Propagation Dynamics of Two Species Competition Models in a Periodic Discrete Habitat

by

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Department of Mathematics and Statistics

Memorial University

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Abstract

Spreading speeds and traveling waves are essential in qualitative studying biological invasions. Sometimes, the invading species can lead to the extinction of the local species competing for resources, and such a phenomenon is called competition exclusion. In this thesis, we study the propagation dynamics of a Lotka-Volterra competition model in a periodic discrete habitat when competition exclusion occurs. First, we present general results on spreading speeds and traveling waves for monotone systems in a periodic discrete habitat. Under appropriate assumptions, we show that a semi-trivial equilibrium is globally stable for the spatially periodic initial value problem when competition exclusion happens. Then we establish the existence of the rightward spreading speed and its coincidence with the minimal wave speed for the spatially periodic rightward traveling waves. We obtain sufficient conditions for the linear determinacy of the rightward spreading speed. Ultimately, we apply all these results to a specific model and conduct numerical simulations to investigate the spreading of the two competing species in a periodic habitat.

To my family

Lay summary

There are usually three cases for biological competition models based on the competitive pattern. Two competitive species coexist when the competition is weak; there is only one species that exist, which we call competition exclusion when there is a superior and inferior species; there is still one species that live, but it depends on the environmental factors when there is strong competition. In [13], we can find more details and descriptions of these three cases. We can utilize the theories and applications of traveling fronts to study these phenomena. This thesis will focus on the propagation dynamics of a Lotka-Volterra competition model in a periodic discrete habitat when the competition exclusion phenomenon occurs.

Based on the theory of monotone systems, we first give some results related to the propagation dynamics for the periodic discrete habitat. Then we show that a semi-trivial equilibrium is globally stable for the spatially periodic initial value problem; one species can persist, and the other will die out in the long run for the spatially periodic initial value. Next, we establish the rightward spreading speed, proving that it equals the minimal wave speed for the spatially periodic rightward traveling waves. Furthermore, we obtain sufficient conditions for the linear determinacy of the rightward spreading speed. Finally, we apply all these results to a model and do some numerical simulations.

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Table of contents

List of figures

Chapter 1

Introduction

Spreading speeds and traveling waves are essential in qualitative studying biological invasions. The invasive species may lead to the extinction of the local species which compete with the invasive ones. Based on the biological models, the heterogeneity of the environment is crucial for the spreading of the invading species. In reality, the rivers or hills can cause the invasion of the terrestrial animals, and the waterfalls or lateral inflow can give rise to the invasion of the aquatic animals. Besides, human activity is a common factor in raising the invasion of the species, like the fleeing pets. Therefore, to maintain the sustainability of different ecosystems, it is important to understand the impact of heterogeneity on extinction, invasion, and competition of the populations in these ecosystems. Due to the pioneering work of Fisher [10] and Kolmogorov, Petrovsky, and Piskunov [18] on traveling waves in reaction-diffusion equations in 1937, there have been lots of related researches on this field. Gärtner and Freidlin [11] investigated the spreading speed in periodic media. Hudson and Zinner [15] established the existence of periodic wavefronts for a set of Fisher-type reaction-diffusion equations with periodic space variable in reaction term. Weinberger [31] proposed the theory of spreading speeds and traveling waves in a periodic habitat for recursion with the periodic order-preserving compact operator. Berestycki [1] studied the pulsating traveling waves of reaction-diffusion-advection equations under periodic domains with periodic diffusion and velocity fields. Berestycki et al. [2] also found some nonlinear propagation phenomena of KPP-type reaction-diffusion equations in periodic habitats. Liang and Zhao [23] developed a general theory of spreading speeds and traveling waves in a periodic habitat for monotone semiflow under the assumption of alpha-contraction compactness. Fang and Zhao [9] further studied traveling waves for monotone semiflows with weak compactness in the case where there may be boundary fixed points between two ordered unstable and stable fixed points. Yu and Zhao [33] considered a more general two-species competition reaction-advection-diffusion model in a periodic habitat:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} =L_1 u_1 + u_1(b_1(x) - a_{11}(x)u_1 - a_{12}(x)u_2), \\ \dfrac{\partial u_2}{\partial t} =L_2 u_2 + u_2(b_2(x) - a_{21}(x)u_1 - a_{22}(x)u_2), \end{cases}$$

where $t > 0, x \in \mathbb{R}$, and $L_i u = d_i(x)\frac{\partial^2 u}{\partial x^2} - g_i(x)\frac{\partial u}{\partial x}$, $i = 1, 2$. Here u_1, u_2 are the population densities of two competition species in an $L-$periodic habitat for some positive number L, $d_i(x)$, $g_i(x)$ and $b_i(x)$ are diffusion, advection and growth rates of the i-th species ($i = 1, 2$), respectively. Besides, $a_{ij}(x)(1 \leqslant i, j \leqslant 2)$ are inter- and intra-specific competition coefficients. More details can be found in [33]. This thesis considers the lattice equation based on the reaction-advection-diffusion system.

The patch or lattice models are investigated in different applications, such as chemical reactions, biological systems, image processing, etc. There are also many real-life examples, like several roads across the forest or the islands in the sea. Sometimes we consider the lattice model as the lattice version of the reaction-advection-diffusion model. The spreading of cancer cells or viruses can also be viewed as a lattice model but not periodic. There are lots of papers that focus on the lattice model. Shigesada et al. [26] first investigated the propagation dynamics of a single species for a reaction-diffusion model in a patchy habitat with periodic mobility and growth rate. Keener [17] proved the propagation and its failure in systems of discrete coupled excitable cells. Zinner [36] investigated the existence of traveling waves for the discrete Nagumo equations. Chow et al. [7] analyzed the existence and stability of traveling waves in lattice dynamical systems. Chen and Guo [4] proved the existence and asymptotic stability of traveling waves for discrete quasilinear monostable equations, and they [5] showed the uniqueness and existence of traveling waves for discrete quasilinear monostable systems. Jin and Zhao [16] considered the spatial dynamics of a class of discrete-time population models in a periodic lattice habitat. Slavik [27] investigated a Lotka-Volterra competition model on a finite number of patches. Chen et al. [3] studied the two-species Lotka-Volterra competition patchy model with asymmetric dispersal and its global dynamical behavior. It seems that there is no research on the competition Lotka-Volterra model in a periodic lattice habitat. Therefore, this thesis introduces a lattice version of (1.0.1). Motivated by the method in [6] and [34], we take the dispersal as follows

$$d_i(j)\left(\sum_{k\in\mathbb{Z}\backslash 0}\alpha_{k;i,j}u_i(j+k)-u_i(j)\right),\ j\in\mathbb{Z},\ i\in\{1,2\},$$

where

(F1) $\sum\limits_{k\in\mathbb{Z}\backslash 0}\alpha_{k;i,j}=1$;

(F2) there is $k_0>0$ such that $\alpha_{k;1,j}=0$, $\alpha_{k;2,j}=0$, for $|k|>k_0$, $j\in\mathbb{Z}$.

Then we have the following system:

$$\begin{cases}\dfrac{\mathrm{d}u_1(t,j)}{\mathrm{d}t}=d_1(j)\left(\sum\limits_{k\in\mathbb{Z}\backslash 0}\alpha_{k;1,j}u_1(j+k)-u_1(j)\right)\\ \qquad\qquad +u_1(t,j)[b_1(j)-a_{11}(j)u_1(t,j)-a_{12}(j)u_2(t,j)],\\ \dfrac{\mathrm{d}u_2(t,j)}{\mathrm{d}t}=d_2(j)\left(\sum\limits_{k\in\mathbb{Z}\backslash 0}\alpha_{k;2,j}u_2(j+k)-u_2(j)\right)\\ \qquad\qquad +u_2(t,j)[b_2(j)-a_{21}(j)u_1(t,j)-a_{22}(j)u_2(t,j)].\end{cases}\tag{1.0.1}$$

Here the coefficients $d_i(j)$, $b_i(j)$, $a_{lm}(j)$ are positive and N-periodic in j with $j\in\mathbb{Z}$, $i,l,m\in\{1,2\}$. We simplify the notation $d_i(j)$, $u_i(t,j)$, $b_i(j)$, $a_{lm}(j)$ into $d_{i,j}$, $u_{i,j}(t)$, $b_{i,j}$ and $a_{lm;j}$, where $d_{i,j}>0$, $a_{lm;j}>0$, with $j\in\mathbb{Z}$, $i,l,m\in\{1,2\}$, respectively. Here u_1, u_2 are the population densities of two competing species in N-periodic discrete habitat for some positive integer $N\geqslant 2$, and $d_{i,j}$, $b_{i,j}$ are dispersal, growth rates of the i-th species ($i=1,2$), at j-th habitat ($j\in\mathbb{Z}$), respectively. Moreover, $\alpha_{k;1,j}$, $\alpha_{k;2,j}$ ($k,j\in\mathbb{Z},k\neq 0$) represent the moving pattern from j-th lattice to $(j+k)$-th lattice of species u_1 and u_2, respectively, for the N-periodic discrete habitat. Besides, $a_{lm;j}$ ($1\leqslant l,m\leqslant 2$, $j\in\mathbb{Z}$) are inter- and intra-specific competition coefficients.

Motivated by the method in [6], we consider a more general equation

$$\begin{cases} w_{1,j}'(t) = d_{1,j}\left(\displaystyle\sum_{k\in\mathbb{Z}\setminus 0}\alpha_{k;1,j}w_{1,j+k} - w_{1,j}\right) + f_{1,j}(w_{1,j}(t), w_{2,j}(t)), \\ w_{2,j}'(t) = d_{2,j}\left(\displaystyle\sum_{k\in\mathbb{Z}\setminus 0}\alpha_{k;2,j}w_{2,j+k} - w_{2,j}\right) + f_{2,j}(w_{1,j}(t), w_{2,j}(t)), \end{cases} \tag{1.0.2}$$

where $w_{i,j}(t) = a_{ii;j}u_{1,j}(t)$, $t \geqslant 0$, $i \in \{1,2\}$, $j \in \mathbb{Z}$,

$$\begin{cases} f_{1,j}(w_{1,j}(t), w_{2,j}(t)) = w_{1,j}(t)[b_{1,j} - w_{1,j}(t) - \gamma_{1,j}w_{2,j}(t)], \\ f_{2,j}(w_{1,j}(t), w_{2,j}(t)) = w_{2,j}(t)[b_{2,j} - w_{2,j}(t) - \gamma_{2,j}w_{1,j}(t)] \end{cases}$$

with $\gamma_{1,j} = \frac{a_{12;j}}{a_{22;j}}$, $\gamma_{2,j} = \frac{a_{21;j}}{a_{11;j}}$, $i \in \{1,2\}$, $j \in \mathbb{Z}$. This thesis is framed as follows. In Chapter 2, we show general results for the competition model in periodic lattice habitat. In Chapter 3, we consider the spatially periodic equilibria of system (1.0.2) under spatially periodic initial value, and we show that one of the semi-trivial equilibria is globally asymptotically stable. In Chapter 4, we change variables for system (1.0.2) to get a cooperative system and investigate its spreading speed and traveling waves. In Chapter 5, we establish sufficient conditions for the linear determinacy of the minimal wave speed for system (1.0.2). In Chapter 6, we apply all the results to a specific model and conduct some numerical simulations.

Chapter 2

Lattice systems in a periodic habitat

In this chapter, we present the lattice version of the results in [9], [21] and [33] on spreading speeds and traveling waves in a periodic habitat.

Let m be a positive integer, $\mathcal{C}$ be the set of all bounded and two-sided sequence in $\mathbb{R}^m$, and $\mathcal{C}_+ := \{u \in \mathcal{C} : u_j \geqslant 0, \forall j \in \mathbb{Z}\}$. Any point in $\mathbb{R}^m$ can be regarded as a constant sequence in $\mathcal{C}$. For $u = (u_1, \cdots, u_m)$, $v = (v_1, \cdots, v_m) \in \mathcal{C}$, we write $u \geqslant v (u \gg v)$ if $u_{i,j} \geqslant v_{i,j} (u_{i,j} > v_{i,j})$, $\forall 1 \leqslant i \leqslant m$, $j \in \mathbb{Z}$, and $u > v$ if $u \geqslant v$ but $u \neq v$. Assume that β is a strongly positive N-periodic sequence in $\mathbb{R}^m$. Set $\mathcal{C}_\beta = \{u \in \mathcal{C} : 0 \leqslant u_j \leqslant \beta_j, \forall j \in \mathbb{Z}\}$, $\mathcal{C}_\beta^P = \{u \in \mathcal{C}_\beta : u_j = u_{j+N}, \forall j \in \mathbb{Z}\}$ and $[1, N]_\mathbb{Z} = \{j \in \mathbb{Z} : j \in [1, N]\}$. Let X be the set of all bounded sequence with N elements in $\mathbb{R}^m$; that is, $X = B([1, N]_\mathbb{Z}, \mathbb{R}^m)$ and equip X with the maximum norm $|\cdot|_X$. Let $X_+ = \{u \in X : u_j \geqslant 0, j \in [1, N]_\mathbb{Z}\}$, $X_\beta = \{u \in X : 0 \leqslant u_j \leqslant \beta_j, \forall j \in [1, N]_\mathbb{Z}\}$. Let $BC(\mathbb{R}, X)$ be the set of all bounded and continuous functions from $\mathbb{R}$ to X. Then we define $\mathcal{X} = BC(\mathbb{R}, X)$, $\mathcal{X}_+ = \{v \in \mathcal{X} : v(x) \in X_+, \forall x \in \mathbb{R}\}$ and $\mathcal{X}_\beta = BC(\mathbb{R}, X_\beta)$. Let $\mathcal{K} = B(N\mathbb{Z}, X)$, and $\mathcal{K}_\beta = B(N\mathbb{Z}, X_\beta)$. In fact, any element in X_β can be seen as a constant function in $\mathcal{X}_\beta$, and any element in $\mathcal{C}_\beta^P$ also corresponds to a constant function in $\mathcal{X}_\beta$. We equip $\mathcal{C}$ and $\mathcal{X}$ with the compact open topology; that is, $u_n \to u$ in $\mathcal{C}$ means that the sequence of $u_{n,j}$ converges to u_j in $\mathbb{R}^m$ for j in any finite sequence; $u_n \to u$ in $\mathcal{X}$ means that the sequence of $u_n(s)$ converges to $u(s)$ in X for s in any compact set. Then We equip $\mathcal{C}$ and $\mathcal{X}$ with the norm $\|\cdot\|_\mathcal{C}$ and

$\|\cdot\|_{\mathcal{X}}$, respectively, which are defined by

$$\|u\|_{\mathcal{C}} = \sum_{k=1}^{\infty} \frac{\max_{|j|\leqslant k} |u_j|}{2^k}, \ \forall u \in \mathcal{C},$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^m$, and

$$\|u\|_{\mathcal{X}} = \sum_{k=1}^{\infty} \frac{\max_{|x|\leqslant k} |u(x)|_X}{2^k}, \ \forall u \in \mathcal{X}.$$

Introduce a translation operator $\mathcal{T}_a$ by $\mathcal{T}_a[u]_j = u_{j+a}$ for any given $a \in N\mathbb{Z}$. Let Q be an operator on $\mathcal{C}_\beta$, where $\beta \in \mathcal{C}$ is strongly positive and N-periodic. To utilize the theory developed in the [9],[21] and [33], we need the following assumptions on Q:

(A1) Q is N-periodic, that is, $\mathcal{T}_a[Q[u]] = Q[\mathcal{T}_a[u]]$, $\forall u \in \mathcal{C}_\beta$, $a \in N\mathbb{Z}$;

(A2) $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is continuous with respect to the compact open topology;

(A3) $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is monotone (order preserving) in the sense that $Q[u] \geqslant Q[w]$ whenever $u \geqslant w$;

(A4) Q admits two N-periodic fixed points 0 and β in $\mathcal{C}_+$, and for any $z \in \mathcal{C}_\beta^P$ with $0 \ll z \leqslant \beta$, there holds $\lim_{n\to\infty} Q^n[z]_j = \beta_j$ uniformly for $j \in \mathbb{Z}$;

(A5) $Q[\mathcal{C}_\beta]$ is precompact in $\mathcal{C}_\beta$ with respect to the compact open topology.

Introduce a homeomorphism $F : \mathcal{C} \to \mathcal{K}$ by

$$F[\phi](i)_m = \phi_{i+m}, \ i \in N\mathbb{Z}, \ m \in [1, N]_{\mathbb{Z}}$$

and an operator $P : \mathcal{K}_\beta \to \mathcal{K}_\beta$ by

$$P = F \circ Q \circ F^{-1}. \tag{2.0.1}$$

Let $v_s \in \mathcal{C}$ be defined by

$$(v_s)_j = v(s + n_j)_{m_j}, \ \forall j = n_j + m_j \in \mathbb{Z}, \ n_j = N[\frac{j-1}{N}], \ m_j \in [1, N]_{\mathbb{Z}}, v \in \mathcal{X},$$

and $\tilde{v}_s \in \mathcal{K}$ be defined by

$$\tilde{v}_s(n)_m = v(n+s)_m, \forall n \in N\mathbb{Z}, m \in [1,N]_{\mathbb{Z}}, s \in \mathbb{R}, v \in \mathcal{X}.$$

Define $\tilde{P} : \mathcal{X} \to \mathcal{X}$ by

$$\tilde{P}[v](s) := P[\tilde{v}_s](0), \, \forall v \in \mathcal{X}, \, s \in \mathbb{R}. \tag{2.0.2}$$

We further claim that

$$\tilde{P}[v](s)_m = Q[v_s]_m, \, \forall v \in \mathcal{X}, \, s \in \mathbb{R}, \, m \in [1,N]_{\mathbb{Z}}. \tag{2.0.3}$$

In fact, since

$$F[\phi](i)_m = \phi_{i+m}, \, F^{-1}[\psi]_j = \psi(n_j)_{m_j},$$

it then shows that

$$\begin{aligned} \tilde{P}[v](s) &= P[\tilde{v}_s](0) = FQF^{-1}[\tilde{v}_s](0) \\ &= FQ[v(n+s)_m](0) = FQ[v_s](0), \end{aligned}$$

so we have

$$\tilde{P}[v](s)_m = FQ[v_s](0)_m = Q[v_s]_m.$$

Let $r \in \mathrm{Int}(\mathrm{X}_+)$. To utilize the conclusions in [9] to $\tilde{P}$, we need to verify the following assumptions for $\tilde{P}$:

(B1) $\mathcal{T}_a[\tilde{P}[u]] = \tilde{P}[\mathcal{T}_a[u]]$, $\forall u \in \mathcal{X}_r$, $a \in \mathbb{R}$;

(B2) $\tilde{P} : \mathcal{X}_r \to \mathcal{X}_r$ is continuous with respect the compact open topology;

(B3) $\tilde{P} : \mathcal{X}_r \to \mathcal{X}_r$ is monotone (order preserving) in the sense that $\tilde{P}[u] \geqslant \tilde{P}[w]$ whenever $u \geqslant w$;

(B4) $\tilde{P}$ admits two fixed points 0 and r in $\bar{X}_r$, and for any $z \in \bar{X}_r$ with $0 \ll z \leqslant r$, there holds $\lim_{n\to\infty} \tilde{P}^n[z] = r$;

(B5) There exists $k \in [0,1)$ such that for any $\mathcal{U} \subset \mathcal{X}_r$, $\alpha(\tilde{P}[\mathcal{U}(0)]) \leqslant k\alpha(\mathcal{U}(0))$, where α denote the Kuratowski measure of noncompactness in $\mathcal{X}_r$.

Proposition 2.0.1. *If $\beta \in \mathcal{C}$ is strongly positive and N-periodic and $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ satisfies assumptions $(A1)$-$(A5)$, then assumptions $(B1)$-$(B5)$ hold for $\tilde{P} : \mathcal{X}_\beta \to \mathcal{X}_\beta$.*

Proof. For any $c \in \mathbb{R}$, let $u(\cdot) = v(\cdot + c)$, $\forall v \in \mathcal{X}$. Then

$$\begin{aligned} T_{-c}\tilde{P}[v](s) &= \tilde{P}[v](s+c) \\ &= Q[v_{s+c}] = Q[u_s] = \tilde{P}[u(\cdot)](s) \\ &= \tilde{P}[T_{-c}v](s), \ \forall v \in \mathcal{X}, \ s \in \mathbb{R}, \end{aligned}$$

so $(B1)$ holds. Furthermore, $(B2)$ can be verified by analogous arguments to those in [22, lemma 2.1], and $(B3)$ can be obtained from $(A3)$. In fact, 0 is the fixed point of $\tilde{P}$ since $Q(0) = 0$. To show (B4), it is necessary to prove that $\beta_{[1,N]_\mathbb{Z}}$ is the fixed point of $\tilde{P}$. Observe that β_j is a constant function in $\mathcal{X}$ with $j \in [1, N]_\mathbb{Z}$, so it follows that

$$(\beta_s)_j = \beta(s + n_j)_{m_j} = \beta_{m_j}, \forall s \in \mathbb{R}, \ j \in \mathbb{Z}.$$

Thus, $\beta_s = \beta \in \mathcal{C}, \forall s \in \mathbb{R}$. Furthermore,

$$\tilde{P}[\beta](s)_m = Q[\beta_s]_m = Q[\beta]_m = \beta_m, \ \forall m \in [1, N]_\mathbb{Z},$$

which shows that $\tilde{P}[\beta] = \beta$ in $\mathcal{X}$. Therefore, $(B4)$ can be obtained from $(A4)$. Next, we prove that $(B5)$ is valid. In fact, for any given $\mathcal{U} \subset \mathcal{X}_\beta$, $\tilde{P}(\mathcal{U})(0)$ is uniformly bounded, so we obtain that $\tilde{P}(\mathcal{U})(0)$ is compact, as it is the set of sequences. Thus, we have $\alpha(\tilde{P}(\mathcal{U})(0)) = 0$, which implies that $(B5)$ is valid with $k = 0$. □

Let $\omega \in X_\beta$ with $0 \ll \omega \ll \beta$, and let $\phi \in \mathcal{X}_\beta$ such that the following properties hold:

(C1) $\phi(s)$ is nonincreasing in s;

(C2) $\phi(s) \equiv 0$ for all $s \geqslant 0$;

(C3) $\phi(-\infty) = \omega$.

Fix c to be a given real number. According to [30], we introduce an operator R_c by

$$R_c[a](s) := \max\{\phi(s), T_{-c}\tilde{P}[a](s)\},$$

and a sequence of functions $a_n(c; s)$ by the recursion

$$a_0(c; s) = \phi(s), \quad a_{n+1}(c; s) = R_c[a_n(c; \cdot)](s).$$

By the similar conclusions to those in [9, Lemma 3.1-Lemma 3.3], we have the following result.

Lemma 2.0.1. *The following statements are valid:*

(1) For each $s \in \mathbb{R}$, $a_n(c; s)$ converges to $a(c; s)$ in X, where $a(c; s)$ is noninccreasing in both c and s with $a(c; \cdot) \in \mathcal{X}_\beta$.

(2) $a(c, -\infty) = \beta$ and $a(c, \infty)$ is a fixed point of $\tilde{P}$.

By [9, 30], we introduce two numbers

$$c_+^* = \sup\{c : a(c, \infty) = \beta\}, \quad \bar{c}_+ = \sup\{c : a(c, \infty) > 0\}. \tag{2.0.4}$$

Since $\{c : a(c, \infty) = \beta\} \subset \{c : a(c, \infty) > 0\}$, it follows that $c_+^* \leqslant \bar{c}_+$. For each $t \geqslant 0$, let P_t and $\tilde{P}_t$ defined as in (2.0.1) and (2.0.2) with $Q = Q_t$, respectively. By [9, Remark 3.2], we have the following result.

Theorem 2.0.1. *Let $\{Q_t\}_{t\geqslant 0}$ be a continuous-time semiflow on $\mathcal{C}_\beta$ with $Q_t[0] = 0$, $Q_t[\beta] = \beta$ for all $t \geqslant 0$ and $\{\tilde{P}_t\}_{t\geqslant 0}$ be define in (2.0.2) for each $t \geqslant 0$ and c_+^* and $\bar{c}_+$ be denoted by (2.0.4) with $\tilde{P} = \tilde{P}_1$. Suppose that Q_t satisfies (A1)-(A5) for each $t > 0$. Then the following statements are valid:*

(i) If $\phi \in \mathcal{C}_\beta$, $0 \leqslant \phi \leqslant \omega \ll \beta$ for some $\omega \in \mathcal{C}_\beta^P$, and $\phi_j = 0$, $\forall j \geqslant H$, for some $H \in \mathbb{Z}$, then $\lim\limits_{t\to\infty, j\geqslant ct} Q_t(\phi)_j = 0$ for any $c > \bar{c}_+$.

(ii) If $\phi \in \mathcal{C}_\beta$ and $\phi_j \geqslant \sigma$, $\forall j \leqslant K$, for some $\sigma \gg 0$ and $K \in \mathbb{Z}$, then

$$\lim_{t\to\infty, j\leqslant ct} (Q_t(\phi)_j - \beta_j) = 0$$

for any $c < c_+^*$.

Proof. As $\{Q_t\}_{t\geqslant 0}$ is a continuous-time semiflow on $\mathcal{C}_\beta$ with $Q_t(0) = 0$ and $Q_t(\beta) = \beta$ for all $t \geqslant 0$, we can obtain that $\{\tilde{P}_t\}_{t\geqslant 0}$ is a continuous-time semiflow on $\mathcal{X}_\beta$ with

$\tilde{P}_t(0) = 0$ and $\tilde{P}_t(\beta) = \beta$ for all $t \geqslant 0$. By Proposition 2.0.1, $\tilde{P}_t$ satisfies $(B1)$-$(B5)$. For $\varphi \in \mathcal{C}_\beta$, $0 \leqslant \phi \leqslant \omega \ll \beta$ with $\omega \in \mathcal{C}_\beta^P$, let

$$u(s)_m = [\phi_{n_s+N+m} - \phi_{n_s+m}]\frac{\theta_s}{N} + \phi_{n_s+m}$$

for $s \in \mathbb{R}$, $s = n_s + \theta_s$, $n_s = N[\frac{s}{N}]$, $\theta_s \in [0, N)$, $m \in [1, N]_{\mathbb{Z}}$. Thus $u \in \mathcal{X}_\beta$ and $0 \leqslant u \leqslant \omega \ll \beta$.

To show statement (i), we assume that there is some $H \in \mathbb{Z}$ such that $\phi_j = 0$, $j \geqslant H$ and $\phi_j \not\equiv 0$ (otherwise, it is trivial), so $u(s) = 0, s \geqslant H + N$. By [9, Remark 3.2], we can obtain that $\lim\limits_{t\to\infty,\, s\geqslant ct} \tilde{P}_t(u)(s) = 0$ in X for any $c > \bar{c}_+$. Furthermore, we have

$$\begin{aligned}\tilde{P}_t[u](n_j)_{m_j} &= Q_t[u_{n_j}]_{m_j} = Q_t[u(n_j + n_i)(m_i)]_{m_j} \\ &= Q_t[\phi(n_j + i)]_{m_j} = Q_t[\phi]_j,\ i, j \in \mathbb{Z},\end{aligned}$$

and for $s \in N\mathbb{Z}$, $\lim\limits_{t\to\infty,\, s\geqslant ct} \tilde{P}_t(u)(s) = 0$ in X is valid for any $c > \bar{c}_+$. Fix a real number $c' \in (\bar{c}_+, c)$, we obtain

$$|Q_t[\phi]_j| \leqslant |\tilde{P}_t[u](n_j)|_X,\ \forall j \geqslant ct,\ t \geqslant \frac{N}{c - c'}, \tag{2.0.5}$$

and $n_j \geqslant ct - N \geqslant c't$. If $t \to \infty$ in (2.0.5), it then follows that $\lim\limits_{t\to\infty, j>ct} Q_t(\phi)_j = 0$ for any $c > \bar{c}$.

By similar arguments to the above, we can prove that statement (ii) is also valid. $\square$

In view of the above theorem, we call $\bar{c}_+$ and c_+^* as the fastest and slowest rightward rightward spreading speeds for $\{Q_t\}_{t\geqslant 0}$ on $\mathcal{C}_\beta$, respectively. If $\bar{c}_+ = c_+^*$, we say that this system admits a single rightward spreading speed.

Next, we show the existence and the non-existence of traveling waves in a discrete periodic habitat for the continuous-time semiflow $\{Q_t\}_{t\geqslant 0}$. Given a continuous-time semiflow $\{Q_t\}_{t\geqslant 0}$ on $\mathcal{C}_\beta$, we regard $V_j(j - ct)$ as an N-periodic rightward traveling wave of $\{Q_t\}_{t\geqslant 0}$ if $\{V_j(j + a)\}_{j\in\mathbb{Z}} \in \mathcal{C}_\beta$, $Q_t[U]_j = V_j(j - ct)$ and $V_j(\xi)$ as an N-periodic function in j, where $a \in \mathbb{R}$, $t \geqslant 0$, $\xi \in \mathbb{R}$, $U_j := V_j(j)$. Furthermore, we say that $V_j(\xi)$ connects β to 0 if $\lim\limits_{\xi\to-\infty} |V_j(\xi) - \beta_j| = 0$ and $\lim\limits_{\xi\to\infty} |V_j(\xi)| = 0$ for $j \in \mathbb{Z}$.

We have the following result by the arguments similar to those in Yu and Zhao [33].

Theorem 2.0.2. *Let $\{Q(t)\}_{t\geqslant 0}$ be a continuous-time semiflow on $\mathcal{C}_\beta$ with $Q_t[0]=0$, $Q_t[\beta]=\beta$ for all $t\geqslant 0$, $\{\tilde{P}\}_{t\geqslant 0}$ be defined as in* (2.0.2) *and c_+^* and $\bar{c}_+$ be denoted by* (2.0.4) *with $\tilde{P}=\tilde{P}_1$. If $Q(t)$ satisfies (A1)-(A5) for each $t>0$, then the following statements are valid:*

(1) For any $c\geqslant c_+^*$, there is an N-periodic traveling wave $W_j(j-ct)$ connecting β to some equilibrium $\beta_1\in\mathcal{C}_\beta^P\setminus\{\beta\}$ with $W_j(\xi)$ be continuous and nonincreasing in $\xi\in\mathbb{R}$.

(2) If, in addition, 0 is an isolated equilibrium of $\{Q_t\}_{t\geqslant 0}$ in $\mathcal{C}_\beta^P$, then for any $c\geqslant\bar{c}_+$, either of the following holds true:

(i) there is an N-periodic traveling wave $W_j(j-ct)$ connecting β to 0 with $W_j(\xi)$ be continuous and nonincreasing in $\xi\in\mathbb{R}$.

(ii) $\{Q_t\}_{t\geqslant 0}$ has two ordered equilibria α_1, $\alpha_2\in\mathcal{C}_\beta^P\setminus\{0,\beta\}$ such that there exist an N-periodic traveling wave $W_{1,j}(j-ct)$ connecting α_1 and 0 and an N-periodic traveling wave $W_{2,j}(j-ct)$ connecting β and α_2 with $W_{i,j}(\xi)$, $i=1,2$ be an continuous and nonincreasing in $\xi\in\mathbb{R}$.

(3) For any $c<c_+^*$, there is no N-periodic traveling wave connecting β, and for any $c<\bar{c}_+$, there is no N-periodic traveling wave connecting β to 0.

Chapter 3

The periodic initial value problem

In this chapter, we study the global dynamics of the spatially periodic and discrete Lotka-Volterra competition system with periodic initial values.

In the rest of the thesis, we always assume that $d_{i,j}$, $b_{i,j}$, and $\gamma_{i,j}$ are N-periodic sequences in j and $d_{i,j} > 0$, $a_{l,m;j} > 0$, for all $i \in \{1, 2\}$, $1 \leqslant l, m \leqslant 2$, $j \in \mathbb{Z}$.

Let Y be the set of all N-periodic sequence from $\mathbb{Z}$ to $\mathbb{R}$, and $Y_+ = \{\psi \in Y : \psi_j \geqslant 0, \forall j \in \mathbb{Z}\}$ be a positive cone of Y. Equip Y with the maximum norm $\| \cdot \|_Y$, that is, $\|\phi\|_Y = \max\limits_{j\in\mathbb{Z}} |\phi_j|$. Indeed, (Y, Y_+) is an ordered Banach space. Next we consider the following ordinary differential equation:

$$u_j'(t) = d_j \left(\sum_{k\in\mathbb{Z}\setminus 0} \alpha_{k;j} u_{j+k} - u_j \right) + u_j(t)[b_j - a_j u_j(t)], \; j \in \mathbb{Z}. \tag{3.0.1}$$

Here b_j, a_j, d_j are N-periodic in $j \in \mathbb{Z}$, $d_j > 0$, $a_j > 0$, and $\sum\limits_{k\in\mathbb{Z}\setminus 0} \alpha_{k;j} = 1$ for all $j \in \mathbb{Z}$. It is easy to see that there exists a positive number $K > 0$, such that

$$K[b_j - a_j K] \leqslant 0, \; j \in \mathbb{Z}.$$

. Let $\hat{K} = \{\hat{K}_j\}_{j\in\mathbb{Z}}$, $\hat{K}_j = K$, and set $W := [0, \hat{K}]$ is an ordered interval in Y. We rewrite system (3.0.1) into this form $u'(t) = Lu + F(u)$, where $u(t) = \{u_j(t)\}_{j\in\mathbb{Z}}$, $F(u) = \{f_j(u(t))\}_{j\in\mathbb{Z}}$, $Lu = \{l_j(u(t))\}_{j\in\mathbb{Z}}$ with $l_j(u) = d_j(\sum\limits_{k\in\mathbb{Z}\setminus 0} \alpha_{k;j} u_{j+k} - u_j)$ and $f_j(u) = u_j(t)[b_j - a_j u_j(t)]$, $j \in \mathbb{Z}$. Let $b := \sup\limits_{j\in\mathbb{Z},\xi\in[0,K]} f_j'(\xi)$. It is clearly that $b < \infty$.

Since $f_j(0) = 0$, $j \in \mathbb{Z}$ and following from [34, (A2), Lemma 4.1, Theorem 4.2], we have the following result.

Proposition 3.0.2. *For any $x \in W$, there exists a unique solution $u(t,x)$ of system* (3.0.1) *with $u(0,x) = x$ and $u(t,x) \in W$ for all $t \in [0,\infty)$ and system* (3.0.1) *admits comparison principle on W. Moreover, let $Q_t(x) = u(t,x)$ for all $t \geqslant 0$ and $x \in W$. Then $\{Q_t\}_{t\geqslant 0}$ is a semiflow on W with respect to the compact open topology.*

Next, we consider the following N-dimensional ordinary differential equation:

$$u_j'(t) = d_j \left(\sum_{k\in\mathbb{Z}\setminus 0} \alpha_{k;j} u_{j+k} - u_j \right) + u_j(t)[b_j - a_j u_j(t)], \tag{3.0.2}$$

where $j \in [1,N]_{\mathbb{Z}}$ and $u_0(t) = u_N(t)$, $u_{N+1}(t) = u_1(t)$, $d_{N+1} = d_1$. For $\tilde{K} = (\hat{K}_j)_{j\in[1,N]_{\mathbb{Z}}}$, $\tilde{W} := [0,\tilde{K}]$ is an ordered interval in $\mathbb{R}^N$. For system (3.0.2), it is easy to see that there exists a unique solution $\tilde{u}(t,x)$ with $\tilde{u}(0,x) = x$ and $\tilde{u}(t,x) \in \tilde{W}$ for all $t \in [0,\infty)$, and system (3.0.2) admits comparison principle on $\tilde{W}$. Moreover, let $\tilde{Q}_t(x) = \tilde{u}(t,x)$ for all $t \geqslant 0$ and $x \in \tilde{W}$. Clearly $\{\tilde{Q}_t\}_{t\geqslant 0}$ is a semiflow on $\tilde{W}$.

Set $p_j(u) = l_j(u) + f_j(u)$, $P = (p_j)_{j\in\mathbb{Z}}$ and $\tilde{P} = (p_j)_{j\in[1,N]_{\mathbb{Z}}}$. Let $\tilde{Q} = \tilde{Q}_1$ and $A = D\tilde{P}(0)$. We can verify that $\tilde{P}$ is cooperative, and the $D\tilde{P}(x)$ is irreducible for every $x \in \mathbb{R}^N_+$. It is easy to verify that $P(0) = 0$ and $p_j(x) \geqslant 0$ for all $x \in \mathbb{R}^N_+$ with $x_j = 0$, $j = 1, 2, \cdots, N$. Then we will show that $\tilde{P}$ is strictly subhomogeneous(sublinear). Let $\tilde{L} = \{l_j\}_{j\in[1,N]_{\mathbb{Z}}}$ and $\tilde{F} = \{f_j\}_{j\in[1,N]_{\mathbb{Z}}}$. In fact,

$$p_j(\lambda\tilde{u}) = l_j(\lambda\tilde{u}) + f_j(\lambda\tilde{u}) > \lambda l_j(\tilde{u}) + \lambda f_j(\tilde{u}) = \lambda p_j(\tilde{u})$$

for all $\tilde{u} \gg 0$ and $\lambda \in (0,1)$ with $j \in [1,N]_{\mathbb{Z}}$, which implies that $\tilde{P}$ is strictly subhomogeneous. We have the following result: by [35, Corollary 3.2, Remark 4.3].

Lemma 3.0.2. *Define the stability modulus of matrix A as*

$$s(A) := \max\{\Re\lambda; \det\Delta(\lambda) = 0\},$$

the following statements are valid.

(a) If $s(A) \leqslant 0$, 0 is a globally stable equilibrium of $\tilde{Q}_t$ in $\tilde{W}$;

(b) If $s(A) > 0$, then there exists a globally stable equilibrium of $\tilde{Q}_t$ in $\tilde{W} \setminus \{0\}$.

Let $\mathbb{P} = P(\mathbb{Z}, \mathbb{R}^2)$ be the N-periodic sequence from $\mathbb{Z}$ to $\mathbb{R}^2$ and $\mathbb{P}_+ = \{\psi \in \mathbb{P} : \psi_j \geqslant 0,\ j \in \mathbb{Z}\}$. Indeed $\mathbb{P}_+$ is a closed cone of $\mathbb{P}$, and it also yields a partial ordering on $\mathbb{P}$. Moreover, we define a norm on $\|\cdot\|_{\mathbb{P}}$ by

$$\|\phi\|_{\mathbb{P}} = \max_{j\in\mathbb{Z}} \sqrt{\phi_{1,j}^2 + \phi_{2,j}^2}.$$

Clearly, $(\mathbb{P}, \|\|_{\mathbb{P}})$ is an ordered Banach space. Let

$$\begin{cases} \tilde{w}_1(t) = (w_{1,1}(t), \cdots, w_{1,j}(t), \cdots, w_{1,N}(t)) \\ \tilde{w}_2(t) = (w_{2,1}(t), \cdots, w_{2,j}(t), \cdots, w_{2,N}(t)) \end{cases}$$

and

$$w(t) = (w_1(t), w_2(t)),\ \tilde{w}(t) = (\tilde{w}_1(t), \tilde{w}_2(t)).$$

We consider the following $2N$ ordinary differential equation

$$\begin{cases} w'_{1,j} = d_{1,j}\left(\sum\limits_{k\in\mathbb{Z}\setminus 0} \alpha_{k;1,j} w_{1,j+k} - w_{1,j}\right) + w_{1,j}[b_{1,j} - w_{1,j} - \gamma_{1,j} w_{2,j}], \\ w'_{2,j} = d_{2,j}\left(\sum\limits_{k\in\mathbb{Z}\setminus 0} \alpha_{k;2,j} w_{2,j+k} - w_{2,j}\right) + w_{2,j}[b_{2,j} - \gamma_{2,j} w_{1,j} - w_{2,j}], \end{cases} \tag{3.0.3}$$

where $j \in [1, N]_{\mathbb{Z}}$, $w_{i,0}(t) = w_{i,N}(t)$, $w_{i,N+1}(t) = w_{i,1}(t)$, $d_{i,N+1} = d_{i,1}$ with $i \in \{1, 2\}$ for $\tilde{w}_1(0) = x \in \mathbb{R}^N$, $\tilde{w}_2(0) = y \in \mathbb{R}^N$. By observation, it is easy to verify that for all $j \in [1, N]_{\mathbb{Z}}$ and $i \in \{1, 2\}$, there is $z_{i,j} \in \mathbb{R}$ such that the righthand side of system (3.0.3) is smaller or equal to 0, that is,

$$d_{1,j}\left(\sum_{k\in\mathbb{Z}\setminus 0} \alpha_{k;1,j} z_{1,j+k} - z_{1,j}\right) + z_{1,j}[b_{1,j} - z_{1,j} - \gamma_{1,j} z_{2,j}] \leqslant 0$$

and

$$d_{2,j}\left(\sum_{k\in\mathbb{Z}\setminus 0} \alpha_{k;2,j} z_{2,j+k} - z_{2,j}\right) + z_{2,j}[b_{2,j} - \gamma_{2,j} z_{1,j} - z_{2,j}] \leqslant 0.$$

Then we let $\tilde{Z}_1 = (z_{1,1}, z_{1,2}, \cdots, z_{1,N})$, $\tilde{Z}_2 = (z_{2,1}, z_{2,2}, \cdots, z_{2,N})$ and $\tilde{Z} = [0, \tilde{Z}_1] \times [0, \tilde{Z}_2]$ be an ordered space. For all the elements w in $\tilde{Z}$, it is easy to see the well-posedness of the system (3.0.3). We can utilize the solution of system (3.0.3) $\tilde{w}(t) =$

$(\tilde{w}_1(t), \tilde{w}_2(t))$ to define the solution map as $T_t(x) = \tilde{w}(t, x)$, where $x \in \tilde{Z}$.

Then we consider the following system,

$$w'_{1,j} = d_{1,j}\left(\sum_{k\in\mathbb{Z}\backslash 0} \alpha_{k;1,j} w_{1,j+k} - w_{1,j}\right) + w_{1,j}[b_{1,j} - w_{1,j}],\ \tilde{w}_1(0) = x \in \mathbb{R}^N,\ j \in [1, N]_{\mathbb{Z}}, \tag{3.0.4}$$

and

$$w'_{2,j} = d_{2,j}\left(\sum_{k\in\mathbb{Z}\backslash 0} \alpha_{k;2,j} w_{2,j+k} - w_{2,j}\right) + w_{2,j}[b_{2,j} - w_{2,j}],\ \tilde{w}_2(0) = y \in \mathbb{R}^N,\ j \in [1, N]_{\mathbb{Z}}, \tag{3.0.5}$$

where $w_{i,0}(t) = w_{i,N}(t)$, $w_{i,N+1}(t) = w_{i,1}(t)$, $d_{i,N+1} = d_{i,1}$ with $i \in \{1, 2\}$. Let

$$p_{i,j}(w_i) = d_{i,j}\left(\sum_{k\in\mathbb{Z}\backslash 0} \alpha_{k;i,j} w_{i,j+k} - w_{i,j}\right) + w_{i,j}[b_{i,j} - w_{i,j}],$$

$P_i = (p_{ij})_{j\in\mathbb{Z}}$ and $\tilde{P}_i = (p_{ij})_{j\in[1,N]_{\mathbb{Z}}}$ with $i \in \{1, 2\}$. Set $A_i = D\tilde{P}_i(0)$. In order to get the two semi-trivial equilibria, we compare the coefficients of (3.0.4), (3.0.5) with those in (3.0.2) and do the similar arguments for system (3.0.2). Motivated by Lemma 3.0.2, we suppose that

(H1) $s(A_i) > 0$ with $i \in \{1, 2\}$.

Therefore we obtain that there exist two positive equilibria $\tilde{w}_1^*$ and $\tilde{w}_2^*$ in $\mathbb{R}^N$ such that $E_1 := (w_1^*, 0)$ and $E_2 := (0, w_2^*)$ are two semi-trivial equilibria for system (1.0.2), where $w_i^* \in Y$, $w_{i,j}^* = \tilde{w}_{i,j}^*$ for $i \in \{1, 2\}$ and $j \in [1, N]_{\mathbb{Z}}$. Then we consider the following system,

$$w'_{1,j} = d_{1,j}\left(\sum_{k\in\mathbb{Z}\backslash 0} \alpha_{k;1,j} w_{1,j+k} - w_{1,j}\right) + w_{1,j}[b_{1,j} - w_{1,j} - \gamma_{1,j}\tilde{w}_{2,j}^*], \tilde{w}_1(0) = x \in \mathbb{R}^N,$$

where $j \in [1, N]_{\mathbb{Z}}$, $w_{1,0}(t) = w_{1,N}(t)$, $w_{1,N+1}(t) = w_{1,1}(t)$, $d_{1,N+1} = d_{1,1}$. Let

$$p_{3j}(w_1) = d_{1,j}\left(\sum_{k\in\mathbb{Z}\setminus 0} \alpha_{k;1,j} w_{1,j+k} - w_{1,j}\right) + w_{1,j}[b_{1,j} - w_{1,j} - \gamma_{1,j}\tilde{w}^*_{2,j}],\ j \in \mathbb{Z},$$

$P_3 = (p_{3j})_{j\in\mathbb{Z}}$ and $\tilde{P}_3 = (p_{3j})_{j\in[1,N]_{\mathbb{Z}}}$. Set $B_1 = D\tilde{P}_3(0)$. We assume that

(H2) $s(B_1) > 0$,

which implies that E_2 linearly unstable. We assume that

(H3) there is no positive equilibrium for system (3.0.3) in $[0, \tilde{w}^*_1] \times [0, \tilde{w}^*_2]$.

For system (3.0.3) and the continuous semiflow T_t, we can verify that it satisfies [14, (H1)-(H4)]. Then we have the following result on the global stability of $E_1 = (\tilde{w}^*_1, 0)$.

Theorem 3.0.3. *If (H1)-(H3) holds, then* $E_1 = (\tilde{w}^*_1, 0)$ *is globally asymptotically stable for all the initial value* $\phi = (\phi_1, \phi_2) \in \mathbb{P}_+ \setminus \{0\}$.

Proof. Let $w(t, \phi)$ be the solution of (1.0.1) with $w_j(0, \phi) = \phi_j$. By virtue of $(A2)$, there is a real positive number $\epsilon_0 \in (0, s(B_1))$. Note that $f_j(w) = b_{1,j} - w_{1,j}(t) - \gamma_{1,j}w_{2,j}(t)$ is uniformly continuous on the set $[0, 1] \times [0, b]$ with $j \in [1, N]_{\mathbb{Z}}$, where $b = \max_{j\in\mathbb{Z}} w^*_{2,j} + 1$. There is $\delta_0 > 0$ such that

$$|f_j(u) - f_j(v)| < \epsilon_0,\ \forall u = (u_1, u_2),\ v = (v_1, v_2) \in [0, 1] \times [0, b], j \in \mathbb{Z},$$

for $|u_i - v_i| < \delta_0$, $i = 1, 2$. Then we have the following result.

Claim. $\limsup_{t\to\infty} \|w(t, \phi) - (0, w^*_2)\|_{\mathbb{P}} \geqslant \delta_0$ for any $\phi \in \mathbb{P}_+$ with $\phi_1 \not\equiv 0$.

By way of contradiction, suppose $\limsup_{t\to\infty} \|w(t, \hat{\phi}) - (0, w^*_2)\|_{\mathbb{P}} < \delta_0$. Then there exists $\hat{\phi} \in \mathbb{P}_+$ with $\hat{\phi}_1 \not\equiv 0$ and $t_0 > 0$ such that

$$\|w_1(t, \hat{\phi})\|_{\mathbb{Y}} < \delta_0,\ \|w_2(t, \hat{\phi}) - w^*_2\|_{\mathbb{Y}} < \delta_0,\ \forall t \geqslant \delta_0.$$

Therefore, we obtain

$$f_{1,j}(w(t, \hat{\phi})) > f_{1,j}(w(t, (0, w^*_2))) - \epsilon_0 = b_{1,j} - \gamma_{1,j}w^*_2(j) - \epsilon_0,\ t \geqslant t_0,\ j \in [1, N]_{\mathbb{Z}}.$$

Let

$$\tilde{F}_1(w) = (f_{11}(w), f_{12}(w), \cdots, f_{1N}(w))$$

and

$$\tilde{L}_1 w = (l_{11}w_1, l_{12}w_1, \cdots, l_{1N}w_1)$$

with $l_{1,j}w_1 = \sum\limits_{k\in\mathbb{Z}\setminus 0} \alpha_{k;1,j}w_{1,j+k} - w_{1,j}$, where $w_{1,N+j}(t) = w_{1,j}(t)$, $j \in \mathbb{Z}$, $d_{1,N+1} = d_{1,1}$. Let $\psi = (\psi_1, \psi_2, \cdots, \psi_N)$ be the eigenvector corresponding to $s(B_1)$. Then we have $B_1 \cdot \psi = s(B_1)\psi$. Since $w_1(0) = \hat{\phi}_1 \not\equiv 0$, we can get the solution of first equation in system (3.0.3) $w_{1,j}(t, \hat{\phi}) > 0$, $\forall j \in [1, N]_{\mathbb{Z}}$. Then there is small $\eta > 0$ such that $w_1(t) > \eta\psi \gg 0$. Therefore, $w_{1,j}(t, \hat{\phi})$ satisfies

$$\begin{cases} w_{1,j}' \geqslant l_{1,j}w_1 + w_{1,j}(b_{1,j} - \gamma_{1,j}w_{2,j}^* - \epsilon_0), \ t \geqslant t_0, j \in [1, N]_{\mathbb{Z}}, \\ w_1(t_0) \geqslant \eta\psi. \end{cases} \tag{3.0.6}$$

From (3.0.6), we obtain that $v_j(t) = \eta \mathrm{e}^{[r(B_1)-\epsilon_0](t-t_0)}\psi_j$, $j \in [1, N]_{\mathbb{Z}}$ satisfies

$$\begin{cases} v_j' = l_{1,j}v + v_j(b_{1,j} - \gamma_{1,j}w_{2,j}^* - \epsilon_0), \ t \geqslant t_0, j \in [1, N]_{\mathbb{Z}}, \\ v_j(t_0) = \eta\psi_j, \ j \in [1, N]_{\mathbb{Z}}. \end{cases} \tag{3.0.7}$$

By (3.0.6), (3.0.7) and the standard comparison principle, we have that

$$w_1(t, \hat{\phi}) \geqslant \eta \mathrm{e}^{[r(B_1)-\epsilon_0](t-t_0)}\psi, t \geqslant t_0.$$

Taking $t \to \infty$, it then follows that $w_1(t, \hat{\phi})$ tends to infinity, a contradiction.

By above claim and $(H3)$, the possibility (c) and (a) can be ruled out in [14, Theorem B]. By the repellence of E_2 in its some neighborhood, [14, Theorem B] shows the global asymptotical stability of E_1 for all initial values $\phi = (\phi_1, \phi_2) \in \mathbb{P}_+$ with $\phi_1 \not\equiv 0$. $\square$

To finish this section, we consider a more specific model. We suppose that there is a positive number $d_1 > 0$, $d_2 > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$ such that $d_{1,j} = d_1$, $d_{2,j} = d_2$, $\gamma_{1,j} = \gamma_1$, $\gamma_{2,j} = \gamma_2$ with $j \in \mathbb{Z}$, where γ_1, γ_2 satisfy

(H) $\gamma_1\gamma_2 \leqslant 1$.

We then have the following system

$$\begin{cases} w'_{1,j} = d_1(\alpha_{1;1,j}w_{1,j+1} + \alpha_{-1;1,j}w_{1,j-1} - w_{1,j}) + w_{1,j}[b_{1,j} - w_{1,j} - \gamma_1 w_{2,j}], \\ w'_{2,j} = d_2(\alpha_{1;2,j}w_{2,j+1} + \alpha_{-1;2,j}w_{2,j-1} - w_{2,j}) + w_{2,j}[b_{2,j} - \gamma_2 w_{1,j} - w_{2,j}]. \end{cases} \tag{3.0.8}$$

By Smith [28, Thorem 4.4.2, Corollary 4.4.3] and Chen et al. [3, Theorem 3.1] for N patches species competition model, we have the following theorem.

Theorem 3.0.4. *If* $(H1)$, $(H2)$ *and* (H) *hold, there is no interior equilibrium for system* (3.0.8) *in* $[0, \tilde{w}_1^*] \times [0, \tilde{w}_2^*]$, *and hence,* E_1 *is globally asymptotically stable for system* (3.0.8).

Chapter 4

Spreading speed and traveling waves

In this chapter, we investigate the spreading speeds and spatially periodic traveling waves for system (1.0.2).

By taking $v_1 = w_1$, $v_2 = w_2^* - w_2$, we obtain the following cooperative system

$$\begin{cases} v_{1,j}' = d_{1,j}\left(\sum_{k\in\mathbb{Z}\setminus 0} \alpha_{k;1,j} v_{1,j+k} - v_{1,j}\right) + v_{1,j}[b_{1,j} - v_{1,j} + \gamma_{1,j}(v_{2,j} - w_{2,j}^*)], \\ v_{2,j}' = d_{2,j}\left(\sum_{k\in\mathbb{Z}\setminus 0} \alpha_{k;2,j} v_{2,j+k} - v_{2,j}\right) \\ \qquad + v_{2,j}[b_{2,j} - \gamma_{2,j} v_{1,j} + (v_{2,j} - 2w_{2,j}^*)] + \gamma_{2,j} v_{1,j}(t) w_{2,j}^*. \end{cases} \tag{4.0.1}$$

The three equilibria of system (1.0.2) become

$$\hat{E}_0 = (0, w_2^*),\ \hat{E}_1 = (w_1^*, w_2^*),\ \hat{E}_2 = (0, 0).$$

Let $\mathcal{C}$ be the set of all two-sided bounded sequences from $\mathbb{Z}$ to $\mathbb{R}^2$ and $\mathcal{C}_+ = \{\phi \in \mathcal{C} : \phi_j \geqslant 0,\ \forall j \in \mathbb{Z}\}$; let β be a strongly positive N-periodic two-sided sequence from $\mathbb{Z}$ to $\mathbb{R}^2$. Set

$$\mathcal{C}_\beta = \{u \in \mathcal{C} : 0 \leqslant u_j \leqslant \beta_j,\ \forall j \in \mathbb{Z}\},\ \mathcal{C}_\beta^P = \{u \in \mathcal{C}_\beta : u_j = u_{j+N}, \forall j \in \mathbb{Z}\}.$$

Let X be all the finite sequences from $[1, N]_{\mathbb{Z}}$ to $\mathbb{R}^2$ equipped with the maximum norm $|\cdot|_X$, and X_+ are all the finite sequences from $[1, N]_{\mathbb{Z}}$ to $\mathbb{R}^2_+$. Let

$$X_\beta = \{u \in X : 0 \leqslant u_j \leqslant \beta_j, \forall j \in [1, N]_{\mathbb{Z}}\}.$$

Let $\mathcal{X} = BC(\mathbb{R}, X)$ be the set containing all the continuous and bounded functions from $\mathbb{R}$ to X, $\mathcal{X}_+ = BC(\mathbb{R}, X_+)$ and $\mathcal{X}_\beta = BC(\mathbb{R}, X_\beta)$. We use the compact open topology on $\mathcal{C}$ and $\mathcal{X}$; that is, $u_n \to u$ in $\mathcal{C}$ means that the sequence of $u_{n,j}$ converges to u_j in $\mathbb{R}^m$ for j in any finite sequence; $u_n \to u$ in $\mathcal{X}$ means that the sequence of $u_n(s)$ converges to $u(s)$ in X for s in any compact set. We equip $\mathcal{C}$ and $\mathcal{X}$ with the norm $\|\cdot\|_{\mathcal{C}}$ and $\|\cdot\|_{\mathcal{X}}$, respectively, which are defined by

$$\|u\|_{\mathcal{C}} = \sum_{k=1}^{\infty} \frac{\max\limits_{|j|\leqslant k} |u_j|}{2^k}, \ \forall u \in \mathcal{C},$$

where $|\cdot|$ denotes the usual norm in $\mathbb{R}^2$ and

$$\|u\|_{\mathcal{X}} = \sum_{k=1}^{\infty} \frac{\max\limits_{|x|\leqslant k} \|u(x)\|_X}{2^k}, \ \forall u \in \mathcal{X}.$$

Let $\beta = (u_1^*, u_2^*)$ and $\mathbb{Y}$ be the set containing all bounded sequences from $\mathbb{Z}$ to $\mathbb{R}$, and $T_1(t)$ and $T_2(t)$ be the uniformly continuous semigroup on $\mathbb{Y}$ generated by the linear lattice equations $v'_{1,j} = d_{1,j}(\sum\limits_{k\in\mathbb{Z}\setminus 0} \alpha_{k;1,j} v_{1,j+k} - v_{1,j}) = l_{1,j}(v)$ and $v'_{2,j} = d_{2,j}(\sum\limits_{k\in\mathbb{Z}\setminus 0} \alpha_{k;2,j} v_{2,j+k} - v_{2,j}) = l_{2,j}(v)$, respectively. It follows that $T_1(t)$ and $T_2(t)$ are compact with respect to the compact open topology for each $t > 0$ (see, e.g., in [34]). For any $v = (v_1, v_2) \in \mathcal{C}_\beta$, define $F : \mathcal{C}_\beta \to \mathcal{C}$, $F = \{F_j(v)\}_{j\in\mathbb{Z}}$ by $F_j(v) = (f_{1,j}(v), f_{2,j}(v))$, where $f_{1,j}(v) = v_{1,j}[b_{1,j} - v_{1,j} + \gamma_{1,j}(v_{2,j} - w^*_{2,j})]$ and $f_{2,j}(v) = v_{2,j}[b_{2,j} - \gamma_{2,j} v_{1,j} + (v_{2,j} - 2w^*_{2,j})] + \gamma_{2,j} v_{1,j}(t) w^*_{2,j}$, for all $j \in \mathbb{Z}$. We write the (4.0.1) into an integral equation

$$\begin{cases} v' = T(t)v(0) + \displaystyle\int_0^t T(t-s)F(v(s))\mathrm{d}s, \ t > 0, \\ v(0) = \phi \in \mathcal{C}_\beta, \end{cases} \tag{4.0.2}$$

where $T(t) = \mathrm{diag}(T_1(t), T_2(t))$.

Note that $F_j(0) = 0$ and there is $\beta \in \mathrm{Int}\mathbb{R}^2$ such that $F_j(\beta) \leqslant 0$, $\forall j \in \mathbb{Z}$. Since $DF_j(\xi)$ is 2×2 matrix with $j \in \mathbb{Z}$, $\xi \in [0, \beta]$, define the norm of $DF_j(\xi)$ as $\|DF_j(\xi)\|_D = \max\left\{|\frac{\partial f_{1,j}(\xi_1)}{\partial v_{1,j}}|, |\frac{\partial f_{1,j}(\xi_2)}{\partial v_{2,j}}|, |\frac{\partial f_{2,j}(\xi_1)}{\partial v_{1,j}}|, |\frac{\partial f_{2,j}(\xi_2)}{\partial v_{2,j}}|\right\}$. Let

$$B := \sup_{\xi \in [0,\beta], j \in \mathbb{Z}} \|DF_j(\xi)\|_D (E_{1,1} + E_{1,2} + E_{2,1} + E_{2,2}) < \infty,$$

where $E_{l,m}$ denote a 2×2 matrix by replacing l, m entry of a zero 2×2 matrix with real number 1, for all $l, m \in \{1, 2\}$. We have the following lemma: motivated by [34, Lemma 4.1, Remark 4.1].

Lemma 4.0.3. *Equation* (4.0.2) *has a unique solution with* $u(0, \phi) = \phi$ *and* $u(t, \phi) \in \mathcal{C}_\beta$ *for all* $t \in [0, \infty)$, *and equation* (4.0.1) *admits the comparison principle.*

Proof. For any real number $h > 0$, $j \in \mathbb{Z}$ and any ϕ, $\psi \in \mathcal{C}_\beta$ with $\phi \geqslant \psi$, we have

$$\phi_j - \psi_j + h(F(\phi) - F(\psi))_j = (I + hDF_j(\xi_j))(\phi_j - \psi_j)$$

for some $\xi \in [\phi, \psi]$. By $|DF_j(\xi))| \leqslant B$, $\forall j \in \mathbb{Z}$, so there is a $h_0 > 0$ such that

$$\phi - \psi + h_0(F(\phi) - F(\psi)) \geqslant 0, \text{in } \mathcal{C},\ \forall h \in [0, h_0],$$

which implies that

$$\mathrm{dist}(\phi - \psi + h(F(\phi) - F(\psi)), \mathcal{C}_+) = 0,\ \forall h \in [0, h_0].$$

Therefore, we have $F : \mathcal{C}_\beta \to \mathcal{C}$ is quasi-monotone. Let $(v^-(t))_j = 0$, $(v^+(t))_j = \beta_j$, $\forall j \in \mathbb{Z}$. Clearly, $[v^-(t), v^+(t)] = \mathcal{C}_\beta$. Letting $\tau = 0$ in [24, Corollary 5], we then obtain the desired conclusion. □

By [34, Theorem 4.1, Remark 4.1], we have the following result.

Theorem 4.0.5. *Let* $Q_t(\phi) = u(t, \phi)$ *for all* $t \geqslant 0$ *and* $\phi \in \mathcal{C}_\beta$. *Then* $\{Q_t\}_{t \geqslant 0}$ *is a monotone semiflow on* $\mathcal{C}_\beta$ *equipped with the compact open topology.*

Proof. Since $F_j(\beta) \leqslant 0$, $\forall j \in \mathbb{Z}$, we can obtain that $Q_t(\mathcal{C}_\beta) \subset \mathcal{C}_\beta$, $\forall t \geqslant 0$. It is easy to verify that $Q_0 = I$ and $Q_t \circ Q_s = Q_{t+s}$, $\forall t, s \geqslant 0$. Therefore, it suffices to show that $Q_t(\phi)$ is jointly continuous at any point $(t_0, \phi) \in [0, \infty) \times \mathcal{C}_\beta$ for compact open topology. By the triangle inequality, we have

$$\|Q_t(\phi) - Q_{t_0}(\psi)\| \leqslant \|Q_t(\phi) - Q_t(\psi)\| + \|Q_t(\psi) - Q_{t_0}(\phi)\|.$$

Thus we only need to show that $Q_t(\psi)$ is continuous at ψ for $t \in [0, t_0 + 1]$.

Since $d_{1,j}$ and $d_{2,j}$ are N-periodic in j, we can take $d = \max\limits_{j\in[1,N]_{\mathbb{Z}}} \{d_{1,j}, d_{2,j}\}$. For linear differential equation $w'(t) = Lw(t) + Bw(t)$, it is easy to see that the operator $L + B$ is bounded, where $(Lw)_j = L_j w$, $L_j = \mathrm{diag}(l_{1,j}, l_{2,j})$ and $(Bw)_j = Bw_j$ for $j \in \mathbb{Z}$. In fact,

$$\begin{aligned}\|(Lw + Bw)_j\|_{\mathbb{P}} \leqslant&\|Lw_j\|_{\mathbb{P}} + \|Bw_j\|_{\mathbb{P}}\\ \leqslant&4d\|w\|_\infty + 2\sqrt{2}\|w\|_\infty\\ \leqslant&(4d + 2\sqrt{2})\|w\|_\infty,\end{aligned}$$

so we yield that $L + B$ is a bounded operator. For any $\phi, \psi \in \mathcal{C}_\beta$, we consider the following equation

$$\overline{\phi}_j = \max\{\phi_j, \psi_j\},\ \underline{\phi}_j = \min\{\phi_j, \psi_j\},\ \forall j \in \mathbb{Z}.$$

Clearly we have $\overline{\phi} = \{\overline{\phi}_j\}_{j\in\mathbb{Z}}$, $\underline{\phi} = \{\underline{\phi}_j\}_{j\in\mathbb{Z}} \in \mathcal{C}_\beta$. Let $w(t) = u(t, \overline{\phi}) - u(t, \underline{\phi})$. Then we have

$$w'(t) \leqslant Lw(t) + Bw(t), \quad \forall t \geqslant 0.$$

Following from comparison principle, we obtain that

$$0 \leqslant w(t) \leqslant \sum_{k=0}^{k=\infty} \frac{t^k}{k!}(L + B)^k w(0),\ \forall t \geqslant 0,$$

so

$$\begin{aligned}0 \leqslant&(u(t, \overline{\phi}) - u(t, \underline{\phi}))_j\\ \leqslant&\left(\sum_{k=0}^{k=\infty} \frac{\bar{t}^k}{k!}(L + B)^k(\overline{\phi} - \underline{\phi})\right)_j\\ \leqslant&\left(\sum_{k=N+1}^{k=\infty} \frac{\bar{t}^k}{k!}\|(L + B)^k\|_\infty \cdot 2\beta\right) + \left(\sum_{k=0}^{k=N} \frac{\bar{t}^k}{k!}(L + B)^k(w(0))\right)_j,\end{aligned}$$

where $\forall j \in \mathbb{Z}$, $t \in [0, \bar{t}], \bar{t} := t_0 + 1$. Indeed $u(t, \underline{\phi}) \leqslant u(t, \phi)$, $u(t, \psi) \leqslant u(t, \overline{\phi})$, $\forall t \geqslant 0$, then we have

$$\begin{aligned} & |u_j(t,\phi) - u_j(t,\psi)| \\ & \leqslant u_j(t,\overline{\phi}) - u_j(t,\underline{\phi}) \\ & \leqslant \max_{0\leqslant t\leqslant \bar{t}} \left(\sum_{k=N+1}^{k=\infty} \frac{(\bar{t}\|L+B\|_\infty)^k}{k!} \cdot 2\beta \right) \\ & + \max_{0\leqslant t\leqslant \bar{t}} \left(\sum_{k=0}^{k=N} \frac{\bar{t}^k}{k!} (L+B)^k (w(0)) \right)_j, \ \forall t \in [0,\bar{t}],\ j \in \mathbb{Z}. \end{aligned} \tag{4.0.3}$$

Since $\sum_{k=0}^{\infty} \frac{(\bar{t}\|L+B\|_\infty)^k}{k!} = \mathrm{e}^{\bar{t}\|L+B\|_\infty}$, for any $\epsilon > 0$, there is $N = N(\epsilon) > 0$ such that

$$\max_{0\leqslant t\leqslant \bar{t}} \left(\sum_{k=N+1}^{k=\infty} \frac{(\bar{t}\|L+B\|_\infty)^k}{k!} \cdot 2\beta \right) < \frac{\epsilon}{2}. \tag{4.0.4}$$

By [34, Lemma 3.2], we obtain that the map $(L+B)^k : \mathcal{C}_{2\beta} \to \mathcal{C}$ is continuous at $\phi = 0$ under compact open topology for any integer $k \geqslant 0$. By [34, Lemma 2.2] with $\mathcal{H} = \mathbb{Z}$, we can get that there is $\delta = \delta(\epsilon, M) > 0$ and an integer $K = K(\epsilon) > 0$ such that

$$\max_{0\leqslant t\leqslant \bar{t}} \left(\sum_{k=0}^{k=N} \frac{\bar{t}^k}{k!} (L+B)^k (w(0)) \right)_j, \forall j \in [-M, M]_{\mathbb{Z}}, \tag{4.0.5}$$

for all $\phi \in \mathcal{C}_\beta$ with $\|\phi_j\|_{\mathbb{P}} < \delta$, $\forall j \in [-M, M]_{\mathbb{Z}}$, any $M > 0$ and above ϵ . Notice that for any $\phi \in \mathcal{C}_\beta$, we can obtain that $\overline{\phi} - \underline{\phi} \in \mathcal{C}_{2\beta}$ with $\overline{\phi}_j - \underline{\phi}_j = |\phi_j - \psi_j|$, $\forall j \in \mathbb{Z}$. Therefore, together with inequality (4.0.3)-(4.0.5), we get

$$|u_j(t,\phi) - u_j(t,\psi)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \forall j \in [-M, M]_{\mathbb{Z}},\ t \in [0, \bar{t}],$$

for all $\phi \in \mathcal{C}_{2\beta}$ with $\|\phi_j - \psi_j\|_{\mathbb{P}} < \delta$, $\forall j \in [-K, K]_{\mathbb{Z}}$. Thus, by [34, Lemma 2.2], it then follows that $Q_t(\psi)$ is continuous at ψ uniformly for $t \in [0, t_0 + 1]$. □

We say $V_j(j - ct)$ is an N-periodic rightward traveling wave of system (4.0.1) if $\{V_j(j - ct)\}_{j\in\mathbb{Z}} \in \mathcal{C}_\beta$, $u_j(t, V(\cdot)) = V_j(j - ct)$, $\forall t \geqslant 0$, and $V_j(\xi)$ is an N-periodic function in j for any fixed $\xi \in \mathbb{R}$. Moreover, we say that $V_j(\xi)$ connect β to 0 if $\lim\limits_{\xi\to-\infty} |V_j(\xi) - \beta_j| = 0$ and $\lim\limits_{\xi\to\infty} |V_j(\xi)| = 0$ uniformly for $j \in \mathbb{Z}$.

Definition 4.0.1. *A function $u_j(t)$ is said to be an upper(a lower) solution of system* (4.0.1) *if it satisfies*

$$u(t) \geqslant (\leqslant) T(t)u(0) + \int_0^t T(t-s)F(u(s))\mathrm{d}s,\ t \geqslant 0.$$

Note that $u_j(t, \phi)$ is a solution of (4.0.1), so is $u_{j-a}(t, \phi)$, $\forall a \in N\mathbb{Z}$, which implies that $(A1)$ holds. By Theorem 3.0.3, we see that $(A4)$ holds for Q_t with $t > 0$. Since $T(t)$ is compact with respect to the compact open topology for each $t > 0$, $(A2)$, $(A3)$ and $(A5)$ then follow from the similar arguments as in [34]. Thus, we have the following observation.

Proposition 4.0.3. *If* $(H1)$*-*$(H3)$ *hold, then for each* $t > 0$, Q_t *satisfies assumptions* $(A1)$*-*$(A5)$ *in Chapter* 2*.*

By utilizing $\{Q_t\}_{t\geqslant 0}$, we can define a family of operators $\{\hat{Q}_t\}_{t\geqslant 0}$ on $\mathcal{X}_\beta$

$$\hat{Q}_t[v](s)_m := Q(v_s)_m,\ \forall v \in \mathcal{X}_\beta,\ s \in \mathbb{R},\ m \in [1, N]_{\mathbb{Z}},\ t \geqslant 0,$$

where $v_s \in \mathcal{C}$ is defined by

$$(v_s)_j = v(s + n_j)_{m_j},\ \forall j = n_j + m_j \in \mathbb{Z},\ n_j = N[\frac{j-1}{N}],\ m_j \in [1, N]_{\mathbb{Z}},\ v \in \mathcal{X}.$$

By Proposition 2.0.1, we can obtain that $\{\hat{Q}_t\}_{t\geqslant 0}$ is a monotone semiflow on $\mathcal{X}_\beta$ and $(B1)$-$(B5)$ hold for $\hat{Q}_t$ in Chapter 2 for each $t > 0$. Now we use the method in Chapter 2 with $m = 2$. Let c_+^*, $\bar{c}_+$ be the form in (2.0.4) with $\tilde{P} = \tilde{Q}_1$. If we want to prove that the minimal wave speed for N-periodic rightward traveling waves of system (4.0.1) connecting β to 0 is $\bar{c}_+$, we need the following assumption:

(H4) $c_{1+}^* + c_{2-}^* > 0$, where c_{1+}^* and c_{2-}^* are the rightward and leftward spreading speeds of (4.0.6) and (4.0.8), respectively.

Theorem 4.0.6. *If* $(H1)$*-*$(H4)$ *hold, then for any* $c \geqslant \bar{c}_+$*, system* (4.0.1) *admits an* N*-periodic rightward traveling wave* $(U_j(j-ct), V_j(j-ct))$ *connecting* β *to* 0*, with wave profile components* $U_j(\xi)$ *and* $V_j(\xi)$ *being continuous and non-increasing in* ξ*; for any* $c < \bar{c}_+$*, there is no such traveling wave connecting* β *to* 0*.*

Proof. By Theorem 2.0.2 (2) and (3), it suffices to prove that the second case in Theorem 2.0.2 (2) does not hold. By way of contradiction, we assume that the statement in Theorem 2.0.2 (2)(ii) holds for some $c \geqslant \bar{c}_+$. Since system (4.0.1) has three N-periodic nonnegative equilibria points and $\hat{E}_0 = (0, u_2^*)$ is the only intermediate N-periodic equilibrium point between $\hat{E}_1 = \beta$ and $\hat{E}_2 = 0$, we obtain $\alpha_1 = \alpha_2 = \hat{E}_0$. Therefore, we consider system (4.0.1) on the order interval $[\hat{E}_0, \hat{E}_1]$ and $[\hat{E}_2, \hat{E}_0]$, respectively. The one spices equation

$$u_j' = l_{1,j}u + u_j(b_{1,j} - u_j),\ j \in \mathbb{Z}, \tag{4.0.6}$$

yields an N-periodic traveling wave $U_j(j - ct)$ connecting u_1^* to 0 with $U_j(\xi)$ being continuous and nonincreasing in ξ, and the other equation

$$v_j' = l_{2,j}v_j + v_j(b_{2,j} - 2w_{2,j}^* + v_j),\ j \in \mathbb{Z}, \tag{4.0.7}$$

also yields an N-periodic traveling wave $V_j(j - ct)$ connecting u_2^* to 0 with $V_j(\xi)$ being continuous and nonincreasing in ξ.

Let $W_j(j - ct) = w_{2,j}^* - V_j(j - ct)$ in (4.0.7), so $W_j(j - ct)$ is an N-periodic traveling wave connecting 0 to w_2^* with $W_j(\xi)$ being continuous and nondecreasing in ξ

$$w_j' = l_{2,j}w_j + w_j(b_{2,j} - w_j),\ j \in \mathbb{Z}. \tag{4.0.8}$$

$W_j(j - ct)$ is an N-periodic leftward traveling wave connecting 0 to u_2^* with wave speed $-c$, and assume that (4.0.6) and (4.0.8) yield rightward spreading speed c_{1+}^* and leftward spreading speed c_{2-}^*, respectively, which are also the rightward and leftward minimal wave speed (see, e.g., and [23, Theorem 5.3]). Then we obtain that $c \geqslant c_{1+}^*$ and $-c \geqslant c_{2-}^*$, which implies $c_{1+}^* + c_{2-}^* \leqslant 0$. □

Let $s(M_2(\mu))$ be the stability modulus of the matrix $M_2(\mu)$, $M_2(\mu) = A_2(\mu) + D_2$ and $A_2(\mu) := [a_{2\mu;i,j}]$, where

$$\begin{cases} a_{2\mu;j,j} = -d_{2,j},\ j = 1, \cdots, N, \\ a_{2\mu;j,j+l} = d_{2,j} \displaystyle\sum_{k-l \in N\mathbb{Z}_+} \alpha_{k;2,j} \mathrm{e}^{-k\mu},\ j = 1, \cdots, N,\ j + l \leqslant N,\ l \in \mathbb{N}, \\ a_{2\mu;j+l,j} = d_{2,j} \displaystyle\sum_{k-l \in N\mathbb{Z}_+} \alpha_{-k;2,j} \mathrm{e}^{k\mu},\ j = 1, \cdots, N,\ j + l \leqslant N,\ l \in \mathbb{N}. \end{cases}$$

Let $D_2 := [d_{2;i,j}]$ be the $N \times N$ diagonal matrix with $d_{2;j,j} = b_{2,j} - w^*_{2,j}$ for all $j = 1, \cdots, N$. To show that system (4.0.1) yields a single rightward spreading speed, we assume:

(H5) $\limsup\limits_{u \to 0^+} \frac{s(M_2(\mu))}{\mu} \leqslant c^*_{1+}$, where c^*_{1+} is the spreading speed of (4.0.6).

Theorem 4.0.7. *If* (*H*1)*-*(*H*5) *hold, then the following statements are valid for system* (4.0.1)*:*

(i) If $\phi \in \mathcal{C}_\beta$, $0 \leqslant \phi \leqslant \omega \ll \beta$ for some $\omega \in \mathcal{C}^P_\beta$, and $\phi_j = 0$, $\forall j \geqslant H$, for some $H \in \mathbb{Z}$, then $\lim\limits_{t\to\infty, j\geqslant ct} u_j(t, \phi) = 0$ for any $c > \bar{c}_+$.

(ii) If $\phi \in \mathcal{C}_\beta$ and $\phi_j \geqslant \sigma$, $\forall j \leqslant K$ for some $\sigma \in \mathbb{R}^2$ with $\sigma \gg 0$ and $K \in \mathbb{Z}$, then $\lim\limits_{t\to\infty, j\leqslant ct}(u_j(t, \phi) - \beta_j) = 0$ for all $c < \bar{c}_+$.

Proof. In view of Theorem 2.0.1, it is enough to prove that $\bar{c}_+ = c^*_+$. If this does not hold, then $\bar{c}_+ > c^*_+$ according to the definition of $\bar{c}_+$ and c^*_+. By (1) and (3) of Theorem 2.0.2, we obtain that system (4.0.1) yields an N-periodic traveling wave $(U_{1,j}(j - c^*_+), U_{2,j}(j - c^*_+))$ connecting (w^*_1, w^*_2) to $(0, w^*_2)$ with $U_{i,j}(\xi)(i = 1, 2)$ being continuous and nonincreasing in ξ. Then, $U_2 \equiv u^*_2$, and $U_{1,j}(j - c^*_+ t)$ is an N-periodic traveling wave connecting u^*_1 to 0, which shows that $c^*_+ \geqslant c^*_{1+}$ with c^*_{1+} being the rightward spreading speed of (4.0.6). Let $s(M_1(\mu))$ be the stability modulus of the matrix $M_1(\mu)$, $M_1(\mu) = A_1(\mu) + D_1$ and $A_1(\mu) := [a_{1\mu;i,j}]$, where

$$\begin{cases} a_{1\mu;j,j} = -d_{1,j},\ j = 1, \cdots, N, \\ a_{1\mu;j,j+l} = d_{1,j} \displaystyle\sum_{k-l\in N\mathbb{Z}_+} \alpha_{k;1,j} \mathrm{e}^{-k\mu},\ j = 1, \cdots, N,\ j + l \leqslant N,\ l \in \mathbb{N}, \\ a_{1\mu;j+l,j} = d_{1,j} \displaystyle\sum_{k-l\in N\mathbb{Z}_+} \alpha_{-k;1,j} \mathrm{e}^{k\mu},\ j = 1, \cdots, N,\ j + l \leqslant N,\ l \in \mathbb{N}. \end{cases}$$

Let $D_1 := [d_{1;i,j}]$ be the $N \times N$ diagonal matrix with $d_{1;j,j} = b_{1,j}$ for all $j = 1, \cdots, N$. It is easy to verify that (4.0.6) satisfies the conditions in [12], so we have $c^*_{1+} = \min\limits_{\mu>0} \frac{s(M_1(\mu))}{\mu}$. For any $c_1 \in (c^*_+, \bar{c}_+)$, there is $\mu_1 > 0$ such that $c_1 = \frac{s(M_1(\mu_1))}{\mu_1}$. Let ϕ^*_1 be the N-periodic positive eigenvector associated with the stability modulus $s(M_1(\mu_1))$

of matrix $M_1(\mu_1)$. Then we have that

$$u_{1,j}(t) := \mathrm{e}^{-\mu_1(j-c_1t)}\phi^*_{1,j} = \mathrm{e}^{-\mu_1 j}\mathrm{e}^{s(M_1(\mu_1))}\phi^*_{1,j},\ t > 0,\ j \in \mathbb{Z},$$

is a solution of the ordinary linear equation

$$w'_{1,j} = l_{1,j}w_{1,j} + u_{1,j}b_{1,j},\ j \in \mathbb{Z}.$$

Because of $c^*_{1+} < c_1$ and $(H5)$, it then follows that there is a smaller number $\mu_2 \in (0, \mu_1)$ such that $c_2 := \frac{s(M_2(\mu_2))}{\mu_2} < c^*_{1+}$. Let ϕ^*_2 be the N-periodic positive eigenvector associated with the stability modulus $s(M_2(\mu_2))$ of matrix $M_2(\mu_2)$. Then we can obtain that

$$u_{2,j}(t) := \mathrm{e}^{-\mu_2(j-c_2t)}\phi^*_{2,j} = \mathrm{e}^{-\mu_2 j}\mathrm{e}^{s(M(\mu_2))t}\phi^*_{2,j},\ t > 0,\ j \in \mathbb{Z},$$

is a solution of the ordinary linear equation

$$w'_{2,j} = l_{2,j}w_{2,j} + w_{2,j}(b_{2,j} - \phi^*_{2,j}),\ j \in \mathbb{Z}.$$

Since $c_1 > c_2$, we can obtain that

$$v_{2,j}(t) := \mathrm{e}^{-\mu_2(j-c_1t)}\phi^*_{2,j} = \mathrm{e}^{\mu_2(c_1-c_2)t}w_{2,j}(t),\ t \geqslant 0,\ j \in \mathbb{Z},$$

satisfies

$$v'_{2,j} \geqslant l_{2,j}v_{2,j} + v_{2,j}(b_{2,j} - \phi^*_{2.j}),\ j \in \mathbb{Z}. \tag{4.0.9}$$

Define the two wave-like functions:

$$\bar{w}_{1,j}(t) := \min\{m_0\mathrm{e}^{-\mu_1(j-c_1t)}\phi^*_{1,j}, w^*_{1,j}\},\ t \geqslant 0,\ j \in \mathbb{Z}$$

and

$$\bar{w}_{2,j}(t) := \min\{q_0\mathrm{e}^{-\mu_2(j-c_1t)}\phi^*_{2,j}, w^*_{2,j}\},\ t \geqslant 0,\ j \in \mathbb{Z},$$

where

$$q_0 = \max_{j\in[1,N]_{\mathbb{Z}}} \frac{w^*_{2,j}}{\phi^*_{2,j}},\ m_0 = \min_{j\in[1,N]_{\mathbb{Z}}} \frac{q_0\phi^*_{2,j}}{\gamma_{2,j}\phi^*_{2,j}}.$$

Then we need to verify that $(\bar{w}_1, \bar{w}_2)$ is an upper solution of (4.0.1). In fact, for all

$j - c_1 t > \frac{1}{\mu_1} \ln(\frac{m_0 \phi^*_{1,j}}{w^*_{1,j}})$, it is easy to see that $\bar{w}_{1,j}(t) = m_0 \mathrm{e}^{-\mu_1(j-c_1t)} \phi^*_{1,j}$, and hence,

$$\begin{aligned}
&\bar{w}'_{1,j} - l_{1,j}\bar{w}_{1,j} - \bar{w}_{1,j}(b_{1,j} - \gamma_{1,j}u^*_{2,j} - \bar{w}_{1,j} + \gamma_{1,j}\bar{w}_{2,j}) \\
&\geqslant \bar{w}'_{1,j} - l_{1,j}w_{1,j} - \bar{w}_{1,j}b_{1,j} = 0,\ j \in \mathbb{Z}.
\end{aligned}$$

For all $j - c_1 t < \frac{1}{\mu_1} \ln(\frac{m_0 \phi^*_{1,j}}{w^*_{1,j}})$, we have $\bar{w}_{1,j}(t) = w^*_{1,j}$, so

$$\begin{aligned}
&\bar{w}'_{1,j} - l_{1,j}\bar{w}_{1,j} - \bar{w}_{1,j}(b_{1,j} - a_{1,j}w^*_{2,j} - \bar{w}_{1,j} + \gamma_{1,j}\bar{u}_{2,j}) \\
&\geqslant \bar{w}'_{1,j} - l_{1,j}w_{1,j} - \bar{w}_{1,j}(b_{1,j} - \bar{w}_{1,j}) = 0,\ j \in \mathbb{Z}.
\end{aligned}$$

For the other equation and all $j - c_1 t > \frac{1}{\mu_2} \ln(\frac{q_0 \phi^*_{2,j}}{w^*_{2,j}}) > 0$, we have

$$\bar{w}_{2,j}(t) = m_0 \mathrm{e}^{-\mu_2(j-c_1t)} \phi^*_{2,j},$$

which satisfies inequality (4.0.9). By utilizing

$$\bar{w}_{1,j}(t) \leqslant m_0 \mathrm{e}^{-\mu_1(j-c_1t)} \phi^*_{1,j}, t \geqslant 0, j \in \mathbb{Z},$$

and $\mu_2 \in (0, \mu_1)$, it then follows that

$$\begin{aligned}
&\bar{w}'_{2,j} - l_{2,j}\bar{w}_{2,j} - \bar{w}_{2,j}(b_{2,j} - 2w^*_{2,j} + \bar{w}_{2,j}) - \gamma_{2,j}\bar{w}_{1,j}(w^*_{2,j} - \bar{w}_{2,j}) \\
&= \bar{w}'_{2,j} - l_{2,j}\bar{w}_{2,j} - \bar{w}_{2,j}(b_{2,j} - u^*_{2,j}) + (\bar{u}_{2,j} - \gamma_{2,j}\bar{w}_{1,j})(w^*_{2,j} - \bar{w}_{2,j}) \\
&\geqslant (w^*_{2,j} - \bar{w}_{2,j})\mathrm{e}^{-\mu_1(j-c_1t)} \phi^*_{1,j}\gamma_{2,j}(\frac{q_0\phi^*_{2,j}}{\gamma_{2,j}\phi^*_{2,j}} - m_0) \\
&\geqslant 0,\ j \in \mathbb{Z}.
\end{aligned}$$

For all $j - c_1 t < \frac{1}{\mu_2} \ln(\frac{q_0 \phi^*_{2,j}}{w^*_{2,j}})$, we have $\bar{w}_{2,j}(t) = w^*_{2,j}$, which implies

$$\begin{aligned}
&\bar{w}'_{2,j} - l_{2,j}\bar{w}_{2,j} - \bar{w}_{2,j}(b_{2,j} - 2w^*_{2,j} + \bar{w}_{2,j}) - \gamma_{2,j}\bar{w}_{1,j}(w^*_{2,j} - \bar{w}_{2,j}) \\
&= -l_{2,j}w^*_{2,j} - w^*_{2,j}(b_{2,j} - w^*_{2,j})) = 0,\ j \in \mathbb{Z}.
\end{aligned}$$

Therefore, we show that $\bar{w} = (\bar{w}_1, \bar{w}_2)$ is a continuous upper solution of system (4.0.1).

Let $\phi \in \mathcal{C}_\beta$ with $\phi_j \geqslant \sigma$, $\forall j \leqslant K$ and $\phi_j = 0$, $\forall j \geqslant H$ for some $\sigma \in \mathbb{R}^2$ with $\sigma \gg 0$ and K, $H \in \mathbb{Z}$. Motivated by [32, Lemma 2.2] and the proof of Theorem 2.0.1, as applied to $\hat{Q}_1$, we can obtain that for any $c < \bar{c}_+$, there is $\delta(c) > 0$ such that

$$\liminf_{n\to\infty, j\leqslant cn} |w_j(n,\phi)| \geqslant \delta > 0. \tag{4.0.10}$$

Further, we can find a sufficiently large positive constant $A \in N\mathbb{Z}$ such that

$$\phi_j \leqslant \bar{w}_{j-A}(0) := \psi_j,\ \forall j \in \mathbb{Z}.$$

By the translation invariance of Q_t, we have that $\bar{w}_{j-A}(t)$ is still an upper solution of system (4.0.1), which means

$$0 \leqslant w_j(t,\phi) \leqslant w_j(t,\psi) \leqslant \bar{w}_{j-A}(t),\ \forall j \in \mathbb{Z},\ \forall t > 0. \tag{4.0.11}$$

Take a rational number $\hat{c} \in (c_1, \bar{c}_+)$. Substituting $t = n_k$, $j = \hat{c}n_k$, where $\{n_k\}_{k=1}^{\infty}$ is a subsequence in $\{n\}_{n=1}^{\infty}$ such that $\hat{c}n_k$ is a integer. Letting $n_k \to \infty$ in (4.0.11), together with (4.0.10), we have

$$0 < \delta(c) \leqslant \limsup_{n_k\to\infty} |w_{\hat{c}n_k}(n_k,\phi)| \leqslant \lim_{n_k\to\infty} |\bar{w}_{\hat{c}n_k - A}(n_k)| = 0,$$

which is a contradiction. Thus $\bar{c}_+ = c_+^*$. □

Note that the leftward case can be addressed similarly. Indeed, by making a change of variable $v_j(t) = w_{-j}(t)$ for system (4.0.1), we obtain a similar result for the rightward case of the resulting system, which is the leftward case of the system (4.0.1).

Remark 4.0.1. *It follows from Lemma* 6.0.4 *that system* (4.0.1) *admits a single rightward spreading speed which coincides with the minimal rightward wave speed provided* $(H1)$-$(H3)$ *hold.*

Chapter 5

Linear determinacy of the spreading speed

In this chapter, we find some sufficient conditions for the rightward spreading speed to be determined by the linearization of system (4.0.1) at $\hat{E} = (0, 0)$:

$$\begin{cases} v'_{1,j} = & d_{1,j}\left(\sum\limits_{k\in\mathbb{Z}\setminus 0} \alpha_{k;1,j} v_{1,j+k} - v_{1,j}\right) + v_{1,j}[b_{1,j} - \gamma_{1,j} w^*_{2,j}], \\ v'_{2,j} = & d_{2,j}\left(\sum\limits_{k\in\mathbb{Z}\setminus 0} \alpha_{k;1,j} v_{2,j+k} - v_{2,j}\right) \\ & +v_{2,j}[b_{2,j} - 2u^*_{2,j})] + \gamma_{2,j} v_{1,j}(t) w^*_{2,j}. \end{cases} \tag{5.0.1}$$

It is clear that if $(H2)$ holds, then the following ordinary equation

$$w' = L_1 w + w(b_{1,j} - \gamma_{1,j} w^*_{2,j} - w),\ t > 0,\ j \in \mathbb{Z}, \tag{5.0.2}$$

admits a rightward spreading speed (by [23, Chapter 7.3], which is also the minimal rightward wave speed). Let $c^0_+ = \min\limits_{\mu>0} \frac{s(M_0(\mu))}{\mu}$, where $s(M_0(\mu))$ is the stability modulus of the matrix $M_0(\mu)$, $M_0(\mu) = A_1(\mu) + D_0$ and $D_0 := [d_{0;i,j}]$ is the $N \times N$ diagonal matrix with $d_{0;j,j} = b_{1,j} - a_{12,j} u^*_{2,j}$ for all $j = 1, \cdots, N$.

Next, we show that c^0_+ is a lower bound of the slowest spreading speed c^*_+ of the system (4.0.1).

Proposition 5.0.4. *Let* $(H1)$*-*$(H3)$ *hold. Then* $c_+^* \geqslant c_+^0$.

Proof. To prove $\bar{c}_+ > c_+^*$, from Theorem 4.0.7, we obtain that $c_+^* \geqslant c_{1+}^*$, where c_{1+}^* is the rightward spreading speed of (4.0.6). Since $M_1(\mu) > M_0(\mu)$, and the off-diagonal entries of $M_1(\mu)$, $M_0(\mu)$ are positive, we have $s(M_1(\mu)) > s(M_0(\mu))$, where $s(M_1(\mu))$ is the stability modulus of matrix $M_0(\mu)$. Thus we have $c_+^* \geqslant c_{1+}^* \geqslant c_+^0$.

For the case $\bar{c}_+ = c_+^*$, let $w(t,\phi) = (w_1(t,\phi), w_2(t,\phi))$ be the solution of system (4.0.1) with $w(0) = \phi = (\phi_1, \phi_2) \in \mathcal{C}_\beta$. Then the positivity of the solution shows

$$w_{1,j}' \geqslant L_1 w_1 + w_{1,j}(b_{1,j} - a_{12}(j) w_{2,j}^* - a_{11}(j) w_{1,j}),\ t > 0, j \in \mathbb{Z}.$$

Let $v_j(t,\phi)$ be the unique solution of (5.0.2) with $v(0) = \phi_1$. By comparison principle, we obtain that

$$w_{1,j}(t,\phi) \geqslant v_j(t,\phi),\ \forall t \geqslant 0,\ j \in \mathbb{Z}. \tag{5.0.3}$$

Since $s(B_1) > 0$, Lemma 3.0.2 shows that there is a unique positive N-periodic equilibrium point $v_{0,j}$ of (5.0.2). Let $\phi^0 = (\phi_1^0, \phi_2^0) \in \mathcal{C}_\beta$, and it satisfies the conditions in (i) and (ii) of Theorem 4.0.7 such that $\phi_1^0 \leqslant v_0$. By way of contradiction, we assume $c_+^* < c_+^0$. Then we can take a rational number $\hat{c} \in (c_+^*, c_+^0)$. Thus, Theorem 4.0.7 implies that $\lim\limits_{t\to\infty,\, j\geqslant \hat{c}t} w_{1,j}(t,\phi^0) = 0$. Let $t = n$, so we have $\lim\limits_{n\to\infty,\, j\geqslant \hat{c}n} w_{1,j}(n,\phi^0) = 0$. Let $(n_k)_{k=1}^\infty$ be a subsequence of $(n)_{n=1}^\infty$ such that $\hat{c}n_k$ is an integer. Thus, we can obtain $\lim\limits_{n_k\to\infty,\, j\geqslant \hat{c}n_k} w_{1,j}(n_k,\phi^0) = 0$. By Theorem 2.0.1, as applied to system (5.0.2), we further obtain $\lim\limits_{t\to\infty,\, j\leqslant \hat{c}t} (v_j(t,\phi^0) - v_{0j}) = 0$. Similar arguments, we have $\lim\limits_{n_k\to\infty,\, j\leqslant \hat{c}n_k} (v_j(n_k,\phi^0) - v_{0j}) = 0$. Since (5.0.3), we have $\lim\limits_{t\to\infty,\, j\geqslant \hat{c}t} v_j(t,\phi^0) = 0$. Analogously we have $\lim\limits_{n_k\to\infty,\, j\geqslant \hat{c}n_k} v_j(n_k,\phi^0) = 0$. However, letting $j = \hat{c}n_k$, we get $\lim\limits_{n_k\to\infty,\, j=\hat{c}n_k} v_j(n_k,\phi^0) = 0$, a contradiction. $\square$

For any given $\mu \in \mathbb{R}$, letting $v_j(t) = \mathrm{e}^{-\mu j} w_j(t)$ in (5.0.1), we then have

$$\begin{cases} w_{1,j}' = d_{1,j}\left(\displaystyle\sum_{k\in\mathbb{Z}\backslash 0} \alpha_{k;1,j} w_{1,j+k} \mathrm{e}^{k\mu} - w_{1,j}\right) + w_{1,j}[b_{1,j} - \gamma_{1,j} w_{2,j}^*], \\ w_{2,j}' = d_{2,j}\left(\displaystyle\sum_{k\in\mathbb{Z}\backslash 0} \alpha_{k;2,j} w_{2,j+k} \mathrm{e}^{k\mu} - w_{2,j}\right) + w_{2,j}[b_{2,j} - 2w_{2,j}^*)] + \gamma_{2,j} w_{1,j}(t) w_{2,j}^*. \end{cases} \tag{5.0.4}$$

Substituting $w_j(t) = \mathrm{e}^{\lambda(t-t_0)}\phi_j$ into (5.0.4), we then obtain the following periodic eigenvalue problem

$$\begin{cases} \lambda\phi_{1,j} = d_{1,j}\left(\sum\limits_{k\in\mathbb{Z}\backslash 0}\alpha_{k;1,j}\phi_{1,j+k}\mathrm{e}^{k\mu} - \phi_{1,j}\right) + \phi_{1,j}[b_{1,j} - \gamma_{1,j}w^*_{2,j}], \\ \lambda\phi_{2,j} = d_{2,j}\left(\sum\limits_{k\in\mathbb{Z}\backslash 0}\alpha_{k;2,j}\phi_{2,j+k}\mathrm{e}^{k\mu} - \phi_{2,j}\right) + \phi_{2,j}[b_{2,j} - 2w^*_{2,j}] + \gamma_{2,j}\phi_{1,j}w^*_{2,j}, \\ \phi_{i,j} = \phi_{i,j+N},\ \forall j\in\mathbb{Z},\ i=1,\,2,\ t>0,\ j\in\mathbb{Z}. \end{cases} \tag{5.0.5}$$

Let $s(\bar{M}(\mu)) = \bar{\lambda}(\mu)$ be the stability modulus of the following eigenvalue problem

$$\begin{cases} \lambda\psi_j = d_{2,j}\left(\sum\limits_{k\in\mathbb{Z}\backslash 0}\alpha_{k;1,j}\psi_{2,j+k}\mathrm{e}^{k\mu} - \psi_{2,j}\right) + \psi_{2,j}[b_{2,j} - 2w^*_{2,j}], \\ \psi_j = \psi_{j+N},\ \forall j\in\mathbb{Z},\ t>0,\ j\in\mathbb{Z}, \end{cases}$$

where $\bar{M}(\mu) = A_2(\mu) + \bar{D}$ and $\bar{D} := [\bar{d}_{i,j}]$ is the $N\times N$ diagonal matrix with $\bar{d}_{j,j} = b_{2,j} - 2w^*_{2,j}$ for all $j = 1,\cdots,N$. Since there exists $\mu_0 > 0$ such that $c^0_+ = \frac{s(M_0(\mu_0))}{\mu_0}$, we impose the following condition:

(D1) $s(M_0(\mu_0)) > s(\bar{M}(\mu_0))$.

Proposition 5.0.5. *If* $(H1)$*-*$(H3)$ *and* $(D1)$ *hold, then the periodic eigenvalue problem* (5.0.5) *with* $\mu = \mu_0$ *has a simple eigenvalue* $s(M_0(\mu_0))$ *associated with a spatially positive* N*-periodic vector* $\phi^* = (\phi^*_1, \phi^*_2)$.

Proof. It is easy to see that there is an N-periodic eigenvector $\phi^*_1 \gg 0$ associated with the stability modulus $s(M_0(\mu_0))$ of matrix $M_0(\mu_0)$. It is enough to prove that $s(M_0(\mu_0))$ has a positive eigenvector $\phi^* = (\phi^*_1, \phi^*_2)$ in (5.0.5), where ϕ^*_2 is to be determined, as the first equation of (5.0.5) is decoupled from the second one. Let $U(t)$ be the solution semigroup generated by the following linear scalar ordinary differential equation

$$\begin{cases} w' = \bar{M}(\mu_0)u, \\ w(0) = \phi\in Y. \end{cases}$$

It is easy to see that $U(t)$ is a positive and compact semigroup on Y with its generator $\bar{M}(\mu_0)$ defined as above. By [25, Theorem 3.12], $\bar{M}(\mu_0)$ is resolvent-positive and

$$(\lambda - \bar{M}(\mu_0))^{-1}\phi = \int_0^\infty \mathrm{e}^{-\lambda t} U(t)\phi \mathrm{d}t,\ \forall \lambda > s(\bar{M}(\mu_0)),\ \phi \in Y.$$

Since $s(M_0(\mu_0)) > s(\bar{M}(\mu_0))$, we have $s(M_0(\mu_0))I - \bar{M}(\mu_0)$ is invertible. Thus we can define $\phi_2^* = [s(M_0(\mu_0))I - \bar{M}(\mu_0)]^{-1}\gamma_2 w_2^* \phi_1^* \gg 0$. It then follows that $\phi^* = (\phi_1^*, \phi_2^*)$ satisfies (5.0.5) with $\mu = \mu_0$. Since $s(M_0(\mu_0))$ is a simple eigenvalue for (5.0.2), so is $s(M_0(\mu_0))$ for (5.0.5). $\square$

From Proposition 5.0.5, we see that for any $M > 0$, the function

$$U_j(t) = M\mathrm{e}^{-\mu_0 j}\mathrm{e}^{s(M_0(\mu_0))t}\phi_j^*,\ t \geqslant 0,\ j \in \mathbb{Z}, \tag{5.0.6}$$

is a positive solution of system (5.0.1). To obtain an explicit formula for the spreading speed $\bar{c}_+$, we need the following condition:

(D2) $\frac{\phi_{1,j}^*}{\phi_{2,j}^*} \geqslant \max\{\gamma_{1,j}, \frac{1}{\gamma_{2,j}}\}$, $j \in \mathbb{Z}$.

We are ready to show that system (4.0.1) yields a single linearly determinate rightward spreading speed $\bar{c}_+$.

Theorem 5.0.8. *If* $(H1)$*-*$(H3)$ *and* $(D1), (D2)$ *are valid, then* $\bar{c}_+ = c_+^* = c_+^0 = \min\limits_{\mu>0} \frac{s(M_0(\mu))}{\mu}$.

Proof. First, we prove that $U_j(t)$ defined in (5.0.6) is an upper solution of system (4.0.1). As $\frac{U_1(t)}{U_2(t)} = \frac{\phi_1^*}{\phi_2^*}$ and $(D2)$ is valid, we obtain that

$$\begin{aligned} U_1' - & L_1 U_1 - U_1(b_1(j) - \gamma_{1,j} w_{2,j}^* - U_1 + \gamma_{1,j} U_2) \\ =& U_1 U_2 \left(\frac{U_1}{U_2} - \gamma_{1,j} \right) \\ =& U_1 U_2 \left(\frac{\phi_1^*}{\phi_2^*} - \gamma_{1,j} \right) \geqslant 0 \end{aligned}$$

and

$$\begin{aligned}
U_2' - & L_2U_2 - U_2[b_2(j) - \gamma_{2,j}U_1 + (U_2 - 2w_{2,j}^*)] - \gamma_{2,j}U_1w_{2,j}^* \\
= & \gamma_{2,j}U_2^2\left(\frac{U_1}{U_2} - \frac{1}{\gamma_{2,j}}\right) \\
= & \gamma_{2,j}U_2^2\left(\frac{\phi_1^*}{\phi_2^*} - \frac{1}{\gamma_{2,j}}\right) \geqslant 0.
\end{aligned}$$

Therefore $U_j(t)$ is an upper solution of (4.0.1). Let $\phi^0 \in \mathcal{C}_\beta$ be the ones in Theorem 4.0.7, (i), (ii), so we can find a sufficiently large number $M_0 > 0$ such that

$$0 \leqslant \phi_j^0 \leqslant M_0 \mathrm{e}^{-\mu_0 j}\phi_j^* = U_j(0),\ j \in \mathbb{Z}.$$

Let $W_j(t)$ be the unique solution of system (4.0.1) with $W(0) = \phi^0$. By the comparison principle and the fact that $c_+^0 w_0 = s(M_0(\mu_0))$, it then follows that

$$0 \leqslant W_j(t) \leqslant U_j(t) = M_0\mathrm{e}^{-\mu_0 j}\mathrm{e}^{s(M_0(\mu_0))t}\phi_j^* = M_0\mathrm{e}^{-\mu_0(j - c_+^0 t)}\phi_j^*,\ j \in \mathbb{Z}.$$

Therefore for any $\epsilon > 0$, we have

$$0 \leqslant W_j(t) \leqslant U_j(t) \leqslant M_0\mathrm{e}^{-\mu_0\epsilon t}\phi_j^*,\ \forall t > 0,\ j \geqslant (c_+^0 + \epsilon)t,$$

then

$$\lim_{t\to\infty, j\geqslant(c_+^0+\epsilon)t} W_j(t) = 0.$$

Following from Theorem 4.0.7 (i), we have $\bar{c}_+ \leqslant c_+^0 + \epsilon$. Letting $\epsilon \to 0$, it then follows that $\bar{c}_+ \leqslant c_+^0$. By contradiction, we assume that $\bar{c}_+ > c_+^*$, but the proof of Proposition 5.0.4 implies that $\bar{c}_+ > c_+^0$, a contradiction. Therefore we have $\bar{c}_+ = c_+^*$. Together with Proposition 5.0.4, we obtain that $c_+^0 = c_+^* = \bar{c}_+$. □

To finish this section, we introduce the lattice version of the Lotka-Volterra competition model

$$\begin{cases} w_{1,j}' = d_1(w_{1,j+1} + w_{1,j-1} - 2w_{1,j}) + r_1w_{1,j}(1 - w_{1,j} - a_1w_{2,j}),\ t \geqslant 0, j \in \mathbb{Z}, \\ w_{2,j}' = d_2(w_{2,j+1} + w_{2,j-1} - 2w_{2,j}) + r_2w_{2,j}(1 - a_2w_{1,j} - w_{2,j}),\ t \geqslant 0, j \in \mathbb{Z}, \end{cases} \tag{5.0.7}$$

where all the parameters are positive constants. This Lotka-Volterra competition model has been investigated in [20], and by straightforward computations, we have a similar result that when $a_1 < 1 \leqslant a_2$, there is no positive equilibrium. Thus, there are three equilibria $E_0(0,0)$, $E_1(1,0)$, $E_1(0,1)$, which implies that $(H3)$ is satisfied. By computation, we have $s(A_1) = r_1 > 0$, $s(A_2) = r_2 > 0$ and $r(B_1) = r_1(1-a_1) > 0$. By replacing the coefficients for system (5.0.7), we see that $(H1)$ and $(H2)$ are also valid. By computation, we can see that $s(M_1(\mu)) > 0$, $c^*_{1+} > 0$. Similarly we have $c^*_{2-} > 0$, where $c^*_{1+} > 0$, $c^*_{2-} > 0$ are defined as in (4.0.6), (4.0.8) by replacing the coefficients for system (5.0.7), which implies that $(H4)$ is satisfied. For $(H5)$, we have $\lim\limits_{\mu \to 0_+} \frac{s(M_2(\lambda))}{\mu} = 0$, which implies that $(H5)$ is satisfied. Therefore system (5.0.7) yields a single spreading speed $\bar{c}_+$ whose llinear determinacy we are not sure of.

Next we give some conditions to make sure $(D1)$-$(D2)$ hold for system (5.0.7). By substituting $d_i(j) = d_i$, $b_i(j) = r_i$, $a_{ii}(j) = r_i$, $a_{12}(j) = r_1 a_1$ and $a_{21}(j) = r_2 a_2$ into system (4.0.1), $i = 1, 2$, we then have $M_0(\mu) = A_1(\mu) + (r_1 - r_1 a_1)I$, $A_1(\mu) := [a_{1\mu;k,j}]$, and $\bar{M}(\mu) = A_2(\mu) - r_2 I$, $A_2(\mu) := [a_{2\mu;k,j}]$, where

$$\begin{cases} a_{1\mu;j,j} = -2d_1,\ j = 1, \cdots, N, \\ a_{1\mu;j,j+1} = d_1 \mathrm{e}^{-\mu},\ j = 1, \cdots, N-1, \\ a_{1\mu;j+1,j} = d_1 \mathrm{e}^{\mu},\ j = 1, \cdots, N-1, \\ a_{1\mu;1,N} = d_1 \mathrm{e}^{\mu},\ a_{1\mu;N,1} = d_1(1) \mathrm{e}^{-\mu}, \\ a_{1\mu;i,j} = 0,\ |i-j| \geqslant 2,\ (i,j) \notin \{(1,N),(N,1)\} \end{cases}$$

and

$$\begin{cases} a_{2\mu;j,j} = -2d_2,\ j = 1, \cdots, N, \\ a_{2\mu;j,j+1} = d_2 \mathrm{e}^{-\mu},\ j = 1, \cdots, N-1, \\ a_{2\mu;j+1,j} = d_2 \mathrm{e}^{\mu},\ j = 1, \cdots, N-1, \\ a_{2\mu;1,N} = d_2 \mathrm{e}^{\mu},\ a_{2\mu;N,1} = d_2 \mathrm{e}^{-\mu}, \\ a_{2\mu;i,j} = 0,\ |i-j| \geqslant 2,\ (i,j) \notin \{(1,N),(N,1)\}. \end{cases}$$

It is easy to see that the two stability modulus

$$s(M_0(\mu)) = r_1(1-a_1) + d_1 \mathrm{e}^{-\mu} + d_1 \mathrm{e}^{\mu} - 2d_1,\ s(\bar{M}(\mu)) = -r_2 + d_2 \mathrm{e}^{-\mu} + d_2 \mathrm{e}^{\mu} - 2d_2$$

has positive constant eigenvectors. By

$$c_+^0 = \min_{\mu>0} \frac{s(M_0(\mu))}{\mu} = \min_{\mu>0} \left\{ \frac{r_1(1-a_1)}{\mu} + \frac{d_1 \mathrm{e}^{-\mu} + d_1 \mathrm{e}^{\mu} - 2d_1}{\mu} \right\},$$

it follows that $c_+^0 = \frac{r_1(1-a_1)}{\mu_0} + \frac{d_1 \mathrm{e}^{-\mu_0} + d_1 \mathrm{e}^{\mu_0} - 2d_1}{\mu_0}$, and μ_0 satisfies the following equation

$$d_1\mu_0(\mathrm{e}^{\mu_0} - \mathrm{e}^{-\mu_0}) = r_1(1-a_1) + d_1\mathrm{e}^{-\mu_0} + d_1\mathrm{e}^{\mu_0} - 2d_1. \tag{5.0.8}$$

Thus, $(D1)$ is equivalent to

$$s(M_0(\mu_0)) = r_1(1-a_1) + d_1(\mathrm{e}^{-\mu_0} + \mathrm{e}^{\mu_0} - 2) > -r_2 + d_2(\mathrm{e}^{-\mu_0} + \mathrm{e}^{\mu_0} - 2) = s(\bar{M}(\mu_0)), \tag{5.0.9}$$

where μ_0 satisfies (5.0.8). Besides, the eigenvalue problem (5.0.5) can be simplified as

$$\begin{cases} \lambda\phi_{1,j} = d_1\phi_{1,j+1}\mathrm{e}^{-\mu} + d_1\phi_{1,j-1}\mathrm{e}^{\mu} - 2d_1\phi_{1,j} + \phi_{1,j}r_1(1-a_1), \\ \lambda\phi_{2,j} = d_2\phi_{2,j+1}\mathrm{e}^{-\mu} + d_2\phi_{2,j-1}\mathrm{e}^{\mu} - 2d_2\phi_{2,j} + r_2a_2\phi_{1,j} - \phi_{2,j}r_2, \\ \phi_{i,j} = \phi_{i,j+N},\ i = 1,\ 2,\ j \in \mathbb{Z}. \end{cases} \tag{5.0.10}$$

Letting $(\phi_1^*, \phi_2^*) = (1, k)$ into the second equation of (5.0.10), we get

$$k = \frac{a_2}{s(M_0(\mu_0)) - s(\bar{M}(\mu_0)) + r_2a_2} > 0.$$

It then follows that $(D2)$ is equivalent to

$$\frac{\phi_1^*}{\phi_2^*} = \frac{s(M_0(\mu_0)) - s(\bar{M}(\mu_0)) + r_2a_2}{a_2} \geqslant \max\left\{a_1, \frac{1}{a_2}\right\},$$

and hence

$$\begin{cases} s(M_0(\mu_0)) - s(\bar{M}(\mu_0)) \geqslant a_2(a_1 - r_2) \\ s(M_0(\mu_0)) - s(\bar{M}(\mu_0)) + r_2a_2 \geqslant 0, \end{cases}$$

that is,

$$s(M_0(\mu_0)) - s(\bar{M}(\mu_0)) \geqslant a_2(a_1 - r_2). \tag{5.0.11}$$

Therefore, by conditions 5.0.9 and 5.0.11, we have

$$\bar{c}_+ = c_+^0 = \frac{r_1(1-a_1)}{\mu_0} + \frac{d_1\mathrm{e}^{-\mu_0} + d_1\mathrm{e}^{\mu_0} - 2d_1}{\mu_0},$$

where μ_0 satisfies (5.0.8).

Remark 5.0.2. *We consider a more general lattice competition model in a periodic habitat:*

$$\begin{cases} u_1' = L_{1,j}w_1 + w_1 f_{1,j}(w_1, w_2), \\ u_2' = L_{2,j}w_2 + w_2 f_{2,j}(w_1, w_2),\ t > 0,\ j \in \mathbb{Z}, \end{cases}$$

where operator $L_{i,j}(\cdot) := d_{i,j}(\sum\limits_{k\in\mathbb{Z}\setminus 0} \alpha_{k;i,j} \cdot_{i,j+k} - \cdot_{i,j})$ with $d_{i,j} > 0$, $\forall j \in \mathbb{Z}$, $i = 1, 2$. Suppose that $d_{i,j}$ and $f_{ij}(w_1, w_2)$ are periodic in j with the same period, and $f_{ij}(w_1, w_2)$ are differentiable with respect to w_1 and w_2, $i = 1, 2$, $j \in \mathbb{Z}$. Furthermore $\partial_{w_1} f_{1,j}(w_1, 0) < 0$ and $\partial_{w_2} f_{2,j}(0, w_2) < 0$, $\forall j \in \mathbb{Z}$, and there are $M_1 > 0$ and $M_2 > 0$ such that $f_{1,j}(M_1, 0) \leqslant 0$, $f_{2,j}(0, M_2) \leqslant 0$, $\partial_{w_2} f_{1,j}(w_1, w_2) < 0$ and $\partial_{w_1} f_{2,j}(w_1, w_2) < 0$ for all $(j, w_1, w_2) \in \mathbb{Z} \times [0, M_1] \times [0, M_2]$. Thus, we can obtain similar conclusions on traveling waves and spreading speeds under analogous assumptions to $(H1)$-$(H5)$ and $(D1)$-$(D2)$.

Chapter 6

An application

In this chapter, we investigate the spatially periodic lattice version of a well-known reaction-diffusion model in [8, 19, 33]:

$$\begin{cases} u_1' = d_1 L_j u_1 + u_1(a_j - u_1 - cu_2), \\ u_2' = d_2 L_j u_2 + u_2(a_j - u_1 - u_2), \ t > 0, \ j \in \mathbb{Z}, \end{cases} \tag{6.0.1}$$

where $L_j \cdot := \cdot_{j+1} + \cdot_{j-1} - 2\cdot_j$, $i = 1, 2$, $j \in \mathbb{Z}$, $0 < d_1 < d_2$, $0 \leqslant c \leqslant 1$ and $\{a_j\}_{j\in\mathbb{Z}}$ is a N-periodic sequence for some positive integer N. Then we will do some numerical simulations for this model. For simplicity, we use the same notation in Sections 3 and 4.

Lemma 6.0.4. *If $(H1)$ and $(H2)$ hold true, then $(H4)$ and $(H5)$ are valid.*

Proof. First, we show that $(H4)$ holds. It follows from [12, Lemma 2.1] that $c_{1+}^* = \min\limits_{\mu>0} \frac{s(M_0(\mu))}{\mu} > 0$. Similarly we also have $c_{2-}^* > 0$, which implies $c_{1+}^* + c_{2-}^* > 0$.

To verify $(H5)$, it suffices to show that $\lim\limits_{\mu\to 0^+} \frac{s(M_2(\mu))}{\mu} = \lim\limits_{\mu\to 0^+} s'(M_2(\mu)) = 0$. From [12, Lemma 2.1], we can get $\lim\limits_{\mu\to 0^+} s'(M_2(\mu)) = 0$. $\square$

Now we impose the following assumption on system (6.0.1)

(M) $\{a_j\}_{j\in\mathbb{Z}}$ is not a constant sequence, and $\bar{a} := \frac{1}{N}\sum\limits_{j=1}^{N} a_j \geqslant 0$.

Lemma 6.0.5. *If (M) holds, then $(H1)$-$(H3)$ hold true.*

Proof. Let ϕ be the positive eigenvector associated with the stability modulus of A_1, that is,

$$d_1(\phi_{j+1}+\phi_{j-1}-2\phi_j)+a_j\phi_j = s(A_1)\phi_j,$$

where $j\in[1,N]_{\mathbb{Z}}$, $\phi_0=\phi_N$, $\phi_{N+1}=\phi_1$. Dividing the equation by ϕ_j and summing both sides of the equation from 1 to N, we get

$$s(A_1)=\frac{1}{N}\sum_{j=1}^{N}a_j-2d_1N+\sum_{j=2}^{N-1}\frac{\phi_{j-1}+\phi_{j+1}}{\phi_j}+\frac{\phi_2+\phi_N}{\phi_1}+\frac{\phi_{N-1}+\phi_1}{\phi_N}.$$

Since a_j is non-constant, by a simple computation, we have ϕ_j also non-constant. Therefore, we have

$$s(A_1)>\frac{1}{N}\sum_{j=1}^{N}a_j-2d_1N\geqslant 0.$$

In fact, it is easy to see that $\sum\limits_{j=2}^{N-1}\frac{\phi_{j-1}+\phi_{j+1}}{\phi_j}+\frac{\phi_2+\phi_N}{\phi_1}+\frac{\phi_{N-1}+\phi_1}{\phi_N}-2d_1N>0$. Similarly, we can show that $s(A_2)>0$, which implies that $(H1)$ holds, so system (6.0.1) has three N-periodic semi-trivial equilibria $E_0:=(0,0)$, $E_1:=(u_{1,j}^*,0)$, $E_2:=(0,u_{2,j}^*)$ in $\mathbb{P}_+$. By observation,

$$d_2L_ju_2^*+u_2^*(a_j-u_{2,j}^*)=0,\ j\in\mathbb{Z},$$

so we have $\lambda(d_2,a-u_2^*)=0$. If $a_j-u_{2,j}^*$, $j\in\mathbb{Z}$ is a constant, by computation, it then follows that u_2^* must be positive constant eigenvector associated with $\lambda(a_2,a-u_2^*)$. Thus, a_j is also a constant, which contradicts the assumption, so we obtain that $a_j-u_{2,j}^*$ is non-constant, where $j\in\mathbb{Z}$.

For the eigenvalue problem (2.0.1) with $d_{1,j}=d$, we set Rayleigh quotient for the stability modulus, so it follows that

$$\lambda(d,h)=\max_{\phi\in\mathbb{R}^N}\frac{-d\sum\limits_{j=1}^{N}(\phi_j-\phi_{j+1})^2+\sum\limits_{j=1}^{N}h_j\phi_j^2}{\sum\limits_{j=1}^{N}\phi_j^2},$$

where $\phi_{N+1}=\phi_1$. It is easy to see that if h_j is non-constant, then $\lambda(d_1,h)>\lambda(d_2,h)$ when $0<d_1<d_2$. Furthermore, we have $\lambda(d_1,a-cu_2^*)>\lambda(d_2,a-u_2^*)=0$; that is, $(H2)$ is valid for $c\in[0,1]$. For $(H3)$, we suppose, by contradiction, that there is a

N-periodic equilibria $(u_0, v_0) \gg 0$ in $\mathbb{P}_+$. It then follows that

$$\begin{cases} d_1 L_j u_0 + u_{0,j}(a_j - u_{0,j} - cv_{0,j}) = 0, \\ d_2 L_j v_0 + v_{0,j}(a_j - u_{0,j} - v_{0,j}) = 0, \ j \in \mathbb{Z}, \end{cases}$$

which shows that $\lambda(d_1, a - u_0 - cv_0) = \lambda(d_2, a - u_0 - v_0) = 0$. Similarly, we obtain that $a - u_0 - cv_0$ is non-constant, which implies

$$\lambda(d_1, a - u_0 - cv_0) > \lambda(d_2, a - u_0 - cv_0) \geqslant \lambda(d_2, a - u_0 - v_0) = 0,$$

where $c \in [0, 1]$, a contradiction. □

As a result of Lemma 6.0.5 and Theorem 3.0.3, we have the following result.

Theorem 6.0.9. *If (M) holds, then $E_1 := (u_1^*, 0)$ is globally asymptotically stable for all initial value $\phi = (\phi_1, \phi_2) \in \mathbb{P}_+$ with $\phi \not\equiv 0$.*

For simplicity, we consider the following cooperative system:

$$\begin{cases} u_1' = d_1 L_j u_1 + u_1(a_j - cu_{2,j}^* - u_1 + cu_2), \\ u_2' = d_2 L_j u_2 + u_1(u_2^* - u_2) + u_2(a_j - 2u_{2,j}^* + u_2), \ t > 0, \ j \in \mathbb{Z}. \end{cases} \tag{6.0.2}$$

Let $u^* = (u_1^*, u_2^*)$ and introduce a family of operators $\{Q_t\}_{t \geqslant 0}$ on $\mathcal{C}_{u^*}$ by $Q_t(\phi) := u(t, \phi)$, where $u(t, \phi)$ is the unique solution of system (6.0.2) with $u(0, \phi) = \phi \in \mathcal{C}_{u^*}$. Let $\{\hat{Q}_t\}_{t \geqslant 0}$ be defined in (2.0.3) and $\bar{c}_+$ be denoted by (2.0.4) with $\tilde{P} = \hat{Q}_1$. By Lemma 6.0.4 and Proposition 5.0.4, it then follows that $\bar{c}_+ \geqslant c_+^0 > 0$.

Because of Theorem 4.0.7 and Remark 4.0.1.

Theorem 6.0.10. *If (M) hold and $u(t, \phi)$ is the solution of system*(6.0.2) *with $u(0) = \phi \in \mathcal{C}_{u^*}$, then the following statements are valid for system* (6.0.2)*:*

(i) If $\phi \in \mathcal{C}_{u^*}$, $0 \leqslant \phi \leqslant \omega \ll \beta$ for some $\omega \in \mathcal{C}_\beta^P$, and $\phi_j = 0$, $\forall j \geqslant H$, for some $H \in \mathbb{Z}$, then $\lim\limits_{t\to\infty, j \geqslant ct} u_j(t, \phi) = 0$ for any $c > \bar{c}_+$.

(ii) If $\phi \in \mathcal{C}_{u^*}$ and $\phi_j \geqslant \sigma$, $\forall j \leqslant K$ for some $\sigma \in \mathbb{R}^2$ with $\sigma \gg 0$ and $K \in \mathbb{Z}$, then $\lim\limits_{t\to\infty, j \leqslant ct} (u_j(t, \phi) - \beta_j) = 0$ for all $c \in (0, \bar{c}_+)$.

By Theorem 4.0.6, we have the following result on periodic traveling waves for system (6.0.1).

Theorem 6.0.11. *If* (M) *hold, then for any* $c \geqslant \bar{c}$*, system* (6.0.1) *yields an* N*-periodic rightward traveling wave* $(U_j(j-ct), V_j(j-ct))$ *connecting* $(0, u_1^*)$ *to* $(u_2^*, 0)$*, with wave profile components* $U_j(\xi)$ *and* $V_j(\xi)$ *being continuous and non-increasing in* ξ*, and for any* $c \in (0, \bar{c})$*, system* (6.0.1) *does not yield* N*-periodic rightward traveling wave connecting connecting* $(0, u_1^*)$ *to* $(u_2^*, 0)$*.*

Last, we study the shape of traveling waves by taking appropriate coefficients. In Figure 6.1 and Figure 6.2, we choose $d_1 = 0.3$, $d_2 = 1$, $c = 0.9$ and $a_j = 0.1 + |\cos(j * \pi/30)|$ for $j \in [1, 90]_{\mathbb{Z}}$ under jump initial function

$$\begin{cases} \phi_j = 0.4, \text{ if } j \in [1, 30]_{\mathbb{Z}}, \\ \phi_j = 0, \text{ if } j \in [31, 90]_{\mathbb{Z}}. \end{cases}$$

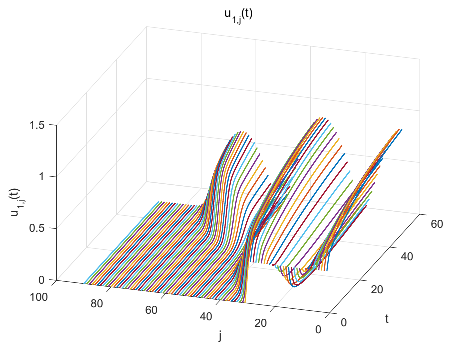


Figure 6.1: Traveling wave solutions of species $u_{1,j}$ for j from 1 to 90.

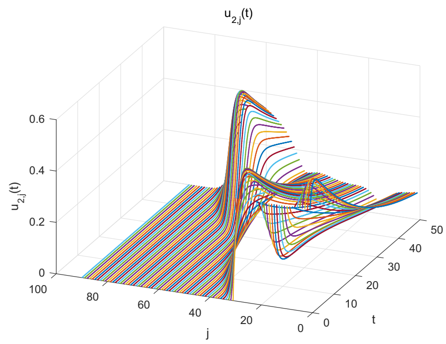
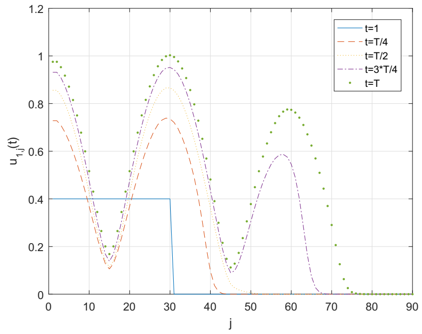


Figure 6.2: Traveling wave solution of species $u_{2,j}$ for j from 1 to 90.

In order to better illustrate the spreading property of u_1 and u_2, we take the cross-section when $t = 1$, $t = T/2$, $t = 3T/4$, $t = T$ in Figure 6.1 and Figure 6.2, where T is biggest coordinate in t axis. We can see the cross-section in Figure 6.3 and 6.4.



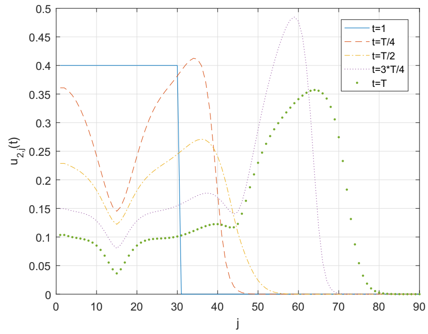
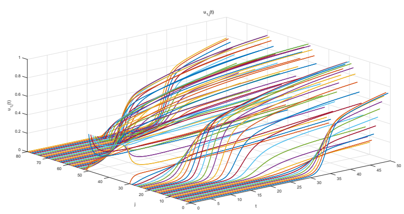


Figure 6.3: Cross-section when $t = 1$, $t = \frac{T}{2}$, $t = \frac{3T}{4}$, $t = T$ in Figure 6.1.

Figure 6.4: Cross-section when $t = 1$, $t = \frac{T}{2}$, $t = \frac{3T}{4}$, $t = T$ in Figure 6.2.

In Figure 6.5 and Figure 6.6, we choose $d_1 = 0.3$, $d_2 = 1$, $c = 0.9$ and $a_j = 0.1 + |\cos(j * \pi/20)|$ for $j \in [1, 80]_{\mathbb{Z}}$ under initial function with compact support

$$\begin{cases} \phi_j = 0.4, \text{ if } j \in [31, 50]_{\mathbb{Z}}, \\ \phi_j = 0, \text{ if } j \in [1, 30]_{\mathbb{Z}} \text{ or } [51, 80]_{\mathbb{Z}}. \end{cases}$$



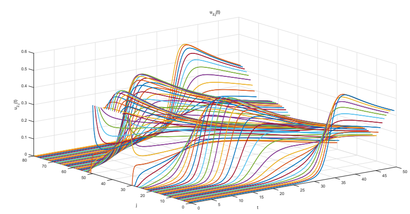


Figure 6.5: Traveling wave solutions of species $u_{1,j}$ for j from 1 to 80.

Figure 6.6: Traveling wave solutions of species $u_{2,j}$ for j from 1 to 80.

For T, we still use the same notation in Figure 6.3 and Figure 6.4. In Figure 6.7 and 6.8, we let $t = 1$, $t = T/2$, $t = 3T/4$, $t = T$ to get the cross-section.

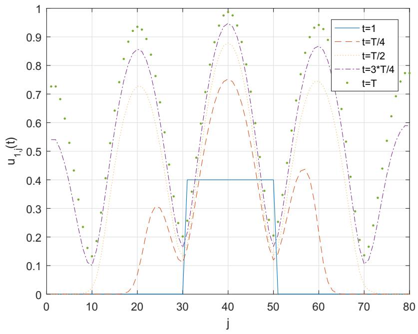


Figure 6.7: Cross-section when $t = 1$, $t = \frac{T}{2}$, $t = \frac{3T}{4}$, $t = T$ in Figure 6.5.

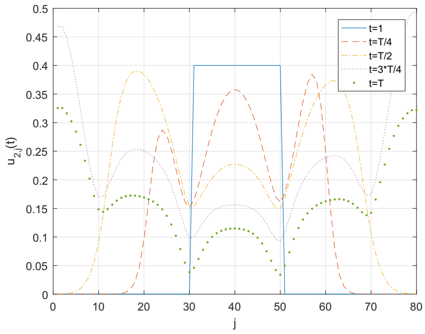


Figure 6.8: Cross-section when $t = 1$, $t = \frac{T}{2}$, $t = \frac{3T}{4}$, $t = T$ in Figure 6.6.

Chapter 7

Future works

Wang et al. [29] showed some new results on the linear and nonlinear selections for the model in Yu and Zhao [33], and they utilized the method of upper and lower bound to estimate the minimal wave speed for the case of nonlinear selection. Since this thesis was motivated by the work in Yu and Zhao [33], we may use some arguments and methods in [29] to study the linear and nonlinear selection for the lattice model in this thesis.

In Section 3, we assume that $(H3)$ hold; that is, there is no interior equilibrium point in $[0, \tilde{w}_1^*] \times [0, \tilde{w}_2^*]$ for system (3.0.3). Chen et al. [3] investigated a more specific model compared with system (1.0.2) and obtained some conditions for the nonexistence of positive equilibrium. Motivated by Chen et al. [3], we will find sufficient conditions for $(H3)$ to hold for system (1.0.2) in the future.

In this thesis, we considered the case where one semi-trivial equilibrium is stable and the other is unstable (i.e., the monostable case) for system (1.0.2). It is also essential for system (1.0.2) to investigate the case that two semi-trivial spatially periodic equilibria are stable (i.e., the bistable case). This motivates us to consider the bistable case for periodic lattice habitat and investigate the existence, uniqueness, and stability of the bistable waves for system (1.0.2).

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