Extended symmetry analysis of (1+2)-dimensional Fokker–Planck equation

by

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Abstract

We carry out the extended symmetry analysis of an ultraparabolic degenerate Fokker–Planck equation with three independent variables, which is also called the Kolmogorov equation and is singled out within the class of such Fokker–Planck equations by its remarkable symmetry properties. In particular, its essential Lie invariance algebra is eight-dimensional, which is the maximum dimension within the above class. We compute the complete and essential point symmetry groups of the Kolmogorov equation using the direct method and analyze their structure. After listing inequivalent one- and two-dimensional subalgebras of the essential Lie invariance algebra of this equation, we exhaustively classify its Lie reductions. As a result, we construct wide families of exact solutions of the Kolmogorov equation, and three of them are parameterized by single arbitrary solutions of the (1+1)-dimensional linear heat equation. We also establish the point similarity of the Kolmogorov equation to the (1+2)-dimensional Kramers equations whose essential Lie invariance algebras are eight-dimensional, which allows us to find wide families of exact solutions of these Kramers equations in an easy way.

Dedicated to Prof. George Bluman, who knows how much suffering brings fascism as much as I now know how much suffering brings russism.

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I also express deepest thanks to the Armed Forces of Ukraine and the civil Ukrainian people for their bravery and courage in defense of peace and freedom in Europe and in the entire world from russism.

Statement of contribution

The thesis is based on the preprint [29]. The general topic of research was suggested by Prof. Roman O. Popovych. Chapter 3 contains only original results. The example of $(1+1)$-dimensional linear heat equation with inverse square potential in Sections 2.4 and 2.5 is also original. The results in Chapter 3 were derived myself independently, except for Section 3.4 which was developed in course of discussing with Prof. Popovych. Throughout the text of the thesis each known theorem has a reference for the corresponding source, all other theorems are original. The text of the manuscript was prepared in collaboration with Dr. Alexander Bihlo and Prof. Roman O. Popovych.

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G	a Lie group
$H < G$	a subgroup H of G
$N \lhd G$	a normal subgroup N of G
$H \ltimes N$	semidirect sum of subgroup $H < G$ with $N \lhd G$
$\mathfrak{g}$	a Lie algebra
$\mathbb{R}$	the field of real numbers
$(\mathbb{R}, +)$	the additive group of the field of real numbers
$\mathbb{Z}$	the ring of integers
$\partial_x = \dfrac{\partial}{\partial x}$	the partial derivative with respect to x
$\mathfrak{l} \Subset \mathfrak{r}$	semidirect sum of a subalgebra $\mathfrak{l}$ of the Lie algebra $\mathfrak{g}$ with an ideal $\mathfrak{r}$ of $\mathfrak{g}$
$\oplus$	direct sum of Lie algebras
$\dot{+}$	direct sum of vector spaces

Chapter 1

Introduction

The idea of symmetries is undoubtedly one of the most conceptual and profound theories, which has found its place among various branches of physics and mathematics. At the intuitive level, the concept of "symmetry" is used to denote the preservation of some properties under transformations. Historically this notion dates back to the Pythagorean school in ancient Greece, where philosophers studied the symmetry of regular polygons [60]. Symmetries play an important role in the construction and study of mathematical models of fundamental laws of nature, which are often formulated in terms of nonlinear (partial) differential equations.

Inspired by Galois's work on algebraic equations, a well-known Norwegian mathematician Sophus Lie started considering the problem of finding a general integration method for arbitrary differential equations. This problem is still open, however Lie developed an approach for computing and using symmetries of differential equations for constructing its solutions, which is based on the theory of continuous groups. Lie showed that the known one-parameter point symmetry group of a given ordinary differential equation, which consists of locally defined parameterized point transformations, allows to reduce the order of the equation by one [35]. In general, if an ordinary differential equation of order n admits an r-parameter solvable Lie point symmetry group, then it can be reduced to an ordinary differential equation of order $n - r$. Applications of Lie groups to differential equations are well explained in [9, 10, 41]. Nowadays, the methods developed by Lie have been formed into an independent section of the theory of differential equations, which began rapidly developing in the 1960s and now is fruitfully used in the study of mathematical models of physical processes and phenomena.

The main strength of this theory is its universality and algorithmicity. It is successfully applied to many systems of differential equations. The main requirement is the non-triviality of the symmetry group admitted by the system of equations. This requirement is not artificial since symmetry is embedded in the initial formulation of most mathematical models in terms of the properties of homogeneity, isotropy of space, symmetry with respect to Lorentz transformations, etc.

For a given system of partial differential equations there exist a number of methods for computing its complete point symmetry group. The most universal among them is the direct method [5, 24–26, 49], which is based on computing of point symmetry transformations directly from the condition of invariance of the system under those actions. In general, the direct approach results in highly-coupled, nonlinear system of partial differential equations for the components of point symmetry transformations, which is usually difficult to solve. However, if it is possible to find a normalized class of equations containing this system, roughly speaking, one can apply general common constraints for admissible transformations (which preserve the equations within this class) for equations from the covering class to the particular system of equations. As a rule, computing certain common restrictions on admissible transformations of the entire normalized class or point symmetry transformations of a single equation from this class have the same level of complexity, e.g., in order to derive the restriction that the transformation component corresponding to t depends only on t, we need to carry out approximately the same operations, independently whether we consider the entire class of $(1+1)$-dimensional evolution equations, any well-defined subclass from this class or any single evolution equation.

Another approach is considering the infinitesimal counterpart of the invariance condition only, which results in the corresponding Lie algebra of infinitesimal generators of one-parameter point symmetry groups of the system. The main drawback of this method is that one can obtain only the identity component of the complete point symmetry group. To obtain the entire group, one can also use the so-called algebraic method [2,3], which is based on the idea that each point symmetry transformation of a system of differential equations induces an automorphism of the corresponding Lie algebra [20–22]. It results in restrictions on the form of point symmetry transformations and then one invokes the direct method in order to complete computations.

In this thesis we carry out the extended symmetry analysis of an ultraparabolic degenerate Fokker–Planck equation with three independent variables, which is singled out within the class of such Fokker–Planck equations by its remarkable symmetry properties. The essential Lie invariance algebra is nonsolvable and is of complicated and specific structure, which makes the classification of its subalgebras nontrivial. Such structure has never appeared before in the literature from the point of view of classification of subalgebras. That is why we also refer to this equation as the remarkable Fokker–Planck equation. The study of this equation was initiated by Kolmogorov in 1934 [27], and that is why it is often called the Kolmogorov equation as well. A preliminary study of symmetry properties of this equation was carried out in [19, 28, 31]. As the examples of the real-world application, such Fokker–Planck equation can be used to model two-dimensional Wiener and Ornstein–Uhlenbeck processes [51], the distribution function for a particle species in plasma physics [52] and the evolution of cell populations [53, 54, 61].

The main goal of this thesis is the classification of codimension-one and -two Lie reductions of the ultraparabolic degenerate $(1 + 2)$-dimensional Fokker–Planck equation. For this purpose we need to find the complete point symmetry group of this equation, describe its properties, then classify one- and two-dimensional subalgebras of the essential Lie invariance algebra of the equation.

The most studied are of course Lie symmetries and other related objects of the Fokker–Planck and Kolmogorov equations in dimension $1 + 1$. The consideration of them was initiated in the seminal paper [34] of Sophus Lie himself in the course of the group analysis of the wider class of second-order linear partial differential equations with two independent variables, including $(1{+}1)$-dimensional second-order linear evolution equations. A number of papers that restate, specify or develop the above Lie's result were published in last decades, see, e.g., [50] for a review of these papers and a modern treatment of the problem. The equivalence of $(1{+}1)$-dimensional second-order linear evolution equations with respect to point transformations was considered in [8, 34, 64] and further in [18, 23, 38]. Darboux transformations between such equations were studied, e.g., in [7, 12, 37, 50]. The group classification problems for the classes of $(1{+}1)$-dimensional Fokker–Planck and Kolmogorov equations and their subclasses were solved in [50] up to the general point equivalence and in [44] with respect to the corresponding equivalence groups using the mapping method of group classification.

The structure of this thesis is as follows.

In Chapter 2 we present some theoretical results on Lie groups (Section 2.1), Lie algebras (Section 2.2) and symmetries of differential equations (Section 2.3) following the classical textbooks and lecture notes [32, 33, 41, 65]. In Section 2.4 we explain direct method for computing the complete point symmetry group and in Section 2.5 we discuss some notions on group-invariant solutions. Throughout Sections 2.4 and 2.5 we consider an example of extended symmetry analysis of the (1+1)-dimensional linear heat equations with inverse square potentials, including the explicit construction of their (real) exact solutions. In view of results of Section 3.5, these solutions directly lead to the solutions of the remarkable Fokker–Planck equation. The last section in Chapter 2 contains rather general results for determining the nature of infinitesimal generators for point symmetry groups of linear partial differential equations.

Chapter 3 contains our main results. In Section 3.2, we present the maximal Lie invariance algebra of the remarkable Fokker–Planck equation (3.2) and describe its key properties. Using the direct method, in Section 3.3 we compute the complete point symmetry group of the equation (3.2) and analyze its structure, including the decomposition of this group and the description of its discrete elements. Section 3.4 is devoted to the classification of one- and two-dimensional subalgebras of the maximal Lie invariance algebra, which are relevant for the framework of Lie reductions. This classification creates the basis for the comprehensive study of the codimension-one and -two Lie reductions of the equation (3.2) in Sections 3.5 and 3.6, respectively, and the subsequent construction of wide families of exact solutions of (3.2). The similarity of Kramers equations with eight-dimensional essential Lie symmetry algebras to the equation (3.2) with respect to point transformations is established in an explicit form in Section 3.7. It results in easily finding wide families of exact solutions of such Kramers equations.

In the last chapter we discuss the obtained results and outline possible directions of further study within the symmetry analysis of $(1+2)$-dimensional ultraparabolic Fokker–Planck equations.

Chapter 2

Theoretical background

2.1 Lie groups

Hereafter by a manifold we understand a topological space equipped with a C^∞-structure. As well we require this space to be T_2-separable and second countable (however in general it is not necessary). In practice in group analysis of differential equations all manifolds are quite nice, i.e., they are Hausdorff and second countable. And by "smooth" we always mean C^∞.

Definition 1. A Lie group is a group G that is a differentiable manifold equipped with a group multiplication map $\mu\colon G \times G \to G$, $\mu(g,h) := g \cdot h$ that is smooth.

Lemma 2 (see [33])**.** *Let G be a Lie group. Denote by $\nu\colon G \to G$, $\nu(g) := g^{-1}$ the inversion map on G. Then ν is smooth and for any $g \in G$ we have $e = \mu(g, \nu(g)) = \mu(\nu(g), g)$, where e is the identity element of the group G.*

Roughly speaking, symmetries of some system of differential equations are some point transformations, defined on the space of dependent and independent variables of the system, which preserve solutions. All these transformations constitute a Lie (pseudo)group with μ being the composition of transformations. However, point symmetries do not exhaust all types of symmetries. For instance, there are contact symmetries, and higher-order or generalized symmetries [41].

In practice one mostly uses local Lie (pseudo)groups, whose elements are sufficiently close to the identity transformation. This allows us to forget about the global structure of the corresponding manifold and define all operations in local coordinates only.

Definition 3. A one-parameter subgroup of a Lie group G is a Lie group homomorphism $\alpha\colon (\mathbb{R},+) \to G$, i.e., a smooth curve $\alpha\colon \mathbb{R} \to G$ such that $\alpha(s+t) = \mu(\alpha(s),\alpha(t))$ for all $s,t \in \mathbb{R}$.

The main tool in computing the group of point symmetry transformations of a system of differential equations is the infinitesimal approach introduced by Sophus Lie. This approach replaces computing this group with computing its Lie algebra, i.e., we switch from differential geometry to linear algebra. And then by using the exponential map we can obtain the entire identity component of the group of point symmetry transformations. That is why below we recall some notions from the theory of abstract Lie algebras and Lie algebras of vector fields.

Consider a differentiable manifold M. The very basic notions in differential geometry are a smooth curve γ on M and a tangent vector v_p at the point $p \in M$ to γ.

Definition 4. A *smooth curve* through the point $p \in M$ is a map $\gamma\colon (-\varepsilon,\varepsilon) \to M$, which is itself smooth, with $\gamma(0) = p$ and $\varepsilon > 0$ to be sufficiently small.

Definition 5. A *tangent vector* to a smooth curve γ at point $p \in M$ is a map from $C^\infty(M) := \{f\colon M \to \mathbb{R} | f \text{ is smooth}\}$ to real numbers, $\mathrm{v}_p(f) := (f \circ \gamma)'(0)$. The set of all tangent vectors at point $p \in M$ is called a *tangent space* in p and is denoted by T_pM.

It is easy to see that a tangent vector v_p is a derivation (it is a linear map, which satisfies the Leibniz rule) and T_pM constitutes a vector space over $\mathbb{R}$ of dimension $\dim M$.

Definition 6. The *tangent bundle* TM to M is a union of all tangent spaces of M, $TM := \sqcup_{p\in M} T_pM$. A *smooth vector field* v is a smooth map $\mathrm{v}\colon M \to TM$, $\mathrm{v}(p) = \mathrm{v}_p \in T_pM$. In the local coordinates $(x_1,\dots,x_n)$, $n = \dim M$, any smooth vector field is of the form

$$\mathrm{v} = \mathrm{v}^1(x)\partial_1 + \dots + \mathrm{v}^n(x)\partial_n,$$

where each v^i is a smooth function of $x = (x_1,\dots,x_n)$ and $\partial_i = \frac{\partial}{\partial x_i}$.

The set of all smooth vector fields on M is denoted by $\mathfrak{X}(M)$. Over the field of real numbers this set constitutes a vector space.

Definition 7. Let M be a manifold and $\mathrm{v} \in \mathfrak{X}(M)$. A smooth map $c\colon (-\varepsilon, \varepsilon) \to M$ is called an *integral curve* of v if the tangent vector to the curve $c(t)$ at each $t \in (-\varepsilon, \varepsilon)$ coincides with $\mathrm{v}(c(t))$.

Given a point $p \in M$ and a smooth vector field $\mathrm{v} \in \mathfrak{X}(M)$, there exists an open neighborhood of $M' \subset M$ of p and a smooth map $\mathrm{Fl}^{\mathrm{v}}\colon (-\varepsilon, \varepsilon) \times M' \to M$ such that $\mathrm{Fl}^{\mathrm{v}}(t, q)$ is a unique integral curve passing through a given p such that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{Fl}^{\mathrm{v}}(t, q) = \mathrm{v}(\mathrm{Fl}^{\mathrm{v}}(t, q)), \quad \text{and} \quad \mathrm{Fl}^{\mathrm{v}}(0, q) = p. \tag{2.1}$$

The curve $\mathrm{Fl}^{\mathrm{v}}(t, q)$ is called a *flow* generated by v and v is called an *infinitesimal generator* of $\mathrm{Fl}^{\mathrm{v}}(t, q)$. We also denote $\mathrm{Fl}^{\mathrm{v}}_t(q) := \mathrm{Fl}^{\mathrm{v}}(t, q)$. Also the flow property holds $\mathrm{Fl}^{\mathrm{v}}_{s+t}(q) = \mathrm{Fl}^{\mathrm{v}}_s(\mathrm{Fl}^{\mathrm{v}}_t(q))$. Thus, the flow generated by a vector field coincides with the local Lie group action of the Lie group $(\mathbb{R}, +)$ on the manifold M. If we endow the manifold M with a Lie group structure, then we obtain according to Definition 3 a one-parameter subgroup of this group. We denote such action as an exponentiation, $\mathrm{Fl}^{\mathrm{v}}_{\varepsilon}(q) = \exp(\varepsilon \mathrm{v})q$, which represents the solution of the system (2.1). Such notation is justified because of properties of the exponential function.

Remark 8. A very important result in the Lie group theory is the Cartan closed-subgroup theorem, which implies that each continuous homomorphism of Lie groups is smooth. Hence a homomorphism α, defined in Definition 3, is smooth and hence it is an exponentiation (due to uniqueness of solution of the initial-value problem 2.1).

Consider two smooth vector fields v and w on M, their *Lie bracket* is defined as a vector field $[\mathrm{v}, \mathrm{w}]$ satisfying $[\mathrm{v}, \mathrm{w}](\mathrm{f}) = \mathrm{v}(\mathrm{w}(\mathrm{f})) - \mathrm{w}(\mathrm{v}(\mathrm{f}))$ for all $f \in C^{\infty}(M)$.

Let $g \in G$, consider left shifts, $L_g\colon G \to G$, $L_g : h \mapsto gh$ for all $h \in G$, which are global diffeomorphisms. A vector field $\mathrm{v} \in \mathfrak{X}(G)$ is said to be *left-invariant* if for all $g, h \in G$ $L_{g*}\mathrm{v}(h) = \mathrm{v}(L_g h) = \mathrm{v}(gh)$. The vector space (over $\mathbb{R}$) of all left-invariant vector fields is denoted by $\mathfrak{X}_L(G)$. Along with the Lie bracket as a "multiplication", $\mathfrak{X}_L(G)$ constitutes a Lie algebra.

Definition 9. The Lie algebra $\mathfrak{g}$ of a Lie group G is the vector space of all left-invariant vector fields along with Lie bracket as an algebra multiplication operation.

Remark 10. There exist a linear isomorphism between the vector spaces T_eG and $\mathfrak{X}_L(G)$, which is given by $v \mapsto L^v$, $v \in T_eG$, $L^v(g) = L_{g*}v$, $L^v \in \mathfrak{X}_L(G)$. Given $v, w \in T_eG$, we can define a Lie bracket on T_eG via $[v, w]_{T_eG} = [L^v, L^w](e)$. The vector space T_eG with the above defined Lie bracket constitutes a Lie algebra. This linear isomorphism preserves a Lie bracket [33], therefore defines the isomorphism of the Lie algebras.

The following theorem determines the correspondence between Lie algebras and their smooth counterparts.

Theorem 11 (The third Lie theorem, [33])**.** *Every finite-dimensional Lie algebra* $\mathfrak{g}$ *over the field of real numbers is the Lie algebra of some simply connected Lie group.*

2.2 Lie algebras

In Definition 9 the notion of the Lie algebra of a Lie group was introduced. The Lie algebra of left-invariant vector fields of the Lie group has all the properties of an abstract Lie algebra, and therefore for its study it is possible to use the developed theory of abstract Lie algebras. Hereafter, the word "abstract" will be omitted.

A vector space $\mathfrak{g}$ over a field F (we fix it to be either $\mathbb{R}$ or $\mathbb{C}$) is called a *Lie algebra* if it is endowed with a bilinear operation (also called a Lie bracket) $[-,-]\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ that is antisymmetric and satisfies the Jacobi identity $[\mathrm{u}, [\mathrm{v}, \mathrm{w}]] + [\mathrm{v}, [\mathrm{w}, \mathrm{u}]] + [\mathrm{w}, [\mathrm{u}, \mathrm{v}]] = 0$ for all $\mathrm{u}, \mathrm{v}, \mathrm{w} \in \mathfrak{g}$.

A *subalgebra* $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ which is closed under the bracket, $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. An *ideal* $\mathfrak{i}$ of $\mathfrak{g}$ is a subalgebra such that $[\mathfrak{i}, \mathfrak{g}] \subset \mathfrak{i}$. Each Lie algebra contains *trivial* ideals that are $\{0\}$ and $\mathfrak{g}$ itself. A Lie algebra which does not contain any nontrivial ideal is called *simple*.

The *derived series* is $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \cdots \supseteq \mathfrak{g}^{(k)} \supseteq \cdots$ with $\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}]$ for $k \in \mathbb{N}_0$. If this sequence terminates on some finite k, i.e., $\mathfrak{g}^{(k)} = 0$, the algebra $\mathfrak{g}$ is called *solvable*.

The *lower central series* is $\mathfrak{g} = \mathfrak{g}^1 \supseteq \cdots \supseteq \mathfrak{g}^k \supseteq \cdots$ with $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$ for $k \in \mathbb{N}$. If this sequence terminates on some finite k, i.e., $\mathfrak{g}^k = 0$, the algebra $\mathfrak{g}$ is called *nilpotent*.

There exist a unique maximal solvable ideal of $\mathfrak{g}$ and there exist a unique maximal nilpotent ideal of $\mathfrak{g}$, which are called *radical* and *nilradical*, respectively.

The following result will be of great use in this work.

Theorem 12 (Levi theorem, [65]). *Any finite-dimensional Lie algebra* $\mathfrak{g}$ *can be decomposed into a semidirect sum* $\mathfrak{g} = \mathfrak{f} \Subset \mathfrak{r}$, *where the (vector space) complement* $\mathfrak{f}$ *of the radical* $\mathfrak{r}$ *in* $\mathfrak{g}$ *is a* semisimple *Lie algebra (direct sum of simple Lie algebras), isomorphic to the factor-algebra* $\mathfrak{g}/\mathfrak{r}$. *The algebra* $\mathfrak{f}$ *is called the* Levi factor *of* $\mathfrak{g}$.

An *automorphism* Φ of a given Lie algebra $\mathfrak{g}$ is a linear map $\Phi\colon \mathfrak{g} \to \mathfrak{g}$ that preserves the Lie bracket, i.e., for all $\mathrm{v}, \mathrm{w} \in \mathfrak{g}$ it is true that $\Phi([\mathrm{v}, \mathrm{w}]) = [\Phi(\mathrm{v}), \Phi(\mathrm{w})]$. The set of all automorphisms of the algebra $\mathfrak{g}$ along with the composition as a multiplication operation constitutes a Lie group, denoted by $\mathrm{Aut}(\mathfrak{g})$, which is a Lie subgroup of the Lie group of nondegenerate linear transformations of the vector space $\mathfrak{g}$, $\mathrm{Aut}(\mathfrak{g}) < \mathrm{GL}(\dim \mathfrak{g}, F)$.

A *derivation* of a given Lie algebra $\mathfrak{g}$ is a linear map $\mathrm{D}\colon \mathfrak{g} \to \mathfrak{g}$ such that for all $\mathrm{v}, \mathrm{w} \in \mathfrak{g}$ it is true that $\mathrm{D}\,[\mathrm{v}, \mathrm{w}] = [\mathrm{D}\,\mathrm{v}, \mathrm{w}] + [\mathrm{v}, \mathrm{D}\,\mathrm{w}]$. If there exists an element $\mathrm{x} \in \mathfrak{g}$ such that for all $\mathrm{v} \in \mathfrak{g}$ $\mathrm{D}\,\mathrm{v} = [\mathrm{x}, \mathrm{v}]$, i.e., $\mathrm{D} = \mathrm{ad}_{\mathrm{x}}$, where ad_{x} is the adjoint representation of Lie algebra on itself, such derivation is called an *inner derivation*. The Lie algebra of the Lie group $\mathrm{Aut}(\mathfrak{g})$ coincides with the algebra of all derivations of $\mathfrak{g}$. A Lie subgroup of $\mathrm{Aut}(\mathfrak{g})$ with the Lie algebra that coincides with the algebra of all inner derivations is called a group of *inner automorphisms*, denoted by $\mathrm{Inn}(\mathfrak{g})$.

A very important problem, which arises in group analysis, is a classification of orbits of $\mathrm{Inn}(\mathfrak{g})$-action on the set of subalgebras of $\mathfrak{g}$. We further refer to such classification as a *subalgebra classification*. We discuss in Section 2.5 that presenting the list of inequivalent subalgebras (modulo $\mathrm{Inn}(\mathfrak{g})$-equivalence) we can find inequivalent group-invariant solutions using effective Lie reduction method. Moreover, the classification is what makes the reduction technique optimal.

2.3 Symmetries of differential equations

Group analysis of differential equations is based on considering a geometrical interpretation of a system of partial differential equations. We can look on this system as a determining system for some variety in some space, which is called a *jet space*. If this variety is a manifold, then we can look for a local Lie group of transformations preserving this manifold. Properties and structure of such groups are of our major interest.

Further we denote $X := \mathbb{R}^p$ and $U := \mathbb{R}^q$ with underlying coordinates $(x_1, \ldots, x_p)$ and $(u^1, \ldots, u^q)$, respectively. The coordinates $(u^1, \ldots, u^q)$ in U are assumed to be smooth functions whose arguments are $(x_1, \ldots, x_p)$. For an arbitrary smooth function $f\colon \mathbb{R}^p \to \mathbb{R}$, $f(x) = f(x_1, \ldots, x_p)$ in p independent variables there exist p_n n-th order partial derivatives, $p_n = C^n_{p+n-1}$ or, equivalently,

$$p_n = \binom{p+n-1}{n}.$$

Let J denote a *multi-index* of order n, i.e., unordered n-tuple $J = (j_1, \ldots, j_n)$ with $1 \leqslant j_i \leqslant p$ for all i. The corresponding partial derivative of f is

$$\partial_J f(x) = \frac{\partial^n f(x)}{\partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_n}}.$$

If $f\colon \mathbb{R}^p \to \mathbb{R}^q$, so $f(x) = (f^1(x), \ldots, f^q(x))$, then we need $q \cdot p_n$ number of $u^\nu_J = \partial_J f^\nu$, $\nu = 1, \ldots, q$ and n-th order multi-index J to represent all possible n-th order derivatives of f at x. Therefore, we set $U_n = \mathbb{R}^{qp_n}$ and its base coordinates u^ν_J. Moreover, we further define $U_{(n)} := U \times U_1 \times \cdots U_n$. The coordinates in $U_{(n)}$ are derivatives of u up to order n. The dimension of $U_{(n)}$ is

$$\dim U_{(n)} = q \binom{p+n}{n} =: qp_{(n)}.$$

We denote elements of $U_{(n)}$ by $u_{(n)}$. They each have $qp_{(n)}$ components u^ν_J, where $\nu \in \{1, \ldots, q\}$ and $J = (j_1, \ldots, j_k)$ is an unordered multi-index, $1 \leqslant j_i \leqslant p$, $0 \leqslant k \leqslant n$. For $k = 0$ the only multi-index is 0 and u^ν_0 is the ν-th component of u.

Definition 13. The n-th jet space $J^n = X \times U_{(n)}$ of the underlying space $X \times U$ is an Euclidean space of dimension $p + qp_{(n)}$, whose coordinates are the independent variables, the dependent variables and the derivatives of the dependent variables up to order n.

Now for any function $f\colon X \to U$ we define its n-th prolongation, $\mathrm{pr}_{(n)} f\colon X \to U_{(n)}$, $u_{(n)} = \mathrm{pr}_{(n)} f$ by

$$u^\nu_J = \partial_J f^\nu(x).$$

For any $x \in X$ the vector $\mathrm{pr}_{(n)} f(x)$ consists of $qp_{(n)}$ components representing the values of f and all its derivatives up to order n at x.

Let $L = \big(L_1(x, u_{(n)}), \dots, L_l(x, u_{(n)})\big)$ be a system of l n-th order partial differential equations in p independent variables $x = (x^1, \dots, x^p)$ and q dependent variables $u = (u^1, \dots, u^q)$,

$$L_\nu(x, u_{(n)}) = 0, \quad \nu = 1, \dots, l,$$

where $u_{(n)} \in U_{(n)}$. The functions L_ν, $\nu = 1, \dots, l$ are assumed to be smooth, so each of them can be viewed as a map from the n-th jet space $X \times U_{(n)}$ to $\mathbb{R}$. System L itself determines a subset L' in $X \times U_{(n)}$ where all functions L_ν, $\nu = 1, \dots, l$ vanish, thus L' forms a subvariety in $X \times U_{(n)}$,

$$L' := \{(x, u_{(n)})\colon L_1(x, u_{(n)}) = 0, \dots, L_l(x, u_{(n)}) = 0\} \subset X \times U_{(n)}.$$

Therefore, we identify the system L with the corresponding variety L'. A *solution* of the system L is a smooth function $u = f(x)$ such that $L_\nu(x, \mathrm{pr}_{(n)} f(x)) = 0$, $\nu = 1, \dots, l$ for all x in the domain of f.

Next, we aim to define point symmetry transformations.

Definition 14. Let M be a manifold. A (local) Lie transformation group G on M consists of a Lie group G, an open subset $\mathcal{U}$ of $G \times M$ with $\{e\} \times M \subseteq \mathcal{U}$ and a smooth map $\Phi\colon \mathcal{U} \to M$ such that

1. If $(h, m) \in \mathcal{U}$, $(g, \Phi(h, m)) \in \mathcal{U}$ and $(gh, m) \in \mathcal{U}$ then $\Phi(g, \Phi(h, m)) = \Phi(gh, m)$.
2. For all $m \in M$, $\Phi(e, m) = m$.
3. If $(g, m) \in \mathcal{U}$ then $(g^{-1}, \Phi(g, m)) = m$.

We will denote $g \cdot m := \Phi(g, m)$.

Definition 15. A *complete point symmetry group* of a system L of (partial) differential equations is a (local) Lie transformation (pseudo)group G acting on some open subset M of the space of independent and dependent variables for the system with the property that whenever $u = f(x)$ is a solution of L, then $u = g \cdot f$ is a solution of L.

Definition 16. Let $M \subset X \times U$ be open and v be a vector field on M, with corresponding one-parameter group $\exp(\varepsilon \mathrm{v})$. The n-th prolongation of v, denoted by $\mathrm{pr}_{(n)}\mathrm{v}$, is a vector field on the n-th jet space $J^{(n)}$. It is defined as the generator of the corresponding prolonged one-parameter group $\mathrm{pr}_{(n)} \exp(\varepsilon \mathrm{v})$. Suppose that v is of the form

$$\mathrm{v} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \eta_l(x,u) \frac{\partial}{\partial u^l}$$

then the n-th prolongation of v is

$$\mathrm{pr}_{(n)}\mathrm{v} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \sum_{J} \eta_l^J(x,u_{(n)}) \frac{\partial}{\partial u_J^l},$$

where the order of J runs from 0 to n. The components η_l^J are determined as

$$\eta_l^J(x,u_{(n)}) = \mathrm{D}_J\big(\eta_l - \sum_{i=1}^{p} \xi^i u_i^l\big) + \sum_{i=1}^{p} \xi^i u_{J,i}^l, \tag{2.2}$$

where $\mathrm{D}_J := \mathrm{D}_{j_1} \cdots \mathrm{D}_{j_n}$ is the J-th total derivative with

$$\mathrm{D}_i P(x,u_{(n)}) = \frac{\partial P}{\partial x_i} + \sum_{l=1}^{q} \sum_{J} u_{J,i}^l \frac{\partial P}{\partial u_J^l}$$

being the total derivative with respect to x_i, and $u_{J,i}^l = \frac{\partial u_J^l}{\partial x_i}$.

In order to state the main theorem, which allows computing point symmetry transformation groups of systems of differential equations the following definition is necessary.

Definition 17. Let $L\colon L_\nu(x,u_{(n)}) = 0$, $\nu = 1,\ldots,l$, be a system of partial differential equations. This system is said to be of *maximal rank* if its $l \times (p + qp_n)$ Jacobian matrix

$$J_L(x,u_{(n)}) = \left(\frac{\partial L_\nu}{\partial x^i}, \frac{\partial L_\nu}{\partial u_l^J} \right)$$

with respect to all the variables $(x,u_{(n)})$ is of rank l whenever $(x,u_{(n)}) \in L'$.

The following theorem provides the necessary and sufficient conditions for a transformation group G to be a symmetry group of some system of partial differential equations.

Theorem 18 (Infinitesimal invariance criteria, see [32, 41])**.** *Let L be a system of differential equations of maximal rank defined on some open subset M of $X \times U$. If G is a local transformation group acting on M and*

$$\mathrm{pr}_{(n)}\mathrm{v}\big[L_\nu(x, u_{(n)})\big] = 0, \nu = 1, \ldots, l, \quad \textit{whenever} \quad (x, u_{(n)}) \in L'$$

for every infinitesimal generator v *of G, then G is a symmetry group of the system.*

2.4 The direct method for computing complete point symmetry groups

Theorem 18 allows to compute the maximal Lie invariance algebra of the system L of partial differential equations. This algebra is the infinitesimal counterpart of the complete point symmetry group of L. Such computation is straightforward since it requires solving an overdetermined system of linear partial differential equations, however, it is sophisticated. After computing the maximal Lie invariance algebra, using exponentiation we can obtain the entire identity component of the complete point symmetry group, although we miss discrete symmetry transformations that are independent up to combining with each other and with continuous point symmetry transformations.

There are two methods in the literature for computing complete point symmetry groups of systems of partial differential equations: the direct method [4, 5, 43, 50] and the algebraic method [3, 20–22]. The first one is based on the definition of a point symmetry transformation and is the most universal.

The starting point of the direct method is to consider a given system of n-th order partial differential equations L as an element of a class $\mathcal{L}|_{\mathcal{S}}$ of a similar system $\mathcal{L}_\theta\colon L(x, u_{(n)}, \theta(x, u_{(n)})) = 0$ parametrized by a tuple of n-th order differential functions (arbitrary elements) $\theta = (\theta^1(x, u_{(n)}), \ldots, \theta^k(x, u_{(n)}))$. In general, the class $\mathcal{L}|_{\mathcal{S}}$ is determined by two objects: the tuple $L = (L_1, \ldots, L_l)$ of l fixed functions depending on x, $u_{(n)}$ and θ, θ running through the set $\mathcal{S}$. By $\mathcal{S}$ we denote the set of solutions of an auxiliary system consisting of a subsystem $S(x, u_n, \theta_{(q)}) = 0$ of differential equations with respect to θ and a non-vanishing condition $\Sigma(x, u_{(n)}, \theta_{(q)}) \neq 0$ with another differential function Σ of θ, the tuple $\theta_{(q)}$ constituted by the derivatives of θ up to order q with respect to n-th jet variables $(x, u_{(n)})$. Up to the gauge equivalence of systems from $\mathcal{L}|_{\mathcal{S}}$, which is usually trivial, the correspondence $\theta \mapsto \mathcal{L}_\theta$ between $\mathcal{S}$ and $\mathcal{L}|_{\mathcal{S}}$ is bijective. Such arguments lead to the following very general definition.

Definition 19. The *equivalence groupoid* $\mathcal{G}^\sim$ of the class $\mathcal{L}|_{\mathcal{S}}$ is the small category with $\mathcal{L}|_{\mathcal{S}}$ or, equivalently, with $\mathcal{S}$ as the set of objects and with the set of point transformations of (x, u), i.e., of (local) diffeomorphisms in the space with the coordinates (x, u), between pairs of systems from $\mathcal{L}|_{\mathcal{S}}$ as the set of morphisms, specifically

$$\mathcal{G}^\sim = \{\mathcal{T} = (\theta, \Phi, \tilde{\theta}) \mid \theta, \tilde{\theta} \in \mathcal{S}, \Phi \in \mathrm{Diff}^{\mathrm{loc}}_{(x,u)} \colon \Phi_* \mathcal{L}_\theta = \mathcal{L}_{\tilde{\theta}}\}.$$

Remark 20. According to the definition of a small category [36], one needs to explicitly define its set of objects and its set of morphisms. However in the above definition we identify the set of morphisms of the groupoid $\mathcal{G}^\sim$ with $\mathcal{G}^\sim$ itself. The set of objects is automatically defined as a domain of admissible transformation $\mathcal{T}$. Elements of $\mathcal{G}^\sim$ are called *admissible transformations* within the class $\mathcal{L}|_{\mathcal{S}}$. The theory of admissible transformations and equivalence groupoids is extensively constructed in [62].

The simplest description of admissible transformations is obtained for *normalized classes* of differential equations. Roughly speaking, a class of (systems of) differential equations is called normalized if any admissible transformation in this class is induced by a transformation from its equivalence group. Different kinds of normalization can be defined depending on what kind of equivalence group (point, contact, usual, etc.) is considered. Thus, the *usual equivalence group* $G^\sim$ of the class $\mathcal{L}|_{\mathcal{S}}$ consists of those point transformations in the space of variables and arbitrary elements, which are projectable on the variable space and preserve the whole class $\mathcal{L}|_{\mathcal{S}}$. The class $\mathcal{L}|_{\mathcal{S}}$ is called normalized in the usual sense if its equivalence groupoid $\mathcal{G}^\sim$ is generated by the usual equivalence group $G^\sim$.

If the class $\mathcal{L}|_{\mathcal{S}}$ is normalized in the certain sense with respect to point transformations, the point symmetry group G_{θ_0} of any equation $\mathcal{L}_{\theta_0}$ from this class is contained in the projection of the corresponding equivalence group of $\mathcal{L}|_{\mathcal{S}}$ onto the space of independent and dependent variables (taken for the value $\theta = \theta_0$) [5]. Such approach is justified, since as a rule computation of common restrictions on admissible transformations of the entire normalized class is simpler than a straightforward computing point symmetry group of a fixed system of equations, although it requires solving nonlinear highly-coupled, and therefore difficult to solve, partial differential equations.

In the way outlined above, it is possible to find a complete point symmetry group of $(1+1)$-dimensional linear parabolic equations, since the corresponding class is sufficiently studied [50]. Further we illustrate the direct method with the computation of the complete point symmetry group of the linear $(1+1)$-dimensional heat equation with inverse square potential [29]. We in purpose choose this equation as an example, since Lie reduction of the equation (3.2), which is of our major interest, with respect to the subalgebra $\mathfrak{s}^0_{1.2} = \langle \mathcal{P}^t \rangle$ leads to reduced equation (see Section 3.5, equation 1.2^0 from the presented list), which is the linear heat equation with potential $V = \mu z_2^{-2}$, where $\mu = \frac{5}{36}$. The linear heat equations with general (nonzero) inverse square potentials $V = \mu z_2^{-2}$, where $\mu \neq 0$, constitute an important case of Lie-symmetry extensions in the class of linear second-order partial differential equations in two independent variables. This result was obtained by Sophus Lie himself [34]. Lie invariant solutions of such heat equations over the complex field were considered in [19]. The essential point symmetry group of these equations was constructed as a by-product in the course of proving Theorem 18 in [44], and a background that can be used for exhaustively classifying Lie invariant solutions of these equations over the real field was developed in the proof of Theorem 51 therein. In this example we revisit the above results, present them in an enhanced and closed form and complete the study of Lie invariant solutions of linear heat equations with (nonzero) inverse square potentials. Recall that Lie reductions of the (1+1)-dimensional linear heat equation, which corresponds to the value $\mu = 0$, were comprehensively studied in [63, Section A], following Examples 3.3 and 3.17 in [41].

Consider a linear heat equation with inverse square potential,

$$u_t = u_{xx} + \frac{\mu}{x^2}u, \tag{2.3}$$

where $\mu \neq 0$. It belongs to the class $\mathcal{E}$ of linear (1+1)-dimensional second-order evolution equations of the general form

$$u_t = A(t,x)u_{xx} + B(t,x)u_x + C(t,x)u + D(t,x) \quad \text{with} \quad A \neq 0. \tag{2.4}$$

Here the tuple of arbitrary elements of $\mathcal{E}$ is $\theta := (A, B, C, D) \in \mathcal{S}_{\mathcal{E}}$, where $S_{\mathcal{E}}$ is the solution set of the auxiliary system consisting of the single inequality $A \neq 0$ with no constraints on B, C and D. By the inequality $A(t,x) \neq 0$ we mean that for each equation form the class $\mathcal{E}$ the corresponding function $A(t,x)$ is not equal to zero on the domain of this equation. Hereafter the analogous class-determining inequalities should be interpreted in the same way.

We start with computing the maximal Lie invariance algebra of this equation using the infinitesimal invariance criteria, Theorem 18. The equation (2.3) is of second order with two independent variables (t,x) and one dependent variable u, so according to our notation in Section 2.3 $p = 2$ and $q = 1$. The $(1+1)$-dimensional linear heat equation can be identified with a subvariety in $X \times U_{(2)}$ determined by the vanishing of $L(t,x,u_{(2)}) = u_t - u_{xx} - \frac{\mu}{x^2}u = 0$

Consider a Lie symmetry vector field of the equation (2.3) of the form

$$\mathrm{v} = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u,$$

which is a vector field of $X \times U$. Then according to Theorem 18

$$\mathrm{pr}_{(2)}\mathrm{v} = \mathrm{v} + \zeta^t\partial_{u_t} + \zeta^x\partial_{u_x} + \zeta^{tt}\partial_{u_{tt}} + \zeta^{tx}\partial_{u_{tx}} + \zeta^{xx}\partial_{u_{xx}},$$

where the components of prolonged vector field are computed according to the general prolongation formula (2.2),

$$\begin{aligned}
\zeta^t &= \mathrm{D}_t\eta - u_t\mathrm{D}_t\tau - u_x\mathrm{D}_t\xi,\\
\zeta^x &= \mathrm{D}_x\eta - u_t\mathrm{D}_x\tau - u_x\mathrm{D}_x\xi,\\
\zeta^{tt} &= \mathrm{D}_t\zeta^t - u_{tt}\mathrm{D}_t\tau - u_{tx}\mathrm{D}_t\xi,\\
\zeta^{tx} &= \mathrm{D}_t\zeta^x - u_{tt}\mathrm{D}_t\tau - u_{tx}\mathrm{D}_t\xi,\\
\zeta^{xx} &= \mathrm{D}_x\zeta^x - u_{tx}\mathrm{D}_t\tau - u_{xx}\mathrm{D}_t\xi,
\end{aligned}$$

where D_t and D_x denote the total derivative operators with respect to t and x, respectively,

$$
\begin{aligned}
&\mathrm{D}_t = \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + \cdots\,,\\
&\mathrm{D}_x = \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + \cdots\,.
\end{aligned}
$$

Evaluating the action of $\mathrm{pr}_{(2)}\mathrm{v}$ on the equation (2.3) we obtain

$$
\zeta^t - \zeta^{xx} + 2\mu x^{-3}u - \mu x^{-2}\eta = 0,
$$

which must be satisfied whenever $u_t - u_{xx} - \frac{\mu}{x^2}u = 0$.

We find the system of determining equations

$$
\begin{aligned}
&\tau_x = \tau_u = 0, \quad \xi_u = 0, \quad \eta_{uu} = 0,\\
&\tau_t - 2\xi_x = 0, \quad \xi_t + 2\eta_{xu} - \xi_{xx} = 0,\\
&\eta_t - \eta_{xx} - \mu x^{-2}\eta = \mu\tau_t x^{-2}u - 2\mu\xi_x x^{-3}u,
\end{aligned}
$$

whose general solution is

$$
\tau = c_1 t + c_2 t^2 + c_4, \quad \xi = \big(\tfrac12 c_1 + c_2 t\big)x, \quad \eta = \big(c_3 - \tfrac14 c_2(x^2+2t)\big)u + h(t,x)
$$

with $c_1c_3 \neq 0$ and $h(t,x)$ be an arbitrary solution of (2.3). The maximal Lie invariance algebra is

$$
\begin{aligned}
\mathfrak g = \big\langle &\mathcal P^t = \partial_t,\ \mathcal D = t\partial_t + \tfrac12 x\partial_x - \tfrac14 u\partial_u,\ \mathcal K = t^2\partial_t + tx\partial_x - \tfrac14(x^2+2t)u\partial_u,\\
&\mathcal I = u\partial_u,\ \mathcal Z(h) = h(t,x)\partial_u\big\rangle.
\end{aligned}
$$

The maximal Lie invariance algebra $\mathfrak g$ contains the infinite-dimensional abelian ideal $\mathfrak g^{\rm lin} := \{h(t,x)\partial_u\}$, where the parameter function h runs through its solution set. This ideal is associated with the linear superposition of solutions of (2.3). The algebra $\mathfrak g$ splits over the ideal $\mathfrak g^{\rm lin}$, $\mathfrak g = \mathfrak g^{\rm ess} \Subset \mathfrak g^{\rm lin}$, where the complement subalgebra $\mathfrak g^{\rm ess}$ is four-dimensional,

$$
\begin{aligned}
\mathfrak g^{\rm ess} = \big\langle &\mathcal P^t = \partial_t,\ \mathcal D = t\partial_t + \tfrac12 x\partial_x - \tfrac14 u\partial_u,\\
&\mathcal K = t^2\partial_t + tx\partial_x - \tfrac14(x^2+2t)u\partial_u,\ \mathcal I = u\partial_u\big\rangle.
\end{aligned}
$$

Up to skew-symmetry of Lie brackets, the nonzero commutation relations of this algebra are exhausted by $[\mathcal{P}^t,\mathcal{D}]=\mathcal{P}^t$, $[\mathcal{D},\mathcal{K}]=\mathcal{K}$, $[\mathcal{P}^t,\mathcal{K}]=2\mathcal{D}$. Therefore, the algebra $\mathfrak{g}^{\rm ess}$ is isomorphic to $\mathrm{sl}(2,\mathbb{R})\oplus A_1$. Here A_1 denotes an one-dimensional Lie algebra (for details see classification [48]).

To find the complete point symmetry group G of the equation (2.3), we start with considering the equivalence groupoid of the class $\mathcal{E}$, which in its turn is a natural choice for a (normalized) superclass for the equation (2.3). We use the papers [44, 50] as reference points for known results on admissible transformations of the class $\mathcal{E}$.

Proposition 21 (see [50]). *The class $\mathcal{E}$ is normalized in the usual sense. Its usual equivalence (pseudo)group $G^\sim_{\mathcal{E}}$ consists of the transformations of the form*

$$\tilde t=T(t),\quad \tilde x=X(t,x),\quad \tilde u=U^1(t,x)u+U^0(t,x), \tag{2.5a}$$

$$\tilde A=\frac{X_x^2}{T_t}A,\quad \tilde B=\frac{X_x}{T_t}\left(B-2\frac{U^1_x}{U^1}A\right)-\frac{X_t-X_{xx}A}{T_t},\quad \tilde C=-\frac{U^1}{T_t}\mathrm{E}\frac{1}{U^1}, \tag{2.5b}$$

$$\tilde D=\frac{U^1}{T_t}\left(D+\mathrm{E}\frac{U^0}{U^1}\right), \tag{2.5c}$$

where T, X, U^0 and U^1 are arbitrary smooth functions of their arguments satisfying $T_tX_xU^1\neq0$, and $\mathrm{E}:=\partial_t-A\partial_{xx}-B\partial_x-C$.

The normalization in usual sense of the class $\mathcal{E}$ means that its equivalence groupoid coincides with the action groupoid of the group $G^\sim_{\mathcal{E}}$ [62].

Theorem 22. *The complete point symmetry (pseudo)group G of the (1+1)-dimensional linear heat equation with inverse square potential* (2.3) *consists of the point transformations of the form*

$$\begin{gathered}\tilde t=\frac{\alpha t+\beta}{\gamma t+\delta},\quad \tilde x=\frac{x}{\gamma t+\delta},\\ \tilde u=\sigma\sqrt{|\gamma t+\delta|}(u+h(t,x))\exp\frac{\gamma x^2}{4(\gamma t+\delta)},\end{gathered} \tag{2.6}$$

where α, β, γ, δ and σ are arbitrary constants with $\alpha\delta-\beta\gamma=1$ and $\sigma\neq0$, and h is an arbitrary solution of (2.3).

Proof. The linear heat equation with inverse square potential (2.3) corresponds to the value $(1,0,\mu x^{-2},0):=\theta^\mu$ of the arbitrary element tuple $\theta:=(A,B,C,D)$ of class $\mathcal{E}$.

Its vertex group $\mathcal{G}_{\theta^\mu} := \mathcal{G}^{\sim}_{\mathcal{E}}(\theta^\mu, \theta^\mu)$ is the set of admissible transformations of the class $\mathcal{E}$ with θ^μ as both their source and target, $\mathcal{G}_{\theta^\mu} = \{(\theta^\mu, \Phi, \theta^\mu) \mid \Phi \in G\}$. This argument allows us to use Proposition 21 in the course of computing G.

We should integrate the equations (2.5), where both the source and target values θ and $\tilde{\theta}$ of the arbitrary-element tuple coincides with θ^μ, with respect to the parameter functions T, X, U^1 and U^0. After a simplification, the equations (2.5b) take the form

$$X_x^2 = T_t, \quad \frac{U^1_x}{U^1} = -\frac{X_t}{2X_x}, \quad \frac{\mu}{X^2} = -\frac{U^1}{T_t}\mathrm{E}\frac{1}{U^1}, \tag{2.7}$$

where $\mathrm{E} := \partial_t - \partial_{xx} - \mu x^{-2}$. The first equation in (2.7) implies that $T_t > 0$, and the first two equations in (2.7) can be easily integrated to

$$X = \varepsilon\sqrt{T_t}x + X^0(t), \quad U^1 = \phi(t)\exp\Big(-\frac{T_{tt}}{8T_t}x^2 - \frac{\varepsilon}{2}\frac{X^0_t}{\sqrt{T_t}}x\Big),$$

where $\varepsilon := \pm 1$ and ϕ is a nonvanishing smooth function of t. By substituting these expressions for X and U^1 into the third equation from (2.7) and splitting the obtained equation with respect to powers of x, we derive that $X^0 = 0$, $4T_t\phi_t + T_{tt}\phi = 0$ and $T_{ttt}/T_t - \frac{3}{2}(T_{tt}/T_t)^2 = 0$. The last equation means that the Schwarzian derivative of T is zero. Therefore, T is a linear fractional function of t, $T = (\alpha t + \beta)/(\gamma t + \delta)$. Since the constant parameters α, β, γ and δ are defined up to a constant nonzero multiplier and $T_t > 0$, we can assume that $\alpha\delta - \beta\gamma = 1$. Then these parameters are still defined up to a multiplier in $\{-1, 1\}$, and hence we can choose them in such a way that $\varepsilon|\gamma t + \delta| = (\gamma t + \delta)$, thus neglecting the parameter ε. The equation $4T_t\phi_t + T_{tt}\phi = 0$ takes the form $2(\gamma t + \delta)\phi_t - \gamma\phi = 0$ and integrates, in view of $\phi \neq 0$, to $\phi = \sigma\sqrt{|\gamma t + \delta|}$ with $\sigma \in \mathbb{R} \setminus \{0\}$.

Finally, the equation (2.5c) takes the form

$$\left(\frac{U^0}{U^1}\right)_t = \left(\frac{U^0}{U^1}\right)_{xx} + \frac{\mu}{x^2}\left(\frac{U^0}{U^1}\right).$$

Therefore, $U^0 = U^1 h$, where $h = h(t, x)$ is an arbitrary solution of (2.3). □

To avoid complicating the structure of the pseudogroup G, we should properly interpret transformations of the form (2.6) and their composition [30]. Given a fixed transformation Φ of the form (2.6), it is natural to assume that its domain $\operatorname{dom}\Phi$ coincides with the relative complement of the set $M_{\gamma\delta} := \{(t,x,u)\in\mathbb{R}^3 \mid \gamma t+\delta=0\}$ with respect to $\operatorname{dom}h\times\mathbb{R}_u$, $\operatorname{dom}\Phi=(\operatorname{dom}h\times\mathbb{R}_u)\setminus M_{\gamma\delta}$. Here $\operatorname{dom}F$ denotes the domain of a function F. Recall that $(\gamma,\delta)\neq(0,0)$, and note that the set $M_{\gamma\delta}$ is the hyperplane defined by the equation $t=-\delta/\gamma$ in $\mathbb{R}^3_{t,x,y,u}$ if $\gamma\neq0$, and $M_{\gamma\delta}=\varnothing$ otherwise. Instead of the standard transformation composition, we use a modified composition for transformations of the form (2.6). More specifically, the domain of the standard composition $\Phi_1\circ\Phi_2 := \tilde\Phi$ of transformations Φ_1 and Φ_2 is usually defined as the preimage of the domain of Φ_1 with respect to Φ_2, $\operatorname{dom}\tilde\Phi=\Phi_2^{-1}(\operatorname{dom}\Phi_1)$. For transformations Φ_1 and Φ_2 of the form (2.6), we have $\operatorname{dom}\tilde\Phi=(\operatorname{dom}\tilde h\times\mathbb{R}_u)\setminus(M_{\gamma_2\delta_2}\cup M_{\tilde\gamma\tilde\delta})$ where $\tilde\gamma=\gamma_1\alpha_2+\delta_1\gamma_2$, $\tilde\delta=\gamma_1\beta_2+\delta_1\delta_2$, $\operatorname{dom}\tilde h=\big((\pi_*\Phi_2)^{-1}\operatorname{dom}h^1\big)\cap\operatorname{dom}h^2$, π is the natural projection onto $\mathbb{R}^2_{t,x}$ in $\mathbb{R}^3_{t,x,u}$, and the parameters with indices 1 and 2 and tildes correspond Φ_1, Φ_2 and $\tilde\Phi$, respectively. As the *modified composition* $\Phi_1\circ^{\mathrm m}\Phi_2$ of transformations Φ_1 and Φ_2, we take the continuous extension of $\Phi_1\circ\Phi_2$ to the set

$$\operatorname{dom}^{\mathrm m}\tilde\Phi := (\operatorname{dom}\tilde h\times\mathbb{R}_u)\setminus M_{\tilde\gamma\tilde\delta},$$

i.e., $\operatorname{dom}(\Phi_1\circ^{\mathrm m}\Phi_2)=\operatorname{dom}^{\mathrm m}\tilde\Phi$. In other words, we set $\Phi_1\circ^{\mathrm m}\Phi_2$ to be the transformation of the form (2.6) with the same parameters as in $\Phi_1\circ\Phi_2$ and with natural domain. It is obvious that we redefine $\Phi_1\circ\Phi_2$ on the set $(\operatorname{dom}\tilde h\times\mathbb{R}_u)\cap M_{\gamma_2\delta_2}$ if $\gamma_1\gamma_2\neq0$; otherwise $\operatorname{dom}^{\mathrm m}\tilde\Phi=\operatorname{dom}\tilde\Phi$ and the extension is trivial.

The point transformations of the form $\tilde t=t$, $\tilde x=x$, $\tilde u=u+h(t,x)$, where the parameter function $h=h(t,x)$ is an arbitrary solution of the equation (2.3), constitute the normal (pseudo)subgroup G^{lin} of G, which is associated with the linear superposition of solutions of (2.3), cf. Section 3.3. Moreover, the group G splits over G^{lin}, $G=G^{\mathrm{ess}}\ltimes G^{\mathrm{lin}}$, where the subgroup G^{ess} consists of the elements of G with $h=0$ and is a four-dimensional Lie group within the framework of the above interpretation. We call this subgroup the *essential point symmetry group* of the equation (2.3). In turn, the group G^{ess} contains the normal subgroup R that is constituted by the point transformations of the form $\tilde t=t$, $\tilde x=x$, $\tilde u=\sigma u$ with $\sigma\in\mathbb{R}\setminus\{0\}$ and is isomorphic to the multiplicative group $\mathbb{R}^\times$ of the field of real numbers. The group G^{ess} splits over R, $G^{\mathrm{ess}}=F\ltimes R$, where the subgroup F is singled out from G^{ess} by the constraint $\sigma=1$ and is isomorphic to the group $\mathrm{SL}(2,\mathbb{R})$.

Corollary 23. *A complete list of discrete point symmetry transformations of the equation* (2.3) *that are independent up to combining with each other and with continuous point symmetry transformations of this equation is exhausted by the single involution* $\mathcal{I}'$ *alternating the sign of* u, $\mathcal{I}'\colon (t,x,u) \mapsto (t,x,-u)$. *Thus, the factor group of the complete point symmetry group* G *of* (2.3) *with respect to its identity component is isomorphic to* $\mathbb{Z}_2$.

Proof. It is obvious that the subgroups G^{lin} and F are connected subgroups of G. Jointly with the splitting of G over R, this implies that elements of the required list can be selected from the subgroup R. Factoring out the elements of the identity component of R, which is isomorphic to $\mathbb{R}^{\times}_{>0}$, we obtain either the identity transformation or $\mathcal{I}'$. □

2.5 Group-invariant solutions

Knowing the Lie symmetry group of a system of partial differential equations, we can construct a family of some of its explicit particular solutions, that are group-invariant. In this section we describe how to carry out such computations in an optimal way (for more details see [41])

Let G be the complete point symmetry group of a system of partial differential equations L in $p > s$ independent variables, defined as in Section 2.3. Each s-parameter subgroup H of the group G allows to reduce the number of independent variables by s (see [41]). Since there is often an infinite number of such subgroups, it is necessary to construct a list of inequivalent subgroups with respect to some equivalence relation. In order to explicitly present such equivalence relation, we have to require the regularity of action of the group of transformations on the underlying manifold in the following sense.

Definition 24. An orbit of an action of a local transformations group G on a manifold M is a minimal nonempty G-invariant subset of M. The group G acts regularly if all the orbits are of the same dimension as submanifolds of M and each point $x \in M$ has an arbitrarily small neighborhood whose intersection with each orbit is a connected subset of M.

Assuming the action of G on the corresponding subset $L' \in X \times U$ to be regular, we can state the following result.

Theorem 25 ([41]). *Let G be the point symmetry group of a system of differential equations L, $H < G$ be an s-parameter subgroup and g an arbitrary element of G. Given an H-invariant solution $u = f(x)$ of the system L, the transformed function $u = \tilde{f}(x) = g \cdot f(x)$ is an $\tilde{H}$-invariant solution, where $\tilde{H} = gHg^{-1}$ is the conjugate subgroup to H under g.*

As a consequence, the problem of classifying group-invariant solutions reduces to the classification of the complete point symmetry group G under conjugation. Due to the correspondence between Lie groups and Lie algebras (Theorem 11), it is possible to state this classification problem in terms of Lie algebras, which in its turn again replaces differential geometry by linear algebra.

Let G be a Lie group. For each $g \in G$ the group conjugation $c_g \colon G \ni h \mapsto ghg^{-1}$ determines a diffeomorphism on G, and the set $\mathrm{Inn}(G) = \{c_g \mid g \in G\}$ is a group with respect to the composition of diffeomorphisms and is called the *group of inner automorphisms* of G.

The pushforward $c_{g*} \colon T_hG \to T_{c_g(h)}G$ preserves the left invariance of vector fields on G, thus giving rise to the linear map

$$\mathrm{v} \in \mathfrak{g} \mapsto \mathrm{Ad}(g)\mathrm{v} = c_{g*}\mathrm{v}$$

on the Lie algebra $\mathfrak{g}$ of G, which is called the *adjoint representation* of G.

For an arbitrary vector field v, which generates a one-parameter subgroup $H = \{\exp(\varepsilon \mathrm{v}) \mid \varepsilon \in \mathbb{R}\} < G$, the vector field $\mathrm{Ad}(g)\mathrm{v}$ generates a one-parameter subgroup $c_g(H)$. This remark can be generalized to higher-dimensional subgroups.

Theorem 26 ([41]). *Let H and $\tilde{H}$ be connected, s-dimensional Lie subgroups of the Lie group G and let $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ be the corresponding Lie subalgebras of the Lie algebra $\mathfrak{g}$ of G. Then the subgroups H and $\tilde{H}$ are conjugate with respect to an element $g \in G$, $\tilde{H} = gHg^{-1}$, if and only if the subalgebras $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ are conjugate with respect to the same element g of G, $\tilde{\mathfrak{h}} = \mathrm{Ad}(g)\mathfrak{h}$.*

Thus, the classification of s-parameter subgroups of G is reduced to the classification of s-dimensional subalgebras of $\mathfrak{g}$ up to the equivalence generated by the adjoint representation of the Lie group G on its Lie algebra $\mathfrak{g}$.

Given a vector field v, which generates the one-parameter subgroup $\{\exp\{\varepsilon \mathrm{v}\} \mid \varepsilon \in \mathbb{R}\}$, then let ad_{v} be the vector field from $\mathfrak{g}$ generating the corresponding subgroup of adjoint actions, where

$$\mathrm{ad}_{\mathrm{v}}\,\mathrm{w} := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mathrm{Ad}(\exp(\varepsilon\mathrm{v}))\mathrm{w} = [\mathrm{v}, \mathrm{w}], \quad \mathrm{w} \in \mathfrak{g}.$$

This action is by definition (see Section 2.2) an $\mathrm{Inn}(\mathfrak{g})$-action on the algebra $\mathfrak{g}$.

Using this $\mathrm{Inn}(\mathfrak{g})$-action of a Lie algebra $\mathfrak{g}$ on itself, one can obtain the corresponding adjoint representation of the underlying Lie group by the standard exponentiation of vector fields via solving the initial-value problem

$$\frac{\mathrm{dw}}{\mathrm{d}\varepsilon} = \mathrm{ad}_{\mathrm{v}}\,\mathrm{w}, \quad \mathrm{w}(0) = \mathrm{w}_0,$$

whose solution is

$$\mathrm{w}(\varepsilon) = \mathrm{Ad}(\exp(\varepsilon\mathrm{v}))\mathrm{w}_0$$

or, alternatively, by conputing Lie series

$$\mathrm{Ad}(\exp(\varepsilon\mathrm{v}))\mathrm{w}_0 = \sum_{i=0}^{\infty} \frac{(-\varepsilon)^i}{i!}(\mathrm{ad}_{\mathrm{v}})^i \mathrm{w}_0.$$

Another approach is based on the fact that we already know that the algebra $\mathfrak{g}$ is the Lie algebra of the complete point symmetry group G, and the actions of G and of the group of inner automorphisms of $\mathfrak{g}$ on $\mathfrak{g}$ coincide. Therefore it requires computing pushforwards of the *elementary transformations* from the complete point symmetry group on basis vector fields of the algebra. A further advantage of this method is that once the complete point symmetry group of the system of partial differential equations has been constructed, the finite form of admitted point symmetries is already known, which is all that is required in this method.

As an example, we continue the study of the $(1+1)$-dimensional linear heat equations with inverse square potential (2.3) [29]. The essential Lie invariance algebra $\mathfrak{g}^{\mathrm{ess}}$ of this equation is four-dimensional and all its $\mathrm{Inn}(\mathfrak{g}^{\mathrm{ess}})$-inequivalent (and therefore G^{ess}-inequivalent) subalgebras were classified in [45], see also [48].

Lemma 27 ([45, 48]). *A complete list of* $\mathrm{Inn}(\mathfrak{g}^{\mathrm{ess}})$*-inequivalent subalgebras of* $\mathfrak{g}^{\mathrm{ess}}$ *is exhausted by the subalgebras*

$$\begin{aligned}
&\mathfrak{s}^{\delta}_{1.1}=\langle\mathcal{P}^t+\delta\mathcal{I}\rangle,\quad \mathfrak{s}^{\nu}_{1.2}=\langle\mathcal{D}+\nu\mathcal{I}\rangle,\ {}_{\nu\geqslant0}\,,\quad \mathfrak{s}^{\nu}_{1.3}=\langle\mathcal{P}^t+\mathcal{K}+2\nu\mathcal{I}\rangle,\quad \mathfrak{s}_{1.4}=\langle\mathcal{I}\rangle,\\
&\mathfrak{s}^{\nu}_{2.1}=\langle\mathcal{P}^t,\mathcal{D}+\nu\mathcal{I}\rangle,\quad \mathfrak{s}_{2.2}=\langle\mathcal{P}^t,\mathcal{I}\rangle,\quad \mathfrak{s}_{2.3}=\langle\mathcal{D},\mathcal{I}\rangle,\quad \mathfrak{s}_{2.4}=\langle\mathcal{P}^t+\mathcal{K},\mathcal{I}\rangle,\\
&\mathfrak{s}_{3.1}=\langle\mathcal{P}^t,\mathcal{D},\mathcal{K}\rangle,\quad \mathfrak{s}_{3.1}=\langle\mathcal{P}^t,\mathcal{D},\mathcal{I}\rangle,\quad \mathfrak{s}_{4.1}=\mathfrak{g}^{\mathrm{ess}},
\end{aligned}$$

where $\delta\in\{-1,0,1\}$*, and* ν *is an arbitrary real constant satisfying indicated constraints. The first number in the subscript of* $\mathfrak{s}$ *denotes the dimension of the corresponding subalgebra.*

We present the exhaustive classification of Lie invariant solutions of the equation (2.3) over the field of real numbers using results from the proof of Theorem 51 in [44] for integrating the obtained reduced equations. Hereafter C_1 and C_2 are arbitrary real constants, $\kappa:=\frac12\sqrt{1-4\mu}$, $\kappa':=\frac12\kappa=\frac14\sqrt{1-4\mu}$, and ν is a constant parameter.

The reduction with respect to the subalgebra $\mathfrak{s}^{\varepsilon}_{1.1}$ with $\varepsilon\in\{-1,1\}$ results in the solution

$$u=\mathrm{e}^{\varepsilon t}\mathsf{Z}_{|\kappa|}(x),$$

where the cylinder function $\mathsf{Z}_{|\kappa|}(x)$ is $\mathcal{Z}_\kappa$, $\mathcal{C}_\kappa$, $\tilde{\mathcal{Z}}_{|\kappa|}$ or $\tilde{\mathcal{C}}_{|\kappa|}$ if $4\mu\leqslant1$ and $\varepsilon=1$, $4\mu\leqslant1$ and $\varepsilon=-1$, $4\mu>1$ and $\varepsilon=1$, $4\mu>1$ and $\varepsilon=-1$, respectively. Here $\mathcal{Z}_\kappa$ and $\mathcal{C}_\kappa$ are linear combination of Bessel functions and linear combination of modified Bessel functions, respectively, $\mathcal{Z}_\kappa=C_1\mathrm{J}_\kappa(x)+C_2\mathrm{Y}_\kappa(x)$, $\mathcal{C}_\kappa=C_1\mathrm{I}_\kappa(x)+C_2\mathrm{K}_\kappa(x)$. The cylinder function $\tilde{\mathcal{Z}}_{|\kappa|}$ is a linear combination of the modifications of the Hankel functions $\mathrm{H}^{(1)}_\kappa$ and $\mathrm{H}^{(2)}_\kappa$,

$$\begin{aligned}
\tilde{\mathcal{Z}}_{|\kappa|}&=C_1\tilde{\mathrm{H}}^{(1)}_\kappa(x)+C_2\tilde{\mathrm{H}}^{(2)}_\kappa(x)\\
&=\frac{C_1}{2}\big(\mathrm{e}^{-\frac12\kappa\pi}\mathrm{H}^{(1)}_\kappa(x)+\mathrm{e}^{\frac12\kappa\pi}\mathrm{H}^{(2)}_\kappa(x)\big)+\frac{C_2}{2\mathrm{i}}\big(\mathrm{e}^{-\frac12\kappa\pi}\mathrm{H}^{(1)}_\kappa(x)-\mathrm{e}^{\frac12\kappa\pi}\mathrm{H}^{(2)}_\kappa(x)\big)
\end{aligned}$$

and the cylinder function $\tilde{\mathcal{C}}_{|\kappa|}$ is a linear combination of the modified Bessel function K_κ

and $\tilde{\mathrm{I}}_\kappa$, which is a modification of modified Bessel function I_κ (for $\kappa\neq0$),

$$\tilde{\mathcal{C}}_{|\kappa|}=C_1\tilde{\mathrm{I}}_\kappa(x)+C_2\mathrm{K}_\kappa(x),\quad \tilde{\mathrm{I}}_\kappa(x):=\frac{\pi\mathrm{i}}{2\sin(\kappa\pi)}\big(\mathrm{I}_\kappa(x)+\mathrm{I}_{-\kappa}(x)\big).$$

Each of the functions $\mathcal{Z}_\kappa$, $\mathcal{C}_\kappa$, $\tilde{\mathcal{Z}}_{|\kappa|}$ or $\tilde{\mathcal{C}}_{|\kappa|}$ is real-valued and represents, for the corresponding values of the parameters κ and ε, the general solution of the equation $\varphi_{\omega\omega}+\mu\omega^{-2}\varphi-\varepsilon\varphi=0$, which is obtained by the Lie reduction of (2.3) with respect to the subalgebra $\langle\mathcal{P}^t+\varepsilon\mathcal{I}\rangle$ using the ansatz $u=\mathrm{e}^{\varepsilon t}\varphi(\omega)$ with $\omega=x$.

The Lie reduction of the equation (2.3) with respect to the subalgebra $\mathfrak{s}^0_{1.1}$ leads to the ansatz $u=\varphi(\omega)$ with $\omega=x$ and the Euler equation $\omega^2\varphi_{\omega\omega}+\mu\varphi=0$ as the corresponding reduced equation. We integrate this equation depending on the value of $\operatorname{sgn}(1-4\mu)\in\{-1,0,1\}$ and obtain the following solutions of (2.3):

$$u=\sqrt{|x|}\big(C_1\cos(|\kappa|\ln|x|)+C_2\sin(|\kappa|\ln|x|)\big),\quad u=\sqrt{|x|}\big(C_1+C_2\ln|x|\big),$$
$$u=\sqrt{|x|}(C_1|x|^\kappa+C_2|x|^{-\kappa}).$$

For each of the subalgebras $\mathfrak{s}^\nu_{1.2}$ and $\mathfrak{s}^\nu_{1.3}$, we first construct an associated ansatz for u, derive the corresponding reduced equation, map this equation to a canonical form, which turns out to be the Whittaker equation

$$\varphi_{\omega\omega}+\left(-\frac14+\frac{a}{\omega}+\frac{1/4-b^2}{\omega^2}\right)\varphi=0$$

with certain a and b, by a point transformation of the invariant independent and dependent variables, and then use this transformation for modifying the ansatz. The general solution of the above Whittaker equation is the general linear combination of the Whittaker functions $\mathrm{W}_{a,b}(z)$ and $\mathrm{M}_{a,b}(z)$, which are linearly independent and whose properties are comprehensively described, e.g., in [58].

Thus, for the subalgebra $\mathfrak{s}^\nu_{1.2}$ with a fixed $\nu\geqslant0$, the used modified ansatz and the values of a and b are

$$u=|t|^\nu|x|^{-\frac12}\mathrm{e}^{-\varepsilon'\frac{x^2}{8t}}\varphi(\omega),\quad \omega=\frac{x^2}{4t},\quad a=-\varepsilon'\nu,\quad b=\kappa'.$$

The representation of the corresponding solutions of (2.3) over the real field depends on the value of sgn$(1-4\mu)$, either sgn$(1-4\mu)\geqslant 0$ or sgn$(1-4\mu)<0$, and is

$$u=|t|^{\nu}|x|^{-\frac12}\mathrm{e}^{-\varepsilon'\frac{x^2}{8t}}\big(C_1\mathrm{M}_{-\varepsilon'\nu,\,\kappa'}(\tfrac{x^2}{4t})+C_2\mathrm{W}_{-\varepsilon'\nu,\,\kappa'}(\tfrac{x^2}{4t})\big),$$

or

$$u=|t|^{\nu}|x|^{-\frac12}\mathrm{e}^{-\varepsilon'\frac{x^2}{8t}}\,\mathrm{Re}\big((C_1-\mathrm{i}C_2)\mathrm{W}_{-\varepsilon'\nu,\,\mathrm{i}|\kappa'|}(\tfrac{x^2}{4t})\big),$$

respectively. We distinguish the case sgn$(1-4\mu)<0$ since then the Whittaker function $\mathrm{W}_{-\varepsilon'\nu,\kappa'}(\omega)$ is complex-valued. However, its real and imaginary parts are linearly independent real solutions of the corresponding Whittaker equation.

For each of the subalgebras $\mathfrak{s}^{\nu}_{1.3}$, we use the modified ansatz

$$u=|x|^{-\frac12}\mathrm{e}^{-\frac{x^2t}{t^2+1}+2\nu\arctan t}\varphi(\omega),\quad \omega=\frac{\mathrm{i}x^2}{2(t^2+1)},$$

and the values of a and b are $a=\mathrm{i}\nu$ and $b=\kappa'$. Hence the corresponding solutions of (2.3) over the real field are

$$u=|x|^{-\frac12}\mathrm{e}^{-\frac{x^2t}{t^2+1}+2\nu\arctan t}\,\mathrm{Re}\left((C_1-\mathrm{i}C_2)\mathrm{W}_{\mathrm{i}\nu,\,\kappa'}\left(\frac{\mathrm{i}x^2}{2(t^2+1)}\right)\right).$$

If $\kappa\in 2\mathbb{N}_0+1$, the solution u can be represented in terms of regular and irregular Coulomb functions,

$$u=|x|^{-\frac12}\mathrm{e}^{-\frac{x^2t}{t^2+1}+2\nu\arctan t}\left(C_1F_{\kappa'-\frac12}\Big(4\nu,\frac{x^2}{4(t^2+1)}\Big)+C_2G_{\kappa'-\frac12}\Big(4\nu,\frac{x^2}{4(t^2+1)}\Big)\right).$$

2.6 On Lie symmetries of linear differential equations

The theory of linear differential equations has developed significantly over the last two centuries. That is why, not unreasonably, the common view is that overall we know too much about linear differential equations and too little about nonlinear differential equations. However, within the group analysis approach, the only classes of linear differential equations that were comprehensively studied are the class $\mathcal{E}$ of $(1+1)$-dimensional second-order linear evolution equations [50] and the class of ordinary

nth order linear differential equations [14]. Moreover, the classification of second-order linear evolution partial differential equations in dimension $(1+2)$ is still an open problem. Nevertheless, it is true that all linear differential equations have some common symmetry properties, which we describe in this section.

Consider a class $\mathcal{A}$ of n-th order differential equations with $n \geqslant 2$ in p independent variables $x = (x_1, \ldots, x_p)$ and one dependent variable u of the form

$$\begin{aligned} &L(x, u_{(n)}) = 0, \\ &\frac{\partial L}{\partial u_J} \neq 0, \end{aligned} \tag{2.8}$$

where L is an arbitrary smooth function of its arguments and J is some multi-index of order n. Moreover, we require each equation from the class $\mathcal{A}$ to satisfy a specific nondegeneracy condition. That is for each equation from the class $\mathcal{A}$ each independent variable is essential, i.e., there is no point transformation which maps this equation to a partial differential equation in $s < p$ independent variables where the remaining $p - s$ variables play the role of parameters. We call an independent variable in partial differential equation a parameter if there is no differentiation with respect to it. This requirement is vital and as for an example for the class $\bar{\mathcal{F}}$ of (1+2)-dimensional ultraparabolic Fokker–Planck equations (3.1) the requirement results in the inequality $B_x \neq 0$ (we describe it more specifically in Section 3.1).

A class $\mathcal{A}^{\rm lin}$ of linear differential equations of order $n \geqslant 2$ in p independent variables $x = (x_1, \ldots, x_p)$ and one dependent variable is singled out from $\mathcal{A}$ by the auxiliary constraints

$$\frac{\partial^2 L}{\partial u_{J_1} \partial u_{J_2}} = 0$$

for all multi-indexes J_1 and J_2 of order less or equal than n. Therefore, each equation from $\mathcal{A}^{\rm lin}$ can be mapped to the linear homogeneous differential equation of the form

$$\begin{aligned} &\frac{\partial^n u}{\partial x_1^n} = \sum_J A_J(x) u_J, \\ &A_{\bar{1}} = 0 \quad \text{with} \quad \bar{1} := (1, 1, \ldots, 1), \end{aligned} \tag{2.9}$$

where the order of J runs from 0 to n, $n \geqslant 2$, $A_J(x) = A_{(j_1, j_2, \ldots, j_n)}(x)$ is symmetric in components of multi-index J and $\bar{1}$ is a multi-index of order n with each of its component equal to 1. That is why hereafter by $\mathcal{A}^{\rm lin}$ we denote the class of equations of the form (2.9) with the corresponding constraints on parameter-functions.

Given an equation of the form (2.9), consider its Lie symmetry vector field

$$\mathrm{v} = \xi^i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u},$$

where ξ^i, $i=1,\dots,p$, and η are smooth functions of (x,u). The infinitesimal invariance criterion (see Theorem 18) implies the condition

$$\mathrm{pr}_{(n)}\mathrm{v}\left(\frac{\partial^n u}{\partial x_1^n} - \sum_J A_J(x)u_J\right) = 0, \quad \text{i.e.,} \quad \eta^{\bar{1}} - \sum_J A_J\eta^J - \sum_J \mathrm{v}(A_J)u_J = 0,$$

where $\partial^n u/\partial x_1^n$ is substituted by the right-hand side of (2.9).

We present an enhanced reformulations of Bluman's theorem [6, Theorems 6 and 7] on properties of Lie symmetry vector fields of equations from the class $\mathcal{A}^{\rm lin}$ depending on (n,p).

Theorem 28 ([6]). *Let $n\geqslant 2$ and $p\geqslant 2$ or $n\geqslant 3$ and $p=1$. Then for an arbitrary Lie symmetry vector field* v *of an arbitrary equation from the class $\mathcal{A}^{\rm lin}$, one has*

$$\frac{\partial \xi^i}{\partial u} = 0, \quad i=1,\dots,p, \quad \text{and} \quad \frac{\partial^2\eta}{\partial u^2} = 0. \tag{2.10}$$

The conditions of Theorem 28 are sufficient, however not necessary.

Consider an arbitrary equation L from the class $\mathcal{A}^{\rm lin}$, whose Lie symmetry vector field components satisfy the constraints (2.10). Then it follows that

$$\xi^i = \xi^i(x), \quad i=1,\dots,p, \quad \text{and} \quad \eta = \eta^1(x)u + \eta^0(x).$$

Moreover, a smooth function $\eta^0(x)$ runs through the solution set of the equation L,

$$\frac{\partial^n \eta^0}{\partial x_1^n} = \sum_J A_J(x)\eta^0_J. \tag{2.11}$$

The maximal Lie invariance algebra $\mathfrak{g}$ of the equation L contains an infinite-dimensional abelian ideal $\mathfrak{g}^{\rm lin} := \{\eta^0\partial_u\}$ associated with the linear superposition of solutions, where $\eta^0 = \eta^0(x)$ runs through the solution set of the equation L. As a rule, a complement subspace to $\mathfrak{g}^{\rm lin}$ in the maximal Lie invariance algebra $\mathfrak{g}$ is finite-dimensional and we call it an essential Lie invariance algebra, denoted by $\mathfrak{g}^{\rm ess}$. Moreover, the maximal Lie invariance algebra $\mathfrak{g}$ splits over its abelian ideal $\mathfrak{g}^{\rm lin}$, $\mathfrak{g} = \mathfrak{g}^{\rm ess} \Subset \mathfrak{g}^{\rm lin}$.

Consider a vector field $\mathrm{w}(\eta^0) = \eta^0(x)\partial_u \in \mathfrak{g}^{\rm lin}$. The point symmetry transformation that is generated by this vector field can be straightforwardly computed via solving the initial-value problem (2.1),

$$\tilde{x} = x, \quad \tilde{u} = u + \varepsilon\eta^0(x).$$

Such transformations correspond to the fundamental linear superposition principle for linear differential equations. Moreover, they constitute a normal subgroup $G^{\rm lin}$ in the complete point symmetry group G of the equation L, $G^{\rm lin} \lhd G$. There exists a complement of the subgroup $G^{\rm lin}$ in G, which we call an *essential point symmetry group* $G^{\rm ess} < G$, such that $G = G^{\rm ess} \ltimes G^{\rm lin}$. The essential Lie invariance algebra $\mathfrak{g}^{\rm ess}$ is the Lie algebra of the essential point symmetry group $G^{\rm ess}$.

More interesting and promising in terms of constructing explicit solutions of the equation L is the study of its essential Lie invariance algebra $\mathfrak{g}^{\rm ess}$. Moreover, there exists a common approach for computing Lie symmetry vector fields of the essential Lie invariance algebra $\mathfrak{g}^{\rm ess}$ of linear equations, which is based on reformulating the infinitesimal invariance condition (Theorem 18) in terms of differential operators. Any infinitesimal generator from $\mathfrak{g}^{\rm ess}$ is of the form

$$\mathrm{v} = \xi^i(x)\frac{\partial}{\partial x_i} + \eta^1(x)u\frac{\partial}{\partial u}.$$

For the vector field v we construct the corresponding differential operator $Q_{\rm v}$ preserving the characteristics of v, $Q_{\rm v}\colon C^\infty(X\times U) \to C^\infty(X\times U)$, of the form

$$Q_{\rm v} := \eta^1(x) - \xi^i(x)\frac{\partial}{\partial x_i},$$

where $X = \mathbb{R}^p$ and $U = \mathbb{R}$ with underlying coordinates $x = (x_1,\dots,x_p)$ and u, respectively. Consider a n-th order differential operator B,

$$\mathrm{B} := \frac{\partial^n}{\partial x_1^n} - \sum_J A_J(x)\partial_J,$$

where the order of J runs from 0 to n, $n \geqslant 2$ and $\partial_0 = 1$, that is $\mathrm{B}u = 0$ coincides with the differential equation L.

Then the infinitesimal invariance criteria (Theorem 18) can be formulated in terms of the above defined differential operators.

Theorem 29 ([17]). *Let L be an equation from the class $\mathcal{A}^{\mathrm{lin}}$, whose Lie symmetry vector field components satisfy the constraints* (2.10)*, defined on some open subset M of $X \times U$. The differential operator B corresponds to the equation L. If G^{ess} is a local transformation group acting on M and*

$$[Q_{\mathrm{v}}, \mathrm{B}] = \lambda\, \mathrm{B} \tag{2.12}$$

for some smooth function $\lambda = \lambda(x)$ and for every infinitesimal generator v of G^{ess}, whose corresponding differential operator is Q_{v}, then G^{ess} is the essential Lie symmetry group of the equation L.

Here $[-,-]$ denotes the commutator of the differential operators. Evaluating the commutator on the left-hand side of (2.12) and equating coefficients for the linearly independent differentials, we obtain the over-determined system of linear partial differential equations for the unknown functions ξ^i, η^1 and λ. Applications of this approach can be found in [39, 40].

Chapter 3

Application for a certain (1+2)-dimensional Fokker–Planck equation

In order to present our results in closed and clear form, hereafter we use notations that differ from those of the other sections of this thesis.

3.1 Introduction

The Fokker–Planck and Kolmogorov equations provide powerful tools for adequately modeling a wide range of natural processes, which involve considering fluctuations of a quantity under the action of a random perturbation. Since the presence of a random noise is a common characteristic of many physical fields, the Fokker–Planck equations have acquired high popularity in applied sciences. However, theoretical studies of these equations have aroused lively interest as well, in particular, in the field of group analysis of differential equations.

The most general form of (1+2)-dimensional ultraparabolic Fokker–Planck equations is

$$
\begin{aligned}
&u_t + B(t,x,y)u_y = A^2(t,x,y)u_{xx} + A^1(t,x,y)u_x + A^0(t,x,y)u + C(t,x,y)\\
&\text{with}\quad A^2 \neq 0,\quad B_x \neq 0.
\end{aligned}
\tag{3.1}
$$

We denote the entire class of these equations by $\bar{\mathcal{F}}$. Thus, the corresponding tuple $\bar\theta := (B, A^2, A^1, A^0, C)$ of arbitrary elements of the class $\bar{\mathcal{F}}$ runs through the solution set of the system of the inequalities $A^2 \neq 0$ and $B_x \neq 0$ with no restrictions on A^0, A^1 and C. The requirement $B_x \neq 0$ is essential since arbitrary equation of the form (3.1) with $B_x = 0$ by an appropriate change of variables can be mapped to the class $\mathcal{E}$ of linear $(1+1)$-dimensional second-order evolution equation (2.4).

A partial preliminary group classification of the class $\bar{\mathcal{F}}$ was carried out in [15]. Some subclasses of the class $\bar{\mathcal{F}}$ were considered within the Lie symmetry framework in [19,31,55–57,66]. Despite the number of papers on this subject, there are still many open problems in the symmetry analysis of the entire class $\bar{\mathcal{F}}$, its subclasses and even particular equations from this class.

In the present thesis, we carry out enhanced symmetry analysis of the equation

$$u_t + xu_y = u_{xx}, \tag{3.2}$$

which is of the simplest form within the class $\bar{\mathcal{F}}$ and corresponds to the values $B = x$, $A^2 = 1$ and $A^1 = A^0 = C = 0$ of the arbitrary elements. This equation is distinguished within the class $\bar{\mathcal{F}}$ by its remarkable symmetry properties. In particular, its essential Lie invariance algebra $\mathfrak{g}^{\rm ess}$ is eight-dimensional, which is the maximum dimension for equations from the class $\bar{\mathcal{F}}$. Moreover, it is, up to the point equivalence, a unique equation in $\bar{\mathcal{F}}$ whose essential Lie invariance algebra is of this dimension. That is why we refer to (3.2) as the remarkable Fokker–Planck equation. The study of the equation (3.2) was initiated by Kolmogorov in 1934 [27], and hence it is often called the Kolmogorov equation as well. In particular, he constructed its fundamental solution,[1]

$$F(t,y,x,t',y',x') = \frac{\sqrt{3}H(t-t')}{2\pi(t-t')^2}\exp\left(-\frac{(x-x')^2}{4(t-t')}-\frac{3\big(y-y'-\frac12(x+x')(t-t')\big)^2}{(t-t')^3}\right), \tag{3.3}$$

where H denotes the Heaviside step function. A preliminary study of symmetry properties of (3.2) was carried out in [19, 31].

The algebra $\mathfrak{g}^{\rm ess}$ is nonsolvable and is of complicated and specific structure, which

[1]The factor before the exponential function in the solution presented in [27] was corrected in later papers.

makes the classification of subalgebras of $\mathfrak{g}^{\rm ess}$ nontrivial. More specifically, this algebra is isomorphic to a semidirect sum ${\rm sl}(2,\mathbb{R}) \ltimes {\rm h}(2,\mathbb{R})$ of the real order-two special linear Lie algebra ${\rm sl}(2,\mathbb{R})$ and the real rank-two Heisenberg algebra ${\rm h}(2,\mathbb{R})$, where the action of the former algebra on the latter is given by the direct sum of the one- and four-dimensional irreducible representations of ${\rm sl}(2,\mathbb{R})$. Such structure has not been studied before in the literature on symmetry analysis of differential equations from the point of view of classifying subalgebras. This was an obstacle for the complete classification of Lie reductions of the equation (3.2), which we successfully overcome in the present thesis.

Moreover, using the direct method, we construct the complete point symmetry group G of this equation and derive a nice representation for transformations from G. This essentially simplifies the subsequent classification of the one- and two-dimensional subalgebras of the algebra up to the G-equivalence, which is required for optimally accomplishing Lie reductions of the equation (3.2). We carefully analyze the structure of the group G and surprisingly find out that it contains only one independent, up to combining with elements from the identity component of G, discrete element. One can choose for such an element the involution that only alternates the sign of u. One more implication of the construction of the group G is that Kolmogorov's fundamental solution of the equation (3.2) is $G^{\rm ess}$-equivalent to the function $u(t,x,y)=1-H(t)$.

The exhaustive classification of Lie reductions of the equation (3.2) on the base of listing $G^{\rm ess}$-inequivalent one- and two-dimensional subalgebras of its essential Lie invariance algebra leads to discovering wide families of its exact solutions, which are in general $G^{\rm ess}$-inequivalent to each other. The most interesting among these families are three families parameterized by single arbitrary solutions of the classical (1+1)-dimensional linear heat equation and one family parameterized by an arbitrary solution of the (1+1)-dimensional linear heat equation with a particular inverse square potential. One more family has three parameters and is expressed in terms of the general solutions of a one-parameter family of Kummer's equations. We show how to construct more general families of solutions of the equation (3.2) using generalized reductions with respect to generalized symmetries that are generated from elements of $\mathfrak{g}^{\rm ess}$ via acting by recursion operators that are counterparts of elements of $\mathfrak{g}^{\rm ess}$ among first-order differential operators in total derivatives. Since the (1+1)-dimensional linear heat equation with a particular inverse square potential arises in the course of a Lie reduction of the equation (3.2), we have exhaustively carried out the classical symmetry analysis of the (1+1)-dimensional linear heat equation with a general inverse square potential, including the construction of its complete point symmetry group by the direct method and the comprehensive classification of its Lie reductions.

Among (1+2)-dimensional ultraparabolic Fokker–Planck equations of some specific form, which are called (1+2)-dimensional Klein–Kramers equations or just Kramers equations, we consider those whose essential Lie invariance algebras are eight-dimensional. We map these equations to the equation (3.2) using point transformations and thus reduce the entire study of these equations within the framework of symmetry analysis, including the construction of exact solutions, to the study of the equation (3.2).

The constructed exact solutions of the equation (3.2) are marked by the bullet • throughout this thesis.

3.2 Lie invariance algebra

The classical infinitesimal approach involves a straightforward algorithm for computing a maximal Lie symmetry algebra of a fixed differential equation. This algorithm is exhaustively described in [9–11, 41] and in Section 2.3. The maximal Lie invariance algebra of the equation (3.2) is (see, e.g., [31])

$$\mathfrak{g} := \langle \mathcal{P}^t,\, \mathcal{D},\, \mathcal{K},\, \mathcal{P}^3,\, \mathcal{P}^2,\, \mathcal{P}^1,\, \mathcal{P}^0,\, \mathcal{I}, \mathcal{Z}(f)\rangle,$$

where

$$\begin{aligned}
&\mathcal{P}^t = \partial_t, \quad \mathcal{D} = 2t\partial_t + x\partial_x + 3y\partial_y - 2u\partial_u,\\
&\mathcal{K} = t^2\partial_t + (tx + 3y)\partial_x + 3ty\partial_y - (x^2 + 2t)u\partial_u,\\
&\mathcal{P}^3 = 3t^2\partial_x + t^3\partial_y + 3(y - tx)u\partial_u, \quad \mathcal{P}^2 = 2t\partial_x + t^2\partial_y - xu\partial_u,\\
&\mathcal{P}^1 = \partial_x + t\partial_y, \quad \mathcal{P}^0 = \partial_y,\\
&\mathcal{I} = u\partial_u, \quad \mathcal{Z}(f) = f(t, x, y)\partial_u.
\end{aligned}$$

Here the parameter function f of (t, x, y) runs through the solution set of the equation (3.2).

The vector fields $\mathcal{Z}(f)$ constitute the infinite-dimensional abelian ideal $\mathfrak{g}^{\rm lin}$ of $\mathfrak{g}$, associated with the linear superposition of solutions of (3.2) (see Section 2.6), $\mathfrak{g}^{\rm lin} := \{\mathcal{Z}(f)\}$. Thus, the algebra $\mathfrak{g}$ can be represented as a semi-direct sum, $\mathfrak{g} = \mathfrak{g}^{\rm ess} \mathbin{\ltimes} \mathfrak{g}^{\rm lin}$, where

$$\mathfrak{g}^{\rm ess} = \langle \mathcal{P}^t, \mathcal{D}, \mathcal{K}, \mathcal{P}^3, \mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^0, \mathcal{I}\rangle \tag{3.4}$$

is an (eight-dimensional) subalgebra of $\mathfrak{g}$, called the essential Lie invariance algebra of (3.2).

Up to the skew-symmetry of the Lie bracket, the nonzero commutation relations between the basis vector fields of $\mathfrak{g}^{\rm ess}$ are the following:

$$\begin{aligned}
&[\mathcal{P}^t, \mathcal{D}] = 2\mathcal{P}^t, \quad [\mathcal{P}^t, \mathcal{K}] = \mathcal{D}, \quad [\mathcal{D}, \mathcal{K}] = 2\mathcal{K},\\
&[\mathcal{P}^t, \mathcal{P}^3] = 3\mathcal{P}^2, \quad [\mathcal{P}^t, \mathcal{P}^2] = 2\mathcal{P}^1, \quad [\mathcal{P}^t, \mathcal{P}^1] = \mathcal{P}^0,\\
&[\mathcal{P}^3, \mathcal{D}] = -3\mathcal{P}^3, \quad [\mathcal{P}^2, \mathcal{D}] = -\mathcal{P}^2, \quad [\mathcal{P}^1, \mathcal{D}] = \mathcal{P}^1, \quad [\mathcal{P}^0, \mathcal{D}] = 3\mathcal{P}^0,\\
&[\mathcal{P}^2, \mathcal{K}] = \mathcal{P}^3, \quad [\mathcal{P}^1, \mathcal{K}] = 2\mathcal{P}^2, \quad [\mathcal{P}^0, \mathcal{K}] = 3\mathcal{P}^1,\\
&[\mathcal{P}^1, \mathcal{P}^2] = -\mathcal{I}, \quad [\mathcal{P}^0, \mathcal{P}^3] = 3\mathcal{I}.
\end{aligned}$$

The algebra $\mathfrak{g}^{\rm ess}$ is nonsolvable. Its Levi decomposition is given by $\mathfrak{g}^{\rm ess} = \mathfrak{f} \mathbin{\ltimes} \mathfrak{r}$, where the radical $\mathfrak{r}$ of $\mathfrak{g}^{\rm ess}$ coincides with the nilradical of $\mathfrak{g}^{\rm ess}$ and is spanned by the vector fields $\mathcal{P}^3$, $\mathcal{P}^2$, $\mathcal{P}^1$, $\mathcal{P}^0$ and $\mathcal{I}$. The Levi factor $\mathfrak{f} = \langle \mathcal{P}^t, \mathcal{D}, \mathcal{K}\rangle$ of $\mathfrak{g}^{\rm ess}$ is isomorphic to $\mathrm{sl}(2, \mathbb{R})$, the radical $\mathfrak{r}$ of $\mathfrak{g}^{\rm ess}$ is isomorphic to the rank-two Heisenberg algebra $\mathrm{h}(2, \mathbb{R})$, and the real representation of the Levi factor $\mathfrak{f}$ on the radical $\mathfrak{r}$ coincides, in the basis $(\mathcal{P}^3, \mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^0, \mathcal{I})$, with the real representation $\rho_3 \oplus \rho_0$ of $\mathrm{sl}(2, \mathbb{R})$. Here ρ_n is the standard real irreducible representation of $\mathrm{sl}(2, \mathbb{R})$ in the $(n + 1)$-dimensional vector space. More specifically, $\rho_n(\mathcal{P}^t)_{ij} = (n - j)\delta_{i,j+1}$, $\rho_n(\mathcal{D})_{ij} = (n - 2j)\delta_{ij}$, $\rho_n(-\mathcal{K})_{ij} = j\delta_{i+1,j}$, where $i, j \in \{1, 2, \dots, n + 1\}$, $n \in \mathbb{N} \cup \{0\}$, and δ_{kl} is the Kronecker delta, i.e., $\delta_{kl} = 1$ if $k = l$ and $\delta_{kl} = 0$ otherwise, $k, l \in \mathbb{Z}$. Thus, the entire algebra $\mathfrak{g}^{\rm ess}$ is isomorphic to the algebra $L_{8,19}$ from the classification of indecomposable Lie algebras of dimensions up to eight with nontrivial Levi decompositions, which was carried out in [59].

Lie algebras whose Levi factors are isomorphic to the algebra $\mathrm{sl}(2, \mathbb{R})$ often arise within the field of group analysis of differential equations as Lie invariance algebras of parabolic partial differential equations. At the same time, the action of Levi factors on the corresponding radicals is usually described in terms of the representations ρ_0, ρ_1, ρ_2 or their direct sums. To the best of our knowledge, algebras similar to $\mathfrak{g}^{\rm ess}$ were never considered in group analysis from the point of view of their subalgebra structure.

3.3 Complete point symmetry group

The identity component of the complete point symmetry group G of the equation (3.2) is generated by the image of the Lie algebra $\mathfrak{g}$ under the exponentiation map. However, there is no way to obtain full list of discrete elements of G without having its complete description. That is why we need to compute G using the direct method.

We start computing the complete point symmetry group G of the equation (3.2) with presenting the exhaustive description of the equivalence groupoid of the class $\bar{\mathcal{F}}$. Then we use special properties of this groupoid for deriving an explicit representation for elements of G. See [2, 47, 62] for definitions and theoretical results on various structures constituted by point transformations within classes of differential equations.

Theorem 30. *The class $\bar{\mathcal{F}}$ is normalized. Its (usual) equivalence (pseudo)group $G^{\sim}_{\bar{\mathcal{F}}}$ consists of the point transformations with the components*

$$\tilde t=T(t,y),\quad \tilde x=X(t,x,y),\quad \tilde y=Y(t,y),\quad \tilde u=U^1(t,x,y)u+U^0(t,x,y), \tag{3.5a}$$

$$\tilde A^0=\frac{A^0}{T_t{+}BT_y}-\frac{A^1}{T_t{+}BT_y}\frac{U^1_x}{U^1}+\frac{A^2}{T_t{+}BT_y}\left(\left(\frac{U^1_x}{U^1}\right)^2-\left(\frac{U^1_x}{U^1}\right)_x\right)+\frac{1}{U^1}\frac{U^1_t{+}BU^1_y}{T_t{+}BT_y}, \tag{3.5b}$$

$$\tilde A^1=A^1\frac{X_x}{T_t+BT_y}-\frac{X_t+BX_y}{T_t+BT_y}+A^2\frac{X_{xx}-2X_xU^1_x/U^1}{T_t+BT_y}, \tag{3.5c}$$

$$\tilde A^2=A^2\frac{X_x^2}{T_t+BT_y},\quad \tilde B=\frac{Y_t+BY_y}{T_t+BT_y}, \tag{3.5d}$$

$$\tilde C=\frac{U^1}{T_t+BT_y}\left(C-E\frac{U^0}{U^1}\right), \tag{3.5e}$$

where T, X, Y, U^1 and U^0 are arbitrary smooth functions of their arguments satisfying the condition $(T_tY_y-T_yY_t)X_xU^1\neq0$, and $E=\partial_t+B\partial_y-A^2\partial_{xx}-A^1\partial_x-A^0$.

The proof of this theorem is based on the straightforward application of the direct method and it is beyond the subject of the present thesis and will be presented elsewhere. The normalization of the class $\bar{\mathcal{F}}$ means that its equivalence groupoid coincides with the action groupoid of the group $G^{\sim}_{\bar{\mathcal{F}}}$.

The remarkable Fokker–Planck equation (3.2) belongs to $\bar{\mathcal{F}}$ with the corresponding value $(x,1,0,0,0) := \bar\theta_0$ of the arbitrary-element tuple $\bar\theta = (B, A^2, A^1, A^0, C)$ of the class $\bar{\mathcal{F}}$. The vertex group $\mathcal{G}_{\bar\theta_0} := \mathcal{G}^{\sim}_{\bar{\mathcal{F}}}(\bar\theta_0, \bar\theta_0)$ is the set of admissible transformations of the class $\bar{\mathcal{F}}$ with $\bar\theta_0$ as both their source and target, $\mathcal{G}_{\bar\theta_0} = \{(\bar\theta_0, \Phi, \bar\theta_0) \mid \Phi \in G\}$. This argument allows us to use Theorem 30 in the course of computing the group G.

Theorem 31. *The complete point symmetry (pseudo)group G of the remarkable Fokker–Planck equation* (3.2) *consists of the transformations of the form*

$$
\begin{aligned}
&\tilde t = \frac{\alpha t+\beta}{\gamma t+\delta}, \quad \tilde x = \frac{\hat x}{\gamma t+\delta} - \frac{3\gamma\hat y}{(\gamma t+\delta)^2}, \quad \tilde y = \frac{\hat y}{(\gamma t+\delta)^3},\\
&\tilde u = \sigma(\gamma t+\delta)^2 \exp\left(\frac{\gamma\hat x^2}{\gamma t+\delta} - \frac{3\gamma^2\hat x\hat y}{(\gamma t+\delta)^2} + \frac{3\gamma^3\hat y^2}{(\gamma t+\delta)^3}\right)\\
&\qquad\times \exp\big(3\lambda_3(y-tx) - \lambda_2 x - (3\lambda_3^2t^3 + 3\lambda_3\lambda_2t^2 + \lambda_2^2t)\big)\big(u + f(t,x,y)\big),
\end{aligned}
\tag{3.6}
$$

where $\hat x := x + 3\lambda_3t^2 + 2\lambda_2t + \lambda_1$, $\hat y := y + \lambda_3t^3 + \lambda_2t^2 + \lambda_1t + \lambda_0$; α, β, γ and δ are arbitrary constants with $\alpha\delta - \beta\gamma = 1$; λ_0, …, λ_3 and σ are arbitrary constants with $\sigma \neq 0$, and f is an arbitrary solution of (3.2)*.*

Proof. We should integrate the system (3.5) with $\tilde{\bar\theta} = \bar\theta = \bar\theta_0$ with respect to the parameter functions T, X, Y and U^1. The equations (3.5d) take the form

$$
X = \frac{Y_t + xY_y}{T_t + xT_y}, \quad X_x^2 = T_t + xT_y.
$$

In view of the first of these equations, the second equation reduces to $(T_tY_y - T_yY_t)^2 = (T_t + xT_y)^5$, which implies $T_y = 0$, $T_t > 0$ and

$$
Y = \varepsilon T_t^{3/2} y + Y^0(t),
$$

where $\varepsilon = \pm1$, and Y^0 is a function of t arising due to the integration with respect to y. This is why we have $X = (Y_t + xY_y)/T_t$, and hence $X_{xx} = 0$. Then the equation (3.5c) simplifies to $X_t + xX_y = -2X_xU^1_x/U^1$ and thus integrates to

$$U^1 = V(t,y)\exp\left(-\frac{T_{tt}}{2T_t}x^2 - 3\frac{2T_{ttt}T_t - T_{tt}^2}{8T_t^2}xy - \frac{(Y^0/T_t)_t}{2T_t^{5/2}}x\right), \tag{3.7}$$

where V is a nonvanishing smooth function of (t,y), the explicit expression for which will be derived below. Substituting the expression for U^1 into the restricted equation (3.5b),

$$\left(\frac{U^1_x}{U^1}\right)^2 - \left(\frac{U^1_x}{U^1}\right)_x + \frac{U^1_t + xU^1_y}{U^1} = 0, \tag{3.8}$$

leads to an equation whose left-hand side is a quadratic polynomial in x. Collecting the coefficients of x^2, we derive the equation $T_{ttt}/T_t - \frac{3}{2}(T_{tt}/T_t)^2 = 0$ meaning that the Schwarzian derivative of T is zero. Therefore, T is a linear fractional function of t, $T = (\alpha t+\beta)/(\gamma t+\delta)$. Since the constants α, β, γ and δ are defined up to a constant nonzero multiplier and $T_t > 0$, we can assume that $\alpha\delta - \beta\gamma = 1$. Then the result of collecting the coefficients of x in the above equation can be represented, up to an inessential multiplier, in the form

$$\frac{V_y}{V} = \frac{6\gamma^3}{(\gamma t+\delta)^3}y + \big((\gamma t+\delta)^3Y^0\big)_{ttt} - 6\gamma^2\big((\gamma t+\delta)Y^0\big)_t.$$

The general solution of the last equation as an equation with respect to V is

$$V = \phi(t)\exp\left(\frac{3\gamma^3}{(\gamma t+\delta)^3}y^2 + \big((\gamma t+\delta)^3Y^0\big)_{ttt}y - 6\gamma^2\big((\gamma t+\delta)Y^0\big)_t y\right),$$

where ϕ is a nonvanishing smooth function of t. Analogously, we collect the summands without x, substitute the above representation for V into the obtained equation, split the result with respect to y and additionally neglect inessential multipliers. This gives the system of two equations

$$\big((\gamma t+\delta)^3Y^0\big)_{tttt} = 0,$$

$$\frac{\phi_t}{\phi} - \frac{2\gamma}{\gamma t+\delta} + \frac{1}{4}(\gamma t+\delta)^4\Big(\big((\gamma t+\delta)Y^0\big)_{tt}\Big)^2 = 0,$$

whose general solution can be represented as

$$Y^0 = \frac{\lambda_3 t^3 + \lambda_2 t^2 + \lambda_1 t + \lambda_0}{(\gamma t + \delta)^3},$$
$$\phi = \sigma(\gamma t + \delta)^2 \exp\left(\frac{\gamma \psi_t^2}{\gamma t + \delta} - \frac{3\gamma^2 \psi_t \psi}{(\gamma t + \delta)^2} + \frac{3\gamma^3 \psi^2}{(\gamma t + \delta)^3}\right) \exp\left(-\lambda_2^2 t - 3\lambda_3\lambda_2 t^2 - 3\lambda_3^2 t^3\right),$$

where ψ is an arbitrary at most cubic polynomial of t, $\psi(t) := \lambda_3 t^3 + \lambda_2 t^2 + \lambda_1 t + \lambda_0$, and $\lambda_0, \ldots, \lambda_3$ and σ are arbitrary constants with $\sigma \neq 0$. Since the constant parameters α, β, γ and δ are still defined up to the multiplier ± 1, we can choose these parameters in such a way that $\varepsilon|\gamma t + \delta| = (\gamma t + \delta)$, and then neglect the parameter ε.

Finally, the equation (3.5e) takes the form

$$\left(\frac{U^0}{U^1}\right)_t + x\left(\frac{U^0}{U^1}\right)_y = \left(\frac{U^0}{U^1}\right)_{xx},$$

and thus $U^0 = U^1 f$, where $f = f(t, x, y)$ is an arbitrary solution of (3.2). □

Accurate analysis of the structure of the (pseudo)group G needs a proper interpretation of the corresponding group operation, which we assume to be done in the same way as that in Section 2.4.

Now we can analyze the structure of G. The point transformations of the form

$$\mathcal{Z}(f)\colon \quad \tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{y} = y, \quad \tilde{u} = u + f(t, x, y),$$

where the parameter function $f = f(t, x, y)$ is an arbitrary solution of the equation (3.2), are associated with the linear superposition of solutions of this equation and, thus, can be considered as trivial. They constitute the normal (pseudo)subgroup $G^{\rm lin}$ of the (pseudo)group G. The group G splits over $G^{\rm lin}$, $G = G^{\rm ess} \ltimes G^{\rm lin}$, where the subgroup $G^{\rm ess}$ of G consists of the transformations of the form (3.6) with $f = 0$ and thus is an eight-dimensional Lie group. Recall that the subgroup $G^{\rm ess}$is called the *essential point symmetry group* of the equation (3.2), see Section 2.6. The subgroup itself splits over R, $G^{\rm ess} = F \ltimes R$. Here R and F are the normal subgroup and the subgroup of $G^{\rm ess}$ that are singled out by the constraints $\alpha = \delta = 1$, $\beta = \gamma = 0$ and $\lambda_3 = \lambda_2 = \lambda_1 = \lambda_0 = 0$, $\sigma = 1$, respectively. They are isomorphic to the groups $\mathrm{H}(2,\mathbb{R}) \times \mathbb{Z}_2$ and $\mathrm{SL}(2,\mathbb{R})$, and their Lie algebras coincide with $\mathfrak{r} \simeq \mathrm{h}(2,\mathbb{R})$

and $\mathfrak{f}\simeq\mathrm{sl}(2,\mathbb{R})$. Here $\mathrm{H}(2,\mathbb{R})$ denotes the rank-two real Heisenberg group. The normal subgroups $R_{\rm c}$ and $R_{\rm d}$ of R that are isomorphic to $\mathrm{H}(2,\mathbb{R})$ and $\mathbb{Z}_2$ are constituted by the elements of R with parameter values satisfying the constraints $\sigma>0$ and $\lambda_3=\lambda_2=\lambda_1=\lambda_0=0$, $\sigma\in\{-1,1\}$, respectively. The isomorphisms of F to $\mathrm{SL}(2,\mathbb{R})$ and of $R_{\rm c}$ to $\mathrm{H}(2,\mathbb{R})$ are established by the correspondences

$$(\alpha,\beta,\gamma,\delta)_{\alpha\delta-\beta\gamma=1}\mapsto\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix},\quad (\lambda_3,\lambda_2,\lambda_1,\lambda_0,\sigma)_{,\,\sigma>0}\mapsto\begin{pmatrix}1&3\lambda_3&-\lambda_2&\ln\sigma\\ 0&1&0&\lambda_0\\ 0&0&1&\lambda_1\\ 0&0&0&1\end{pmatrix}.$$

Thus, F and $R_{\rm c}$ are connected subgroups of $G^{\rm ess}$, but $R_{\rm d}$ is not. The natural conjugacy action of the group F on the normal subgroup R is given by $(\tilde\lambda_3,\tilde\lambda_2,\tilde\lambda_1,\tilde\lambda_0,\tilde\sigma)=(\lambda_3,\lambda_2,\lambda_1,\lambda_0,\sigma)\,A$ in the parameterization (3.6) of G, where $A=\varrho_3(\alpha,\beta,\gamma,\delta)\oplus(1)$, and ϱ_3 is the standard real irreducible four-dimensional representation of $\mathrm{SL}(2,\mathbb{R})$, which can be identified with the action of $\mathrm{SL}(2,\mathbb{R})$ on binary cubics,

$$\varrho_3\colon\ (\alpha,\beta,\gamma,\delta)_{\alpha\delta-\beta\gamma=1}\mapsto\begin{pmatrix}\alpha^3&3\alpha^2\beta&3\alpha\beta^2&\beta^3\\ \alpha^2\gamma&2\alpha\beta\gamma+\alpha^2\delta&2\alpha\beta\delta+\beta^2\gamma&\beta^2\delta\\ \alpha\gamma^2&2\alpha\gamma\delta+\beta\gamma^2&2\beta\gamma\delta+\alpha\delta^2&\beta\delta^2\\ \gamma^3&3\gamma^2\delta&3\gamma\delta^2&\beta\delta^2\end{pmatrix}.$$

Summing up, the group $G^{\rm ess}$ is isomorphic to $\big(\mathrm{SL}(2,\mathbb{R})\ltimes_\varphi\mathrm{H}(2,\mathbb{R})\big)\times\mathbb{Z}_2$, where the antihomomorphism $\varphi\colon\mathrm{SL}(2,\mathbb{R})\to\mathrm{Aut}(\mathrm{H}(2,\mathbb{R}))$ is defined, in the chosen local coordinates, by $\varphi(\alpha,\beta,\gamma,\delta)=(\lambda_3,\lambda_2,\lambda_1,\lambda_0,\sigma)\mapsto(\lambda_3,\lambda_2,\lambda_1,\lambda_0,\sigma)A$.

Transformations from the one-parameter subgroups of $G^{\rm ess}$ that are generated by the basis elements of $\mathfrak{g}^{\rm ess}$ given in (3.4) are of the following form:

$$\begin{array}{lllll}
\mathcal{P}^t(\epsilon)\colon & \tilde t=t+\epsilon, & \tilde x=x, & \tilde y=y, & \tilde u=u,\\
\mathcal{D}(\epsilon)\colon & \tilde t={\rm e}^{2\epsilon}t, & \tilde x={\rm e}^{\epsilon}x, & \tilde y={\rm e}^{3\epsilon}x, & \tilde u={\rm e}^{-2\epsilon}u,\\
\mathcal{K}(\epsilon)\colon & \tilde t=\dfrac{t}{1-\epsilon t}, & \tilde x=\dfrac{x}{1-\epsilon t}+\dfrac{3\epsilon y}{(1-\epsilon t)^2}, & \tilde y=\dfrac{y}{(1-\epsilon t)^3}, & \tilde u=(1-\epsilon t)^2{\rm e}^{\frac{\epsilon x^2}{1-\epsilon t}+\frac{3\epsilon^2xy}{(1-\epsilon t)^2}+\frac{3\epsilon^3y^2}{(1-\epsilon t)^3}}u,\\
\mathcal{P}^3(\epsilon)\colon & \tilde t=t, & \tilde x=x+3\epsilon t^2, & \tilde y=y+\epsilon t^3, & \tilde u={\rm e}^{3\epsilon(y-tx)-3\epsilon^2t^3}u,\\
\mathcal{P}^2(\epsilon)\colon & \tilde t=t, & \tilde x=x+2\epsilon t, & \tilde y=y+\epsilon t^2, & \tilde u={\rm e}^{-\epsilon x+\epsilon^2t}u,\\
\mathcal{P}^1(\epsilon)\colon & \tilde t=t, & \tilde x=x+\epsilon, & \tilde y=y+\epsilon t, & \tilde u=u,
\end{array}$$

$$\mathcal{P}^0(\epsilon)\colon\;\; \tilde t=t,\; \tilde x=x,\; \tilde y=y+\epsilon,\; \tilde u=u,$$
$$\mathcal{I}(\epsilon)\colon\;\;\; \tilde t=t,\; \tilde x=x,\; \tilde y=y, \qquad \tilde u=\mathrm{e}^{\epsilon}u,$$

where ϵ is an arbitrary constant. At the same time, using this basis of $\mathfrak{g}^{\rm ess}$ in the course of studying the structure of the group $G^{\rm ess}$ hides some of its important properties and complicates its study.

Although the pushforward of the group G by the natural projection of $\mathbb{R}^4_{t,x,y,u}$ onto $\mathbb{R}_t$ coincides with the group of linear fractional transformations of t and is thus isomorphic to the group $\mathrm{PSL}(2,\mathbb{R})$, the subgroup F of G is isomorphic to the group $\mathrm{SL}(2,\mathbb{R})$, and its Iwasawa decomposition is given by the one-parameter subgroups of G respectively generated by the vector fields $\mathcal{P}^t+\mathcal{K}$, $\mathcal{D}$ and $\mathcal{P}^t$. The first subgroup, which is associated with $\mathcal{P}^t+\mathcal{K}$, consists of the point transformations

$$\begin{gathered}\tilde t=\frac{t\cos\epsilon-\sin\epsilon}{t\sin\epsilon+\cos\epsilon},\; \tilde x=\frac{x}{t\sin\epsilon+\cos\epsilon}-\frac{3y\sin\epsilon}{(t\sin\epsilon+\cos\epsilon)^2},\; \tilde y=\frac{y}{(t\sin\epsilon+\cos\epsilon)^3},\\ \tilde u=(t\sin\epsilon+\cos\epsilon)^2\exp\left(\frac{x^2\sin\epsilon}{t\sin\epsilon+\cos\epsilon}-\frac{3xy\sin^2\epsilon}{(t\sin\epsilon+\cos\epsilon)^2}+\frac{3y^2\sin^3\epsilon}{(t\sin\epsilon+\cos\epsilon)^3}\right)u,\end{gathered}\tag{3.9}$$

where ϵ is an arbitrary constant parameter, which is defined by the corresponding transformation up to a summand $2\pi k$, $k\in\mathbb{Z}$.

The equation (3.2) is invariant with respect to the involution $\mathscr{J}$ simultaneously alternating the sign of (x,y),

$$\mathscr{J}\colon (t,x,y,u)\mapsto(t,-x,-y,u).$$

In the context of the one-parameter subgroups of $G^{\rm ess}$ that are generated by the basis elements of $\mathfrak{g}^{\rm ess}$ listed in (3.4), the involution $\mathscr{J}$ looks like a discrete point symmetry transformation of (3.2), but in fact this is not the case. It belongs to the one-parameter subgroup of G generated by $\mathcal{P}^t+\mathcal{K}$. More precisely, it coincides with the transformation (3.9) with $\epsilon=\pi$. The value $\epsilon=\pi/2$ corresponds to the transformation

$$\mathcal{K}'\colon\quad \tilde t=-\frac1t,\quad \tilde x=\frac xt-3\frac{y}{t^2},\quad \tilde y=\frac{y}{t^3},\quad \tilde u=t^2\mathrm{e}^{\frac{x^2}{t}-\frac{3xy}{t^2}+\frac{3y^2}{t^3}}u,$$

which also deceptively looks, in the above context, like a discrete point symmetry transformation of (3.2) being independent with $\mathscr{J}$ since the factorization $\mathscr{J}\circ\mathcal{K}'=$

$\mathcal{P}^t(1)\circ\mathcal{K}(1)\circ\mathcal{P}^t(1)$ is not intuitive. This is why it is relevant to accurately describe discrete point symmetries of the equation (3.2).

Corollary 32. *A complete list of discrete point symmetry transformations of the remarkable Fokker–Planck equation* (3.2) *that are independent up to combining with each other and with continuous point symmetry transformations of this equation is exhausted by the single involution $\mathcal{I}'$ alternating the sign of u, $\mathcal{I}'\colon (t,x,y,u)\mapsto(t,x,y,-u)$. Thus, the factor group of the complete point symmetry group G with respect to its identity component is isomorphic to $\mathbb{Z}_2$.*

Proof. It is obvious that the entire subgroup $G^{\rm lin}$ is contained in the connected component of the identity transformation in G. The same claim holds for the subgroups F and $R_{\rm c}$ in view of their isomorphisms to the groups ${\rm SL}(2,\mathbb{R})$ and ${\rm H}(2,\mathbb{R})$, respectively. Therefore, without loss of generality a complete list of independent discrete point symmetry transformations of (3.2) can be assumed to consist of elements of the subgroup $R_{\rm d}$. Thus, the only discrete point symmetry transformation of (3.2) that is independent in the above sense is the transformation $\mathcal{I}'$. □

Below, the transformations $\mathcal{P}^t(\epsilon)$, $\mathcal{D}(\epsilon)$, $\mathcal{K}(\epsilon)$, $\mathcal{P}^3(\epsilon)$, $\mathcal{P}^2(\epsilon)$, $\mathcal{P}^1(\epsilon)$, $\mathcal{P}^0(\epsilon)$, $\mathcal{I}(\epsilon)$, $\mathcal{J}$, $\mathcal{K}'$, $\mathcal{J}'$ and $\mathcal{Z}(f)$ will be called elementary point symmetry transformations of the equation (3.2).

One of the many remarkable properties of the equation (3.2) is that its fundamental solution (3.3) is $G^{\rm ess}$-equivalent to the trivial constant solution $u=1$. More specifically, the following assertion, which is obvious in view of Theorem 31, holds true.

In view of Theorem 31, the formal application of the point symmetry transformation $\Phi:=\mathcal{I}(\ln\frac{\sqrt3}{2\pi})\circ\mathcal{P}^0(y')\circ\mathcal{P}^1(x')\circ\mathcal{P}^t(t')\circ\mathcal{K}'$ of the equation (3.2),

$$
\begin{aligned}
\Phi\colon\quad &\tilde t=-\frac1t+t',\quad \tilde x=\frac xt-3\frac{y}{t^2}+x',\quad \tilde y=\frac{y}{t^3}+y',\\
&\tilde u=\frac{\sqrt3}{2\pi}t^2\exp\left(\frac{x^2}{t}-3\frac{xy}{t^2}+3\frac{y^2}{t^3}+3x'x-3x'\frac yt+3(x')^2t\right)u,
\end{aligned}
\tag{3.10}
$$

maps the function $u(t,x,y)=1-H(t)$ to the fundamental solution (3.3) of the equation (3.2). Recall that H denotes the Heaviside step function. Note that attempts to interpret this fundamental solution within the framework of group analysis of differential equations were made in [19, 31].

3.4 Classification of inequivalent subalgebras

In order to carry out Lie reductions of codimension one and two for the equation (3.2) in the optimal way, we need to classify one- and two-dimensional subalgebras of $\mathfrak{g}^{\rm ess}$ up to $G^{\rm ess}$-equivalence. In the course of this classification, we use the Levi decomposition of $\mathfrak{g}^{\rm ess}$, $\mathfrak{g}^{\rm ess}=\mathfrak{f}\Subset\mathfrak{r}$, and the fact that the Levi factor $\mathfrak{f}$ of $\mathfrak{g}^{\rm ess}$ is isomorphic to $\mathrm{sl}(2,\mathbb{R})$. This allows us to apply the technique of classifying the subalgebras of an algebra with a proper ideal suggested in [46]. This technique becomes simpler if the algebra under consideration can be represented as the semi-direct sum of a subalgebra and an ideal, which are $\mathfrak{f}$ and $\mathfrak{r}$ for the algebra $\mathfrak{g}^{\rm ess}$, respectively. An additional simplification is that an optimal list of subalgebras of $\mathrm{sl}(2,\mathbb{R})$ is well known (see, e.g., [45,48]). Thus, for the realization $\mathfrak{f}$ of $\mathrm{sl}(2,\mathbb{R})$, this list consists of the subalgebras $\{0\}$, $\langle\mathcal{P}^t\rangle$, $\langle\mathcal{D}\rangle$, $\langle\mathcal{P}^t+\mathcal{K}\rangle$, $\langle\mathcal{P}^t,\mathcal{D}\rangle$ and $\mathfrak{f}$ itself. The subalgebras $\mathfrak{s}_1$ and $\mathfrak{s}_2$ of $\mathfrak{g}^{\rm ess}$ are definitely $G^{\rm ess}$-inequivalent if their projections $\pi_\mathfrak{f}\mathfrak{s}_1$ and $\pi_\mathfrak{f}\mathfrak{s}_2$ are F-inequivalent, see Section 3.3. Here and in what follows $\pi_\mathfrak{f}$ and $\pi_\mathfrak{r}$ denote the natural projections of $\mathfrak{g}^{\rm ess}$ onto $\mathfrak{f}$ and $\mathfrak{r}$ according to the decomposition $\mathfrak{g}^{\rm ess}=\mathfrak{f}\dotplus\mathfrak{r}$ of the vector space $\mathfrak{g}^{\rm ess}$ as the direct sum of its subspaces $\mathfrak{f}$ and $\mathfrak{r}$. In other words, we can use the projections $\pi_\mathfrak{f}\mathfrak{s}$ of the subalgebras $\mathfrak{s}$ of $\mathfrak{g}^{\rm ess}$ of the same dimension for partitioning the set of these subalgebras into subsets such that each subset contains no subalgebra being equivalent to a subalgebra from another subset.

Let us specifically describe the application of the above technique to the classification of one- and two-dimensional subalgebras of $\mathfrak{g}^{\rm ess}$ up to $G^{\rm ess}$-equivalence. We fix the dimension d of subalgebras $\mathfrak{s}$ to be classified, either $d=1$ or $d=2$, and consider those subalgebras $\mathfrak{s}_\mathfrak{f}$ of $\mathfrak{f}$ from the above list with dimension d' less than or equal to d. For each of these subalgebras, we take the set of d-dimensional subalgebras of $\mathfrak{g}^{\rm ess}$ with $\pi_\mathfrak{f}\mathfrak{s}=\mathfrak{s}_\mathfrak{f}$ and construct a complete list of $G^{\rm ess}$-inequivalent subalgebras in this set.

One usually classifies subalgebras of a Lie algebra up to their equivalence generated by the group of inner automorphisms of this algebra, and this group is computed via summing up Lie series or solving Cauchy problems with the adjoint action of the algebra on itself, see, e.g., [41, Section 3.3]. At the same time, we already know that the algebra $\mathfrak{g}^{\rm ess}$ is the Lie algebra of the group $G^{\rm ess}$, and the actions of $G^{\rm ess}$ and of the group of inner automorphisms of $\mathfrak{g}^{\rm ess}$ on $\mathfrak{g}^{\rm ess}$ coincide. Moreover, recall that for the purpose of finding Lie invariant solutions of the equation (3.2), subalgebras of $\mathfrak{g}^{\rm ess}$ have in fact to be classified up to the $G^{\rm ess}$-equivalence. This is why following, e.g., [16], we directly compute the action of $G^{\rm ess}$ on $\mathfrak{g}^{\rm ess}$ via pushing forward vector fields from $\mathfrak{g}^{\rm ess}$ by transformations from $G^{\rm ess}$. The non-identity pushforwards of basis elements of $\mathfrak{g}^{\rm ess}$ by the elementary transformations from $G^{\rm ess}$ are the following:

$$
\begin{aligned}
&\mathscr{P}^t(\epsilon)_*\mathcal{D}=\mathcal{D}-2\epsilon\mathcal{P}^t, && \mathscr{K}(\epsilon)_*\mathcal{D}=\mathcal{D}+2\epsilon\mathcal{K},\\
&\mathscr{P}^t(\epsilon)_*\mathcal{K}=\mathcal{K}-\epsilon\mathcal{D}+\epsilon^2\mathcal{P}^t, && \mathscr{K}(\epsilon)_*\mathcal{P}^t=\mathcal{P}^t+\epsilon\mathcal{D}+\epsilon^2\mathcal{K},\\
&\mathscr{P}^t(\epsilon)_*\mathcal{P}^3=\mathcal{P}^3-3\epsilon\mathcal{P}^2+3\epsilon^2\mathcal{P}^1-\epsilon^3\mathcal{P}^0, && \mathscr{K}(\epsilon)_*\mathcal{P}^2=\mathcal{P}^2+\epsilon\mathcal{P}^3,\\
&\mathscr{P}^t(\epsilon)_*\mathcal{P}^2=\mathcal{P}^2-2\epsilon\mathcal{P}^1+\epsilon^2\mathcal{P}^0, && \mathscr{K}(\epsilon)_*\mathcal{P}^1=\mathcal{P}^1+2\epsilon\mathcal{P}^2+\epsilon^2\mathcal{P}^3,\\
&\mathscr{P}^t(\epsilon)_*\mathcal{P}^1=\mathcal{P}^1-\epsilon\mathcal{P}^0, && \mathscr{K}(\epsilon)_*\mathcal{P}^0=\mathcal{P}^0+3\epsilon\mathcal{P}^1+3\epsilon^2\mathcal{P}^2+\epsilon^3\mathcal{P}^3,
\end{aligned}
$$

$$
\begin{aligned}
&\mathscr{D}(\epsilon)_*\mathcal{P}^t=e^{2\epsilon}\mathcal{P}^t, && \mathscr{D}(\epsilon)_*\mathcal{P}^3=e^{-3\epsilon}\mathcal{P}^3, && \mathscr{D}(\epsilon)_*\mathcal{P}^1=e^{\epsilon}\mathcal{P}^1,\\
&\mathscr{D}(\epsilon)_*\mathcal{K}=e^{-2\epsilon}\mathcal{K}, && \mathscr{D}(\epsilon)_*\mathcal{P}^2=e^{-\epsilon}\mathcal{P}^2, && \mathscr{D}(\epsilon)_*\mathcal{P}^0=e^{3\epsilon}\mathcal{P}^0,
\end{aligned}
$$

$$
\begin{aligned}
&\mathscr{P}^3(\epsilon)_*\mathcal{P}^t=\mathcal{P}^t+3\epsilon\mathcal{P}^2, && \mathscr{P}^0(\epsilon)_*\mathcal{D}=\mathcal{D}-3\epsilon\mathcal{P}^0,\\
&\mathscr{P}^3(\epsilon)_*\mathcal{D}=\mathcal{D}+3\epsilon\mathcal{P}^3, && \mathscr{P}^0(\epsilon)_*\mathcal{K}=\mathcal{K}-3\epsilon\mathcal{P}^1,\\
&\mathscr{P}^3(\epsilon)_*\mathcal{P}^0=\mathcal{P}^0+3\epsilon\mathcal{I}, && \mathscr{P}^0(\epsilon)_*\mathcal{P}^3=\mathcal{P}^3-3\epsilon\mathcal{I},
\end{aligned}
$$

$$
\begin{aligned}
&\mathscr{P}^2(\epsilon)_*\mathcal{P}^t=\mathcal{P}^t+2\epsilon\mathcal{P}^1-\epsilon^2\mathcal{I}, && \mathscr{P}^1(\epsilon)_*\mathcal{P}^t=\mathcal{P}^t+\epsilon\mathcal{P}^0,\\
&\mathscr{P}^2(\epsilon)_*\mathcal{D}=\mathcal{D}+\epsilon\mathcal{P}^2, && \mathscr{P}^1(\epsilon)_*\mathcal{D}=\mathcal{D}-\epsilon\mathcal{P}^1,\\
&\mathscr{P}^2(\epsilon)_*\mathcal{K}=\mathcal{K}-\epsilon\mathcal{P}^3, && \mathscr{P}^1(\epsilon)_*\mathcal{K}=\mathcal{K}-2\epsilon\mathcal{P}^2-\epsilon^2\mathcal{I},\\
&\mathscr{P}^2(\epsilon)_*\mathcal{P}^1=\mathcal{P}^1-\epsilon\mathcal{I}, && \mathscr{P}^1(\epsilon)_*\mathcal{P}^2=\mathcal{P}^2+\epsilon\mathcal{I},
\end{aligned}
$$

$$
\begin{aligned}
&\mathscr{J}_*(\mathcal{P}^3,\mathcal{P}^2,\mathcal{P}^1,\mathcal{P}^0)=(-\mathcal{P}^3,-\mathcal{P}^2,-\mathcal{P}^1,-\mathcal{P}^0),\\
&\mathscr{K}'_*(\mathcal{P}^t,\mathcal{D},\mathcal{K},\mathcal{P}^3,\mathcal{P}^2,\mathcal{P}^1,\mathcal{P}^0)=(\mathcal{K},-\mathcal{D},\mathcal{P}^t,\mathcal{P}^0,-\mathcal{P}^1,\mathcal{P}^2,-\mathcal{P}^3).
\end{aligned}
$$

The result of the classification of one- and two-dimensional subalgebras of $\mathfrak{g}^{\rm ess}$ up to $G^{\rm ess}$-equivalence is presented in the two subsequent lemmas.

Lemma 33. *A complete list of $G^{\rm ess}$-inequivalent one-dimensional subalgebras of $\mathfrak g^{\rm ess}$ is exhausted by the subalgebras*

$$\begin{aligned}&\mathfrak s_{1.1}=\langle\mathcal P^t+\mathcal P^3\rangle,\quad \mathfrak s^{\delta}_{1.2}=\langle\mathcal P^t+\delta\mathcal I\rangle,\quad \mathfrak s^{\nu}_{1.3}=\langle\mathcal D+\nu\mathcal I\rangle,\quad \mathfrak s^{\mu}_{1.4}=\langle\mathcal P^t+\mathcal K+\mu\mathcal I\rangle,\\ &\mathfrak s^{\varepsilon}_{1.5}=\langle\mathcal P^2+\varepsilon\mathcal P^0\rangle,\quad \mathfrak s_{1.6}=\langle\mathcal P^1\rangle,\quad \mathfrak s_{1.7}=\langle\mathcal P^0\rangle,\quad \mathfrak s_{1.8}=\langle\mathcal I\rangle,\end{aligned}$$

where $\varepsilon\in\{-1,1\}$, $\delta\in\{-1,0,1\}$, and μ and ν are arbitrary real constants with $\nu\geqslant0$.

Proof. A complete set of F-inequivalent d'-dimensional subalgebras of $\mathfrak f$ with $d'\leqslant1$ consists of the subalgebras $\langle\mathcal P^t\rangle$, $\langle\mathcal D\rangle$, $\langle\mathcal P^t+\mathcal K\rangle$ and $\{0\}$. Therefore, without loss of generality we can consider only one-dimensional subalgebras of $\mathfrak g^{\rm ess}$ with basis vector fields Q of the general form

$$Q=\hat Q+a_3\mathcal P^3+3a_2\mathcal P^2+3a_1\mathcal P^1+a_0\mathcal P^0+b\mathcal I,$$

where $\hat Q\in\{\mathcal P^t,\mathcal D,P^t+\mathcal K,0\}$ and a_0, …, a_3 and b are real constants. We factor out the multiplier 3 in the coefficients of $\mathcal P^1$ and $\mathcal P^2$ since then the coefficients a_3, a_2, a_1 and a_0 are changed under the action of F in the same way as the coefficients of the real cubic binary form $a_3x^3+3a_2x^2y+3a_1xy^2+a_0y^3$ are changed under the standard action of the group of linear fractional transformations in such forms, cf. [42, Example 2.22]. We need to further simplify the vector fields Q of the above form by acting with elements of $G^{\rm ess}$ on them and/or scaling them.

Let $\hat Q=\mathcal P^t$. If $a_3\neq0$, then we successively set $a_3=1$ and $a_2=0$, $a_1=0$, $a_0=0$ and $b=0$ using $\mathscr D(e^{\epsilon_4})_*$ simultaneously with rescaling Q, $\mathscr P^3(\epsilon_3)_*$, $\mathscr P^2(\epsilon_2)_*$, $\mathscr P^1(\epsilon_1)_*$ and $\mathscr P^0(\epsilon_0)_*$, with appropriate constants ϵ_i, $i=0,\dots,4$, respectively. This gives the subalgebra $\mathfrak s_{1.1}$. Similarly, for $a_3=0$ we apply $\mathscr P^3(\epsilon_2)_*$, $\mathscr P^2(\epsilon_1)_*$ and $\mathscr P^1(\epsilon_0)_*$ with appropriate constants ϵ_i, $i=1,2,3$, to get rid of a_2, a_1 and a_0. Then, the action of $\mathscr D(e^{\epsilon_b})_*$ with appropriate ϵ_b and rescaling Q allow us to set $b\in\{-1,1\}$ if $b\neq0$. Thus, we obtain the subalgebras $\mathfrak s^{\delta}_{1.2}$.

For $\hat Q=\mathcal D$ and $\hat Q=\mathcal P^t+\mathcal K$, removing the summands with $\mathcal P^3$, $\mathcal P^2$, $\mathcal P^1$ and $\mathcal P^0$ is analogous. The only further possible simplification is that in the case $\hat Q=\mathcal D$, the sign of the obtained value of b can be alternated using $\mathscr K'_*$ and alternating the sign of Q. This results in the collections of the subalgebras $\mathfrak s^{\nu}_{1.3}$ and $\mathfrak s^{\mu}_{1.4}$, respectively.

Let $\hat Q=0$. The classification of real binary cubics presented in the second table on page 26 in [42] implies that up to F-equivalence and rescaling Q, we have $a_3 = 0$ and $(a_2, a_1, a_0) \in \{(1, \varepsilon, 0), (0, 1, 0), (0, 0, 1), (0, 0, 0)\}$. If $(a_3, a_2, a_1, a_0) \neq (0, 0, 0, 0)$, then using one of the actions $\mathcal{P}^3(\epsilon)_*$, $\mathcal{P}^2(\epsilon)_*$, $\mathcal{P}^1(\epsilon)_*$ and $\mathcal{P}^0(\epsilon)_*$ we can set $b = 0$; otherwise we set $b = 1$ by rescaling Q. Thus, we obtain the rest of subalgebras from the list given in the lemma's statement. □

Lemma 34. *A complete list of G^{ess}-inequivalent two-dimensional subalgebras of $\mathfrak{g}^{\text{ess}}$ is given by*

$$
\begin{aligned}
&\mathfrak{s}_{2.1}^{\mu} = \langle \mathcal{P}^t, \mathcal{D} + \mu\mathcal{I} \rangle, \quad \mathfrak{s}_{2.2}^{\delta} = \langle \mathcal{P}^t + \delta\mathcal{I}, \mathcal{P}^0 \rangle, \quad \mathfrak{s}_{2.3} = \langle \mathcal{P}^t, \mathcal{P}^0 + \mathcal{I} \rangle, \\
&\mathfrak{s}_{2.4}^{\mu} = \langle \mathcal{D} + \mu\mathcal{I}, \mathcal{P}^1 \rangle, \quad \mathfrak{s}_{2.5}^{\mu} = \langle \mathcal{D} + \mu\mathcal{I}, \mathcal{P}^0 \rangle, \\
&\mathfrak{s}_{2.6} = \langle \mathcal{P}^0, \mathcal{P}^1 \rangle, \quad \mathfrak{s}_{2.7} = \langle \mathcal{P}^0, \mathcal{P}^2 \rangle, \quad \mathfrak{s}_{2.8}^{\varepsilon} = \langle \mathcal{P}^1, \mathcal{P}^3 + \varepsilon\mathcal{P}^0 \rangle, \\
&\mathfrak{s}_{2.9} = \langle \mathcal{P}^t + \mathcal{P}^3, \mathcal{I} \rangle, \quad \mathfrak{s}_{2.10}^{\delta} = \langle \mathcal{P}^t, \mathcal{I} \rangle, \quad \mathfrak{s}_{2.11} = \langle \mathcal{D}, \mathcal{I} \rangle, \quad \mathfrak{s}_{2.12} = \langle \mathcal{P}^t + \mathcal{K}, \mathcal{I} \rangle, \\
&\mathfrak{s}_{2.13}^{\varepsilon} = \langle \mathcal{P}^2 + \varepsilon\mathcal{P}^0, \mathcal{I} \rangle, \quad \mathfrak{s}_{2.14} = \langle \mathcal{P}^1, \mathcal{I} \rangle, \quad \mathfrak{s}_{2.15} = \langle \mathcal{P}^0, \mathcal{I} \rangle,
\end{aligned}
$$

where $\varepsilon \in \{-1, 1\}$, $\delta \in \{-1, 0, 1\}$, and μ is an arbitrary real constant.

Proof. A complete set of F-inequivalent d'-dimensional subalgebras of $\mathfrak{f}$ with $d' \leqslant 2$ consists of the subalgebras $\langle \mathcal{P}^t, \mathcal{D} \rangle$, $\langle \mathcal{P}^t \rangle$, $\langle \mathcal{D} \rangle$, $\langle \mathcal{P}^t + \mathcal{K} \rangle$ and $\{0\}$. Therefore, without loss of generality we can consider only two-dimensional subalgebras of $\mathfrak{g}^{\text{ess}}$ with basis vector fields of the general form $Q_i = \hat{Q}_i + \check{Q}_i$, $i = 1, 2$, where

$$
(\hat{Q}_1, \hat{Q}_2) \in \big\{(\mathcal{P}^t, \mathcal{D}),\ (\mathcal{P}^t, 0),\ (\mathcal{D}, 0),\ (\mathcal{P}^t + \mathcal{K}, 0),\ (0, 0)\big\},
$$

and $\check{Q}_i = a_{i3}\mathcal{P}^3 + 3a_{i2}\mathcal{P}^2 + 3a_{i1}\mathcal{P}^1 + a_{i0}\mathcal{P}^0 + b_i\mathcal{I}$ with real constants a_{ij} and b_i, $i = 1, 2$, $j = 0, \dots, 3$. We further need to simplify $\check{Q}_i$ by linearly recombining the vector fields Q_i and acting on them by elements from G^{ess}, when simultaneously maintaining the closedness of the subalgebras with respect to the Lie bracket, $[Q_1, Q_2] \in \langle Q_1, Q_2 \rangle$, under the condition $\dim\langle Q_1, Q_2 \rangle = 2$ and preserving $\hat{Q}_i$.

We separately consider each of the values of $(\hat{Q}_1, \hat{Q}_2)$ in the above set.

$\boldsymbol{(\mathcal{P}^t, \mathcal{D})}$**.** Then according to Lemma 33, up to the G^{ess}-equivalence the vector field Q_1 is equal to either $\mathcal{P}^t + \mathcal{P}^3$ or $\mathcal{P}^t + \delta\mathcal{I}$. The first value is not appropriate since then $[Q_1, Q_2] = \mathcal{P}^t - 3\mathcal{P}^3 + \tilde{Q}$ with $\tilde{Q} \in \langle \mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^0, \mathcal{I} \rangle$ and thus $[Q_1, Q_2] \notin \langle Q_1, Q_2 \rangle$ for any $\check{Q}_2 \in \mathfrak{r}$. For the second value, requiring the condition $[Q_1, Q_2] \in \langle Q_1, Q_2 \rangle$ implies

$a_{23} = a_{22} = a_{21} = \delta = 0$. Acting by $\mathcal{P}^0(a_{20})_*$ on Q_2, we can set $a_{20} = 0$. Thus, we obtain the subalgebra $\mathfrak{s}^{\mu}_{2.1}$ with $\mu = b_2$.

In all the other cases, we have $\hat{Q}_2 = 0$. If in addition $a_{2j} = 0$, $j = 0, \dots, 3$, then $Q_2 = \mathcal{I}$, and thus the basis elements Q_1 and Q_2 commute, then corresponding canonical forms of Q^1 modulo the $G^{\rm ess}$-equivalence are given by the basis elements of the one-dimensional algebras $\mathfrak{s}_{1.1}$, ..., $\mathfrak{s}_{1.7}$ from Lemma 33 up to neglecting the summands with $\mathcal{I}$ due to the possibility of linearly combining Q_1 with Q_2. This results in the subalgebras $\mathfrak{s}_{2.9}$, ..., $\mathfrak{s}_{2.15}$. Hence below we assume that $(a_{23}, a_{22}, a_{21}, a_{20}) \neq (0, 0, 0, 0)$ and, moreover, $\operatorname{rank}(a_{ij})^{i=1,2}_{j=0,\dots,3} = 2$ if $\hat{Q}_1 = \hat{Q}_2 = 0$.

$\boldsymbol{(\mathcal{P}^t, 0)}$**.** The closedness of $\langle Q_1, Q_2\rangle$ with respect to the Lie bracket implies $a_{23} = a_{22} = a_{21} = a_{13}a_{20} = 0$ and thus $a_{13} = 0$ since then $a_{20} \neq 0$. Hence $Q_1 = \mathcal{P}^t + b_1\mathcal{I}$ and $Q_2 = \mathcal{P}^0 + b_2\mathcal{I}$.

If $b_2 = 0$, we can rescale b_1 to δ by combining $\mathcal{D}(\epsilon)_*$ with scaling the entire vector field Q^1. This results in the subalgebras $\mathfrak{s}^{\delta}_{2.2}$.

Let $b_2 \neq 0$. Simultaneously acting by $\mathcal{D}(\epsilon)_*$ with the suitable value of ϵ, rescaling b_2 and, if $b_2 < 0$, acting by $\mathcal{J}_*$ and alternating the sign of Q^2, we can set $b_2 = 1$. Then the pushforward $\mathcal{P}^1(b_1)_*$ preserves Q^2, and $\mathcal{P}^1(b_1)_*Q_1 = \mathcal{P}^t + b_1Q^2$. After linearly combining Q_1 with Q^2, we obtain the subalgebras $\mathfrak{s}_{2.3}$.

$\boldsymbol{(\mathcal{D}, 0)}$**.** In view of Lemma 33, we can reduce Q_1 to $\mathcal{D}+\mu\mathcal{I}$ modulo the $G^{\rm ess}$-equivalence. The condition $[Q_1, Q_2] \in \langle Q_1, Q_2\rangle$ and the commutation relations of $\mathcal{D}$ with basis elements of $\mathfrak{r}$ imply that up to scaling of Q^2, we have

$$(a_{23}, a_{22}, a_{21}, a_{20}, b_2) \in \{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0)\},$$

where the first and the second values are reduced by $\mathcal{K}'_*$ to the third and the fourth values, respectively. Thus, we obtain the subalgebras $\mathfrak{s}^{\mu}_{2.4}$ and $\mathfrak{s}^{\mu}_{2.5}$.

$\boldsymbol{(\mathcal{P}^t + \mathcal{K}, 0)}$**.** This pair is not appropriate since from the closedness of $\langle Q_1, Q_2\rangle$ with respect to the Lie bracket, we straightforwardly derive $a_{2j} = 0$, $j = 0, \dots, 3$.

$\boldsymbol{(0, 0)}$**.** Using Lemma 33, we can set from the very beginning that $Q_1 \in \{\mathcal{P}^2 + \varepsilon\mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^0\}$. Since $\operatorname{rank}(a_{ij})^{i=1,2}_{j=0,\dots,3} = 2$ here, simultaneously with $b_1 = 0$ we can set $b_2 = 0$ using the suitable pushforward $\mathcal{P}^1(\epsilon)_*$.

For $Q=a_3\mathcal{P}^3+3a_2\mathcal{P}^2+3a_1\mathcal{P}^1+a_0\mathcal{P}^0$ with $\bar a:=(a_0,a_1,a_2,a_3)\neq(0,0,0,0)$, we denote

$$\operatorname{Discr}(Q):=a_0^2a_3^2-6a_0a_1a_2a_3+4a_0a_2^3-3a_1^2a_2^2+4a_1^3a_3.$$

The association with cubic binary forms implies that $Q\in\{\mathcal{P}^1,\mathcal{P}^0\}$ modulo the $G^{\rm ess}$-equivalence if and only if $\operatorname{Discr}(Q)=0$, cf. [42, Example 2.22]. To distinguish inequivalent subalgebras in the present classification case, we will consider lines in the space run by $\bar a$, where $\operatorname{Discr}(Q)=0$. We call such lines singular. It is obvious that two-dimensional subalgebras in $\langle\mathcal{P}^0,\mathcal{P}^1,\mathcal{P}^2,\mathcal{P}^3\rangle$ are $G^{\rm ess}$-inequivalent if they possess different numbers of singular lines associated with their elements or, more precisely, different sets of multiplicities of such singular lines.

If $Q_1=\mathcal{P}^0$, then the condition $[Q_1,Q_2]\in\langle Q_1,Q_2\rangle$ implies $a_{23}=0$, and we can set $a_{30}=0$ at any step, linearly combining Q_2 with Q_1.

The case $a_{32}=0$ corresponds to the subalgebra $\mathfrak{s}_{2.6}$. $\operatorname{Discr}(Q)=0$ for any $Q\in\mathfrak{s}_{2.6}$, i.e., the subalgebra $\mathfrak{s}_{2.6}$ possesses an infinite number of singular lines.

For $a_{32}\neq0$, we multiply Q_2 by $a_{32}\neq0$, act by $\mathcal{P}^t(\epsilon)_*$ with $\epsilon=a_{22}/(2a_{32})$ and reset $a_{30}=0$, linearly combining Q_2 with Q_1, which gives the basis of the subalgebra $\mathfrak{s}_{2.7}$. For an arbitrary vector field $Q=a_0\mathcal{P}^0+3a_2\mathcal{P}^2$ in $\mathfrak{s}_{2.7}$ we obtain $\operatorname{Discr}(Q)=4a_0a_3^3$, and thus the subalgebra $\mathfrak{s}_{2.7}$ possesses one singular line of multiplicity one and one singular line of multiplicity three.

If $Q_1=\mathcal{P}^1$, then the condition $[Q_1,Q_2]\in\langle Q_1,Q_2\rangle$ implies $a_{22}=0$, and we can set $a_{31}=0$ at any step, linearly combining Q_2 with Q_1. Then $a_{32}\neq0$ since otherwise we again obtain the subalgebra $\mathfrak{s}_{2.6}$. Rescaling the basis element Q_2, we set $a_{32}=1$. The subalgebra $\langle\mathcal{P}^1,\mathcal{P}^3\rangle$ is mapped by $\mathcal{K}'_*$ to the subalgebra $\mathfrak{s}_{2.7}$. This is why the coefficient a_{30} should be nonzero as well. The action by $\mathcal{P}^1(\epsilon)_*$ with the suitable value of ϵ allows us to set $a_{30}=\varepsilon$, i.e., we derive the subalgebras $\mathfrak{s}_{2.8}^\varepsilon$. For an arbitrary vector field $Q=3a_1\mathcal{P}^1+a_3(\mathcal{P}^3+\varepsilon\mathcal{P}^0)$ in $\mathfrak{s}_{2.8}^\varepsilon$ we obtain $\operatorname{Discr}(Q)=a_3(4a_1^3+a_3^3)$. This means that each of the subalgebras $\mathfrak{s}_{2.8}^\varepsilon$ possesses two singular line of multiplicity one.

The consideration of the case $Q_1=\mathcal{P}^2+\varepsilon\mathcal{P}^0$ is the most difficult. The span $\langle Q_1,Q_2\rangle$ is closed with respect to the Lie bracket if and only if $a_{21}=-3\varepsilon a_{23}$. The coefficient a_{22} is set to zero by linearly combining Q_2 with Q_1. If $a_{23}=0$, then $a_{21}=0$ as well, and we again obtain the subalgebra $\mathfrak{s}_{2.7}$. Hence $a_{23}\neq0$, and we scale Q_2 to set $a_{23}=1$, which leads to $Q_2=\mathcal{P}^3-3\varepsilon\mathcal{P}^1+\alpha\mathcal{P}^0$ with $\alpha:=a_{20}$. We intend to show that depending on the value of α, the subalgebra $\mathfrak{s}:=\langle Q_1,Q_2\rangle$ is $G^{\rm ess}$-equivalent to either the subalgebra $\mathfrak{s}_{2.7}$ or the subalgebra $\mathfrak{s}_{2.8}$.

We compute that

$$\mathrm{Discr}(Q)=12\varepsilon a_2^4+4\alpha a_3a_2^3+24a_3^2a_2^2+12\varepsilon\alpha a_3^3a_2+a_3^4\alpha^2-4a_3^4\varepsilon$$

for a vector field $Q=3a_2Q_1+a_3Q_2$ in $\mathfrak{s}$. If $a_3=0$ and thus $a_2\neq0$, then $\mathrm{Discr}(Q)=12\varepsilon a_2^4\neq0$, i.e., the corresponding line is not singular. Further we can assume without loss of generality that $a_3=1$. Thus, the finding singular lines associated with the subalgebra $\mathfrak{s}$ reduces to solving the quartic equation $\mathrm{Discr}(Q)=0$ with respect to a_2. The substitution $a_2=z-\varepsilon\alpha/12$ and the scaling of the left-hand side reduce this equation to its monic depressed counterpart $z^4+pz^2+qz+r=0$, where

$$p:=-\frac{1}{24}\alpha^2+2\varepsilon,\quad q:=\frac{\varepsilon}{216}\alpha^3+\frac{2}{3}\alpha,\quad r:=-\frac{1}{6912}\alpha^4+\frac{\varepsilon}{72}\alpha^2-\frac{1}{3}.$$

To find the number of (real) roots of this equation, following [1, p. 53] we compute

$$\delta:=256r^3-128p^2r^2+144pq^2r+16p^4r-27q^4-4p^3q^2=-\frac{1}{432}(\alpha^2+16\varepsilon)^4,$$
$$L:=8pr-9q^2-2p^3=-\frac{\varepsilon}{12}(\alpha^2+16\varepsilon)^2.$$

Thus, there are only two cases of the number of roots of the equation $\mathrm{Discr}(Q)=0$.

1. If $\varepsilon=1$ or ($\varepsilon=-1$ and $\alpha\neq\pm4$), then $\delta<0$, and hence the equation $\mathrm{Discr}(Q)=0$ has two roots of multiplicity one, which corresponds to two singular lines of multiplicity one for the subalgebra $\mathfrak{s}$. Therefore, this algebra is $G^{\rm ess}$-equivalent to one of the subalgebras $\mathfrak{s}_{2.8}^{\varepsilon}$.

2. If $\varepsilon=-1$ and $\alpha=\pm4$, then $\delta=L=0$ and $p<0$, and hence equation $\mathrm{Discr}(Q)=0$ has one root of multiplicity one and one root of multiplicity three, which corresponds to one singular line of multiplicity one and one singular line of multiplicity three for the subalgebra $\mathfrak{s}$. Therefore, this algebra is $G^{\rm ess}$-equivalent to one of the subalgebras $\mathfrak{s}_{2.7}$. □

We can also classify one and two-dimensional subalgebras of the entire algebra $\mathfrak{g}$ up to the G-equivalence and show that only subalgebras of $\mathfrak{g}^{\rm ess}$ are essential in the course of classifying Lie reductions of the equation (3.2). Thus, according to the decomposition $G = G^{\rm ess} \ltimes G^{\rm lin}$ (see Section 3.3) an arbitrary transformation Φ from G can be represented in the form $\Phi = \mathcal{F} \circ \mathcal{Z}(f)$, where $\mathcal{F} \in G^{\rm ess}$ and $\mathcal{Z}(f) \in G^{\rm lin}$. To exhaustively describe the adjoint action of G on the algebra $\mathfrak{g}$, in view of the decomposition $\mathfrak{g} = \mathfrak{g}^{\rm ess} \mathbin{\ooalign{$\in$\cr\hidewidth$\supset$\hidewidth}} \mathfrak{g}^{\rm lin}$ it is suffices to supplement the adjoint action of $G^{\rm ess}$ on $\mathfrak{g}^{\rm ess}$ with the adjoint actions of $G^{\rm ess}$ on $\mathfrak{g}^{\rm lin}$ and of $G^{\rm lin}$ on $\mathfrak{g}^{\rm ess}$,

$$\mathcal{Z}(f)_*Q = Q + Q[f]\partial_u, \quad \mathcal{F}_*\mathscr{Z}(f) = \mathscr{Z}(\mathcal{F}_*f),$$

whereas the adjoint action of $G^{\rm lin}$ on $\mathfrak{g}^{\rm lin}$ is trivial. Here Q is an arbitrary vector field from $\mathfrak{g}^{\rm ess}$, $\mathcal{Q}[f]$ denotes the evaluation of the characteristic $Q[u]$ of Q at $u = f$, and $\mathcal{F}$ is an arbitrary transformation from $G^{\rm ess}$.

The classification of subalgebras of $\mathfrak{g}$ is based on the classification of subalgebras of $\mathfrak{g}^{\rm ess}$. This is due to the fact that subalgebras $\mathfrak{s}_1$ and $\mathfrak{s}_2$ of $\mathfrak{g}$ are definitely G-inequivalent if $\pi_{\mathfrak{g}^{\rm ess}}\mathfrak{s}_1$ and $\pi_{\mathfrak{g}^{\rm ess}}\mathfrak{s}_2$ are $G^{\rm ess}$-inequivalent. Here $\pi_{\mathfrak{g}^{\rm ess}}$ denotes the natural projection of $\mathfrak{g}$ onto $\mathfrak{g}^{\rm ess}$ under the decomposition $\mathfrak{g} = \mathfrak{g}^{\rm ess} \dotplus \mathfrak{g}^{\rm lin}$ in the sense of vector spaces. In the course of classifying the subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ of a fixed (finite) dimension, we partition the set of these subalgebras into subsets such that each subset consists of the subalgebras with the same (up to the $G^{\rm ess}$-equivalence) projection $\pi_{\mathfrak{g}^{\rm ess}}\mathfrak{s}$. As a result, each of these subset contains no subalgebra being equivalent to a subalgebra from another subset.

The classification of one and two-dimensional subalgebras of the $\mathfrak{g}$ up to the G-equivalence is given in the following two assertions.

Lemma 35. *A complete list of G-inequivalent one-dimensional subalgebras of $\mathfrak{g}$ consists of the one-dimensional subalgebras of $\mathfrak{g}^{\rm ess}$ listed in Lemma 33 and the subalgebras of the form $\langle \mathscr{Z}(f) \rangle$, where the function f belongs to a fixed complete set of $G^{\rm ess}$-inequivalent nonzero solutions of the equation* (3.2).

Proof. Consider an arbitrary one-dimensional subalgebra $\mathfrak{s}$ of $\mathfrak{g}$. Let Q be its basis element. We can represent Q as $Q = \hat{Q} + \mathscr{Z}(f)$ for some $\hat{Q} \in \mathfrak{g}^{\rm ess}$ and some solution f of the equation (3.2).

If $\hat{Q} \neq 0$, then Lemma 33 implies that, modulo the $G^{\rm ess}$-equivalence, the vector field $\hat{Q}$ can be assumed to be the basis element of one of the subalgebras $\mathfrak{s}_{1.1}, \dots, \mathfrak{s}_{1.8}$ of $\mathfrak{g}^{\rm ess}$. Then we set f identically equal to zero via pushing Q forward with $\mathcal{Z}(h)$, where h is a solution of the equation (3.2) that in addition satisfies the constraint $\hat{Q}[h] + f = 0$.[2]

If $\hat{Q} = 0$, then $Q = \mathcal{Z}(f)$. It suffices to note that the G- and $G^{\rm ess}$-equivalences coincide on $\mathfrak{g}^{\rm lin}$. □

Lemma 36. *A complete list of G-inequivalent two-dimensional subalgebras of $\mathfrak{g}$ consists of*

1. *the two-dimensional subalgebras of $\mathfrak{g}^{\rm ess}$ listed in Lemma 34,*
2. *the subalgebras of the form $\langle\hat{Q}, \mathcal{Z}(f)\rangle$, where $\hat{Q}$ is the basis element of one of the one-dimensional subalgebras $\mathfrak{s}_{1.1}$, …, $\mathfrak{s}_{1.7}$ of $\mathfrak{g}^{\rm ess}$ listed in Lemma 33, and the function f belongs to a fixed complete set of $\mathrm{St}_{G^{\rm ess}}(\langle\hat{Q}\rangle)$-inequivalent nonzero $\langle\hat{Q} + \lambda\mathcal{I}\rangle$-invariant solutions of the equation (3.2) with $\mathrm{St}_{G^{\rm ess}}(\langle\hat{Q}\rangle)$ denoting the stabilizer subgroup of $G^{\rm ess}$ with respect to $\langle\hat{Q}\rangle$ under the action of $G^{\rm ess}$ on $\mathfrak{g}^{\rm ess}$ and with $\lambda \in \{0, 1\}$ if $\hat{Q} \in \{\mathcal{P}^1, \mathcal{P}^0\}$, $\lambda \in \{-1, 0, 1\}$ if $\hat{Q} = \mathcal{P}^t$, $\lambda \geqslant 0$ if $\hat{Q} = \mathcal{P}^2 + \varepsilon\mathcal{P}^0$ and $\lambda \in \mathbb{R}$ otherwise,*
3. *the subalgebras of the form $\langle\mathcal{I}, \mathcal{Z}(f)\rangle$, where the function f belongs to a fixed complete set of $G^{\rm ess}$-inequivalent nonzero solutions of the equation (3.2), and*
4. *the subalgebras of the form $\langle\mathcal{Z}(f^1), \mathcal{Z}(f^2)\rangle$, where the function pair (f^1, f^2) belongs to a fixed complete set of $G^{\rm ess}$-inequivalent (up to linearly recombining components) pairs of linearly independent solutions of the equation (3.2).*

Proof. Consider an arbitrary two-dimensional subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, and let Q_1 and Q_2 be its basis elements, $\mathfrak{s} = \langle Q_1, Q_2\rangle$ with $[Q_1, Q_2] \in \mathfrak{s}$. Due to the decomposition $\mathfrak{g} = \mathfrak{g}^{\rm ess} \Subset \mathfrak{g}^{\rm lin}$, each Q_i, $i = 1, 2$, can be represented in the form $Q_i = \hat{Q}_i + \mathcal{Z}(f^i)$, where $\hat{Q}_i \in \mathfrak{g}^{\rm ess}$ and the function f^i is a solution of the equation (3.2). As the principal G-invariant value in the course of classifying two-dimensional subalgebras of $\mathfrak{g}$, we choose the dimension $d := \dim \pi_{\mathfrak{g}^{\rm ess}}\mathfrak{s}$ of the projection of $\mathfrak{s}$ onto $\mathfrak{g}^{\rm ess}$.

[2]For $\hat{Q} = \mathcal{I}$, the constraint $\hat{Q}[h] + f = 0$ just means $h = -f$. For the other values of $\hat{Q}$ and locally analytical solutions f of the equation (3.2), the existence of locally analytical solutions of the system $h_t + xh_y = h_{xx}$, $\hat{Q}[h] + f = 0$ with respect to h follows from Riquier's theorem and the fact that $\hat{Q}$ is a Lie symmetry vector field of (3.2). We suppose this existence in the smooth case as well. In fact, the proof of this existence reduces to the existence of a solution of linear inhomogeneous partial differential equation in two independent variables, whose homogeneous counterpart coincides with a reduction of the equation (3.2) with respect to $\langle\hat{Q}\rangle$.

***d* = 2.** Up to the G^{ess}-equivalence, the pair $(\hat{Q}_1, \hat{Q}_2)$ can be assumed to be the chosen basis of one of the subalgebras $\mathfrak{s}^{\mu}_{2.1}$, …, $\mathfrak{s}_{2.15}$ of $\mathfrak{g}^{\mathrm{ess}}$ listed in Lemma 34.

For the subalgebras $\mathfrak{s}_{2.9}$, …, $\mathfrak{s}_{2.15}$, we have $Q_2 = \mathcal{I}$. Pushing $\mathfrak{s}$ forward with $\mathcal{Z}(h)$, where $h = -f^2$, we can set $f^2 = 0$. Then the commutation relation $[Q_1, Q_2] = 0$ implies $f^1 = 0$.

For the subalgebras $\mathfrak{s}^{\mu}_{2.1}$, …, $\mathfrak{s}^{\varepsilon}_{2.8}$, let κ_1 and κ_2 be the constants such that $[\hat{Q}_1, \hat{Q}_2] = \kappa_1\hat{Q}_1 + \kappa_2\hat{Q}_2$. Then the functions f^1 and f^2 satisfy the constraint $\hat{Q}_1[f^2] - \hat{Q}_2[f^1] = \kappa_1 f^1 + \kappa_2 f^2$. Hence we can set them to be identically zero using the push-forward $\mathcal{Z}(h)_*$, where h is a solution of the overdetermined system $h_t + xh_y = h_{xx}$, $\hat{Q}_1[h] + f^1 = 0$ and $\hat{Q}_2[h] + f^2 = 0$ with respect to h.[3]

In total, this results in the first family of subalgebras from the lemma's statement.

***d* = 1.** Without loss of generality, we can assume $\hat{Q}_2 = 0$ up to linearly recombining the basis elements Q_1 and Q_2 and thus $f^2 \neq 0$, whereas modulo the G^{ess}-equivalence, the vector field $\hat{Q}_1$ can be assumed to be the basis element of one of the subalgebras $\mathfrak{s}_{1.1}$, …, $\mathfrak{s}_{1.8}$ of $\mathfrak{g}^{\mathrm{ess}}$ that are listed in Lemma 33. Then Lemma 33 also implies that $f^1 = 0$ up to the G^{ess}-equivalence.

For the subalgebras $\mathfrak{s}_{1.1}$, …, $\mathfrak{s}_{1.7}$, we have $\hat{Q}_1 \notin \langle\mathcal{I}\rangle$. Following the proof of Lemma 35, we can set the function f^1 to be identically zero. Then the closedness of the subalgebra $\mathfrak{s}$ with respect to the Lie bracket implies the constraint $\hat{Q}_1[f^2] = \lambda f^2$. In other words, the function f^2 is a $\langle\hat{Q}_1 + \lambda\mathcal{I}\rangle$-invariant solution of the equation (3.2). Transformations from the group $\mathrm{St}_{G^{\mathrm{ess}}}(\langle\hat{Q}_1\rangle)$ allows us to set the restrictions on λ depending on the value of $\hat{Q}_1$, which are presented in item 2 of the lemma's statement. The transformations from $\mathrm{St}_{G^{\mathrm{ess}}}(\langle\hat{Q}_1\rangle)$ that preserve λ also induce an equivalence relation on the corresponding set run by f^2.

Since $\hat{Q}_1 = \mathcal{I}$ for the subalgebra $\mathfrak{s}_{1.8}$, in the way described in the proof of Lemma 35 we can set the function f^1 to be identically zero. Then the only commutation relation of the algebra $\mathfrak{s}$ is $[Q_1, Q_2] = Q_2$, which implies no constraint for the function f^2, and $\mathrm{St}_{G^{\mathrm{ess}}}(\langle\mathcal{I}\rangle) = G^{\mathrm{ess}}$. Hence in this case the parameter function f^2 should run through a fixed complete set of G^{ess}-inequivalent nonzero solutions of the equation (3.2).

[3]Similarly to the proof of Lemma 35, for locally analytical solutions f^1 and f^2 of the equation (3.2) that satisfy the above constraint, the existence of locally analytical solutions of the last system follows from Riquier's theorem and the fact that $\hat{Q}_1$ and $\hat{Q}_2$ are Lie symmetry vector fields of (3.2), and we assume this existence in the smooth case as well.

Thus, we obtained the second and third families of subalgebras listed in the lemma's statement.

$\boldsymbol{d=0}$**.** This case is obvious since then $\hat{Q}_1=\hat{Q}_2=0$. In other words, the subalgebra $\langle Q_1,Q_2\rangle$ is of the form $\langle \mathcal{Z}(f^1),\mathcal{Z}(f^2)\rangle$, where f^1 and f^2 are linearly independent solutions of the equation (3.2), which should be chosen modulo the $G^{\rm ess}$-equivalence and linearly recombining. □

3.5 Lie reductions of codimension one

The classification of Lie reductions of the equation (3.2) to partial differential equations with two independent variables is based on the classification of one-dimensional subalgebras of the algebra $\mathfrak{g}^{\rm ess}$, which is given in Lemma 33. In Table 3.1, for each of the subalgebras listed therein, we present an associated ansatz for u and the corresponding reduced equation. Throughout this section, the subscripts 1 and 2 of functions depending on (z_1,z_2) denote derivatives with respect to z_1 and z_2, respectively. In this and the next sections, we mark Lie reductions and all related objects with complex labels $d.i^*$, where d, i and $*$ are the dimension of the corresponding subalgebra, its number in the list of d-dimensional subalgebras of the algebra $\mathfrak{g}^{\rm ess}$, given in Lemmas 33 or 34 for $d=1$ or $d=2$, and the list of subalgebra parameters for a subalgebra family, respectively. We omit the superscript in the label when it is not essential.

Each of the reduced equations $1.i$ presented in Table 3.1, $i\in\{1,2^\delta,3^\nu,4^\mu,5^\varepsilon,6,7\}$, is a linear homogenous partial differential equation in two independent variables. Hence its maximal Lie invariance algebra $\mathfrak{g}_{1.i}$ contains the infinity-dimensional abelian ideal $\{h(z_1,z_2)\partial_w\}$, which is associated with the linear superposition of solutions of reduced equation $1.i$ and thus assumed to be a trivial part of $\mathfrak{g}_{1.i}$. Here the parameter function $h=h(z_1,z_2)$ runs through the solution set of reduced equation $1.i$. Moreover, the algebra $\mathfrak{g}_{1.i}$ is the semi-direct sum of the above ideal and a finite-dimensional subalgebra $\mathfrak{a}_i$ called the essential Lie invariance algebra of reduced equation $1.i$, cf. Section 3.2. The algebras $\mathfrak{a}_i$ are the following:

$$\mathfrak{a}_1=\mathfrak{a}_3=\mathfrak{a}_4=\langle w\partial_w\rangle,$$

Table 3.1: G-inequivalent Lie reductions of codimension one

no.	u	z_1	z_2	Reduced equation
1.1	$\mathrm{e}^{\frac{3}{10}t(t^4+5tx-10y)}w$	$y-\frac14t^4$	$x-t^3$	$z_2w_1=w_{22}-3z_1w$
1.2^δ	$\mathrm{e}^{\delta t}w$	y	x	$z_2w_1=w_{22}-\delta w$
1.3	$\lvert t\rvert^{\frac12\nu-1}w$	$\lvert t\rvert^{-\frac32}y$	$\lvert t\rvert^{-\frac12}x$	$\chi(z_1,z_2)w_1=2w_{22}+z_2\varepsilon'w_2-(\nu-2)\varepsilon'w$
1.4	$\mathrm{e}^{-\psi(t,x,y,\mu)}w$	$\dfrac{y}{(t^2+1)^{\frac32}}$	$\dfrac{(t^2+1)x-3ty}{(t^2+1)^{\frac32}}$	$z_2w_1=3z_1w_2+w_{22}+(\mu+z_2^2)w$
1.5	$\lvert t\rvert^{-\frac12}\mathrm{e}^{-\frac{x^2}{4t}}w$	$\frac13t^3+2\varepsilon t-t^{-1}$	$2y-(t+\varepsilon t^{-1})x$	$w_1=w_{22}$
1.6	w	$\frac13t^3$	$y-tx$	$w_1=w_{22}$
1.7	w	t	x	$w_1=w_{22}$

Here $\varepsilon'=\operatorname{sgn}t$, $\varepsilon\in\{-1,1\}$, $\delta\in\{-1,0,1\}$, $w=w(z_1,z_2)$ is the new unknown function of the new independent variables (z_1,z_2), and

$$\psi(t,x,y,\mu)=\frac{3t^3y^2+t(2x(t^2+1)-3ty)^2}{4(t^2+1)^3}+\mu\arctan t,\quad \chi(z_1,z_2)=2z_2-3\varepsilon'z_1.$$

$$\begin{aligned}
&\mathfrak a_2^0=\langle\partial_{z_1},\,3z_1\partial_{z_1}+z_2\partial_{z_2},\,9z_1^2\partial_{z_1}+6z_1z_2\partial_{z_2}-(6z_1+z_2^3)w\partial_w,\,w\partial_w\rangle,\\
&\mathfrak a_2^\delta=\langle\partial_{z_1},\,w\partial_w\rangle\quad\text{if}\quad\delta\neq0,\\
&\mathfrak a_5=\mathfrak a_6=\mathfrak a_7=\langle\partial_{z_1},\,\partial_{z_2},\,2z_1\partial_{z_1}+z_2\partial_{z_2},\,2z_1\partial_{z_2}-z_2w\partial_w,\\
&\qquad\qquad 4z_1^2\partial_{z_1}+4z_1z_2\partial_{z_2}-(2z_1+z_2^2)w\partial_w,\,w\partial_w\rangle.
\end{aligned}$$

The equation (3.2) admits hidden symmetries with respect to a Lie reduction if the corresponding reduced equation possesses Lie symmetries that are not induced by Lie symmetries of the original equation (3.2). To check which Lie symmetries of reduced equation $1.i$ are induced by Lie symmetries of (3.2), we compute the normalizer of the subalgebra $\mathfrak s_{1.i}$ in the algebra $\mathfrak g^{\rm ess}$:

$$\begin{aligned}
&\mathrm N_{\mathfrak g^{\rm ess}}(\mathfrak s_{1.1})=\langle\mathcal P^t+\mathcal P^3,\mathcal I\rangle,\\
&\mathrm N_{\mathfrak g^{\rm ess}}(\mathfrak s_{1.2}^0)=\langle\mathcal P^t,\mathcal D,\mathcal P^0,\mathcal I\rangle,\quad \mathrm N_{\mathfrak g^{\rm ess}}(\mathfrak s_{1.2}^\delta)=\langle\mathcal P^t,\mathcal P^0,\mathcal I\rangle\ \text{ if }\ \delta\neq0,\\
&\mathrm N_{\mathfrak g^{\rm ess}}(\mathfrak s_{1.3}^\nu)=\langle\mathcal D,\mathcal I\rangle,\quad \mathrm N_{\mathfrak g^{\rm ess}}(\mathfrak s_{1.4}^\mu)=\langle\mathcal P^t+\mathcal K,\mathcal I\rangle,\\
&\mathrm N_{\mathfrak g^{\rm ess}}(\mathfrak s_{1.5}^\varepsilon)=\langle\mathcal P^2,\mathcal P^0,\mathcal P^3-3\varepsilon\mathcal P^1,\mathcal I\rangle,\\
&\mathrm N_{\mathfrak g^{\rm ess}}(\mathfrak s_{1.6})=\langle\mathcal D,\mathcal P^3,\mathcal P^1,\mathcal P^0,\mathcal I\rangle,\quad \mathrm N_{\mathfrak g^{\rm ess}}(\mathfrak s_{1.7})=\langle\mathcal P^t,\mathcal D,\mathcal P^2,\mathcal P^1,\mathcal P^0,\mathcal I\rangle.
\end{aligned}$$

The algebra of induced Lie symmetries of reduced equation $1.i$ is isomorphic to the quotient algebra $\mathrm{N}_{\mathfrak{g}^{\rm ess}}(\mathfrak{s}_{1.i})/\mathfrak{s}_{1.i}$. Thus, all Lie symmetries of the reduced equation $1.i$ are induced by Lie symmetries of the original equation (3.2) if and only if $\dim\mathfrak{a}_i = \dim\mathrm{N}_{\mathfrak{g}^{\rm ess}}(\mathfrak{s}_{1.i})-1$. Comparing the dimensions of $\mathrm{N}_{\mathfrak{g}^{\rm ess}}(\mathfrak{s}_{1.i})$ and $\mathfrak{a}_i$, we conclude that all Lie symmetries of reduced equations 1.1, 1.3 and 1.4 are induced by Lie symmetries of (3.2). The subalgebras of induced symmetries in the algebras $\mathfrak{a}_2^\delta$ and $\mathfrak{a}_5$–$\mathfrak{a}_7$ are

$$
\begin{aligned}
&\tilde{\mathfrak{a}}_2^0=\langle\partial_{z_1},\,3z_1\partial_{z_1}+z_2\partial_{z_2},\,w\partial_w\rangle,\quad \tilde{\mathfrak{a}}_2^\delta=\langle\partial_{z_1},\,w\partial_w\rangle \ \text{ if }\ \delta\neq0,\\
&\tilde{\mathfrak{a}}_5=\langle\partial_{z_2},\,2z_1\partial_{z_2}-z_2w\partial_w,\,w\partial_w\rangle,\\
&\tilde{\mathfrak{a}}_6=\langle\partial_{z_2},\,2z_1\partial_{z_1}+z_2\partial_{z_2},\,2z_1\partial_{z_2}-z_2w\partial_w,\,w\partial_w\rangle,\\
&\tilde{\mathfrak{a}}_7=\langle\partial_{z_1},\,\partial_{z_2},\,2z_1\partial_{z_1}+z_2\partial_{z_2},\,2z_1\partial_{z_2}-z_2w\partial_w,\,w\partial_w\rangle.
\end{aligned}
$$

For each $i\in\{2_{\delta=0},5,6,7\}$, the elements of the complement $\mathfrak{a}_i/\tilde{\mathfrak{a}}_i$ are nontrivial hidden symmetries of the equation (3.2) that are associated with reduction $1.i$.

We discuss reduced equations 1.1–1.7 and present solutions of (3.2) that can be found using the known solutions of the reduced equations.

Using point transformations, we can further map reduced equations 1.1–1.4 to (1+1)-dimensional linear heat equations with potentials, which are of the general form $w_1=w_{22}+Vw$, where V is an arbitrary function of (z_1,z_2). These transformations and mapped equations are

1.1. $\tilde z_1=\frac94z_1,\quad \tilde z_2=|z_2|^{\frac32},\quad \tilde w=|z_2|^{\frac14}w\colon\quad \tilde w_1=\tilde w_{22}-\frac19\big(\frac{16}{3}\tilde z_1\tilde z_2^{-\frac23}-\frac54\tilde z_2^{-2}\big)\tilde w;$

1.2^δ. $\tilde z_1=\frac94z_1,\quad \tilde z_2=|z_2|^{\frac32},\quad \tilde w=|z_2|^{\frac14}w\colon\quad \tilde w_1=\tilde w_{22}-\frac19\big(4\delta\tilde z_2^{-\frac23}-\frac54\tilde z_2^{-2}\big)\tilde w;$

1.3. $\tilde z_1=\frac94\tilde\varepsilon z_1,\quad \tilde z_2=|z_2-\frac32\varepsilon'z_1|^{\frac32},\quad \tilde w=|2z_2-3\varepsilon'z_1|^{\frac14}\mathrm{e}^{\frac18z_2(4\varepsilon'z_2-9z_1)}w$ with $\tilde\varepsilon:=\operatorname{sgn}(z_2-\frac32\varepsilon'z_1)$:
$\tilde w_1=\tilde w_{22}+\frac{1}{144}\tilde z_2^{-\frac23}\big(16\tilde z_1^2-40(\tilde z_2^{\frac23}+\frac23\varepsilon'\tilde\varepsilon\tilde z_1)^2+20\tilde z_2^{-\frac43}-32\varepsilon'\nu\big)\tilde w;$

1.4. $\tilde z_1=\frac94\tilde\varepsilon z_1,\quad \tilde z_2=|z_2|^{\frac32},\quad \tilde w=|z_2|^{\frac14}\mathrm{e}^{\frac32z_1z_2}w$ with $\tilde\varepsilon:=\operatorname{sgn}z_2$:
$\tilde w_1=\tilde w_{22}+\frac19\big(\frac54\tilde z_2^{-2}+10\tilde z_2^{\frac23}+4(\mu-\frac49\tilde z_1^2)\tilde z_2^{-\frac23}\big)\tilde w.$

The essential Lie invariance algebras of reduced equations 1.1, 1.3 and 1.4 are trivial, and thus the further consideration of these equations with the classical framework gives no results.

Mapped reduced equation 1.2^0 coincides with the linear heat equation with potential $V=\mu z_2^{-2}$, where $\mu=\frac{5}{36}$. Hence we construct the following family of solutions of the equation (3.2):

- $u=|x|^{\frac14}\theta^\mu(\frac94 y,|x|^{\frac32})$ with $\mu=\frac{5}{36}$,

where $\theta^\mu=\theta^\mu(z_1,z_2)$ is an arbitrary solution of the equation $\theta^\mu_1=\theta^\mu_{22}+\mu z_2^{-2}\theta^\mu$. A complete collection of inequivalent Lie invariant solutions of such equations with $\mu\neq0$ is presented in Section 2.5.

The essential Lie invariance algebra of reduced equation 1.2^δ with $\delta\neq0$ is induced by the subalgebra $\langle\mathcal P^0,\mathcal I\rangle$ of $\mathfrak g^{\rm ess}$. The subalgebras of $\mathfrak a_2^\delta$ that are appropriate for Lie reduction of reduced equation 1.2^δ are exhausted by the subalgebras of the form $\langle\partial_{z_1}+\nu w\partial_w\rangle$, which are respectively induced by the subalgebras $\langle\mathcal P^0+\nu\mathcal I\rangle$ of $\mathfrak g^{\rm ess}$. This is why the family of solutions of reduced equation 1.2^δ that are invariant with respect to $\langle\partial_{z_1}+\nu w\partial_w\rangle$ gives a family of solutions of the equation (3.2) that is contained, up to the $G^{\rm ess}$-equivalence, in the family of $\langle\mathcal P^0\rangle$-invariant solutions, see reduction 1.7.

Each of reduced equations 1.5–1.7 themselves coincides with the classical (1+1)-dimensional linear heat equation, which leads to the following families of solutions of the equation (3.2):

- $u=|t|^{-\frac12}{\rm e}^{-\frac{x^2}{4t}}\theta(z_1,z_2)$, where $z_1=\frac13t^3+2\varepsilon t-t^{-1}$, $z_2=2y-(t+\varepsilon t^{-1})x$,
- $u=\theta(z_1,z_2)$, where $z_1=\frac13t^3$, $z_2=y-tx$,
- $u=\theta(z_1,z_2)$, where $z_1=t$, $z_2=x$.

Here $\theta=\theta(z_1,z_2)$ is an arbitrary solution of the (1+1)-dimensional linear heat equation, $\theta_1=\theta_{22}$. An enhanced complete collection of inequivalent Lie invariant solutions of this equation was presented in [63, Section A], following Examples 3.3 and 3.17 in [41]. These solutions exhaust, up to their linear superposition, the set of known exact solutions of this equation that are expressed in closed form in terms of elementary and special functions.

3.6 Lie reductions of codimension two

Based on the list of G^{ess}-inequivalent two-dimensional subalgebras from Lemma 34, we carry out the exhaustive classification of codimension-two Lie reductions of the equation (3.2). In Table 3.2, for each of these subalgebras, we present an associated ansatz for u and the corresponding reduced ordinary differential equation.

Table 3.2: G-inequivalent Lie reductions of codimension two

no.	u	ω	Reduced equation
2.1	$\lvert y\rvert^{\mu-2}\varphi$	$y^{-1}x^3$	$9\omega\varphi_{\omega\omega}+(\omega+6)\varphi_\omega=(\mu-2)\varphi$
2.2^δ	$\mathrm{e}^{\delta t}\varphi$	x	$\varphi_{\omega\omega}=\delta\varphi$
2.3	$\mathrm{e}^{y}\varphi$	x	$\varphi_{\omega\omega}=\omega\varphi$
2.4	$\lvert t\rvert^{\frac12\mu-1}\varphi$	$\lvert t\rvert^{-\frac32}(y-tx)$	$2\varphi_{\omega\omega}=\varepsilon'(\mu-2)\varphi-3\varepsilon'\omega\varphi_\omega$
2.5	$\lvert t\rvert^{\frac12\mu-1}\varphi$	$\lvert t\rvert^{-\frac12}x$	$2\varphi_{\omega\omega}=\varepsilon'(\mu-2)\varphi-\varepsilon'\omega\varphi_\omega$
2.6	φ	t	$\varphi_\omega=0$
2.7	$\mathrm{e}^{-\frac{x^2}{4t}}\varphi$	t	$2\omega\varphi_\omega+\varphi=0$
2.8	$\mathrm{e}^{-\frac32\frac{(y-tx)^2}{2t^3-1}}\varphi$	t	$(2\omega^3-1)\varphi_\omega+3\omega^2\varphi=0$

Here $\varepsilon'=\operatorname{sgn}t$, $\varphi=\varphi(\omega)$ is the new unknown function of the invariant independent variable ω.

We find (nonzero) exact solutions of the constructed reduced equations and present the associated exact solutions of the equation (3.2). Hereafter, C_1 and C_2 are arbitrary constants.

2.1. Only this reduction results in new exact solutions of (3.2). The substitution $\varphi=\lvert\omega\rvert^{\frac13}\mathrm{e}^{-\frac{\omega}{9}}\tilde\varphi(\omega)$ maps reduced equation 2.1 to well-known Kummer's equation

$$9\tilde\varphi_{\omega\omega}+(12-\omega)\tilde\varphi_\omega=(\mu-1)\tilde\varphi,$$

whose general solution is $\tilde\varphi=C_1M(\mu-1,\frac43,\frac19\omega)+C_2U(\mu-1,\frac43,\frac19\omega)$, where $M(a,b,z)$ and $U(a,b,z)$ are the standard solutions of Kummer's equation $z\varphi_{zz}+(b-z)\varphi_z-a\varphi=0$. After taking $\kappa:=\mu-1$ instead of μ as an arbitrary constant parameter, the corresponding family of particular solutions of the equation (3.2) can be represented in the form

- $u=x\lvert y\rvert^{\kappa-\frac43}\mathrm{e}^{-\tilde\omega}\Big(C_1M\big(\kappa,\frac43,\tilde\omega\big)+C_2U\big(\kappa,\frac43,\tilde\omega\big)\Big)$ with $\tilde\omega:=\frac19y^{-1}x^3$.

The rest of reductions only lead to solutions of the equation (3.2) each of which is $G^{\rm ess}$-equivalent to a solution of form 1.6 or 1.7 given in the previous Section 3.5. Indeed, each of the corresponding subalgebras $\mathfrak{s}_{2.2}$–$\mathfrak{s}_{2.8}$ contains, up to the $G^{\rm ess}$-equivalence for the subalgebra $\mathfrak{s}_{2.3}$, at least one of the vector fields $\mathcal{P}_0$ and $\mathcal{P}_1$. Nevertheless, we discuss these reductions below in order to explicitly present solutions of (3.2) that are invariant with respect to two-dimensional algebras of Lie-symmetry vector fields of (3.2).

2.2$^\delta$. The reduced equations 2.2^δ trivially integrates to

$$\varphi=\begin{cases} C_1{\rm e}^{\omega}+C_2{\rm e}^{-\omega} & \text{if}\ \ \delta=1,\\ C_1\omega+C_2 & \text{if}\ \ \delta=0,\\ C_1\sin\omega+C_2\cos\omega & \text{if}\ \ \delta=-1.\end{cases}$$

2.3. Reduced equation 2.3 is the Airy equation, whose general solution is $\varphi = C_1\,{\rm Ai}(\omega)+C_2\,{\rm Bi}(\omega)$.

2.4, 2.5. Substituting $\varphi=\omega{\rm e}^{-\frac34\omega^2}\tilde\varphi(\omega)$, we obtain the equations

$$2\omega\tilde\varphi_{\omega\omega}+(4-3\varepsilon'\omega^2)\tilde\varphi_\omega=\varepsilon'(\mu+4)\omega\tilde\varphi,$$
$$2\omega\tilde\varphi_{\omega\omega}+(4-\varepsilon'\omega^2)\tilde\varphi_\omega=\varepsilon'\mu\omega\tilde\varphi,$$

whose general solutions are respectively

$$\tilde\varphi=C_1M\big(\tfrac{\mu}{6}+\tfrac23,\tfrac32,\tfrac34\epsilon'\omega^2\big)+C_2U\big(\tfrac{\mu}{6}+\tfrac23,\tfrac32,\tfrac34\epsilon'\omega^2\big),$$
$$\tilde\varphi=C_1M\big(\tfrac{\mu}{2},\tfrac32,\tfrac14\epsilon'\omega^2\big)+C_2U\big(\tfrac{\mu}{2},\tfrac32,\tfrac14\epsilon'\omega^2\big).$$

2.6–2.8. Here the reduced equations are ordinary first-order differential equations, which are trivially integrated to

$$2.6.\ \varphi=C_0,\quad 2.7.\ \varphi=\frac{C_0}{\sqrt{\omega}}\,,\quad 2.8.\ \varphi=\frac{C_0}{\sqrt{2\omega^3-1}}\,.$$

Substituting the above general solutions of reduced equations 2.2–2.8 into the corresponding ansatzes, we construct the following families of particular solutions of the equation (3.2), which are contained (as mentioned earlier, up to the G^{ess}-equivalence for the fourth family) in the solution families obtained in the previous Section 3.5 using Lie reductions 1.6 and 1.7:

- $u = C_1\mathrm{e}^{t+x} + C_2\mathrm{e}^{t-x}$,
- $u = C_1x + C_2$,
- $u = C_1\mathrm{e}^{-t}\sin x + C_2\mathrm{e}^{-t}\cos x$,
- $u = C_1\mathrm{e}^{y}\,\mathrm{Ai}(x) + C_2\mathrm{e}^{y}\,\mathrm{Bi}(x)$,
- $u = |t|^{\frac{\mu-5}{2}}(y-tx)\mathrm{e}^{-\tilde\omega}\Big(C_1M\big(\frac{\mu}{6}+\frac{2}{3},\frac{3}{2},\epsilon'\tilde\omega\big) + C_2U\big(\frac{\mu}{6}+\frac{2}{3},\frac{3}{2},\epsilon'\tilde\omega\big)\Big)$ with $\tilde\omega := \frac{3}{4}|t|^{-3}(y-tx)^2$,
- $u = |t|^{\frac{\mu-3}{2}}x\mathrm{e}^{-\tilde\omega}\Big(C_1M\big(\frac{\mu}{2},\frac{3}{2},\epsilon'\tilde\omega\big) + C_2U\big(\frac{\mu}{2},\frac{3}{2},\epsilon'\tilde\omega\big)\Big)$ with $\tilde\omega := \frac{1}{4}|t|^{-1}x^2$,
- $u = C_0$,
- $u = \dfrac{C_0}{\sqrt{t}}\mathrm{e}^{-\frac{x^2}{4t}}$,
- $u = \dfrac{C_0}{\sqrt{2t^3-1}}\mathrm{e}^{-\frac{3}{2}\frac{(y-tx)^2}{2t^3-1}}$.

3.7 On Kramers equations

An important subclass of the class $\bar{\mathcal F}$ of (1+2)-dimensional ultraparabolic Fokker–Planck equations, which are of the general form (3.1), is the class $\mathcal K$ of Kramers equations (more commonly called the Klein–Kramers equations)

$$u_t + xu_y = F(y)u_x + \gamma(xu+u_x)_x, \tag{3.11}$$

which describe the evolution of the probability density function $u(t,x,y)$ of a Brownian particle in the phase space (y,x) in one spatial dimension. Here the variables t, y and x play the role of the time, the position and the momentum, respectively, F is an arbitrary smooth function of y that is the derivative of the external potential with respect to y, and γ is an arbitrary nonzero constant, which is related to the friction coefficient. The subclass $\mathcal K$ is singled out from the class $\bar{\mathcal F}$ by the auxiliary differential constraints $C=0$, $A^0=A^1_x=A^2$, $A^0_t=A^0_x=A^0_y=0$ and $A^1_t=0$.

A preliminary group classification of the class $\mathcal K$ was presented in [57]. In particular, it was proved that the essential Lie invariance algebra of an equation from the class $\mathcal K$ is eight-dimensional if and only if $F(y)=ky$, where $k=-\frac{3}{4}\gamma^2$ or $k=\frac{3}{16}\gamma^2$.

Any equation from the class $\bar{\mathcal F}$ with eight-dimensional essential Lie symmetry algebra is $\mathcal G_{\bar{\mathcal F}}$-equivalent to the remarkable Fokker–Planck equation (3.2). (We will present the proof of this fact as well as the complete group classification of the class $\bar{\mathcal{F}}$ in a future paper.) Therefore, there exist point transformations that map the equations

$$u_t + xu_y = \gamma u_{xx} + \gamma(x - \tfrac{3}{4}\gamma y)u_x + \gamma u, \tag{3.12}$$

$$u_t + xu_y = \gamma u_{xx} + \gamma(x + \tfrac{3}{16}\gamma y)u_x + \gamma u \tag{3.13}$$

to the equation (3.2).

Bases of the essential Lie invariance algebras $\mathfrak{g}^{\rm ess}_{(3.12)}$ and $\mathfrak{g}^{\rm ess}_{(3.13)}$ of the corresponding equations (3.12) and (3.13) consist of the vector fields

$$\begin{aligned}
&\hat{\mathcal{P}}^t = \mathrm{e}^{-\gamma t}\left(\tfrac{1}{\gamma}\partial_t - \tfrac{3}{2}y\partial_y + \tfrac{1}{2}(3\gamma y - x)\partial_x - (\tfrac{1}{4}x^2 + \tfrac{3}{4}\gamma xy + \tfrac{9}{16}\gamma^2y^2 - \tfrac{3}{2})u\partial_u\right),\\
&\hat{\mathcal{K}} = \mathrm{e}^{\gamma t}\left(\tfrac{1}{\gamma}\partial_t + \tfrac{3}{2}y\partial_y + \tfrac{1}{2}(3\gamma y + x)\partial_x - (\tfrac{3}{4}x^2 + \tfrac{3}{4}\gamma xy - \tfrac{9}{16}\gamma^2y^2 + \tfrac{1}{2})u\partial_u\right),\\
&\hat{\mathcal{D}} = \tfrac{2}{\gamma}\partial_t + 3u\partial_u,\\
&\hat{\mathcal{P}}^3 = \mathrm{e}^{\frac{3}{2}\gamma t}\left(\tfrac{1}{\gamma}\partial_y + \tfrac{3}{2}\partial_x + \tfrac{3}{4}(\gamma y - 2x)u\partial_u\right), \quad \hat{\mathcal{P}}^2 = \mathrm{e}^{\frac{1}{2}\gamma t}\left(\tfrac{1}{\gamma}\partial_y + \tfrac{1}{2}\partial_x\right),\\
&\hat{\mathcal{P}}^1 = \mathrm{e}^{-\frac{1}{2}\gamma t}\left(\tfrac{1}{\gamma}\partial_y - \tfrac{1}{2}\partial_x + \tfrac{1}{4}(3\gamma y + 2x)u\partial_u\right), \quad \hat{\mathcal{P}}^0 = \mathrm{e}^{-\frac{3}{2}\gamma t}\left(\tfrac{1}{2\gamma}\partial_y - \tfrac{3}{4}\partial_x\right),\\
&\hat{\mathcal{I}} = u\partial_u
\end{aligned}$$

and

$$\begin{aligned}
&\check{\mathcal{P}}^t = \mathrm{e}^{-\frac{1}{2}\gamma t}\left(\tfrac{2}{\gamma}\partial_t - \tfrac{3}{2}y\partial_y + \tfrac{1}{4}(3\gamma y - 2x)\partial_x + u\partial_u\right), \quad \check{\mathcal{D}} = \tfrac{4}{\gamma}\partial_t + 4u\partial_u,\\
&\check{\mathcal{K}} = \mathrm{e}^{\frac{1}{2}\gamma t}\left(\tfrac{2}{\gamma}\partial_t + \tfrac{3}{2}\gamma y\partial_y + \tfrac{1}{4}(3\gamma y + 2x)\partial_x - \tfrac{1}{2}(x^2 + \tfrac{3}{2}\gamma xy + \tfrac{9}{16}\gamma^2y^2)u\partial_u\right),\\
&\check{\mathcal{P}}^3 = \tfrac{1}{\sqrt{2}}\mathrm{e}^{\frac{3}{4}\gamma t}\left(\tfrac{4}{\gamma}\partial_y + 3\partial_x - \tfrac{3}{4}(4x + \gamma y)u\partial_u\right),\\
&\check{\mathcal{P}}^2 = \tfrac{1}{\sqrt{2}}\mathrm{e}^{\frac{1}{4}\gamma t}\left(\tfrac{4}{\gamma}\partial_y + \partial_x - (4x + 3\gamma y)u\partial_u\right),\\
&\check{\mathcal{P}}^1 = \tfrac{1}{\sqrt{2}}\mathrm{e}^{-\frac{1}{4}\gamma t}\left(\tfrac{4}{\gamma}\partial_y - \partial_x\right), \quad \check{\mathcal{P}}^0 = \tfrac{1}{\sqrt{2}}\mathrm{e}^{-\frac{3}{4}\gamma t}\left(\tfrac{4}{\gamma}\partial_y - 3\partial_x\right), \quad \check{\mathcal{I}} = u\partial_u,
\end{aligned}$$

respectively. We choose the basis elements of the algebra $\mathfrak{g}^{\rm ess}_{(3.12)}$ (resp. $\mathfrak{g}^{\rm ess}_{(3.13)}$) in such a way that the commutation relations between them coincide with those between the basis elements of the algebra $\mathfrak{g}^{\rm ess}$ presented in Section 3.2. Thus, the fact that the algebras $\mathfrak{g}^{\rm ess}$, $\mathfrak{g}^{\rm ess}_{(3.12)}$ and $\mathfrak{g}^{\rm ess}_{(3.13)}$ are isomorphic becomes obvious.

Point transformations $\Phi_{(3.12)}$ and $\Phi_{(3.13)}$ that respectively map the equations (3.12) and (3.13) to the remarkable Fokker–Planck equation (3.2) are constructed using the conditions that $\Phi_{(3.12)*}\hat{\mathcal{V}}=\mathcal{V}$ and $\Phi_{(3.13)*}\check{\mathcal{V}}=\mathcal{V}$ for $\mathcal{V}\in\{\mathcal{P}^t,\mathcal{D},\mathcal{K},\mathcal{P}^3,\mathcal{P}^2,\mathcal{P}^1,\mathcal{P}^0,\mathcal{I}\}$, Thus, the pushforwards $\Phi_{(3.12)*}$ and $\Phi_{(3.13)*}$ establish the isomorphisms of the algebras $\mathfrak{g}^{\rm ess}_{(3.12)}$ and $\mathfrak{g}^{\rm ess}_{(3.13)}$ to the algebra $\mathfrak{g}^{\rm ess}$, $\Phi_{(3.12)*}\mathfrak{g}^{\rm ess}_{(3.12)}=\mathfrak{g}^{\rm ess}$ and $\Phi_{(3.13)*}\mathfrak{g}^{\rm ess}_{(3.13)}=\mathfrak{g}^{\rm ess}$. As a result, we obtain the point transformations

$$
\begin{aligned}
&\Phi_{(3.12)}: \quad \tilde t={\rm e}^{\gamma t},\quad \tilde x={\rm e}^{\frac12\gamma t}(\tfrac32\gamma y+x),\quad \tilde y=\gamma{\rm e}^{\frac32\gamma t}y,\quad \tilde u={\rm e}^{-\frac1{16}(3\gamma y+2x)^2-\frac32\gamma t}u,\\
&\Phi_{(3.13)}: \quad \tilde t={\rm e}^{\frac12\gamma t},\quad \tilde x=\tfrac{1}{2\sqrt2}{\rm e}^{\frac14\gamma t}(\tfrac34\gamma y+x),\quad \tilde y=\tfrac{1}{2\sqrt2}\gamma{\rm e}^{\frac34\gamma t}y,\quad \tilde u={\rm e}^{-\gamma t}u,
\end{aligned}
$$

where the variables with tildes correspond to the equation (3.2), and the variables without tildes correspond to the source equation (3.12) or (3.13). By the direct computation, we check that indeed the transformations $\Phi_{(3.12)}$ and $\Phi_{(3.13)}$ respectively map the equations (3.12) and (3.13) to the equation (3.2). In other words, an arbitrary solution $\tilde u=f(t,x,y)$ of (3.2) is pulled back by $\Phi_{(3.12)}$ and $\Phi_{(3.13)}$ to the solutions $u=\Phi^*_{(3.12)}f$ and $u=\Phi^*_{(3.13)}f$ of the equations (3.12) and (3.13), respectively,

$$
\begin{aligned}
&u=\Phi^*_{(3.12)}f={\rm e}^{\frac1{16}(3\gamma y+2x)^2+\frac32\gamma t}f\Big({\rm e}^{\gamma t},{\rm e}^{\frac12\gamma t}\big(\tfrac32\gamma y+x\big),\gamma{\rm e}^{\frac32\gamma t}y\Big),\\
&u=\Phi^*_{(3.13)}f={\rm e}^{\gamma t}f\Big({\rm e}^{\frac12\gamma t},\tfrac{1}{2\sqrt2}{\rm e}^{\frac14\gamma t}\big(\tfrac34\gamma y+x\big),\tfrac{1}{2\sqrt2}\gamma{\rm e}^{\frac34\gamma t}y\Big).
\end{aligned}
$$

Using the formulas and families of solutions of the equation (3.2) that have been constructed in Sections 3.5 and 3.6, we find the following families of solutions for the equation (3.12):

$$
\begin{aligned}
&u={\rm e}^{\frac1{16}(3\gamma y+2x)^2+\frac{13}8\gamma t}|\tfrac32\gamma y+x|^{\frac14}\,\theta^\mu\big(\tfrac94\gamma{\rm e}^{\frac32\gamma t}y,\ {\rm e}^{\frac32\gamma t}|\tfrac32\gamma y+x|^{\frac32}\big)\quad\text{with}\quad \mu=\tfrac5{36},\\
&u={\rm e}^{\gamma t}\,\theta\Big(\tfrac13{\rm e}^{3\gamma t}+2\varepsilon{\rm e}^{\gamma t}-{\rm e}^{-\gamma t},\ {\rm e}^{-\frac12\gamma t}\big(\tfrac12\gamma({\rm e}^{2\gamma t}-3\varepsilon)y-({\rm e}^{2\gamma t}+\varepsilon)x\big)\Big),\\
&u={\rm e}^{\frac1{16}(3\gamma y+2x)^2+\frac32\gamma t}\,\theta\big(\tfrac43{\rm e}^{3\gamma t},\ {\rm e}^{\frac32\gamma t}(\gamma y+2x)\big),\\
&u={\rm e}^{\frac1{16}(3\gamma y+2x)^2+\frac32\gamma t}\,\theta\big(4{\rm e}^{\gamma t},\ {\rm e}^{\frac12\gamma t}(3\gamma y+2x)\big),\\
&u={\rm e}^{\frac12\tilde\zeta-\frac{x\zeta^2}{24\gamma y}+\frac32\gamma\kappa t}\zeta|y|^{\kappa-\frac43}\Big(C_1M\big(\kappa,\tfrac43,\tilde\zeta\big)+C_2U\big(\kappa,\tfrac43,\tilde\zeta\big)\Big)\\
&\qquad\text{with}\quad \zeta:=3\gamma y+2x,\quad \tilde\zeta:=\tfrac1{72}(\gamma y)^{-1}\zeta^3.
\end{aligned}
$$

In the same way, we can also obtain solutions of the equation (3.13),

$$
\begin{aligned}
&u={\rm e}^{\frac{17}{16}\gamma t}|\tfrac34\gamma y+x|^{\frac14}\,\theta^\mu\Big(18\gamma{\rm e}^{\frac34\gamma t}y,\ {\rm e}^{\frac38\gamma t}|\tfrac34\gamma y+x|^{\frac32}\Big)\quad\text{with}\quad \mu=\tfrac5{36},\\
&u={\rm e}^{-\frac1{32}(\frac34\gamma y+x)^2+\frac34\gamma t}\,\theta\Big(\tfrac83{\rm e}^{\frac32\gamma t}+16\varepsilon{\rm e}^{\frac12\gamma t}-8{\rm e}^{-\frac12\gamma t},{\rm e}^{-\frac14\gamma t}\big(\tfrac14\gamma(5{\rm e}^{\gamma t}-3\varepsilon)y-({\rm e}^{\gamma t}+\varepsilon)x\big)\Big),
\end{aligned}
$$

$$u = \mathrm{e}^{\gamma t}\,\theta\big(\tfrac{128}{3}\mathrm{e}^{\frac{3}{2}\gamma t},\, \gamma\mathrm{e}^{\frac{3}{4}\gamma t}(\gamma y - 4x)\big), \quad u = \mathrm{e}^{\gamma t}\,\theta\big(128\mathrm{e}^{\frac{1}{2}\gamma t},\, \mathrm{e}^{\frac{1}{4}\gamma t}(3\gamma y + 4x)\big),$$
$$u = \mathrm{e}^{-\zeta^3+\frac{1}{4}\gamma t(3\kappa+1)}\zeta|y|^{\kappa-1}\Big(C_1 M\big(\kappa, \tfrac{4}{3}, \zeta^3\big) + C_2 U\big(\mu - 1, \tfrac{4}{3}, \zeta^3\big)\Big)$$
$$\text{with} \quad \zeta := (9\gamma y)^{-\frac{1}{3}}\big(\tfrac{3}{4}\gamma y + x\big).$$

Recall that $\theta^\mu = \theta^\mu(z_1, z_2)$ and $\theta = \theta(z_1, z_2)$ denote arbitrary solutions of the (1+1)-dimensional linear heat equation with potential μz_2^{-2} and of the (1+1)-dimensional linear heat equation, $\theta^\mu_1 = \theta^\mu_{22} + \mu z_2^{-2}\theta^\mu$ and $\theta_1 = \theta_{22}$, respectively. $M(a, b, \omega)$ and $U(a, b, \omega)$ are the standard solutions of Kummer's equation $\omega\varphi_{\omega\omega} + (b-\omega)\varphi_\omega - a\varphi = 0$.

Chapter 4

Conclusion

4.1 Summary of results

The ultraparabolic Fokker–Planck equation (3.2) had been intensively studied in the mathematical literature from various points of view, including the classical group analysis of differential equations. Nevertheless, there were no exhaustive and accurate results even on its complete point symmetry group and on classification of its Lie reductions, and the present paper has filled up this gap. Moreover, some of the obtained results were unexpected and may affect the entire field of classical group analysis of differential equations.

The complete point symmetry group G of the remarkable Fokker–Planck equation (3.2) is given in Theorem 31. To simplify the computation of this group, we have applied the two-step version of the direct method. On the first step, we have considered the class $\bar{\mathcal{F}}$ of (1+2)-dimensional ultraparabolic Fokker–Planck equations of the general form (3.1), which contains the equation (3.2), have proven its normalization in the usual sense and have found its equivalence group $G^{\sim}_{\bar{\mathcal{F}}}$, see Theorem 30. Exhaustively describing the equivalence groupoid of the class $\bar{\mathcal{F}}$ in this way, we have derived the principal constraints for point symmetries of the equation (3.2). On the second step, we have in fact looked for admissible transformations of the class $\bar{\mathcal{F}}$ that preserve the equation (3.2), i.e., that constitute its vertex group. This has led to a highly coupled overdetermined system of nonlinear partial differential equations for the transformation components. We have successfully found its general solution and constructed a nice representation (3.6) for elements of G. A similar splitting in the course of computing point symmetries with the direct method was used earlier, e.g., in [5], but there the equivalence groupoid of the corresponding class had been known. In the present paper, we have first found the equivalence groupoid of a class of differential equations in order to compute the point symmetry group of a single element of this class, and this is indeed the optimal way of computing in spite of looking peculiar.

The representation (3.6) for the elements of G has allowed us to comprehensively analyze the structure of G. The group G contains the abelian normal (pseudo)subgroup $G^{\rm lin}$, which is associated with the linear superposition of solutions. Moreover, the group G splits over $G^{\rm lin}$, $G = G^{\rm ess} \ltimes G^{\rm lin}$. Here $G^{\rm ess}$ is a subgroup of G, which is a (finite-dimensional) Lie group and admits the factorization $G^{\rm ess} = (F \ltimes R_{\rm c}) \times R_{\rm d}$, where $F \simeq {\rm SL}(2, \mathbb{R})$, $R_{\rm c} \simeq {\rm H}(2, \mathbb{R})$ and $R_{\rm d} \simeq \mathbb{Z}_2$. Since ${\rm SL}(2, \mathbb{R})$ and ${\rm H}(2, \mathbb{R})$ are connected Lie groups and $G^{\rm lin}$ is a connected Lie pseudogroup, the above factorizations directly imply Corollary 32, which states the surprising fact that the equation (3.2) admits a single independent discrete point symmetry transformation. The simplest choice for such a transformation is the involution $\mathcal{I}'\colon (t, x, y, u) \mapsto (t, x, y, -u)$, which generates the group $R_{\rm d}$.

In addition, we have constructed the point transformation (3.10), which maps the function $u = 1 - H(t)$ to the fundamental solution of the equation (3.2). The similar construction arises in the Lie symmetry analysis of linear (1+1)-dimensional heat equation.

The accurate consideration of the representation of the subgroup F of G on the radical $\mathfrak{r}$ of the algebra $\mathfrak{g}^{\rm ess}$, which coincides with the representation of ${\rm SL}(2, \mathbb{R})$ on the space of real binary cubic forms, has allowed us to successfully classify $G^{\rm ess}$-inequivalent one- and two-dimensional subalgebras of $\mathfrak{g}^{\rm ess}$, which for a long time had been a stumbling block in the course of constructing inequivalent Lie invariant solutions of the equation (3.2) before.

Using this classification, we have found all inequivalent Lie reductions of codimensions one and two and constructed Lie invariant solutions of the equation (3.2). It has been reduced to the (1+1)-dimensional linear heat equation, the heat equation with an inverse quadratic potential, Kummer's equation, the Airy equation and first-order differential equations. The (1+1)-dimensional linear heat equation had been comprehensively studied within the framework of Lie reduction. In Section 2.5, we have constructed many particular solution of linear (1+1)-dimensional heat equation with inverse square potential over the real field. For each of the reduced ordinary differential equations, we have found its general solution. It has also been shown that Lie reductions to algebraic equations, which are of codimension three, give no new solutions in comparison with the ones constructed using Lie reductions of less codimensions.

Using this classification we found all inequivalent Lie reductions of codimension one and two and constructed Lie invariant solutions of the equation (3.2). The remarkable Fokker–Planck equation can be reduced to the (1+1)-dimensional linear heat equation, heat equation with inverse square potential, Kummer's equation, Airy equation and first order differential equations. The solutions of the reduced equations are found and mapped to the solutions of the original equation (3.2). Also in Sections 2.4 and 2.5 we computed the complete point symmetry group and constructed many particular solution of linear (1+1)-dimensional heat equation with inverse square potential.

Studying Kramers equations, which constitute the subclass $\mathcal{K}$ of the class $\bar{\mathcal{F}}$, we have found nontrivial point transformations that map all equations from the class $\mathcal{K}$ with eight-dimensional essential Lie invariance algebras, i.e., the equations (3.12) and (3.13) up to shifts of y, to the equation (3.2). Moreover, the presented formulas for mapping solutions of the equation (3.2) to solutions of the equations (3.12) and (3.13) gives the simplest way of obtaining explicit solutions of the latter equations.

4.2 Future work

Results obtained in the thesis can be extended in many directions. In particular, we can construct more general families of exact solutions of the equation (3.2) using reduction modules [13] or more promising recursion operators on solutions. Since the equation (3.2) is linear and homogenous, each first-order linear differential operator in total derivatives that is associated with an essential Lie symmetry of this equation is its recursion operator. Thus, it is obvious that the equation (3.2) admits, in particular, the generalized vector fields $\big((\mathfrak{P}^2+\varepsilon\mathfrak{P}^0)^n u\big)\partial_u$, $\big((\mathfrak{P}^1)^n u\big)\partial_u$ and $\big((\mathfrak{P}^0)^n u\big)\partial_u$ as its generalized symmetries. Here $\mathfrak{P}^2 := 2t\mathrm{D}_x + t^2\mathrm{D}_y - xu\mathrm{D}_u$, $\mathfrak{P}^1 := \mathrm{D}_x + t\mathrm{D}_y$, $\mathfrak{P}^0 := \mathrm{D}_y$ are the differential operators in total derivatives that are associated with the Lie-symmetry vector fields $\mathcal{P}^2$, $\mathcal{P}^1$ and $\mathcal{P}^0$ of the equation (3.2), and D_x, D_y and D_u

denote the operators of total derivatives with respect to x, y and u respectively. The corresponding invariant solutions are constructed via solving the equation (3.2) with the associated invariant surface condition, $(\mathfrak{P}^2+\varepsilon\mathfrak{P}^0)^n u=0$, $(\mathfrak{P}^1)^n u=0$ or $(\mathfrak{P}^0)^n u=0$, respectively. For $n=2$, it is easy to derive the general form of such solutions in terms of solutions of the (1+1)-dimensional linear heat equation,

- $u=|t|^{-\frac12}\mathrm{e}^{-\frac{x^2}{4t}}\big(\frac12 t^{-1}x\theta^1-(t-\varepsilon t^{-1})\theta^1_2+\theta^0\big)$,
  where $z_1=\frac13 t^3+2\varepsilon t-t^{-1}$, $z_2=2y-(t+\varepsilon t^{-1})x$,
- $u=x\theta^1-t^2\theta^1_2+\theta^0$, where $z_1=\frac13 t^3$, $z_2=y-tx$,
- $u=y\theta^1_{22}+\frac14 x^2\theta^1_2-\frac14 x\theta^1+\theta^0$, where $z_1=t$, $z_2=x$.

Here $\theta^i=\theta^i(z_1,z_2)$ is an arbitrary solution of the (1+1)-dimensional linear heat equation, $\theta^i_1=\theta^i_{22}$, $i=0,1$. As in Section 3.5, the subscripts 1 and 2 of functions depending on (z_1,z_2) denote derivatives with respect to z_1 and z_2, respectively. It is possible but quite challenging to construct similar explicit expressions in the case of general n, which are combinations of derivatives of arbitrary solutions of the (1+1)-dimensional linear heat equation with coefficients being polynomials in (x,y) and in (t,x), respectively.

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