb{C}$ normalized to have $\|\psi(0)\|_\eta^2 = 1$, that is,

$$1 = \Big(\frac{x_1+x_2}{2}\Big)\Big(\frac{a_2}{a_1}|A|^2 + \frac{a_1}{a_2}|B|^2\Big) + (x_1 - x_2)\mathrm{Re}(AB^*). \tag{6.17}$$

The dynamics is given by

$$|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle = e^{-it\nu}A(t)|e_{\rm S}\rangle + e^{-it\nu}B(t)|e_{\rm B}\rangle, \tag{6.18}$$

where

$$\begin{aligned} A(t) &= A\cos(\omega t) - iB\frac{a_1}{a_2}\sin(\omega t), \\ B(t) &= B\cos(\omega t) - iA\frac{a_2}{a_1}\sin(\omega t). \end{aligned} \tag{6.19}$$

The normalization

$$\|\psi(t)\|_\eta^2 = x_1\big|\langle v_+^*|\psi(t)\rangle\big|^2 + x_2\big|\langle v_-^*|\psi(t)\rangle\big|^2 = 1 \tag{6.20}$$

holds for all $t$, as $e^{-itH}$ acts unitarily on $\mathcal{H}_1$ equipped with the inner product $\langle\cdot|\cdot\rangle_\eta$. The relation (6.20) is the same as (6.17) with $A$ and $B$ replaced by $A(t)$ and $B(t)$.

We now introduce the reduction of the system to the single oscillator ($a^*$, $a$). The average of a system observable $O_\mathrm{S}$ (observable of the single oscillator) in the state $|\psi(t)\rangle$ given by (6.18) evolves according to

$$\langle\psi(t)|\eta O_\mathrm{S}|\psi(t)\rangle = \mathrm{tr}_\mathrm{S}\big(\bar\rho_H(t)O_\mathrm{S}\big), \tag{6.21}$$

where the reduced system state is

$$\bar\rho_H(t) = \mathrm{tr}_\mathrm{B}\,\rho_H(t) = \mathrm{tr}_\mathrm{B}\big(|\psi(t)\rangle\langle\psi(t)|\eta\big). \tag{6.22}$$

For the partial trace we have the identities $\mathrm{tr}_\mathrm{B}|e_\mathrm{S}\rangle\langle e_\mathrm{S}| = |1_\mathrm{S}\rangle\langle 1_\mathrm{S}|$, $\mathrm{tr}_\mathrm{B}|e_\mathrm{B}\rangle\langle e_\mathrm{B}| = |0_\mathrm{S}\rangle\langle 0_\mathrm{S}|$ and $\mathrm{tr}_\mathrm{B}|e_\mathrm{S}\rangle\langle e_\mathrm{B}| = 0 = \mathrm{tr}_\mathrm{B}|e_\mathrm{B}\rangle\langle e_\mathrm{S}|$. Using (6.18) and $\eta$ of the form (6.14), we obtain after a calculation

$$\begin{aligned}\bar\rho_H(t) \;=\; & \Big(\frac{x_1+x_2}{2}\frac{a_1}{a_2}|B(t)|^2 + \frac{x_1-x_2}{2}A(t)B(t)^*\Big)\,|0_\mathrm{S}\rangle\langle 0_\mathrm{S}| \\ & + \Big(\frac{x_1+x_2}{2}\frac{a_2}{a_1}|A(t)|^2 + \frac{x_1-x_2}{2}A(t)^*B(t)\Big)\,|1_\mathrm{S}\rangle\langle 1_\mathrm{S}|.\end{aligned} \tag{6.23}$$

This matrix is diagonal in the basis $\{|0_\mathrm{S}\rangle, |1_\mathrm{S}\rangle\}$ and the two diagonal entries are its eigenvalues. One checks directly that $\mathrm{tr}_\mathrm{S}(\bar\rho_H(t)) = 1$ (the sum of the diagonal elements equals $\|\psi(t)\|_\eta^2 = 1$). However, the eigenvalues of $\bar\rho_H(t)$ are complex, in general, unless the metric is chosen to satisfy $x_1 = x_2$. Indeed, the imaginary part of the first eigenvalue is $\frac{x_1-x_2}{2}\mathrm{Im}(A(t)B(t)^*)$. If $A, B$, the coefficients in the initial state (6.16), are real then this quantity becomes[1] $-(x_1-x_2)(\frac{1}{x_1+x_2} - \frac{a_2}{a_1}A^2 - \frac{x_1-x_2}{x_1+x_2}AB)\cos(\omega t)\sin(\omega t)$. Unless $x_1 = x_2$ or the initial condition satisfies $(x_1+x_2)\frac{a_2}{a_1}A^2 + (x_1-x_2)AB = 1$, the eigenvalues of $\bar\rho_H(t)$ will not be real except at the discrete set of times $t$ when $\sin(\omega t)\cos(\omega t) = 0$.

[1] Use the equations (6.19) and write $B$ as a function of $A$ according to the normalization condition (6.17).

We require $\bar{\rho}_H(t)$ to be a density matrix (and in particular to have non-negative eigenvalues) for all times. To do so with a metric that does not depend on the initial conditions we therefore must choose $x_1 = x_2$. We thus take

$$x_1 = x_2 = x > 0$$

for the remainder of this chapter. In the basis $\{|e_{\rm S}\rangle, |e_{\rm B}\rangle\}$ the metric $\eta$ is diagonal,

$$\eta = x \begin{pmatrix} a_2/a_1 & 0 \\ 0 & a_1/a_2 \end{pmatrix}, \tag{6.24}$$

see (6.15). As explained after (6.15), this is equivalent to $\eta$ being of product form. With this choice, $\bar{\rho}_H(t)$ given by (6.23) is $^*$-Hermitian. According to (6.23) and (6.19) we have

$$\bar{\rho}_H(t) = p(t)\,|0_{\rm S}\rangle\langle 0_{\rm S}| + (1-p(t))\,|1_{\rm S}\rangle\langle 1_{\rm S}|, \tag{6.25}$$

where

$$\begin{aligned} p(t) &= x\frac{a_1}{a_2}|B(t)|^2 \\ &= x\Big(\frac{a_1}{a_2}|B|^2\cos^2(\omega t) + \frac{a_2}{a_1}|A|^2\sin^2(\omega t) - 2\sin(\omega t)\cos(\omega t)\mathrm{Im}(A^*B)\Big) \\ &= \frac{1}{2} + \Big(\frac{1}{2} - x\frac{a_2}{a_1}|A|^2\Big)\cos(2\omega t) - x\sin(2\omega t)\mathrm{Im}(A^*B). \end{aligned} \tag{6.26}$$

In the last step, we used the normalization condition (6.17), resulting in $\frac{a_1}{a_2}|B|^2 = \frac{1}{x} - \frac{a_2}{a_1}|A|^2$, and the trigonometric identities $\sin(\omega t)\cos(\omega t) = \frac{1}{2}\sin(2\omega t)$, $\cos^2(\omega t) = \frac{1}{2}(1+\cos(2\omega t))$ and $\sin^2(\omega t) = \frac{1}{2}(1-\cos(2\omega t))$. In view of (6.26) it is natural to introduce the parameter

$$\alpha \equiv x\frac{a_2}{a_1}|A|^2 \in [0,1]. \tag{6.27}$$

Equations (6.25) and (6.26) show the following.

**Properties of $p(t)$:**

1. $p(t)$ and $\bar{\rho}_H(t)$ depend on time unless $\alpha = \frac{1}{2}$ and $A^*B \in \mathbb{R}$, in which case $p(t) = \frac{1}{2}$ and $\bar{\rho}_H(t) = \frac{1}{2}\mathbb{1}$.

2. Otherwise $p(t)$ and $\bar{\rho}_H(t)$ are periodic in time, with period $\pi/\omega$, and the mean

value of $p(t)$ is

$$p_0 = \frac{\omega}{\pi}\int_0^{\pi/\omega} p(t)dt = \frac{1}{2}. \tag{6.28}$$

### 6.1.3 Reduced Hermitian system dynamics

Next, we turn our attention to the density matrix of the Hermitian system, which according to (1.17) is

$$\rho_{h_W}(t) = S\rho_H(t)S^{-1} = W\sqrt{\eta}\rho_H(t)\tfrac{1}{\sqrt{\eta}}W^* = W\sqrt{\eta}\,|\psi(t)\rangle\langle\psi(t)|\sqrt{\eta}\,W^*. \tag{6.29}$$

We keep $W$ in the notation $h_W$ to highlight that the choice of $h$ depends on $W$, see (1.11). Again choosing a metric $\eta$ of the form (6.14) with $x_1 = x_2 = x > 0$, we use (6.18) to obtain

$$\sqrt{\eta}|\psi(t)\rangle = e^{-it\nu}\gamma(t)|e_{\rm S}\rangle + e^{-it\nu}\delta(t)|e_{\rm B}\rangle, \tag{6.30}$$

where

$$\gamma(t) = \sqrt{x\frac{a_2}{a_1}}A(t), \qquad \delta(t) = \sqrt{x\frac{a_1}{a_2}}B(t). \tag{6.31}$$

We then obtain

$$\sqrt{\eta}\,|\psi(t)\rangle\langle\psi(t)|\sqrt{\eta} = \begin{pmatrix} |\gamma(t)|^2 & \gamma(t)\delta(t)^* \\ \gamma(t)^*\delta(t) & |\delta(t)|^2 \end{pmatrix}, \tag{6.32}$$

written in matrix form in the ordered basis $\{|e_{\rm S}\rangle, |e_{\rm B}\rangle\}$ of $\mathcal{H}_1$. Next, we take a general (time-independent) unitary on $\mathcal{H}_1$, expressed in the same basis as

$$W = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ac^* + bd^* = 0, \quad |a|^2 + |b|^2 = 1 = |c|^2 + |d|^2. \tag{6.33}$$

Using (6.29), (6.32), (6.33) and writing momentarily $\delta, \gamma$ for $\delta(t), \gamma(t)$, we get

$$\rho_{h_W}(t) = \begin{pmatrix} |a\gamma + b\delta|^2 & ac^*|\gamma|^2 + bc^*\gamma^*\delta + ad^*\gamma\delta^* + bd^*|\delta|^2 \\ a^*c|\gamma|^2 + b^*c\gamma\delta^* + a^*d\gamma^*\delta + b^*d|\delta|^2 & |c\gamma + d\delta|^2 \end{pmatrix}. \tag{6.34}$$

We observe that $\rho_{h_W}(t)$ is periodic in time, with period $\pi/\omega$. This follows from (6.31) together with (6.19), since $\cos^2(\omega t)$, $\sin^2(\omega t)$ and $\sin(\omega t)\cos(\omega t)$ all have period $\pi/\omega$.

Next we calculate the reduced density matrix $\bar{\rho}_{h_W}(t)$ of S by taking the partial

trace of $\rho_{h_W}(t)$ over B,

$$\bar{\rho}_{h_W}(t) = q(t)\,|0_{\rm S}\rangle\langle 0_{\rm S}| + \big(1-q(t)\big)\,|1_{\rm S}\rangle\langle 1_{\rm S}|, \tag{6.35}$$

where $q(t) = \big|c\gamma(t) + d\delta(t)\big|^2$ and $\gamma(t), \delta(t)$ are given in (6.31). We get the expression

$$q(t) = x\Big|c\sqrt{\frac{a_2}{a_1}}A(t) + d\sqrt{\frac{a_1}{a_2}}B(t)\Big|^2, \tag{6.36}$$

where $A(t)$ and $B(t)$ are given in (6.19). Expanding the square and using $|d|^2 = 1-|c|^2$ as well as the normalization (6.17) (which is valid for $A, B$ replaced by $A(t), B(t)$ at any time), we obtain

$$\begin{aligned} q(t) &= |c|^2 + (1-2|c|^2)x\frac{a_1}{a_2}|B(t)|^2 + 2x{\rm Re}\big(cd^*A(t)B(t)^*\big) \\ &= |c|^2 + (1-2|c|^2)p(t) + 2x{\rm Re}\big(cd^*A(t)B(t)^*\big), \end{aligned} \tag{6.37}$$

where $p(t)$ is the population of $\bar{\rho}_H(t)$ evaluated above in (6.26). Expanding the real part term in (6.37) using (6.19), we arrive at

$$\begin{aligned} q(t) &= |c|^2 + (1-2|c|^2)p(t) \\ &\quad - \sin(2\omega t)\big(1 - 2x\frac{a_2}{a_1}|A|^2\big){\rm Im}(cd^*) - 2x\cos(2\omega t){\rm Im}(AB^*){\rm Im}(cd^*) \\ &\quad + 2x{\rm Re}(AB^*){\rm Re}(cd^*). \end{aligned} \tag{6.38}$$

We take into account (6.26) to rewrite

$$\begin{aligned} q(t) &= \tfrac{1}{2} + 2x{\rm Re}(AB^*){\rm Re}(cd^*) \\ &\quad + \cos(2\omega t)\Big[(\tfrac{1}{2}-\alpha)(1-2|c|^2) - 2x{\rm Im}(AB^*){\rm Im}(cd^*)\Big] \\ &\quad - \sin(2\omega t)\Big[(1-2\alpha){\rm Im}(cd^*) - x(1-2|c|^2){\rm Im}(AB^*)\Big], \end{aligned} \tag{6.39}$$

where we recall that $\alpha$ is given in (6.27). Using the explicit form (6.39) of $q(t)$, we obtain the following information.

**Properties of $q(t)$:**

1. $q(t)$ and $\bar{\rho}_{h_W}(t)$ depend on time unless both factors of the cosine and sine terms

in (6.39) vanish. Thus $q(t)$ is time-independent and hence equal to

$$q_0 = \frac{\omega}{\pi}\int_0^{\pi/\omega} q(t)dt = \frac{1}{2} + 2x\mathrm{Re}(AB^*)\mathrm{Re}(cd^*) \tag{6.40}$$

if and only if:

- $\alpha = 1/2$, $|c| = 1/\sqrt{2}$ and $\mathrm{Im}(AB^*)\mathrm{Im}(cd^*) = 0$, or
- $\alpha = 1/2$, $|c| \neq 1/\sqrt{2}$ and $\mathrm{Im}(AB^*) = 0$, or
- $\alpha \neq 1/2$, $|c| = 1/\sqrt{2}$ and $\mathrm{Im}(cd^*) = 0$, or
- $\alpha \neq 1/2$, $|c| \neq 1/\sqrt{2}$ and $\mathrm{Im}(A^*B) = \frac{1}{2x}|1-2\alpha|$ and $\mathrm{Im}(cd^*) = \frac{1}{2}(1 - 2|c|^2)\mathrm{sgn}(1-2\alpha)$, where $\mathrm{sgn}(x) = |x|/x$.

2. Otherwise $q(t)$ and $\bar{\rho}_{h_W}(t)$ are periodic in time, with period $\pi/\omega$, and the mean value of $q(t)$ is $q_0$ from (6.40).

**Generic initial states and unitaries.** As $x > 0$, the average $q_0$ equals $\frac{1}{2}$ exactly when $\mathrm{Re}(AB^*)\mathrm{Re}(cd^*) = 0$. This is a condition on the initial state (via $A, B$) and the unitary $W$ (via $c, d$). We call the initial state and the unitary *generic*, respectively, when

$$AB^* \notin \mathbb{R} \quad \text{and} \quad cd^* \notin \mathbb{R}. \tag{6.41}$$

In other words, for generic initial states and unitaries, the average $q_0$ of the population of $\bar{\rho}_{h_W}(t)$ differs from the average $p_0 = \frac{1}{2}$ of the population of $\bar{\rho}_H(t)$. As we show in the next section, this deviation from the value $\frac{1}{2}$ causes the entropy of $\bar{\rho}_{h_W}(t)$ to oscillate with exactly *half* the frequency of the entropy of $\bar{\rho}_H(t)$.

## 6.2 Entropy

Recall that the states $\bar{\rho}_H(t)$ and $\bar{\rho}_{h_W}(t)$ are given in (6.25) and (6.35), with associated populations $p(t)$, $q(t)$ evaluated in (6.26) and (6.39). Their von Neumann entropy is given by

$$\begin{aligned} \mathcal{E}(\bar{\rho}_H(t)) &= -p(t)\ln p(t) - (1-p(t))\ln(1-p(t)), \\ \mathcal{E}(\bar{\rho}_{h_W}(t)) &= -q(t)\ln q(t) - (1-q(t))\ln(1-q(t)). \end{aligned} \tag{6.42}$$

We show in Appendix B that

$$\mathcal{E}(\bar{\rho}_H(t)) = \mathcal{E}(\bar{\rho}_{h_W}(t)) \text{ for all } t \geq 0 \text{ and initial conditions } (A, B) \quad \Longleftrightarrow \quad cd = 0.$$

If $\mathrm{Re}(cd^*) \neq 0$, then according to (6.28) and (6.40) the averages $p_0$ and $q_0$ around which the populations $p(t)$ and $q(t)$ oscillate are *different* for all generic initial conditions, i.e., all coefficients $A, B$ satisfying $\mathrm{Re}(AB^*) \neq 0$. This translates into a modification of the period of the entropy of $\bar{\rho}_{h_W}(t)$ as a function of time $t$ relative to that of $\bar{\rho}_H(t)$, as we explain now.

### 6.2.1 Period doubling of von Neumann entropy in the non-Hermitian versus the Hermitian system

Consider the function

$$\mathcal{E}(Q) = -Q \ln Q - (1-Q)\ln(1-Q), \quad Q \in [0,1].$$

Suppose now that $Q = Q(t)$ depends periodically on time and has average $Q_0$,

$$Q(t) = Q_0 + \Delta(t) \in [0,1], \tag{6.43}$$

with $\Delta(t)$ having period $\pi/\omega$ and zero average. This setup incorporates both cases $p(t)$ and $q(t)$ in one. As Figure 6.1 illustrates, if $Q_0 = \frac{1}{2}$, which is the value where $\mathcal{E}(Q)$ takes its maximum, then as $Q(t)$ moves over one period, the entropy $\mathcal{E}(Q(t))$ moves over *two* periods.

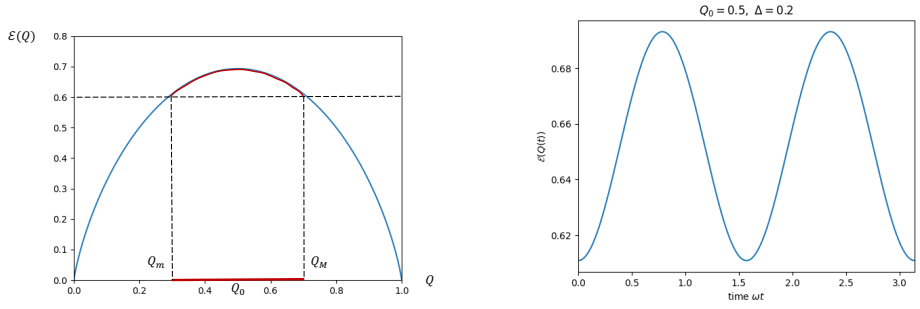



Figure 6.1: The period of the entropy $\mathcal{E}(Q(t))$ relative to the period of $Q(t)$, considering parameter choices $Q_0 = 0.5$ and $\Delta \equiv \max_t \Delta(t) = 0.2$.

According to (6.43), $Q(t)$ starts at $Q_m = 0.3$ (at a time we take to be $t = 0$) and moves to $Q_M = 0.7$ at time $\omega t = \pi/2$, and then back to $Q_m$ at time $\omega t = \pi$ (left panel), so the value of the entropy $\mathcal{E}(Q(t))$ evolves through two periods (right panel). In each period, the entropy has two local minima (counting minima at the endpoints of the considered intervals once).

On the other hand, if $Q_0 \neq \frac{1}{2}$, then the period of the entropy $\mathcal{E}(Q(t))$ is *not* doubled relative to that of $Q(t)$, as Figures 6.2 and 6.3 show.

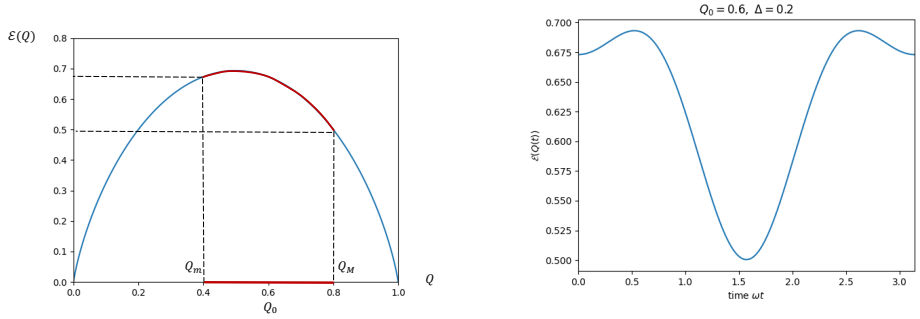



Figure 6.2: The period of the entropy $\mathcal{E}(Q(t))$ relative to the period of $Q(t)$, considering parameter choices $Q_0 = 0.6$, $\Delta = \max_t \Delta(t) = 0.2$.

In this case $Q(t)$ starts at $Q_m = 0.4$ when $t = 0$ (upon a possible shift of the time axis) and moves to $Q_M = 0.8$ at time $\omega t = \pi/2$ and back to $Q_m$ at time $\omega t = \pi$ (left panel). The value of the entropy $\mathcal{E}(Q(t))$ evolves through one single period (right panel). In each period, the entropy has two local minima (counting minima at the endpoints once).

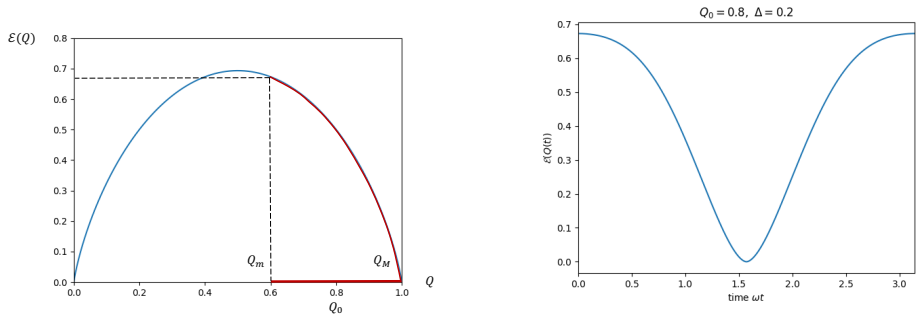



Figure 6.3: The period of the entropy $\mathcal{E}(Q(t))$ relative to the period of $Q(t)$, considering parameter choices $Q_0 = 0.8$, $\Delta = \max_t \Delta(t) = 0.2$.

In this case $Q(t)$ starts at $Q_m = 0.6$ when $t = 0$ (after possibly shift the time axis) and moves to $Q_M = 1.0$ at time $\omega t = \pi/2$ and back to $Q_m$ at time $\omega t = \pi$ (left panel). The value of the entropy $\mathcal{E}(Q(t))$ evolves through one single period (right panel). Since 0.5 is not in the interval $(Q_m, Q_M)$, the graph of the entropy has only one local minimum in each period, instead of two when the interval contains the value 0.5 for $Q$.

We draw the following conclusions:

- The period of the von Neumann entropy of the *non-Hermitian* system $\mathcal{E}(\bar{\rho}_H(t))$ is $\frac{1}{2}\pi/\omega$, regardless of the initial condition (except for the stationary state).
- Regardless of the metric (parameter $x$), the period of the von Neumann entropy of the *Hermitian* system $\mathcal{E}(\bar{\rho}_{h_W}(t))$ is:
  - $\pi/\omega$, provided $\mathrm{Re}(AB^*)\mathrm{Re}(cd^*) \neq 0$ (generic case),
  - $\frac{1}{2}\pi/\omega$, provided $\mathrm{Re}(AB^*)\mathrm{Re}(cd^*) = 0$ (special case).

This means that for generic initial conditions (meaning $\mathrm{Re}(AB^*) \neq 0$) and generic choices of the unitary $W$ (meaning $\mathrm{Re}(cd^*) \neq 0$), **the period of the von Neumann entropy of the Hermitian system is double that of the non-Hermitian system**. That is, the entropy of the non-Hermitian system oscillates faster. This is so even though the populations in both cases have the same frequency $\pi/\omega$. The change of the period is due to the shift of the average in the population induced by $W$, as given in (6.40).

### 6.2.2 Numerical illustration of the period doubling

We plot the populations and entropies for parameters in the regime

$$x = x_1 = x_2 > 0 \qquad \text{(metric } \eta\text{, (6.24))} \tag{6.44}$$

$$c, d \geq 0 \qquad \text{(unitary } W\text{, (6.33))} \tag{6.45}$$

$$A, B \geq 0 \qquad \text{(initial state } |\psi(0)\rangle\text{, (6.16))} \tag{6.46}$$

According to (6.45) and the unitarity of $W$, we have $d = \sqrt{1-c^2}$. Moreover, $x^2A^2B^2 = \alpha(1-\alpha)$, where $\alpha = x\frac{a_2}{a_1}A^2 \in [0,1]$. The population $q(t)$ of the Hermitian system reduced density matrix, given in (6.39), then becomes

$$q(t) = q_0 + \Delta\cos(2\omega t) \tag{6.47}$$

with

$$q_0 = \tfrac{1}{2} + 2\sqrt{c^2(1-c^2)}\sqrt{\alpha(1-\alpha)}, \tag{6.48}$$

$$\Delta = \tfrac{1}{2}(1-2c^2)(1-2\alpha). \tag{6.49}$$

Here, $\alpha = x\frac{a_2}{a_1}A^2 \in [0,1]$ and $c \in [0,1]$ can be chosen freely. The population $p(t)$ of the non-Hermitian system, in (6.26), is simply the expression (6.47) with $c = 0$.

Note that the change $\alpha \mapsto 1-\alpha$ leaves $q_0$ invariant and flips the sign of $\Delta$. It then suffices to plot graphs for $\alpha \in [0, 1/2]$. For $\alpha = 1/2$ we get $\Delta = 0$, which gives a stationary state (for all $c$). The same invariance of $q_0$ and sign flip of $\Delta$ is induced by $c^2 \mapsto 1-c^2$.

In Figure 6.4, we compare the von Neumann entropies of the two density matrices $\bar{\rho}_H(t)$ and $\bar{\rho}_{h_W}(t)$, directly seeing the doubling of the period.

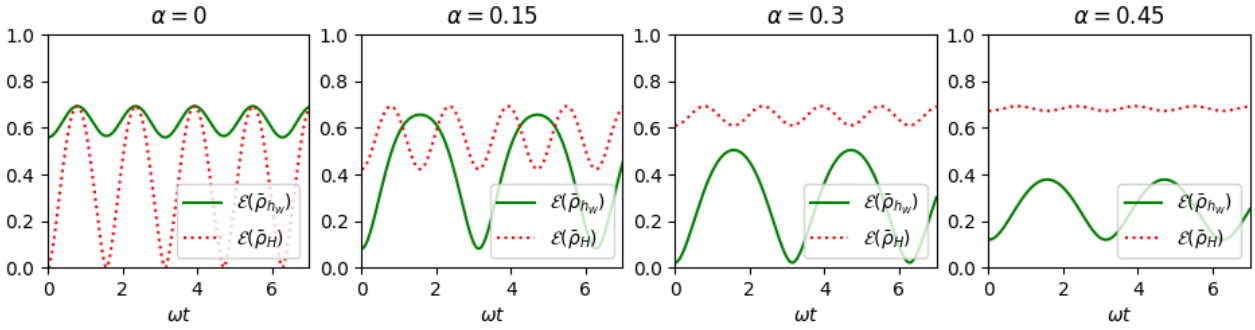



Figure 6.4: Comparing the entropies of $\bar{\rho}_{h_W}(t)$ and of $\bar{\rho}_H(t)$ for the values $c = 0.5$ and $\alpha = 0, 0.15, 0.3, 0.45$.

The doubling of the period for $\alpha \neq 0$ is manifest. The oscillations decrease as $\alpha$ approaches 0.5, which gives the stationary state.

## 6.3 Entanglement of system and bath oscillators

The total Hilbert space $\mathcal{H} = \mathcal{H}_{\rm S} \otimes \mathcal{H}_{\rm B}$ in (6.1) is bipartite, one part being the singled-out oscillator (system), the other being the remaining $N$ oscillators (bath). We say that a nonzero vector $|\psi\rangle \in \mathcal{H}$ is of *product form*, or *disentangled*, if $|\psi\rangle = |\psi_{\rm S}\rangle \otimes |\psi_{\rm B}\rangle$ for some $|\psi_{\rm S}\rangle \in \mathcal{H}_{\rm S}$ and some $|\psi_{\rm B}\rangle \in \mathcal{H}_{\rm B}$. We call a nonzero $|\psi\rangle \in \mathcal{H}$ *entangled* if it is not of product form. The notion of being entangled or not does not depend on the metric determining the inner product of $\mathcal{H}$. Nevertheless, the physical interpretation of entanglement in terms of independence of the subsystems S and B does depend on the metric. The physical manifestation of disentangled states is the independence of the two subsystems S and B. Namely, if the inner product of $\mathcal{H}$ is given by a metric of the form $\eta = \Lambda_{\rm S} \otimes \Lambda_{\rm B}$ (a particular example being $\mathbb{1}_{\rm S} \otimes \mathbb{1}_{\rm B}$), then *measurement outcomes* of observables on either of the subsystems are *independent random variables*. This follows because expectation values of observables $O_{\rm S} \otimes O_{\rm B}$ in a state $|\psi_{\rm S}\rangle \otimes |\psi_{\rm B}\rangle$ split into products,

$$\langle \psi_{\rm S} \otimes \psi_{\rm B} | \eta (O_{\rm S} \otimes O_{\rm B}) \psi_{\rm S} \otimes \psi_{\rm B} \rangle = \langle \psi_{\rm S} | \Lambda_{\rm S} O_{\rm S} \psi_{\rm S} \rangle \langle \psi_{\rm B} | \Lambda_{\rm B} O_{\rm B} \psi_{\rm B} \rangle .$$

However, those random variables become dependent (correlated) if $\eta$ is not of product form, because then their average will not split into a product of a system term times a bath term.

In the model defined in Section 6.1, the metrics $\eta$ we consider are restrictions to the subspace $\mathcal{H}_1$ of product metrics $\Lambda_{\rm S} \otimes \Lambda_{\rm B}$ of $\mathcal{H}$ (c.f. (6.15) and Appendix A). The physical meaning of SB entanglement in terms of subsystem independence does therefore not depend on the choice of $\eta$ within this class. In other words, measurement outcomes of system and bath observables in $|\psi\rangle \in \mathcal{H}_1$ are independent or dependent, according to whether $|\psi\rangle \in \mathcal{H}_1$ is disentangled or not, regardless of the choice of $\eta$. It is then sensible to investigate the SB entanglement in pure states belonging to the subspace $\mathcal{H}_1$ for all $\eta$.

Any vector $|\psi\rangle \in \mathcal{H}_1$ is of the form

$$|\psi\rangle = A|e_{\rm S}\rangle + B|e_{\rm B}\rangle, \qquad A, B \in \mathbb{C}. \tag{6.50}$$

In accordance with (6.6), we may write $|e_{\rm S}\rangle = |10\rangle = |1\rangle \otimes |0\rangle \in \mathcal{H}_{\rm S} \otimes \mathcal{H}_{\rm B}$ and

$|e_{\rm B}\rangle = |01\rangle = |0\rangle \otimes |1\rangle \in \mathcal{H}_{\rm S} \otimes \mathcal{H}_{\rm B}$. Explicitly, $|0\rangle, |1\rangle \in \mathcal{H}_{\rm S}$ are the ground state and first excited state of the system oscillator, and $|0\rangle, |1\rangle \in \mathcal{H}_{\rm B}$ are the ground state of all the $N$ bath oscillators and the distributed excitation state $\frac{1}{\sqrt{N}} \sum_{n=1}^{N} |1_n\rangle$, respectively; see (6.6).

We now show that

$$|\psi\rangle \text{ of the form (6.50) is disentangled} \quad \Longleftrightarrow \quad AB = 0. \tag{6.51}$$

To see that the implication $\Rightarrow$ in (6.51) holds, let $\rho_{\rm S} \equiv {\rm tr}_{\rm B}|\psi\rangle\langle\psi|$ (partial trace over B). On the one hand, (6.50) gives $\rho_{\rm S} = |A|^2|1\rangle\langle 1| + |B|^2|0\rangle\langle 0|$. On the other hand, if $|\psi\rangle$ is disentangled, then $\rho_{\rm S}$ must have rank one, since $\rho_{\rm S} = {\rm tr}_{\rm B}(|\psi_{\rm S}\rangle\langle\psi_{\rm S}| \otimes |\psi_{\rm B}\rangle\langle\psi_{\rm B}|) = |\psi_{\rm S}\rangle\langle\psi_{\rm S}| \, \|\psi_{\rm B}\|^2$. This forces either $A = 0$ or $B = 0$. Conversely, to see the implication $\Leftarrow$ in (6.51), we note that if either of $A$ or $B$ vanish, then $|\psi\rangle$ is is proportional to $|e_{\rm B}\rangle$ or $|e_{\rm S}\rangle$, so $|\psi\rangle$ is disentangled.

**Entanglement in the non-Hermitian system.** An initial state $|\psi(0)\rangle = A|e_{\rm S}\rangle + B|e_{\rm B}\rangle$ evolves into $|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle = A(t)|e_{\rm S}\rangle + B(t)|e_{\rm B}\rangle$, where the time dependent coefficients are given in (6.19). According to (6.51), $|\psi(t)\rangle$ is disentangled exactly if $A(t)B(t) = 0$. Let us first analyze the condition $A(t) = 0$. This equality is equivalent to the two equations

$$\begin{aligned} ({\rm Re}A)\cos(\omega t) + ({\rm Im}B)\frac{a_1}{a_2}\sin(\omega t) = 0, \\ ({\rm Im}A)\cos(\omega t) - ({\rm Re}B)\frac{a_1}{a_2}\sin(\omega t) = 0. \end{aligned} \tag{6.52}$$

The condition $B(t) = 0$ is the same as (6.52) but with $A \leftrightarrow B$ swapped and $a_1 \leftrightarrow a_2$ swapped.

Suppose $|\psi(0)\rangle$ is disentangled, so $AB = 0$. Then exactly one of $A$ or $B$ vanish and the equations (6.52) are satisfied for $\omega t \in \pi\mathbb{Z}$ (if $A = 0$) or for $\omega t = \frac{\pi}{2}(2\mathbb{Z} + 1)$ (if $B = 0$). We conclude that $|\psi(t)\rangle$ is entangled except periodically at discrete moments in time where it is disentangled.

On the other hand, if $|\psi(0)\rangle$ is entangled, then both $A$ and $B$ do not vanish. If $A$ and $B$ are both real or both purely imaginary, then (6.52) is not satisfied for any $t$. For all other $A$ and $B$ (6.52) is satisfied for discrete, periodically repeating values of $t$.

We conclude:

(a) If the initial state $|\psi(0)\rangle$ is entangled and both $A, B$ are either purely real or purely imaginary, then $|\psi(t)\rangle$ is entangled for all times $t$.

(b) With the exception of case (a) and regardless of the entanglement in the initial state $|\psi(0)\rangle$, the state $|\psi(t)\rangle$ is entangled except at periodically repeating instants.

**Entanglement in the Hermitian systems.** The Hermitian system pure state vector is given by (see also (1.15))

$$|\phi(t)\rangle = S|\psi(t)\rangle = W\sqrt{\eta}|\psi(t)\rangle = \widetilde{A}(t)|e_{\rm S}\rangle + \widetilde{B}(t)|e_{\rm B}\rangle,$$

a normalized vector in $\mathcal{H}$ (with the original inner product), where

$$\begin{pmatrix}\widetilde{A}(t)\\ \widetilde{B}(t)\end{pmatrix} = T\begin{pmatrix}A(t)\\ B(t)\end{pmatrix}, \qquad T = \sqrt{x\frac{a_2}{a_1}}\begin{pmatrix}a & ba_1/a_2\\ c & da_1/a_2\end{pmatrix}, \qquad \det T = x\det W \neq 0 \tag{6.53}$$

satisfy $|\widetilde{A}(t)|^2 + |\widetilde{B}(t)|^2 = 1$; c.f. (6.31) and (6.33). It follows from (6.51) that $|\phi(t)\rangle$ is entangled if and only if $\widetilde{A}(t)\widetilde{B}(t) = 0$. An analysis of the latter equality along the lines of that carried out after (6.52) shows that $|\phi(t)\rangle$ is entangled except at isolated, periodically reoccurring instants in time, just like the state of the non-Hermitian system.

**Effect of choice of $W$ on entanglement.** Given a state $|\psi\rangle = A|e_{\rm S}\rangle + B|e_{\rm B}\rangle$ of the non-Hermitian system, the associated Hermitian system state is $|\phi\rangle = W\sqrt{\eta}|\psi\rangle$. For the choice $W = \mathbb{1}$ we have

$$|\phi\rangle = \sqrt{\eta}|\psi\rangle = \sqrt{xa_2/a_1}A|e_{\rm S}\rangle + \sqrt{xa_1/a_2}B|e_{\rm B}\rangle. \tag{6.54}$$

Hence for $W = \mathbb{1}$, $|\phi\rangle$ is entangled if and only if $|\psi\rangle$ is entangled (recall (6.51)). The metric $\eta$ does not alter the property of being entangled. Choosing a different $W$ to build $|\phi\rangle$ from $|\psi\rangle$, however, changes this. It is not hard to see that the unitaries $W$ that map every product state (that is $|e_{\rm S}\rangle$ and $|e_{\rm B}\rangle$) into another product state are exactly the diagonal and the off-diagonal $W$. Furthermore, given an entangled $\sqrt{\eta}|\psi\rangle$

as in (6.54), one can always find unitaries $W$ such that $W\sqrt{\eta}|\psi\rangle$ is not entangled. Those $W$ are precisely the ones with $|a| = |d| = \sqrt{a_1/(xa_2)}|B|$ and $|b| = |c| = \sqrt{xa_2/a_1}|A|$.[2]

We now examine the effect of $W$ on the *time-averaged density matrix*

$$\langle\rho\rangle = \frac{\omega}{\pi}\int_0^{\pi/\omega} |\phi(t)\rangle\langle\phi(t)|\,dt \tag{6.55}$$

where we integrate $|\phi(t)\rangle\langle\phi(t)| = \rho_{h_W}(t)$ over one period, see (6.34). A direct calculation yields

$$\langle\rho\rangle = \begin{pmatrix} q_0 & z \\ z^* & 1-q_0 \end{pmatrix}, \quad z = (bc^* + ad^*)x\mathrm{Re}(AB^*), \tag{6.56}$$

with $q_0 = \frac{1}{2} + 2x\mathrm{Re}(AB^*)\mathrm{Re}(cd^*)$, cf. (6.40). The density matrix (6.56) is written in the basis $\{|e_{\mathrm{S}}\rangle \equiv |10\rangle, |e_{\mathrm{B}}\rangle \equiv |01\rangle\}$ of $\mathcal{H}_1$, using the same notation as after (6.50). We view $\mathcal{H}_1$ as a subspace of the four-dimensional space of two qubits, spanned by the vectors $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. In this basis, the density matrix (6.56) takes the form

$$\langle\rho\rangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1-q_0 & z^* & 0 \\ 0 & z & q_0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6.57}$$

We calculate the *concurrence* [41, 24] of $\langle\rho\rangle$ to be[3]

$$\mathcal{C}\big(\langle\rho\rangle\big) = 2x|\mathrm{Re}(AB^*)|. \tag{6.58}$$

The concurrence of any two qubit density matrix is bounded below by 0 (separable state) and above by 1 (maximally entangled state). Both inequalities in

$$2x|\mathrm{Re}(AB^*)| \le 2x|A||B| \le x\Big(\frac{a_2}{a_1}|A|^2 + \frac{a_1}{a_2}|B|^2\Big) = 1$$

(the last equality is the normalization (6.17), with $x = x_1 = x_2 > 0$) are saturated exactly if $AB^* \in \mathbb{R}$ and $|A| = \frac{a_1}{a_2}|B|$.

[2]This follows from the characterization (6.51) of product states and the normalization $\|\sqrt{\eta}|\psi\rangle\|_{\mathcal{H}} = 1$.

[3]In the present case, the square of the concurrence is the difference between the two non-zero eigenvalues of the squared matrix $\langle\rho\rangle^2$.

We conclude that the concurrence of $\langle\rho\rangle$ is the same for all choices $W$, so it only depends on the initial state. The state $\langle\rho\rangle$ is separable if and only if $\mathrm{Re}(AB^*) = 0$, and is maximally entangled if and only if $\mathrm{Im}(AB^*) = 0$ and $|A| = \frac{a_1}{a_2}|B|$.

# Chapter 7

# Summary and Future research

## 7.1 Summary

We demonstrated that, for a $PT$ symmetric Hamiltonian $H$ operating on the Hilbert space $\mathcal{H} = \mathbb{C}^2$, at the transition point, $H$ is either non-diagonalizable or is real proportional to the identity. Additionally, we determined all $PT$ symmetric Hamiltonians $H$ acting on $\mathcal{H} = \mathbb{C}^2$ when $T$ is chosen as the complex conjugation operator.

We studied the existence and characteristics of metrics associated with $H(\varkappa)$. The focal point of this investigation is Theorem 5.2.2, which established a crucial condition for the diagonalizability of the Hamiltonian $H(\varkappa_0)$ at the transition point $\varkappa_0$: the diagonalizability of $H(\varkappa_0)$ is equivalent to the existence of a time-independent metric $\eta$ that solves the equation $H^*(\varkappa_0)\eta = \eta H(\varkappa_0)$.

In our analysis, it was demonstrated that within the $PT$ symmetry broken regime, the existence of time-independent metrics is absent, as indicated by Proposition 5.2.5. Moreover, as outlined in Theorem 5.2.3, in the $PT$ symmetry unbroken regime, any potential time-dependent metric remains bounded over time. Conversely, within the $PT$ symmetry broken regime, it was shown that a time-dependent metric characterized by exponential growth over time can always be identified.

Generally, the Dyson map assigns to a given quasi-Hermitian quantum system an associated Hermitian system in a non-unique way. We quantified the non-uniqueness employing a metric operator $\eta$ and a unitary map $W$. The physical properties of the Hermitian systems depend on the choice of $W$, and it is not obvious how to capture

the dynamics of the original quasi-Hermitian system in its Hermitian counterparts – unless there happens to be some universality throughout the Hermitian family. We described an aspect of universality for a quasi-Hermitian open system consisting of a single oscillator coupled to a bath of $N$ oscillators. We showed that there is a unique metric operator for which the reduced state of the system (single oscillator) is a well-defined density matrix. Using this metric, we construct all Hermitian systems obtained from the quasi-Hermitian one by varying $W$. We found that the entropy of the single oscillator in the Hermitian system evolves periodically in time with exactly *double the period* of the corresponding entropy of the quasi-Hermitian system, independently of $W$. We further showed that averaged over one time period, the entanglement between the oscillator and the bath is independent of the choice of $W$.

## 7.2 Future research

(1) **Behaviour of metric across the exceptional point.** Consider a $PT$ symmetric $H(\varkappa)$ with an exceptional point $\varkappa_0$. We know that for $\varkappa < \varkappa_0$, in the unbroken regime, there are time-independent metrics $\eta_-(\varkappa)$. In the broken region, $\varkappa > \varkappa_0$, there are only time-dependent metrics $\eta_+(t, \varkappa)$. We want to set up a simple generic model in which we can study the transition of the metric from a time-independent to a time-dependent one as $\varkappa$ crosses $\varkappa_0$. For instance: consider a model in which $H(\varkappa)$ has simple eigenvalues for $\varkappa < \varkappa_0$ which collapse for $\varkappa = \varkappa_0$, resulting in a non-diagonalizable $H(\varkappa)$ with a single eigenvalue, and such that for $\varkappa > \varkappa_0$, the operator $H(\varkappa)$ has again purely simple spectrum consisting of non-real, complex conjugate pairs (broken regime). Let $\varkappa \mapsto \eta(\varkappa)$ be a continuous family of time-independent metrics for $\varkappa < \varkappa_0$. Can we link this family continuously to a family of time-dependent metrics $\eta(t, \varkappa)$ as $\varkappa$ crosses $\varkappa_0$? If so, is this continuation unique?

(2) **Infinite dimensions.** The original Hamiltonians in the $PT$ symmetry literature are Schrödinger operators with complex potentials [3, 6], which are unbounded operators with discrete spectrum. On the other hand, the mathematically rigorous results in this field – like our theorems in Chapter 5 – are proven for finite-dimensional operators [30, 6, 32, 43]. We are interested in extending the mathematically rigorous definitions and results from finite to infinite

dimensions:

(i) In a first round, consider the operators $P$ and $T$ bounded, and $H$ unbounded, having discrete spectrum. Can we generalize our Theorems in Chapter 5 to this situation?

(ii) One way of characterizing the unbroken symmetry regime for finite-dimensional Hamiltonians $H$ is that all eigenvectors of $H$ are also eigenvectors of $PT$ see Theorem 2.2.2. What if we have an $H$ that has a continuous spectrum? Then the notion of an eigenvector does not make sense. We can still define the phases in terms of the nature of the spectrum (purely real or invariant under complex conjugation). Can one still prove results about metrics $\eta$ analogous to our Theorems in Chapter 5?

(3) **Physical observables, non-uniqueness of pairing $h(t) \leftrightarrow H$.** The relation between the Hermitian $h(t)$ and the non-Hermitian $H$, given by the Dyson equation

$$h(t) = S(t)HS(t)^{-1} + i\dot{S}(t)S(t)^{-1}, \tag{7.1}$$

is not uniquely defined. It involves the metric $\eta(t)$, which is a solution of the quasi-Hermiticity equation

$$H^*\eta(t) - \eta(t)H = i\partial_t\eta(t), \tag{7.2}$$

which involves the choice of an initial condition $\eta(0)$, and a not uniquely defined square root $S(t)$ of the metric $\eta(t)$. What does this non-uniqueness represent physically? We are unclear about the answer to this question at the moment. To elucidate it, we need to clarify the notion of physically observable quantities (measurements) in non-Hermitian systems. We have come across this notion in different papers [6, 31], but have never seen that researchers ask how the non-uniqueness $h(t) \leftrightarrow H$ affects an observable. More precisely, if $A$ is an observable of the non-Hermitian system, then (most probably?) $A'(t) = [S(t)^{-1}]^* A S(t)^{-1}$ is to be interpreted as an observable of the associated Hermitian system (recall that $\phi(t) = S(t)\psi(t)$). But $A'(t)$ depends on the choice of $\eta(t)$, $S(t)$. In what sense (if any) is $A'$ insensitive to changing $\eta(t)$ and $S(t)$?

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# Appendix A

# Diagonal form of $\eta$ is equivalent to product form

We have seen above in (6.24) that $\eta$ must be diagonal for the populations of $\bar{\rho}_H(t)$ to be non-negative and that conversely if $\eta$ is diagonal, then the populations of $\bar{\rho}_H(t)$ are positive. As it turns out, $\eta$ being diagonal is also equivalent to $\eta$ being of product form. More precisely, the following two statements are equivalent:

(1) $\eta$ is diagonal in the basis $|e_{\mathrm{S}}\rangle, |e_{\mathrm{B}}\rangle$ of $\mathcal{H}_1$.

(2) There are metrics $\Lambda_{\mathrm{S}}$ and $\Lambda_B$ on $\mathcal{H}_{\mathrm{S}}$ and $\mathcal{H}_{\mathrm{B}}$, respectively, such that $\Lambda_{\mathrm{S}} \otimes \Lambda_{\mathrm{B}}$ leaves $\mathcal{H}_1$ invariant and $\eta = \Lambda_{\mathrm{S}} \otimes \Lambda_{\mathrm{B}} \restriction_{\mathcal{H}_1}$ is the restriction of this product to $\mathcal{H}_1$.

Given $\eta$, the $\Lambda_{\mathrm{S}}$ and $\Lambda_{\mathrm{B}}$ are not unique.

*Proof of (1) $\Leftrightarrow$ (2).*

Consider a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{B}}$ with an orthonormal basis $|v_{ij}\rangle = |e_i\rangle \otimes |f_j\rangle$, the $|e_i\rangle$ and $|f_j\rangle$ being orthonormal bases of $\mathcal{H}_{\mathrm{S}}$ and $\mathcal{H}_{\mathrm{B}}$, respectively. Let $\mathcal{H}_1$ be the two-dimensional subspace $\mathcal{H}_1 = \mathrm{span}\{|v_{11}\rangle, |v_{22}\rangle\}$. In this setup, $|v_{11}\rangle$ is identified with $|e_{\mathrm{S}}\rangle$ and $|v_{22}\rangle$ with $|e_{\mathrm{B}}\rangle$. Let $\eta > 0$ be a strictly positive operator on $\mathcal{H}_1$.

We first show (1) $\Rightarrow$ (2). Assume that $\eta$ is diagonal, that is, $\eta = a|v_{11}\rangle\langle v_{11}| +$

$b|v_{22}\rangle\langle v_{22}|$, where $a, b > 0$. Set

$$\begin{aligned}\Lambda_{\rm S} &= \alpha_1|e_1\rangle\langle e_1| + \alpha_2|e_2\rangle\langle e_2| + \Lambda_{\rm S}^{\perp} \\ \Lambda_{\rm B} &= \beta_1|f_1\rangle\langle f_1| + \beta_2|f_2\rangle\langle f_2| + \Lambda_{\rm B}^{\perp},\end{aligned}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ satisfy $\alpha_1\beta_1 = a$, $\alpha_2\beta_2 = b$ and $\Lambda_{\rm S}^{\perp}$, $\Lambda_{\rm B}^{\perp}$ are arbitrary positive operators on the orthogonal complements of $\mathrm{span}\{|e_1\rangle, |e_2\rangle\}$ and $\mathrm{span}\{|f_1\rangle, |f_2\rangle\}$ in $\mathcal{H}_{\rm S}$ and $\mathcal{H}_{\rm B}$, respectively. Then $\Lambda_{\rm S} \otimes \Lambda_{\rm B}$ is a metric on $\mathcal{H}$ which leaves $\mathcal{H}_1$ invariant and satisfies $(\Lambda_{\rm S} \otimes \Lambda_{\rm B})|v_{jj}\rangle = \eta|v_{jj}\rangle$ for $j = 1, 2$.

Next we prove that (2) $\Rightarrow$ (1). The orthogonal projection onto $\mathcal{H}_1$ is given by

$$\pi = |v_{11}\rangle\langle v_{11}| + |v_{22}\rangle\langle v_{22}| = p_{11} \otimes q_{11} + p_{22} \otimes q_{22},$$

where $p_{ij} = |e_i\rangle\langle e_j|$ and $q_{ij} = |f_i\rangle\langle f_j|$. Since $\Lambda_{\rm S} \otimes \Lambda_{\rm B}$ leaves $\mathcal{H}_1$ invariant, we have $(\Lambda_{\rm S} \otimes \Lambda_{\rm B})\pi = \pi(\Lambda_{\rm S} \otimes \Lambda_{\rm B})\pi$. Now

$$(\Lambda_{\rm S} \otimes \Lambda_{\rm B})\pi = \Lambda_{\rm S}p_{11} \otimes \Lambda_{\rm B}q_{11} + \Lambda_{\rm S}p_{22} \otimes \Lambda_{\rm B}q_{22} \tag{A.1}$$

and, with $[\Lambda_{\rm S}]_{ij} = \langle e_i|\Lambda_{\rm S}e_j\rangle$ and similarly for $\Lambda_{\rm B}$,

$$\begin{aligned}\pi(\Lambda_{\rm S} \otimes \Lambda_{\rm B})\pi &= [\Lambda_{\rm S}]_{11}[\Lambda_{\rm B}]_{11}\ p_{11} \otimes q_{11} + [\Lambda_{\rm S}]_{12}[\Lambda_{\rm B}]_{12}\ p_{12} \otimes q_{12} \\ &= [\Lambda_{\rm S}]_{21}[\Lambda_{\rm B}]_{21}\ p_{21} \otimes q_{21} + [\Lambda_{\rm S}]_{22}[\Lambda_{\rm B}]_{22}\ p_{22} \otimes q_{22}. \end{aligned} \tag{A.2}$$

Taking the partial trace over B in (A.1) and (A.2) and equating the two results gives

$$[\Lambda_{\rm B}]_{11}\Lambda_{\rm S}p_{11} + [\Lambda_{\rm B}]_{22}\Lambda_{\rm S}p_{22} = [\Lambda_{\rm S}]_{11}[\Lambda_{\rm B}]_{11}\ p_{11} + [\Lambda_{\rm S}]_{22}[\Lambda_{\rm B}]_{22}\ p_{22}. \tag{A.3}$$

Since $[\Lambda_{\rm B}]_{11}, [\Lambda_{\rm B}]_{22} > 0$ we get from (A.3) that $\Lambda_{\rm S}p_{jj} = [\Lambda_{\rm S}]_{jj}p_{jj}$ for $j = 1, 2$, so $\Lambda_{\rm S}|e_j\rangle = [\Lambda_{\rm S}]_{jj}|e_j\rangle$. Hence the restriction of $\Lambda_{\rm S}$ to $\mathrm{span}\{|e_1\rangle, |e_2\rangle\}$ is diagonal, $\Lambda_{\rm S} = [\Lambda_1]_{11}p_{11} + [\Lambda_{\rm S}]_{22}p_{22} + \Lambda_{\rm S}^{\perp}$, where $\Lambda_{\rm S}^{\perp}$ is the block acting on the orthogonal complement of that span. By taking the partial trace over S in (A.1) and (A.2) and proceeding analogously, we see that $\Lambda_{\rm B} = [\Lambda_{\rm B}]_{11}q_{11} + [\Lambda_{\rm B}]_{22}q_{22} + \Lambda_{\rm B}^{\perp}$. It follows that $(\Lambda_{\rm S} \otimes \Lambda_{\rm B})|v_{jj}\rangle = [\Lambda_{\rm S}]_{jj}[\Lambda_{\rm B}]_{jj}\,|v_{jj}\rangle$ for $j = 1, 2$, so

$$\eta = [\Lambda_{\rm S}]_{11}[\Lambda_{\rm B}]_{11}\,|v_{11}\rangle\langle v_{11}| + [\Lambda_{\rm S}]_{22}[\Lambda_{\rm B}]_{22}\,|v_{22}\rangle\langle v_{22}|$$

is diagonal.

# Appendix B

# Criteria for the equivalence of entropy and density matrices in both Hermitian and non-Hermitian systems

Conditions for $\bar{\rho}_H(t) = \bar{\rho}_{h_W}(t)$ and $\mathcal{E}(\bar{\rho}_H(t)) = \mathcal{E}(\bar{\rho}_{h_W}(t))$

In this section, we assume the metric $\eta$ is of the form (6.14) with $x_1 = x_2 = x > 0$. and we want to study Recall the formulas (6.25) and (6.35) for the quasi-Hermitian and Hermitian density matrices.

First we ask when the two reduced density matrices coincide. We show that the following statements are equivalent:

1. $\bar{\rho}_H(t) = \bar{\rho}_{h_W}(t)$ for all $t$ in an open interval $I \subset \mathbb{R}$ and all $A, B \in \mathbb{C}$;
2. $\bar{\rho}_H(t) = \bar{\rho}_{h_W}(t)$ for all $t \in \mathbb{R}$ and all $A, B \in \mathbb{C}$;
3. There are two real phases $\Phi_1$, $\Phi_2$, such that

$$W = \begin{pmatrix} e^{i\Phi_1} & 0 \\ 0 & e^{i\Phi_2} \end{pmatrix}.$$

1. $\Rightarrow$ 3. : Assume that $\bar{\rho}_H(t) = \bar{\rho}_{h_W}(t)$ for $t \in I$. Then $p(t) = q(t)$ for $t \in I$,

where these quantities are given in (6.26) and (6.39), respectively. Their equality is equivalent with

$$\xi_1 + \xi_2 \cos(2\omega t) + \xi_3 \sin(2\omega t) = 0 \quad \text{for all } t \in I, \tag{B.1}$$

with $\xi_1 = 2x\mathrm{Re}(AB^*)\,\mathrm{Re}(cd^*)$, $\xi_2 = -|c|^2(1-2\alpha) - 2x\mathrm{Im}(AB^*)\mathrm{Im}(cd^*)$ and $\xi_3 = -(1-2\alpha)\mathrm{Im}(cd^*) + 2x(1-|c|^2)\mathrm{Im}(AB^*)$. As the constant function and the sine and cosine are three independent functions, and since (B.1) holds for all $t$ in an interval, we conclude that $\xi_1 = \xi_2 = \xi_3 = 0$. Since $\xi_1 = 0$ for all $A, B$ and $x > 0$, we have $\mathrm{Re}(cd^*) = 0$, that is, $\mathrm{Im}(cd^*) = cd^*$. The coefficients $A, B$ and $\alpha$ are related by (6.17) and (6.27), resulting in $A = \sqrt{\alpha}\sqrt{\frac{a_1}{xa_2}}e^{if_1}$ and $B = \sqrt{1-\alpha}\sqrt{\frac{a_2}{xa_1}}e^{if_2}$, where $f_1, f_2 \in \mathbb{R}$ are phases. This gives $\mathrm{Im}(AB^*) = \sqrt{\alpha(1-\alpha)}\frac{1}{x}\,\mathrm{Im}\,e^{i(f_1-f_2)}$. Then $\xi_2 = 0$ for all $A, B$ implies that

$$|c|^2(1-2\alpha) = -2\sqrt{\alpha(1-\alpha)}cd^*\,\mathrm{Im}\,e^{i(f_1-f_2)} \quad \text{for all } f_2, f_2 \in \mathbb{R},\ \alpha \in [0,1].$$

This forces $|c|^2(1-2\alpha) = 0 = \alpha(1-\alpha)cd^*$ for all $\alpha \in [0,1]$. Hence $c = 0$. Then due to (6.33), $|d| = 1$ and $bd^* = 0$, so $b = 0$, and statement 3. holds.

3. $\Rightarrow$ 2. : Suppose $c = 0$. Then $|d| = 1$ and from (6.36) we have $q(t) = x\frac{a_1}{a_2}|B(t)|^2$, which equals the population of $|0_\mathrm{S}\rangle$ in $\bar{\rho}_H(t)$, see (6.23). Therefore 2. holds.

2. $\Rightarrow$ 1. : Obvious.

This completes the proof of the equivalence of the three statements 1.-3.

Next we ask when the entropies of the two density matrices coincide. We show that the following statements 4.-6. are equivalent:

4. $\mathcal{E}(\bar{\rho}_H(t)) = \mathcal{E}(\bar{\rho}_{h_W}(t))$ for all $t$ in an open interval $I \subset \mathbb{R}$ and all $A, B \in \mathbb{C}$;

5. $\mathcal{E}(\bar{\rho}_H(t)) = \mathcal{E}(\bar{\rho}_{h_W}(t))$ for all $t \in \mathbb{R}$ and all $A, B \in \mathbb{C}$;

6. There are two real phases $\Phi_1$, $\Phi_2$ such that $W$ is of either of the two forms

$$W = \begin{pmatrix} e^{i\Phi_1} & 0 \\ 0 & e^{i\Phi_2} \end{pmatrix} \qquad \text{or} \qquad W = \begin{pmatrix} 0 & e^{i\Phi_1} \\ e^{i\Phi_2} & 0 \end{pmatrix}.$$

4. $\Rightarrow$ 6. : Start by looking at the function $\mathcal{E}(q) = -q\ln(q) - (1-q)\ln(1-q)$,

for $q \in [0,1]$. It is clear from the graph of $\mathcal{E}(q)$ (see the left panel of Figure 6.1) that $\mathcal{E}(q) = \mathcal{E}(q')$ exactly if either $q = q'$ or $q = 1 - q'$. Consequently, if $\mathcal{E}(\bar{\rho}_{h_W}(t)) = \mathcal{E}(\bar{\rho}_H(t))$ for all $t \in I$, then for each $t \in I$ *individually*, we have either $p(t) = q(t)$ or $p(t) = 1 - q(t)$. We now show that the same alternative must happen for *all* $t \in I$.

Suppose first that $p(t_0) \neq 1 - q(t_0)$ for some $t_0 \in I$. Then by the continuity of $p(t)$ and $q(t)$, we have $p(t) \neq 1 - q(t)$ for all $t$ in an open interval $I_0 \subset I$ around $t_0$, so we must have $p(t) = q(t)$ for $t \in I_0$. But this means that $\bar{\rho}_H(t) = \bar{\rho}_{h_W}(t)$ for all $t \in I_0$. Hence, as 2. and 3. are equivalent, $W$ is of the diagonal form as given in point 3. above. Similarly, if $p(t_0) \neq q(t_0)$ for some $t_0 \in I$, we obtain $p(t) = 1 - q(t)$ on an interval around $t_0$. Proceeding as in the proof of the implication 1. $\Rightarrow$ 3. above, this implies that $a = d = 0$, so $W$ is of the off-diagonal form given in statement 6. above.

6. $\Rightarrow$ 5. : If $W$ is of the diagonal form, we already showed that $p(t) = q(t)$ when we proved 3. $\Rightarrow$ 2. In the same way, if $W$ is off-diagonal, then one sees that $p(t) = 1 - q(t)$. In either case, $\mathcal{E}(\bar{\rho}_H(t)) = \mathcal{E}(\bar{\rho}_{h_W}(t))$.

5. $\Rightarrow$ 4. : Obvious.