NUMERICAL ANALYSIS OF DELAY DIFFERENTIAL AND INTEGRO-DIFFERENTIAL EQUATIONS

WENKUI ZHANG

0-612-42489-8

Numerical Analysis of Delay Differential and Integro-differential Equations

by

© Wenkui Zhang

A thesis submitted to the School of Graduate Studies

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics

Memorial University of Newfoundland

July 1998

St.John's Newfoundland Canada

Abstract

In this thesis, we analyze the collocation-based continuous Runge-Kutta methods for delay differential equations and delay Volterra integro-differential equations. We will look at the global convergence and local superconvergence properties of collocation solutions. We also consider the possible extensions of these results to neutral type delay equations and higher order equations.

In Chapter 2, we give the resolvent representations for solutions to Volterra integral and integro-differential equations with constant delay, and discuss their relevance for the superconvergence order problem. We prove that the resolvent representation does not exist for the proportional delay case. We then analyze the impact of discontinuities in solutions on our numerical methods. We show that discontinuities occur in higher order derivatives for delay integro-differential equations than for delay differential equations. We also prove that discontinuities arising in solutions to neutral delay integro-differential equations are different from those for neutral delay differential equations. Similar results hold for delay Volterra integral equation and delay Volterra integro-differential equation. We also give the discontinuity properties for solutions to state-dependent delay equations.

In Chapter 3, we discuss collocation solutions to various equations with constant delay, and survey global and local convergence results. Some extensions to neutral type constant delay problems are also described.

In Chapter 4, we introduce collocation methods for differential and integro-differential equations with variable delay, especially proportional delay. We prove that the global convergence order equals the number of collocation parameters used for first order differential equations with proportional delay. We give concrete representations for collocation solutions after the first step, and conduct some numerical experiments which suggest that superconvergence does exist in the proportional delay case. An extension to second order DDE is also given.

In Chapter 5, we suggest a new approach, standard embedding, to the superconvergence order problem of collocation solutions to differential equations with proportional delay, and are able to prove that superconvergence results again do exist under certain conditions.

Acknowledgment

I would like to thank my supervisor, Dr. Hermann Brunner, without whose advice and assistance, this work could not have been completed. I am also grateful to him for providing me with the opportunity and the means to attend two meetings during the course of my study.

I would also like to thank the School of Graduate Studies and the Department of Mathematics and Statistics for their financial supports to make my stay at Memorial University possible; Drs. Bruce Watson and Herb Gaskill, former and present heads of the department, for providing me with the facilities I needed in preparation of this thesis, and a friendly atmosphere in which to work. I am also grateful to the Dean of Graduate Studies, the Dean of Science and the Graduate Students Union for their financial support to my participation in two meetings.

Finally, I would like to thank my wife Gaiye and my daughter Yang for their unfailing support, and welcome distractions from mathematics.

Contents

List of Tables

Abbreviations

DDE	– delay differential equation
DE	– differential equation
DVIDE	– delay Volterra integro-differential equation
DVIE	– delay Volterra integral equation
IVP	– initial value problem
ODE	– ordinary differential equation
VIDE	– Volterra integro-differential equation
VIE	– Volterra integral equation

Symbols

$C^m(I)$	– m times differentiable functions on I
$\mathbb{C}$	– complex numbers
$\mathbb{N}$	– positive integers
$\mathbb{N}_0$	– positive integers and zero
Π_N	– a partition of an interval
$\mathbb{R}$	– real numbers
$\mathbb{R}^m$	– m-dimensional vector space
$\mathbb{R}^{m\times m}$	– $m \times m$ matrix space
$\Re$	– the real part of a complex number
$S_\mu^{(d)}$	– space of piecewise polynomials of degree μ and continuity class C^d
S_τ	– $I \times [-\tau, T-\tau]$
X_N	– the set of collocation points

Chapter 1

Introduction

In this chapter, we introduce the problems we are concerned with, the methods we are going to use, some basic definitions, a short history of collocation methods, and an outline of this thesis.

In the numerical analysis of initial-value problem for ordinary differential equations, three principal questions have to be answered:

1. Does the numerical method converge as "$h \to 0^+$"?
2. What is the optimal order of convergence (globally, on the prescribed interval; or locally, at the mesh points) of the method?
3. Does the numerical method mimic the stability properties of the given problem ("$h > 0$" fixed and "$t \to \infty$")?

Analogous questions arise for integral equations and integro-differential equations of Volterra type (functional equations with memory terms), since they may be viewed as generalized initial-value problems. In this thesis, we concentrate on collocation methods for delay differential equations, delay integral equations and delay Volterra integro-differential equations, namely

$$\begin{aligned} y'(t) &= f(t, y(t), y(\theta(t))), && (1.0.1) \\ y(t) &= g(t, y(t)) + \int_0^t k(t, s, y(s), y(\theta(s)))ds, && (1.0.2) \end{aligned}$$

and

$$y'(t) = g(t, y(t)) + \int_0^t k(t, s, y(s), y(\theta(s)))ds. \qquad (1.0.3)$$

We shall be concerned in particular with the cases $\theta(t) = t - \tau$, $\tau > 0$, and $\theta(t) = qt$, $0 < q < 1$, although there are some results about discontinuities for equations with more general delay, in particular, state-dependent delay.

1.1 Collocation methods

Let $I := [0, T]$ be the interval on which the given initial-value problem is to be solved, and let $\Pi_N : 0 = t_0 < t_1 < \cdots < t_N = T$ be a (not necessarily uniform) mesh for I. We set

$$h_n := t_{n+1} - t_n, \quad h := \max_{0 \leq n \leq N-1} \{h_n\}, \quad I_n := [t_n, t_{n+1}],$$

for $n = 0, 1, \cdots, N-1$, and denote by

$$S_\mu^{(d)}(\Pi_N) := \{u \in C^d(I) : u|_{I_n} \in \pi_\mu, \quad 0 \leq n \leq N-1\},$$

the space of (real) piecewise polynomials (or splines) of degree at most μ that have continuous derivatives of order d on I, with $-1 \leq d < \mu$. In the case where $d = -1$, the elements of $S_{\mu}^{(-1)}(\Pi_N)$ in general have (finite) jump discontinuities at the interior mesh points; we then set $I_n = (t_n, t_{n+1}]$ for $n = 1, \cdots, N-1$.

Theorem 1.1.1 *(See [21]) The dimension of the piecewise polynomial space $S_{\mu}^{(d)}(\Pi_N)$ is given by*

$$\dim S_{\mu}^{(d)}(\Pi_N) = N(\mu - d) + (d+1).$$

In particular, if $\mu = m + d$ $(m \geq 1)$, we have

$$\dim S_{m+d}^{(d)}(\Pi_N) = Nm + (d+1), \quad d \geq -1. \tag{1.1.1}$$

The "classical" spline spaces are given by $d = \mu - 1$; thus,

$$\dim S_{\mu}^{(\mu-1)}(\Pi_N) = N + \mu.$$

The proof of this result is straightforward and therefore omitted. Interested readers may look at [20] or [21] for more details about the basic setting of collocation methods.

The basic idea of a collocation method is to approximate the exact solution of a given functional equation in a suitably chosen finite-dimensional function space (often, but not always, a subspace of the space containing the solution) such that the approximating element satisfies the functional equation on a certain finite subset (consistent with the dimension of the approximating space) of the interval on which the equation is to be solved. This element will, in general, not satisfy the equation at a point not belonging to this finite subset (the set of collocation points).

The result of Theorem 1.1.1 indicates, in the context of collocation, that the natural choice of d in (1.1.1) will be governed by the nature of the functional equation to be solved: if the equation under consideration is a differential or integro-differential equation of order κ, then $d = \kappa - 1$. For problem (1.0.2), we certainly take $d = -1$, i.e., the collocation solution is found in the space $S_m^{(-1)}(\Pi_N)$. For problems (1.0.1) and (1.0.3), we take $d = 0$, i.e., we solve them in space $S_m^{(0)}(\Pi_N)$. The above theorem also suggests an obvious way of placing these collocation points if they are of multiplicity one: each of the N subintervals I_n contains $\mu - d$ collocation points (or, in the case of the space $S_{m+d}^{(d)}(\Pi_N)$, m such points for all $d \geq -1$). We denote the collocation points in I_n by $t_n + c_i h_n$, and set

$$X_N := \{t_n + c_i h_n : 0 \leq c_1 < \cdots < c_{\mu-d} \leq 1, \quad n = 0, 1, \cdots, N-1\}.$$

Note that if $c_1 = 0$ and $c_{\mu-d} = 1$, then the corresponding collocation solution u is in $C^{d+1}(I)$, that is, $S_{m+d}^{(d+1)}(\Pi_N)$, provided the data are continuous.

For differential equations, unlike integral equations in general, collocation leads directly to a set of algebraic equations for the parameters. The most common sets of functions used are global or piecewise polynomials, even though some researchers use non-polynomial splines instead, see [13], [32] and [68]. Some care is needed in the choice of collocation points if an effective algorithm is to be obtained.

1.2 Historical survey of collocation methods

In the late 1960s, Loscalzo [74], and Loscalzo and Talbot [75] introduced collocation methods in the classical spline space $S_m^{(m-1)}(\Pi_N)$ for initial-value problems of first-order ODEs; see also [55]. Callender [31] employed $S_m^{(1)}(\Pi_N)$ as the approximating space for collocation. A general analysis of polynomial spline collocation (including multiple collocation points) is due to Mülthei [82], see also the references therein. Keller [68] studied collocation methods in certain nonpolynomial spline spaces. While these papers are concerned with the global order of collocation approximations, Guillou and Soulé [50] had shown in 1969 that collocation in $S_m^{(0)}(\Pi_N)$ yields an m-stage implicit Runge-Kutta method, and has a local superconvergence order of $2m$ if the collocation points are the Gauss points.

Polynomial spline collocation methods in $S_m^{(m-1)}(\Pi_N)$ for initial-value problems for second-order ODEs were analyzed by Harvey [54] and Micula [78], see also the bibliography [79] for a comprehensive list of references. Collocation in $S_m^{(m-1)}(\Pi_N)$, $m \geq 3$, for second-order IVPs was done by Hung [56].

For (1.0.1), [9] deals with the one-step collocation method with continuous piecewise polynomial functions; primary discontinuities are studied in [42] and [46]; Runge-Kutta methods for vanishing delay differential equations has been studied in [43]; numerical investigation of the long time dynamical behavior of the solution has been conducted in [47], [59], [72], [73], [87] and [95], etc.

In recent years, various aspects of numerical methods for (1.0.2) have been studied. Convergence property results have been found for collocation and continuous implicit Runge-Kutta methods [16]; iterated collocation method [17]; continuous Volterra-Runge-Kutta methods [3]; Euler's method, the trapezoidal and midpoint method for (1.0.2) with pure delay [88]; Hermite-type collocation for (1.0.2) [44]; direct quadrature methods for (1.0.2) with state-dependent delay [29]; extension of ODE Runge-Kutta methods to (1.0.2) [2]; and general Runge-Kutta methods and their natural extensions for (1.0.2) [89].

It appears that VIDEs with delay arguments like (1.0.3) were first introduced by Volterra [93] in the late 1920s. More recently, delay VIDEs, and more general Volterra functional equations, have come to play an important role in the mathematical modeling of biological (see [38]) and physical phenomena and, not surprisingly, there has been a growing interest in the numerical solution of such equations; compare the survey papers [10] and [65]. Linear multistep methods and direct quadrature methods were studied in [4] for ordinary VIDEs, [64] and [67] for neutral Volterra functional equations and VIDEs; compare also [63] and the survey [65] for an analysis of one-step methods for neutral Volterra functional differential equations. Collocation methods were discussed in [77] for delay VIDEs and [14] for neutral VIDEs of order $r \geq 1$, see also [9] for delay differential equations, [16] and [89] for Volterra integral equations with delay.

Delay equations arise from many areas, including automatic control, physics, technology, and even certain areas of economics and the biological sciences. See [69] and [59] for comprehensive lists of references.

1.3 Outline of thesis

In Chapter 2, we will provide some basic theory for various types of Volterra equations relevant to the subsequent numerical analysis, especially the analysis of local superconvergence; we also point out some difficulties due to the discontinuities of the solutions and their derivatives. Resolvent representation is a classical approach to prove superconvergence results for many types of initial-value problems. However, it does not work in the proportional delay case as shown in Section 2.1. Discontinuities may have a negative impact on the convergence properties of numerical solutions. This problem and related theorems can be found in Section 2.2. In this chapter, our main contributions are Theorems 2.1.5–2.1.7, 2.1.9, 2.2.3, 2.2.5 and 2.2.8.

In Chapter 3, we will look in detail at the collocation methods for several kinds of equations with constant delay. We survey various known results related to constant delay in order to compare them with similar results for equations with proportional delay in Chapter 4. We introduce the collocation method for these equations in Section 3.1. We review the global convergence results in Section 3.2. Local convergence results are covered in Section 3.3. In Section 3.4, results about delay Volterra integro-differential equations of neutral type are discussed.

In Chapter 4, we will develop the collocation methods for various equations with proportional delay. Its global convergence properties are discussed in Sections 4.1 and 4.2. In Section 4.3, we discuss the order of local convergence of our test equation. In Section 4.4, we extend the results to second-order DDEs. Some numerical examples will be provided as a further illustration for these results. Our main contributions in this chapter are Theorems 4.2.1, 4.2.2, 4.3.3 and 4.4.2–4.4.3.

In Section 5.1, we will propose a new approach to the superconvergence order problem of collocation solutions to differential equations with proportional delay, and prove the result under certain conditions. In Section 5.2, we look at some potential research projects. In this chapter, our main contributions are Theorem 5.1.3 and 5.1.4.

We assume that the reader is familiar with the theory of ODEs and the methods for their numerical solutions. Representative books about this subject are [27], [39], [51], [52] and [70], see also the references therein. For an introduction to Volterra integral and differential equations, reader may consult [21], [26], [34], [48] and [80]. Classical treatments of integral equations may also be found in [90], [91] and [92].

Chapter 2

Mathematical Background

In this chapter, we present some analytic results which are crucial for the rest of this thesis. Resolvent representation is a classical approach to prove superconvergence results for many types of initial-value problems. However, such a representation does not exist in general in the variable delay case as shown in Section 2.1. Discontinuities may have a negative impact on the convergence properties of numerical solutions. This problem and related theorems can be found in Section 2.2.

Consider the (linear) ordinary differential equation,

$$y'(t) = a(t)y(t) + g(t), \quad t \in I, \quad y(0) = y_0, \tag{2.0.1}$$

and the integro-differential equation,

$$y'(t) = a(t)y(t) + g(t) + \int_0^t K(t,s)y(s)ds, \quad t \in I, \quad y(0) = y_0. \tag{2.0.2}$$

Since resolvent representations of solutions are the key to the proof of superconvergence results in collocation approximations, we will study this issue in next section.

2.1 Resolvent Results

2.1.1 Equations without delay

Definition 2.1.1 If the solution of an equation given above can be expressed in the form

$$y(t) = R(t,0)y(0) + \int_0^t R(t,s)g(s)ds, \quad t \in I, \tag{2.1.1}$$

where $R(t,s)$ depends only on the data in the homogeneous part of the given equation, then (2.1.1) is called the resolvent representation of the solution. The function $R(t,s)$ is called the resolvent kernel.

Theorem 2.1.1 *If $a, g \in C(I)$, then the resolvent representation of the solution for (2.0.1) is given by*

$$y(t) = R(t,0)y(0) + \int_0^t R(t,s)g(s)ds, \quad t \in I,$$

where $R(t,s)$ solves the resolvent equation

$$\frac{\partial R(t,s)}{\partial t} = a(t)R(t,s), \quad s \le t,$$

with $R(t,t) = 1$.

In this case, we can directly write down the resolvent kernel as

$$R(t,s) = \exp(\int_s^t a(x)dx).$$

A number of reasons motivate us to look for such representations of a solution. For example, if we want to solve an equation whose solution has a resolvent representation numerically by collocation methods, this eventually leads to an equation for the collocation error $e(t) := y(t) - u(t)$ (where $y(t)$ and $u(t)$ are the exact and collocation solutions, respectively) which differs from the original equation only in the nonhomogeneous term: $g(t)$ is replaced by the defect term $\delta(t)$ which, by definition, vanishes at the collocation points $\{t_{n,i} := t_n + c_i h,\ 0 \le c_1 < \cdots < c_n \le 1, \quad (n = 0, 1, \cdots, N-1)\}$, see page 51 for the precise definition. If the solution of the equation has a resolvent representation (2.1.1), then it follows that

$$e(t) = R(t,0)e(0) + \int_0^t R(t,s)\delta(s)ds = \int_0^t R(t,s)\delta(s)ds. \qquad (2.1.2)$$

Setting $t = t_n$ in (2.1.2) leads to

$$e(t_n) = h\sum_{i=0}^{n-1} \int_0^1 R(t_n, t_i + sh)\delta(t_i + sh)ds. \qquad (2.1.3)$$

This integral form allows us to derive superconvergence results for the mesh points $t = t_n$. For (2.0.1), it was shown in [50], see also [51], that the attainable order of local superconvergence on Π_N is related to the degree of precision of the (interpolatory) m-point quadrature formula having the $\{c_i\}$ as abscissas:

Theorem 2.1.2 *(See [50]) If the collocation parameters $\{c_i\}$ satisfy*

$$\int_0^1 s^{k-1} M(s) ds = 0 \quad \textit{for} \quad k = 1, \ldots, r,$$

with $M(s) := \Pi_{i=1}^m (s - c_i)$, and if u is the corresponding collocation solution in $S_m^{(0)}(\Pi_N)$ for (2.0.1), then

$$\max_{1 \le n \le N} |y(t_n) - u(t_n)| \le C h^{m+r}, \quad r \le m, \tag{2.1.4}$$

for some constant C whenever the solution y is sufficiently smooth.

The following remark can serve as an informal proof for the result (2.1.4) in Theorem 2.1.2.

Remark 2.1.1 Using an m-point interpolatory quadrature formula with abscissas $\{t_i + c_l h : l = 1, \cdots, m\}$, weights $\{w_l\}$, and quadrature errors $E_{n,i}$, the integrals in (2.1.3) may be written as

$$\int_0^1 R(t_n, t_i + sh) \delta(t_i + sh) ds = \sum_{l=1}^m w_l R(t_n, t_i + c_l h) \delta(t_i + c_l h) + E_{n,i}.$$

That is,

$$\int_0^1 R(t_n, t_i + sh) \delta(t_i + sh) ds = E_{n,i},$$

because $\delta(t) = 0$ when $t = t_i + c_l h \in X_N$. This implies that the convergence order of $e(t_n)$ is totally determined by the order of the quadrature errors $E_{n,i}$ which in turn depends on the degree of smoothness of the integrand. Indeed, if the given functions are sufficiently smooth, these orders are equal to $2m$ when we take the Gauss points, i.e., the zeros of the (shifted) Legendre polynomial $P_m(2s-1)$, since the quadrature error at m Gauss points always has an order of $2m$.

This simple idea, which was used first in [50], has potentially great impact on the analysis of superconvergence results for more general differential, or integro-differential equations, even with delay.

We may ask the following question: If $u \in S_m^{(0)}(\Pi_N)$ is the collocation solution of (2.0.1) or of (2.0.2) for the Gauss points, we know that

$$e(t_n) = \mathcal{O}(h^{2m}),$$

but what about $e'(t_n)$? Can we get the same order for $e'(t_n)$? The answer is not very encouraging: for the Gauss points, we can only get the lower order of m, i.e.,

$$e'(t_n) = \mathcal{O}(h^m).$$

As we shall see below, $e(t_n) = \mathcal{O}(h^{p^*})$ and $e'(t_n) = \mathcal{O}(h^{p^*})$, in which $p^* = 2m-1$, is possible for the Radau II points, i.e., the zeros of $P_m(2s-1) - P_{m-1}(2s-1)$ where P is the Legendre polynomial.

However, when the iterated collocation solution is introduced, the result is much better. In fact, we again get the same order of $2m$. The iterated collocation solution of (2.0.1) is defined by

$$u_{it}'(t) := a(t)u(t) + g(t), \quad t \in I,$$

where $u(t)$ is the collocation solution we already have. Accordingly, the iterated error for the derivative is $e'_{it}(t_n) := y'(t_n) - u'_{it}(t_n)$. We summarize the above analysis as follows:

Theorem 2.1.3 *For equation (2.0.1), if $a(t), g(t) \in C^{2m}(I)$, then we have*

$$\max_{1 \le n \le N} \{|e(t_n)|, |e'_{it}(t_n)|\} \le Ch^{p^*},$$

for some constant C and $p^* \le 2m$. We have $p^* = 2m$ if, and only if, the $\{c_i\}$ are the Gauss points.

A more general result can be found in [18], Theorem 3.2.

Another natural question arises, namely, for which $\{c_i\}$ do we have

$$e(t_n) = \mathcal{O}(h^{p^*}),$$

and

$$e'(t_n) = \mathcal{O}(h^{p^*}),$$

with $p^* > p = m$ at the same time? A necessary condition is $c_m = 1$ (compare [21]). From (2.1.2), we have

$$e'(t) = R(t,t)\delta(t) + \int_0^t \frac{\partial R(t,s)}{\partial t}\delta(s)ds, \quad t \in I.$$

In order to have an expression similar to (2.1.3), we need $\delta(t_n) = 0$, that is, $t_{n-1} + c_m h = t_n$, i.e., $c_m = 1$ (Note that in general we have $\delta(t_n) = \mathcal{O}(h^m)$). This is equivalent to saying that the last collocation point in each subinterval coincides with its right end-point. Therefore, obviously the answer cannot be true for Gauss points where $c_m < 1$.

We have the following theorem concerning the resolvent representation of the solution of (2.0.2) (see [21] and [49]).

Theorem 2.1.4 *If $a(t)$, $g(t)$ and $K(t,s)$ are continuous on their own domains, then the resolvent representation of solution (2.0.2) is given by*

$$y(t) = R(t,0)y(0) + \int_0^t R(t,s)g(s)ds,$$

and $R(t,s)$ solves the resolvent equation

$$\frac{\partial R(t,s)}{\partial t} = a(t)R(t,s) + \int_s^t K(t,x)R(x,s)dx, \quad s \le t,$$

with $R(t,t) = 1$.

2.1.2 Equations with delay

For delay Volterra integral equations of the form

$$\begin{aligned} y(t) &= g(t) + \int_0^t K_1(t,s)y(s)ds + \int_0^{t-\tau} K_2(t,s)y(s)ds, \quad t \in I, \quad (2.1.5) \\ y(t) &= \phi(t), \quad t \in [-\tau, 0], \end{aligned}$$

we have the following two results which are slight extensions of similar results in [17]:

Theorem 2.1.5 *Assume g, K_1 and K_2 in (2.1.5) are continuous, and $\xi_M = T$ for some $M \in \mathbb{N}$. Then for $t \in [\xi_\mu, \xi_{\mu+1}]$, $\xi_\mu := \mu\tau$, $0 \le \mu \le M-1$, the solution y to (2.1.5) has the resolvent representation*

$$y(t) = \bar{g}_\mu(t) + \int_0^t R(t,s)\bar{g}_\mu(s)ds,$$

where $\bar{g}_\mu(t) := g(t) + \int_0^{t-\tau} K_2(t,s)y_\mu(s)ds$, $y_\mu(t)$ *is the solution of (2.1.5) on* $[-\tau, \xi_\mu]$ *with* $y_0(t) = \phi(t)$, *and* $R(t,s)$ *solves*

$$R(t,s) = K_1(t,s) + \int_s^t K_1(t,x)R(x,s)dx, \quad 0 \le s \le t \le T. \tag{2.1.6}$$

Proof: For $\mu = 0$, suppose

$$y(t) = \bar{g}(t) + \int_0^t R(t,s)\bar{g}(s)ds, \quad t \in [0,\tau], \tag{2.1.7}$$

where $R(t,s)$ is to be determined and $\bar{g}(t) := g(t) + \int_0^{t-\tau} K_2(t,s)\phi(s)ds$. Substituting (2.1.7) back to (2.1.5), we have

$$\begin{aligned} \int_0^t R(t,s)\bar{g}(s)ds &= \int_0^t K_1(t,s)\bar{g}(s)ds + \int_0^t \int_0^s K_1(t,s)R(s,x)\bar{g}(x)dxds \\ &= \int_0^t K_1(t,s)\bar{g}(s)ds + \int_0^t \int_s^t K_1(t,x)R(x,s)dx\bar{g}(s)ds. \end{aligned}$$

Hence, $R(t,s)$ must solve (2.1.6).

For $\mu > 0$, we use the same argument but work with $[-\tau, \xi_\mu]$ and $y_\mu(t)$ instead of $[-\tau, 0]$ and $\phi(t)$. □

The collocation error $e := y - u$ satisfies

$$e(t) = \delta(t) + \int_0^t K_1(t,s)e(s)ds + F(t),$$

where $F(t) = \int_0^{t-\tau} K_2(t,s)e(s)ds$, and most importantly, we have

Theorem 2.1.6 *The error e has a resolvent representation of the form*

$$e(t) = \bar{\delta}_\mu(t) + \int_0^t R(t,s)\bar{\delta}_\mu(s)ds, \tag{2.1.8}$$

on $[\xi_\mu, \xi_{\mu+1}]$, $0 \le \mu \le M-1$, *where* $\bar{\bar{\delta}}_\mu(t) := \delta(t) + \int_0^{t-\tau} K_2(t,s) e_\mu(s) ds$, $e_\mu(t)$ *is the error on* $[0, \xi_\mu]$ *with* $e_0(t) \equiv 0$ *and* $R(t,s)$ *satisfies (2.1.6) provided that each known function in (2.1.5) is continuous.*

Due to the presence of $\delta(t)$ in (2.1.8), the local superconvergence order is largely determined by the nature of the defect term $\delta(t)$. A more detailed discussion about the superconvergence order for (2.1.5), using the resolvent approach, can be found in [17].

Consider delay Volterra integro-differential equations of the form

$$\begin{aligned} y'(t) &= g(t) + \int_0^t K_1(t,s) y(s) ds + \int_0^{t-\tau} K_2(t,s) y(s) ds, \quad t \in I, \quad (2.1.9) \\ y(t) &= \phi(t), \quad t \in [-\tau, 0]. \end{aligned}$$

Theorem 2.1.7 *Let* g, K_1 *and* K_2 *in (2.1.9) be continuous and* $\xi_M = T$ *for some* $M \in \mathbf{N}$. *Then for* $t \in [\xi_\mu, \xi_{\mu+1}]$, $\xi_\mu := \mu\tau$, $0 \le \mu \le M-1$, *the solution* y *to (2.1.9) can be expressed in the form*

$$y(t) = R(t,0) y(0) + \int_0^t R(t,s) \bar{g}_\mu(s) ds,$$

where $\bar{g}_\mu(t) := g(t) + \int_0^{t-\tau} K_2(t,s) y_\mu(s) ds$, $y_\mu(t)$ *is the solution of (2.1.9) on* $[-\tau, \xi_\mu]$, *and* $R(t,s)$ *solves*

$$\frac{\partial R}{\partial t}(t,s) = \int_s^t K_1(t,x) R(x,s) dx, \quad 0 \le s \le t \le T. \qquad (2.1.10)$$

Proof: Similar to that of Theorem 2.1.5. □

Since the collocation error satisfies $e(0) = 0$, we obtain,

Corollary 2.1.1 *If g, K_1 and K_2 in (2.1.9) are continuous, and $\xi_M = T$ for some $M \in \mathbf{N}$. Then for $t \in [\xi_\mu, \xi_{\mu+1}]$, $0 \le \mu \le M-1$, the collocation error $e(t)$ to (2.1.9) can be expressed in the form*

$$e(t) = \int_0^t R(t,s)\bar{\delta}_\mu(s)ds, \tag{2.1.11}$$

where $\bar{\delta}_\mu(t) := \delta(t) + \int_0^{t-\tau} K_2(t,s)e_\mu(s)ds$, $e_\mu(t)$ is the error on $[0, \xi_\mu]$ with $e_0(t) \equiv 0$ and $R(t,s)$ solves (2.1.10).

Since we have the integral expression of error $e(t)$ in the form of (2.1.11), the convergence order of $e(t_n)$ is again determined by the order of the quadrature error which in turn, depends on the smoothness of the integrand $R(t,s)\bar{\delta}_\mu(s)$. This argument leads to the following superconvergence result.

Theorem 2.1.8 *(See [18], Theorem 3.2) Assume that the given functions in (2.1.9) are sufficiently smooth on their domains, i.e., $g \in C^{2m}(I)$, $K_1 \in C^{2m}(S)$, $K_2 \in C^{2m}(S_\tau)$ where $S_\tau := [-\tau, T-\tau]$ and $\phi(t) \in C^{2m}[-\tau, 0]$. If the collocation points are Gauss points, and $h = \tau/r$ (constrained mesh Π_N) is sufficiently small, then*

$$\max_{1 \le n \le N} |y(t_n) - u(t_n)| \le Ch^{2m},$$

for some finite constant C.

Certainly, the most challenging problem is to establish the local superconvergence properties of solutions in the proportional delay case. Unfortunately, the resolvent approach does not work for this case, since the resolvent representation does not exist. As a result, a new approach has to be found. We make this clear by the following result.

Consider integro-differential equations with proportional delay,

$$y'(t) = a(t)y(t) + g(t) + \int_0^{qt} K(t,s)y(s)ds, \quad t \in I, \quad y(0) = y_0, \quad (2.1.12)$$

with $0 < q < 1$.

Theorem 2.1.9 *There is no resolvent representation of the form (2.1.1) for the solution of (2.1.12).*

Proof: We prove it by contradiction. Rewrite (2.1.12) as

$$y'(x) = a(x)y(x) + g(x) + \int_0^{qx} K(x,s)y(s)ds, \qquad (2.1.13)$$

multiply by $R(t,x)$ and integrate from 0 to t on both sides of (2.1.13):

$$\begin{aligned} R(t,t)y(t) = {} & R(t,0)y(0) + \int_0^t R(t,s)g(s)ds + \int_0^t \left(\frac{\partial R(t,s)}{\partial s} + R(t,s)a(s)\right) y(s)ds \\ & + \int_0^{qt} \left(\int_{s/q}^t R(t,x)K(x,s)dx\right) y(s)ds \\ = {} & R(t,0)y(0) + \int_0^t R(t,s)g(s)ds - \int_{qt}^t \left(\int_{s/q}^t R(t,x)K(x,s)dx\right) y(s)ds \\ & + \int_0^t \left(\frac{\partial R(t,s)}{\partial s} + R(t,s)a(s) + \int_{s/q}^t R(t,x)K(x,s)dx\right) y(s)ds. \end{aligned}$$

If the resolvent kernel $R(t,s)$ satisfies the resolvent equation

$$\frac{\partial R(t,s)}{\partial s} = -R(t,s)a(s) - \int_{s/q}^t R(t,x)K(x,s)dx, \quad s \le t, \qquad (2.1.14)$$

with $R(t,t) = 1$, then

$$y(t) = R(t,0)y(0) + \int_0^t R(t,s)g(s)ds - \int_{qt}^t \left(\int_{s/q}^t R(t,x)K(x,s)dx\right) y(s)ds.$$

This is no longer a resolvent representation since $y(t)$ itself is also involved in the last term on the right side. This indicates that we cannot have the resolvent equation (2.1.14) and the resolvent representation (2.1.1) at the same time unless $q = 1$ (no delay) or $K(t, s) \equiv 0$ (no integral term).

$\square$

Another proof of Theorem 2.1.9: Assume a resolvent representation holds for solution of (2.1.12):

$$y(t) = R(t, 0)y(0) + \int_0^{\gamma t} R(t, s)g(s)ds, \tag{2.1.15}$$

with either $\gamma = 1$ or $\gamma = q$. Substitute it back to (2.1.12),

$$\begin{aligned}
&\frac{\partial R(t,0)}{\partial t}y(0) + \gamma R(t, \gamma t)g(\gamma t) + \int_0^{\gamma t} \frac{\partial R(t,s)}{\partial t}g(s)ds \\
= \; & a(t)R(t,0)y(0) + \int_0^{qt} K(t,s)R(s,0)y(0)ds + g(t) \\
& + \int_0^{\gamma t} a(t)R(t,s)g(s)ds + \int_0^{qt}\int_0^{\gamma s} K(t,s)R(s,x)g(x)dxds \\
= \; & a(t)R(t,0)y(0) + \int_0^{qt} K(t,s)R(s,0)y(0)ds + g(t) \\
& + \int_0^{\gamma t} a(t)R(t,s)g(s)ds + \int_0^{\gamma qt}\left(\int_{s/\gamma}^{qt} K(t,x)R(x,s)dx\right) g(s)ds.
\end{aligned}$$

In either case, we cannot derive the resolvent equation:

$$\frac{\partial R(t,s)}{\partial t} = a(t)R(t,s) + \int_s^{qt} K(t,x)R(x,s)dx, \quad s \le qt,$$

unless $q = 1$. Hence, the solution of (2.1.12) does not have a resolvent representation of the form (2.1.15). $\square$

Similar conclusions hold for other type equations with proportional delay. For example, Chambers [33] proved that solutions to the following integral equations do not have a resolvent representation:

$$y(t) = g(t) + \int_0^{qt} K(t,s)y(s)ds, \quad t \in I. \tag{2.1.16}$$

Using the Picard iteration method, the iterative solution is given by

$$\begin{aligned} y_{n+1}(t) &= g(t) + \int_0^{qt} K(t,s)y_n(s)ds, \\ y_0(t) &= \phi(t). \end{aligned}$$

Theorem 2.1.10 *([33]) When $0 < q \leq 1$, (2.1.16) has a unique solution, and it can be expressed as*

$$y(t) = g(t) + \sum_{m=1}^{\infty} \int_0^{q^m t} K_m(t,s)g(s)ds, \tag{2.1.17}$$

where the K_m are defined iteratively by

$$K_{m+1}(t,s) = \int_{sq^{-m}}^{qt} K(t,x)K_m(x,s)dx, \quad m \geq 1,$$

and $K_1(t,s) = K(t,s)$.

However, the above solution does not have a resolvent representation. By change of variable,

$$\begin{aligned} y(t) &= g(t) + \sum_{m=1}^{\infty} \int_0^t q^m K_m(t, q^m s)g(q^m s)ds \\ &= g(t) + \int_0^t \sum_{m=1}^{\infty} q^m K_m(t, q^m s)g(q^m s)ds. \end{aligned}$$

It is clear that $g(s)$ can not be separated from the summation. As a result, the solution to (2.1.16) does not have a resolvent representation. If $q = 1$ ("classical case"), then (2.1.17) can be written as

$$y(t) = g(t) + \int_0^t R(t,s)g(s)ds,$$

where the resolvent kernel is $R(t,s) = \sum_{m=1}^{\infty} K_m(t,s)$; this is the resolvent representation for the solution of (2.1.16) with $q = 1$.

Remark 2.1.2 The origin of proportional delay integral equations can be traced back to as early as 1897. In [92], Volterra studied the existence and uniqueness of solutions to equation

$$\int_{qt}^{t} K(t,s)y(s)ds = g(t), \quad t \in I,$$

with $0 < q < 1$, $K, g \in C^1(I)$; $K(t,t) \neq 0$, for $t \in I$. Differentiating it gives

$$K(t,t)y(t) - qK(t,qt)y(qt) + \int_{qt}^{t} \frac{\partial}{\partial t} K(t,s)y(s)ds = g'(t), \quad t \in I.$$

This is a second kind Volterra integral equation with proportional delay. We can get a neutral VIDE with proportional delay by further differentiation.

2.2 Primary discontinuities in solutions

Consider the first-order delay differential equation of the form

$$\begin{aligned} y'(t) &= f(t, y(t), y(t-\tau)), \quad t \geq 0, \\ y(t) &= \phi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{2.2.1}$$

where $\tau > 0$ is a constant. The theory of existence and uniqueness of solutions to (2.2.1) does not present substantial additional difficulties with respect to the classical (non-delay) case. This is also true when we consider differential equations with more general delay afterwards, as long as the delay is uniformly strictly positive and does not depend on the solution y itself. We refer the reader to [53] for a comprehensive introduction to the theory of DDEs.

In this section, we discuss the possible sources for discontinuities and prove discontinuity properties for solutions of delay integral and integro-differential equations, and review some known results about (2.2.1). See also [9], [16], [22],[46], [69] and [97] for additional details.

Regarding the analytical solution of (2.2.1), the most natural method, see also [42], is called the method of steps (or the method of successive integrations). It consists of determining the solution $y(t)$ from the differential equation without delay,

$$\begin{aligned} y'(t) &= f(t, y(t), \phi(t-\tau)), \quad t \in [0, \tau], \\ y(0) &= \phi(0), \end{aligned}$$

since for $0 \leq t \leq \tau$, the argument $t - \tau$ varies in the initial interval $[-\tau, 0]$ and, consequently, the third argument $y(t-\tau)$ of the function f equals the initial function $\phi(t-\tau)$. Assuming the existence of a solution $y = \phi_1(t)$ of this initial value problem on the whole interval $[0, \tau]$, analogously we obtain:

$$\begin{aligned} y'(t) &= f(t, y(t), \phi_n(t-\tau)), \\ &\qquad \text{for} \quad t \in [n\tau, (n+1)\tau], \quad \text{with} \quad y(n\tau) = \phi_n(n\tau), \end{aligned}$$

where $n = 1, 2, \cdots$ and $\phi_n(t)$ is the solution of the considered initial value problem on the interval $[(n-1)\tau, n\tau]$.

Definition 2.2.1 If the solution of a DDE (or a DVIDE) and its derivatives of order μ are continuous at some point in the time interval, but the derivative of order $\mu + 1$ is not, then such a point is called a *primary discontinuity* of the given problem.

Theorem 2.2.1 *The points $\xi_\mu := \mu\tau$, $\mu = 0, 1, \cdots$, are the primary discontinuities of problem (2.2.1). More precisely, $y^{(\mu)}$ is continuous at ξ_μ but $y^{(\mu+1)}$ is, in general, not, even if the functions ϕ and f have continuous derivatives of all orders.*

Proof: See [42]. □

Note that, as t increases, the solution becomes smoother. In fact, at the initial point $t = 0$, the first derivative $y'(t)$ has a primary discontinuity, since the integrable equation

$$y'(t) = f(t, y(t), \phi(t-\tau)), \quad t \in [0, \tau],$$

may satisfy the condition $y(0) = \phi(0)$, but it is unlikely to satisfy the additional condition $y'(0+) = \phi'(0-)$. Only for special choices of the initial function $\phi(t)$ is it possible to guarantee continuity of the derivative of the solution at point 0, for such a function must satisfy the condition $\phi'(0-) = f(0, \phi(0), \phi(-\tau))$.

Example 2.2.1 Consider

$$\begin{aligned} y'(t) &= ay(t-\tau), \quad t \in [0, +\infty), \\ y(t) &= 1, \quad t \in [-\tau, 0]. \end{aligned}$$

Using the method of steps, it is easy to see that the solution $y(t)$ is a piecewise polynomial. On each subinterval $[i\tau, (i+1)\tau]$, $y(t)$ is an $(i+1)$-th order polynomial, i.e.,

$$y(t) = \sum_{j=0}^{i+1} \frac{a^j}{j!}(t - (j-1)\tau)^j, \quad i \in \mathbb{N}_0.$$

It is also clear that integer multiples of τ are primary discontinuities for this particular problem.

The method of steps can be extended to differential equations with other types of delays, such as multiple delays, variable delay and even state-dependent delay. The difficulty is to locate the primary discontinuities. As a generalization of (2.2.1), we consider

$$\begin{aligned} y'(t) &= f(t, y(t), y(t-\tau(t))), \quad t \geq 0, \\ y(t) &= \phi(t), \quad t \in [\overline{a}, 0], \end{aligned} \tag{2.2.2}$$

where $t - \tau(t)$ is a strictly increasing function and

$$0 < \tau(t) \leq t, \quad \overline{a} = \inf_{t \geq 0}(t - \tau(t)).$$

Remark 2.2.1 Throughout this thesis, when the delay τ depends on time t, we will make this clear by the notation $\tau(t)$. Otherwise, τ is a positive constant.

Theorem 2.2.2 *(see [97]) The primary discontinuities of problem (2.2.2) are generated inductively by the recursion*

$$\xi_k - \tau(\xi_k) = \xi_{k-1}, \quad k \geq 1, \tag{2.2.3}$$

where $\xi_0 = 0$.

Because of the hypotheses made, a strictly increasing sequence $\{\xi_k\}_{k\geq 0}$ is determined which can be actually computed a priori by using (2.2.3). In this way, a sequence of intervals $[\xi_{k-1}, \xi_k]$ is also defined, see also [97].

Remark 2.2.2 If the functions $\phi(t)$ and $\tau(t)$ in (2.2.2) have some discontinuities with respect to t in some of their derivatives, then such discontinuities are also propagated by the delay argument $t - \tau(t)$ following the rule (2.2.3). These discontinuities are called secondary discontinuities.

Example 2.2.2 Consider

$$\begin{aligned} y'(t) &= ay(t-\tau), \quad t \in [0, +\infty), \\ y(t) &= \phi(t), \quad t \in [-\tau, 0], \end{aligned}$$

where

$$\phi(t) = \begin{cases} 0, & t \in [-\tau, -\tau/2), \\ 1, & t \in [-\tau/2, 0]. \end{cases}$$

On $[0, \tau]$,

$$y(t) = y(0) + a\int_0^t \phi(s-\tau)ds = \begin{cases} 1, & t \in [0, \tau/2), \\ 1 + at, & t \in [\tau/2, \tau]. \end{cases}$$

Obviously, in addition to $\xi_n = n\tau$, the points $n\tau + \tau/2$ are also discontinuities. However, they are secondary discontinuities as they inherit this property from the initial function $\phi(t)$.

If the initial function $\phi(t)$ is changed to

$$\phi(t) = \begin{cases} 0, & t \in [-\tau, -\theta\tau), \\ 1, & t \in [-\theta\tau, 0], \end{cases}$$

where $0 < \theta < 1$, then on $[0, \tau]$, we have

$$y(t) = y(0) + a\int_0^t \phi(s-\tau)ds = \begin{cases} 1, & t \in [0, \tau - \theta\tau), \\ 1 + at, & t \in [\tau - \theta\tau, \tau]. \end{cases}$$

Clearly, $n\tau$ and $n\tau - \theta\tau$ $(n \geq 1)$ are both discontinuities. $n\tau$ is primary, and $n\tau - \theta\tau$ is secondary.

More discussion about this topic, especially the state-dependent delay case, can be found on the following pages. See also [42] for extension to the multiple delay case.

The existence of primary and secondary discontinuities may lead to a loss of accuracy (reduction of order) or to numerical instability if the mesh underlying a discretization method does not take into account these discontinuities. For a detailed discussion of this problem, see for example, [97].

Similar discontinuity results hold for Volterra integro-differential equation with constant delay.

$$\begin{aligned} y'(t) &= f(t, y(t)) + \int_0^t K(t, s, y(s), y(s-\tau))ds, \quad t \geq 0, \qquad (2.2.4) \\ y(t) &= \phi(t), \quad t \in [-\tau, 0]. \end{aligned}$$

However, we shall see (compare Theorems 2.2.3 and 2.2.1) that there are fundamental differences between the regularity of solutions to (2.2.4) and those to (2.2.1).

Remark 2.2.3 If the delay occurs in one of the limits of integration, for example,

$$y'(t) = f(t, y(t)) + \int_0^t K_1(t, s, y(s))ds + \int_0^{t-\tau} K_2(t, s, y(s))ds, \qquad (2.2.5)$$

or

$$y'(t) = f(t, y(t)) + \int_{t-\tau}^t K(t, s, y(s))ds, \qquad (2.2.6)$$

we can always convert the equations into the form of (2.2.4) by a suitable change of variables. For example, in (2.2.5), we may write

$$\begin{aligned} y'(t) &= f(t, y(t)) + \int_0^t K_1(t, s, y(s))ds + \int_\tau^t K_2(t, v-\tau, y(v-\tau))dv \\ &= f(t, y(t)) + \int_0^t \{K_1(t, s, y(s)) + K_3(t, s-\tau, y(s-\tau))\}ds, \end{aligned}$$

where

$$K_3 = \begin{cases} K_2, & v \in [\tau, t], \\ 0, & v \in [0, \tau]. \end{cases}$$

Therefore, we can change (2.2.5) to the form of (2.2.4). For (2.2.6), we have

$$y'(t) = f(t, y(t)) + \int_0^{t-\tau} K_1(t, s, y(s))ds,$$

where

$$K_1(t, s, y(s)) = \begin{cases} 0, & s \in [0, t-\tau], \\ -K(t, s, y(s)), & s \in [t-\tau, t]. \end{cases}$$

Then, following the steps for (2.2.5), we can again change it to the form of (2.2.4). Hence, without loss of generality, we only need to consider (2.2.4).

Theorem 2.2.3 *The primary discontinuities of problem (2.2.4) are the points $\xi_\mu := \mu\tau$, $\mu = 0, 1, \cdots$. To be more precise, the derivative $y^{(2\mu+1)}(t)$ is discontinuous at the point ξ_μ, but lower order derivatives are continuous under the assumption that the functions f, K and ϕ are sufficiently smooth.*

Proof: Basically, we use the method of steps. In the first interval $[0, \tau]$,

$$y'(t) = f(t, y(t)) + \int_0^t K(t, s, y(s), \phi(s - \tau))ds.$$

It is possible to satisfy the condition $y(0) = \phi(0)$, but not, in general, also the condition $y'(0+) = \phi'(0-)$. The continuity of the derivative of the solution can be guaranteed at the initial point 0 only for special choices of $\phi(t)$ satisfying the condition $\phi'(0-) = f(0, \phi(0))$.

At the point $t = \tau$, the first derivative of the solution is already continuous. In fact, the derivative

$$y'(t) = f(t, y(t)) + \int_0^t K(t, s, y(s), y(s - \tau))ds,$$

and the right-hand part are continuous functions of t at the point τ, since $y(t)$ is continuous at the point 0. The second derivative

$$y''(t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}y'(t) + K + \int_0^t \frac{\partial K}{\partial t}(t, s, y(s), y(s - \tau))ds,$$

is continuous where we have written $K = K(t, t, y(t), y(t - \tau))$. However,

$y'''(t)$ is not continuous at τ, since it includes $y'(t-\tau)$ as a factor, and $y'(t-\tau)$ is not continuous at τ because $y'(t)$ is not continuous at 0.

At the point $t = 2\tau$, $y^{(4)}(t)$ is continuous, and $y^{(5)}(t)$ is not. At $t = \mu\tau$, we suppose that $y^{(2\mu+1)}(t)$ is not continuous, while all lower order derivatives are. At $t = (\mu+1)\tau$, differentiate (2.2.4) $2\mu+1$ and $2\mu+2$ times, respectively to obtain

$$y^{(2\mu+2)}(t) = \frac{\partial f}{\partial y} y^{(2\mu+1)}(t) + \frac{\partial K}{\partial y(t-\tau)} y^{(2\mu)}(t-\tau) + \text{lower order terms},$$

and

$$y^{(2\mu+3)}(t) = \frac{\partial f}{\partial y} y^{(2\mu+2)}(t) + \frac{\partial K}{\partial y(t-\tau)} y^{(2\mu+1)}(t-\tau) + \text{lower order terms}.$$

According to the hypothesis, $y^{(2\mu+1)}(t)$ is continuous at $t = (\mu+1)\tau$, as is $y^{(2\mu)}(t-\tau)$. As a result, $y^{(2\mu+2)}(t)$ is continuous at $t = (\mu+1)\tau$. Unfortunately, $y^{(2\mu+3)}(t)$ will lose the continuity at $t = (\mu+1)\tau$ as $y^{(2\mu+1)}(t)$ is not continuous at $t = \mu\tau$. By induction, we know that the derivative $y^{(2\mu+1)}(t)$ is not continuous at the point $\mu\tau$, but lower order derivatives are continuous under the smoothness assumption for f and K. $\square$

Remark 2.2.4 The difference between Theorem 2.2.1 and Theorem 2.2.3 certainly has some numerical implications. When the mesh Π_N is not constrained, i.e., $h \neq \tau/r$ for some $r \in \mathbb{N}$, we can expect a higher convergence order for (2.2.4) than for (2.2.1), due to the better regularity property of the solution for the former problem.

Consider now

$$
\begin{aligned}
y'(t) &= f(t, y(t)) + \int_0^t K(t, s, y(s), y(s-\tau(s)))ds, \quad t \geq 0, \quad (2.2.7) \\
y(t) &= \phi(t), \quad t \in [\overline{a}, 0],
\end{aligned}
$$

where $\overline{a} = \inf_{t \geq 0}(t - \tau(t)) < 0$. Here, $0 < \tau(t) < t$ and $t - \tau(t) < t$ are strictly increasing. A result similar to Theorem 2.2.2 holds for (2.2.7).

Theorem 2.2.4 *The primary discontinuities of problem (2.2.7) are generated inductively by the recursion*

$$
\xi_k - \tau(\xi_k) = \xi_{k-1}, \quad k \geq 1,
$$

with $\xi_0 = 0$.

We can also give an analogous result for Volterra integral equations with constant delay of the form

$$
\begin{aligned}
y(t) &= g(t) + \int_0^t K(t, s, y(s), y(s-\tau))ds, \quad t \geq 0, \qquad (2.2.8) \\
y(t) &= \phi(t), \quad t \in [-\tau, 0].
\end{aligned}
$$

The given functions g and K are assumed to be sufficiently smooth.

Theorem 2.2.5 *The primary discontinuities of problem (2.2.8) are located at points* $\xi_\mu := \mu\tau$, $\mu = 0, 1, \cdots$. *More precisely,* $y^{(\mu-1)}$ *and lower order derivatives are continuous at* ξ_μ *but* $y^{(\mu)}$ *is, in general, not, even if the functions* ϕ *and* g *have continuous derivatives of all orders.*

Proof: The solution $y(t)$ of (2.2.8) may be not continuous at the initial point $t = 0$, since in general, $g(0+) \neq \phi(0-)$, unless we make a contrary assumption in advance.

For $t = \tau$, the first derivative is

$$y'(t) = g'(t) + K(t, t, y(t), y(t-\tau)) + \int_0^t \frac{\partial K}{\partial t}(t, s, y(s), y(s-\tau))ds.$$

Clearly, $y'(t)$ is not continuous at τ provided that $y(t)$ is not continuous at 0. The remaining argument is similar to that in the proof of Theorem 2.2.3. We leave the details to the reader. □

Remark 2.2.5 It is worth noticing that, in contrast to Theorem 2.2.1, the primary discontinuities of the integral equation (2.2.8) happen to lower order derivatives. For the integro-differential equation (2.2.4), such discontinuities occur in higher order derivatives as shown in Theorem 2.2.3.

Consider now the neutral Volterra integro-differential equation with constant delay,

$$\begin{aligned} y'(t) &= f(t, y(t)) + \int_0^t K(t, s, y(s), y(s-\tau), y'(s-\tau))ds, \quad t \geq 0, \quad (2.2.9) \\ y(t) &= \phi(t), \quad t \in [-\tau, 0]. \end{aligned}$$

Theorem 2.2.6 *The primary discontinuities of problem (2.2.9) are the points $\xi_\mu := \mu\tau$, $\mu = 0, 1, \cdots$. To be more precise, the derivative $y^{(\mu+1)}(t)$ is discontinuous at the point ξ_μ, but lower order derivatives are continuous whenever the functions f, K and ϕ are sufficiently smooth.*

Proof: Similar to that of Theorem 2.2.3. $\square$

Remark 2.2.6 For the neutral DDE,

$$\begin{aligned} y'(t) &= f(t, y(t), y(t-\tau), y'(t-\tau)), \quad t \geq 0, \qquad (2.2.10) \\ y(t) &= \phi(t), \quad t \in [-\tau, 0], \end{aligned}$$

we do not have results similar to Theorem 2.2.6. Rather, there are two notable differences. First, the initial function $\phi(t)$ for the solution of equation (2.2.10) must be not merely continuous, but also differentiable (or piecewise differentiable), since the last term of (2.2.10) involves the derivative of $\phi(t)$ when $t \in [0, \tau]$. Second, the solution of equation (2.2.10) is not smoothed. In fact, the left-hand derivative $\phi'(0-)$ is not only not equal to $y'(0+)$ at the point 0, but $y'(t)$ is in general discontinuous at the point τ because of the discontinuity of the last argument $y'(t-\tau)$ at $t = \tau$. This line of reasoning shows that the solution $y(t)$ has discontinuities for $t = \mu\tau$, $\mu = 0, 1, 2, \cdots$. Therefore, no smoothing happens to the solution of the neutral delay equation (2.2.10).

Consider the state-dependent delay differential equation

$$y'(t) = f(t, y(t), y(\theta(t, y(t)))), \quad t \geq 0, \qquad (2.2.11)$$

$$y(t) = \phi(t), \quad t \in [\overline{a}, 0], \qquad (2.2.12)$$

where $\overline{a} = \inf_{t \geq 0} \theta(t, y(t)) < 0$ and $\theta(t, y(t)) \leq t$ for $t \geq 0$. θ is called the retarding function. Some classical treatments for (2.2.11) can be found in

[5], [6], [46] and [83]. (2.2.11) is said to be of continuity class $p \geq 1$, if the followings hold over appropriate domains:

1. The partial derivatives $f_{i,j,k}$ are continuous for all $i+j+k \leq p$;

2. The partial derivatives $\theta_{i,j}$ are continuous for all $i+j \leq p$;

3. $\phi \in C^p[\overline{a}, 0]$.

$C_l^p[L-\delta, L+\delta]$ is defined by

$$C_l^p[L-\delta, L+\delta] = C^p[L-\delta, L] \cap C^p[L, L+\delta] \cap C^l[L-\delta, L+\delta].$$

Theorem 2.2.7 *([46]) Let problem (2.2.11) have continuity class $p \geq 1$. For $L \geq 0$, let the integer $l \in [1, p]$ be such that $y \in C_{l-1}^p[L-\delta, L+\delta]$ for some $\delta > 0$. Assume that there exists a least number $Z > L$, such that Z is a zero of multiplicity $m \geq 1$ of $\theta(t, y(t)) - L$. Then $y \in C_{\mu l}^p[Z-\delta, Z+\delta]$ for some $\delta > 0$ where $\mu = p$ if m is even, and $\mu = \min(p, ml)$ if m is odd.*

When (2.2.11) has continuity class p, we expect the solution $y(t)$ has $p+1$ continuous derivatives except at the various derivative jump points.

The idea behind Theorem 2.2.7 is the following. Suppose θ is the retarding function, and L is a discontinuity point. We try to get another point $Z > L$, $\theta(Z, y(Z)) - L = 0$ and an interval $[Z-\eta, Z+\eta]$, such that the range of $\theta(t, y(t))$ for $t \in [Z-\eta, Z+\eta]$ covers $[L-\xi, L+\xi]$, a neighborhood of L. When we calculate the derivative on both side of (2.2.11), and evaluate it at $t = Z$, the discontinuity appears on the right-hand side because

$L = \theta(Z, y(Z))$ is such a point. As a result, the left-hand side is discontinuous at $t = Z$ with a higher order (at least one order higher).

When $\theta(t, y(t)) = t - \tau$, where τ is a positive constant, Theorem 2.2.7 reduces to Theorem 2.2.1. In such a case, the discontinuities are $\mu\tau$ for $\mu = 0, 1, \cdots$, and $y \in C^p_\mu[\mu\tau - \delta, \mu\tau + \delta]$ for some $\delta \in (0, \tau)$.

When $\theta(t, y(t)) = qt$, Theorem 2.2.7 tells us that no discontinuities will occur since we cannot find any $t \in [0, +\infty)$ other than zero such that $qt \leq 0$. See also [59].

We now generalize Theorem 2.2.7 to Volterra integral equations of the form:

$$
\begin{aligned}
y(t) &= g(t) + \int_a^t K(t, s, y(\theta(s, y(s))))ds, \quad t \geq 0, && (2.2.13)\\
y(t) &= \phi(t), \quad t \in [\overline{a}, 0], && (2.2.14)
\end{aligned}
$$

where $\overline{a} = \inf_{t \geq 0} \theta(t, y(t))$ and $\theta(t, y(t)) \leq t$ for $t \geq 0$. Again, by continuity class $p \geq 1$, we mean that the following holds over appropriate domains:

1. The partial derivatives $K_{i,j,k}$ are continuous for all $i + j + k \leq p$;

2. The partial derivatives $\theta_{i,j}$ are continuous for all $i + j \leq p$;

3. $g(t) \in C^p[0, +\infty]$ and $\phi \in C^p[\overline{a}, 0]$.

Theorem 2.2.8 *Let the data in (2.2.13) be in C^p, $p \geq 1$. For $L \geq 0$, let integer $l \in [1, p]$ be such that $y \in C^p_{l-1}[L - \delta, L + \delta]$ for some $\delta > 0$. Assume that there exists a least number $Z > L$, such that Z is a zero of multiplicity*

$m \geq 1$ *of* $\theta(t, y(t)) - L$. *Then* $y \in C^p_\mu[Z-\delta, Z+\delta]$ *for some* $\delta > 0$ *where* $\mu = p$ *if* m *is even, and* $\mu = \min(p, ml)$ *if* m *is odd.*

Remark 2.2.7 Note that (2.2.11) and (2.2.13) are not identical. Differentiation of (2.2.13) leads to

$$y'(t) = g'(t) + \int_a^t K_1'(t, s, y(\theta(s, y(s))))ds + K(t, t, y(\theta(t, y(t)))), \quad (2.2.15)$$

where $K_1' = \partial K / \partial t$. There is an additional integral term on the right-hand side.

Proof of Theorem 2.2.8: Suppose $L - \xi \leq \theta(t, y(t)) \leq L + \xi$ for $t \in [Z-\eta, Z+\eta]$. Let $w(t) = \theta(t, y(t))$ and

$$\mathbf{W}(t) = (t, t, y(\theta(t, y(t)))).$$

Then $\mathbf{W}^{(1)}(t) = (1, 1, y^{(1)}(\theta)\theta^{(1)})$, and

$$\mathbf{W}^{(k)}(t) = (0, 0, \sum_{Q=1}^{k} v_{kQ} y^{(Q)}(\theta)),$$

for $k \geq 2$. Here, $\theta^{(1)} = d\theta(t, y(t))/dt$, and

$$v_{kQ} = \sum \frac{k!}{j_1! \cdots j_k!} (\frac{\theta^{(1)}(t)}{1!})^{j_1} \cdots (\frac{\theta^{(k)}(t)}{k!})^{j_k}.$$

The sum is taken over all k-tuples of nonnegative integers $(j_1, \cdots, j_k)$ that satisfy $j_1 + \cdots + j_k = Q$ and $j_1 + 2j_2 + \cdots + kj_k = k$. Denote scalar function $K_1^{(i)}(t, t, y(\theta(t, y(t))))$ simply by $K_1^{(i)}$, that is, $K_1^{(i)} := K_1^{(i)}(\mathbf{W}(t))$. Then

$$y''(t) = g''(t) + \int_a^t K_1'' ds + K' + K_1',$$

$$y^{(3)}(t) = g^{(3)}(t) + \int_a^t K_1'''ds + K'' + K_1'' + K_1'',$$

$$y^{(k+1)}(t) = g^{(k+1)}(t) + \int_a^t K_1^{(k+1)}ds + K_1^{(k)} + \sum_{i=1}^{k} K_1^{(i-1)(k-i+1)},$$

and

$$y^{(k+1)}(t) = g^{(k+1)}(t) + \int_a^t K_1^{(k+1)}ds + K_1^{(k)} \tag{2.2.16}$$
$$+ \sum_{i=1}^{k}\sum_{j=1}^{k-i+1}\sum \sigma_{k-i+1} \nabla\left(\cdots \nabla\left(\nabla K_1^{(i-1)} \circ \mathbf{W}^{(i_1)}\right) \circ \mathbf{W}^{(i_2)} \cdots\right) \circ \mathbf{W}^{(i_j)}.$$

Observe that the highest-order derivative of $\mathbf{W}$ occurs when $i = j = 1$ in (2.2.16). The term is $\nabla K \circ \mathbf{W}^{(k)}$. Consequently the highest order derivative of y in any term on the right side of (2.2.16) is the k-th derivative. Since $y(t)$ is continuous, it follows from (2.2.15) that $y'(t) = y^{(1)}(t)$ is the composition of continuous functions, hence itself continuous at $t = Z$. Since (2.2.13) has continuity class $p \geq 1$, it is easy to show by induction from (2.2.16) that $y^{(k)}$ is continuous at $t = Z$ for all $k \leq l$ since $l \leq p$. The induction terminates at the l-th derivative because $y(\theta(t, y(t)))$ need not necessarily have more than $l - 1$ derivatives at $t = Z$. This bound on k can be improved.

Let m be even. Then $\theta(t, y(t)) - L$ remains either nonnegative or nonpositive in some neighborhood of $t = Z$. In other words, $\theta(t, y(t))$ for t in a neighborhood of Z does not range over intervals containing the jump point at $t = L$. Hence $y(\theta(t, y(t)))$ for t in a neighborhood of Z could have more than l continuous derivatives. It is easy to show by induction from (2.2.16) that $y^{(p)}$ is continuous at $t = Z$. This establishes the first case.

Let m be odd. Then either $\theta(t, y(t)) - L$ changes sign at Z or Z is a cluster point of zeros of $\theta(t, y(t)) - L$. In either case Z may be a derivative jump point. It is easy to verify that the derivatives up to order $ml - 1$ of $y(\theta(t, y(t)))$ that could be discontinuous at $t = Z$ in (2.2.16) are actually multiplied by appropriate derivatives of order up to $m - 1$ of $\theta(t, y(t))$ which are continuous and which by hypotheses vanish at $t = Z$. Thus the effect of the discontinuities at $t = Z$ are nullified. This completes the proof of the remaining case. $\square$

We add an example, also to indicate that Theorem 2.2.8 remains valid for more general DVIEs.

Example 2.2.3 Consider

$$\begin{aligned} y(t) &= 1 + \int_1^t \frac{1}{s} y(s) y(\ln y(s)) ds, \quad \text{when} \quad t \geq 1, \qquad (2.2.17) \\ y(t) &= 1, \quad \text{when} \quad t \leq 1. \end{aligned}$$

By the step method, we get the solution

$$y(t) = \begin{cases} t, & \text{when} \quad 1 \leq t \leq e, \\ \exp(t/e), & \text{when} \quad e \leq t \leq e^2. \end{cases}$$

It is clear that $y \in C_0^\infty[1 - \delta, 1 + \delta]$ with $0 < \delta \ll 1$ since $y'(1-) = 0$ and $y'(1+) = 1$. The root of $\theta(t, y(t)) - L = \ln y(t) - 1 = 0$ is $y(t) = e$, i.e., $t = e$ with single multiplicity. According to Theorem 2.2.8, $y \in C_1^\infty[e - \delta, e + \delta]$ with $0 < \delta \ll 1$. In fact, $y'(e) = 1$, $y''(e-) = 0$ and $y''(e+) = 1/e$.

The proof of Theorem 2.2.8 can be readily modified to establish an analogous result for Volterra integro-differential equations with state-dependent delay,

$$
\begin{aligned}
y'(t) &= f(t, y(t)) + \int_a^t K(t, s, y(\theta(s, y(s))))ds, \quad t \geq 0, &(2.2.18)\\
y(t) &= \phi(t), \quad t \in [\overline{a}, 0], &(2.2.19)
\end{aligned}
$$

where $\overline{a} = \inf_{t\geq 0} \theta(t, y(t))$ and $\theta(t, y(t)) \leq t$ for $t \geq 0$.

Theorem 2.2.9 *Let problem (2.2.18), (2.2.19) have continuity class $p \geq 1$. For $L \geq 0$, let integer $l \in [1, p]$ be such that $y \in C^p_{l-1}[L-\delta, L+\delta]$ for some $\delta > 0$. Assume that there exists a least number $Z > L$, such that Z is a zero of multiplicity $m \geq 1$ of $\theta(t, y(t)) - L$. Then $y \in C^p_\mu[Z-\delta, Z+\delta]$ for some $\delta > 0$ where $\mu = p$ if m is even, and $\mu = \min(p, ml)$ if m is odd.*

Chapter 3

Collocation for Differential and Volterra Integro-differential Equations with Constant Delay

In this chapter, we review some known results related to constant delay problems in order to compare them with similar results for equations with proportional delay. We describe the collocation method for constant delay equations in Section 3.1. We present the global convergence results in Section 3.2. Local convergence results are examined in Section 3.3. In Section 3.4, we survey results about delay Volterra integro-differential equations of neutral type.

Consider the first-order differential equation of the form

$$\begin{aligned} y'(t) &= f(t, y(t), y(t-\tau)), \quad t \in I, \\ y(t) &= \phi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{3.0.1}$$

where $\tau > 0$ is a constant and $I := [0, T]$, the Volterra integro-differential equations with constant delay,

$$y'(t) = f(t, y(t)) + \int_0^t k(t, s, y(s), y(s-\tau))ds, \quad t \in I, \tag{3.0.2}$$

and

$$y'(t) = f(t, y(t)) + \int_{t-\tau}^t k(t, s, y(s))ds, \quad t \in I, \tag{3.0.3}$$

with initial conditions as in (3.0.1).

Our primary goal is to find the collocation solutions u in $S_m^{(0)}(\Pi_N)$ for (3.0.1), (3.0.2) and (3.0.3) with respect to the Gauss points, and study certain aspects related to such an approximation. Note that (3.0.3) can always be changed into the form of (3.0.2) by Remark 2.2.3.

The reader who is interested in Volterra integral equations with constant delay may find results and references in [16]. Collocation methods for (classical) Volterra integral and integro-differential equations are described in [21].

3.1 Collocation methods

For ease of exposition, we choose a uniform mesh Π_N on I, given by $t_n := nh$, $n = 0, 1, \cdots, N$, $t_N = T$, and set $\Pi_N := \{t_0, t_1, \cdots, t_N\}$, $I_0 := [t_0, t_1]$, $I_n := (t_n, t_{n+1}]$, $n \geq 1$. We assume that Π_N is a constrained mesh, i.e.,

$$h = \tau/r, \qquad \text{for some} \quad r \in \mathbf{N}. \tag{3.1.1}$$

The motivation for choosing such meshes is to include the primary discontinuities of the solution in the mesh. The use of arbitrary meshes will in general result in a loss of order of convergence due to the presence of primary discontinuities.

3.1.1 Collocation for delay differential equations

We solve (3.0.1) in space $S_m^{(0)}(\Pi_N)$. For given real numbers $\{c_j\}$ with $0 \leq c_1 < c_2 < \cdots < c_m \leq 1$, define the set $X_N := \{t_{n,j}\}$ of collocation points by

$$t_{n,j} := t_n + c_j h, \quad j = 1, 2, \cdots, m, \quad n = 0, 1, \cdots, N-1. \tag{3.1.2}$$

The collocation solution $u \in S_m^{(0)}(\Pi_N)$ of (3.0.1) is defined by

$$v'(t_n + c_j h) = f\left(t_n + c_j h, v(t_n + c_j h), v(t_{n-r} + c_j h)\right), \tag{3.1.3}$$

for $j = 1, 2, \cdots, m$, $n = 0, 1, \cdots, N-1$, subject to the initial condition $v(t) = \phi(t)$, when $t \in [-\tau, 0]$. We may write

$$v'(t_n + sh) = \sum_{l=1}^{m} V_{n,j} L_l(s), \quad s \in [0, 1],$$

where $V_{n,j} := v'(t_n + c_j h)$. Upon integrating and setting

$$\alpha_j(t) := \int_0^t L_j(s)ds, \tag{3.1.4}$$

where

$$L_j(t) := \prod_{i \neq j}^{m} \frac{t - c_i}{c_j - c_i}, \quad j = 1, 2, \cdots, m,$$

are the Lagrange fundamental polynomials with respect to the collocation parameters $\{c_i\}$, we have the (local) representation

$$v(t_n + sh) = v(t_n) + h \sum_{l=1}^{m} \alpha_l(s) V_{n,l}, \quad s \in [0, 1], \tag{3.1.5}$$

where $V_{n,l}$ is determined by (3.1.3), namely

$$V_{n,j} = f\left(t_n + c_j h, v_n + h \sum_{l=1}^{m} V_{n,l}\alpha_{j,l}, v_{n-r} + h \sum_{l=1}^{m} V_{n-r,l}\alpha_{j,l}\right),$$

where $\alpha_{i,j} := \alpha_j(c_i), \quad i, j = 1, 2, \cdots, m.$

We introduce some other notations and properties associated with Lagrange fundamental polynomials for future use.

$$L_j(c_i) = \delta_{ij}, \quad \delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases} \quad i, j = 1, 2, \cdots, m,$$

and

$$\sum_{i=1}^{m} L_i(s) = 1, \quad \sum_{j=1}^{m} \alpha_{i,j} = c_i, \quad \text{for} \quad i = 1, 2, \cdots, m.$$

3.1.2 Collocation for delay integro-differential equations

Consider the Volterra integro-differential equation with constant delay τ,

$$y'(t) = f(t, y(t)) + \int_{t-\tau}^{t} k(t, s, y(s))ds, \quad t \in I, \tag{3.1.6}$$

with initial condition

$$y(t) = \phi(t), \quad t \in [-\tau, 0]. \tag{3.1.7}$$

Here, ϕ is a given C^1-function.

The collocation solution $u \in S_m^{(0)}(\Pi_N)$ to (3.1.6) and (3.1.7) is given by

$$u'(t) = f(t, u(t)) + \int_{t-\tau}^{t} k(t, s, u(s))ds, \quad t \in X_N, \tag{3.1.8}$$

with $u(t) = \phi(t)$, $t \in [-\tau, 0]$. Define

$$F_n(t) := \Phi(t) + \int_{0}^{t_n} k(t, s, u(s))ds, \quad t \in [t_n, t_{n+1}], \quad \text{if} \quad 0 \leq n < r, \tag{3.1.9}$$

and

$$F_n(t) := \int_{t-\tau}^{t_n} k(t, s, u(s))ds, \quad t \in [t_n, t_{n+1}], \quad \text{if} \quad r \leq n \leq N-1, \tag{3.1.10}$$

where r is as in (3.1.1), and $\Phi(t)$ denotes the delay integral

$$\Phi(t) := \int_{t-\tau}^{0} k(t, s, \phi(s))ds, \quad t \in [0, \tau]. \tag{3.1.11}$$

Using the notation introduced in Section 3.1.1, equation (3.1.8) defining the exact collocation solution $u \in S_m^{(0)}(\Pi_N)$ to (3.1.6) and (3.1.7) may be rewritten as

$$Y_{n,j} = f(t_n + c_j h, U_{n,j}) + Z_{n,j} + F_n(t_n + c_j h), \quad j = 1, \cdots, m,$$

with

$$U_{n,j} = u(t_n + c_j h) = y_n + h \sum_{l=1}^{m} \alpha_{j,l} Y_{n,l}, \quad \alpha_{j,l} := \alpha_l(c_j),$$

and

$$Z_{n,j} := h \int_0^{c_j} k(t_n + c_j h, t_n + vh, u(t_n + vh)) dv.$$

On the interval $[t_n, t_{n+1}]$, $0 \leq n \leq N-1$, the collocation solution is given by

$$u(t_n + vh) = y_n + h \sum_{j=1}^{m} \alpha_j(v) Y_{n,j}, \quad v \in [0,1].$$

In contrast to collocation for ODEs or DDEs, which yields an m-stage implicit Runge-Kutta method, the above method for delay VIDE involves integrals which in general cannot be calculated analytically, and thus an additional discretization step is necessary. The resulting method is a continuous implicit Volterra-Runge-Kutta method with m stages: if the discretization of these integrals employs m-point interpolatory quadrature formulas based on the collocation parameters $\{c_j\}$, then this method is described by the equations (3.1.12)-(3.1.17):

$$\hat{u}(t_n + vh) = \hat{y}_n + h \sum_{j=1}^{m} \alpha_j(v) \hat{Y}_{n,j}, \quad v \in [0,1], \tag{3.1.12}$$

where

$$\hat{Y}_{n,j} = f(t_n + c_j h, \hat{U}_{n,j}) + \hat{Z}_{n,j} + \hat{F}_n(t_n + c_j h), \tag{3.1.13}$$

for $j = 1, 2, \cdots, m$, with

$$\hat{U}_{n,j} = \hat{y}_n + h \sum_{l=1}^{m} \alpha_{j,l} \hat{Y}_{n,l}, \tag{3.1.14}$$

and

$$\hat{Z}_{n,j} := h \sum_{\mu=1}^{m} w_{j,\mu} k\left(t_n + c_j h, t_n + \xi_{j,\mu} h, \hat{u}(t_n + \xi_{j,\mu} h)\right). \tag{3.1.15}$$

Here, $w_{j,\mu} := c_j b_\mu$, with $b_\mu := \alpha_\mu(1)$ and $\xi_{j,\mu} := c_j c_\mu$. The lag term approximations $\hat{F}_n(t)$ in (3.1.13) corresponding, respectively, to the exact lag terms (3.1.9) and (3.1.10) are

$$\hat{F}_n(t_{n,j}) := \Phi(t_{n,j}) + h \sum_{i=0}^{n-1} \sum_{\mu=1}^{m} b_\mu k(t_{n,j}, t_{i,\mu}, \hat{U}_{i,\mu}), \tag{3.1.16}$$

if $0 \le n < r$, and

$$\begin{aligned} \hat{F}_n(t_{n,j}) \; &:= \; h \sum_{\mu=1}^{m} \tilde{w}_{j,\mu} k(t_{n,j}, t_{n-r} + \eta_{j,\mu} h, \hat{u}(t_{n-r} + \eta_{j,\mu} h)) \\ &\quad + h \sum_{i=n-r+1}^{n-1} \sum_{\mu=1}^{m} b_\mu k(t_{n,j}, t_{i,\mu}, \hat{U}_{i,\mu}), \end{aligned} \tag{3.1.17}$$

if $r \le n \le N-1$. Here, $\tilde{w}_{j,\mu} := (1 - c_j) b_\mu$ and $\eta_{j,\mu} := c_j + (1 - c_j) c_\mu$.

3.2 Global convergence of collocation solutions

For the sake of later comparison, we recall the following convergence result for DDEs of the form (3.0.1) from [9].

Theorem 3.2.1 *Suppose $f(t, v, w)$ and ϕ in (3.0.1) have derivatives of order m which are piecewise continuous on their domains; $y(t)$ is the exact solution* *of (3.0.1) and $u \in S_m^{(0)}(\Pi_N)$ is the corresponding constrained-mesh collocation solution with collocation parameters $0 \leq c_1 < \cdots < c_m \leq 1$. Then*

$$||y - u||_\infty \leq Ch^p, \quad p = m.$$

Proof: See [9] where the author gave a proof for an even more general case, $\tau = \tau(t) \geq 0$. □

Remark 3.2.1 It should be pointed out that in [9], the author claimed a higher order of global convergence, i.e., $m+1$ under the condition $\tau = \tau(t) \geq 0$. However, in general, this is not true.

We now turn to the delay integro-differential equation (3.1.6). Let $u \in S_m^{(0)}(\Pi_N)$ denote the exact collocation solution to (3.1.6) defined by (3.1.8)–(3.1.11).

For ease of exposition, we choose the linear version of (3.1.6),

$$y'(t) = f(t)y(t) + g(t) + \int_{t-\tau}^{t} k(t,s)y(s)ds, \quad t \in I, \tag{3.2.1}$$

with $y(t) = \phi(t)$ for $t \in [-\tau, 0]$, where $k \in C(S_\tau)$, $S_\tau := I \times [-\tau, T-\tau]$.

Theorem 3.2.2 *(see [18]) Assume that the given functions in (3.2.1) and (3.1.7) satisfy $f \in C^m(I)$, $k \in C^m(S_\tau)$, $\phi \in C^m[-\tau, 0]$, and that, for $t \in [0, \tau]$, the integral*

$$\Phi(t) := \int_{t-\tau}^{0} k(t,s)\phi(s)ds, \quad t \in [0, \tau], \tag{3.2.2}$$

is known exactly. Then for all sufficiently small $h = \tau/r$, $r \in \mathbf{N}$, *the constrained-mesh collocation solution* $u \in S_m^{(0)}$ *to (3.2.1) satisfies*

$$||y - u||_\infty \le Ch^m,$$

for some finite C *not depending on* h. *This estimate holds for all collocation parameters* $\{c_j\}$ *with* $0 \le c_1 < \cdots < c_m \le 1$.

3.3 Local superconvergence on Π_N

In Section 3.2, we saw that globally, we can expect a convergence order of m if the collocation solution is in $S_m^{(0)}(\Pi_N)$. When we focus on some special points, (i.e., the mesh points), we certainly expect to attain a higher order.

Definition 3.3.1 Let $y(t)$ and $u(t)$ be the exact solution and corresponding collocation solution of (3.0.1) respectively. If

$$\max_{1\le n\le N} |y(t_n) - u(t_n)| \le Ch^{p^*},$$

where $p^* > p$, with p as in Theorem 3.2.1, then p^* is called the local superconvergence order of the collocation solution.

Theorem 3.3.1 *Suppose* $f(t, v, w)$ *and* ϕ *in (3.0.1) have derivatives of order* $2m$ *which are continuous on their domains. If* $y(t)$ *is the exact solution of (3.0.1) and* $u \in S_m^{(0)}(\Pi_N)$ *is the corresponding constrained-mesh collocation solution with collocation parameters* $0 \le c_1 < \cdots < c_m \le 1$, *then*

$$\max_{1\le n\le N} |y(t_n) - u(t_n)| \le Ch^{p^*}, \quad p^* \le 2m,$$

i.e., the superconvergence order p^* is at most $2m$. More precisely, we have:

(i) If the collocation parameters $\{c_j\}$ are the Gauss points in $(0,1)$, i.e., the zeros of the (shifted) Legendre polynomial $P_m(2s-1)$, then $p^* = 2m$, while u' possesses a lower order of convergence on the mesh Π_N:

$$\max_{1\leq n\leq N} |y'(t_n) - u'(t_n)| \leq C_1 h^m.$$

(ii) If the $\{c_j\}$ are the Radau II points, which are zeros of $P_m(2s-1) - P_{m-1}(2s-1)$, then

$$\max_{1\leq n\leq N} |y^{(l)}(t_n) - u^{(l)}(t_n)| \leq C_l h^{2m-1}, \quad l = 0, 1.$$

(iii) If the $\{c_j\}$ are the Radau I points which are zeros of $P_m(2s-1) + P_{m-1}(2s-1)$, i.e., $0 = c_1 < c_2 < \cdots < c_m < 1$, then

$$\max_{1\leq n\leq N} |y(t_n) - u(t_n)| \leq C_0 h^{2m-1},$$

and

$$\max_{1\leq n\leq N} |y'(t_n) - u'(t_n)| \leq C_1 h^m.$$

Proof: See [9]. □

The local superconvergence results of Theorem 3.3.1 remain true for delay integro-differential equations (3.2.1):

Theorem 3.3.2 *Assume that the given functions in (3.2.1) are sufficiently smooth on their domains, i.e., they are in C^{m+d} for some d with $0 \leq d \leq$*

m, and let the delay integral Φ in (3.2.2) be known exactly. Then for all sufficiently small $h = \tau/r$, $r \in \mathbb{N}$, the constrained-mesh collocation solution $u \in S_m^{(0)}(\Pi_N)$ to (3.2.1) is uniquely defined, and has the following properties:

(i) If the collocation parameters $\{c_j\}$ are the Gauss points in $(0,1)$, then

$$\max_{1\leq n\leq N} |y(t_n) - u(t_n)| \leq C_0 h^{2m}, \tag{3.3.1}$$

for some finite constant C_0, provided $d = m$, while u' possesses a lower order of convergence on the mesh Π_N:

$$\max_{1\leq n\leq N} |y'(t_n) - u'(t_n)| \leq C_1 h^{m}. \tag{3.3.2}$$

(ii) If the $\{c_j\}$ are the Radau II points and $d = m-1$, then

$$\max_{1\leq n\leq N} |y^{(l)}(t_n) - u^{(l)}(t_n)| \leq C_l h^{2m-1}, \quad l = 0, 1. \tag{3.3.3}$$

(iii) If the $\{c_j\}$ are the Radau I points and $d = m-1$, i.e., $0 = c_1 < c_2 < \cdots < c_m < 1$, then

$$\max_{1\leq n\leq N} |y(t_n) - u(t_n)| \leq C_0 h^{2m-1}, \tag{3.3.4}$$

and

$$\max_{1\leq n\leq N} |y'(t_n) - u'(t_n)| \leq C_1 h^{m}. \tag{3.3.5}$$

Proof: We proceed along the lines of [18]. The collocation error, $e(t) := y(t) - u(t)$, is the solution of the initial-value problem

$$e'(t) = f(t)e(t) + \delta(t) + G(t) + \int_0^t k(t,s)e(s)ds, \quad t \in I, \tag{3.3.6}$$

where

$$G(t) = \int_0^{t-\tau} H(t,s)e(s)ds = -\int_0^{t-\tau} k(t,s)e(s)ds, \tag{3.3.7}$$

$e(t) = 0$ on $[-\tau, 0]$. The defect (or residual) function $\delta(t)$ given by

$$\delta(t) := -u'(t) + f(t)u(t) + g(t) + \int_{t-\tau}^{t} k(t,s)u(s)ds,$$

vanishes on the set X_N of collocation points and satisfies $\delta(t) = 0$ for $t < 0$.

Setting $z_1(t) = e(t)$, $z_2(t) = e'(t)$, $\mathbf{z}(t) = (z_1(t), z_2(t))^T$, and writing $z_1(t) = \int_0^t z_2(s)ds$, the VIDE (3.3.6) may be written as a system of two Volterra integral equations of the second kind,

$$\mathbf{z}(t) = \mathbf{D}(t) + \int_0^t \mathbf{k}(t,s)z(s)ds, \quad t \in I, \tag{3.3.8}$$

with

$$\mathbf{D}(t) = \begin{pmatrix} 0 \\ \delta(t) + G(t) \end{pmatrix}, \quad \mathbf{k}(t,s) = \begin{pmatrix} 0 & 1 \\ k(t,s) & f(t) \end{pmatrix}.$$

Let

$$R(t,s) = \begin{pmatrix} R_{11}(t,s) & R_{12}(t,s) \\ R_{21}(t,s) & R_{22}(t,s) \end{pmatrix}$$

denote the resolvent of $\mathbf{k}(t,s)$ in (3.3.8). Note that by definition of R, its smoothness is governed by the smoothness of k and f. The solution of (3.3.8) is then given by

$$\mathbf{z}(t) := \mathbf{D}(t) + \int_0^t R(t,s)\mathbf{D}(s)ds, \quad t \in I,$$

and hence we obtain the representations

$$e(t) := \int_0^t R_{12}(t,s)\{\delta(s) + G(s)\}ds, \tag{3.3.9}$$

and

$$e'(t) := \delta(t) + G(t) + \int_0^t R_{22}(t,s)\{\delta(s) + G(s)\}ds, \tag{3.3.10}$$

where $t \in [0, T]$. For $t \in [0, \tau]$, we have $G(t) = 0$, since by assumption the delay integrals in the lag term (3.1.9),

$$\Phi(t) := \int_{t-\tau}^0 H(t,s)\phi(s)ds,$$

are evaluated analytically.

Now, we shall show that (3.3.9) and (3.3.10) can be rewritten to yield representations of the collocation error and its derivative in terms of the defect function $\delta(t)$. Since this is key to the proof of Theorem 3.3.2, we summarize the result in Lemma 3.3.1 ([18])

Lemma 3.3.1 *Let $\xi_\mu = \mu\tau$, $\mu = 0, 1, \cdots, M$, and assume, without loss of generality, that $\xi_M = T$ for some $M \in N$. If $t \in [\xi_\mu, \xi_{\mu+1}]$, $\mu = 0, 1, \cdots, M-1$, then*

$$e(t) = \sum_{i=0}^{\mu} \int_0^{t-i\tau} Q_{\mu,i}^{[0]}(t,s)\delta(s)ds, \tag{3.3.11}$$

and

$$e'(t) = \delta(t) + \sum_{i=0}^{\mu} \int_0^{t-i\tau} Q_{\mu,i}^{[1]}(t,s)\delta(s)ds, \tag{3.3.12}$$

where the $Q^{[l]}_{\mu,i}$ (with $Q^{[l]}_{\mu,0} = R_{l2}(t,s)$, $l = 0,1$), are functions which depend on the given kernel function k and on f in (3.2.1), and whose smoothness is determined by that of these given functions.

To prove Lemma 3.3.1, let $t \in [\tau, 2\tau]$ $(= [\xi_1, \xi_2])$. It follows from (3.3.7), (3.3.9) and (3.3.10), that

$$\begin{aligned} G(t) &= \int_0^{t-\tau} H(t,s)e(s)ds = \int_0^{t-\tau} H(t,s)\int_0^s R_{12}(s,v)\delta(v)dvds \\ &= \int_0^{t-\tau}\left(\int_v^{t-\tau} H(t,s)R_{12}(s,v)ds\right)\delta(v)dv, \end{aligned}$$

or

$$G(t) = \int_0^{t-\tau} Q_0(t,s)\delta(s)ds, \quad t \in [\tau, 2\tau],$$

with obvious meaning of $Q_0(t,s)$. Since on $[\tau, 2\tau]$ the solution of the error equation (3.3.6) is given by (3.3.9) and (3.3.10), and the defect function $\delta(t)$ vanishes on the interval $[-\tau, 0]$, we find that for $k = 1, 2$,

$$\begin{aligned} \int_0^t R_{k2}(t,s)G(s)ds &= \int_0^t R_{k2}(t,s)\int_0^{s-\tau} Q_0(s,v)\delta(v)dvds \\ &= \int_0^{t-\tau}\left(\int_{v+\tau}^t R_{k2}(t,s)Q_0(s,v)ds\right)\delta(v)dv. \end{aligned}$$

The inductive extension of these results to an arbitrary interval $[\xi_\mu, \xi_{\mu+1}]$ is now straightforward.

Consider (3.3.11), (3.3.12) and choose $t = t_n \in [\xi_\mu, \xi_{\mu+1}]$. Note that $t_n - i\tau = t_{n-ir}$, since $\tau = rh$ from (3.1.1). Setting

$$\Psi^{[l]}_{n,i}(t_k + vh) := Q^{[l]}_{\mu,i}(t_n, t_k + vh)\delta(t_k + vh), \quad 0 \le i \le \mu,$$

we may write (3.3.11) and (3.3.12) as

$$e^{(l)}(t) := l \cdot \delta(t_n) + h \sum_{i=0}^{\mu} \sum_{k=0}^{n-ir-1} \int_0^1 \Psi_{n,i}^{[l]}(t_k + vh)dv, \quad l = 0, 1. \quad (3.3.13)$$

Replace each integral over $[0, 1]$ by the sum of its (interpolatory) m-point quadrature formula (with the collocation points as abscissas) and the corresponding quadrature error $E_{i,k}^{(n,l)}$. Note that, by our assumption on the exact delay integral $\Phi(t)$, we have $E_{i,k}^{(n,l)} = 0$ for $0 \leq n < r$. Since the defect function $\delta(t)$ vanishes at $t = t_k + c_j h \in X_N$, we have $\Psi_{n,i}^{[l]}(t_k + c_j h) = 0$, and thus the above expression (3.3.13) for $e^{(l)}(t_n)$, $l = 0, 1$, reduces to

$$e^{(l)}(t) := l \cdot \delta(t_n) + h \sum_{i=0}^{\mu} \sum_{k=0}^{n-ir-1} E_{i,k}^{(n,l)}, \quad (3.3.14)$$

where $l = 0, 1$, $M\tau = T$ and $0 \leq \mu \leq n \leq \mu + 1 \leq M$. Since by assumption the integrands $\Psi_{n,i}^{[l]}(t_k + vh)$ are in $C^{m+d}[0, 1]$, it follows from Peano's Theorem [86] that, for sufficiently smooth integrands, the quadrature errors $E_{i,k}^{(n,l)}$ associated with the interpolatory quadrature formulas employed in (3.3.11) and (3.3.12) can be bounded by

$$|E_{i,k}^{(n,l)}| \leq Q_l h^{m+d},$$

where $d = m$ for the Gauss points and $d = m - 1$ for the Radau II points. Hence, for $l = 0$, (3.3.13) yields the uniform estimate

$$|e(t_n)| \leq h \sum_{i=0}^{\mu} \sum_{k=0}^{n-ir-1} |E_{i,k}^{(n,0)}| \leq Q_0 h^{m+d} \cdot M \cdot N \cdot h =: C_0 h^{m+d},$$

$n = 1, 2, \cdots, N$, since $Nh = T$, and $M = T/\tau$ is a fixed integer. Thus, the statements (3.3.1) and (3.3.3) (with $l = 0$) follow readily.

We note that (3.3.3), when $l = 0$, is also valid for the Radau I points which are zeros of $P_m(2s-1) + P_{m-1}(2s-1)$. So we have (3.3.4).

Consider now (3.3.14) with $l = 1$. If the collocation parameters $\{c_j\}$ are such that $c_m = 1$, then $t_n \in X_N$ and hence $\delta(t_n) = 0$. This holds in particular for the Radau II points, and hence we obtain (3.3.3) with $l = 1$. If the $\{c_j\}$ are the Gauss points or Radau I points, then $c_m < 1$ and thus, in general, $\delta(t_n) \neq 0$ in (3.3.14):

$$|e'(t)| \leq |\delta(t_n)| + C_1 h^{m+d}, \quad n = 1, 2, \cdots, N.$$

It follows from the global convergence analysis (cf. Theorem 3.2.2) that in these two cases the defect δ behaves like $\delta(t_n) = \mathcal{O}(h^m)$ in general, implying the results (3.3.2) and (3.3.5). $\square$

Remark 3.3.1 The local superconvergence results of Theorem 3.3.2 remain true for nonlinear delay VIDEs of the form

$$y'(t) = f(t, y(t)) + (Vy)(t), \quad t \in I, \tag{3.3.15}$$

where the operator V is given by

$$(Vy)(t) := \int_0^t k_1(t, s, y(s))ds + \int_0^{t-\tau} k_2(t, s, y(s))ds.$$

This can be verified by using linearization techniques (see, for example, [15] and [51]). Since the equation for the collocation error contains the terms $(Vy)(t) - (Vu)(t)$ we may write, under the standard smoothness and boundedness hypotheses on the kernels k_1 and k_2,

$$k_i(t,s,y(s)) - k_i(t,s,u(s)) = \frac{\partial k_i}{\partial y} \cdot e(s) + \mathcal{O}(h^{2m}),$$

where the partial derivative of k is evaluated at $(t, s, y(s))$. The $\mathcal{O}(h^{2m})$-term stems from terms involving $e^2(s)$ and makes use of the nonlinear version of the global convergence result in Theorem 3.2.2. The delay VIDE (3.1.6) is a particular case of (3.3.15): it corresponds to the choice $k_2(t, s, y) = -k_1(t, s, y)$ in the above operator V.

3.4 Extension of results to neutral DVIDEs

Consider the neutral Volterra integro-differential equation with constant delay τ,

$$\begin{aligned} y'(t) &= f(t,y(t)) + \int_{t-\tau}^{t} k(t,s,y(s),y'(s))ds, \quad t \in I, & (3.4.1) \\ y(t) &= \phi(t), \quad t \in [-\tau, 0]. & (3.4.2) \end{aligned}$$

Here, ϕ is a given C^1-function. The discretization of such problem is studied in [18].

3.4.1 Collocation for neutral DVIDEs

The collocation solution $u \in S_m^{(0)}(\Pi_N)$ to (3.4.1) and (3.4.2) is given by the equation

$$u'(t) = f(t,u(t)) + \int_{t-\tau}^{t} k(t,s,u(s),u'(s))ds, \quad t \in X_N, \qquad (3.4.3)$$

subject to initial condition $u(t) = \phi(t)$, $t \in [-\tau, 0]$. Define

$$F_n(t) := \Phi(t) + \int_0^{t_n} k(t, s, u(s), u'(s))ds, \quad t \in [t_n, t_{n+1}], \quad \text{if } 0 \le n < r, \tag{3.4.4}$$

and

$$F_n(t) := \int_{t-\tau}^{t_n} k(t, s, u(s), u'(s))ds, \quad t \in [t_n, t_{n+1}], \quad \text{if } r \le n \le N-1, \tag{3.4.5}$$

where r is as in (3.1.1), and $\Phi(t)$ denotes the delay integral

$$\Phi(t) := \int_{t-\tau}^{0} k(t, s, \phi(s), \phi'(s))ds, \quad t \in [0, \tau].$$

As in Section 3.1.1, we set

$$u'(t_n + vh) = \sum_{l=1}^{m} L_l(v) Y_{n,l}, \quad Y_{n,l} := u'(t_n + c_l h),$$

and

$$u(t_n + vh) = y_n + h \sum_{l=1}^{m} \alpha_l(v) Y_{n,l}, \quad v \in [0, 1], \quad y_n := u(t_n).$$

Then (3.4.3) may be rewritten as

$$Y_{n,j} = f(t_n + c_j h, U_{n,j}) + Z_{n,j} + F_n(t_n + c_j h), \tag{3.4.6}$$

for $j = 1, \cdots, m$, with

$$U_{n,j} = u(t_n + c_j h) = y_n + h \sum_{l=1}^{m} \alpha_{j,l} Y_{n,l}, \quad \alpha_{j,l} := \alpha_l(c_j),$$

and

$$Z_{n,j} := h \int_0^{c_j} k\left(t_n + c_j h, t_n + vh, u(t_n + vh), u'(t_n + vh)\right) dv.$$

On the interval $[t_n, t_{n+1}]$, $0 \leq n \leq N-1$, the collocation solution is given by

$$u(t_n + vh) = y_n + h \sum_{j=1}^{m} \alpha_j(v) Y_{n,j}, \quad v \in [0,1]. \tag{3.4.7}$$

The above method for neutral delay VIDE involves again integrals which cannot be calculated analytically, and thus an additional discretization step is necessary. If the discretization of these integrals employs m-point interpolatory quadrature formulas based on the collocation parameters $\{c_j\}$, then this method is described by the equations (3.4.8)-(3.4.13) (compare with (3.1.12)-(3.1.17)):

$$\hat{u}(t_n + vh) = \hat{y}_n + h \sum_{j=1}^{m} \alpha_j(v) \hat{Y}_{n,j}, \quad v \in [0,1], \tag{3.4.8}$$

where

$$\hat{Y}_{n,j} = f(t_n + c_j h, \hat{U}_{n,j}) + \hat{Z}_{n,j} + \hat{F}_n(t_n + c_j h), \tag{3.4.9}$$

for $j = 1, \cdots, m$, with

$$\hat{U}_{n,j} = \hat{y}_n + h \sum_{l=1}^{m} \alpha_{j,l} \hat{Y}_{n,l}, \tag{3.4.10}$$

and

$$\hat{Z}_{n,j} = h \sum_{\mu=1}^{m} w_{j,\mu} k\left(t_n + c_j h, t_n + \xi_{j,\mu} h, \hat{u}(t_n + \xi_{j,\mu} h), \hat{u}'(t_n + \xi_{j,\mu} h)\right). \tag{3.4.11}$$

Here, $w_{j,\mu} := c_j b_\mu$, with $b_\mu := \alpha_\mu(1)$ and $\xi_{j,\mu} := c_j c_\mu$. The lag term approximations $\hat{F}_n(t)$ in (3.4.9) corresponding, respectively, to the exact lag terms (3.4.4) and (3.4.5) are

$$\hat{F}_n(t_{n,j}) := \Phi(t_{n,j}) + h \sum_{i=0}^{n-1} \sum_{\mu=1}^{m} b_\mu k(t_{n,j}, t_{i,\mu}, \hat{U}_{i,\mu}, \hat{Y}_{i,\mu}), \tag{3.4.12}$$

if $0 \leq n < r$, and

$$\begin{aligned}\hat{F}_n(t_{n,j}) \;&:=\; h\sum_{\mu=1}^{m}\bar{w}_{j,\mu}k(t_{n,j},t_{n-r}+\eta_{j,\mu}h,\hat{u}(t_{n-r}+\eta_{j,\mu}h),\hat{u}'(t_{n-r}+\eta_{j,\mu}h)) \\ &\quad +h\sum_{i=n-r+1}^{n-1}\sum_{\mu=1}^{m}b_\mu k(t_{n,j},t_{i,\mu},\hat{U}_{i,\mu},\hat{Y}_{i,\mu}), \qquad (3.4.13)\end{aligned}$$

if $r \leq n \leq N-1$. Here, $\bar{w}_{j,\mu} := (1-c_j)b_\mu$ and $\eta := c_j + (1-c_j)c_\mu$.

3.4.2 Convergence results for neutral DVIDEs

Let $u \in S_m^{(0)}(\Pi_N)$ denote the exact collocation solution to (3.4.1) defined by (3.4.6)–(3.4.7). For ease of exposition, we choose the linear version of (3.4.1),

$$y'(t) = f(t)y(t) + q(t) + (Vy)(t), \quad t \in I, \qquad (3.4.14)$$

with

$$(Vy)(t) := \int_0^t \sum_{\mu=0}^{1} K_\mu(t,s)y^{(\mu)}(s)ds + \int_0^{t-\tau}\sum_{\mu=0}^{1} H_\mu(t,s)y^{(\mu)}(s)ds,$$

subject to initial condition (3.4.2).

Theorem 3.4.1 *([18]) Assume the given functions in (3.4.14) and (3.4.2) satisfy $f \in C^m(I)$, $K \in C^m(S_\tau)$, $\phi \in C^m[-\tau,0]$, and for $t \in [0,\tau]$, the integral*

$$\Phi(t) := \int_{t-\tau}^{0}\{H_0(t,s)\phi(s) + H_1(t,s)\phi'(s)\}ds, \quad t \in [0,\tau], \qquad (3.4.15)$$

is known exactly. Then for all sufficiently small $h = \tau/r$, $r \in \mathbb{N}$, the constrained-mesh collocation solution $u \in S_m^{(0)}$, $m \geq 1$, to (3.4.14) satisfies

$$||y^{(l)} - u^{(l)}|| \leq C_l h^m,$$

for some finite constant C_1 not depending on h and $l = 0, 1$. This estimate holds for all collocation parameters $\{c_j\}$ with $0 \leq c_1 < \cdots < c_m \leq 1$.

Theorem 3.4.2 *([18]) Assume that the given functions in (3.4.14) are sufficiently smooth on their domains, i.e., they are in C^{m+d} for some d with $0 \leq d \leq m$, and let the delay integral Φ in (3.4.15) be known exactly. Then for all sufficiently small $h = \tau/r$, $r \in \mathbb{N}$, the constrained-mesh collocation solution $u \in S_m^{(0)}$ to (3.4.14) is uniquely defined, and has the following properties:*

(i) If the collocation parameters $\{c_j\}$ are the Gauss points in $(0,1)$, then

$$\max_{1\leq n\leq N} |y(t_n) - u(t_n)| \leq C_0 h^{2m},$$

for some finite constant C_0, provided $d = m$, while u' possesses a lower order of convergence on the mesh Π_N:

$$\max_{1\leq n\leq N} |y'(t_n) - u'(t_n)| \leq C_1 h^m.$$

(ii) If the $\{c_j\}$ are the Radau II points and $d = m-1$, then

$$\max_{1\leq n\leq N} |y^{(l)}(t_n) - u^{(l)}(t_n)| \leq C_l h^{2m-1}, \quad l = 0, 1.$$

(iii) If the $\{c_j\}$ are the Radau I points and $d = m-1$, i.e., $0 = c_1 < c_2 < \cdots < c_m < 1$, then

$$\max_{1\leq n\leq N} |y(t_n) - u(t_n)| \leq C_0 h^{2m-1},$$

and

$$\max_{1\le n\le N} |y'(t_n) - u'(t_n)| \le C_1 h^m.$$

Proof: A detailed proof of this theorem can be found in [18].

□

While the collocation methods to integral, differential and integro-differential equations with constant delay are well understood, the numerical analysis of these equations with proportional delay is significantly more difficult. Indeed, the results to date are incomplete and their derivation calls for new mathematical techniques.

It is known that the collocation method in $S_m^{(0)}(\Pi_N)$ for constant delay problems has a global convergence order m and a local superconvergence order p^*, $m < p^* \le 2m$. The question is:

> if we apply the collocation method to variable delay problems, can we get a global convergence order m and a superconvergence order p^* $(m < p^* \le 2m)$ using m collocation points?

It is the scope of next chapter to investigate proportional delay problems and answer this question.

Chapter 4

Collocation for Differential and Volterra Integro-differential Equations with Proportional Delay qt $(0 < q < 1)$

In this chapter, we concentrate on the discretization analysis of differential equations with proportional delay. The collocation method and its global convergence properties are discussed in Sections 4.1 and 4.2. In Section 4.3, we discuss the local convergence of collocation solution to first order DDE and DVIDE. In Section 4.4, we extend the results to second-order DDEs. Some numerical examples are provided as a further illustration for these results.

Consider the first-order delay differential equation

$$y'(t) = f(t, y(t), y(qt)), \quad t \in I, \quad y(0) = y_0, \tag{4.0.1}$$

and the delay Volterra integro-differential equation

$$y'(t) = f(t, y(t)) + \int_{qt}^{t} k(t, s, y(s))ds, \quad t \in I, \quad y(0) = y_0, \tag{4.0.2}$$

with $0 < q < 1$.

Many special cases of (4.0.1) and (4.0.2) are encountered in applications: collection of current by electric locomotives [84], number theory [76], probability theory on algebraic structures [85], nonlinear dynamical systems [41], absorption of light by interstellar matter [1].

Theoretical and numerical results on (4.0.1) and (4.0.2) may be found, for example, in [8], [19], [21], [23], [24], [25], [40], [47], [59], [61], [62], and [81].

There are remarkable differences, both analytically and numerically, between differential equations with constant delay and those with proportional delay, see also [73]. In the case of proportional delay, the discontinuity property as discussed in Section 2.1 disappears, that is, for smooth data, the analytic solution is smooth, see [19] and [59] (but see also Remark 4.0.2 below). Hence, there is no need to keep track in the numerical solution of the primary discontinuities. In the case of constant delay, the solution possesses discontinuities even for smooth data (see Section 2.2). In this sense, the proportional delay problem is simpler to solve numerically since there is no need to use a constrained mesh. However, this is offset by the considerably more complicated form of discretized equations as we will see in Section 4.1.1, a form that renders them difficult to analyze.

Remark 4.0.1 When $q > 1$, the uniqueness of solutions to (4.0.1) and (4.0.2) may not hold. A detailed discussion can be found in [66], and for VIEs in [33]. As a result, we only consider the case of $0 < q \leq 1$.

Remark 4.0.2 If the initial point $t = t_0$ is not equal to zero, primary discontinuities may exist. Baker *et al* [5] give the following example,

$$y'(t) = ay(qt) \quad \text{for} \quad t \geq 1, \quad y(t) = 2 \quad \text{for} \quad t < 1, \quad y(1) = 0,$$

with $q \in (0, 1]$, which has primary discontinuities at $t = 1/q, 1/q^2, \cdots$.

4.1 Collocation and continuous Runge-Kutta methods

In order to exhibit the essential features of the collocation method, we only consider a special case of (4.0.1),

$$y'(t) = f(t, y(qt)), \quad t \in I, \quad y(0) = y_0. \tag{4.1.1}$$

Let Π_N be a uniform mesh on the interval $I := [0, T]$, given by $t_n := nh$, $n = 0, 1, \cdots, N$; $t_N = T$. The set

$$X_N := \{t_{n,i} := t_n + c_i h, \quad i = 1, 2, \cdots, m, \quad n = 0, 1, \cdots, N-1\},$$

with $0 \leq c_1 < c_2 < \cdots < c_m \leq 1$, denotes the Nm collocation points in $[0, T]$. Define

$$q_{n,i} := [q(n + c_i)] \in \mathbb{N}_0, \quad \gamma_{n,i} := q(n + c_i) - q_{n,i} \in [0, 1), \tag{4.1.2}$$

for $i = 1, \cdots, m$, where $[x]$ denotes the greatest integer not exceeding $x \in \mathbb{R}$. With this notation,

$$qt_{n,i} = q(t_n + c_i h) = q_{n,i}h + \gamma_{n,i}h = t_{q_{n,i}} + \gamma_{n,i}h.$$

This is a typical relation in collocation and Runge-Kutta methods for proportional delay problems of the form (4.0.1).

4.1.1 Collocation for differential equations with proportional delay

The approximation $u \in S_m^{(0)}(\Pi_N)$ to the exact solution of (4.1.1) is determined by the collocation equation

$$v'(t_n + c_j h) = f(t_n + c_j h, v(t_{q_{n,j}} + \gamma_{n,j}h)), \quad j = 1, 2, \cdots, m, \tag{4.1.3}$$

for $n = 0, 1, \cdots, N-1$, subject to the initial condition $v(0) = y_0$. We write (compare Section 3.1.1)

$$v'(t_n + sh) = \sum_{l=1}^{m} V_{n,j} L_l(s), \quad s \in [0, 1],$$

where $V_{n,j} := v'(t_n + c_j h)$. Upon integrating, we obtain

$$v(t_n + sh) = v(t_n) + h \sum_{l=1}^{m} \alpha_l(s) V_{n,l}, \quad s \in [0, 1], \tag{4.1.4}$$

where $V_{n,l}$ is determined by (4.1.3), namely

$$V_{n,j} = f\left(t_n + c_j h, v_{q_{n,j}} + h\sum_{l=1}^{m} V_{q_{n,j},l}\alpha_l(\gamma_{n,j})\right), \quad j = 1, 2, \cdots, m. \quad (4.1.5)$$

Thus, (4.1.4)–(4.1.5) define a continuous *m-stage implicit continuous Runge-Kutta method* for the first-order delay initial-value problem (4.1.1).

Illustration:

$$y'(t) = ay(t) + by(qt), \quad t \in I, \quad y(0) = y_0, \quad (4.1.6)$$

where $0 < q < 1$. On $[t_n, t_{n+1}]$, the collocation solution for (4.1.6) is determined by

$$v'(t_{n,i}) = av(t_{n,i}) + bv(qt_{n,i}), \quad i = 1, \cdots, m. \quad (4.1.7)$$

On this subinterval, v may be written as

$$v(t_n + sh) = v_n + h\sum_{j=1}^{m} \alpha_j(s)V_{n,j}, \quad s \in [0,1], \quad (4.1.8)$$

where

$$v_n := v(t_n), \quad v_0 = y_0; \quad V_{n,j} := v'(t_n + c_j h).$$

Thus, using (4.1.8) and (4.1.2) we readily find that the quantities $\{V_{n,i}\}$ in (4.1.8) are defined by the solution of the linear system

$$V_{n,i} = ah\sum_{j=1}^{m} \alpha_{i,j}V_{n,j} + bh\sum_{j=1}^{m} \alpha_j(\gamma_{n,i})V_{q_{n,i},j} + av_n + bv_{q_{n,i}}, \quad (4.1.9)$$

$i = 1, \cdots, m$, with $\alpha_{i,j} := \alpha_j(c_i)$. Once the $\{V_{n,i}\}$ have been found, the approximation at the next mesh point t_{n+1} is

$$v_{n+1} = v_n + h\sum_{j=1}^{m} b_i V_{n,i},$$

where we have set $b_i := \alpha_i(1)$.

Now, we give a more concrete computational formulation for (4.1.8) and (4.1.9).

Case I: $q_{n,i} = n$ for all $i = 1, 2, \cdots, m$.

We write (4.1.9) in the form:

$$V = h(aA + bB)V + (a + b)v_n e, \tag{4.1.10}$$

where $V = (V_{n,1}, V_{n,2}, \cdots, V_{n,m})^T$, $A = (\alpha_{i,j})_{i,j=1,2,\cdots,m}$, $B = (\alpha_j(\gamma_{n,i}))_{i,j=1,2,\cdots,m}$, and $e = (1, 1, \cdots, 1)^T$. Hence,

$$V = (a + b)v_n(I - h(aA + bB))^{-1}e.$$

Therefore, in this case, (4.1.8) is equivalent to

$$v(t_n + sh) = v_n + v_n(a + b)h\alpha(s)^T(I - h(aA + bB))^{-1}e, \tag{4.1.11}$$

where $\alpha(s) = (\alpha_1(s), \alpha_2(s), \cdots, \alpha_m(s))^T$.

Case II: $q_{n,i} < n$ if $i = 1, 2, \cdots, \mu$; $q_{n,i} = n$ if $i = \mu + 1, \cdots, m$ for some μ with $1 \leq \mu < m$.

(4.1.9) is equivalent to

$$V = haAV + hbB_\mu V + av_n e + b\bar{v} + bh \sum_{i=\mu+1}^{m} B^{(i)} V_i, \tag{4.1.12}$$

where B_μ is an $m \times m$ matrix whose i-th row is

$$(\alpha_1(\gamma_{n,i}), \alpha_2(\gamma_{n,i}), \cdots, \alpha_m(\gamma_{n,i})),$$

for $i = 1, 2, \cdots, \mu$, and which has zero vectors for all other rows. $B^{(i)}$ is an $m \times m$ matrix whose i-th row is

$$(\alpha_1(\gamma_{n,i}), \alpha_2(\gamma_{n,i}), \cdots, \alpha_m(\gamma_{n,i})),$$

and all other rows are zero vectors.

$$\bar{v} = \left(v_{q_{n,1}}, v_{q_{n,2}}, \cdots, v_{q_{n,m}}\right)^T, \quad V_i = \left(v_{q_{n,i},1}, v_{q_{n,i},2}, \cdots, v_{q_{n,i},m}\right)^T.$$

Hence,

$$\begin{aligned} V &= hb(I - h(aA + bB_\mu))^{-1} \sum_{i=\mu+1}^{m} B^{(i)} V_i + av_n(I - h(aA + bB_\mu))^{-1} e \\ &+ b(I - h(aA + bB_\mu))^{-1} \bar{v}. \end{aligned}$$

Therefore, (4.1.8) becomes

$$\begin{aligned} v(t_n + sh) &= v_n + bh^2 \alpha(s)(I - h(aA + bB_\mu))^{-1} \sum_{i=\mu+1}^{m} B^{(i)} V_i \\ &\quad + ahv_n \alpha(s)(I - h(aA + bB_\mu))^{-1} e \\ &\quad + bh\alpha(s)(I - h(aA + bB_\mu))^{-1} \bar{v}. \end{aligned}$$

Case III: $q_{n,i} < n$ for all $i = 1, 2, \cdots, m$.

In this case, (4.1.9) can be written as

$$V = haAV + av_n e + b\bar{v} + bh \sum_{i=1}^{m} B^{(i)} V_i. \tag{4.1.13}$$

Hence,

$$V = hb(I - haA)^{-1} \sum_{i=1}^{m} B^{(i)} V_i + av_n(I - haA)^{-1} e + b(I - haA)^{-1} \bar{v}.$$

Therefore, in this case, (4.1.8) takes the form

$$\begin{aligned} v(t_n + sh) &= v_n + bh^2\alpha(s)(I - haA)^{-1} \sum_{i=\mu+1}^{m} B^{(i)}V_i \\ &+ ahv_n\alpha(s)(I - haA)^{-1}e + bh\alpha(s)(I - haA)^{-1}\bar{v}. \end{aligned}$$

Remark 4.1.1 It is worth noticing that, in the above three cases, the ODE parts remain the same while the DDE parts change according to the values of q and n.

4.1.2 Collocation for integro-differential equations with proportional delay

Consider now the delay Volterra integro-differential equation

$$y'(t) = f(t, y(t)) + \int_{qt}^{t} k(t, s, y(s))ds, \quad t \in I, \quad 0 < q < 1, \qquad (4.1.14)$$

with initial condition $y(0) = y_0$.

The collocation solution $u \in S_m^{(0)}(\Pi_N)$ to (4.1.14) is given by

$$u'(t) = f(t, u(t)) + \int_{qt}^{t} k(t, s, u(s))ds, \quad t \in X_N, \qquad (4.1.15)$$

with $u(0) = y_0$. Define

$$F_n(t) := \int_{0}^{t_n} k(t, s, u(s))ds, \qquad (4.1.16)$$

and set

$$u'(t_n + vh) = \sum_{l=1}^{m} L_l(v)Y_{n,l}, \quad Y_{n,l} := u'(t_n + c_l h),$$

then

$$u(t_n + vh) = y_n + h\sum_{l=1}^{m} \alpha_l(v) Y_{n,l}, \quad v \in [0,1], \quad y_n := u(t_n),$$

where $L_l(v)$ and $\alpha_l(v)$ are defined in Section 3.1.1. Thus (4.1.15) may be rewritten as

$$\begin{aligned} Y_{n,j} &= f(t_n + c_j h, U_{n,j}) + Z_{n,j} + F_n(t_n + c_j h) \\ &\quad - \bar{Z}_{n,j} - F_{q_{n,j}}(t_n + c_j h), \quad j = 1, \cdots, m, \end{aligned}$$

with

$$U_{n,j} := u(t_n + c_j h) = y_n + h\sum_{l=1}^{m} \alpha_{j,l} Y_{n,l}, \quad \alpha_{j,l} := \alpha_l(c_j),$$

$$Z_{n,j} := h\int_0^{c_j} k(t_n + c_j h, t_n + vh, u(t_n + vh))dv,$$

and

$$\bar{Z}_{n,j} := h\int_0^{\gamma_{n,j}} k\left(t_n + c_j h, t_{q_{n,j}} + vh, u(t_{q_{n,j}} + vh)\right) dv.$$

See (4.1.2) for the definitions of $q_{n,i}$ and $\gamma_{n,i}$.

On the interval $[t_n, t_{n+1}]$, $0 \le n \le N-1$, the collocation solution is given by

$$u(t_n + vh) = y_n + h\sum_{j=1}^{m} \alpha_j(v) Y_{n,j}, \quad v \in [0,1].$$

The above method for proportional delay VIDEs involves integrals which cannot be calculated analytically, and thus an additional discretization step is necessary. If the discretization of these integrals employs m-point interpolatory quadrature formulas based on the collocation parameters $\{c_j\}$, then this method is described by (4.1.17)-(4.1.23):

$$\hat{u}(t_n + vh) = \hat{y}_n + h\sum_{j=1}^{m} \alpha_j(v)\hat{Y}_{n,j}, \quad v \in [0,1], \tag{4.1.17}$$

where

$$\hat{Y}_{n,j} = f(t_n + c_j h, \hat{U}_{n,j}) + \hat{Z}_{n,j} + \hat{F}_n(t_n + c_j h) \tag{4.1.18}$$

$$-\overline{Z}_{n,j} - \hat{F}_{q_{n,j}}(t_n + c_j h), \quad j = 1, \cdots, m, \tag{4.1.19}$$

with

$$\hat{U}_{n,j} = \hat{y}_n + h\sum_{l=1}^{m} \alpha_{j,l}\hat{Y}_{n,l}, \tag{4.1.20}$$

$$\hat{Z}_{n,j} := h\sum_{\mu=1}^{m} w_{j,\mu} k\left(t_n + c_j h, t_n + \xi_{j,\mu}h, \hat{u}(t_n + \xi_{j,\mu}h)\right), \tag{4.1.21}$$

and

$$\overline{Z}_{n,j} := h\sum_{\mu=1}^{m} \overline{w}_{j,\mu} k\left(t_n + c_j h, t_{q_{n,j}} + \overline{\xi}_{j,\mu}h, \hat{u}(t_{q_{n,j}} + \overline{\xi}_{j,\mu}h)\right). \tag{4.1.22}$$

Here, $w_{j,\mu} := c_j b_\mu$, $\overline{w}_{j,\mu} := \gamma_{n,j} b_\mu$, with $b_\mu := \alpha_\mu(1)$, $\xi_{j,\mu} := c_j c_\mu$ and $\overline{\xi}_{j,\mu} := \gamma_{n,j} c_\mu$. The lag term approximations $\hat{F}_n(t)$ in (4.1.19) corresponding to the exact lag term (4.1.16) is

$$\hat{F}_n(t_{n,j}) := h\sum_{i=0}^{n-1}\sum_{\mu=1}^{m} b_\mu k(t_{n,j}, t_{i,\mu}, \hat{U}_{i,\mu}). \tag{4.1.23}$$

4.2 Global convergence of collocation solutions

In this section, we give the global convergence results for collocation approximations to solutions of delay differential equations of first order. The discrete version of Gronwall-type inequality is the essential tool for the proof. It should be mentioned that similar results hold for higher order delay differential or integro-differential equations with properly modified proofs. In [100], a proof for global convergence of collocation solutions to Volterra integro-differential equations with proportional delay qt $(0 < q < 1)$ was given.

Consider the delay differential equation

$$y'(t) = a(t)y(t) + b(t)y(qt), \quad t \in I, \quad y(0) = y_0, \tag{4.2.1}$$

where $0 < q < 1$. We know the solution of (4.2.1) is smooth if a and b are smooth. We seek a collocation solution u for (4.2.1) in $S_m^{(0)}(\Pi_N)$, and give the global convergence order for such a numerical solution.

Theorem 4.2.1 *Suppose a, $b \in C^m[0,T]$ in (4.2.1). Then for any (uniform) mesh Π_N with sufficiently small $h = T/N$, the collocation solution $u \in S_m^{(0)}(\Pi_N)$ to (4.2.1) is uniquely defined. For every choice of the collocation parameters $\{c_j\}$ with $0 \le c_1 < \cdots < c_m \le 1$, the error $e := y - u$ satisfies*

$$||e||_\infty \le C_0 h^m, \qquad ||e'||_\infty \le C_1 h^m,$$

with C_0 and C_1 denoting suitable finite constants depending on the $\{c_j\}$.

Proof: The Taylor expansion of the analytic solution $y(t)$ for (4.2.1) is

$$y(t_n + sh) = \sum_{l=0}^{m} \alpha_{n,l} s^l + h^{m+1} R_n(s),$$

where $\alpha_{n,l} = h^l y^{(l)}(t_n)/l!$, $R_n(s) = y^{(m+1)}(\xi_n) s^{m+1}/(m+1)!$, $\xi_n \in (t_n, t_{n+1})$. Also, the collocation solution u is of the form

$$u(t_n + sh) = \sum_{l=0}^{m} \hat{\alpha}_{n,l} s^l, \quad s \in [0, 1].$$

Hence, the error e satisfies

$$e(t_n + sh) = h^{m+1}\{\beta_{n,0} + \sum_{l=1}^{m} \beta_{n,l} s^l + R_n(s)\}, \quad s \in [0, 1], \qquad (4.2.2)$$

where $h^{m+1}\beta_{n,l} = \alpha_{n,l} - \hat{\alpha}_{n,l}$, $l = 0, 1, \cdots, m$. Again, the error satisfies the following equation

$$\begin{aligned} e'(t_n + sh) &= a(t_n + sh)e(t_n + sh) \\ &\quad + b(t_n + sh)e(q(t_n + sh)) + \delta(t_n + sh). \qquad (4.2.3) \end{aligned}$$

Computing derivatives on both sides of (4.2.2), we get

$$e'(t_n + sh) = h^m\{\sum_{l=1}^{m} l\beta_{n,l} s^{l-1} + R_n'(s)\}, \quad s \in [0, 1]. \qquad (4.2.4)$$

Substituting (4.2.2) and (4.2.4) into (4.2.3) yields

$$\begin{aligned} \sum_{l=1}^{m} l\beta_{n,l} c_i^{l-1} &= ha \sum_{l=0}^{m} \beta_{n,l} c_i^l + hb \sum_{l=0}^{m} \beta_{q_{n,i},l} \gamma_{n,i}^l \\ &+ haR_n(c_i) + hbR_{q_{n,i}}(\gamma_{n,i}) - R_n'(c_i), \qquad (4.2.5) \end{aligned}$$

with $a = a(t_n + c_i h)$ and $b = b(t_n + c_i h)$, where $q_{n,i}$ and $\gamma_{n,i}$ are defined in (4.1.2). We may rewrite (4.2.5) as

$$\begin{aligned} &\sum_{l=1}^{m} l\beta_{n,l}c_i^l - ha\sum_{l=1}^{m}\beta_{n,l}c_i^l \\ = \quad & ha\beta_{n,0} + hb\sum_{l=1}^{m}\beta_{q_{n,i},l}\gamma_{n,i}^l + hb\beta_{q_{n,i},0} + \triangle_{n,i}, \end{aligned} \tag{4.2.6}$$

$i = 1, 2, \cdots, m$; $n = 0, 1, \cdots, N-1$, with

$$\triangle_{n,i} = haR_n(c_i) + hbR_{q_{n,i}}(\gamma_{n,i}) - R_n'(c_i).$$

The continuity of the approximating polynomial spline at the knots Π_N yields an additional relationship between $\beta_{n,0}$ and the vectors β_i, $i < n$, namely,

$$\beta_{n,o} = \beta_{n-1,0} + \sum_{l=1}^{m}\beta_{n-1,l} + R_{n-1}(1),$$

$n = 1, 2, \cdots, N-1$. Furthermore, we have

$$\beta_{n,o} = \sum_{i=0}^{n-1}\sum_{l=1}^{m}\beta_{i,l} + \sum_{k=0}^{n-1}R_k(1). \tag{4.2.7}$$

Combining (4.2.6) and (4.2.7), we get

$$\sum_{l=1}^{m} l\beta_{n,l}c_i^l - ha\sum_{l=1}^{m}\beta_{n,l}c_i^l - hb\sum_{l=1}^{m}\beta_{q_{n,i},l}\gamma_{n,i}^l = \Phi_{n,i} + \rho_{n,i}, \tag{4.2.8}$$

where

$$\begin{aligned} \Phi_{n,i} &= ha\sum_{j=0}^{n-1}\sum_{l=1}^{m}\beta_{j,l} + hb\sum_{j=0}^{q_{n,i}-1}\sum_{l=1}^{m}\beta_{j,l}, \\ \rho_{n,i} &= ha(R_0(1) - R_n(0)) + hb(R_0(1) - R_{q_{n,i}}(0)) + \triangle_{n,i}. \end{aligned}$$

We now need to consider three cases according to the value of $q_{n,i}$ (see also the three cases in Section 4.1.1):

Case I: $q_{n,i} = n$ for all $i = 1, 2, \cdots, m$.

In this case, we can switch (4.2.8) to a more compact form, namely

$$\beta_n = (A - hD)^{-1}V,$$

where $A = (lc_i^{l-1}) \in \mathbb{R}^{m\times m}$, $D = (ac_i^l + b\gamma_{n,i}^l) \in \mathbb{R}^{m\times m}$ and $V = (\Phi_{n,i} + \rho_{n,i})^T \in \mathbb{R}^m$. The matrices A and D are invertible for sufficiently small $h > 0$. Setting $||\beta_n||_1 := \sum_{i=1}^m |\beta_{n,i}|$, we obtain

$$||\beta_n||_1 \le hC \sum_{j=0}^{n-1} ||\beta_j||_1 + R, \quad n = 0, 1, \cdots, N-1, \tag{4.2.9}$$

where C and R have obvious meanings. This is a discrete Gronwall-type inequality, and thus we obtain (see Chapter 1 of [21])

$$||\beta_n||_1 \le e^{NhC} R = e^{CT} R.$$

Hence, by (4.2.2) and (4.2.7),

$$|e_n(t_n + sh)| \le h^m (B + M_m), \quad t_n + sh \in I_n,$$

where $M_m = \max\{|y^{(m)}(t)|/m! : t \in I\}$.

Case II: $q_{n,i} < n$ if $i = 1, 2, \cdots, \mu$; $q_{n,i} = n$ if $i = \mu + 1, \cdots, m$ for some μ with $1 \le \mu < m$. In this case,

$$\beta_n = (A - hD_0)^{-1}(h \sum_{i=1}^{\mu} D_i \beta_{q_{n,i}} + V),$$

where D_i $(1 \le i \le \mu)$ is an $m \times m$ matrix whose i-th row is $(b\gamma_{n,i}^1, \cdots, b\gamma_{n,i}^m)$, and all other rows are zero vectors. D_0 is also an $m \times m$ matrix: its j-th row is $(ac_j^1, \cdots, ac_j^m)$, when $j = 1, \cdots, \mu$, and is $(ac_j^1 + b\gamma_{n,j}^1, \cdots, ac_j^m + b\gamma_{n,j}^m)$, when $j = \mu + 1, \cdots, m$. The rest of the proof is similar to that of Case I.

Case III: $q_{n,i} < n$ for all $i = 1, 2, \cdots, m$.

The derivation of the analogue of (4.2.9) is straightforward, since in this case, $D_0 = (ac_i^j)_{m \times m}$. $\square$

Theorem 4.2.1 remains valid for the nonlinear DVIDE (4.1.14).

Theorem 4.2.2 *Let $f := f(t, y)$ and $k := k(t, s, y)$ in (4.1.14) be m times continuously differentiable on their respective domains, and assume that f_y and k_y are bounded. Then there exists $\bar{h} > 0$ such that the collocation equation (4.1.15) defines for each $h \in (0, \bar{h})$ a unique approximation $u \in S_m^{(0)}(\Pi_N)$. For every choice of the collocation parameters $\{c_j\}$ with $0 \le c_1 < \cdots < c_m \le 1$, the error $e := y - u$ satisfies*

$$||e||_\infty \le C_0 h^m, \quad ||e'||_\infty \le C_1 h^m,$$

with C_0 and C_1 denoting suitable finite constants depending on the $\{c_j\}$.

The proof is similar to that of Theorem 4.2.1 using the linearization techniques described in Remark 3.3.1. The reader may consult [100] for further details. See also [21] for analogous results for Volterra integro-differential equation without delay.

Remark 4.2.1 Theorem 4.2.2 is also valid for Volterra integro-differential equation with pure delay

$$y'(t) = g(t) + \int_0^{qt} k(t, s, y(s))ds, \quad t \in I, \quad y(0) = y_0,$$

where $0 < q < 1$.

4.3 Local superconvergence

In Section 4.2, we investigated the global convergence order of collocation method. In this section, we want to know what order of convergence we can get if we only look at the mesh points instead of the whole interval. Intuitively, we expect a higher order of convergence as in the case of constant delay equations.

First, we provide some properties of the analytic solution of our problem. This gives us some ideas about what could happen to the numerical solution of the problem. Then we look at the problem specifically at $t = h$ for the DDE and the DVIDE. We also provide a numerical example.

Consider the first-order equation

$$y'(t) = by(qt), \quad y(0) = 1, \tag{4.3.1}$$

with $b \in \mathbb{C}$ and $0 < q < 1$. The analytic solution of (4.3.1) is

$$y(t) := \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{k!}(bt)^k. \tag{4.3.2}$$

Detailed descriptions of its properties may be found in [47], [66], and in [59]. We only mention the following result from [59].

Theorem 4.3.1 *The solution of (4.3.1) cannot be uniformly bounded for $t \geq 0$, regardless of the value of $b \in \mathbb{C}\backslash\{0\}$ and $q \in (0,1)$.*

All these shows that the DDE (4.3.1) has property that is very different from that of its classical (non-delay) counterpart. We may expect this difference carries over to the collocation solution.

Padé approximants to the exact solution play an important role in the numerical analysis of initial value problems [27] and [61]. Basically, Padé approximants are optimal rational approximants to a function possessing a power series. The following definition makes this more precise.

Definition 4.3.1 Let $f(z)$ have a power series in a neighborhood of $z = 0$. If polynomials $P(z)$ and $Q(z)$, of degrees p and q respectively, can be found such that

$$f(z) - \frac{P(z)}{Q(z)} = O(|z|^{p+q+1}),$$

with $Q(0) = 1$, then $P(z)/Q(z)$ is a Padé approximant to $f(z)$. When $p = q$, $P(z)/Q(z)$ is called a diagonal Padé approximant to $f(z)$.

The following examples are given for illustrative purpose, and we set $z := bh$. The first two diagonal Padé approximants for (4.3.2) are (see [19])

Example 4.3.1

$$R_{1,1}(z;q) = \frac{1 + (1 - \frac{q}{2})z}{1 - \frac{q}{2}z}$$

Thus, for $q = \frac{1}{2}$,

$$R_{1,1}(z;\frac{1}{2}) = \frac{1 + \frac{3}{4}z}{1 - \frac{1}{4}z},$$

compared with

$$R_{1,1}(z) = R_{1,1}(z;1) = \frac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z},$$

for $f(z;1) = \exp(z)$.

Example 4.3.2

$$R_{2,2}(z;q) = \frac{1 + \frac{6-4q-2q^2+q^4}{2(3-2q)}z + \frac{q(18-24q+10q^3-3q^4)}{12(3-2q)}z^2}{1 + \frac{q^2(q^2-2)}{2(3-2q)}z + \frac{q^4(4-3q)}{12(3-2q)}z^2}$$

In particular,

$$R_{2,2}(z;\frac{1}{2}) = \frac{1 + \frac{57}{64}z + \frac{113}{768}z^2}{1 - \frac{7}{64}z + \frac{5}{768}z^2},$$

compared with

$$R_{2,2}(z) = R_{2,2}(z;1) = \frac{1 + \frac{1}{2}z + \frac{1}{12}z^2}{1 - \frac{1}{2}z + \frac{1}{12}z^2},$$

for $f(z;1) = \exp(z)$.

The collocation equation of (4.3.1) is given by

$$v_{n+1} = v_n + h\sum_{i=1}^{m} V_{n,i}\alpha_i(1), \quad V_{n,i} = b\{v_{q_{n,i}} + h\sum_{j=1}^{m}\alpha_j(\gamma_{n,i})V_{q_{n,i},j}\},$$

where $V_{n,i} := y'(t_n + c_i h)$. When $n = 0$, $t_0 = 0$, $t_1 = h$,

$$v(h) = 1 + h\sum_{i=1}^{m} V_{0,i}\alpha_i(1),$$

$$V_{0,i} = b\{1 + h\sum_{j=1}^{m}\alpha_j(qc_i)V_{0,j}\}, \quad i = 1, 2, \cdots, m.$$

Theorem 4.3.2 *Let $v \in S_m^{(0)}(\Pi_N)$ be the collocation solution to the DDE (4.3.1). Then for $q \in (0,1)$,*

$$v(h) = \frac{P_{m,m}(z;q)}{Q_{m,m}(z;q)},$$

where

$$P_{m,m}(z;q) := \sum_{j=0}^{m} q^{j(2m+1-j)/2} N^{(m-j)}(q^{j-m-1}) z^j$$

and

$$Q_{m,m} := \sum_{j=0}^{m} q^{j(2m+1-j)/2} N^{(m-j)}(0) z^j,$$

with

$$N(t) := \frac{1}{m!} \prod_{i=1}^{m} (t - c_i).$$

For a brief proof of this theorem and next example, reader may consult [19].

Example 4.3.3 For $m = 1$ we get

$$v(h) = \frac{1 + (1 - qc_1)z}{1 - qc_1 z}.$$

It is easy to verify that $|y(h) - v(h)| = \mathcal{O}(h^2)$ if and only if $c_1 = 1/2$ which is a Gauss point.

For $m = 2$, collocation for DDE (4.3.1) at the Gauss points yields

$$v(h) = \frac{1 + (1 - \frac{q^2}{2})z + \frac{q}{2}(1 - q + \frac{q^2}{6})z^2}{1 - \frac{q^2}{2}z + \frac{q^3}{12}z^2},$$

with $y(h) - v(h) = \mathcal{O}(h^4)$ for all $0 < q < 1$.

We now extend the above theorem to a special case of the VIDE (4.1.14) with proportional delay,

$$y'(t) = -\int_0^{qt} \frac{b^2}{q} y(s)ds, \quad y(0) = 1, \tag{4.3.3}$$

with $b \in \mathbb{C}$ and $0 < q < 1$. The analytic solution of (4.3.3) is

$$y(t) := \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)}}{(2k)!} (bt)^{2k}, \tag{4.3.4}$$

which, for $q = 1$, reduces to $y(t) = \cos(bt)$.

The computational form for the collocation solution to (4.3.3) is given by

$$u(t_{n+1}) = u(t_n) + h\sum_{j=1}^{m} U_{n,j}\alpha_j(1),$$

with

$$\begin{aligned} U_{n,i} &= -\gamma_{n,i}\frac{b^2h}{q}u(t_{q_{n,i}}) - \frac{b^2h^2}{q}\sum_{j=1}^{m} U_{q_{n,i},j}\beta_j(\gamma_{n,i}) \\ &\quad -\frac{b^2h}{q}\sum_{j=0}^{q_{n,i}-1} u(t_j) - \frac{b^2h^2}{q}\sum_{j=0}^{q_{n,i}-1}\sum_{l=1}^{m}\beta_l(1)U_{j,l}, \quad i = 1, 2, \cdots, m, \end{aligned}$$

where

$$\beta_j(t) := \int_0^t (t-s)L_j(s)ds. \tag{4.3.5}$$

Lemma 4.3.1 *The collocation solution u of (4.3.3) satisfies*

$$u(h) = (1 - z^2\hat{\beta}^T(I + \frac{1}{q}z^2\hat{\mathbf{A}}_0(q))^{-1}\hat{\mathbf{C}})y_0, \tag{4.3.6}$$

where $\hat{\mathbf{C}}^T := (c_1, c_2, \cdots, c_m)$, $\hat{\beta}^T := (\alpha_1(1), \cdots, \alpha_m(1))$, and

$$\hat{\mathbf{A}}_0(q) := (\beta_j(qc_i))_{i,j=1,2,\cdots,m}.$$

Theorem 4.3.3 *Let $u \in S_m^{(0)}(\Pi_N)$ be the collocation solution to the DVIE (4.3.3), and set $n := \lceil (m+1)/2 \rceil$. Then for $q \in (0,1)$,*

$$u(h) = \frac{P_{2m,2m}(z;q)}{Q_{2m,2m}(z;q)},$$

where

$$P_{2m,2m}(z;q) := \sum_{j=0}^{n-1}(-1)^j q^{j(j+1)} N^{(2n-2j-1)}(q^{j-n})z^{2j}, \tag{4.3.7}$$

and

$$Q_{2m,2m}(z;q) := \sum_{j=0}^{n-1}(-1)^j q^{j(j+1)} N^{(2n-2j-1)}(0) z^{2j}, \tag{4.3.8}$$

with

$$N(t) := \frac{1}{m!}\prod_{i=1}^{m}(t-c_i).$$

Proof: In order to establish the above result, we generalize the approach first introduced by Nørsett for ODEs (see, for example, [61]). Assume, without loss of generality, that $h=1$. Since on $[0,h]=[0,1]$, the collocation solution u is a polynomial of degree m, we set

$$u'(t)+\int_0^{qt}\frac{b^2}{q}u(s)ds = K\cdot N(t), \quad \text{with} \quad N(t) := \frac{1}{m!}\prod_{i=1}^{m}(t-c_i),$$

with the constant K to be determined. Successive differentiation and replacing of $u'(q^jt)$ by the corresponding expressions involving only $u'(q^{j+1}t)$ and derivatives of N leads to

$$0 \equiv u^{(2n)}(t) - (-1)^n b^{2n} q^{n(n-1)} u(q^n t) - K\sum_{j=0}^{n-1}(-1)^j b^{2j} q^{j(j+1)} N^{(2n-2j-1)}(q^j t),$$

with $n=\lceil (m+1)/2\rceil$. If we now set $t=0$ and $t=1/q^n$ in the above equation and replace b by z $(=bh)$ we readily obtain the result of Theorem 4.3.3.

□

Example 4.3.4 For $m = 1$, we find

$$u(h) = \frac{1 - (2c_1 - qc_1^2)\frac{z^2}{2}}{1 + qc_1^2\frac{z^2}{2}}.$$

In order to get $y(h) - u(h) = \mathcal{O}(h^{2m})$, we must have $c_1 = 1/2$ which is a Gauss point. When $q = 1$ and $c_1 = 1/2$,

$$u(h) = \frac{1 - \frac{3}{8}z^2}{1 + \frac{1}{8}z^2}.$$

When $m = 2$, c_1 and c_2 are the Gauss points, we have

$$u(h) = \frac{1 + (\frac{5}{36}q^2 - \frac{1}{12}q - \frac{1}{2})z^2 + \frac{1}{12}q(\frac{1}{36}q^4 - \frac{1}{3}q + \frac{1}{2})z^4}{1 + (\frac{5}{36}q - \frac{1}{12})qz^2 + \frac{1}{432}q^5z^4}. \tag{4.3.9}$$

Note that $y(h) - u(h) = \mathcal{O}(h^4)$ holds for all $0 < q < 1$. If $q = 1$, then

$$u(h) = \frac{1 - \frac{4}{9}z^2 + \frac{7}{432}z^4}{1 + \frac{1}{18}z^2 + \frac{1}{432}z^4}. \tag{4.3.10}$$

As an illustration, see also [19], consider the linear DDE (4.2.1) with $a = -1$, $b = -1/2$, and let $m = 2$; i.e., the collocation solution v is in $S_2^{(0)}$. The collocation parameters are the Gauss points, $c_1 = (3 - \sqrt{3})/6$, $c_2 = (3 + \sqrt{3})/6$.

$t = t_n$	h	$y(t) - v(t)$ q=0.9	$y(t) - u_{it}(t)$ q=0.9	$y(t) - v(t)$ q=0.5	$y(t) - u_{it}(t)$ q=0.5
h	0.2	-1.78E-6	4.95E-5	5.78E-6	2.43E-5
	0.1	-3.87E-8	6.79E-6	4.48E-7	1.75E-6
	0.05	3.75E-10	8.80E-7	3.10E-8	1.17E-7
	0.025	1.20E-10	1.12E-7	2.04E-9	7.60E-9
		$(p^* = 2)$	$(p^* = 3)$	$(p^* = 4)$	$(p^* = 4)$
1.0	0.2	-4.33E-6	-4.88E-6	-1.55E-8	3.48E-6
	0.1	-1.88E-7	-2.00E-7	-3.33E-7	-2.39E-7
	0.05	-1.18E-8	-1.20E-8	-2.07E-8	-6.20E-8
	0.025	-7.57E-10	-9.51E-10	-1.30E-9	-3.87E-9
		$(p^* = 4)$	$(p^* = 4)$	$(p^* = 4)$	$(p^* = 4)$
5.0	0.2	-3.13E-8	-2.78E-8	6.00E-7	1.22E-6
	0.1	-1.51E-9	-1.95E-9	1.36E-9	1.01E-7
	0.05	-8.97E-11	-1.08E-10	9.53E-11	-8.38E-9
	0.025	-5.65E-12	-6.80E-12	6.20E-12	-5.21E-10
		$(p^* = 4)$	$(p^* = 4)$	$(p^* = 4)$	$(p^* = 4)$

Table 4.1: Numerical results for equation (4.2.1)

The numbers between parentheses for p^* indicate the observed order of local superconvergence. These results suggest that, in spite of the non-optimal order at $t_1 = h$, the conjectured (exact) optimal order $p^* = 2m = 4$ is being recovered as the integration progresses (see also Chapter 5).

4.4 Extension of results to second-order DDE

In this section, we extend part of our results in Section 4.3 to second-order differential equations with proportional delay, especially the superconvergence results at $t = h$. See also [100].

In analogy to first-order ODEs and delay DEs, see [24], [25] and [61], consider the test equation

$$y''(t) = -b^2 y(qt), \quad y(0) = 1, \quad y'(0) = 0, \tag{4.4.1}$$

with $b \in \mathbb{C}$ and $0 < q < 1$. The analytic solution of (4.4.1) is (compare (4.3.4))

$$y(t) := \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)}}{(2k)!} (bt)^{2k}, \tag{4.4.2}$$

which, for $q = 1$, reduces to $y(t) = \cos(bt)$.

Theorem 4.4.1 *([8]) The solution of (4.4.1) is an entire function of order zero, and hence cannot be uniformly bounded for $t \geq 0$, regardless of the value of $b \in \mathbb{C}\backslash\{0\}$. Also, $y(t)$ possesses infinitely many zeros for any $q \in (0,1)$.*

The first part can be proved by calculating the order of (4.4.2), which is an entire function, in complete analogy to a result for the first-order DDE (4.3.1) (see [59]). The second part is obtained by applying a result from [94].

The collocation solution to (4.4.1) is given by

$$v_{n+1} = v_n + hv_n' + h^2 \sum_{i=1}^{m} V_{n,i}\beta_i(1),$$

$$v'_{n+1} = v'_n + h\sum_{i=1}^{m} V_{n,i}\alpha_i(1),$$

$$V_{n,i} = -b^2\{v_{q_{n,i}} + \gamma_{n,i} h v'_{q_{n,i}} + h^2 \sum_{j=1}^{m} \beta_j(\gamma_{n,i}) V_{q_{n,i},j}\},$$

where $V_{n,i} := y''(t_n + c_i h)$, and $\beta_j(t)$ is defined by (4.3.5). When $n = 0$, $t_0 = 0$, $t_1 = h$,

$$v(h) = 1 + h^2 \sum_{i=1}^{m} V_{0,i}\beta_i(1),$$

$$V_{0,i} = -b^2\{1 + h^2 \sum_{j=1}^{m} \beta_j(qc_i)V_{0,j}\}, \quad i = 1, 2, \cdots, m.$$

Theorem 4.4.2 *Let $v \in S^{(1)}_{m+1}(\Pi_N)$ be the collocation solution to the DDE (4.4.1), and $n := \lceil (m+2)/2 \rceil$. Then for $q \in (0,1)$,*

$$v(h) = \frac{P_{2m,2m}(z;q)}{Q_{2m,2m}(z;q)},$$

where

$$P_{2m,2m}(z;q) := \sum_{j=0}^{n-1} (-1)^j q^{j(2n-j-1)} \hat{N}^{(2n-2j-2)}(q^{j-n}) z^{2j}, \tag{4.4.3}$$

and

$$Q_{2m,2m} := \sum_{j=0}^{n-1} (-1)^j q^{j(2n-j-1)} \hat{N}^{(2n-2j-2)}(0) z^{2j}, \tag{4.4.4}$$

where

$$c_0 = \frac{\sum_{j=0}^{n-1}(2n-2j-1)(-1)^j b^{2j} q^{j(2n-j)} N^{(2n-2j-2)}(0)}{\sum_{j=0}^{n-1}(-1)^j b^{2j} q^{j(2n-j)} N^{(2n-2j-1)}(0)}, \tag{4.4.5}$$

and

$$\hat{N}(t) := (t - c_0)N(t), \quad N(t) := \frac{1}{m!}\prod_{i=1}^{m}(t - c_i).$$

Proof: Assume $h = 1$. Since on $[0, h] = [0, 1]$, the collocation solution v is a polynomial of degree $m+1$, we set

$$v''(t) + b^2 v(qt) = K \cdot \hat{N}(t), \quad \text{with} \quad \hat{N}(t) := \frac{1}{m!}\prod_{i=0}^{m}(t - c_i),$$

with the constant K to be determined. Successive differentiation and replacing of $v''(q^j t)$ by the corresponding expressions involving only $v''(q^{j+1}t)$ and derivatives of $\hat{N}$ leads to

$$0 \equiv v^{(2n)}(t) - (-1)^n b^{2n} q^{n(n-1)} v(q^n t) - K \sum_{j=0}^{n-1} (-1)^j b^{2j} q^{j(2n-j-1)} \hat{N}^{(2n-2j-2)}(q^j t),$$

with $n := \lceil (m+2)/2 \rceil$. After setting $t = 0$ and $t = 1/q^n$ in the above equation and substituting b by $z = bh$ we obtain (4.4.3) and (4.4.4). The number c_0 is determined by further differentiation and use of the initial condition $y'(0) = 0$.

$$\sum_{j=0}^{n-1} (-1)^j b^{2j} q^{j(2n-j)} \hat{N}^{(2n-2j-1)}(0) = 0,$$

$$\sum_{j=0}^{n-1} (-1)^j b^{2j} q^{j(2n-j)} \left[-c_0 N^{(2n-2j-1)}(0) + (2n-2j-1) N^{(2n-2j-2)}(0) \right] = 0.$$

Hence, (4.4.5) holds. □

It should be noted that the derivatives in (4.4.3) and (4.4.4) are of even orders, while those in (4.3.7) and (4.3.8) are of odd orders.

Example 4.4.1 When $m = 1$, we get

$$v(h) = \frac{1 - (1 - c_1^2 q^2)\frac{z^2}{2}}{1 + c_1^2 q^2 \frac{z^2}{2}}.$$

If $q = 1$, $c_1 = 1/2$ (Gauss point), then

$$v(h) = \frac{1 - \frac{3}{8}z^2}{1 + \frac{1}{8}z^2}.$$

When $m = 2$, collocation for the DDE (4.4.1) at the Gauss points yields

$$v(h) = \frac{1 - (\frac{1}{2} + \frac{1}{12}q^2 - \frac{5}{36}q^3)z^2 + (\frac{1}{12} - \frac{5}{72}q + \frac{1}{432}q^3)q^2z^4}{1 - z^2q^2(\frac{1}{12} - \frac{5}{36}q) + \frac{1}{432}q^5z^4}, \tag{4.4.6}$$

with $y(h) - v(h) = \mathcal{O}(h^4)$ for all $0 < q < 1$. If $q = 1$,

$$v(h) = \frac{1 - \frac{4}{9}z^2 + \frac{7}{432}z^4}{1 + \frac{1}{18}z^2 + \frac{1}{432}z^4}. \tag{4.4.7}$$

Remark 4.4.1 While (4.4.6) and (4.3.9) are not identical, (4.4.7) and (4.3.10) coincide.

Theorem 4.4.3 *Assume that $v \in S^{(1)}_{m+1}(\Pi_n)$ and $u \in S^{(0)}_m(\Pi_n)$ are, respectively, the collocation solutions for the DDE (4.4.1) and the DVIDE (4.3.3), using the same collocation parameters $\{c_1, c_2, \cdots, c_m\}$. Then at $t = t_1 = h$, $v(h) \neq u(h)$ whenever $0 < q \leq 1$.*

If $f(z)$ is given by (4.4.2), one expects its Padé approximant to contain only even order terms. In the following, we give the first two diagonal Padé approximants of $f(z)$, also those of $\cos z$ (corresponding to $q = 1$), and make comparisons between them.

Example 4.4.2 The first two diagonal Padé approximants for (4.4.2) are

$$R_{2,2}(z;q) = \frac{1 - \frac{1}{2}(1 - \frac{1}{6}q^2)z^2}{1 + \frac{1}{12}q^2z^2},$$

and

$$R_{4,4}(z;q)=\frac{1+\frac{1}{2q^2-5}(\frac{5}{2}-q^2-\frac{1}{6}q^4+\frac{1}{28}q^8)z^2-\frac{1}{2q^2-5}(\frac{5}{24}-\frac{1}{6}q^2+\frac{59}{2520}q^6-\frac{1}{336}q^8)q^2z^4}{1-\frac{1}{2q^2-5}(\frac{1}{6}-\frac{1}{28}q^4)q^4z^2+\frac{1}{2q^2-5}(-\frac{1}{180}+\frac{1}{336}q^2)q^8z^4}$$

Example 4.4.3 The first two diagonal Padé approximants for $\cos z$ are

$$R_{2,2}(z) = \frac{1-\frac{5}{12}z^2}{1+\frac{1}{12}z^2} = R_{2,2}(z;1)$$

and

$$R_{4,4}(z) = \frac{1-\frac{115}{252}z^2+\frac{313}{15120}z^4}{1+\frac{11}{252}z^2+\frac{13}{15120}z^4} = R_{4,4}(z;1).$$

Based on the examples in this and the previous sections, we have the following result.

Theorem 4.4.4 *For $0 < q \le 1$, the diagonal Padé approximants of solutions of (4.4.1) at $t = h$ are not equal to the collocation solutions of (4.4.1) at $t = h$ corresponding to the Gauss points. This is true in particular for $q = 1$ where the solution of (4.4.1) is* $\cos z$.

Remark 4.4.2 Since the diagonal Padé approximant is unique and has an order of $2m+1$, Theorem 4.4.4 suggests that the optimal order of collocation method is less than $2m+1$. This observation is also supported by the following numerical example.

We now provide some numerical results for problem (4.4.1), choosing $m = 2$, $c_1 = (3-\sqrt{3})/6$, $c_2 = (3+\sqrt{3})/6$.

$t = t_n$(nh)	h		y(t)-u(t) for		
		q=1.0	q=0.99	q=0.5	q=0.1
h(n=1)	0.1	-2.31E-10	-2.40E-10	-9.40E-11	-3.02E-13
	0.05	-3.61E-12	-3.75E-12	-1.47E-12	-4.73E-15
	0.025	-5.65E-14	-5.87E-14	-2.30E-14	(0)
	0.0125	-8.88E-16	(0)	-3.61E-16	(0)
		($p^* = 6$)	($p^* = 6$)	($p^* = 6$)	($p^* = 6$)
1.0	0.1	-1.95E-8	-2.04E-8	-1.63E-8	-6.94E-10
	0.05	-1.22E-9	-1.29E-9	-1.02E-9	-4.34E-11
	0.025	-7.61E-11	-8.82E-11	-6.36E-11	-2.71E-12
	0.0125	-4.76E-12	(0)	-3.97E-12	-1.69E-13
		($p^* = 4$)	($p^* = 4$)	($p^* = 4$)	($p^* = 4$)
5.0	0.1	1.11E-7	1.84E-7	-1.98E-7	-1.72E-8
	0.05	6.94E-9	1.14E-8	-1.24E-8	-1.07E-9
	0.025	4.34E-10	7.32E-10	-7.72E-10	-6.65E-11
	0.0125	2.71E-11	(0)	-4.83E-11	-4.15E-12
		($p^* = 4$)	($p^* = 4$)	($p^* = 4$)	($p^* = 4$)

Table 4.2: Numerical illustration for equation (4.4.1)

From the above table, we see that $p^* = 6$ when $n = 1$ and $p^* = 4$ when $n > 1$. This example suggests that the convergence order at t_n is at least $2m$. We are still curious about the result of the first step ($n = 1$), because the numerical result at this step suggests a higher order of convergence, $2m + 2$.

We showed in Table 4.2 that local superconvergence of order $p^* = 2m+2$ occurs at $t = t_1 = h$ if collocation is at the m Gauss points. In contrast to

DDEs with constant delay and VIDEs, the analysis of local superconvergence at *all* mesh points $t = t_n$, $n \leq N$, $t_N = T$, is much more complex in the case of the proportional delay qt, $0 < q < 1$. See also Section 5.1.

The problem of local superconvergence in collocation methods for differential and Volterra functional equations with state-dependent delay remains open.

Chapter 5

New Approach and Outlook

In Section 5.1, we propose a new approach to the superconvergence order problem of collocation solutions to differential equations with proportional delay. The reader may look at [45] in which embedding techniques for delay equations are discussed. In Section 5.2, we present some potential research projects.

5.1 New approach

As shown in Section 2.1.2, Theorem 2.1.9 in particular, the classical resolvent approach does not work for establishing local superconvergence results in the proportional delay case. In this section, we shall outline a new approach to this problem, and obtain some initial results.

In [45], the authors proposed a standard embedding scheme for delay differential equations. The basic idea is to convert the given delay differential equation into an infinite-dimensional ODE system, and then to truncate it at some point. The property of the solution to the truncated system largely reflects that of the solution to the original problem.

In this section, we first embed our proportional delay problem into an infinite-dimensional ODE system, and then truncate it. The truncated system is finite-dimensional. The classical superconvergence results hold for this system. Then, we find the error between the collocation solution to the original problem and the collocation solution to the truncated system. By doing so, we are able to measure the collocation error of the proportional delay problem at mesh points. Thus we determine the superconvergence order.

5.1.1 Embedding techniques

Consider delay differential equation

$$y'(t) = f(t, y(t), y(\theta(t))), \quad y(0) = y_0, \tag{5.1.1}$$

where f is a smooth function and the differentiable delay function θ satisfies $\theta(0) = 0$, $0 \le \theta(t) \le t$ for $t > 0$. Let θ_n be the n-th iterate of the function θ:

$$\theta_0(t) = t, \quad \theta_n(t) = \overbrace{\theta \circ \theta \circ \cdots \circ \theta}^{n \text{ times}}, \quad n \in \mathbb{N},$$

and define

$$z_n(t) = y(\theta_n(t)), \quad n \in \mathbb{N}_0.$$

Since $z_n'(t) = \theta_n'(t) y'(\theta_n(t))$, it follows from (5.1.1) that the functions $\{z_n\}_{n=0}^{\infty}$ obey the infinite-dimensional ODE system:

$$z_n'(t) = \theta_n'(t) f(\theta_n(t), z_n(t), z_{n+1}(t)), \quad z_n(0) = y_0, \quad n \in \mathbb{N}_0. \qquad (5.1.2)$$

We call (5.1.2) a standard embedding of the delay equation (5.1.1) (see also [45]).

Choose a (large) M and let $z_M = y_0$. For every $i = M-1, M-2, \cdots, 0$, solve the scalar ODE

$$z_i'(t) = \theta_i'(t) f(\theta_i(t), z_i(t), z_{i+1}(t)), \quad z_i(0) = y_0.$$

Consider a special case of (5.1.1), $f(t, u, v) = au + bv$ and $\theta(t) = qt$, the pantograph equation:

$$y'(t) = ay(t) + by(qt), \quad y(0) = y_0. \qquad (5.1.3)$$

For problem (5.1.3), we are able to derive explicitly the functions z_i by back substitution and compare them to the known exact solution (see [47])

$$y(t) = y_0(-w; q)_\infty \sum_{i=0}^{\infty} \frac{(-w)^i}{[q]_i} e^{aq^i t},$$

where $w = b/a$ and

$$(d; q)_n := \prod_{k=0}^{n-1} (1 - q^k d), \quad [q]_n := (q; q)_n.$$

Eventually $z := z_0$ is given by

$$\begin{aligned} z(t) &= y_0 \left[(-w)^M + \sum_{i=0}^{M-1} \frac{(-w)^i}{[q]_i} \left(\sum_{j=0}^{M-i-1} \frac{q^{j(j-1)/2}}{[q]_j} w^j \right.\right. \\ &\quad \left.\left. + \frac{q^{(M-i)(M-i-1)/2}}{[q]_{M-i-1}} w^{M-i} \right) e^{q^i at} \right]. \end{aligned} \qquad (5.1.4)$$

Clearly, $z(t) \to y(t)$ as $n \to \infty$ and if $\Re(a) \leq 0$, then this convergence is uniform for $t \geq 0$. See [45].

The main difficulty is to estimate the difference between the collocation solution to the truncated system and the collocation solution to the original problem.

For test purpose, consider the delay differential equation

$$y'(t) = by(qt), \quad y(0) = y_0, \tag{5.1.5}$$

with $0 < q < 1$. Its analytic solution is (compare (4.3.2))

$$y(t) := y_0 \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{k!}(bt)^k. \tag{5.1.6}$$

Note that (5.1.6) is not a special case of (5.1.4) in that we cannot simply let $a = 0$ in (5.1.4) (since $w = b/a$) even though (5.1.3) includes (5.1.5).

First, we embed (5.1.5) into an infinite system of ODEs. Let

$$z_i(t) = y(q^i t), \quad i \in \mathbb{N}_0.$$

Then (5.1.5) is equivalent to the infinite system

$$z_i'(t) = bq^i z_{i+1}(t), \quad i \in \mathbb{N}_0. \tag{5.1.7}$$

Observe that, after a finite number of steps, the quantity on the right side of (5.1.7) can be very small. This fact motivates us to truncate (5.1.7) after a certain number of steps to get a finite system of ODEs, namely,

$$z_i'(t) = bq^i z_{i+1}(t), \quad z_i(0) = y_0, \quad i = 0, 1, 2, \cdots, M-1, \tag{5.1.8}$$

and

$$z'_M(t) = 0, \quad z_M(0) = y_0.$$

Our purpose is to understand how large the difference is between the exact solution and the collocation approximation of (5.1.5) at certain points (mesh points). But first, we introduce the following compact notations:

$$\begin{aligned} Z_M(t) &= (Z_{M,0}(t), Z_{M,1}(t), \cdots, Z_{M,M}(t))^T \\ &= (z_0(t), z_1(t), \cdots, z_M(t))^T, \end{aligned}$$

$$Z_M(0) = y_0(1, 1, \cdots, 1)^T \in \mathbb{R}^{M+1},$$

$$P = (e_{ij})_{(M+1)\times(M+1)},$$

where

$$e_{ij} = \begin{cases} bq^{i-1}, & j = i+1, \\ 0, & j \neq i+1. \end{cases}$$

Then (5.1.8) may be rewritten as

$$Z'_M(t) = PZ_M(t), \quad Z_M(0) = y_0(1, 1, \cdots, 1)^T. \tag{5.1.9}$$

From the classical ODE theory we know that the analytic solution of (5.1.9) is

$$Z_M(t) = \exp(tP) \cdot Z_M(0).$$

Its i-th component is given by

$$Z_{M,i}(t) = z_i(t) = y_0 \sum_{k=0}^{M-i} \frac{q^{k(k-1)/2+ik}}{k!}(bt)^k,$$

where $i = 0, 1, 2, \cdots, M$. In particular,

$$Z_{M,0}(t) = z_0(t) = y_0 \sum_{k=0}^{M} \frac{q^{k(k-1)/2}}{k!}(bt)^k. \tag{5.1.10}$$

Theorem 5.1.1 *The difference between the analytic solution of (5.1.5) and that of (5.1.9) is bounded by $Ce^{|bt|} \cdot |bt|^{M+1}$, i.e.,*

$$|y(t) - Z_{M,0}(t)| \le Ce^{|bt|} \cdot |bt|^{M+1}, \tag{5.1.11}$$

where $C = |y_0|/(M+1)!$.

Proof: The result is proved by subtracting (5.1.10) from (5.1.6). $\square$

Remark 5.1.1 We can make $C \cdot |bt|^{M+1}$ arbitrarily small as long as M is big enough and t is finite.

5.1.2 Collocation solution of the truncated system

Rather than dealing with the problem of establishing local superconvergence result for the proportional delay differential equation (5.1.5) directly, we first concentrate on the collocation solution of (5.1.8).

Definition 5.1.1

$$\hat{S}_\mu^{(d)}(\Pi_N) := \overbrace{S_\mu^{(d)}(\Pi_N) \times S_\mu^{(d)}(\Pi_N) \times \cdots \times S_\mu^{(d)}(\Pi_N)}^{M+1 \text{ times}},$$

the Cartesian product of $S_\mu^{(d)}(\Pi_N)$ which is defined in Section 1.1.

We denote the collocation solution of (5.1.9) by

$$V_M(t) := (V_{M,0}(t), V_{M,1}(t), \cdots, V_{M,M}(t))^T \in \hat{S}_m^{(0)}(\Pi_N),$$

and set

$$V_M'(t_{n,i}) := \left(V_{M,0}'(t_{n,i}), V_{M,1}'(t_{n,i}), \cdots, V_{M,M}'(t_{n,i})\right)^T \in \mathbb{R}^{M+1},$$

with $t_{n,i} := t_n + c_i h$, for $i = 1, 2, \cdots, m$. Then,

$$V_M(t_n + sh) = V_M(t_n) + hV'(t_n)B_s, \tag{5.1.12}$$

where

$$V'(t_n) = (V_M'(t_{n,1}), \cdots, V_M'(t_{n,m}))_{(M+1)\times m},$$

$$B_s = (\alpha_1(s), \alpha_2(s), \cdots, \alpha_m(s))^T.$$

Note that $V(\cdot)$ also depends on M. For the definition of $\alpha_i(s)$ and related properties, see Section 3.1.1. Setting $V(t_n) = (V_M(t_n), \cdots, V_M(t_n))_{(M+1)\times m}$, we have

$$V_M'(t_n + c_i h) = PV_M(t_n + c_i h) = PV_M(t_n) + hPV'(t_n)B_{c_i}, \quad i = 1, 2, \cdots, m.$$

Hence,

$$V'(t_n) = PV(t_n) + hPV'(t_n)B, \tag{5.1.13}$$

where

$$B = (B_{c_1}, B_{c_2}, \cdots, B_{c_m}) = (\alpha_i(c_j))_{i,j=1,2,\cdots,m}.$$

Iterating (5.1.13), we find

$$\begin{aligned} V'(t_n) &= PV(t_n) + hPV'(t_n)B = PV(t_n) + hP(PV(t_n) + hPV'(t_n)B)B \\ &= PV(t_n) + hP^2V(t_n)B + h^2P^2V'(t_n)B^2 \\ &= PV(t_n) + hP^2V(t_n)B + h^2P^3V(t_n)B^2 + h^3P^4V(t_n)B^3 \\ &\quad + \cdots + h^{M-1}P^MV(t_n)B^{M-1} + \mathcal{O}(h^M), \end{aligned}$$

because rank$P = M$, $P^{M+1} = 0$. Letting $s = 1$ in (5.1.12), we obtain

$$\begin{aligned} V_M(t_{n+1}) &= V_M(t_n) + hV'(t_n)B_1 \\ &= V_M(t_n) + hPV(t_n)B_1 + h^2P^2V(t_n)BB_1 + h^3P^3V(t_n)B^2B_1 \\ &+ \cdots + h^MP^MV(t_n)B^{M-1}B_1 + \mathcal{O}(h^{M+1}). \end{aligned} \tag{5.1.14}$$

Its first component is

$$\begin{aligned} V_{M,0}(t_{n+1}) &= V_{M,0}(t_n) + bh\hat{V}_1(t_n)B_1 + h^2b^2q\hat{V}_2(t_n)BB_1 + h^3b^3q^3\hat{V}_3(t_n)B^2B_1 \\ &+ \cdots + h^Mb^Mq^{M(M-1)/2}\hat{V}_M(t_n)B^{M-1}B_1 + \mathcal{O}(h^{M+1}), \end{aligned} \tag{5.1.15}$$

where $\hat{V}_i(t_n) = V_{M,i}(t_n)e^T \in \mathbb{R}^m$ and $e = (1, 1, \cdots, 1)^T \in \mathbb{R}^m$.

The following is a classical result (see, for example, [39] and [51]).

Theorem 5.1.2 *For any finite M, the collocation solution $V_M(t) \in \hat{S}_m^{(0)}(\Pi_N)$ to the system of ODEs (5.1.9) has superconvergence order of $2m$ if the collocation parameters $\{c_i : i = 1, 2, \cdots, m\}$ are Gauss points. In other words,*

$$|Z_{M,0}(t_n) - V_{M,0}(t_n)| \leq C_M h^{2m}, \tag{5.1.16}$$

where C_M is a finite constant.

5.1.3 Superconvergence results

Consider now the m-stage collocation solution for (5.1.5).

$$v'(t_n + sh) = \sum_{j=1}^{m} v'(t_n + c_jh)L_j(s),$$

$$v'(t_n + c_i h) = bv(q(t_n + c_i h)) = bv(t_{q_{n,i}} + \gamma_{n,i} h),$$

$$v(t_n + sh) = v(t_n) + h \sum_{j=1}^{m} v'(t_n + c_j h)\alpha_j(s).$$

In particular, when $s = 1$, we have

$$v(t_{n+1}) = v(t_n) + h \sum_{j=1}^{m} v'(t_n + c_j h)\alpha_j(1), \tag{5.1.17}$$

$$v'(t_n + c_i h) = bv(t_{q_{n,i}}) + bh \sum_{j=1}^{m} v'(t_{q_{n,i}} + c_j h)\alpha_j(\gamma_{n,i}), \tag{5.1.18}$$

for $i = 1, 2, \cdots, m$.

In order to highlight the difficulties of the problem, we use the following diagram to describe them:

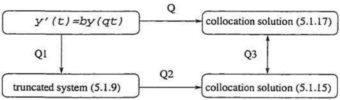

Figure 5.1: New approach

Q: Does the collocation solution (5.1.17) have a superconvergence order of $2m$ for the Gauss points? i.e., does

$$\max_{0 \le n \le N} |y(t_n) - v(t_n)| \le Ch^{2m}$$

hold?

We break down this question into three subquestions:

Q1: Can the analytic solution of the truncated system be arbitrarily close to the analytic solution of (5.1.5)?

This question is answered by Theorem 5.1.1.

Q2: Does the collocation solution to the truncated system have a superconvergence order of $2m$ for the Gauss points?

This question is answered by Theorem 5.1.2.

Q3: Can the difference between collocation solutions (5.1.17) and (5.1.15) be bounded by Ch^{2m}? i.e., does

$$\max_{0\leq n\leq N} |V_{M,0}(t_n) - v(t_n)| \leq Ch^{2m}$$

hold?

If we have an answer to Q3, problem Q is solved by

$$|y(t)-v(t)| \leq |y(t)-Z_{M,0}(t)|+|Z_{M,0}(t)-V_{M,0}(t)|+|V_{M,0}(t)-v(t)|, \quad t\in \Pi_N.$$

We can explicitly connect $V_{M,0}(t_n)$ to the initial condition by iterating (5.1.14). However, it is hard to do so for (5.1.17).

For illustration purpose, we consider the collocation solution of (5.1.5) with $m = 1$. Since we have only a single collocation parameter c_1, and hence a single value $q_{n,1}$, we only need to consider *Case I* and *III* (recall Section 4.1, page 67).

Case I: $q_{n,1} = n$. Only one value of n satisfies $q_{n,1} = n$, it is $n = 0$. In this case,

$$|y(h) - v(h)| \leq Ch^{2m},$$

when collocation is at the Gauss point $c = 1/2$. It is even true for $m > 1$. See Examples 4.3.3-4.3.4, 4.4.1 and [19].

Case III: $q_{n,1} < n$. This is the case for all $n \geq 1$.

In this case, (5.1.18) becomes

$$v'(t_{n,1}) = bv(t_{q_{n,1}}) + bhv'(t_{q_{n,1},1})\gamma_{n,1}.$$

From (5.1.17), we find

$$v(t_{n+1}) = v(t_n) + bh(1 - \gamma_{n,1})v(t_{q_{n,1}}) + bh\gamma_{n,1}v(t_{q_{n,1}+1}). \qquad (5.1.19)$$

It is hard to get an explicit expression which connects $v(t_n)$ and $v(0)$ from (5.1.19) for general $q \in (0,1)$. However, when q is a reciprocal of a positive integer, we are able to prove the following result (see also [99]):

Theorem 5.1.3 *When $q = 1/l$, $l \in \mathbb{N}$, the one-point ($m = 1$) collocation solution of (5.1.5) in $S_1^{(0)}(\Pi_N)$ possesses the superconvergence order $p^* = 2m = 2$ if and only if collocation is at the Gauss point, i.e., $c_1 = c = 1/2$.*

Remark 5.1.2 In [25], among other results, a sufficient condition for stability of one-point collocation solution of (5.1.3) with $q = 1/2$ and $y_0 = 1$ is given.

Proof: The one-point collocation solution of (5.1.5) is of the form (5.1.19). Let $z := bh$. When $q = 1/l$, $l \in \mathbf{N}$, we have

$$q_{lk+i-1,1} = k, \quad \gamma_{lk+i-1,1} = \frac{i-1+c}{l} =: l_i, \quad \text{for} \quad i = 1, 2, \cdots, l. \quad (5.1.20)$$

Then (5.1.19) can be rewritten as

$$v(t_{lk+i}) = v(t_{lk+i-1}) + (1 - l_i) z v(t_k) + l_i z v(t_{k+1}), \quad (5.1.21)$$

for $i = 1, 2, \cdots, l$. For ease of exposition, we choose $l = 2$, then (5.1.20) becomes

$$q_{2k,1} = q_{2k+1,1} = k, \quad \gamma_{2k,1} = \frac{c}{2}, \quad \gamma_{2k+1,1} = \frac{1+c}{2},$$

and (5.1.21) is simplified as

$$\begin{aligned} v(t_{2k+1}) &= v(t_{2k}) + \frac{2-c}{2} z v(t_k) + \frac{c}{2} z v(t_{k+1}), && (5.1.22) \\ v(t_{2k+2}) &= v(t_{2k+1}) + \frac{1-c}{2} z v(t_k) + \frac{1+c}{2} z v(t_{k+1}). && (5.1.23) \end{aligned}$$

We claim that

$$\begin{aligned} v(t_{2k}) &= \left(1 + 2kz + k(k + c - \frac{1}{2}) z^2\right) v(0) + \mathcal{O}(2k \cdot z^3), && (5.1.24) \\ v(t_{2k+1}) &= \left(1 + (2k+1) z + (k + \frac{1}{2})(k + c) z^2\right) v(0) + \mathcal{O}((2k+1) z^3). && (5.1.25) \end{aligned}$$

We prove the claim by induction. It is clear that, when $k = 0$,

$$v(t_1) = v(0) + (1 - \frac{c}{2}) z v(0) + \frac{c}{2} z v(t_1).$$

Hence,

$$\begin{aligned} v(t_1) &= \frac{1+(1-c/2)z}{1-cz/2}v(0) \\ &= (1+z+cz^2/2)v(0)+\mathcal{O}(z^3), \end{aligned}$$

where $\mathcal{O}(z^3)=\mathcal{O}((2k+1)\cdot z^3)$. Suppose (5.1.24) and (5.1.25) are true for all $k \le n$. When $k=n+1$,

$$\begin{aligned} v(t_{2k}) &= v(t_{2n+2}) = v(t_{2n+1}) + \frac{1-c}{2}zv(t_n) + \frac{1+c}{2}zv(t_{n+1}) \\ &= \left(1+(2n+1)z+(n+1/2)(n+c)z^2\right)v(0) + \frac{1-c}{2}zv(t_n) \\ &\quad + \frac{1+c}{2}zv(t_{n+1}) + \mathcal{O}(2k\cdot z^3). \end{aligned}$$

If $n=2l$, we have

$$\begin{aligned} v(t_{2k}) &= \left(1+(4l+2)z+(1/2+c+3l+2lc+4l^2)z^2\right)v(0)+\mathcal{O}(2k\cdot z^3) \\ &= \left(1+2kz+k(k+c-1/2)z^2\right)v(0)+\mathcal{O}(2k\cdot z^3). \end{aligned}$$

The same argument can be used for $n=2l+1$. In either case, we have proved (5.1.24). Similarly, we can prove (5.1.25).

For the exact solution of (5.1.5), we have (compare (5.1.6))

$$\begin{aligned} y(t) &= y_0\left(1+bt+\frac{qb^2t^2}{2}\right)+\mathcal{O}((bt)^3) \\ &= y_0\left(1+bt+\frac{b^2t^2}{4}\right)+\mathcal{O}((bt)^3). \end{aligned}$$

Hence,

$$|y(t_{2k})-v(t_{2k})| = \left|k^2-k(k+c-1/2)\right|z^2+\mathcal{O}(2k\cdot z^3),$$

where $2k \cdot z^3 = 2b^3h^2 \cdot kh \leq 2b^3h^2 \cdot t_N = Ch^2$. Thus $|y(t_{2k}) - v(t_{2k})|$ is of order 2 if and only if $k^2 = k(k + c - 1/2)$ which implies $c = 1/2$. Similarly, $|y(t_{2k+1}) - v(t_{2k+1})| = \mathcal{O}(h^2)$ if and only if $c = 1/2$. This concludes the proof.

$\square$

Remark 5.1.3 If $c \neq 1/2$, we will have $|y(t_n) - v(t_n)| = \mathcal{O}(h)$. When $l = 1$, hence $q = 1$, this theorem includes the classical superconvergence result. We expect that the problem becomes much harder for $q \neq 1/l$ ($l \in \mathbb{N}$) and $m > 1$.

When $m = 1$, the collocation solution of (4.3.3) is given by

$$u(t_n + sh) = u(t_n) + shu'(t_n + ch), \tag{5.1.26}$$

where $u'(t_n + ch)$ is determined by

$$\begin{aligned}
u'(t_n + ch) &= -\frac{b^2}{q}\int_0^{t_{q_{n,1}}+\gamma_{n,1}h} u(s)ds \\
&= -\frac{b^2}{q}\int_{t_{q_{n,1}}}^{t_{q_{n,1}}+\gamma_{n,1}h} u(s)ds - \frac{b^2}{q}\sum_{i=0}^{q_{n,1}-1}\int_{t_i}^{t_{i+1}} u(s)ds \\
&= -\frac{b^2h}{q}\left[\int_0^{\gamma_{n,1}} u(t_{q_{n,1}}+sh)ds + \sum_{i=0}^{q_{n,1}-1}\int_0^1 u(t_i+sh)ds\right] \\
&= -\frac{b^2h}{2q}\Bigg[2\gamma_{n,1}u(t_{q_{n,1}}) + \gamma_{n,1}^2 hu'(t_{q_{n,1}} + ch) \\
&\qquad + \sum_{i=0}^{q_{n,1}-1}\left[2u(t_i) + hu'(t_i + ch)\right]\Bigg].
\end{aligned} \tag{5.1.27}$$

Let $z := bh$ and eliminate $u'(t_n + ch)$, we get

$$\begin{aligned}
u(t_{n+1}) &= u(t_n) - \frac{z^2}{2q}\Bigg[\gamma_{n,1}(2 - \gamma_{n,1})u(t_{q_{n,1}}) \\
&\qquad + \gamma_{n,1}^2 u(t_{q_{n,1}+1}) + \sum_{i=0}^{q_{n,1}-1}\left[u(t_{i+1}) + u(t_i)\right]\Bigg].
\end{aligned} \tag{5.1.28}$$

Theorem 5.1.4 *When $q = 1/l$, $l \in \mathbf{N}$, the one-point ($m = 1$) collocation solution of (4.3.3) in $S_1^{(0)}(\Pi_N)$ possesses the superconvergence order $p^* = 2m = 2$ if and only if collocation is at the Gauss point, i.e., $c_1 = c = 1/2$.*

Proof: When $q = 1/l$, $l \in \mathbf{N}$, (5.1.20) holds, and (5.1.28) becomes

$$u(t_{lk+i}) = u(t_{lk+i-1}) - \frac{z^2}{2q}\Bigg[l_i(2-l_i)u(t_k) + l_i^2 u(t_{K+1}) + \sum_{j=0}^{k-1}[u(t_{j+1}) + u(t_j)]\Bigg], \tag{5.1.29}$$

for $i = 1, 2, \cdots, l$. For ease of exposition, we choose $l = 2$, then (5.1.20) becomes

$$q_{2k,1} = q_{2k+1,1} = k, \quad \gamma_{2k,1} = \frac{c}{2}, \quad \gamma_{2k+1,1} = \frac{1+c}{2},$$

and (5.1.29) is simplified as

$$u(t_{2k+1}) = u(t_{2k}) - z^2\left[(c-\frac{c^2}{4})u(t_k) + \frac{c^2}{4}u(t_{k+1}) + \sum_{i=0}^{k-1}[u(t_{i+1}) + u(t_i)]\right] \tag{5.1.30}$$

$$u(t_{2k+2}) = u(t_{2k+1}) - z^2\left[\frac{3+2c-c^2}{4}u(t_k) + \frac{1+2c+c^2}{4}u(t_{k+1}) + \sum_{i=0}^{k-1}[u(t_{i+1}) + u(t_i)]\right]. \tag{5.1.31}$$

We claim that

$$u(t_k) = \left(1 - \frac{k}{2}(k + 2c - 1)z^2\right)u(0) + \mathcal{O}(h^4), \tag{5.1.32}$$

and prove it by induction. When $k = 0$, from (5.1.28) we know

$$u(t_1) = u(0) - z^2\left[\frac{c}{2}(2 - \frac{c}{2})u(0) + \frac{c^2}{4}u(t_1)\right].$$

Hence,

$$\begin{aligned} u(t_1) &= \frac{1-(c-\frac{c^2}{4})z^2}{1+\frac{c^2}{4}z^2}u(0) \\ &= (1-cz^2)u(0)+\mathcal{O}(h^4). \end{aligned}$$

It is also clear that,

$$u(t_2) = u(t_1) - z^2\left[\frac{1+c}{2}(2-\frac{c}{2})u(0) + (\frac{1+c}{2})^2 u(t_1)\right].$$

Therefore,

$$\begin{aligned} u(t_2) &= \left[1-(\frac{1+c}{2})^2 z^2\right]u(t_1) - \frac{1+c}{2}\left(2-\frac{1+c}{2}\right)z^2 u(0) \\ &= \left(1-(1+2c)z^2\right)u(0)+\mathcal{O}(h^4). \end{aligned}$$

So, (5.1.32) is true for $k=1,2$. Suppose (5.1.32) is true for all $k \le 2n$. When $k = 2n+1$, from (5.1.30) we know

$$\begin{aligned} u(t_{2n+1}) &= \left(1-n(2n+2c-1)z^2\right)u(0) - (c+2n)z^2u(0) + \mathcal{O}(h^4) \\ &= \left(1-(2n+1)(n+c)z^2\right)u(0)+\mathcal{O}(h^4) \\ &= \left(1-\frac{2n+1}{2}(2n+1+2c-1)z^2\right)u(0)+\mathcal{O}(h^4). \end{aligned}$$

Hence (5.1.32) is true for $k = 2n+1$. The same argument can be used for $k = 2n+2$. This completes the proof of (5.1.32).

For the exact solution of (4.3.3), we have (compare (4.3.4))

$$y(t) = y_0\left(1-\frac{b^2t^2}{2}\right)+\mathcal{O}((bt)^4).$$

Hence,

$$|y(t_k) - u(t_k)| = \left|\frac{k^2}{2} - \frac{k}{2}(k + 2c - 1)\right| z^2 + \mathcal{O}(h^4).$$

Thus $|y(t_k) - u(t_k)|$ is of order 2 if and only if $\frac{k^2}{2} = \frac{k}{2}(k + 2c - 1)$ which implies $c = 1/2$. This finishes the proof.

□

Remark 5.1.4 The technique used in the above proofs appears to work in the case of $m > 1$ and $q = 1/l$, $l \in \mathbb{N}$. But more complex formulations are expected.

The numerical experiments suggest that the superconvergence results also hold for $q \neq 1/l$ ($l \in \mathbb{N}$) and $m > 1$ (see also Tables 4.1 and 4.2):

> The collocation solutions of problems (4.3.3), (4.4.1) and (5.1.3), for general $q \in (0, 1)$ and $m > 1$, all have superconvergence order of $2m$,
>
> $$\max_{0 \leq n \leq N} |y(t_n) - v(t_n)| \leq Ch^{2m},$$
>
> provided collocation is at the Gauss points.

5.2 Future projects

Based on previous work, some of our potential research projects include the following:

5.2.1 Stability analysis of collocation methods for DEs with constant delay

In order to describe the open numerical stability problems for DVIDEs and DVIEs, we first provide a short survey of stability results for Runge-Kutta and collocation methods for DDEs.

In the recent years, stability properties of numerical methods for delay differential equations have been studied by numerous authors, for example, see [57], [97], [98] and the references therein. In this section, we introduce some relevant numerical stability concepts for collocation methods based on several test equations, and survey some known numerical stability results.

Consider

$$\begin{aligned} y'(t) &= ay(t) + by(t-\tau), \quad t \geq 0, \\ y(t) &= \phi(t), \quad t \leq 0, \end{aligned} \tag{5.2.1}$$

and

$$\begin{aligned} y'(t) &= a(t)y(t) + b(t)y(t-\tau), \quad t \geq 0, \\ y(t) &= \phi(t), \quad t \leq 0. \end{aligned} \tag{5.2.2}$$

Theorem 5.2.1 *([87]) If ϕ is continuous and $\Re(a) + |b| < 0$, then the exact solution of (5.2.1) is asymptotically stable for every τ.*

Theorem 5.2.2 *([87]) The analytic solution of (5.2.2) is bounded by $\phi(t)$,* *provided that,*

$$|b(t)| \leq -\Re(a(t)), \quad t \geq 0. \tag{5.2.3}$$

Recall Barwell's definitions of P-stability and GP-stability (see [7]) for numerical methods.

Definition 5.2.1 A numerical method for DDEs is P-stable if for all a, b satisfying $\Re(a) + |b| < 0$, the numerical solution y_n of (5.2.1) satisfies $\lim_{n\to\infty} y_n = 0$ for every stepsize $h > 0$ such that

$$h = \tau/r, \tag{5.2.4}$$

where r is a positive integer. A mesh with this property is called a constrained mesh.

In other words, a numerical method for DDEs is P-stable if it preserves the asymptotic stability properties of the solution $y(t)$ of (5.2.1) under the constraint (5.2.4) on the stepsize.

Definition 5.2.2 A numerical method for DDEs is GP-stable if, under condition $\Re(a) + |b| < 0$, $\lim_{n\to\infty} y_n = 0$ for every stepsize $h > 0$.

It is clear that a GP-stable method is P-stable too. Definitions of P-stability and GP-stability regions can be found in, for example, [97].

Theorem 5.2.3 *([96]) A Runge-Kutta method for DDEs is P-stable if, when used for ODEs, it is A-stable.*

Theorem 5.2.4 *The one-step collocation method at Gauss points for DDEs is P-stable.*

Proof: See [95] for a direct proof. □

It is proved in [58] that no one-step collocation method with abscissae in $[0, 1)$ can be GP-stable.

GP-stability was also studied for the θ-method in [71].

Definition 5.2.3 A numerical method for DDEs is PN-stable if, under the condition (5.2.3), the numerical solution y_n of (5.2.2) is such that

$$|y_n| \leq \max_{t \leq 0} |\phi(t)|, \tag{5.2.5}$$

for every n and every stepsize h such that $h = \tau/r$, where r is a positive integer.

Definition 5.2.4 A numerical method for DDEs is GPN-stable if, under condition (5.2.3), the numerical solution y_n of (5.2.2) satisfies (5.2.5) for every n and every stepsize $h > 0$.

We observe that a GPN-stable method is also PN-stable and that a PN-stable method for DDEs is AN-stability for ODEs.

PN-stability and GPN-stability are stronger concepts than P-stability and GP-stability in that they are based on a more general test equation, to the same extent that AN-stability is a stronger stability concept than

A-stability for ODEs. Yet another difference is that PN-stability and GPN-stability are demands for contractivity of the numerical solutions, whereas P-stability and GP-stability are demands for asymptotic stability (convergence to zero).

While the A-stability of the numerical method for ODEs is sufficient to assure P-stability and GP-stability provided a suitable interpolation procedure is employed, PN-stability and GPN-stability cannot be guaranteed even if the numerical method for ODEs is AN-stable [97].

The collocation methods at Gauss points, which are A-stable, are P-stable when applied to DDEs [96]. However, not all AN-stable Gauss collocation methods are PN-stable. It is shown in [87] that how the one-stage Gauss collocation method, which is AN-stable, gives rise to a numerical solution y_n which blows up as $n \to \infty$ for (5.2.2) with $a(t) = -b(t) \leq 0$. As a result, a stronger stability concept for ODEs methods has to be introduced.

Consider the test equation

$$\begin{aligned} y'(t) &= a(t)y(t) + \Re(a(t))g(t), \quad t \geq 0, \qquad (5.2.6) \\ y(0) &= y_0, \end{aligned}$$

where $g(t)$ is continuous.

Definition 5.2.5 ([97]) A numerical method is AN_f-stable if the numerical solution y_n of (5.2.6) satisfies

$$|y_{n+1}| \leq \max\{|y_n|, \max_{1 \leq i \leq m} |g(t_n + c_i h)|\},$$

whenever $\Re(a(t)) \leq 0$, $t \geq 0$, and for any mesh Π_N.

It is obvious that requiring a numerical method to be AN$_f$-stable is more than requiring that it be AN-stable. In fact, AN-stability is obtained as a particular case when the forcing term $g(t)$ is identically zero in the test equation (5.2.6).

The link between PN-stability, GPN-stability and AN$_f$-stability is established in the following result.

Theorem 5.2.5 *([97]) If the Runge-Kutta method for DDEs is PN-stable, then the method for ODEs is AN$_f$-stable. Conversely, if the Runge-Kutta method for ODEs is AN$_f$-stable, then the method for DDEs is GPN-stable.*

The stability of collocation methods and direct quadrature methods for DVIEs with constant delay have been studied by Vermiglio [89] and Cahlon [28], respectively. See also [30] for theoretical stability results for a more general test equation.

Consider the following delay integral equation:

$$\begin{aligned} y(t) &= f(t)+\int_0^t K(t,s,y(s))\,ds+\int_0^{t-\tau} H(t,s,y(s))\,ds, \quad t \geq 0, \quad (5.2.7) \\ y(t) &= \phi(t), \quad t \in [-\tau, 0], \end{aligned}$$

and the test equation corresponding to (5.2.7),

$$\begin{aligned} y(t) &= 1+a\int_0^t y(s)\,ds+b\int_0^{t-\tau} y(s)\,ds, \quad t \geq 0, \qquad (5.2.8) \\ y(t) &= \phi(t), \quad t \in [-\tau, 0], \end{aligned}$$

where a, b are complex parameters. We observe that, by differentiating (5.2.8), we obtain (5.2.1). The P-stability is defined similar to Definition 5.2.1 using test equation (5.2.8).

Theorem 5.2.6 *([89]) If the collocation parameters $\{c_i\}$ are such that they yield an A-stable collocation method for an ODE, then the corresponding (discretized) collocation method for delay integral equation (5.2.7) is P-stable.*

Stability properties of exact and discretized collocation methods for Volterra integral and integro-differential equations without delay are studied in [21], [35], [36] and [37].

5.2.2 Stability analysis of collocation methods for equations with proportional delay

For numerical solution of the proportional delay problem (the mesh is not required to be constrained), the concepts of P-stability and PN-stability are no longer feasible. We only need to consider the classical asymptotic stability (the numerical solution y_n tends to zero as $n \to \infty$).

While the stability analysis of numerical methods for the constant delay problem is rather developed, only a very limited number of stability results are known for proportional delay problems. Several open problems are addressed in this section.

Consider the following two test equations,

$$y'(t) = ay(t) + by(qt), \quad t \geq 0, \quad y(0) = 1, \tag{5.2.9}$$

and

$$y'(t) = ay(t) + by(qt) + cy'(qt), \quad t \geq 0, \quad y(0) = 1. \tag{5.2.10}$$

Theorem 5.2.7 *([59]) The analytic solution of (5.2.10) is asymptotically stable if and only if $\Re(a) < 0$ and $|b| < |a|$ while c has no bearing.*

Corollary 5.2.1 *The analytic solution of (5.2.9) is asymptotically stable if and only if $\Re(a) < 0$ and $|b| < |a|$.*

The stability analysis is difficult in the proportional delay case because the delay is not fixed. Instead, the lag term $(1-q)t$ becomes bigger as t increases. However, some work has been done for (5.2.9) and (5.2.10) when $q = 1/2$, for example,

Theorem 5.2.8 *([25]) If $|b| < |a|$ and let*

$$|\frac{1}{2}hb\frac{1-d}{1-hda}| < 1$$

or

$$|\frac{1}{2}hdb\frac{1}{1-hda}| < 1$$

hold, depending on whether $d < 1/2$ or $d \geq 1/2$ where d is the only collocation point. Then the one-stage collocation solution y_n of (5.2.9) with $q = 1/2$ is square-summable, so in particular $\lim_{n\to\infty} y_n = 0$.

A sufficient condition for the asymptotic stability of the numerical solution to problem (5.2.10) with a particular value of q, i.e., $q = 1/2$, is given in [24]. Let $q = 1/2$, and consider the following numerical scheme to (5.2.10)

$$y_{2n} = \alpha y_{2n-1} + \frac{1}{4}(y_{n-1} + 3y_n) + \gamma(y_n - y_{n-1}), \tag{5.2.11}$$

$$y_{2n+1} = \alpha y_{2n} + \frac{1}{4}(3y_n + y_{n+1}) + \gamma(y_{n+1} - y_n), \tag{5.2.12}$$

where

$$\alpha := \frac{1+\frac{1}{2}ha}{1-\frac{1}{2}ha}, \quad \beta := \frac{hb}{1-\frac{1}{2}ha}, \quad \gamma := \frac{c}{1-\frac{1}{2}ha}.$$

Theorem 5.2.9 *([24]) The numerical solution y_n of (5.2.10) with $q = 1/2$, defined by (5.2.11)-(5.2.12), is asymptotically stable if $\Re(a) < 0$, $|b| < |a|$ and*

$$\max\{|c + \frac{1}{4}hb|, |c - \frac{1}{4}hb|\} < |c - \frac{1}{2}ha|. \tag{5.2.13}$$

These conditions coincide with conditions for asymptotic stability of exact solution of (5.2.10) (see Theorem 5.2.7) except the stepsize h need be restricted. It is also pointed out in [24] that the conclusion of Theorem 5.2.9 holds when q is a reciprocal of an integer with the last condition (5.2.13) replaced by

$$\max\{|c + \frac{1}{2}qhb|, |c - \frac{1}{2}qhb|\} < |c - \frac{1}{2}ha|.$$

But the approach used there fails for general $q \in (0, 1)$.

The stability properties of m-stage collocation solutions to (5.2.9) and (5.2.10) are still unknown for $m = 1$ and $q \neq 1/2$, or $m \geq 2$ and $0 < q < 1$.

In [11], contractivity conditions are found for Runge-Kutta methods as applied to DDE of the type

$$y'(t) = f(t, y(t), y(\theta(t))), \quad t \geq t_0, \tag{5.2.14}$$

$$y(t) = \phi(t), \quad t \leq t_0. \tag{5.2.15}$$

where $\theta(t) \leq t$.

The asymptotic stability of exact solution to VIDEs with proportional delay of the form

$$y'(t) = ay(t) + \int_0^1 y(qt)d\mu(q) + \int_0^1 y'(qt)d\nu(q), \quad t > 0, \quad y(0) = y_0, \tag{5.2.16}$$

where the integrals being considered are of Riemann-Stieltjes type, is investigated in [60]. (5.2.16) includes many interesting equations, for example, (5.2.9) when $d\mu(q) = b\delta(q-p)dq$ and $d\nu(q) \equiv 0$ where δ is the Dirac function.

Theorem 5.2.10 *([60]) If $\Re(a) < 0$, $\int_0^1 |d\mu(q)| < |a|$, and*

$$\lim_{h \to 0} \int_0^h |d\mu(q)| = \lim_{h \to 0} \int_0^h |d\nu(q)| = 0,$$

then the analytic solution of (5.2.16) is asymptotically stable.

The stability analysis of numerical solutions to (5.2.16) is open.

The paper [73] gave the first stability analysis of the θ-method used for the numerical solution to (5.2.9).

Theorem 5.2.11 *([73]) If $\Re(a) < 0$, then the numerical solution y_n of a given θ-method (applied to equation (5.2.9))*

1. *tends to* 0 *as* $n \to \infty$ *provided that* $(2\theta - 1)|a| > |b|$ *and* $\lim_{n\to\infty} h_n = \infty$,

2. *is uniformly bounded provided that* $(2\theta - 1)|a| = |b|$ *and* $\sum_{n=0}^{\infty} h_n^{-1} < \infty$.

The stability analysis of θ-method for neutral functional-differential equation (5.2.10) is accomplished in [72] (constant stepsize) and [12] (constrained variable stepsize).

Theorem 5.2.12 *([72]) The numerical solution of (5.2.10) tends to zero for any constant stepsize as long as* $\Re(a) < 0$ *and* $|a| > |b|$, *if and only if* $\theta \geq 1/2$.

The stability analysis of collocation method for differential, integral and integro-differential equations with proportional delay is one of our future projects. In particular, we want to know if the conclusions of the Theorems 5.2.3 and 5.2.6 hold for the collocation solutions of (4.0.1) and (4.0.2) respectively.

5.2.3 Convergence of collocation methods for VIDE with state-dependent delay

The convergence and local superconvergence analysis for collocation methods when applied to VIDE with state-dependent delay of the form:

$$\begin{aligned} y'(t) &= g(t) + \int_0^t K(t-s, y(s), y(\theta(y(s))))ds, \quad t \in I, \\ y(t) &= \phi(t), \quad -\tau \leq t < 0, \end{aligned}$$

where τ is a positive constant, is at present an open problem.

Since the positions of the primary discontinuities in solution $y(t)$ depend on $y(t)$ itself, it is difficult to predict a priori where they may arise.

However, similar DVIEs with state-dependent delay are studied in [29]. Consider Volterra integral equations of the form

$$\begin{aligned} y(t) &= f(t) + \int_0^t H(t, s, y(s), y(\theta(y(s))))\, ds, \quad t \in I, \\ y(t) &= \phi(t), \quad -\tau \le t < 0. \end{aligned}$$

The determination of the solution y requires knowledge of $y(t) = \phi(t)$ for some initial set of negative t. The question of existence of the solution y is approached using a fixed-point theorem; and numerical methods for determining an approximate solution involve the replacement of $\int_0^t k(t,s)y(s)\,ds$ by $\sum_{j=0}^{n} w_{n,j}(t)y(t_{n,j})$ in order to discretize the case where $H(t,s,y,z) = k(t,s)k_1(t,s,y,z)$. The convergence of this numerical method is proved.

We state two questions followed by corresponding conjectures:

1. Do the collocation methods in $S_m^{(0)}(\Pi_N)$ for state-dependent DVIDEs have a global convergence order of m?

2. Is local superconvergence possible for VIDEs with state-dependent delay; i.e., is a convergence order of p^* with $p^* = 2m$ possible at the mesh points?

Conjectures:

1. The collocation methods for VIDE with state-dependent delay have a

global convergence order of m provided that the given functions are sufficiently smooth.

2. Local superconvergence order of p^{*} with $p^{*} = 2m$ is possible for state-dependent DVIDE when collocation is at the Gauss points.

Bibliography

[1] V.A. Ambartsumian. On the fluctuation of the brightness of the Milky Way. *Doklady Akad. Nauk USSR*, 44:223–226, 1944.

[2] H. Arndt and C.T.H. Baker. Runge-Kutta formulae applied to Volterra functional equations with a fixed delay. In *Numerical Treatment of Differential Equations (Halle, 1987)*, volume 104 of *Teubner-Texte Math.*, pages 19–30. Teubner, Leipzig, 1988.

[3] N. Baddour and H. Brunner. Continuous Volterra-Runge-Kutta methods for integral equations with pure delay. *Computing*, 50(3):213–227, 1993.

[4] C.T.H. Baker and N.J. Ford. Asymptotic error expansions for linear multistep methods for a class of delay integro-differential equations. *Bull. Greek Math. Soc.*, 31:5–18, 1990.

[5] C.T.H. Baker and C.A.H. Paul. Pitfalls in parameter estimation for delay differential equations. *SIAM J. Sci. Comput.*, 18(1):305–314,

1997.

[6] C.T.H. Baker, C.A.H. Paul, and D.R. Willé. Issues in the numerical solution of evolutionary delay differential equations. *Adv. Comp. Math.*, 3:171–196, 1995.

[7] V.K. Barwell. Special stability problems for functional differential equations. *BIT*, 15:130–135, 1975.

[8] J. Bélair. Sur une équation différentielle fonctionnelle analytique. *Canad. Math. Bull.*, 24(1):43–46, 1981.

[9] A. Bellen. One-step collocation for delay differential equations. *J. Comput. Appl. Math.*, 10(3):275–283, 1984.

[10] A. Bellen. Constrained mesh methods for functional-differential equations. In *Delay Equations, Approximation and Application (Mannheim, 1984)*, volume 74 of *Internat. Schriftenreihe Numer. Math.*, pages 52–70. Birkhäuser, Basel, 1985.

[11] A. Bellen. Contractivity of continuous Runge-Kutta methods for delay differential equations. *Appl. Numer. Math.*, 24(2-3):219–232, 1997. Volterra centennial (Tempe, AZ, 1996).

[12] A. Bellen, N. Guglielmi, and L. Torelli. Asymptotic stability properties of θ-methods for the pantograph equation. *Appl. Numer. Math.*, 24(2-3):279–293, 1997. Volterra centennial (Tempe, AZ, 1996).

[13] H. Brunner. Nonpolynomial spline collocation for Volterra equations with weakly singular kernels. *SIAM J. Numer. Anal.*, 20(6):1106–1119, 1983.

[14] H. Brunner. The approximate solution of initial-value problems for general Volterra integro-differential equations. *Computing*, 40:125–137, 1988.

[15] H. Brunner. On discrete superconvergence properties of spline collocation methods for nonlinear Volterra integral equations. *J. Comput. Math.*, 10(4):348–357, 1992.

[16] H. Brunner. Collocation and continuous implicit Runge-Kutta methods for a class of delay Volterra integral equations. *J. Comput. Appl. Math.*, 53(1):61–72, 1994.

[17] H. Brunner. Iterated collocation methods for Volterra integral equations with delay arguments. *Math. Comp.*, 62(206):581–599, 1994.

[18] H. Brunner. The numerical solution of neutral Volterra integro-differential equations with delay arguments. *Ann. Numer. Math.*, 1(1-4):309–322, 1994. Scientific computation and differential equations (Auckland, 1993).

[19] H. Brunner. On the discretization of differential and Volterra integral equations with variable delay. *BIT*, 37(1):1–12, 1997.

[20] H. Brunner. The use of splines in the numerical solution of Volterra integral and integro-differential equations. In S. Dubuc, editor, *Splines and the Theory of Wavelets I, II*, CRM Proceedings and Lecture Notes. American Mathematical Society, Providence, R.I., 1998.

[21] H. Brunner and P.J. Van der Houwen. *The Numerical Solution of Volterra Equations.* CWI Monograph 3, North-Holland, Amsterdam, 1986.

[22] H. Brunner and W. Zhang. Primary discontinuities in solutions for delay integro-differential equations. *Methods and Applications of Analysis, submitted*, 1997.

[23] M.D. Buhmann and A. Iserles. Numerical analysis of functional equations with a variable delay. In *Numerical Analysis 1991 (Dundee, 1991)*, volume 260 of *Pitman Res. Notes Math. Ser.*, pages 17–33. Longman Sci. Tech., Harlow, 1992.

[24] M.D. Buhmann and A. Iserles. On the dynamics of a discretized neutral equation. *IMA J. Numer. Anal.*, 12:339–363, 1992.

[25] M.D. Buhmann, A. Iserles, and S.P. Nørsett. Runge-Kutta methods for neutral differential equations. In *Contributions in Numerical Mathematics*, volume 2 of *World Sci. Ser. Appl. Anal.*, pages 85–98. World Sci. Publishing, River Edge, NJ, 1993.

[26] T.A. Burton. *Volterra Integral and Differential Equations*, volume 167 of *Mathematics in Science and Engineering*. Academic Press, 1983.

[27] J.C. Butcher. *The Numerical Analysis of Ordinary Differential Equations*. Wiley, London, 1987.

[28] B. Cahlon. On the numerical stability of Volterra integral equations with delay argument. *J. Comput. Appl. Math.*, 33(1):97–104, 1990.

[29] B. Cahlon and L.J. Nachman. Numerical solutions of Volterra integral equations with a solution dependent delay. *J. Math. Anal. Appl.*, 112(2):541–562, 1985.

[30] B. Cahlon and D. Schmidt. Stability criteria for certain delay integral equations of Volterra type. *J. Comput. Appl. Math.*, 84(2):161–188, 1997.

[31] E.D. Callender. Single step methods and low order splines for solutions of ordinary differential equations. *SIAM J. Numer. Anal.*, 8:61–66, 1971.

[32] G.M. Cerezo. *Solution Representation and Identification for Singular Neutral Functional Differential Equations*. PhD thesis, Virginia Polytechnic Institute and State University, Blacksburg, Virginia, December 1996.

[33] L.G. Chambers. Some properties of the functional equation $\phi(x) = f(x) + \int_0^{\lambda x} g(x, y, \phi(y))dy$. *Internat. J. Math. & Math. Sci.*, 14(1):27–44, 1991.

[34] C. Corduneanu. *Integral Equations and Applications.* Cambridge University Press, Cambridge, 1991.

[35] M.R. Crisci, E. Russo, and A. Vecchio. On the stability of the one-step exact collocation method for the second kind Volterra integral equation with degenerate kernel. *Computing*, 40(4):315–328, 1988.

[36] M.R. Crisci, E. Russo, and A. Vecchio. Stability of collocation methods for Volterra integro-differential equations. *J. Integral Equations Appl.*, 4(4):491–507, 1992.

[37] M.R. Crisci, E. Russo, and A. Vecchio. Stability results for one-step discretized collocation methods in the numerical treatment of Volterra integral equations. *Math. Comp.*, 58(197):119–134, 1992.

[38] J.M. Cushing. *Integro-Differential Equations and Delay Models in Population Dynamics, Lecture Notes in Biomathematics, Vol. 20.* Springer-Verlag, New York, 1977.

[39] K. Dekker and J.G. Verwer. *Stability of Runge-Kutta Methods for Stiff Nonlinear Differential Equations.* North Holland, Amsterdam, 1984.

[40] G.A. Derfel. On the asymptotic behavior of the solutions of a class of functional-differential equations. In *Asymptotic Behavior of Solutions of Functional-differential Equations*, pages 58–65. Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 1978.

[41] G.A. Derfel. Kato problem for functional-differential equations and difference Schrödinger operators. *Operator Theory: Advances and Applications*, 34:319–321, 1990.

[42] L.E. El'sgol'ts and S.B. Norkin. *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*. Academic Press, New York, 1973.

[43] W.H. Enright and M. Hu. Interpolating Runge-Kutta methods for vanishing delay differential equations. *Computing*, 55:223–236, 1995.

[44] R. Esser. Numerische Behandlung einer Volterraschen Integralgleichung. *Computing*, 19:269–284, 1978.

[45] A. Feldstein, A. Iserles, and D. Levin. Embedding of delay equations into an infinite-dimensional ODE system. *J. Differential Equations*, 117(1):127–150, 1995.

[46] A. Feldstein and K.W. Neves. High order methods for state-dependent delay differential equations with nonsmooth solutions. *SIAM J. Numer. Anal.*, 21(8):844–863, 1984.

[47] L. Fox, D.F. Mayers, J.R. Ockendon, and A.B. Tayler. On a functional differential equation. *J. Inst. Math. Appl.*, 8:271–307, 1971.

[48] G. Gripenberg, S.-O. Londen, and O. Staffans. *Volterra Integral and Functional Equations*, volume 34 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1990.

[49] S.I. Grossman and R.K. Miller. Perturbation theory for Volterra integro-differential systems. *J. Differential Equations*, 8:457–474, 1970.

[50] A. Guillou and J.L. Soulé. La résolution numérique des problèmes différentiels aux conditions initiales par des méthodes de collocation. *R.I.R.I.*, 3:17–44, 1969.

[51] E. Hairer, S.P. Nørsett, and G. Wanner. *Solving Ordinary Differential Equations I, Nonstiff Problems.* Springer-Verlag, Berlin, 1993.

[52] E. Hairer and G. Wanner. *Solving Ordinary Differential Equations II, Stiff and Differential Algebraic Problems.* Springer-Verlag, Berlin, 1987.

[53] J.K. Hale and S.M. Verduyn Lunel. *Introduction to Functional-differential Equations*, volume 99 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1993.

[54] D. Harvey. Global approximations of initial value problems for special second order ordinary differential equations. *M.A. thesis, Dalhousie University, Halifax, Nova Scotia*, 1971.

[55] H.S. Hung. Application of linear spline functions to the numerical solution of Volterra integral equations of the second kind. *Comp. Sci. Techn. Report 27, University of Wisconsin, Madison*, 1970.

[56] H.S. Hung. The numerical solution of differential and integral equations by spline functions. *MRC Techn. Summ. Report 1053, University of Wisconsin, Madison*, 1970.

[57] K.J. in 't Hout. Stability analysis of Runge-Kutta methods for system of delay differential equations. *IMA J. Numer. Anal.*, 17:17–27, 1997.

[58] K.J. in 't Hout and M.N. Spijker. Stability analysis of numerical methods for delay differential equations. *Numer. Math.*, 59:807–814, 1991.

[59] A. Iserles. On the generalized pantograph functional-differential equation. *European J. Appl. Math.*, 4(8):1–38, 1993.

[60] A. Iserles and Y. Liu. On pantograph integro-differential equations. *J. Integral Equations Appl.*, 6:213–237, 1994.

[61] A. Iserles and S.P. Nørsett. *Order Stars.* Chapman & Hall, 1991.

[62] A. Iserles and J. Terjéki. Stability and asymptotic stability of functional-differential equation. *J. London Math. Soc.*, 51(2):559–572, 1995.

[63] Z. Jackiewicz. The numerical solution of Volterra functional differential equations of neutral type. *SIAM J. Numer. Anal.*, 18:615–626, 1981.

[64] Z. Jackiewicz. Quasilinear multistep methods and variable step predictor-corrector methods for neutral functional differential equations. *SIAM J. Numer. Anal.*, 23:423–452, 1986.

[65] Z. Jackiewicz and M. Kwapisz. The numerical solution of functional differential equations, a survey. *Mat. Stos.*, 33:57–78, 1991.

[66] T. Kato and J.B. McLeod. The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$. *Bull. Amer. Math. Soc.*, 77:891–937, 1971.

[67] N.G. Kazakova and D.D. Bainov. An approximate solution of the initial value problem for integro-differential equations with a deviating argument. *Math. J. Toyama Univ.*, 13:9–27, 1990.

[68] G. Keller. Numerical solution of initial-value problems by collocation methods using generalized piecewise functions. *Computing*, 28:199–211, 1982.

[69] V. Kolmanovskii and A. Myshkis. *Applied Theory of Functional Differential Equations.* Kluwer Academic Publishers, Dordrecht, 1992.

[70] J.D. Lambert. *Numerical Methods for Ordinary Differential Equations.* Wiley, Chichester, 1991.

[71] M.Z. Liu and M.N. Spijker. The stability of the θ-methods in the numerical solution of delay differential equations. *IMA J. Numer. Anal.*, 10(1):31–48, 1990.

[72] Y. Liu. Stability analysis of θ-methods for neutral functional-differential equations. *Numer. Math.*, 70:473–485, 1995.

[73] Y. Liu. On the θ-method for delay differential equations with infinite lag. *J. Comput. Appl. Math.*, 71:177–190, 1996.

[74] F.R. Loscalzo. An introduction to the application of spline functions to initial value problems. In *Theory and Applications of Spline Functions (Proceedings of Seminar, Math. Research Center, Univ. of Wisconsin, Madison, Wis., 1968)*, pages 37–64, New York, 1969. Academic Press.

[75] F.R. Loscalzo and T.D. Talbot. Spline function approximations for solutions of ordinary differential equations. *SIAM J. Numer. Anal.*, 4:433–445, 1967.

[76] K. Mahler. On a special functional equation. *J. London Math. Soc.*, 15:115–123, 1940.

[77] A. Makroglou. A block-by-block method for the numerical solution of Volterra delay integro-differential equations. *Computing*, 30:49–62, 1983.

[78] G. Micula. Approximate solution of the differential equation $y'' = f(x, y)$ with spline functions. *Math. Comp.*, 27:807–816, 1973.

[79] G. Micula and R. Gorenflo. Theory and Applications of Spline Functions. *Preprint No. A-91-33 Part I and Part II, Fachbereich Mathematik, Serie A, Freie Universität, Berlin*, 1991.

[80] R.K. Miller. *Nonlinear Volterra Integral Equations*. Benjamin, Menlo Park (California), 1971.

[81] G.R. Morris, A. Feldstein, and E.W. Bowen. The Phragmén-Lindelöf principle and a class of functional differential equations. In *Ordinary Differential Equations (Proc. Conf., Math. Res. Center, Naval Res. Lab., Washington, D.C., 1971)*, pages 513–540, New York, 1972. Academic Press.

[82] H.N. Mülthei. Numerische Lösung gewöhnlicher Differentialgleichungen mit Splinefunktionen. *Computing*, 25:317–335, 1980.

[83] K.W. Neves and A. Feldstein. Characterization of jump discontinuities for state dependent delay differential equations. *J. Math. Anal. Appl.*, 56(8):689–707, 1976.

[84] J.R. Ockedon and A.B. Taylor. The dynamics of a current collection system for an electric locomotive. *Proc. Royal Soc. Series A*, 322:447–468, 1971.

[85] E.Y. Romanenko and A.N. Sharkovskiĭ. Asymptotic solutions of differential-functional equations. In *Asymptotic Behavior of Solutions of Differential Equations*, pages 5–39. Inst. Math. Akad. Nauk UkrSSR Press, 1978.

[86] A.H. Stroud. *Numerical Quadrature and Solution of Ordinary Differential Equations*. Springer-Verlag, New York, 1974.

[87] L. Torelli. Stability of numerical methods for delay differential equations. *J. Comput. Appl. Math.*, 25:15–26, 1989.

[88] P. Vâtă. Convergence theorems of some numerical approximation schemes for the class of nonlinear integral equations. *Bul. Univ. Galati Fasc. II Mat. Fiz. Mec. Teoret.*, 1:25–33, 1978.

[89] R. Vermiglio. On the stability of Runge-Kutta methods for delay integral equations. *Numer. Math.*, 61(4):561–577, 1992.

[90] V. Volterra. Sulla inversione degli integrali definiti, Nota I. *Atti R. Accad. Sci. Torino*, 31:311–323, 1896.

[91] V. Volterra. Sulla inversione degli integrali definiti, Nota II. *Atti R. Accad. Sci. Torino*, 31:400–408, 1896.

[92] V. Volterra. Sopra alcune questioni di inversione di integrali definite. *Ann. Mat. Pura Appl.*, 25:139–178, 1897.

[93] V. Volterra. Sur la théorie mathématique des phénomènes héréditaires. *J. Math. Pures Appl.*, 7(9):249–298, 1928.

[94] P. Waltman. A note on an oscillation criterion for an equation with a functional argument. *Canad. Math. Bull.*, 11:593–595, 1968.

[95] M. Zennaro. On the P-stability of one-step collocation for delay differential equations. *ISNM*, 74:334–343, 1985.

[96] M. Zennaro. P-stability properties of Runge-Kutta methods for delay differential equations. *Numer. Math.*, 49:305–318, 1986.

[97] M. Zennaro. Delay differential equations: theory and numerics. In *Theory and Numerics of Ordinary and Partial Differential Equations (Leicester, 1994)*, Adv. Numer. Anal., IV, pages 291–333. Oxford Univ. Press, New York, 1995.

[98] M. Zennaro. Asymptotic stability analysis of Runge-Kutta methods for nonlinear systems of delay differential equations. *Numer. Math.*, 77:549–563, 1997.

[99] W. Zhang. Superconvergence of collocation solution of differential and integro-differential equations with proportional delay. *Computing, submitted*, 1998.

[100] W. Zhang and H. Brunner. Collocation approximations for second-order differential equations and Volterra integro-differential equations with variable delay. *Canadian Applied Mathematics Quarterly*, 6(3):1–17, 1998.

Index