Cops and Robber Games on Graphs: An approach from Large Scale Geometry

by

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Abstract

We introduce two variations of the cops and robber game for graphs which define two invariants for infinite connected graphs: the weak-cop number and the strong-cop number. We exhibit examples of graphs with arbitrary weak-cop number, and examples with arbitrary strong-cop number. We prove that these invariants are preserved by quasi-isometry; for example, this allow us to show that the square-grid, triangular-grid and hexagonal-grid have the same weak-cop number and the same strong-cop number. We also prove that hyperbolic graphs have strong cop number one; for example this implies that any graph arising as a regular tiling of the hyperbolic plane has strong cop-number one. Our main result is that one-ended non-amenable locally-finite vertex-transitive graphs have infinite weak-cop number. This last result includes graphs arising as regular tilings of the hyperbolic plane.

To my loving parents

Lay summary

A graph is a collection of vertices (nodes) and edges (links), where each edge consists of two vertices. A property of a graph is the *cop number*; this number comes from playing a game of Cops and Robber in the graph, where both, the cops and the robber move between adjacent vertices in a strategy to try to capture the robber (in the case of the cops) and try to escape from the cops (in the case of the robber). The cop number is the smallest number of cops which are able to always capture a robber. Several variations of this game have been studied, for example providing a range (of capture) to the cops or changing the speeds on how both cops and robber move, in particular we introduce two variations and our objective is to provide some results for these variations on different families of graphs.

In this document we study graphs with geometric properties such as *hyperbolicity*, *ends* or *amenability*. We provide three main results, the first one is that the cop-number given by our variations is preserved through the group theoretic notion of *quasi-isometry* and there is an inequality in the notion of *quasi-retraction*. The second is that for hyperbolic graphs there is only one cop needed for one of the variations. And the third result is on one-ended non-amenable graphs, that tell us that those graphs need an infinite number of cops to win in one of our variations. Finally these results together with a family of graphs we denote Θ_n-extensions, we manage to exhibit examples with arbitrary cop number for each of the variations.

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Statement of contribution

Chapters 3 and 4 were suggested by Dr. Martinez-Pedroza. Chapters 5, 6 and 7 came as result from the discussed ideas with Dr. Martinez-Pedroza. The preparation of the thesis document was done by myself under the supervision of Dr. Martinez-Pedroza.

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Chapter 1

Introduction

The Cops and Robber game was introduced in [Quilliot, 1978] and [Nowakowski and Winkler, 1983]. This is a two player game on a graph, where one player controls a set of cops and the other one controls a single robber. On the graph each cop and the robber chooses a vertex to occupy, with the cops choosing first. The game then alternates between cops and the robber moving along adjacent vertices, with the cops moving first. The cops win if, after a finite number of rounds, a cop occupies the same vertex as the robber; and the robber wins if he can avoid capture indefinitely. Different variations of this game have been studied especially on finite graphs, some classical results can be found in [Bonato and Nowakowski, 2011]. A particular variation of the game was studied in [Chalopin et al., 2010] where the cops and the robber have different speeds, that is, at each turn cops and robber are allowed to move to vertices a distance at most their speeds. Their results show connections between graphs that are cop-win in this variation and the notion of Gromov's hyperbolicity. On the other hand, a version of the game involving the concept of *radius of capture* was studied in [Bonato and Chiniforooshan, 2009] and [Bonato et al., 2010]; in this variation, the cop captures if the distance to the robber is less than the radius of capture. In this work we introduce two variations of the game which involve different speeds for the cops and the robber, as well as an extra parameter called the cop's reach which works similarly to the radius of capture. Our variations of the cops and robber game define invariants of infinite graphs called the weak-cop number and the strong-cop number.

Let us give an informal definition of these invariants, for a formal definition we refer the reader to Section 2.2. Let Γ be a graph. For positive integers σ, ρ, ψ and R, we say that Γ is $\mathsf{CopWin}(n, \sigma, \rho, \psi, R)$ if, for any vertex u_0 of Γ, n cops with speed σ and reach ρ can eventually protect the closed ball $\overline{B_R}(u_0)$ from a robber with speed ψ. The definitions of the weak-cop number and the strong-cop number differ in the order the parameters of the game are chosen, for the weak-cop number the robber has an information advantage, and for the strong-cop number the cops have the advantage. More precisely, these invariants are defined as follows:

- We say that Γ is *n-weak-cop win* if there exists a $\sigma \in \mathbb{Z}_{>0}$ and a $\rho \in \mathbb{Z}_{\geq 0}$ such that for any $\psi, R \in \mathbb{Z}_{>0}$ and any $u_0 \in V(\Gamma)$, Γ is $\mathsf{CopWin}(n, \sigma, \rho, \psi, R)$. In symbols,

$$\Gamma \text{ is } n\text{-weak-cop win} \iff \exists\ \sigma, \rho\ \forall\ \psi, R\colon \Gamma \text{ is } \mathsf{CopWin}(n, \sigma, \rho, \psi, R).$$

- We say that Γ is *n-strong-cop win* if there exists $\sigma \in \mathbb{Z}_{>0}$ such that for any $\psi \in \mathbb{Z}_{>0}$, there is $\rho \in \mathbb{Z}_{\geq 0}$ so that for any $R \in \mathbb{Z}_{>0}$, and $u_0 \in V(\Gamma)$, Γ is $\mathsf{CopWin}(n, \sigma, \rho, \psi, R)$. In symbols,

$$\Gamma \text{ is } n\text{-strong-cop win} \iff \exists\ \sigma\ \forall\ \psi\ \exists\ \rho\ \forall\ R\colon \Gamma \text{ is } \mathsf{CopWin}(n, \sigma, \rho, \psi, R).$$

Observe that if the robber chooses the parameters ψ and R after the cops chose the parameters σ and ρ, then the robber has an information advantage.

The *strong-cop number* $\mathsf{sCop}(\Gamma)$ of a graph Γ is defined as the smallest value of n such that Γ is n-strong-cop-win, with $\mathsf{sCop}(\Gamma) = \infty$ if there is no such n. In the case that $\mathsf{sCop}(\Gamma) = \infty$, we say that Γ is *robber-win*. The *weak-cop number* $\mathsf{wCop}(\Gamma)$ is defined analogously, and if $\mathsf{wCop}(\Gamma) = \infty$ we say that Γ is *fast-robber win*. Observe that

$$\mathsf{sCop}(\Gamma) \leq \mathsf{wCop}(\Gamma).$$

Let us summarize the result of the thesis. We exhibit examples of graphs with arbitrary weak cop number, and examples with arbitrary strong cop number, see Corollary 6.10. We prove that these invariants are preserved by quasi-isometry, for example, this allow us to show that the square-grid, triangular-grid and hexagonal-grid have the same weak-cop number and the same strong-cop number. We also prove that hyperbolic graphs have strong cop number one, see Theorem 4.6; for example this implies that any graph arising as a regular tiling of the hyperbolic plane has strong cop-number one. Our main result is that one-ended non-amenable locally-finite vertex-transitive graphs have infinite weak-cop number, Theorem 5.1. This last result includes graphs arising as regular tilings of the hyperbolic plane. The rest of the introduction states the results.

Cops numbers as large scale invariants

Our first result gives an extension to the concept of retractions on graphs, using the notion of quasi-retractions introduced in [Alonso, 1994]. This notion is weaker than retractions of graphs, as a quasi-isometry does not necessarily involve a subgraph. The second section provides a formal definition of quasi-retraction and proves the following result.

Theorem A (Theorem 3.6)**.** *Let Γ and Δ be connected graphs. If Δ is a quasi-retract of Γ then:*

$$\mathsf{wCop}(\Delta) \leq \mathsf{wCop}(\Gamma) \quad \textit{and} \quad \mathsf{sCop}(\Delta) \leq \mathsf{sCop}(\Gamma).$$

An illustration of this result can be found in Corollary 3.8, that give us a lower bound on the weak and strong cop number of the product of graphs in terms of their factors. We recover that the weak cop number is a quasi-isometric invariant, a result first proved in [Lee, 2019]; this result is an immediate corollary that is also extended to strong-cop numbers as follows:

Corollary B (Corolary 3.14)**.** *If Γ and Δ are connected quasi-isometric graphs, then:*

$$\mathsf{wCop}(\Delta) = \mathsf{wCop}(\Gamma) \quad \textit{and} \quad \mathsf{sCop}(\Delta) = \mathsf{sCop}(\Gamma).$$

Let us remark that there are similar results to this corollary for other persuit and evasion games in the literature. For example, it is shown in [Dyer et al., 2017] that for a variation of Hartnell's firefighter game [Hartnell, 1995], the existence of a winning strategy for the firefighters is a quasi-isometic invariant.

Strong-cop number of hyperbolic graphs

The notion of a δ-hyperbolic graph is a generalization of the notion of a tree, moreover, trees are 0-hyperbolic graphs. A graph is δ-hyperbolic if every geodesic triangle is δ-slim in the sense that the δ-neighborhood of any two sides contains all sides of the triangle (see Figure 4.1), and we refer the reader to Section 4 for the formal definition. The connection between cop-win graphs and Gromov's hyperbolicity has been previously studied for variations of the cops and robber game, see for example in [Chalopin et al., 2010]. For the variations introduced in this work there is also a strong relation given by the following result which is proved in Section 4.

Theorem C (Theorem 4.6)**.** *If* Γ *is a hyperbolic graph then* $\mathsf{sCop}(\Gamma) = 1$.

It was remarked in [Lee, 2019] that trees have weak-cop number one, a first illustration of our result. Some examples of hyperbolic graphs that are not trees arise as regular tilings of the hyperbolic plane, see Figure 4.2 for some illustrations.

Corollary D. *Any graph arising as a regular tiling of the hyperbolic plane has strong-cop-number one.*

In contrast the weak-cop number of an arbitrary hyperbolic graph is not necessarily one, as our next result will show.

Non-amenable one-ended graphs

Now we state the main result of this work. Definitions of the terminology used in the statement as well as the proof are the contents of Section 5. Let us briefly describe some of the terminology in the theorem, one-ended graphs and non-amenable graphs.

A connected graph Γ is *one-ended* if for any finite subset of vertices the induced subgraph $\Gamma \setminus K$ has only one unbounded connected component. An example of a one-ended graph is the infinite square grid.

Intuitively, a connected graph Γ is *non-amenable* if does not have large bottlenecks. More formally, for a subset of vertices K of Γ, let ∂K be the set of edges of Γ with one endpoint in K and the other endpoint not in K. The graph Γ is *non-amenable* if its Cheeger constant $h(\Gamma)$ is nonzero, that is

$$0 < h(\Gamma) = \inf \left\{ \frac{|\partial K|}{|K|} \mid K \subset V(\Gamma),\ |K| < \infty \right\}$$

The Cheeger constant can be interpreted as how difficult is to remove large subsets of vertices with small boundary.

Theorem E (Theorem 5.1)**.** *If Γ is a connected, one-ended, non-amenable, locally finite, vertex transitive graph then* $\mathsf{wCop}(\Gamma) = \infty$*.*

While this result looks technical due to the number of hypotheses, the class of graphs where the theorem applies is large as the following corollaries illustrate.

Corollary F. *Every graph arising as a regular tiling of the hyperbolic plane has infinite weak-cop number.*

Corollary G (Corollary 7.31)**.** *Let Γ be the Cayley graph of a one-ended hyperbolic group. Then* $\mathsf{wCop}(\Gamma) = \infty$ *and* $\mathsf{sCop}(\Gamma) = 1$*.*

Range of the weak and strong cop numbers.

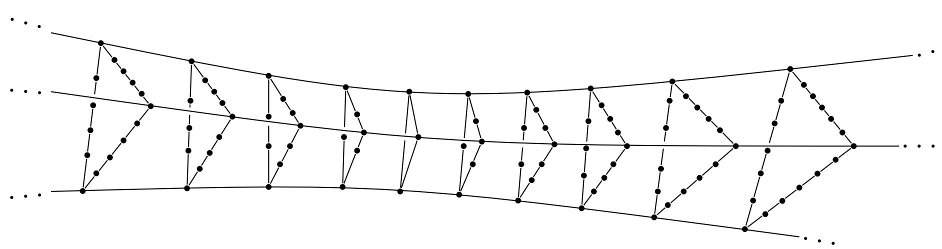

Figure 1.1: Θ_3-extension of the infinite length path

Given a graph Γ and a positive integer n, we introduce the Θ_n-extension $\Theta_n(\Gamma)$ of Γ. As an example, the graph in Figure 1.1 is the Θ_3-extension of the infinite path. The graph $\Theta_n(\Gamma)$ quasi-retracts to Γ and allows us to produce a variety of examples of graphs with different weak and strong cop numbers. The construction of $\Theta_n(\Gamma)$ is defined in Section 6 where the following results are proved.

Theorem H (Corollary 6.10)**.** *For any connected graph* Γ *and for any integer* $n > 0$*,*

$$\mathsf{wCop}(\Gamma) \leq \mathsf{wCop}(\Theta_n(\Gamma)) \leq n \cdot \mathsf{wCop}(\Gamma)$$

and

$$\mathsf{sCop}(\Gamma) \leq \mathsf{sCop}(\Theta_n(\Gamma)) \leq n \cdot \mathsf{sCop}(\Gamma).$$

Since trees have weak and strong cop number one, the following result shows that there exists graphs with arbitrary weak-cop number, and with arbitrary strong-cop number. Theorem I is not proved in this thesis, it appears in an the following article [Lee et al., 2022].

Theorem I. *[Lee et al., 2022] Let* Γ *be a connected infinite graph. If* $\mathsf{wCop}(\Gamma) = 1$ *then* $\mathsf{wCop}(\Theta_n(\Gamma)) = n$*. Analogously, if* $\mathsf{sCop}(\Gamma) = 1$ *then* $\mathsf{sCop}(\Theta_n(\Gamma)) = n$*.*

The Θ_n-extensions allow to construct graphs that have different weak and strong cop numbers. However, we do not have examples addressing the following question.

Question J. *Let* n, m *be arbitrary positive integers, such that* $n > m$*. Does there exist a graph* Γ *with* $\mathsf{wCop}(\Gamma) = n$ *and* $\mathsf{sCop}(\Gamma) = m$*?*

Finally in section 7 we define an extension of the weak and strong cop numbers to finitely generated groups using Cayley graphs, and provide additional applications of our theorems.

Chapter 2

Preliminaries

2.1 Graph theory preliminaries

A *graph* Γ is defined as a pair (V, E), where V is the set and E is a set subsets of V of cardinality two. Elements of V and E are *vertices* and *edges* respectively. A pair of vertices u and v are *adjacent* in Γ if $\{u, v\}$ is an an edge of Γ. A *path* in Γ is a sequence of vertices $v_0, v_1 \ldots, v_n$ such that v_i and v_{i+1} are adjacent; n is called the *length* of the path; and v_0 and v_n are called the *initial* and *terminal* vertices of the path respectively. A graph is said to be *connected* if there exists a path between any two vertices. The *degree* of a vertex v is the number of edges containing v. If every vertex of Γ has finite degree, we say that the graph is *locally finite*. If there is $N > 0$ such that every vertex has degree at most N we say that the graph has *uniform bounded degree*.

We can define a metric on the vertex set of a connected graph Γ by defining the distance $\mathsf{dist}_\Gamma(u, v)$ between vertices u and v as the minimum of the lengths of all paths from u to v. If the graph is understood from the context we denote this metric by $\mathsf{dist}(u, v)$. More generally the *distance between subgraphs* Δ_1, Δ_2 is defined as $\min\{\mathsf{dist}(u, v) : u \in \Delta_1 \text{ and } v \in \Delta_2\}$. A path from u to v of minimal length is called a *geodesic*. In this metric, the *closed ball* of size N with center the vertex u is denoted as $\overline{B_N(u)}$ and $\mathsf{diam}(\Gamma) = \sup\{\mathsf{dist}(x, y) : x, y \in V(\Gamma)\}$ is the *diameter* of Γ.

A *graph automorphism* is a function $\sigma\colon V(\Gamma) \to V(\Gamma)$ such that for any pair of vertices u, v of Γ, $\{u, v\} \in E$ if and only if $\{\sigma(u), \sigma(v)\} \in E$. A graph is *vertex transitive* if given any two vertices on the graph, there is an automorphism that maps one into the other. We say that Δ is a *retract* of Γ if there is a homomorphism f from Γ onto Δ so that $f(v) = v$ for $v \in V(\Gamma)$; that is f is the identity on Δ. The map f is sometimes called a *graph retraction*.

2.1.1 Products of graphs

There are many different products between graphs, some of them are listed below.

Cartesian product of graphs: Let $\Gamma = (V(\Gamma), E(\Gamma))$ and $\Delta = (V(\Delta), E(\Delta))$ be two graphs. We can define the *Cartesian product*, denoted as $\Gamma \,\square\, \Delta$, as:

- $V(\Gamma \,\square\, \Delta) = V(\Gamma) \times V(\Delta)$.
- Any two vertices $(x_1, y_1), (x_2, y_2) \in V(\Gamma \,\square\, \Delta)$, are adjacent if and only if:
    - $x_1 = x_2$ and y_1 is adjacent to y_2 in Δ, or
    - $y_1 = y_2$ and x_1 is adjacent to x_2 in Γ.

Strong product of graphs: Let $\Gamma = (V(\Gamma), E(\Gamma))$ and $\Delta = (V(\Delta), E(\Delta))$ be two graphs. We can define the *Strong product*, denoted as $\Gamma \boxtimes \Delta$, as:

- $V(\Gamma \boxtimes \Delta) = V(\Gamma) \times V(\Delta)$.
- Any two vertices $(x_1, y_1), (x_2, y_2) \in V(\Gamma \boxtimes \Delta)$, are adjacent if and only if:
    - $x_1 = x_2$ and y_1 is adjacent to y_2 in Δ, or
    - $y_1 = y_2$ and x_1 is adjacent to x_2 in Γ, or
    - x_1 is adjacent to x_2 in Γ and y_1 is adjacent to y_2 in Δ.

Lexicographic product of graphs: Let $\Gamma = (V(\Gamma), E(\Gamma))$ and $\Delta = (V(\Delta), E(\Delta))$ be two graphs. We can define the *Lexicographic product*, denoted as $\Gamma(\Delta)$, as:

- $V(\Gamma(\Delta)) = V(\Gamma) \times V(\Delta)$.
- Any two vertices $(x_1, y_1), (x_2, y_2) \in V(\Gamma(\Delta))$, are adjacent if and only if:
    - x_1 is adjacent to x_2 in Γ, or
    - $x_1 = x_2$ and y_1 is adjacent to y_2 in Δ.

Rooted product of graphs: Let $\Gamma = (V(\Gamma), E(\Gamma))$ and $\Delta = (V(\Delta), E(\Delta))$ be two graphs and y be a fixed vertex in $V(\Delta)$. We can define the *Rooted product*, denoted as $\Gamma \circ_y \Delta$, using a three stage process:

1. For each $v \in V(\Gamma)$ create a copy of Δ.
2. Identify the vertex correspond to y in its copy of Δ, Denote this vertex y_v,
3. replace the vertex v with y_v.

2.2 Definitions of the cops and robber games

We will present two games, the parameters and instructions are the same for both of them.

2.2.1 Parameters of the games

Let Γ be a connected graph and n be a positive integer representing the number of cops in the game, ρ be a positive integer called *cop reach*, and σ and ψ be positive integers called the *cop speed* and *robber speed*, respectively. Let u_0 be a fixed vertex of Γ and R be a positive integer. The closed ball $\overline{B_R}(u_0)$ of radius R will be referred to as the ball that the cops aim to protect.

After the parameters have been chosen, the n cops choose their initial positions $c_{1,1}, \ldots, c_{n,1}$. Then the robber, knowing the initial positions of the cops, choose his initial position r_1.

The players take alternating turns, starting with the cops. At the beginning of the j-th turn, the positions of the n cops are denoted by $c_{i,j}$ with i corresponding to the i-th cop. Similarly r_j is the position of the robber at the beginning of the same turn.

2.2.2 Instructions of the game

At the beginning of a turn, the cops positions are $(c_{1,j}, ..., c_{n,j})$. Moving to the positions $(c_{1,j+1}, ..., c_{n,j+1})$ is valid if there is a path with length less than or equal to σ from each $c_{i,j}$ to $c_{i,j+1}$.

After the cop's movement on the same turn, the position of the robber is r_j. He moves to the position r_{j+1} if there is a path γ from r_j to r_{j+1} that does not intersect the reach ρ of the cops, and whose length is at most the speed of the robber ψ.

It is the objective of the cops to protect the closed ball $\overline{B_R}(u_0)$. If the robber is at distance less than or equal to ρ from the one of the cops, the robber is said to be caught, or captured, and the ball is said to be protected if the robber has been caught or, after some point in the game, the position of the robber stays outside the closed ball $\overline{B_R}(u_0)$ for the rest of the game.

2.2.3 Definition of cop numbers

For values $\sigma, \rho, \psi, R \in \mathbb{Z}_{>0}$, and $u_0 \in V$ we say that Γ is $\mathsf{CopWin}(n, \sigma, \rho, \psi, u_0, R)$ if n cops with speed σ and reach ρ can eventually protect the closed ball $\overline{B_R}(u_0)$ from a robber with speed ψ, this means, either the robber is captured at some stage, or there exists N such that $\mathsf{dist}(r_k, u_0) > R$ for every $k \geq N$.

- We say that Γ is *n-weak-cop win* if there exists a $\sigma \in \mathbb{Z}_{>0}$ and a $\rho \in \mathbb{Z}_{\geq 0}$ such that for any $\psi, R \in \mathbb{Z}_{>0}$ and any $u_0 \in V(\Gamma)$, Γ is $\mathsf{CopWin}(n, \sigma, \rho, \psi, u_0, R)$. In symbols,

$$\Gamma \text{ is } n\text{-weak-cop win} \iff \exists\, \sigma, \rho\ \forall\, \psi, R, u_0 \colon \Gamma \text{ is } \mathsf{CopWin}(n, \sigma, \rho, \psi, u_0, R).$$

- We say that Γ is *n-strong-cop win* if there exists $\sigma \in \mathbb{Z}_{>0}$ such that for any $\psi \in \mathbb{Z}_{>0}$, there is $\rho \in \mathbb{Z}_{\geq 0}$ so that for any $u_0 \in V(\Gamma)$ and any $R \in \mathbb{Z}_{>0}$, Γ is $\mathsf{CopWin}(n, \sigma, \rho, \psi, u_0, R)$. In symbols,

$$\Gamma \text{ is } n\text{-strong-cop win} \iff \exists\, \sigma\ \forall\, \psi\ \exists\, \rho\ \forall\, R, u_0 \colon \Gamma \text{ is } \mathsf{CopWin}(n, \sigma, \rho, \psi, u_0, R).$$

The smallest value of n such that the graph is n-strong-cop-win (resp. n-weak-cop win) is called the *strong-cop number* (resp. *weak-cop number*) of the graph; we denote it as $\mathsf{sCop}(\Gamma)$ (resp. $\mathsf{wCop}(\Gamma)$). If there is no n such that the graph is n-strong-cop-win (resp. n-weak-cop win), then the graph is said to be *robber-win* (resp. *fast-robber win*) or $\mathsf{sCop}(\Gamma) = \infty$ (resp. $\mathsf{wCop}(\Gamma) = \infty$).

Remark 2.1. As the main difference between both games is the order of the parameters, ρ can depend on ψ for the strong cops. Moreover:

$$\mathsf{sCop}(\Gamma) \leq \mathsf{wCop}(\Gamma).$$

Remark 2.2. If Γ is a connected finite graph, then $\mathsf{wCop}(\Gamma) = \mathsf{sCop}(\Gamma) = 1$. Indeed, a single cop with reach the diameter of the graph can capture the robber on turn one.

Remark 2.3. We consider only connected graphs for this game, note that, if Γ is a non-connected graph, then

$$\mathsf{wCop}(\Gamma) = \max\{\mathsf{wCop}(\Gamma_i) \,:\, \Gamma_i \text{ is a connected component of } \Gamma\},$$

and

$$\mathsf{sCop}(\Gamma) = \max\{\mathsf{sCop}(\Gamma_i) \,:\, \Gamma_i \text{ is a connected component of } \Gamma\}.$$

As neither the robber nor the cops can escape from the connected component they lie on.

Proposition 2.4. *[Lee, 2019, Proposition 3.1] Trees are* 1*-weak-cop win.*

Proposition 2.5. *[Lee, 2019, Proposition 3.5] The infinite square grid has infinite weak-cop-number.*

This previous result motivates the following question for which we do not have an answer.

Question 2.6. Does the infinite square grid have infinite strong-cop-number?

Chapter 3

Cop numbers and quasi-retractions

Results by Jonathan Lee [Lee, 2019] show that the behaviour of wCop on infinite graphs relates to the large scale structure rather than to the local structure. Indeed, the main theorem of his Master's project shows that *quasi-isometric graphs* have the same weak-cop-number. We study a weaker relation between graphs called *quasi-retraction*, and show that this relation implies inequalities between the weak and strong cop numbers of the related graphs. Our main theorem generalizes and recovers Lee's results.

Let $C \geq 1$ and $D \geq 0$ be integers, a function $f : X \to Y$ between two metric spaces (X, dist_X) and (Y, dist_Y) is (C, D)-*Lipschitz* if for any $x_1, x_2 \in X$, $\mathsf{dist}_Y(f(x_1), f(x_2)) \leq C\,\mathsf{dist}_X(x_1, x_2) + D$.

Definition 3.1 (Quasi-retraction)**.** Given two metric spaces (X, dist_X) and (Y, dist_Y), we say that X is a *quasi-retract* of Y if there exists two (C, D)-Lipschitz functions $f : X \to Y$ and $g : Y \to X$ such that:

$$\mathsf{dist}_X(g(f(x)), x) \leq D,$$

for any $x \in X$. The pair (f, g) is called a quasi-retraction of Y into X.

Definition 3.2 (Quasi-retraction of graphs)**.** A connected graph Γ determines a metric on its vertex set $V(\Gamma)$ by defining the distance between vertices u and v as the minimum of the lengths of all paths from u to v. A path from u to v of minimal length is called a *geodesic*. A graph Δ is a *quasi-retract* of a graph Γ if the metric space determined by Δ is a quasi-retract of the metric space determined by Γ.

Example 3.3 (Quasi-retractions and finite graphs)**.** If Γ and Δ are connected graphs and Δ is finite, then Δ is a quasi-retract of Γ. Specifically, let $f\colon V(\Delta) \to V(\Gamma)$ and $g\colon V(\Gamma) \to V(\Delta)$ be constant functions, and observe that (f, g) is a quasi-retraction of Γ into Δ with constants $C = 1$ and $D = \operatorname{diam}(\Delta)$.

Example 3.4 (Graph products and quasi-retractions)**.** Let Γ, Δ be two connected graphs. If Λ denotes the Cartesian product $\Gamma \,\square\, \Delta$, the strong product $\Gamma \boxtimes \Delta$, or the rooted product $\Gamma \circ_y \Delta$; then Λ quasi-retracts onto Γ and also onto Δ. The lexicographic product $\Gamma(\Delta)$ quasi-retracts onto Γ, but not necessarily onto Δ. To prove this, note that the pair $(\imath, r)$ is a quasi-retraction, where r is the retraction from Λ into Γ and $\imath$ is one of the natural inclusions from Δ onto Λ.

Example 3.5 (Graph retractions and quasi-retractions)**.** It is an observation that a graph retraction $r\colon \Gamma \to \Delta$ induces a quasi-retraction $(\imath, r)$ of Γ into Δ. However quasi-retractions into subgraphs are not necessarily graph retractions. For example, consider the graph Γ in Figure 3.1 and let Δ be the subgraph spammed by the vertices a, b. It can easily be seen that there is no graph retraction from Γ to Δ. However, there is a quasi-retraction (f, g) of Γ into Δ where $g\colon V(\Gamma) \to V(\Delta)$ is given by $g(x) = b$ if $x \neq a$ and $g(a) = a$ and $f\colon \{a, b\} \to V(\Gamma)$ is the inclusion map.

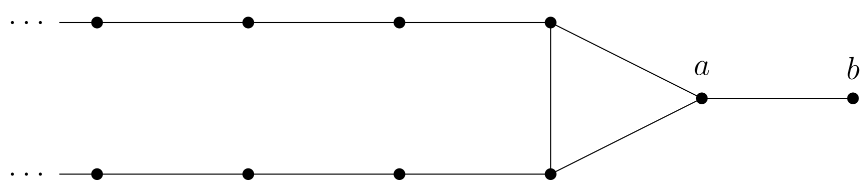

Figure 3.1: The graph in the illustration quasi-retracts to the subgraph spammed by the vertices a, b but does not retract to that subgraph

The main result of this section is the following.

Theorem 3.6. *Let Γ and Δ be connected graphs. If Δ is a quasi-retract of Γ then:*

$$\mathsf{wCop}(\Delta) \leq \mathsf{wCop}(\Gamma) \quad \textit{and} \quad \mathsf{sCop}(\Delta) \leq \mathsf{sCop}(\Gamma).$$

We give an example which illustrates that the quasi-retraction hypothesis in Theorem 3.6 is not superfluous.

Example 3.7. Let Δ be the infinite square grid. It was proved in [Lee, 2019, Prop. 3.5] that $\mathsf{wCop}(\Delta) = \infty$. Let Γ be the graph obtained by adding to Δ a new vertex

w which is adjacent to every vertex of Δ. Note that for any $u, v \in V(\Gamma)$, $d(u,v) \leq 2$, and hence $\mathsf{wCop}(\Gamma) = 1$. Since Δ has vertices at arbitrary large distance, there is no Lipschitz map from $V(\Gamma) \to V(\Delta)$. In particular, Δ is not a quasi-retract of Γ.

Corollary 3.8. *Let Γ and Δ be connected graphs, and let $y \in V(\Delta)$. Suppose*

$$\Lambda \in \{\Gamma \,\square\, \Delta, \Gamma \boxtimes \Delta, \Gamma \circ_y \Delta, \Gamma(\Delta)\}.$$

Then

$$\mathsf{wCop}(\Gamma) \leq \mathsf{wCop}(\Lambda) \quad \textit{and} \quad \mathsf{sCop}(\Gamma) \leq \mathsf{sCop}(\Lambda).$$

Definition 3.9 (Quasi-isometry)**.** Given two metric spaces $(X, \mathsf{dist}_X), (Y, \mathsf{dist}_Y)$, we say that X and Y are *quasi-isometric* if there exists two (C, D)-Lipschitz functions $f : X \to Y$ and $g : Y \to X$ such that:

$$\mathsf{dist}_X(g(f(x)), x) \leq D \quad \text{ and } \mathsf{dist}_Y(f(g(y)), y) \leq D,$$

for any $x \in X$ and any $y \in Y$. The pair (f, g) is called a *quasi-isometry* between X and Y.

Remark 3.10 (Quasi-isometry in terms of quasi-retractions)**.** Note that a pair (f, g) is a a quasi-isometry between the metric spaces X and Y if and only if (f, g) and (g, f) are quasi-retractions from X into Y and Y into X respectively.

Example 3.11. Regular tilings of the euclidean plane are quasi-isometric between each other. See Figure 3.2 for some illustrations. The explanation of this statement is beyond the scope of this work, for the interested reader, it is a consequence of Ŝvarc-Milnor Lemma, see [Löh, 2017].

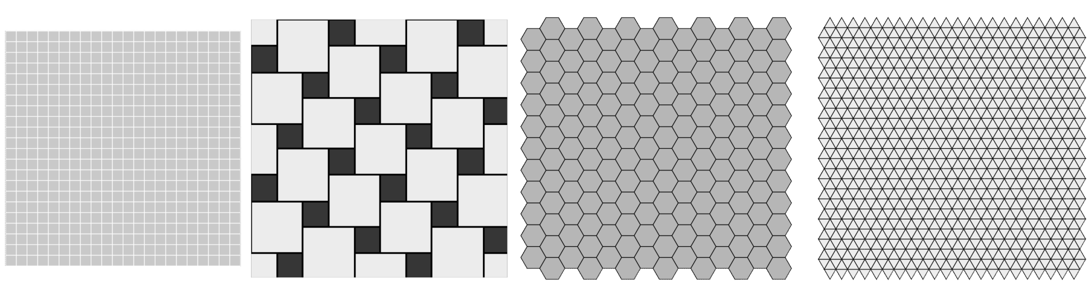

Figure 3.2: Some quasi-isometric tilings, from [Ruen, 2013]

Example 3.12. Let Γ, Δ be two connected graphs such that Δ is finite. If Λ denotes the Cartesian product $\Gamma \,\square\, \Delta$, the strong product $\Gamma \boxtimes \Delta$, the lexicographic product $\Gamma(\Delta)$, or the rooted product $\Gamma \circ_y \Delta$; then Λ and Γ are quasi-isometric.

For this, note that Example 3.4 shows that Λ quasi-retracts to Γ, moreover note that if we take $D = \mathsf{diam}(\Delta)$, then we have the inequality that we need to show the quasi-isometry.

Example 3.13 (A non-example)**.** As mentioned before, if Δ is the infinite square grid then $\mathsf{wCop}(\Delta) = \infty$ [Lee, 2019, Prop. 3.5]. Moreover if Γ is a tree then $\mathsf{wCop}(\Gamma) = 1$ [Lee, 2019, Proposition 3.1]. As a consequence of the following result, these two graphs are not quasi-isometric.

The following corollary appears as the main result of Jonathan Lee Master's project.

Corollary 3.14. *[Lee, 2019] If Γ and Δ are quasi-isometric connected graphs, then*

$$\mathsf{wCop}(\Delta) = \mathsf{wCop}(\Gamma) \quad \text{and} \quad \mathsf{sCop}(\Delta) = \mathsf{sCop}(\Gamma).$$

The rest of Chapter 3 consists of two subsections. The first subsection establishes important observations regarding quasi-retractions. The second subsection contains the proof of Theorem 3.6.

3.1 Quasi-retractions of subgraphs

Most examples of quasi-retractions that we have discussed are quasi-retracts into subgraphs. The following proposition shows that any quasi-retractions of a graph Γ into a graph Δ can be thought as quasi-retraction into a subgraph.

Theorem 3.15. *If (f, g) is a quasi-retraction from Γ into Δ, then there exists a quasi-retraction $(\imath, g')$ of Γ onto a subgraph Δ' of Γ, where $\imath$ is the natural inclusion; and a quasi-isometry $h\colon \Delta \to \Delta'$ such that the following diagram commutes*

$$
\begin{tikzcd}
\Delta' \arrow[rr] \arrow[dr, hook] & & \Delta' \arrow[dd, "\tilde{g}"] \\
& \Gamma \arrow[ur, "g'"] \arrow[dr, "g"] & \\
\Delta \arrow[uu, "\tilde{f}"] \arrow[ur, "f"] \arrow[rr] & & \Delta
\end{tikzcd}
$$

To prove Theorem 3.15 we require the following lemma.

Lemma 3.16. *If X is a quasi-retract of X' and X' is a quasi-retract of X'', then X is itself a quasi-retract of X''.*

Proof. Let consider a quasi-retraction (f, g) from $X \to X'$ and a quasi-retraction (f', g') from $X' \to X''$. Now, define:

$$f'' = f' \circ f \qquad g'' = g \circ g'.$$

Note that the composition of Lipschitz functions, is indeed, another Lipschitz function, hence both f'' and g'' are Lipschitz, moreover:

$$
\begin{aligned}
d_X(g''(f''(x)), x) &= d_X(g(g'(f'(f(x)))), x) \\
&\leq d_X(g(g'(f'(f(x)))), g(f(x))) + d_X(g(f(x)), x) \\
&\leq (C d_X(g'(f'(f(x))), f(x)) + D) + D \leq D(C+2),
\end{aligned}
$$

and the result follows. $\square$

Proof of Theorem 3.15. Suppose that Δ is a quasi-retract of Γ with constants (C, D) in the following way:

$$\Delta \xrightarrow{f} \Gamma \xrightarrow{g} \Delta,$$

Hence we want to exhibit a subgraph $\Delta' \subseteq \Gamma$ and functions $\tilde{f} : \Delta' \to \Delta$ and $\tilde{g} : \Delta \to \Delta'$ that makes the diagram commute.

For this, define the vertex set of Δ'. First consider two adjacent vertices $x, y \in V(\Delta)$, as f is (C, D)-Lipschitz, $\mathsf{dist}_\Gamma(f(x), f(y)) \leq C + D$, hence, there exists a path γ of vertices $v_0 = f(x), v_1, \ldots, v_{n-1}, v_n = f(y)$ such that $n \leq C + D$. We denote $\gamma_{x,y} = \{v_0, \ldots, v_n\}$. Hence we get the following sets for vertices and edges.

$$V(\Delta') = \{v_i \in \gamma_{x,y} \, : x, y \in V(\Delta) \text{ with } \mathsf{dist}_\Delta(x,y) = 1\},$$

and,

$$E(\Delta') = \{(v_i, v_{i+1}) \, : v_i \in \gamma_{x,y} \text{ for all } x, y \in V(\Delta) \text{ with } \mathsf{dist}_\Delta(x,y) = 1\}.$$

Clearly Δ' is a subgraph of Γ. Note that f and g induce a quasi-isometry between Δ and $\Delta' \subset \Gamma$. Let us denote them $\tilde{f} : \Delta' \to \Delta$ and $\tilde{g} : \Delta \to \Delta'$, and Lemma 3.16 tells us that then, $\Delta' \subset \Gamma$ is indeed a quasi-retract of Γ. What is more, as all functions have finite distance with the others, we can modify the constants such that:

$$\Delta' \hookrightarrow \Gamma \xrightarrow{g'} \Delta',$$

is a quasi-retract and hence the diagram commutes.

□

3.2 Proof of Theorem 3.6

The argument relies on the following proposition.

Proposition 3.17. *Given two graphs Γ and Δ such that (f,g) is a quasi-retraction with constants (C,D) of Γ into Δ.*

If Γ is $\mathsf{CopWin}(n, \sigma, \rho, \psi, f(u_0), R)$ then Δ is $\mathsf{CopWin}(n, \sigma_\Delta, \rho_\Delta, \psi_\Delta, u_0, R_\Delta)$, where:

$$\sigma_\Delta = C\sigma + D, \quad \rho_\Delta = C\rho + 2D, \quad R = CR_\Delta + D, \quad \psi = (C+D)\psi_\Delta.$$

Proof. Fix the parameters $(n, \sigma, \rho, \psi, u_0, R)$ for the game in Γ, and let the parameters for the game on Δ be defined as in the statement of the proposition.

Suppose that Γ is $\mathsf{CopWin}(n, \sigma, \rho, \psi, f(u_0), R)$. To show that the graph Δ is $\mathsf{CopWin}(n, \sigma_\Delta, \rho_\Delta, \psi_\Delta, u_0, R_\Delta)$ we describe a winning strategy for Δ based on the winning strategy for Γ. We will be playing simultaneous games on Δ and Γ. The cops in Γ will move accordingly to a winning strategy. The moves of the cops in Γ will determine the moves of the cops in Δ, and the move of the robber on Δ will determine the move of the robber on Γ. We will show that this translates to a winning strategy for the cops on Δ.

In the set up of the game on Γ, the initial position of the i-th cop is denoted as $c^\Gamma_{(i,0)}$. For the game on Δ, choose the initial position of the i-th cop as

$$c_{(i,0)} = g(c^\Gamma_{(i,0)}).$$

In this way, each cop in Δ corresponds to a cop in Γ. At this point the robber in Δ chooses an initial position $r_0 \in \Delta$. Let the initial position of the robber in Γ be

$$r^\Gamma_0 = f(r_0).$$

For the game in Γ, let $c^\Gamma_{(1,l)}, c^\Gamma_{(2,l)}, \dots, c^\Gamma_{(n,l)}$ and r^Γ_l denote the locations of the the cops and the robber respectively after l turns; similarly for the game on Δ, let $c_{(1,l)}, c_{(2,l)}, \dots, c_{(n,l)}$ and r_l denote the locations of the the cops and the robber after l turns. The positions of the cops in Δ are defined by

$$c_{(i,l)} = g(c^\Gamma_{(i,l)}),$$

for $1 \leq i \leq n$. The movement from the position $c_{(i,l)}$ at the stage l to the position $c_{(i,l+1)}$ at stage $l+1$ is valid since

$$\begin{aligned} \mathsf{dist}_\Delta(c_{(i,l)}, c_{(i,l+1)}) = \mathsf{dist}_\Delta(g(c^\Gamma_{i,l}), g(c^\Gamma_{i,l+1})) &\leq C\, \mathsf{dist}_\Gamma(c^\Gamma_{i,l}, c^\Gamma_{i,l+1}) + D \\ &\leq C\sigma + D = \sigma_\Delta. \end{aligned}$$

At turn $l+1$, after the cops have made their movement in Δ, if the robber in Δ has not been captured, the robber moves from r_l to a position r_{l+1}.

Now the robber in Γ moves from r^Γ_l to

$$r^\Gamma_{l+1} = f(r_{l+1}).$$

We need to show that this is a valid move, that is, there is a path in Γ from r^Γ_l to r^Γ_{l+1} of length at most ψ which has every vertex a distance larger than ρ from every cop.

Since the move of the robber from r_l to r_{l+1} in Δ was valid, there is a path $[r_l = w_0, w_1, \dots, w_k = r_{l+1}]$ of length $k \leq \psi_\Delta$ such that $\rho_\Delta < \mathsf{dist}_\Delta(w_i, c_{i,l+1})$ for each

$1 \leq i \leq n$. It follows that

$$\begin{aligned}
\rho_\Delta &< \mathsf{dist}_\Delta(w_i, c_{(i,l+1)}) \\
&\leq \mathsf{dist}_\Delta(w_i, g(f(w_i)) + \mathsf{dist}_\Delta(g(f(w_i)), g(c^\Gamma_{(i,l+1)})) \\
&\leq 2D + C\,\mathsf{dist}_\Gamma(f(w_i), c^\Gamma_{(i,l+1)})).
\end{aligned}$$

Since $\rho_\Delta = C\rho + C + 3D$, we have that

$$\rho + C + D < \mathsf{dist}_\Gamma(f(w_i), c^\Gamma_{(i,l+1)}).$$

On the other hand,

$$\mathsf{dist}_\Gamma(f(w_i), f(w_{i+1})) \leq C\,\mathsf{dist}_\Delta(w_i, w_{i+1}) + D = C + D.$$

These last two inequalities imply that every vertex in a geodesic from $f(u_i)$ to $f(u_{i+1})$ is at distance larger than ρ from the cops positions $c^\Gamma_{i,l+1}$ during turn $l+1$. Hence a path from r^Γ_l to r^Γ_{l+1} obtained as a concatenation of geodesic paths between consecutive vertices of the sequence $r^\Gamma_l = f(w_0), f(w_1), \ldots, f(w_k) = r^\Gamma_{l+1}$ has length at most

$$(C + D)k \leq (C + D)\psi_\Delta = \psi$$

and every vertex is at distance larger than ρ from every from the cops positions $c^\Gamma_{i,l+1}$ during turn $l+1$. Hence the move of the robber in Γ during the $(l+1)$-turn is valid.

Through the rest of the game, the moves of the cops in Δ are given by the moves of the cops in Γ, and the moves of the robber in Γ are given by the moves of the robber in Δ as described above. The conclusion of the proposition then follows from the following two claims.

Claim 1: *Once the robber is captured in the game on Γ, the robber will be captured in the game on Δ.*

Suppose that the robber is captured in Γ during the l-turn. This means that

$\mathsf{dist}_\Gamma(c^\Gamma_{(i,l)}, r^\Gamma_l) \leq \sigma + \rho$ for some i. It follows that

$$\begin{aligned}
\mathsf{dist}_\Delta(c_{(i,l)}, r_l) &= \mathsf{dist}_\Delta(g(c^\Gamma_{i,l}), r_l) \\
&\leq \mathsf{dist}_\Delta(g(c^\Gamma_{i,l}), g(f(r_l)) + \mathsf{dist}_\Delta(g(f(r_l), r_l) \\
&\leq C\,\mathsf{dist}_\Gamma(c^\Gamma_{i,l}, f(r_l)) + 2D \\
&\leq C\,\mathsf{dist}_\Gamma(c^\Gamma_{i,l}, r^\Gamma_l) + 2D \\
&\leq C(\sigma + \rho) + 2D \\
&\leq \sigma_\Delta + \rho_\Delta
\end{aligned}$$

This shows that at the end of the round l the robber in Δ is at distance less than $\sigma_\Delta + \rho_\Delta$ from at least one cop, and hence it has been captured.

Claim 2: *If the robber is forced out of the ball in the game on Γ, the robber will be forced out of the ball in the game on Δ.*

Assume the robber is forced outside the ball in the game on Γ at stage l, that is, $\mathsf{dist}_\Gamma(r^\Gamma_l, f(u_0)) > R$. Then

$$\begin{aligned}
\mathsf{dist}_\Delta(r_l, u_0) &\geq \frac{1}{C}(\mathsf{dist}_\Gamma(f(r_l), f(u_0)) - D) \\
&= \frac{1}{C}(\mathsf{dist}_\Gamma(r^\Gamma_l, f(u_0)) - D) \\
&= \frac{1}{C}(\mathsf{dist}_\Gamma(r^\Gamma_l, f(u_0)) - D) \\
&\geq \frac{1}{C}(R - D) \\
&= R_\Delta
\end{aligned}$$

Hence, if the robber in Γ is outside the ball $B(f(u_0), R)$ in Γ, then the robber in Δ is outside the ball $B(u_0, R_\Delta)$.

This two claims together, implies the result.

□

Proof of Theorem 3.6. Let us suppose that (f, g) is a (C, D)-quasi-retraction from Γ into Δ. Suppose Γ is $\mathsf{CopWin}(n, \sigma, \rho, \psi, f(u), R)$. Proposition 3.17 implies that Δ is $\mathsf{CopWin}(n, \sigma_\Delta, \rho_\Delta, \psi_\Delta, u_0, R_\Delta)$ where ψ_Δ and R_Δ are the constants provided by

Proposition 3.17. Hence $\mathsf{wCop}(\Delta) \leq n$. The argument for $\mathsf{sCop}(\Delta) \leq \mathsf{sCop}(\Gamma)$ is analogous. $\square$

Chapter 4

Strong cop number of hyperbolic graphs

Is well known that *hyperbolicity* is a quasi-isometric invariant of metric spaces, hence study the behaviour of wCop and sCop on graphs with this property. There are several equivalent definitions of a hyperbolic space; we refer the reader to [Bridson and Haefliger, 1999] for a survey. For our purposes we use the following:

Definition 4.1. [Bridson and Haefliger, 1999, Chapter III.H, Definition 1.1] Let X be a geodesic metric space and let $\delta \geq 0$. A triangle is said to be δ-*slim* if each one of its sides is contained in the δ-neighborhood of the union of the other two sides. A geodesic metric space X is said to be δ-*hyperbolic* (or just *hyperbolic*) if every triangle whose sides are geodesics, is δ-slim. An example of this condition is given in Figure 4.1

A connected graph can be considered a geodesic metric space by regarding each edge as a segment of length one and imposing the path metric, that is, the distance between any two points is the length of a shortest path.

Definition 4.2. A connected graph is said to be *hyperbolic* if it is a hyperbolic metric space with the path metric.

Proposition 4.3. *If T is a tree, then T is hyperbolic for $\delta = 0$.*

Proof. As T contains no cycles, any triangle is a subtree with at most one vertex of degree three. It is immediate that two sides of the triangle contain the third side. $\square$

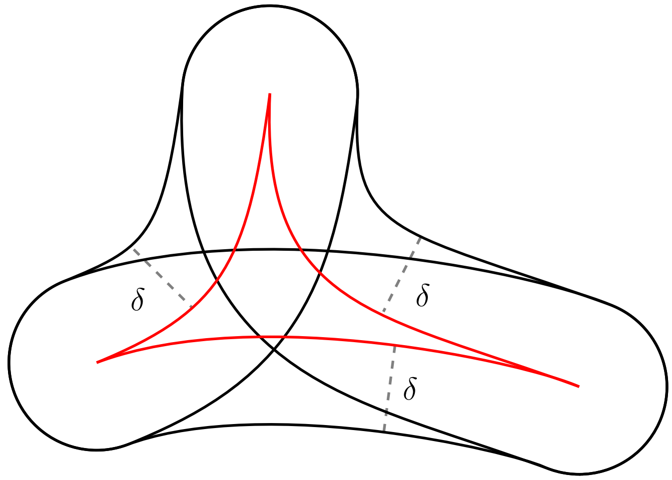

Figure 4.1: δ-slim condition

Example 4.4. Any regular tiling of the hyperbolic plane, is hyperbolic. Some examples of these are in Figure 4.2. The explanation of this statement is beyond the scope of this work, it is a consequence of Ŝvarc-Milnor Lemma, for this see [Löh, 2017]. Some examples of these are given in Figure 4.2.

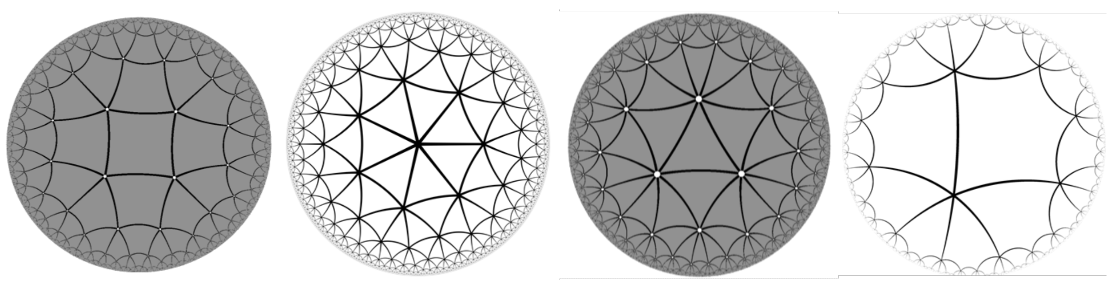

Figure 4.2: Regular tilings of the hyperbolic plane, from [Ruen, 2013]

Example 4.5. A particular class of examples of hyperbolic graphs are Cayley graphs of hyperbolic groups. Hyperbolic groups include groups with a cyclic group of finite index, free groups, groups that act properly discontinuously on locally finite trees, and fundamental groups of oriented closed connected surfaces of genus at least two. Understanding these examples goes beyond the scope of this document. We refer the interested reader to [Löh, 2017] and [Bridson and Haefliger, 1999].

It was proven in [Lee, 2019, Proposition 3.1] that if Γ is a tree, $\mathsf{wCop}(\Gamma) = 1$, hence $\mathsf{sCop}(T) = 1$ as well. Trees are the first examples of hyperbolic graphs, hence, we initially thought to extend the property of being 1-weak-cop win to hyperbolic graphs, but, as we will show, this is not true. Nevertheless, this holds for the strong cop number through the following result.

Theorem 4.6. *Hyperbolic graphs are* 1*-strong-cop win.*

To establish this, we use the following lemma.

Lemma 4.7. *[Bridson and Haefliger, 1999, Chapter III.H, Proposition 1.6] Let X be a δ-hyperbolic geodesic space, let C be a continuous rectifiable path in X, if $[p, q]$ is a geodesic segment containing the endpoints of C, then for all $x \in [p, q]$:*

$$\mathsf{dist}(x, C) \leq \delta |\log_2(l(C))| + 1,$$

with $l(C)$ the length of C.

Proof of Theorem 4.6. Note that we can assume that δ is an integer by taking $\lceil \delta \rceil$, this is the smallest integer greater or equal to δ.

Consider a hyperbolic graph, and the game for a robber with speed ψ and one cop with speed $\sigma = 2\delta + 1$ and reach $\rho = \delta|\log_2(\psi)| + \delta + \psi + 1$. The cop must protect the closed ball $\overline{B_R}(u_0)$. Let us assume the initial location of the cop is $c_1 = u_0$ and the initial position of the robber is r_1 and let g_1 be a geodesic path from u_0 to r_1. At the beginning of the n-th turn, the position of the cop is denoted by c_n and the position of the robber is r_n. We describe a strategy for the cop that guaranties the following two conditions for every n:

1. If after the cop moves on the n-th turn, and the robber has not been captured; then c_{n+1} is a vertex of a geodesic from u_0 to r_n,
2. $\mathsf{dist}(u_0, c_n) < \mathsf{dist}(u_0, c_{n+1})$.

Let us recall that at the beginning of every turn, the cop moves first. The strategy for the cop is defined below:

- In the first turn, the cop moves in the direction of the robber along g_1.
- If at the beginning of the $(n + 1)$-th turn the robber has not been captured, then the movement of the cop is described as follows: Consider a δ-slim triangle

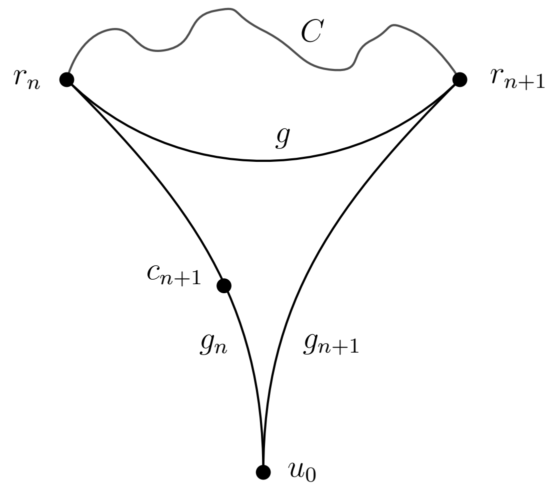

Figure 4.3: Illustration of the thin triangle defining the strategy of the cops on the proof of Theorem 4.6

with vertices r_n, r_{n+1} and u_0 and with sides g_n, g_{n+1} and g, where g is defined as a geodesic path between r_n and r_{n+1}. We assume that c_{n+1} is a vertex of the geodesic g_n.

The assumption that the robber was not captured during the n-th turn implies that $\mathsf{dist}(c_{n+1}, g) > \delta$. Indeed, suppose that $\mathsf{dist}(c_{n+1}, g) \leq \delta$. Then there exists a vertex x in the geodesic g such that $\mathsf{dist}(c_{n+1}, x) \leq \delta$. Suppose that the robber moved from r_n to r_{n+1} along the path C, and hence its length satisfies $l(C) \leq \psi$. Lemma 4.7 implies that

$$\mathsf{dist}(c_{n+1}, C) \leq \mathsf{dist}(c_{n+1}, x) + \mathsf{dist}(x, C) \leq \delta + \delta|\log_2(l(C))| + 1 \leq \rho,$$

and therefore the robber was captured. This contradicts our hypothesis and therefore $\mathsf{dist}(c_{n+1}, g) > \delta$.

Since the triangle is δ-thin there is a vertex y on g_{n+1} at distance less than or equal to δ from c_{n+1}. The cop will move first to y. If the robber has not been captured yet, then $\mathsf{dist}(y, r_{n+1}) > \rho > \delta + 1$ and the cop moves $\delta + 1$ units along the geodesic g_{n+1} in the direction of r_{n+1}. Note that the cop can do this total movement due to the assumption that $2\delta + 1 \leq \sigma$. Moreover,

$$\begin{aligned}
\mathsf{dist}(u_0, c_{n+2}) &= \mathsf{dist}(u_0, y) + \delta + 1 \\
&\geq \mathsf{dist}(u_0, c_{n+1}) - \mathsf{dist}(c_{n+1}, y) + \delta + 1 \\
&\geq \mathsf{dist}(u_0, c_{n+1}) - \delta + \delta + 1 \\
&> \mathsf{dist}(u_0, c_{n+1}).
\end{aligned}$$

Let us verify that this is a winning strategy for the cop. Assume the cop follows the strategy and the robber is never captured. After the cop moves, on the n-th turn, $\mathsf{dist}(u_0, r_n) = \mathsf{dist}(u_0, c_{n+1}) + \mathsf{dist}(c_{n+1}, r_n)$ due to condition (1). In particular $\mathsf{dist}(u_0, r_n) \geq \mathsf{dist}(u_0, c_{n+1})$. After a finite number of steps, let us say m, the cop will be located at distance at least R from u_0, this is $\mathsf{dist}(u_0, c_n) \geq R$ for every $n \geq m$, due to condition (2). It follows that $\mathsf{dist}(u_0, r_n) \geq R$ for any $n \geq m$, thus the cop has been able to protect the ball.

$\square$

Chapter 5

Weak cop number of one-ended non-amenable graphs

The objective of this section is to prove the following result.

Theorem 5.1. *If* Γ *is a connected, one-ended, non-amenable, vertex transitive graph with uniform bounded degree, then* $\mathsf{wCop}(\Gamma) = \infty$.

In order to do this, we divide the chapter into 3 sections. The first section establish the definitions and some results on *ends of graphs* and *amenability*, respectively. The third is meant to construct the lemmas and propositions we use to proof the main result.

5.1 One-ended graphs

The *ends of a graph* for an infinite graph can be interpreted as the set of "directions" when the graph goes to infinity. There are different ways to define the ends of a graph. The concept is more generally defined using the concept of *geodesic rays* but a more intuitive definition can be done when the graph is locally finite and connected, see [Diestel and Kühn, 2003].

As we consider a graph as a metric space, we can consider the *ball of radius* r *with center* v denoted as $B_r(v)$. We denote the number of connected, unbounded components on the complement of $B_r(v)$ as $||\Gamma \setminus B_r(v)||$. For unbounded component we refer to a subgraph with vertex at arbitrary distance.

Definition 5.2. Let Γ be a connected, locally finite graph, and for any vertex $v \in V(\Gamma)$ then the *number of ends* of Γ is

$$\mathcal{E}(\Gamma) = \lim_{r\to\infty} ||\Gamma \setminus B_r(v)||$$

Note that the sequence $||\Gamma \setminus B_r(v)||$ for $r = 1, 2, 3\dots$ is non-decreasing and moreover the base point is irrelevant; hence the limit is well defined.

Example 5.3. The following examples illustrate better the concept of ends:

1. If Γ is a finite graph, then $\mathcal{E}(\Gamma) = 0$.
2. Let Γ be the infinite square grid, $||\Gamma \setminus B_r(v)|| = 1$ for any r, hence $\mathcal{E}(\Gamma) = 1$.
3. Let Γ be the infinite length path, $||\Gamma \setminus B_r(v)|| = 2$ for any $r \geq 0$, hence $\mathcal{E}(\Gamma) = 2$.
4. Let Γ be the 4-regular infinite tree, $||\Gamma \setminus B_r(v)|| = 4(3)^r$, hence $\mathcal{E}(\Gamma) = \infty$. A local view of this is shown in Figure 5.1.

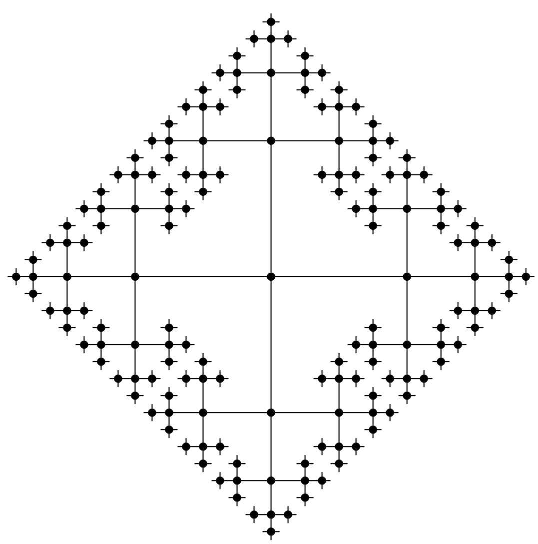

Figure 5.1: Local view of 4-regular infinite tree

Definition 5.4. A locally finite graph is said to be *almost symmetrical* if its automorphism group have finitely many orbits.

Theorem 5.5. *[Mohar, 1991] Let Γ be a connected, locally finite and almost symmetrical graph, then $\mathcal{E}(\Gamma) \leq 2$ or $\mathcal{E}(\Gamma) = \infty$.*

5.2 Non-amenable graphs

There are many different ways to define the concept of amenability. Most of those are beyond the scope of this document, but more information can be found on [Löh, 2017]. We present the following definition.

Definition 5.6. For a subset of vertices K of a graph Γ, let $|K|$ denote number of vertices in K, and let ∂K be the set of edges of Γ with one endpoint in K and the other endpoint not in K. Hence $|\partial K|$ is the number of edges with the given characteristic.

The *Cheeger constant* of a graph Γ is defined as:

$$h(\Gamma) = \inf_K \frac{|\partial K|}{|K|},$$

where K is any non-empty finite subset of vertices. We say that a graph Γ with uniform bounded degree is *non-amenable* if $h(\Gamma) > 0$, otherwise we say that the graph is *amenable.*

Remark 5.7 (Convention)**.** When we consider a non-amenable graph we always assume that the graph is connected.

Example 5.8. The infinite length path is amenable since $|\partial K| = 2$ for any connected non-empty finite subset of vertices.

Theorem 5.9. *The* 4*-regular infinite tree (Figure 5.1), is non-amenable.*

In order to prove this consider the following results.

Lemma 5.10. *Let T be a finite tree such that every vertex is either a leaf or has degree at least* 3*. If T has ℓ leaves, then T has at most $2\ell - 3$ edges.*

Proof. Let T be a tree as in the statement and denote by $e(T)$ and $\ell(T)$ the number of edges and leaves respectively. Note that there is a tree Δ such that every vertex of Δ is either a leaf or has degree 3, and a simplicial map (i.e. the image of an edge is another edge or a vertex), $\Delta \to T$ which, when restricted to the set of leaves, is a bijection. In particular, $e(T) \leq e(\Delta)$ and $\ell(T) = \ell(\Delta)$. Hence, it is enough to prove the statement for trees for which every interior vertex has degree 3.

Let T be a tree such that every vertex is either a leaf or has degree three. If $\ell(T) \leq 2$ the statement of the lemma is trivial. Suppose that $\ell(T) > 2$. Let u be a leaf adjacent to v. Since $\ell(T) > 2$, v is a vertex of degree three. Let T' be the tree obtained by deleting u and v, and then adding an edge between the two remaining vertices that were adjacent to v. Then T' is a tree such that every vertex is either a leaf or has degree three. Since $e(T') + 2 = e(T)$ and $\ell(T') + 1 = \ell(T)$, by induction we have that

$$e(T) = e(T') + 2 \leq 2\ell(T') - 1 = 2\ell(T) - 3.$$

□

Lemma 5.11. *If T is a finite tree with ℓ leaves and d vertices of degree 2, if e denotes the number of edges of T, then:*

$$e \leq 2\ell - 3 + d.$$

Proof. This is a direct consequence of Lemma 5.10. □

Proof of theorem 5.9: Let K be a finite subset of vertices and let T be the induced subgraph. We will show that $|\partial T|/|T| \geq 1/4$. Let $e(T)$, $\ell(T)$ and $d(T)$ denote the number of edges, leaves and vertices of degree 2 of T respectively.

First suppose that T is connected. If T has no edges, then it is a single vertex and $|\partial T|/|T| = 4$. Suppose that T has edges. Then $|T| = e(T) + 1 \leq 2e(T)$ and therefore

$$\frac{|\partial T|}{|T|} \geq \frac{\ell(T) + d(T)}{2e(T)} \geq \frac{1}{2e(T)} \left(\frac{e(T) - d(T) + 3}{2} + d(T) \right) \geq \frac{1}{4} \left(\frac{e(T) + 3 + d}{e(T)} \right) \geq \frac{1}{4},$$

where the second inequality is given by Lemma 5.11.

Suppose that T is not connected. Then $T = T_1 \cup T_2 \cup \cdots \cup T_k$ where each T_i is a connected component of T. Since T is an induced subgraph, if $i \neq j$ then $\mathsf{dist}(T_i, T_j) \geq 2$ and in particular $\partial T_i \cap \partial T_j = \emptyset$. It follows that

$$|\partial T| = \sum_{i=1}^{k} |\partial T_i|.$$

Define subgraphs $T_1', \ldots, T_k'$ such that T_j' is isomorphic to T_j and $\mathsf{dist}(\bigcup_{i=1}^{j-1} T_i', T_j') = 1$ for $1 \leq j \leq k$. The existence of the subgraphs T_j' can be argued by induction on

j by letting $T'_1 = T_1$ and observing that if $T'_1, \ldots, T'_{j-1}$ have been defined there is a subgraph T'_j with the required properties. Let T^* be the subgraph induced by the union of the sets of vertices of all T'_j. Note that T^* is a tree arising as the union of $\bigcup_{j=1}^k T'_j$ and $k-1$ additional edges. In particular, $|T^*| = |T|$ and by induction one verifies that

$$|\partial T^*| = \sum_{j=1}^{k} |\partial T'_j| - 2(k-1) = |\partial T| - 2(k-1).$$

Since T^* is connected, it follows that

$$\frac{|\partial T|}{|T|} \geq \frac{|\partial T^*|}{|T^*|} \geq \frac{1}{4}.$$

□

5.3 One-ended non-amenable implies fast-robber win

In order to prove the Theorem 5.1, we use several lemmas and concepts that are shown/introduced this section. First, we provide a definition and some remarks.

Definition 5.12 (Rips's Graph [Gromov, 1987])**.** Let Γ be a graph and let $\Delta_1, \ldots, \Delta_n$ be a collection of n subgraphs. Define $\mathsf{Rips}_\Gamma(\Delta_1, \ldots, \Delta_n; L)$ as the graph with vertex set $\{\Delta_1, \ldots, \Delta_n\}$ and edge set $\{\{\Delta_i, \Delta_j\} \mid 0 < \mathsf{dist}_\Gamma(\Delta_i, \Delta_j) \leq L\}$. One example of this is given in Figure 5.2.

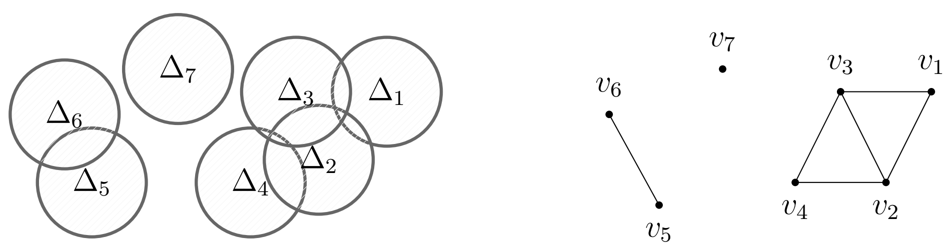

Figure 5.2: Schematic of a graph $\mathsf{Rips}_\Gamma(\Delta_1, \ldots, \Delta_7; L)$

Remark 5.13. If $\mathsf{Rips}_\Gamma(\Delta_1, \ldots, \Delta_n; L)$ is disconnected then, after re-enumerating the Δ_i's, there is $1 \leq k < n$ such that $\mathsf{Rips}_\Gamma(\Delta_1, \ldots, \Delta_n; L)$ is the disjoint union of $\mathsf{Rips}_\Gamma(\Delta_1, \ldots, \Delta_k; L)$ and $\mathsf{Rips}_\Gamma(\Delta_{k+1}, \ldots, \Delta_n; L)$.

Remark 5.14 (Notation for cardinality of spheres and balls)**.** Let v be a fixed vertex of Γ. Let $\alpha(n)$ denote the number of vertices of Γ at distance exactly n from v, and let $\beta(n)$ denote the number of vertices of Γ at distance at most n from v. If Γ is vertex transitive, $\alpha(n)$ and $\beta(n)$ are independent of the choice of v, and we refer to them as the *size of the spheres of radius* n, and the *size of the balls of radius* n, respectively.

For our purposes, since Γ is necessarily an infinite graph, we have that $\beta\colon \mathbb{Z}_+ \to \mathbb{Z}_+$ is an increasing function and in particular $\lim_{n\to\infty} \beta(n) = \infty$.

5.3.1 Existence of Robber Speed

The purpose of this subsection is to prove the following proposition.

Proposition 5.15 (Undistorted Embedding)**.** *Let Γ be a connected, vertex transitive, locally finite graph. For any pair of integers m and n, there is an integer $L_{m,n} = L(\Gamma, m, n)$ with the following property: Let Δ be a subgraph of Γ with at most m vertices and such that $\Gamma - \Delta$ is connected. Suppose that Δ is the union of pairwise disjoint connected subgraphs $\Delta_1, \Delta_2, \ldots, \Delta_n$. If a, b are vertices in $\Gamma - \Delta$ then:*

$$\mathsf{dist}_{\Gamma-\Delta}(a, b) \leq L_{m,n} \,\mathsf{dist}_{\Gamma}(a, b).$$

This proposition will allow us to define a robber speed in the main theorem. The proof of the proposition is divided into a series of lemmas. The definition of the constants $L_{m,n}$ is given as part of Lemma 5.18. That the constants $L_{m,n}$ satisfy the statement of the proposition is proved by induction on n, where Lemma 5.19 provides the case $n = 1$, and then Lemma 5.20 concludes the proof of the proposition.

Lemma 5.16. *Let m, n and L be positive integers and let Γ be a locally finite, vertex transitive graph. Then, up to the action by $\mathsf{Aut}(\Gamma)$, there are finitely many subgraphs Δ such that*

- $|\Delta| \leq m$,
- *Δ is the union of n pairwise disjoint connected subgraphs $\Delta = \Delta_1 \cup \cdots \cup \Delta_n$,*
- $0 < \mathsf{dist}_{\Gamma}(\Delta_1, \Delta_i) \leq L$ *for* $1 < i \leq n$.

Proof. Fix a vertex u of Γ. Since Γ is locally finite, there are finitely many subgraphs Δ as in the statement that contain the vertex u. Since Γ is vertex transitive, lemma follows. $\square$

Lemma 5.17. *Let m, n and L be positive integers and let Γ be a locally finite, vertex transitive graph. Then, up to the action by $\mathsf{Aut}(\Gamma)$, there are finitely many subgraphs Δ such that*

- $|\Delta| \leq m$,
- *Δ is the union of n pairwise disjoint connected subgraphs $\Delta = \Delta_1 \cup \cdots \cup \Delta_n$,*
- *$\mathsf{Rips}_\Gamma(\Delta_1, \ldots, \Delta_n; L)$ is connected.*

Proof. Since each Δ_i is connected with at most m vertices, it follows that each Δ_i has diameter at most $m-1$. Since $\mathsf{Rips}_\Gamma(\Delta_1, \ldots, \Delta_n; L)$ is connected, we have that $0 < \mathsf{dist}_\Gamma(\Delta_1, \Delta_i) \leq (m+n)L$ for $1 < i \leq n$. Then the statement follows from Lemma 5.16. $\square$

Lemma 5.18 (Definition of $L_{m,n}$)**.** *Let Γ be a locally finite, vertex transitive graph. For integers $m > 0, n \geq 0$, let $L_{m,0} = 0$ and*

$$\begin{aligned} L_{m,n} = \max\{\, & \mathsf{dist}_{\Gamma-\Delta}(a,b) \mid \\ & 1 \leq k \leq n, \\ & \Delta_1, \ldots, \Delta_k \text{ are disjoint connected subgraphs of } \Gamma, \\ & \Delta = \Delta_1 \cup \ldots \cup \Delta_k, \\ & \mathsf{Rips}_\Gamma(\Delta_1, \ldots, \Delta_k; L_{m,k-1}) \text{ is connected}, \\ & \Gamma - \Delta \text{ is connected}, \\ & |\Delta| \leq m, \\ & a, b \in V(\Gamma) - V(\Delta), \text{ and} \\ & \mathsf{dist}_\Gamma(a, \Delta) = 1 \text{ and } \mathsf{dist}_\Gamma(b, \Delta) = 1\}. \end{aligned}$$

Then $L_{m,n}$ are well-defined integers such that:

$$0 \leq L_{m,1} \leq L_{m,2} \leq \cdots L_{m,k} \leq L_{m,k+1} \leq \cdots .$$

Proof. By Lemma 5.17, for $m > 0$ and $n > 0$, $L_{m,n}$ is the maximum of a finite number of integers and hence it is well defined. That $L_{m,k} \leq L_{m,k+1}$ is immediate from the definition. □

Lemma 5.19 (Base Case)**.** *Let Γ be a locally finite, vertex transitive graph, for any connected subgraph Δ such that $|\Delta| \leq m$ and $\Gamma - \Delta$ is connected, if a, b are vertices in $\Gamma - \Delta$ then*

$$\mathsf{dist}_{\Gamma-\Delta}(a,b) \leq L_{m,1}\,\mathsf{dist}_{\Gamma}(a,b).$$

Proof. Let a and b be vertices of $\Gamma - \Delta$ and let p be a geodesic in Γ between them. Then one can replace maximal sub-paths of p with all interior vertices in Δ with sub-paths in $\Gamma - \Delta$ of length at most $L_{m,1}$ showing that $\mathsf{dist}_{\Gamma-\Delta}(a,b) \leq L_{m,1}\,\mathsf{dist}_{\Gamma}(a,b)$. □

Lemma 5.20 (Inductive Step)**.** *Let Γ be a locally finite, vertex transitive graph, let Δ be a subgraph such that $|\Delta| \leq m$, $\Gamma - \Delta$ is connected, and Δ is the union of n pairwise disjoint connected subgraphs $\Delta = \Delta_1 \cup \cdots \cup \Delta_n$. If a, b are vertices in $\Gamma - \Delta$ then*

$$\mathsf{dist}_{\Gamma-\Delta}(a,b) \leq L_{m,n}\,\mathsf{dist}_{\Gamma}(a,b).$$

Proof. Fix m. We argue by induction on n that $L_{m,n}$ satisfies the property of the statement of the lemma. The base case $n = 1$ is Lemma 5.19. Hence, suppose $n > 1$ and that the result holds for all $L_{m,k}$ if $k < n$. We consider two cases:

Case 1. Suppose that $\mathsf{Rips}_{\Gamma}(\Delta_1, \ldots, \Delta_n; L_{m,n-1})$ is disconnected. By Remark 5.13, we can assume that there is $1 \leq k < n$ such that the graph $\mathsf{Rips}_{\Gamma}(\Delta_1, \ldots, \Delta_n; L_{m,n-1})$ is the disjoint union of $\mathsf{Rips}_{\Gamma}(\Delta_1, \ldots, \Delta_k; L_{m,n-1})$ and $\mathsf{Rips}_{\Gamma}(\Delta_{k+1}, \ldots, \Delta_n; L_{m,n-1})$.

Let $\Lambda_1 = \Delta_1 \cup \cdots \cup \Delta_k$ and $\Lambda_2 = \Delta_{k+1} \cup \cdots \cup \Delta_n$, and observe that

$$\mathsf{dist}_{\Gamma}(\Lambda_1, \Lambda_2) > L_{m,n-1}.$$

Note that the subgraph $\Gamma - \Lambda_i$ is connected, $|\Lambda_i| \leq m$, and Λ_i has at most $n-1$ connected components. By induction, the constant $L_{m,n-1}$ applies to both Λ_i. More specifically, suppose that $a, b \in V(\Gamma)$ satisfy that $\mathsf{dist}(a, \Lambda_1) = \mathsf{dist}(b, \Lambda_1) = 1$. By definition of $L_{m,n-1}$, any geodesic path q in $\Gamma - \Lambda_1$ between a and b has length at most $L_{m,n-1}$ and hence it does not intersect the subgraph Λ_2. It follows that q is a path in $\Gamma - \Delta$. This reasoning is symmetric in Λ_1 and Λ_2.

Let v and w be vertices of $\Gamma - \Delta$ and let p be a geodesic in Γ between them. Then each maximal sub-path of p with all internal vertices in Λ_1 (respectively, Λ_2) can be replaced with a sub-path in $\Gamma - \Delta$ of length at most $L_{m,n-1}$. In this way, one can obtain a path between v and w of length at most $L_{m,n-1}\,\mathsf{dist}_\Gamma(v,w)$ in $\Gamma - \Delta$. Hence

$$\mathsf{dist}_{\Gamma-\Delta}(v,w) \leq L_{m,n-1}\,\mathsf{dist}_\Gamma(v,w) \leq L_{m,n}\,\mathsf{dist}_\Gamma(v,w).$$

Case 2. Suppose that $\mathsf{Rips}_\Gamma(\Delta_1,\ldots,\Delta_n;L_{m,n-1})$ is connected. Let v and w be vertices of $\Gamma - \Delta$. Let p be a geodesic path in Γ between v and w. Then, by definition of $L_{m,n}$, one can replace each maximal subpath of p with all interior vertices in Δ with subpaths in $\Gamma - \Delta$ of length at most $L_{m,n}$. This produces a path in $\Gamma - \Delta$ between v and w of length at most $L_{m,n}\,\mathsf{dist}_\Gamma(v,w)$, and hence $\mathsf{dist}_{\Gamma-\Delta}(v,w) \leq L_{m,n}\,\mathsf{dist}_\Gamma(v,w)$.

□

5.3.2 Proof of Theorem 5.1

Before proving the main theorem, we need additional lemmas.

Remark 5.21 (Estimations using the Cheeger constant)**.** Let Γ be a non-amenable connected graph with uniform bounded degree and Cheeger constant $h(\Gamma)$. Let K be a finite subset of vertices and consider the subgraph $\Gamma - K$. If Δ is a connected component of $\Gamma - K$ then $\partial\Delta \subset \partial K$. Therefore

1. If Δ is a finite connected component, then

$$|\Delta| \leq \frac{1}{h(\Gamma)}|\partial\Delta| \leq \frac{1}{h(\Gamma)}|\partial K|.$$

2. The number of connected components of $\Gamma - K$ is at most $|\partial K|$. As a consequence, the number of vertices of $\Gamma - K$ that belong to a finite connected component is at most $\frac{1}{h(\Gamma)}|\partial K|^2$.

3. If Δ is a connected subgraph of Γ disjoint from K, and $|\Delta| > \dfrac{1}{h(\Gamma)}|\partial K|$; then Δ is a subgraph of an unbounded connected component of $\Gamma - K$.

Remark 5.22. Let Γ be a connected, locally finite, infinite graph. A connected component Δ is finite if and only if it is bounded. Indeed, observe that in a locally finite graph, any ball of finite radius has finitely many vertices. Hence a bounded locally finite graph is finite. On the other hand, observe that a graph with finitely many vertices is bounded.

Lemma 5.23. *Let Γ be a one-ended, non-amenable, vertex transitive connected graph and Cheeger constant $h(\Gamma)$. For any positive integers n and ρ, there is an integer $m = m(\Gamma, n, \rho) > 0$ with the following property. If $\{c_1, \ldots, c_n\}$ is any collection of n vertices of Γ, and Λ is the unbounded connected component of the subgraph $\Gamma - \bigcup_{i=1}^n B_\rho(c_i)$, then*

$$\mathsf{dist}_\Lambda(a, b) \leq L_{m,n}\, \mathsf{dist}_\Gamma(a, b).$$

for any pair of vertices a, b of Λ, where $L_{m,n}$ is defined in Lemma 5.18 and:

$$m = n\beta(\rho) + \frac{(\alpha(\rho) \cdot d \cdot n)^2}{h(\Gamma)}$$

where d is the degree of each vertex in Γ.

Proof. Let Δ be the subgraph of Γ induced by all vertices in $\bigcup_{i=1}^n B_\rho(c_i)$ together with all vertices that belong to bounded components of $\Gamma - \bigcup_{i=1}^n B_\rho(c_i)$. Since Γ is one-ended, the graph $\Gamma - \Delta$ is the unbounded connected component of $\Gamma - \bigcup_{i=1}^n B_\rho(c_i)$. In particular, $\Lambda = \Gamma - \Delta$ is connected.

Let us argue that Δ has at most m vertices. Observe that

$$\left|\bigcup_{i=1}^n B_\rho(c_i)\right| \leq n\beta(\rho) \quad \text{and} \quad \left|\partial \bigcup_{i=1}^n B_\rho(c_i)\right| \leq \alpha(\rho) \cdot d \cdot n.$$

Hence, by Remark 5.21, the number of vertices in $\Gamma - \bigcup_{i=1}^n B_\rho(c_i)$ that belong to bounded components is at most $\frac{1}{h(\Gamma)} \left(\alpha(\rho) \cdot n \cdot d\right)^2$ and therefore

$$|\Delta| \leq n\beta(\rho) + \frac{1}{h(\Gamma)} \left(\alpha(\rho) \cdot d \cdot n)\right)^2 = m.$$

Now we argue that Δ has at most n connected components. First observe that the subgraph of Γ induced by a ball $B_\rho(c_i)$ is connected. Since Γ is connected, every connected component of $\Gamma - \bigcup_{i=1}^n B_\rho(c_i)$ has a vertex adjacent to a vertex in $\bigcup_{i=1}^n B_\rho(c_i)$.

It follows that every vertex of Δ is in a connected component containing a ball $B_\rho(c_i)$, and therefore Δ has at most n connected components.

To summarize Δ is a subgraph of Γ with at most m vertices, at most n connected components, and such that $\Gamma - \Delta$ is a connected subgraph. By Proposition 5.15, for any vertices $a, b \in \Gamma - \Delta$, we have that

$$\mathsf{dist}_\Lambda(a,b) = \mathsf{dist}_{\Gamma-\Delta}(a,b) \leq L_{m,n}\,\mathsf{dist}_\Gamma(a,b).$$

□

Lemma 5.24 (Safe Distance)**.** *Let Γ be a connected, vertex transitive, non-amenable graph with uniform bounded degree. For any integers n and ρ there exists an integer $\lambda = \lambda(\rho, n)$ such that for any collection of $(n+1)$ vertices, denoted as r and $(c_1, \ldots, c_n)$, if $\mathsf{dist}(r, c_i) > \lambda$ for every i, then r lies in an unbounded component of $\Gamma - \bigcup_{i=1}^n B_\rho(c_i)$.*

Proof. Let d be the degree of vertices of Γ. Define λ to be the least integer such that

$$\beta(\lambda) > \frac{1}{h(\Gamma)}(\alpha(\rho) \cdot d \cdot n).$$

Let r and $(c_1, \ldots, c_n)$ be $n+1$ arbitrary vertices of Γ such that $\mathsf{dist}(r, c_i) > \lambda$ for every i. Let $K = \bigcup_{i=1}^n B_\rho(c_i)$ and observe that

$$|\partial K| \leq \sum_{i=1}^n |\partial B_\rho(c_i)| \leq n \cdot d \cdot \alpha(\rho).$$

Let Δ be the subgraph induced by $B_\lambda(r)$. Observe that Δ is connected and $|\Delta| = \beta(\lambda) > \frac{1}{h(\Gamma)}|\partial K|$. By the third item of Remark 5.21, the subgraph Δ lies inside the unbounded connected component of $\Gamma - K$ that contains the vertex r. □

Lemma 5.25 (Robber's strategy safe points)**.** *Let Γ be a connected non-amenable vertex transitive graph, and let u_0 be a fixed vertex. For any positive integers n, σ and ρ, there exists integers R and $D > \rho$, and there are $(n+1)$ vertices $\{s_1, \ldots, s_{n+1}\}$ such that the following properties hold:*

1. *For every i, $\mathsf{dist}(u_0, s_i) \leq R$.*
2. *For any collection of n vertices $\{c_1, \ldots, c_n\}$, there is a vertex s in $\{s_1, \ldots, s_{n+1}\}$ such that $\mathsf{dist}(s, \{c_1, \ldots, c_n\}) > D + \sigma$.*

3. If $s \in \{s_1, \ldots, s_{n+1}\}$ satisfies $\mathsf{dist}(s, \{c_1, \ldots, c_n\}) > D$, then s belongs to an unbounded component of $\Gamma - \bigcup_{i=1}^{n} B_\rho(c_i)$.

Proof. Let $\lambda(\rho, n)$ be the constant provided by Lemma 5.24. Let

$$D > \max\{\lambda(\rho, n), \rho\}.$$

Since Γ is a locally finite, infinite and connected graph, we can let $s_1 = u_0$ and choose inductively vertices s_i such that $\mathsf{dist}(s_i, \{s_1, \ldots, s_{i-1}\}) > 2D+2\sigma$ to obtain a collection of $n+1$ vertices $s_1, \ldots, s_n, s_{n+1}$ with the property that $\mathsf{dist}(s_i, s_j) > 2D + 2\sigma$ if $i \neq j$. Let $R = \max\{\mathsf{dist}(u_0, s_i) \mid 1 \leq i \leq n+1\}$. The first item of the lemma is immediate.

Let $\{c_1, \ldots, c_n\}$ be n vertices. First we prove that there is $s \in \{s_1, \ldots, s_n\}$ such that $\mathsf{dist}(s, \{c_1, \ldots, c_n\}) > D + \sigma$. Suppose by contradiction that this is not the case, that is, for each $1 \leq i \leq n+1$, there exists at least one c_j such that $\mathsf{dist}(s_i, c_j) \leq D+\sigma$. By the pigeon-hole argument, there must be c_i and two distinct s_j and s_k such that $\mathsf{dist}(s_j, c_i) \leq D + \sigma$ and $\mathsf{dist}(s_k, c_i) \leq D + \sigma$. It follows that

$$\mathsf{dist}(s_j, s_k) \leq \mathsf{dist}(s_j, c_i) + \mathsf{dist}(s_k, c_i) \leq 2D + 2\sigma,$$

which contradicts the properties of the set $\{s_1, \ldots, s_{n+1}\}$.

Let $s \in \{s_1, \ldots, s_n\}$ such that $\mathsf{dist}(s, c_i) > D$ for $1 \leq i \leq n$. Since $D > \lambda_\Gamma(\rho, n)$, Lemma 5.24 implies that s lies in an unbounded component of $\Gamma - \bigcup_{i=1}^{n} B_\rho(c_i)$. $\square$

Now we have all the necessary tools to prove the main result.

Proof of Theorem 5.1. Let Γ be the given graph and let σ, ρ, u_0, be fixed parameters. We will prove that for every n there exists an R and ψ such that Γ is not n-weak-cop win. First, we construct the parameters that we are going to use, and then we will prove that using these R and ψ, then Γ indeed is not n-weak-cop win.

Consider a collection of vertices $\{s_1, \ldots, s_{n+1}\}$ and the integers R and D provided by Lemma 5.25 for the parameters u_0, n, σ, ρ. Let

$$\psi = 2RL_{m,n}$$

where $L_{m,n}$ is the constant provided by Proposition 5.15 for

$$m = n\beta(\rho) + \frac{(\alpha(\rho) \cdot n \cdot d)^2}{h(\Gamma)}.$$

Consider the game in Γ with parameters $(n, \sigma, \rho, \psi, u_0, R)$. We will prove that the robber has an strategy such it is never captured and at any stage of the game. His position is on a vertex in $\{s_1, \dots, s_{n+1}\}$. Since $\mathsf{dist}_\Gamma(u_0, \{s_1, \cdots, s_{n+1}\}) < R$, this will be a winning strategy for the robber.

The robber will move in the following way:

- Suppose the n cops have chosen their initial positions, say $c_{1,0}, \dots c_{n,0}$. By the second item of Lemma 5.25, there is a vertex

  $$r_0 \in \{s_1, \dots, s_{n+1}\}$$

  such that

  $$\mathsf{dist}_\Gamma(s_i, \{c_{1,0}, \dots, c_{n,0}\}) > D + \sigma.$$

  Let r_0 be the initial position of the robber. Since Γ is one-ended, $\Gamma - \bigcup_{i=1}^n B_\rho(c_{i,0})$ has one unbounded connected component. The third item of Lemma 5.25 implies that r_0 is in the unbounded component of $\Gamma - \bigcup_{i=1}^n B_\rho(c_{i,0})$.

- Let r_k and $c_{1,k}, \dots, c_{n,k}$ denote the positions of the robber and the cops respectively at the end of the k-turn. Suppose that

  $$r_k \in \{s_1, \dots, s_{n+1}\},$$

  and

  $$\mathsf{dist}_\Gamma(r_k, \{c_{1,k}, \dots, c_{n,k}\}) > D + \sigma.$$

  Then, at the beginning of the $(k+1)$-turn, the cops move first. The last inequality implies that

  $$\mathsf{dist}_\Gamma(r_k, \{c_{1,k+1}, \dots, c_{n,k+1}\}) > D > \rho$$

  and hence the robber has not been captured. By the second item of Lemma 5.25, there is a vertex $r_{k+1} \in \{s_1, \ldots, s_{n+1}\}$ such that

$$\mathsf{dist}_\Gamma(r_{k+1}, \{c_{1,k+1}, \ldots, c_{n,k+1}\}) > D + \sigma > \rho.$$

Now we argue that the robber has a valid move from r_k to r_{k+1}. Since Γ is one-ended, $\Gamma - \bigcup_{i=1}^{n} B_\rho(c_{i,k+1})$ has only one unbounded connected component that we denote by Λ. The last two inequalities of the previous paragraph together with the the third item of Lemma 5.25 imply that both r_k to r_{k+1} are both elements of the unbounded component Λ of the subgraph $\Gamma - \bigcup_{i=1}^{n} B_\rho(c_{i,k+1})$. By Lemma 5.23, we have that

$$\mathsf{dist}_\Lambda(r_k, r_{k+1}) \leq L_{m,n} \, \mathsf{dist}_\Gamma(r_k, r_{k+1}) \leq 2R \cdot L_{m,n} = \psi$$

where the second inequality follows $\mathsf{dist}_\Gamma(u_0, \{s_1, \ldots, s_{n+1}\}) < R$ as it was assumed. It follows that there is a path in $\Lambda \subset \Gamma - \bigcup_{i=1}^{n} B_\rho(c_{k+1,i})$ from r_k to r_{k+1} of length at most ψ, and hence this is valid move for the robber.

$\square$

Chapter 6

Θ_n-extensions

In this chapter we look at the weak and strong cop numbers of an specific extension of a graph. We present the following result due to Lee.

Proposition 6.1. *[Lee, 2019, Proposition 3.2].*

For each $n \in \mathbb{Z}_{>0}$, there exists a connected graph with weak-cop number n.

Figure 6.1 gives an example of a graph provided by Lee with have weak-cop-number 3.

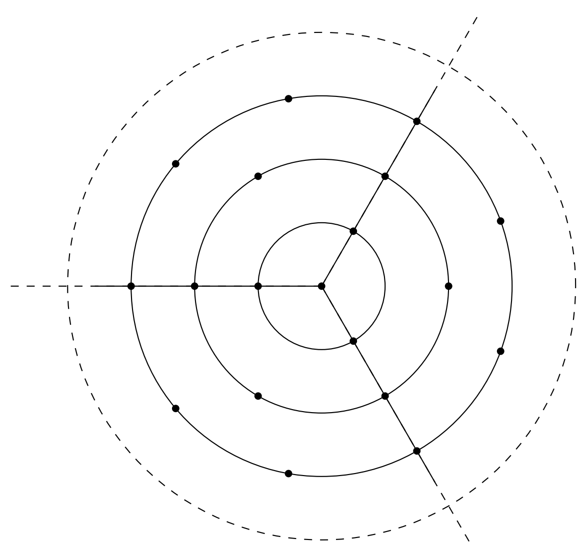

Figure 6.1: A graph with weak-cop-number 3

To establish this result, Lee exhibits examples A_n for any $n > 0$ of connected graphs such that $\mathsf{wCop}(A_n) = n$. It is possible to show that $\mathsf{sCop}(A_n) = n$ as well. These examples are a specific case of a construction that we call Θ_n-extensions. This will allow us to give an examples of graphs Γ such that $\mathsf{wCop}(\Gamma) = \infty$ and $\mathsf{sCop}(\Gamma) = n$ for any $n \in \mathbb{N}$ (see Corollary 7.31).

Definition 6.2. Let Γ be a connected graph and $n > 1$, we can define the Θ_n-*extension* of the graph, denoted as $\Theta_n(\Gamma)$, in the following way:

- Let Γ_i for $1 \leq i \leq n-1$ be graphs such that $\Gamma_i \cong \Gamma$ i.e. there is an fixed isomorphism $\eta_i : \Gamma \to \Gamma_i$. To simplify notation, let $\Gamma = \Gamma_0$.
- Let $u_0 \in \Gamma_0$ be a fixed vertex, and $\eta_i(u_0) = u_i \in \Gamma_i$. We will call this vertex the *center* of Γ_0 (resp. center of Γ_i).
- For any $r \geq 0$, let $S_r = \{x \in \Gamma_0 : \mathsf{dist}_{\Gamma_0}(x, u_0) = r\}$.
- For any $x_0 \in S_r$, denote $x_i = \eta_i(x_0)$. We will construct *bridges* between the image of the same vertex, of length $r+1$. Specifically, for any x_i, x_j let $B_{i,j}$ be the graph defined as:
    - Construct r additional vertices $v_1, \ldots, v_r$. Denoting $x_i = v_0$ and $x_j = v_{r+1}$ $V(B_{i,j}) = \{v_0, \ldots, v_{r+1}\}$.
    - $E(B_{i,j}) = \{\{v_i, v_{i+1}\} : 0 \leq i \leq r+1\}$

  The union of all $B_{i,j}$ for all vertices on S_r is denoted as $\mathcal{B}_r$.

After adding all possible bridges we obtain:

- $V(\Theta_n(\Gamma)) = \bigcup_{i=0}^{n-1} V(\Gamma_i) \cup \bigcup_{r=0}^{\infty} V(\mathcal{B}_r)$.
- $E(\Theta_n(\Gamma)) = \bigcup_{i=0}^{n-1} E(\Gamma_i) \cup \bigcup_{r=0}^{\infty} E(\mathcal{B}_r)$.

Definition 6.3. Let Γ be a graph, consider $\Theta_n(\Gamma)$, for any $v \in V(\Gamma_i)$ there exists a vertex $u \in V(\Gamma_0)$ such that $\eta_i(u) = v$. We denote the "*shadows*" of v to be the set:

$$\mathcal{W}(v) = \{u, \eta_1(u), \eta_2(u), \ldots, \eta_n(u)\}.$$

Moreover, if a vertex t belongs to a bridge, then there exists a vertex $u \in \Gamma_0$ and integers i, j such that the bridge connects $\eta_i(u)$ and $\eta_j(u)$. We will define $\mathcal{W}(t) = \mathcal{W}(u)$.

Remark 6.4. The Θ_n-extension of a graph, depends on the vertex defined as the center of the graph, but if the initial graph is vertex transitive, then all Θ_n-extensions are isomorphic.

Remark 6.5. The graphs exhibited by [Lee, 2019] are the Θ_n-extension of the infinite ray with centre the initial vertex with an additional central vertex.

Example 6.6. Consider the infinite length path, the Θ_2-extension and Θ_3-extension of this graph can be shown on Figures 6.2 and 6.4 respectively.

Remark 6.7. If a graph Γ has one of the following properties: connected, has n-ends, or has uniform bounded degree; then for any $n > 0$, then $\Theta_n(\Gamma)$ has them as well.

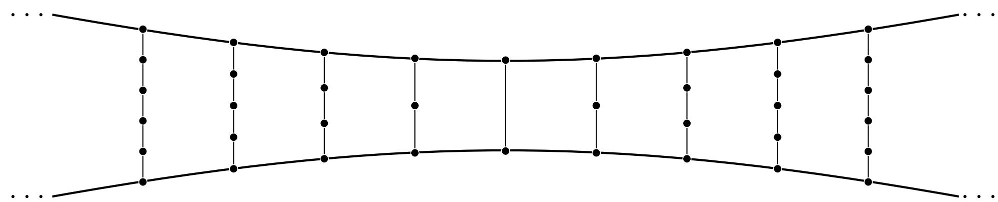

Figure 6.2: Θ_2-extension of the infinite path

Proposition 6.8. *For any connected graph Γ and any $n > 1$, Γ is a quasi-retract of $\Theta_n(\Gamma)$.*

Proof. We will use the notation of Definition 6.2. Let consider the following:

$$\Gamma \overset{\iota}{\hookrightarrow} \Theta_n(\Gamma) \overset{g}{\to} \Gamma,$$

with the function g that send $\eta_i(v) \to v$ for any i and all paths on the bridges that are connected to v or to $\eta_i(v)$, also to v. Note that this function is $(1, 0)$-Lipschitz, and moreover (ι, g) is a quasi-retraction. $\square$

Figure 6.3 is an explicit example of the quasi-retraction for Γ the infinite length path and $n = 2$ provided by the previous proposition.

Proposition 6.9. *For any connected graph Γ and any integer $N > 0$,*

$$\mathsf{wCop}(\Theta_N(\Gamma)) \leq N \cdot \mathsf{wCop}(\Gamma),$$

and

$$\mathsf{sCop}(\Theta_N(\Gamma)) \leq N \cdot \mathsf{sCop}(\Gamma).$$

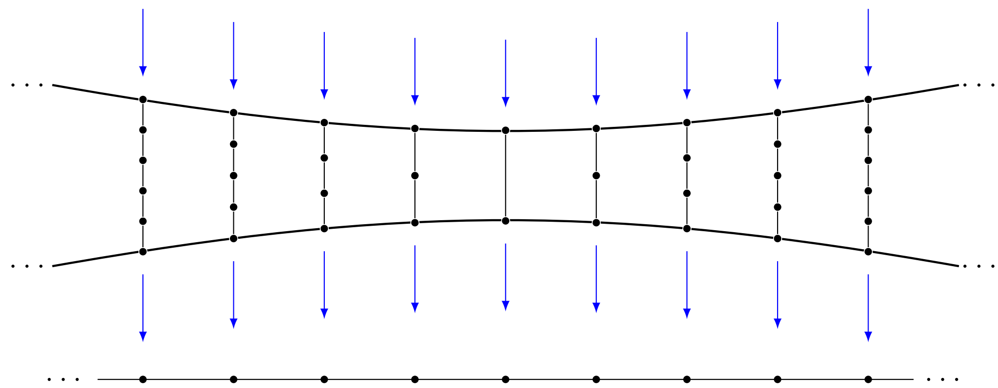

Figure 6.3: Explicit quasi-retraction from Θ_2-extension

Proof. To prove these inequalities it is enough to show that if Γ is $\mathsf{CopWin}(n, \sigma, \rho, \psi, v, R)$ then $\Theta_N(\Gamma)$ is $\mathsf{CopWin}(nN, \sigma, \rho, \psi, \eta_0(v), R)$. Suppose that Γ is $\mathsf{CopWin}(n, \sigma, \rho, \psi, v, R)$.

We will describe an strategy for a game on $\Theta_N(\Gamma)$ with those parameters by playing a parallel game on Γ. The $n \cdot N$ cops in the game on $\Theta_N(\Gamma)$ will be indexed by pairs i, k where $0 \leq i < n$ and $0 \leq k < N$, and their positions after the end of the j-turn will be denoted by $c^k_{i,j}$ where $0 \leq i < n$ and $0 \leq k < N$. While playing on $\Theta_N(\Gamma)$, the cops play a parallel game on Γ with parameters $(n, \sigma, \rho, \psi, v, R)$ using a winning strategy. The positions of n cops in Γ after the end of the j-turn are denoted by $c_{i,j}$. The movements of the cops in Γ will determine the moves of the cops in $\Theta_N(\Gamma)$ according to the rule

$$\eta_k(c_{i,j}) = c^k_{i,j}.$$

The moves of the robber in $\Theta_N(\Gamma)$ will determine the moves of the robber in the game on Γ by considering its shadow on Γ.

It is an observation that a movement of the robber in $\Theta_N(\Gamma)$ determines a valid move of the robber in Γ. Indeed if p is a path in $\Theta_N(\Gamma)$ such that its shadow in Γ has a vertex at distance less than ρ, then p has a vertex at distance less than ρ from a cop in $\Theta_N(\Gamma)$.

Note that if the robber on $\Theta_N(\Gamma)$ lies on a bridge that connects $\eta_i(x)$ and $\eta_j(x)$ and there are cops on those two vertices, the robber is trapped between those vertices and the cops can now capture the robber after a finite number of turns. In this situation we will say that the robber is in "*zugzwang*".

Observe that if the robber in Γ is captured by the end of the j-turn, then the robber in $\Theta_N(\Gamma)$ is in zugzwang by the end of the j-turn. Hence after a finite number of turns the robber in $\Theta_N(\Gamma)$ is captured.

On the other hand, if at any turn the robber in Γ is at distance larger than R from v, then the robber in $\Theta_N(\Gamma)$ is at distance larger than R from $\eta_0(v)$.

The statements of the last two paragraphs show that following the strategy on Γ yields an strategy on $\Theta_N(\Gamma)$ so that if the robber is captured in Γ then it is also captured in $\Theta_N(\Gamma)$; and if the robber in Γ is pushed away from the R-ball centered at v, then the robber in $\Theta_N(\Gamma)$ is also pushed away from the R-ball centered at $\eta_0(v)$. $\square$

Corollary 6.10. *For any connected graph* Γ *and for any integer* $N > 0$*,*

$$\mathsf{wCop}(\Gamma) \leq \mathsf{wCop}(\Theta_N(\Gamma)) \leq N \cdot \mathsf{wCop}(\Gamma)$$

and

$$\mathsf{sCop}(\Gamma) \leq \mathsf{sCop}(\Theta_N(\Gamma)) \leq N \cdot \mathsf{sCop}(\Gamma).$$

Proof. The left inequalities are consequences of Proposition 6.8 and Theorem 3.6. The right inequalities follow from Lemma 6.9. $\square$

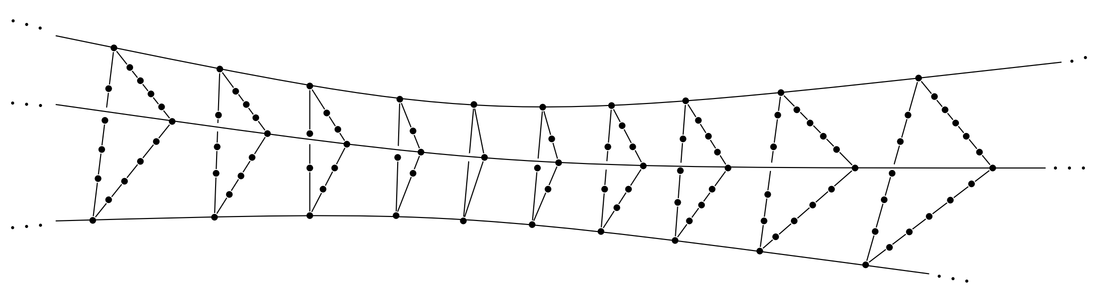

Figure 6.4: Θ_3-extension of the infinite path

Chapter 7

Cop numbers and groups

A very interesting family of graphs is given by groups through Cayley graphs, this allow us to extend the concepts of weak and strong cop numbers to finitely generated groups. In this chapter we provide some results on this, using the theorems and corollaries that were studied in the whole document.

Definition 7.1. Let G be group, and S a finite generating set, we define the *Cayley graph* of G with respect to S, denoted as $\mathrm{Cay}(G, S)$, in the following way:

- the of vertices of $\mathrm{Cay}(G, S)$ are the elements of G,
- the of edges of $\mathrm{Cay}(G, S)$ are:

$$\{\{g, g \cdot s\} | g \in G \text{ and } s \in (S \cup S^{-1})\}.$$

Example 7.2. Some Cayley graphs are isomorphic to graphs that we have already worked with, for example:

- If $G = \mathbb{Z}$, with generator set $S = \{1\}$, then $\mathrm{Cay}(G, S)$ is isomorphic to the infinite length path.
- If $G = \mathbb{Z}^2$ with generator set $\{(1, 0), (0, 1)\}$, then $\mathrm{Cay}(G, S)$ is isomorphic to the infinite square grid.
- Consider a set S, the free group F generated by S consists on all reduced words on $S \cup S^{-1}$, where S^{-1} are the formal inverses of the set S; by reduced we refer

as not having elements of S next to its formal inverse (for example $ab^{-1}bc = ac$). A well known interpretation of a free group is to think on F as the set of all possible words using S as an alphabet. The isomorphism class of a free group is determined by the cardinality of S which is called the rank of the free group. We denote by $\mathbb{F}_n$ the free group of rank n. The Cayley graph of the free group $\mathbb{F}_2$ with respect a free generating set is isomorphic to the infinite regular tree of degree four. An illustration of this graph is given in Figure 5.1.

Observe that if G is a finitely generated group, the Cayley Graph $\mathrm{Cay}(G, S)$ with respect to any finite generating set S, is a locally finite, vertex transitive graph.

Proposition 7.3. *[Löh, 2017, Proposition 5.2.5] Let G be a finitely generated group, with finite generating sets S and T. The Cayley graphs $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, T)$ are quasi-isometric.*

As a consequence of the quasi-isometry invariance of the weak and strong cop-number of graphs, we can define the weak-cop-number and strong-cop-number of a finitely generated group and they become quasi-isometric invariants of finitely generated groups.

Definition 7.4. Let G be a finitely generated group, with finite generating set S, we say that $\mathsf{wCop}(G) = n$ (resp. $\mathsf{sCop}(G) = n$) if and only if $\mathsf{wCop}(\mathrm{Cay}(G, S)) = n$ (resp. $\mathsf{sCop}(\mathrm{Cay}(G, S)) = n$).

Proposition 7.5. *Quasi-isometric groups have the same weak-cop-number and strong-cop-number.*

Example 7.6. From Example 7.2 and the results from [Lee, 2019], we have:

- If $G = \mathbb{Z}$ then $\mathsf{wCop}(G) = \mathsf{sCop}(G) = 1$.
- If $G = \mathbb{Z}^2$ then $\mathsf{wCop}(G) = \infty$.
- If $G = \mathbb{F}_2$ then $\mathsf{wCop}(G) = \mathsf{sCop}(G) = 1$.

Quasi-isometries of groups is a well studied area in geometric group theory. Some of the following results on cop numbers follow from classic results that can be found in [Löh, 2017]. For example, a finitely generated group is quasi-isometric to any subgroup of finite index [Löh, 2017, Corollary 5.3.10], therefore we get the following.

Proposition 7.7. *Let G be a finitely generated group, and H be a finite index subgroup, then H and G have the same weak and strong cop-numbers.*

7.1 Ends of groups and Cop numbers

Definition 7.8. A finitely generated group have n ends if and only if $\mathrm{Cay}(G, S)$ has n ends under a finite generating set.

Remark 7.9. As all Cayley graphs of a group are quasi-isometric no matter the finite generating set, the ends of a group is a well defined invariant on finitely generated groups.

Remark 7.10. Let Γ be a finitely generated group, then $\mathcal{E}(\Gamma) \leq 2$ or $\mathcal{E}(\Gamma) = \infty$ due to Theorem 5.5.

The following statement is a classical result on ends of groups, for more information see [Löh, 2017].

Proposition 7.11. *A group has two ends if and only if it has a subgroup of finite index isomorphic to the group $\mathbb{Z}$.*

As a consequence of the above proposition we have:

Corollary 7.12. *If a group G is two-ended then* $\mathsf{wCop}(G) = \mathsf{sCop}(G) = 1$.

Remark 7.13. Note that the previous result is not necessarily true in general for graphs. The Θ_2-extension of the infinite length path is two ended and has weak and strong cop number two, see Theorem I.

7.2 Products of groups and cop numbers

In this section we recall the notions of direct product, free product and amalgamated free product of groups. It will turn out that these products quasi-retract to their factors and as consequence of Theorem 3.6 we have that cop numbers of the factors are bounded from above by the cop numbers of the products, see Corollary 7.24.

Definition 7.14. Let G and H be finitely generated groups. We say that G *quasi-retracts* to H if a Cayley graph of G with respect to a finite generating set quasi-retracts to a Cayley graph of H with respect to a finite generating set.

It is an observation that the above definition is independent of the choice of generating sets for G and H. An immediate consequence of Theorem 3.6 is the following.

Theorem 7.15. *Let G and H be finitely generated groups. If G quasi-retracts to H then*

$$\mathsf{wCop}(H) \leq \mathsf{wCop}(G) \quad \textit{and} \quad \mathsf{sCop}(H) \leq \mathsf{sCop}(G).$$

The following is a well known fact in geometric group theory.

Proposition 7.16. *[Alonso, 1994] Let G be a finitely generated group and H a subgroup. If $r\colon G \to H$ is a retraction of groups, then H is finitely generated and G quasi-retracts to H.*

7.2.1 Direct products

Definition 7.17. Given two groups with their binary operations $(G, *)$ and $(H, \cdot)$, the *direct product*, denoted as $G \oplus H$, is defined as:

1. the elements of the group are ordered pairs (g, h) such that $g \in G$ and $h \in H$,
2. if we consider two elements (g_1, h_1) and (g_2, h_2), the product $(g_1, h_1)(g_2, h_2) = (g_1 * g_2, h_1 \cdot h_2)$.

Proposition 7.18. *Let G, H be two finitely generated groups, the direct product $G \oplus H$ quasi-retracts to both, G and H.*

Proof. Let G and H be two finitely generated groups with finite generating sets S, T respectively. Consider $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(H, T)$. Observe that $\mathrm{Cay}(G \oplus H, (S, e_S) \cup (e_G, T)) = \mathrm{Cay}(G, S) \,\Box\, \mathrm{Cay}(H, T)$. Therefore immediate that $\mathrm{Cay}(G \oplus H, (S, e_S) \cup (e_G, T))$ retracts to $\mathrm{Cay}(G, S)$. $\square$

7.2.2 Free products and amalgamated free products

Definition 7.19. Given two groups G and H, we can define *free product* of these as the group $G * H$ with homomorphisms $\iota_G : G \to G * H$ and $\iota_H : H \to G * H$ so that given any other group K with homomorphisms $f_G : G \to K$ and $f_H : H \to K$ there is a unique homomorphism $\varphi : G * H \to K$. The diagram that refers to this property is the following.

$$\begin{tikzcd}
G \arrow[dr, hook, "\iota_G"] \arrow[ddr, "f_G"'] & & H \arrow[dl, hook', "\iota_H"'] \arrow[ddl, "f_H"] \\
& G * H \arrow[d, dashed, "\varphi"] & \\
& K &
\end{tikzcd}$$

Proposition 7.20. *Let G and H be finitely generated groups, the free product $G * H$ quasi-retracts to both, G and H.*

Proof. In the definition of free product of $G * H$, take $K = H$ and the functions $f_G(g) = e_H$, for all $g \in G$ and $f_H(h) = h$ for any $h \in H$. Then the resulting function φ give us a retraction from $G * H \to H$ and therefore $G * H$ quasi-retracts to H by Proposition 7.16. Analogously, we can get a quasi-retraction from $G * H \to G$. □

There exists another quasi-retraction from the free product $G * H$ to G, different from than one provided by the previous retraction. For example, one can regard the elements of the free product as words using the generating sets H and T of G and H respectively. Then we can consider $\pi_G : G * H \to G$ as the function that sends an element of the free product to the first syllable of its normal form if its first syllable belong to G, and to the identity if the first syllable belongs to H. This yields a quasi-retraction of the corresponding Cayley graphs, as the length of every word in $G * H$ is reduced in G to length of the first syllable or to 1. This construction of quasi-retractions can be extended to show that amalgamated free products over finite products quasi-retract to their factors as stated in the next proposition. The details of such argument belong to the area of combinatorial group theory and are beyond the scope of this work, so we will not provide a proof here.

Definition 7.21. Suppose a pair of group homomorphisms $\varphi_G : N \to G$ and $\varphi_H : N \to H$. We define the amalgamated free product as the quotient $G *_N H = (G * H)/N$

where N is a normal subgroup of $G * N$ that is generated by the elements of the form $\varphi_G(n)\varphi_H(n)^{-1}$.

Proposition 7.22. *Let G and H be finitely generated groups, and N a finite group, then $G *_N H$ quasi-retracts to both, G and H.*

Example 7.23. For the interested reader, it can be shown that the result is not necessarily true when we amalgamate over an infinite group, for example taking $\mathsf{BS}(1,2) *_{\mathbb{Z}} \mathbb{Z}^2$.

Corollary 7.24. *Let G and H be finitely generated groups, and let K be a finite group. Then*

1. $\mathsf{wCop}(H) \leq \mathsf{wCop}(G \oplus H)$ *and* $\mathsf{sCop}(H) \leq \mathsf{sCop}(G \oplus H)$.
2. $\mathsf{wCop}(H) \leq \mathsf{wCop}(G * H)$ *and* $\mathsf{sCop}(H) \leq \mathsf{sCop}(G * H)$.
3. $\mathsf{wCop}(H) \leq \mathsf{wCop}(G *_K H)$ *and* $\mathsf{sCop}(H) \leq \mathsf{sCop}(G *_K H)$.

Proof. This follows from Theorem 7.15 and Proposition 7.16. The statements uses Propositions 7.18, 7.20 and 7.22 respectively. □

Example 7.25. The following statements are direct consequences of Theorem 3.6, and Propositions 7.18 and 7.20.

- The group $\mathbb{Z}^2 * \mathbb{Z}$ quasi-retracts to $\mathbb{Z}^2$ so $\mathsf{wCop}(\mathbb{Z}^2 * \mathbb{Z}) = \infty$. More generally, $\mathsf{wCop}(\mathbb{Z}^2 * G) = \infty$ for any finitely generated group G.
- All free abelian groups of any range greater or equal than two have infinite weak-cop number since they retract to $\mathbb{Z}^2$.

7.3 Hyperbolic Groups and Cop numbers

Definition 7.26. A group G is said to be *hyperbolic* if $\mathrm{Cay}(G,S)$ is hyperbolic as a metric space. Similarly a group is said to be *non-amenable* if if $\mathrm{Cay}(G,S)$ is non-amenable as a metric space.

Example 7.27. The groups $\mathbb{Z}$ and $\mathbb{F}_2$ are hyperbolic, as their Cayley graphs are trees.

The following proposition follows directly from Theorem 4.6.

Proposition 7.28. *All hyperbolic groups have strong cop number equal to one.*

Example 7.29. The group $\mathbb{F}_2$ is non-amenable, this follows from Example 5.9.

Proposition 7.30. *One-ended hyperbolic groups are non-amenable.*

Sketch of the proof. This proof is away from the scope of this paper, but the idea is that every one-ended hyperbolic group contains a free subgroup of rank two by the ping pong argument. Since free groups of rank at least two are non-amenable, it follows that one-ended hyperbolic groups are non-amenable. The definitions and theorems needed for this proof can be found on [Löh, 2017]. □

The following corollary follows from putting together the previous proposition and our Theorems 4.6 and 5.1.

Corollary 7.31. *Let* Γ *be the Cayley-graph of a one-ended hyperbolic group. Then* $\mathsf{wCop}(\Gamma) = \infty$ *and* $\mathsf{sCop}(\Gamma) = 1$.

Corollary 7.32. *Let* Γ *be the Cayley graph of a one-ended hyperbolic group. Then* $\mathsf{wCop}(\Theta_n(\Gamma)) = \infty$ *and* $\mathsf{sCop}(\Theta_n(\Gamma)) = n$.

This is a consequence of the previous Corollary and Theorem I. An interesting extension that could be done for this last corollary is given by the following question.

Question 7.33. Let Γ be a connected, one-ended, hyperbolic, locally finite, vertex transitive graph. Does $\mathsf{wCop}(\Gamma) = \infty$ hold?

The point here is that the Cayley graph of a one-ended discrete hyperbolic group is non-amenable. We do not know whether this is true in the framework of the question. A positive answer to this question could be implicit in deep work by Caprace, Cournullier, Monod and Tessera [Caprace et al., 2015]. They have a classification of amenable locally compact hyperbolic groups. It could be the case that for a locally finite hyperbolic graph the locally compact group $\mathsf{Aut}(\Gamma)$ is never amenable.

Question 7.34. Is there a finitely generated group G such that the weak-cop number is different from one or infinite? Similarly for the strong-cop number.

A classic result is that the number of ends of Cayley graphs of a finitely generated group is $0, 1, 2$ or infinite, [Löh, 2017, Theorem 8.2.8]. We have shown that if a finitely generated group has two ends, then its weak and strong cop numbers are both one. For one ended non-amenable groups, the weak cop number is infinite. We could not show a stronger result for amenable groups, but as Lee proved (see [Lee, 2019]) the infinite square grid also have infinite weak cop number, and a classic result (see [Löh, 2017]) is that all abelian groups are amenable. Hence we have an example of a one-ended amenable group with infinite weak-cop number. Finally for groups with infinite ends, we have $\mathbb{F}_2$ such that $\mathsf{wCop}(\mathbb{F}_2) = \mathsf{sCop}(\mathbb{F}_2) = 1$ and also $\mathsf{wCop}(\mathbb{Z}^2 * \mathbb{Z}) = \infty$, hence we need more properties in order to characterize cop numbers of groups in terms of ends.

Chapter 8

Conclusions

This chapter summarizes our main results and examples.

We first introduce two variations of the classical game of Cops and Robber in graphs, with the difference that our variations have different speed for the Cops and the Robber and also a parameter reach for the cop. This games define two quasi-isometric invariant of graphs, the weak-cop number and the strong-cop number. They have the property (Theorem 3.6) that if, Δ is a quasi-retract of Γ, then:

$$\mathsf{wCop}(\Delta) \leq \mathsf{wCop}(\Gamma) \quad \text{and} \quad \mathsf{sCop}(\Delta) \leq \mathsf{sCop}(\Gamma).$$

We also study these new graph invariants parallel to other quasi-isometric invariants of graphs as hyperbolicity, amenability and the number of ends. For this we have two main results, the first one (Theorem 4.6) on hyperbolic graphs was that, one of our invariants, the strong-cop number, is always one. The second result (Theorem 5.1) is on one-ended amenable graphs, where the other invariant, the weak-cop number, is infinite.

Finally, given a graph Γ and an integer n, we construct a graph $\Theta_n(\Gamma)$ that we call the Θ_n-extension of Γ. The construction allows us to exhibit examples of graphs with arbitrary weak-cop number and examples with arbitrary strong-cop number (Theorem H). We also observe that the weak and strong cop number are invariants of groups and compute these invariants in some cases. In particular this is used to provide a graph with infinite weak-cop number and arbitrary finite strong-cop number (Corollary 7.31).

Let us summarize some of our examples in the following tables:

Groups	Weak-cop-number	Strong-cop-number
Finite groups	1	1
$\mathbb{Z}$	1	1
$\mathbb{F}^2$	1	1
$\mathbb{Z}^n : n \geq 2$	∞	unknown
$\mathbb{Z}^2 * G$	∞	unknown
One-ended non-amenable	∞	unknown
One-ended hyperbolic groups	∞	unknown
Two-ended groups	1	1

Table 8.1: wCop and sCop for some groups

Graphs	Weak-cop-number	Strong-cop-number
$\Theta_n(\mathbb{Z})$	n	n
$\Theta_n(\mathbb{F}_2)$	n	n
$\Theta_n(\mathbb{Z}^n) : k \geq 2$	∞	unknown
$\Theta_n(G) : G$ is one-ended non-amenable	∞	unknown
$\Theta_n(G) : G$ is one-ended hyperbolic	∞	n

Table 8.2: wCop and sCop for some Θ_n-extensions of groups

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