The Evolution of Marginally Stable MOTS in Spherically Symmetric Spacetimes

by

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Abstract

The main objective of this thesis is to study the time evolutionary behaviour of a dynamical black hole horizon characterized by a marginally outer trapped tube (MOTT), a quasi-local model of a black hole horizon defined as a 3-dimensional hypersurface foliated by marginally outer trapped surfaces (MOTS). Motivated by numerical simulations of a binary black hole merger which predict that during the time evolution of the system a MOTT will suddenly appear or disappear and exhibit non-smooth evolutionary behaviour, we work in a spherically symmetric setting and build on established results about the existence of MOTTs based on a stability criteria and derive a local geometric condition which will allow us to distinguish the type of evolution and identify MOTTs with the same behaviour as the numerical model.

Lay summary

In general relativity (GR), a black hole can be defined as a region of spacetime from which nothing can escape. This definition satisfies our intuition for what a black hole should be, however it is teleological and requires a complete understanding of the time development to actually classify black holes. We instead characterize black holes by more local geometric objects called marginally outer trapped surfaces (MOTS). If we take a snapshot of the spacetime at a single instant of time then a MOTS is a 2-surface in the snapshot that we can identify as the boundary of a region from which light can't escape. This characterization provides a framework to study the time evolution of dynamical black holes, an active area of research in GR.

Recent numerical works simulating binary black hole mergers and collisions have observed that as two disjoint MOTSs corresponding to distinct black holes get closer, they influence each other and cause bizarre phenomena such as the MOTSs merging or the sudden appearance of a common outer horizon enclosing the original MOTSs. Although such behaviour has been observed numerically, it corresponds to non-smooth evolutions of the MOTS and is not well understood analytically.

In this thesis, while working in the context of a spherically symmetric spacetime, a spacetime with some convenient symmetries, we will construct a geometric condition which only relies on quantities dependant on the choice of the snapshot and the MOTS that will allow us to distinguish such non-smooth evolutions.

Acknowledgements

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$(\mathcal{M}, g_{\alpha\beta})$	Ambient 4-dimensional spacetime
(Σ, h_{ab})	3-dimensional spacelike hypersurface in $(\mathcal{M}, g)$
$(\mathcal{S}, q_{AB})$	2-dimensional closed spacelike surface in (Σ, h)
$e^{\alpha}{}_{a}$, $e^{\alpha}{}_{A}$	Holonomic basis vectors
$g_{\alpha\beta}$	Metric on $\mathcal{M}$
$h_{ab} = e^{\alpha}{}_{a} e^{\beta}{}_{b} g_{\alpha\beta}$	Induced metric on Σ
$q_{AB} = e^{\alpha}{}_{A} e^{\beta}{}_{B} g_{\alpha\beta}$	Induced metric on $\mathcal{S}$
$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$	Metric on unit 2-sphere
n^{α}	Unit timelike normal to (Σ, h)
s^{α}	Unit spacelike normal to $(\mathcal{S}, q)$
∇_{α}	Covariant derivative on $(\mathcal{M}, g)$
$\mathcal{D}_a$	Covariant derivative on (Σ, h)
$\mathring{\partial}_A$	Covariant derivative on $(\mathcal{S}, q)$
$K_{ab} = e^{\alpha}{}_{a} e^{\beta}{}_{b} \nabla_{\alpha} n_{\beta}$	Extrinsic curvature of (Σ, h)
$J_{AB} = e^{\alpha}{}_{A} e^{\beta}{}_{B} \nabla_{\alpha} s_{\beta}$	Extrinsic curvature of $(\mathcal{S}, q)$
$\mathscr{L}_u A^{\alpha}$	Lie derivative of A^{α} along u^{α}
${}^{\mathcal{M}}R$	Scalar curvature of $(\mathcal{M},g)$
${}^{\Sigma}R$	Scalar curvature of (Σ,h)
${}^{\mathcal{S}}R$	Scalar curvature of $(\mathcal{S},q)$

Chapter 1

Introduction

1.1 General Relativity and Black Holes

The field of general relativity formally began in 1915 when Albert Einstein published the historic papers [11, 12] in which he developed the mathematical and physical framework to determine the gravitational effects of astronomical bodies to a much higher degree of accuracy then could be obtained using Newton's law of universal gravitation.

In his theory, Einstein proposed that spacetime is described by a pseudo-Riemannian metric. This spacetime metric need not be flat as in special relativity, and in fact the curvature accounts for the physical effects normally attributed to a gravitational field. Einstein postulated that curvature was related to the matter distribution of the spacetime, represented by an energy-momentum tensor, by a tensorial equation called the *Einstein Field Equation* (EFE).

The first non-trivial solution of the EFE was found in 1916 by Karl Schwarzschild [28]. The Schwarzschild metric describes the spacetime around a static, uncharged, spherically symmetric body in a vacuum. The Schwarzschild solution is important to the theory of general relativity as it is the simplest black hole solution, and by a result called Birkhoff's theorem, any spherically symmetric vacuum solution is isometric to the Schwarzschild metric [26].

Although the Schwarzschild metric is simple and elegant, it is hardly perfect. In the general spherical form originally expressed by Schwarzschild given by (4.1), there is a supposed singularity when $r = 2M$, where r is a radial coordinate and M is a real parameter representing the mass of the black hole. Performing a coordinate transformation on the metric however reveals that it was merely a coordinate singularity and that the original coordinates didn't cover the entire manifold.

While this apparent singularity in the Schwarzschild metric isn't a physical singularity like the one at $r = 0$ which can't be eliminated through re-parameterization, it indicates the presence of something on the surface $r = 2M$ causing the breakdown. That something is a *black hole*, in particular the surface $r = 2M$ acts as a boundary separating the black hole and the rest of spacetime. The world tube developing from this surface is called the *event horizon*. We can define a black hole as a region of spacetime from which no signal can ever escape. Although this definition works intuitively, it's teleological and not physically reasonable for identifying black holes as it requires an observer to know the complete time development of the black hole to make sure that anything that falls in never escapes. For obvious reasons, such a global understanding can only be obtained by an omniscient observer, so we need a more local definition. We instead use a quasi-local geometric characterization in the form of *trapped surfaces*, in particular *marginally outer trapped surfaces* (MOTS). One can think of a MOTS as a closed 2-dimensional submanifold embedded in the spacetime such that the area of a light sphere emanating outward from the surface doesn't increase.

Along with allowing us to use geometric tools in our analysis, MOTS also allow us to study black holes through the perspective of an initial value formulation as they are contained in constant time slices of the spacetime. Results from [1] have shown that if a MOTS satisfies some stability criteria, then the MOTS will evolve smoothly into a *marginally outer trapped tube* (MOTT), a 3-dimensional manifold foliated by MOTSs. The result was generalized in [3], showing that if the MOTS only satisfied a weaker stability condition but also satisfied a proposed genericity condition, then the MOTS will also evolve smoothly into a MOTT, tangent to the time slice containing the MOTS. The MOTT being tangent to the time slice allows for some interesting behaviour. For example, it would be possible for the MOTT to exist entirely in the future of the time slice or only in the past. Although the result allows for such possibilities, the original statement doesn't mention a local way to distinguish them.

One of the objectives in this thesis is to derive a local geometric condition which

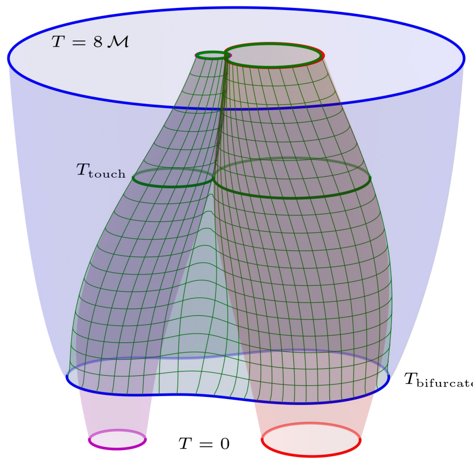


Figure 1.1: At time $T = 0$, there are two distinct MOTSs corresponding to the two black holes, then at $T_{\text{bifurcate}}$, a new MOTS surrounding the original two forms and immediately bifurcates into the future. Figure from [25].

will give us the ability to distinguish the behaviour. The need for such a condition is motivated by recent numerical simulations of the merger of binary black holes [25]. As seen in fig. 1.1, the spacetime initially contains two disjoint MOTS corresponding to distinct black holes, however on the time slice $T_{\text{bifurcate}}$, a new MOTS enveloping the original two forms. The new MOTS evolves smoothly into a MOTT existing in the future of the time slice.

1.2 Outline

In chapter 2, we begin by describing the geometric structure of spacetime and defining the relevant notation. We discuss the postulates of general relativity as proposed by Einstein, from which we construct the initial value problem for general relativity. Following that, we discuss a geometric characterization of a black hole in the form of trapped surfaces and their world tubes, introducing the stability problem which serves as a basis for the work done in this thesis. The chapter ends with a brief discussion regarding results about the existence of trapped surface world tubes depending on a stability condition.

In chapter 3, we give the construction for a spherically symmetric spacetime and discuss how the symmetries simplify the variations defining the stability condition in chapter 2. Utilizing the simplifications to the stability criteria, we obtain analogous statements to the existence results in the context of spherical symmetry which are more restrictive in that they require certain metric symmetries, however one of which becomes a stronger result compared to the general statement.

In chapter 4, we study an example of a spherically symmetric spacetime containing a dynamical black hole, namely the Vaidya spacetime. Using the construction from chapter 2, we identify a MOTS in Vaidya satisfying our stability criteria, then apply the result we obtained in chapter 3 to show the existence of a MOTT passing through it. Following this, we state and prove one of the main results of the thesis theorem 4.2.1, concluding with a few examples.

Chapter 2

Preliminary Information

Over the course of this thesis, we will consider a quasi-local geometric characterization of black holes using the notion of trapped surfaces originally defined by Penrose in [21], in particular we will be investigating their time evolution and how it relates to the evolution of a dynamical black hole. Before considering any results regarding the evolution of black holes and trapped surfaces, we must first discuss and introduce some notation and background information on the theory of black holes. The following discussions are motivated in large part by [8, 14, 15, 17, 20, 24, 31].

2.1 Notation

2.1.1 Spacetime

We denote a spacetime by a pair $(\mathcal{M}, g)$ where $\mathcal{M}$ is a 4-dimensional smooth manifold and g is a metric with Lorentzian signature. At a point $p \in \mathcal{M}$, non-zero tangent vectors $X \in T_p\mathcal{M}$ are classified by the metric into one of three groups depending on the sign of $g(X, X)$:

1. Timelike if $g(X, X) < 0$
2. Spacelike if $g(X, X) > 0$
3. Null if $g(X, X) = 0$.

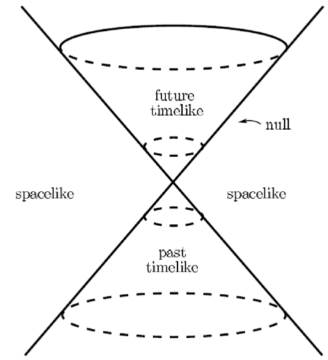


Figure 2.1: [20] Null cone in $T_p\mathcal{M}$ separating spacelike and timelike vectors.

Similarly, we call a curve in the manifold spacelike, timelike, or null depending on the class of its tangent vector provided the tangent vector remains of the same class at all points along the curve. While every tangent vector can be classified as spacelike, timelike, or null, the same is not true for curves. Objects moving through space travel along timelike or null curves. If the object has mass, then it travels along a timelike curve and if it has no mass then it follows a null curve, meaning it travels at the speed of light. Null and timelike curves and vectors are called *causal* or *non-spacelike* as points connected by such curves can influence one another.

Given two points joined by a causal curve, we would like to say one is in the future of the other. At each point $p \in \mathcal{M}$, the null vectors at p form a double-sided cone in $T_p\mathcal{M}$ called the *null cone* as seen in fig. 2.1. We can designate half the null cone as the future and the other half as the past. If a continuous choice can be made as p varies over $\mathcal{M}$, then we say that $\mathcal{M}$ is *time-orientable.*

Lemma 2.1.1 ([14, 31])**.** *A Lorentzian manifold $(\mathcal{M}, g)$ is time-orientable if and only if it admits a smooth timelike vector field T.*

For a point $p \in \mathcal{M}$, we define the set $J^+(p)$ to be the *causal future* of p containing all points that can be reached from p by future-directed causal curves. We similarly denote the *causal past* of p by $J^-(p)$. Similarly, we denote by $I^+(p)$ and $I^-(p)$ the *chronological future* and *chronological past* of p defined as the set of points that can be reached from p by future-directed and past-directed timelike curves respectively.

Proposition 2.1.2 ([20])**.** *Given a spacetime $(\mathcal{M}, g)$, let $p, q \in \mathcal{M}$ such that $q \in J^+(p) \setminus I^+(p)$. Then there exists a future null geodesic from p to q.*

Definition 2.1.3 ([20])**.** Given a spacetime $(\mathcal{M}, g)$, we say that a smooth hypersurface Σ is a *Cauchy hypersurface* if every inextendible causal curve must pass through Σ exactly once. A spacetime containing a Cauchy hypersurface is called *globally hyperbolic.*

In definition 2.1.3, we refer to an *inextendible* causal curve $\gamma : I \to \mathcal{M}$ as one that can't be extended to a causal curve on a larger domain. The hypothesis of global hyperbolicity plays a role in spacetime geometry similar to that of completeness in Riemannian geometry [20].

Proposition 2.1.4 ([16, 20])**.** *If Σ is a Cauchy hypersurface in $(\mathcal{M}, g)$, then there exists a homeomorphism between $\mathcal{M}$ and $\mathbb{R}\times\Sigma$ that provides a foliation of $\mathcal{M}$ by Cauchy hypersurfaces. Moreover, any Cauchy hypersurface in $\mathcal{M}$ must be homeomorphic to Σ.*

2.1.2 Curvature

As we just saw, the path of test particles is determined by the spacetime data $(\mathcal{M}, g)$. In particular, the infinitesimal geometry of the manifold is described by the *curvature* or *Riemann* tensor. The Riemann tensor is a way of capturing a measure of the intrinsic curvature of a manifold. It captures the failure of second covariant derivatives to commute.

Definition 2.1.5 ([20, 31])**.** The *Riemann curvature tensor* of a connection ∇ is a rank-4 tensor defined by

$$R_{\alpha\beta\mu}{}^{\nu}\omega_{\nu} = (\nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha})\omega_{\mu} \tag{2.1}$$

for any 1-form ω_{μ}.

The *Ricci tensor* is a symmetric rank-2 tensor defined as the trace of the Riemann tensor over its first and third or second and fourth indices:

$$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta} = R_{\beta\alpha}. \tag{2.2}$$

The *scalar curvature* is defined as the trace of the Ricci tensor:

$$R = g^{\alpha\beta} R_{\alpha\beta}. \tag{2.3}$$

Locally, the scalar curvature measures the deviation of the volume of infinitesimally small geodesic balls from the volume of balls in Euclidean space.

In the context of general relativity, the Einstein field equations relate the local spacetime curvature, expressed using the Ricci and scalar curvature, with the local matter distribution, expressed by the stress-energy tensor.

Definition 2.1.6 ([24]). The distribution of matter in a spacetime is modelled by the stress-energy tensor $T_{\alpha\beta}$, as such it must satisfy certain *energy conditions* which attempt to capture the belief that "energy should be positive." The energy conditions are:

Weak The energy density for any matter distribution, as measured by an observer in spacetime, must be non-negative. For any future-directed timelike vector field v^α, we must have

$$T_{\alpha\beta} v^\alpha v^\beta \geq 0. \tag{2.4}$$

Null For any future-directed null vector field k^α, we must have

$$T_{\alpha\beta} k^\alpha k^\beta \geq 0. \tag{2.5}$$

Strong For any future-directed timelike vector field v^α,

$$\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right) v^\alpha v^\beta \geq 0. \tag{2.6}$$

Dominant In addition to the weak energy condition holding, for every future-directed causal vector field X^α, the vector field $-T^\alpha{}_\beta X^\beta$ must be a future-directed causal vector.

Remark 2.1.1. If the stress-energy tensor is that of a perfect fluid, then there are some implications among the energy conditions. In particular, the dominant energy condition implies the weak form which in turn implies the null form. In addition, the strong energy condition also implies the null form, however it should be noted that the strong energy condition does not imply the weak form.

2.1.3 Field Equations

The general theory of relativity, as proposed by Einstein, can be presented as four postulates regarding the structure of spacetime [17].

Postulate 1: Spacetime is modelled by the pair $(\mathcal{M}, g)$ where $\mathcal{M}$ is a four-dimensional differentiable manifold equipped with Lorentzian metric g which provides a Levi-Civita connection.

Postulate 2: Free particles travel along non-spacelike geodesics.

Postulate 3: The energy, momentum, and stresses of the matter content of the spacetime are described by a symmetric tensor $T_{\alpha\beta}$ called the *stress-energy tensor* that is conserved: $\nabla^{\alpha}T_{\alpha\beta}$=0.

Postulate 4: The curvature of spacetime is related to the stress-energy tensor by way of the *Einstein field equation*:

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta}. \tag{2.7}$$

We choose units such that $G = c = 1$. The field equations (2.7) are a set of ten coupled non-linear PDEs in the metric and its first and second derivatives. The covariant divergence of both sides vanish identically, thus the field equations are really only six independent differential equations for the metric. Einstein later published a correction to the field equations in the form

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{2.8}$$

where Λ is the cosmological constant. For our purposes, we will only be considering the case $\Lambda = 0$.

If in some region of spacetime the stress-energy tensor vanishes, then the field equations are called the *vacuum field equations* and can be written as

$$G_{\alpha\beta} = 0, \tag{2.9}$$

solutions of which are called vacuum solutions. Due to the form of the Einstein tensor, (2.9) is equivalent to

$$R_{\alpha\beta} = 0. \tag{2.10}$$

Examples of vacuum solutions include flat Minkowski space, the Schwarzschild solution, and the Kerr solution.

2.2 Initial Data Formulation

Due to the complexity of the Einstein equations, solutions can be difficult to derive. As a result, it is natural to consider an initial value problem for the vacuum Einstein equations (2.9), that is, given information about the metric at a fixed time, we would like to know how the system evolves with time. It turns out that general relativity admits a well posed initial value formulation. The problem amounts to specifying the metric on a fixed spacelike hypersurface of the spacetime and its "time derivative." We will be following the construction of [20, 24, 26, 31] in our discussion.

We denote the ambient spacetime by the triple $(\mathcal{M}, g_{\alpha\beta}, \nabla_\alpha)$ where ∇_α denotes the Levi-Civita connection on $\mathcal{M}$ and we use Greek letters $\{\alpha, \beta, \gamma, \dots\}$ as abstract indices on $\mathcal{M}$. We will similarly use lower-case Latin letters $\{a, b, c, \dots\}$ as abstract indices on 3-dimensional hypersurfaces Σ embedded in $\mathcal{M}$ and upper-case Latin letters $\{A, B, C \dots\}$ on 2-dimensional surfaces $\mathcal{S}$ embedded in Σ.

Definition 2.2.1. In a 4-dimensional spacetime manifold, a *hypersurface* is a 3-dimensional submanifold that is locally described by an equation of the form:

$$\Phi(x^\alpha) = 0. \tag{2.11}$$

A hypersurface can be either spacelike or timelike depending on the causality of its normal, $n_\alpha = \nabla_\alpha \Phi$. That is, the hypersurface is spacelike if its normal is everywhere timelike and vice-versa.

As we will be talking about tensors in different coordinate charts, we need a way to transform between them. Let e denote the pullback operator with indices indicating the spaces being operated on. For example, if we wish to change coordinate charts in a hypersurface defined by the parametric equations

$$x^\alpha = x^\alpha(y^a) \tag{2.12}$$

where y^a $(a = 1, 2, 3)$ are coordinates intrinsic to the hypersurface, then the pull-back/pushforward operator is

$$e^\alpha{}_a = \frac{\partial x^\alpha}{\partial y^a}. \tag{2.13}$$

Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime. The objective is to perform a 3+1 decomposition, splitting the spacetime into a sequence of "time slices." By proposition 2.1.4, we can foliate $\mathcal{M}$ by Cauchy surfaces Σ_t, parameterized by a global time function t. Denote by $(\Sigma, h_{ab}, \mathcal{D}_a)$ a 3-dimensional spacelike hypersurface embedded in the spacetime with unit timelike normal n^α and metric connection $\mathcal{D}_a$. The metric h_{ab} on each Σ_t is induced by the spacetime metric $g_{\alpha\beta}$ and takes the form

$$h_{ab} = e^\alpha{}_a e^\beta{}_b g_{\alpha\beta}. \tag{2.14}$$

Taking the inverse of this metric and pushing forward to $\mathcal{M}$, we obtain the transverse projection tensor, taking the form

$$\begin{aligned} h_{\alpha\beta} &= g_{\alpha\beta} + n_\alpha n_\beta \\ &= g_{\alpha\mu} g_{\beta\nu} e^\mu{}_a e^\nu{}_b h^{ab}. \end{aligned} \tag{2.15}$$

To work with arbitrary tensor fields, we introduce some additional notation:

$$e^a{}_\alpha = g_{\alpha\beta} h^{ab} e^\beta{}_b. \tag{2.16}$$

The projection operator projects an arbitrary (p, q) tensor field T on $\mathcal{M}$ down to the hypersurface such that only its components tangential to Σ_t survive

$$\hat{T}_{a_1 \ldots a_q}{}^{b_1 \ldots b_p} = e^{\alpha_1}{}_{a_1} \ldots e^{\alpha_q}{}_{a_q} e^{b_1}{}_{\beta_1} \ldots e^{b_p}{}_{\beta_p} T_{\alpha_1 \ldots \alpha_q}{}^{\beta_1 \ldots \beta_p}. \tag{2.17}$$

In particular, for any vector field V on $\mathcal{M}$, it can be decomposed into components tangential and normal to Σ_t:

$$\begin{aligned} V^\alpha &= h^\alpha{}_\beta V^\beta - n^\alpha n_\beta V^\beta \\ &= e^\alpha{}_a \hat{V}^a - V_n n^\alpha \end{aligned} \tag{2.18}$$

for $\hat{V}$ on Σ_t and $V_n \in \mathbb{R}$.

Let T^α be a vector field on $\mathcal{M}$ such that $T^\alpha \nabla_\alpha t = 1$, that is, T^α is the vector

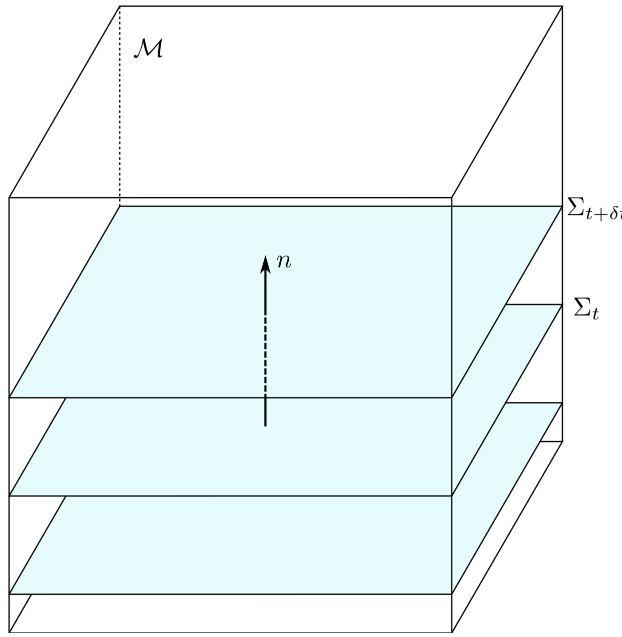


Figure 2.2: The spacetime $\mathcal{M}$ is foliated by a family of spacelike hypersurfaces $\{\Sigma_t\}_{t\in\mathbb{R}}$.

field dual to the 1-form dt. As we did before, this vector field can be decomposed into components normal and tangential to the hypersurface Σ_t:

$$T^\alpha = N^\alpha + fn^\alpha \tag{2.19}$$

where f is the *lapse function* and N^α is the *shift vector* defined as

$$f = -T^\alpha n_\alpha = (n^\alpha \nabla_\alpha t)^{-1} \tag{2.20}$$

$$N^\alpha = h^\alpha{}_\beta T^\beta \tag{2.21}$$

see fig. 2.3.

The vector field T^α represents the "flow of time" in the spacetime. What we mean is that we can identify the hypersurfaces Σ_t and $\Sigma_{t+\delta t}$ with the diffeomorphism resulting from following the integral curves of T^α, thus we can view the effect of "moving forward in time" as changing the spatial metric on an abstract three-dimensional manifold Σ from $h_{ab}(t)$ to $h_{ab}(t+\delta t)$. Hence a globally hyperbolic spacetime $(\mathcal{M}, g_{\alpha\beta})$ can be viewed as representing the time development of a Riemannian metric on a fixed manifold.

Let us now consider how tensor fields are differentiated. There are two ways we

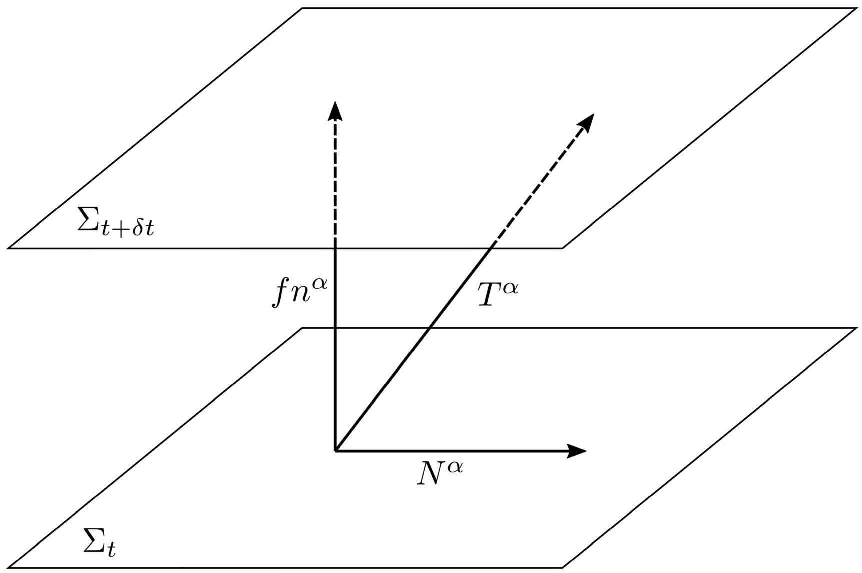


Figure 2.3: The spacetime vector T^α can be decomposed into a component tangential to Σ_t denoted by the shift vector N^α and a component normal to Σ_t denoted fn^α.

could go about defining the induced covariant derivative on Σ:

1. as the projection of the covariant derivative with respect to the connection compatible with the spacetime metric $g_{\alpha\beta}$, i.e. for a tensor A_α, $\hat{\nabla}_b A_a = e^\alpha{}_a e^\beta{}_b \nabla_\beta A_\alpha$;

2. or as the covariant derivative with respect to the connection compatible with the spatial metric h_{ab}, i.e. $\mathcal{D}_a$.

It turns out that $\hat{\nabla}_a$ and $\mathcal{D}_a$ are the same [31].

With that out of the way, we can now discuss the "time-derivative" of the spatial metric h_{ab}. We define the *extrinsic curvature* or *second fundamental form* of Σ as the symmetric rank-2 tensor field K_{ab} given by

$$K_{ab} = e^\alpha{}_a e^\beta{}_b \nabla_\beta n_\alpha = \frac{1}{2}(\mathscr{L}_n g_{\alpha\beta}) e^\alpha{}_a e^\beta{}_b. \tag{2.22}$$

While the spatial metric h_{ab} is concerned with the intrinsic geometry of Σ, K_{ab} is concerned with the extrinsic geometry, the way Σ is embedded in the spacetime manifold. The two equivalent relations defining the extrinsic curvature give rise to two interpretations:

- $K_{ab} = \frac{1}{2}(\mathscr{L}_n g_{\alpha\beta}) e^\alpha{}_a e^\beta{}_b$: the rate of change of the geometry if Σ is evolved along n;
- $K_{ab} = e^\alpha{}_a e^\beta{}_b \nabla_\beta n_\alpha$: the rate of change of n when moved around Σ.

Using the decomposition of T^α (2.19), it turns out that the extrinsic curvature tensor can be expressed in the form

$$K_{ab} = \frac{1}{2f}(\partial_t h_{ab} - \mathcal{D}_a N_b - \mathcal{D}_b N_a), \tag{2.23}$$

thus giving us our "time-derivative" of h_{ab}.

From the above, it appears that appropriate initial data should consist of the triple (Σ, h_{ab}, K_{ab}) where Σ is a 3-dimensional manifold, h_{ab} is a Riemannian metric on Σ, and K_{ab} is a symmetric rank-2 tensor field on Σ. An important result due to Choquet-Bruhat and Geroch has shown that given such initial data, subject to certain initial value constraints, there exists a globally hyperbolic spacetime $(\mathcal{M}, g_{\alpha\beta})$ satisfying the field equations which possesses a Cauchy surface diffeomorphic to Σ on which the induced metric is h_{ab} and the induced extrinsic curvature is K_{ab}; see theorem 2.2.2.

So far, we have seen that the spacetime metric $g_{\alpha\beta}$ induces a Riemannian metric h_{ab} on Σ which uniquely determines an intrinsic derivative operator denoted by $\mathcal{D}_a$. In addition, this derivative operator defines a purely intrinsic curvature tensor by the relation

$$\mathcal{D}_a \mathcal{D}_b \omega_c - \mathcal{D}_b \mathcal{D}_a \omega_c = R_{abc}{}^d \omega_d \tag{2.24}$$

for a 1-form ω. Just as the intrinsic derivative was related to the spacetime covariant derivative, we can similarly obtain a relation between the curvature of Σ and the spacetime curvature by way of the *Gauss–Codazzi* equations. For a general rank-2 tensor field T_{ab} on Σ,

$$\mathcal{D}_a T_{bc} = e^\alpha{}_a e^\beta{}_c e^\gamma{}_c \nabla_\alpha T_{\beta\gamma} = e^\alpha{}_a e^\beta{}_b e^\gamma{}_c \nabla_\alpha (e^i{}_\beta e^j{}_\gamma T_{ij}). \tag{2.25}$$

A brief calculation gives us

$$\mathcal{D}_a \mathcal{D}_b \omega_c = e^\alpha{}_a e^\beta{}_b e^\gamma{}_c \nabla_\alpha \nabla_\beta \omega_\gamma - K_{ab}(e^\gamma{}_c n^\mu \nabla_\mu \omega_\gamma) + K_{ac} K_b{}^i \omega_i. \tag{2.26}$$

We thus obtain the *Gauss relation*

$$R_{abcd} = e^{\alpha}{}_{a} e^{\beta}{}_{b} e^{\mu}{}_{c} e^{\nu}{}_{d} R_{\alpha\beta\mu\nu} + K_{bc}K_{ad} - K_{ac}K_{bd}. \tag{2.27}$$

A similar calculation gives us the *Codazzi relation*

$$\mathcal{D}_a K^a{}_b - \mathcal{D}_b K^a{}_a = e^{\alpha}{}_{a} R_{\alpha\mu} n^{\mu}. \tag{2.28}$$

The Gauss–Codazzi equations (2.27) and (2.28) can be written in a contracted form in terms of the Einstein tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{^{\mathcal{M}}R}{2} g_{\alpha\beta}$. We can express the spacetime Ricci tensor in the form

$$\begin{aligned} R_{\alpha\beta} &= g^{\mu\nu} R_{\mu\alpha\nu\beta} \\ &= (h^{mn} e^{\mu}{}_{m} e^{\nu}{}_{n} - n^{\mu} n^{\nu}) R_{\mu\alpha\nu\beta} \end{aligned} \tag{2.29}$$

and the Ricci scalar as

$$\begin{aligned} {}^{\mathcal{M}}R &= g^{\alpha\beta} R_{\alpha\beta} \\ &= (h^{ab} e^{\alpha}{}_{a} e^{\beta}{}_{b} - n^{\alpha} n^{\beta})(h^{mn} e^{\mu}{}_{m} e^{\nu}{}_{n} - n^{\mu} n^{\nu}) R_{\mu\alpha\nu\beta} \\ &= h^{ab} h^{mn} e^{\mu}{}_{m} e^{\alpha}{}_{a} e^{\nu}{}_{n} e^{\beta}{}_{b} R_{\mu\alpha\nu\beta} - 2 h^{ab} e^{\alpha}{}_{a} e^{\beta}{}_{b} n^{\mu} n^{\nu} R_{\mu\alpha\nu\beta} \\ &= h^{ab} h^{mn} (R_{manb} + K_{mn} K_{ab} - K_{mb} K_{an}) - 2 R_{\mu\nu} n^{\mu} n^{\nu} \\ &= ({}^{\Sigma}R + K^2 - \|K\|^2) - 2 R_{\mu\nu} n^{\mu} n^{\nu} \end{aligned} \tag{2.30}$$

using the Gauss relation in the second last line. From the form of the Einstein tensor, we find that

$$\begin{aligned} {}^{\mathcal{M}}R + 2R_{\mu\nu} n^{\mu} n^{\nu} &= {}^{\mathcal{M}}R + 2G_{\mu\nu} n^{\mu} n^{\nu} + {}^{\mathcal{M}}R g_{\mu\nu} n^{\mu} n^{\nu} \\ &= 2G_{\mu\nu} n^{\mu} n^{\nu} \end{aligned} \tag{2.31}$$

thus the contracted Gauss relation takes the form:

$$2G_{\mu\nu} n^{\mu} n^{\nu} = {}^{\Sigma}R + K^2 - \|K\|^2. \tag{2.32}$$

Similarly, the Codazzi relation takes the form:

$$e^{\alpha}{}_{a} G_{\alpha\beta} n^{\beta} = \mathcal{D}_a K^a{}_b - \mathcal{D}_b K. \tag{2.33}$$

Equations (2.32) and (2.33) are called the *Hamiltonian constraint* and the *momentum constraint* respectively and they form the basis of the initial value problem in general relativity. In particular, the Hamiltonian constraint reveals that the intrinsic curvature on Σ must be related to the matter distribution by way of (2.32) so we're not free to arbitrarily specify h_{ab} and K_{ab}.

To summarize, the initial value problem of general relativity starts with the selection of a spacelike hypersurface Σ representing a 'moment of time', on which we specify initial data consisting of the triple (Σ, h_{ab}, K_{ab}) where h_{ab} is the pull-back of the spacetime metric to Σ and K_{ab} is a symmetric rank-2 tensor which carries information about the derivative of the metric in the direction normal to Σ. By Einstein's equations, these tensors must satisfy the constraint equations (2.32) and (2.33). We then have the following fundamental result by Choquet-Bruhat and Geroch:

Theorem 2.2.2 ([10]). *Let* (Σ, h_{ab}, K_{ab}) *be an initial data set satisfying the vacuum Hamiltonian and momentum constraints (i.e. letting* $G_{\alpha\beta} = 0$ *in* (2.32) *and* (2.33)*). Then there exists a unique (up to diffeomorphism) spacetime* $(\mathcal{M}, g_{\alpha\beta})$*, called the* maximal Cauchy development *of* (Σ, h_{ab}, K_{ab}) *such that:*

1. $(\mathcal{M}, g_{\alpha\beta})$ *satisfies the vacuum Einstein equation* (2.9)*;*
2. $(\mathcal{M}, g_{\alpha\beta})$ *is globally hyperbolic with Cauchy surface* Σ*;*
3. *the induced metric and extrinsic curvatures of* Σ *are* h_{ab} *and* K_{ab} *respectively;*
4. *any other spacetime satisfying the above conditions is isometric to a subset of* $(\mathcal{M}, g_{\alpha\beta})$.

Building on the work above, a similar construction can be established in the non-vacuum case, however it will be omitted here.

Let $(\mathcal{S}, q_{AB}, \mathring{\partial}_A)$ be a closed 2-dimensional spacelike surface embedded in Σ with unit spacelike normal s^a and connection $\mathring{\partial}_A$. The metric q_{AB} is induced by the ambient spacetime metric and takes the form

$$q_{AB} = e^{\alpha}{}_{A} e^{\beta}{}_{B} g_{\alpha\beta} = e^{a}{}_{A} e^{b}{}_{B} h_{ab}. \tag{2.34}$$

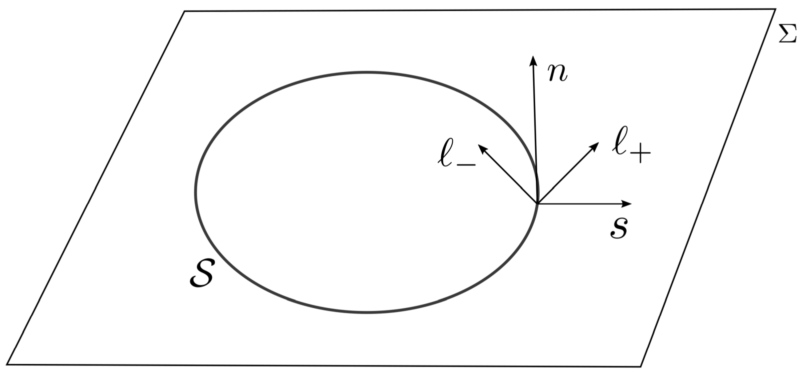


Figure 2.4: 2-dimensional surface $\mathcal{S}$ embedded in Σ, a 3-dimensional slice of the ambient spacetime $\mathcal{M}$. $\mathcal{S}$ has unit spacelike normal s and null normals ℓ and k, the slice Σ has unit timelike normal n. Adapted from figure in [15] to fit our notation.

It can also be treated as a tensor in the ambient spacetime, taking the form

$$\begin{aligned} q_{\alpha\beta} &= h_{\alpha\beta} - s_\alpha s_\beta \\ &= g_{\alpha\beta} + n_\alpha n_\beta - s_\alpha s_\beta. \end{aligned} \tag{2.35}$$

When talking about the curvature of $\mathcal{S}$ in Σ, we will usually reference the mean curvature J which can be viewed as the divergence of the unit normal s_a

$$\begin{aligned} J &= q^{ab}\mathcal{D}_a s_b \\ &= \frac{1}{\sqrt{\det q}}\partial_a(\sqrt{\det q}\, s^a) \end{aligned} \tag{2.36}$$

where we make use of the fact we can express the divergence using partial derivatives. We remark that the mean curvature is the trace of the extrinsic curvature tensor of $\mathcal{S}$ with respect to s^a.

Since the spacetime is time-orientable, we can assemble a pair of future pointing null vectors normal to $\mathcal{S}$ which we denote by $\ell_\pm$. We construct this pair of null normals from the previously discussed normals in the following way:

$$\ell_\pm = n \pm s. \tag{2.37}$$

With this normalization, we have $g(\ell_+, \ell_-) = -2$.

2.3 Trapped Surfaces

Following the construction above, we have a spacelike 2-surface $\mathcal{S}$ embedded in a spacetime $(\mathcal{M}, g)$ and at any point p on $\mathcal{S}$, there are two distinct future pointing, outward and inward directed unit null normals denoted $\ell_\pm$.

Definition 2.3.1. The *null expansion scalars* of $\mathcal{S}$, $\Theta_{(\pm)}$, are defined as the divergence of light rays emanating orthogonally from $\mathcal{S}$ and are of the form

$$\Theta_{(\pm)} = \operatorname{div}_{\mathcal{S}} \ell_\pm \tag{2.38}$$

The null expansions can be expressed in terms of the initial data as

$$\Theta_{(\pm)} = \operatorname{tr}_q K \pm J \tag{2.39}$$

where $\operatorname{tr}_q K$ is the trace of the extrinsic curvature tensor K_{ab} with respect to q, the induced metric on $\mathcal{S}$, viewed as a tensor on Σ, and J is the mean curvature of $\mathcal{S}$.

Remark 2.3.1. For a 2-sphere in Minkowski space, a short calculation reveals that $\Theta_{(-)} < 0$ and $\Theta_{(+)} > 0$ which we interpret to mean light rays emanating outward from $\mathcal{S}$ expand while those travelling inward contract.

The Minkowski case agrees with our intuition, however there are cases where both $\Theta_{(\pm)} < 0$ and the light rays are said to be trapped. In such cases, we call $\mathcal{S}$ a *trapped surface*, and below we further classify $\mathcal{S}$ based on the signs of $\Theta_{(\pm)}$.

Definition 2.3.2 ([21])**.** A *trapped surface* is a closed spacelike 2-surface such that $\Theta_{(\pm)} < 0$. If there is a consistent notion of an 'outward' direction, say along ℓ_+, then we call $\mathcal{S}$ *outer trapped* if $\Theta_{(+)} < 0$ with no additional constraint on $\Theta_{(-)}$. We can further classify $\mathcal{S}$ by:

- weakly outer trapped if $\Theta_{(+)} \leq 0$,
- marginally outer trapped if $\Theta_{(+)} = 0$.

The inequalities must hold on all of $\mathcal{S}$.

Remark 2.3.2. Trapped surfaces describe the interior of the event horizon for stationary spacetimes, a region from which light cannot escape. The same can not be said in the case of dynamical black hole spacetimes; see [5, 6]. The idea of a trapped surface is that the ingoing and outgoing congruences of light rays emanating from $\mathcal{S}$ are converging, meaning that any signal originating from the surface is trapped inside a shrinking region.

Remark 2.3.3. Considering the initial data form of the null expansions (2.39), in the case of time symmetric spacetimes where $K_{ab} = 0$ like in Minkowski, then marginally outer trapped surfaces have vanishing mean curvature. Such surfaces are called *minimal surfaces.*

Definition 2.3.3 ([24])**.** We call any hypersurface Σ with vanishing extrinsic curvature $K_{ab} = 0$ a *moment of time symmetry* in spacetime. Since the extrinsic curvature is essentially the 'time derivative' of the metric, a moment of time symmetry corresponds to a turning point of the metric's evolution at which its 'time derivative' vanishes.

The following theorem from Penrose [21] states that if $\mathcal{S}$ is a trapped surface, then the ambient spacetime $\mathcal{M}$ is *future null geodesically incomplete*, meaning that there exists a null geodesic that is of finite length and cannot be extended. Null geodesics represent light rays so this statement is saying that after some finite time or affine parameter, the light ray abruptly ends and cannot be extended.

Theorem 2.3.4 ([14, 21])**.** *Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime containing a non-compact Cauchy hypersurface and satisfies the null energy condition. If $\mathcal{M}$ contains a trapped surface $\mathcal{S}$, then $(\mathcal{M}, g)$ is future null geodesically incomplete.*

In this thesis, we will exclusively consider outer trapped surfaces, thus the following variant of the singularity theorem is more useful for us.

Theorem 2.3.5 ([14])**.** *Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime satisfying the null energy condition, with smooth spacelike Cauchy surface Σ. Let $\mathcal{S}$ be a smooth closed hypersurface in Σ which separates Σ into an "inside" U and an "outside" V, i.e. $U, V \subset \Sigma$ are connected disjoint sets such that $\Sigma \setminus \mathcal{S} = U \cup V$. Suppose that $\bar{V}$ is non-compact. If $\mathcal{S}$ is outer trapped, then $(\mathcal{M}, g)$ is future null geodesically incomplete.*

Through the singularity theorem, Penrose proved that provided a spacetime satisfied some causality and energy condition, then once the gravitational field becomes strong enough to cause the appearance of trapped surfaces, the spacetime must come to an abrupt end in the future, signalling the presence of a singularity in the spacetime, a point where the metric breaks down. The existence of *naked singularities* [22], singularities visible to observers in the spacetime, suggests a failure of the Einstein equations as a physical theory as the presence would make the spacetime unpredictable. To attempt to rectify this problem, Penrose [23] proposed the *weak cosmic censorship hypothesis*, expressed informally as:

Conjecture 1 ([23, 31]). The complete gravitational collapse of a body always results in a black hole rather than a naked singularity, i.e. all singularities of gravitational collapse are "hidden" within black holes where they can't be seen by observers at $\mathscr{I}^+$, future null infinity.

Although weak cosmic censorship has not been proven, some evidence in favour of the hypothesis is that outer trapped surfaces also signal the existence of black holes, as seen in the following:

Proposition 2.3.6 ([17])**.** *Let $(\mathcal{M}, g)$ be a* regular predictable *space developing from a partial Cauchy surface Σ satisfying the null energy condition. Then an outer trapped surface $\mathcal{S}$ in $D^+(\Sigma)$, the future Cauchy development of Σ, does not intersect $J^-(\mathscr{I}^+, \bar{\mathcal{M}})$, the event horizon.*

Definition 2.3.7 ([17, 20])**.** Suppose the spacetime $(\mathcal{M}, g)$ is conformally compactifiable, that is there is a spacetime $(\tilde{\mathcal{M}}, \tilde{g})$ into which $(\mathcal{M}, g)$ is conformally embedded as a manifold with boundary where the boundary $\partial\mathcal{M}$ consists of two null surfaces $\mathscr{I}^{\pm}$ which represent future and past null infinity.

We say that spacetime is *strongly future asymptotically predictable* from a partial Cauchy surface Σ if $\mathscr{I}^+$ is contained in the closure of $D^+(\Sigma)$ in $\bar{\mathcal{M}}$, and $J^+(\Sigma) \cap \bar{J}^-(\mathscr{I}^+, \bar{\mathcal{M}})$ is contained in $D^+(\Sigma)$.

$(\mathcal{M}, g)$ is called a *regular predictable space* if it is strongly future asymptotically predictable from a partial Cauchy surface Σ and satisfies the following conditions:

- $\Sigma \cap \bar{J}^-(\mathscr{I}^+, \bar{\mathcal{M}})$ is homeomorphic to $\mathbb{R}^3$;
- Σ is simply connected;
- for sufficiently large t, $\Sigma_t \cap \bar{J}^-(\mathscr{I}^+, \bar{\mathcal{M}})$ is contained in $\bar{J}^+(\mathscr{I}^-, \bar{\mathcal{M}})$.

Proposition 2.3.6 states that an outer trapped surface in an asymptotically flat, globally hyperbolic spacetime satisfying the null energy condition can't intersect the domain of outer communication, meaning it can't be seen by observers at $\mathscr{I}+$, and hence must be contained in a black hole.

2.4 Stability of Marginally Outer Trapped Surfaces

Definition 2.4.1 ([20])**.** A *minimal surface* is defined as a 2-surface with everywhere vanishing mean curvature, however it can equivalently be characterized as a critical point of the volume functional.

Remark 2.4.1. Despite the terminology, a minimal surface need not be a local minimum of the volume functional.

Definition 2.4.2 ([20])**.** A minimal surface $\mathcal{S}$ is called *stable* if its second variation of volume is non-negative for all deformations and *strictly stable* if the second variation is strictly positive.

Definition 2.4.3 ([1])**.** A MOTS $\mathcal{S}$ is called *locally outermost* in Σ if and only if there exists a two-sided neighbourhood U of $\mathcal{S}$ such that the part of U outside of $\mathcal{S}$ does not contain any weakly outer trapped surfaces.

To verify when $\mathcal{S}$ is locally outermost, we need to examine how the value of $\Theta_{(+)}$ changes just outside of it. To do this, we consider a one-parameter family of deformations of $\mathcal{S}$ denoted by

$$\mathcal{S}_t = \{\exp(t\psi s)\} \tag{2.40}$$

where exp is the exponential map of Σ. This family of deformations is depicted in fig. 2.5. Let $\Theta(t)$ denote the null expansion of $\mathcal{S}_t$ with respect to $\ell_t = n + s_t$. To observe how $\Theta_{(+)}$ changes on this family of surfaces, we consider the variation

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \Theta(t) = L\psi \tag{2.41}$$

where $L : C^\infty(\mathcal{S}) \to C^\infty(\mathcal{S})$ is a second-order linear elliptic operator defined by

$$L\psi = -\Delta_{\mathcal{S}}\psi + 2q(\omega, \mathring{\partial}\psi) + \left(\frac{{}^{\mathcal{S}}R}{2} - G(n, \ell_+) - \frac{1}{2}\|\chi_+\|^2 + \operatorname{div}_{\mathcal{S}}\omega - \|\omega\|^2\right)\psi. \quad (2.42)$$

In the above, $\Delta_{\mathcal{S}}$, $\mathring{\partial}$, and $\operatorname{div}_{\mathcal{S}}$ are respectively the Laplacian, gradient, and divergence on $\mathcal{S}$, $\|\cdot\|$ is the point-wise norm induced by q, ω is the vector field on $\mathcal{S}$ dual to the one-form $K(s, \cdot)|_{\mathcal{S}}$, ${}^{\mathcal{S}}R$ is the scalar curvature of $\mathcal{S}$, χ_+ is the null second fundamental form of $\mathcal{S}$ with respect to ℓ_+, that is, $\chi^+_{AB} = e^\alpha{}_A e^\beta{}_B \nabla_\alpha \ell^+_\beta$, and finally G is the Einstein tensor of the spacetime $(\mathcal{M}, g)$. The operator L is called the *MOTS stability operator* and in the time-symmetric case where the extrinsic curvature of Σ vanishes, it reduces to the usual stability operator for minimal surfaces [15]. See appendix A for a derivation of the operator in index form.

Let us now consider the eigenvalue problem

$$L\psi = \lambda\psi. \quad (2.43)$$

In general, L is not self-adjoint with respect to the L^2 inner product due to the presence of its first-order term, and so $\lambda \in \mathbb{C}$. It turns out however that at least one λ will be real.

Definition 2.4.4 ([13])**.** The *principal eigenvalue* λ_p for the operator L is an eigenvalue such that for any other eigenvalue $\lambda \in \mathbb{C}$ of L,

$$\operatorname{Re}(\lambda) \geq \operatorname{Re}(\lambda_p). \quad (2.44)$$

Lemma 2.4.5 ([1])**.** *The principal eigenvalue λ_p of L is real and simple. Moreover, the corresponding principal eigenfunction ψ_p is either everywhere positive or everywhere negative.*

If we use the principal eigenfunction ψ_p to define the variation, then we obtain from (2.41)

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \Theta(t) = \lambda_p \psi_p. \quad (2.45)$$

For $\mathcal{S}$ to be locally outermost, we require $\frac{\partial \Theta}{\partial t}|_{t=0} \geq 0$ for small $t > 0$, so from (2.45), this condition is satisfied provided $\lambda_p \geq 0$.

Definition 2.4.6 ([1, 2])**.** A MOTS $\mathcal{S}$ is called *stable* if $\lambda_p \geq 0$ and *strictly stable* if

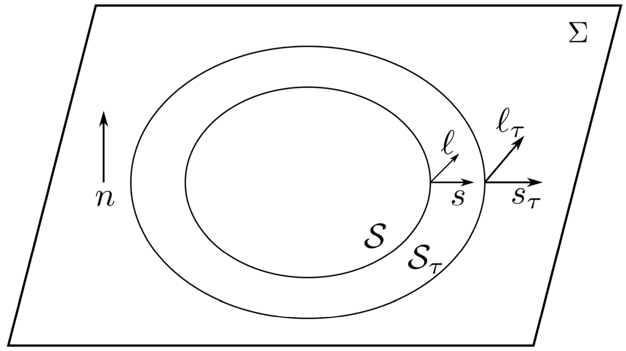


Figure 2.5: $\mathcal{S}_t$ is a surface obtained from deforming $\mathcal{S}$ an amount $t\psi$ along the spacelike normal s.

$\lambda_p > 0$.

Remark 2.4.2. This definition is an analogue of the notion of stability from minimal surface theory seen in definition 2.4.2.

To conclude this discussion on stability, we have the following result which shows the relation between a MOTS being locally outermost and its stability.

Proposition 2.4.7 ([1]). *Let $\mathcal{S}$ be a MOTS. Then:*

1. *If $\mathcal{S}$ is strictly stable, then it is locally outermost;*

2. *If $\mathcal{S}$ is locally outermost, then it is stable.*

2.5 Marginally Outer Trapped Tubes

A common problem studied in numerical relativity is the merger of two black holes. As seen in the previous section, marginally trapped surfaces signal the existence of a black hole and act as a quasi-local characterization in a time slice. To study a merger, we identify the initial state of the black holes with two disjoint marginally trapped surfaces and track their world tubes, called marginally trapped tubes (MTTs) as they evolve and eventually merge. Following the conventions of [4], a spacelike MTT is called a *dynamical horizon*, a timelike MTT is called a *timelike membrane*, and a null MTT is called a *non-expanding horizon* provided the spacetime Ricci tensor is such that $-R_{\alpha\beta}\ell^\beta$ is future causal, or *isolated horizon* if the intrinsic connection and matter fields are time-independent. As defined, an isolated horizon provides a quasi-local characterization of a black hole which has reached equilibrium, and a dynamical horizon represents an evolving black hole. There is a general expectation that under physically reasonable conditions, dynamical horizons 'settle down' to isolated horizons in asymptotic future. One does not associate a timelike membrane with the surface of a black hole even quasi-locally.

Definition 2.5.1 ([17])**.** A set $\mathcal{V}$ is said to be *achronal* if $I^+(\mathcal{V}) \cap \mathcal{V} = \emptyset$, in other words if there are no two points of $\mathcal{V}$ with timelike separation.

Definition 2.5.2 ([4])**.** A dynamical horizon $\mathscr{H}$ is said to be *regular* if:

1. $\mathscr{H}$ is achronal;

2. $\mathscr{H}$ satisfies a "genericity condition" that $W = \|\chi_+\|^2 + T_{\alpha\beta}\ell_+^\alpha \ell_+^\beta$ never vanishes on $\mathscr{H}$.

Since dynamical horizons are spacelike, $\mathscr{H}$ is automatically locally achronal, the regularity condition however requires $\mathscr{H}$ to be globally achronal.

The proposed genericity condition is obtained from $\mathscr{L}_-\Theta_{(+)}$, that is the directional derivative of $\Theta_{(+)}$ along the inward pointing null normal ℓ_-. In particular, the genericity condition is satisfied if $\mathscr{L}_-\Theta_{(+)} < 0$ which encodes the idea that $\mathscr{H}$ is "outer" as deformations along the inward normal ℓ_-^α make the 2-surface trapped.

Following the construction of [9], let X^α be a vector field that:

1. is tangential to $\mathscr{H}$;

2. is everywhere orthogonal to the foliation by MTSs;

3. generates a flow which preserves the foliation.

For some smooth function C on $\mathscr{H}$, we can express the tangent vector as

$$X^\alpha = \ell_+^\alpha - C\ell_-^\alpha. \tag{2.46}$$

The causal character of $\mathscr{H}$ is dependent on the sign of C; in particular, $\mathscr{H}$ is null, spacelike, or timelike if C is respectively zero, positive, or negative. We know that

$\Theta_{(+)}$ vanishes everywhere on the horizon, giving us

$$\begin{aligned} 0 &= \mathscr{L}_X\Theta_{(+)} \\ &= \mathscr{L}_+\Theta_{(+)} - C\mathscr{L}_-\Theta_{(+)} \\ \mathscr{L}_-\Theta_{(+)} &= -\frac{1}{C}(\|\chi_+\|^2 + R_{\alpha\beta}\ell_+^\alpha\ell_+^\beta) \end{aligned} \tag{2.47}$$

where in the last line we used the Raychaudhuri equation for a null congruence [24].

As shown by Hayward [18], if the null energy condition holds, then a dynamical horizon $\mathscr{H}$ satisfies the genericity condition if and only if it's a future outer trapping horizon (FOTH).

Acronym	Name	Dimension	$\Theta_{(-)}$	$\mathscr{L}_-\Theta_{(+)}$	Other
MOTS	Marginally outer trapped surface	2			closed topology
MOTT	Marginally outer trapped tube	3			foliated by MOTSs
MTS	Marginally trapped surface	2	< 0		closed topology
MTT	Marginally trapped tube	3	< 0		foliated by MTSs
TH	Trapping horizon	3	$\neq 0$	$\neq 0$	foliated by MOTSs
FOTH	Future outer trapping horizon	3	< 0	< 0	foliated by MTSs
DH	Dynamical horizon	3	< 0		foliated by MTSs

Table 2.1: Summary of quasi-local horizons with $\Theta_{(+)} = 0$ and their definitions. From [8].

Let $(\mathcal{M}, g)$ be a spacetime foliated by spacelike hypersurfaces Σ_t. On some initial hypersurface Σ_0, we let $\mathcal{S}$ be a closed, spacelike 2-surface with $\Theta_{(+)} = 0$; that is, $\mathcal{S}$ is a MOTS. A technique used in the study of dynamical black holes is to study the evolution of a MOTS over time. This motivates the idea of a marginally outer trapped tube (MOTT), a hypersurface $\mathscr{H}$ foliated by MOTSs. When given a spacetime foliation, we can define the MOTT adapted to the foliation by letting the leaves of $\mathscr{H}$ be the intersection of $\mathscr{H}$ with the spacetime slices, that is, $\mathcal{S}_t = \mathscr{H} \cap \Sigma_t$. Although this definition is fine for most cases, a problem arises in cases where the intersection $\mathscr{H} \cap \Sigma_t$ has multiple components as is found in [25].

An alternative definition is to let $\mathscr{H}$ be a hypersurface foliated by surfaces $\mathcal{S}_\kappa$ such that each $\mathcal{S}_\kappa$ lies inside a time slice $\mathcal{S}_\kappa \subset \Sigma_{t(\kappa)}$ and is a MOTS. This would mean that $\mathscr{H} = \bigcup_\kappa \mathcal{S}_\kappa$ and $\mathcal{S}_\kappa \subset \mathscr{H} \cap \Sigma_{t(\kappa)}$. The difference between the two definitions occurs when $t(\kappa)$ is not monotone such as in fig. 4.1.

2.6 AMMS Results

A MOTS is a useful tool in studying the behaviour of a black hole in a single hypersurface corresponding to an instant in time, however as seen in section 2.5, they can also be used in the study of dynamical black holes. In numerical relativity, dynamical black holes can be studied by tracking the evolution of a MOTS in an initial data set, then if the MOTS evolves smoothly in time, one can consider the smooth hypersurface $\mathscr{H}$ formed by tracing the MOTSs in successive time slices. Although MOTSs are a useful tool in studying properties of black hole spacetimes, there is a level of uncertainty about their behaviour under time evolutions as they have been found to exhibit non-smooth evolutions such as the sudden creation or annihilation of a pair of MOTS. In [17], Hawking and Ellis proposed the well known "pair of pants" model of the merger of two black holes, depicted in fig. 2.6, which details the merger of two distinct event horizons into a single one encompassing the originals. Adapting to the language of outer trapped surfaces, the model for the merger of two MOTSs was established numerically in [25]. The model initially considered two disjoint MOTSs corresponding to the two approaching black holes, then when the black holes were sufficiently close, it was observed that a common MOTS enclosing the original MOTSs was formed and then immediately bifurcated into an outer and inner component. Before the MOTSs could make contact, the formation of the common horizon caused the apparent horizon to jump discontinuously, as observed in fig. 2.7.

The authors of [1] set out to develop tools to study evolutions of black hole spacetimes analytically. The situation modelled in [25] and other non-smooth evolutions occasionally occur in numerical simulations, motivating the need to study the evolutions analytically.

Theorem 2.6.1 ([1, 2]). *Let $(\mathcal{M}, g)$ be a spacetime foliated by spacelike hypersurfaces Σ_t. Assume that in some leaf Σ_0, there is a MOTS $\mathcal{S}$. If $\mathcal{S}$ is strictly stable, then $\mathcal{S}$ is part of a horizon $\mathscr{H}$ whose marginally outer trapped leaves lie in Σ_t.*

To prove the existence of $\mathscr{H}$, the authors use the strict stability condition on $\mathcal{S}$ considered in section 2.4, making the stability operator L invertible. Then the implicit function theorem for Banach spaces [19] proves the local existence of a horizon $\mathscr{H}$ such that $\mathcal{S}_t = \mathscr{H} \cap \Sigma_t$ near $\mathcal{S}$ are MOTS. Since the result uses the implicit function theorem to prove the existence of the horizon $\mathscr{H}$, L being invertible is a sufficient

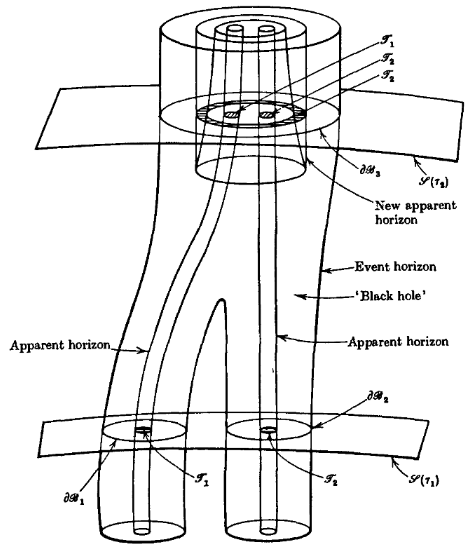


Figure 2.6: The "pair of pants" model for the merger of two black holes proposed by Hawking and Ellis in [17]. At time τ_1, there are apparent horizons $\partial\mathcal{T}_1$, $\partial\mathcal{T}_2$ inside the event horizons $\partial\mathcal{B}_1$, $\partial\mathcal{B}_2$ respectively. By time τ_2, the event horizons have merged to form a single event horizon; a third apparent horizon has now formed surrounding both the previous apparent horizons.

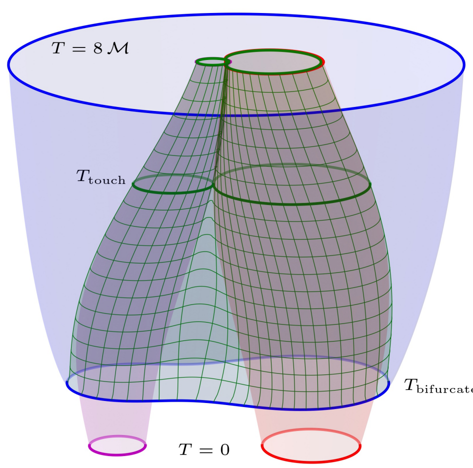


Figure 2.7: At time $T = 0$, there are two distinct MOTSs corresponding to the two black holes, then at $T_{\text{bifurcate}}$, a new MOTS surrounding the original two forms and immediately bifurcates into the future. Figure from [25].

condition. The constraint that $\mathcal{S}$ is strictly stable guarantees that no eigenvalue of L vanishes, hence proving L is invertible.

The above result only accounts for the case when $\mathcal{S}$ is strictly stable, however in [3], a similar result is proven in the case when $\mathcal{S}$ is only marginally stable and is stated in the following.

Theorem 2.6.2 ([3])**.** *Similar to before, let $(\mathcal{M}, g)$ be a spacetime, satisfying the null energy condition, foliated by spacelike hypersurfaces (Σ_t, h_t, K_t) and assume that $\mathcal{S} \subseteq \Sigma_0$ is a marginally stable MOTS satisfying the genericity assumption*

$$W = \|\chi_+\|^2 + R_{\alpha\beta}\ell_+^\alpha \ell_+^\beta \not\equiv 0. \tag{2.48}$$

Then there exists a spacelike MOTT $\mathscr{H}$ containing $\mathcal{S}$ which is tangent to Σ_0 at $\mathcal{S}$. For some neighbourhood U of $\mathcal{S}$, all MOTS in $U \cap \Sigma_t$ are contained in $\mathscr{H}$.

The genericity condition in consideration is precisely (2.47), so the above result can be summarized as saying that provided the variation of the outer null expansion along the null normal ℓ_- doesn't vanish identically, then the marginally stable MOTS $\mathcal{S}$ will evolve into a MOTT $\mathscr{H}$ tangent to Σ_0 at $\mathcal{S}$. Since the MOTT is tangent to the time slice, it's possible for the MOTT to exist entirely to the future of Σ_0, in which case $\mathscr{H}$ bifurcates into two distinct horizons, the situation observed in fig. 2.7. In fact, it's even possible for the MOTT to lie entirely inside Σ_0; see example 4.1.1.

Chapter 3

Spherically Symmetric Spacetimes

Having introduced and defined most of the notation and concepts we'll need for our analysis in chapter 2, we are now ready to discuss some of the main results of the thesis. We begin with the construction of a *spherically symmetric* spacetime and show how variations and the stability problem are simplified due to symmetries of the spacetime.

3.1 General Spherical Spacetimes

A spacetime is spherically symmetric if its metric is invariant under rotations, that is it has an isometry group containing a subgroup isomorphic to $SO(3)$, the group of 3-dimensional rotations. In the usual spherical coordinates, a general spherically symmetric spacetime can be paired with a metric of the form

$$ds^2 = -F(r,t)dt^2 + 2G(r,t)dtdr + H(r,t)dr^2 + R^2(r,t)d\Omega^2 \tag{3.1}$$

where F, G, H are arbitrary functions, R is the areal function, and $d\Omega^2$ is the metric on the unit 2-sphere.

Given a spacetime $\mathcal{M}$ foliated by spacelike hypersurfaces Σ_t with spherically symmetric initial data sets (Σ_t, h_t, K_t), the 2-sphere of constant coordinate radius r embedded in Σ_t has constant outward null expansion which we represent by the function $\Theta : (0,\infty) \times \mathbb{R} \to \mathbb{R}$. In the time slice Σ_{t_0}, the sphere of radius r_0 is a MOTS if

$\Theta(r_0, t_0) = 0$. This construction provides us with the natural characterization of a MOTT as the zero set of $\Theta(r, t)$. If the MOTT is smooth, we can also characterize it as a disjoint union of smooth curves $(r(s), t(s))$ in the $(r, t)-$plane such that $\Theta(r(s), t(s)) = 0$ for all s.

Because the initial data set is spherically symmetric, the first order term in the MOTS stability operator (2.42) drops out and L becomes self-adjoint. We however don't need the explicit form of the stability operator to determine whether the MOTS satisfies the stability condition. As we will see in theorem 3.1.1, due to spherical symmetry, constant speed variations simplify to taking partial derivatives, and in particular, the constant speed variation along the normal used to define the stability operator simplifies to a radial derivative. The MOTS stability condition therefore reduces to a simple sign condition on the first derivative of the null expansion.

Theorem 3.1.1. *Let $\mathcal{S}$ be a spherical MOTS in a spherically symmetric spacetime $\mathcal{M}$. Then the normal variation of the null expansion Θ with respect to a constant speed deformation corresponds to a radial partial derivative. Moreover, the principal eigenvalue of the stability operator for $\mathcal{S}$ and $\partial_r\Theta$ differ by a positive constant.*

Proof. It was previously mentioned that in a spherically symmetric background, the MOTS stability condition reduced to a sign condition on the first derivative of the null expansion. To see where this comes from, we recall a MOTS (r_0, t_0) is called stable when the principal eigenvalue of the stability operator L is non-negative, that is $\lambda_p \geq 0$. Due to spherical symmetry, it turns out that L is a self-adjoint operator, from which we immediately know that the eigenvalues are all real [13]. Moreover, the eigenvalue problem for L is simply a Helmholtz equation and the corresponding eigenvalues are simply the usual eigenvalues of the Laplacian, given by $\lambda^\ell = \ell(\ell+1)$ for $\ell = 0, 1, 2, \dots,$ shifted by a constant. This tells us that the eigenvalues have multiplicity $2\ell + 1$ with the principal eigenvalue being simple and that the corresponding eigenfunctions will be linear combinations of spherical harmonics. In particular the principal eigenfunction is just a positive constant on the sphere.

Denote by $\mathcal{S}$ the MOTS (r_0, t_0) corresponding to a 2-sphere of radius r_0. We let $\mathcal{S}_\mu$ denote the 1-parameter family of constant speed deformations of $\mathcal{S}$ an amount $a\mu$ in the outward normal direction ν, that is $\mathcal{S}_\mu$ are 2-spheres of radius $r(\mu) = r_0 + a\mu$ where we choose $a(r_0, t_0) = H(r_0, t_0)^{-1/2}$ so that the deformation has unit speed. Then since the deformation has unit speed, we find that the first variation of the null expansion is given by

$$\begin{aligned} L(1) &= \frac{\partial}{\partial \mu}\Theta(r(\mu),t)\bigg|_{\mu=0,t=t_0} \\ &= \frac{\partial \Theta}{\partial r}\frac{\mathrm{d}r(\mu)}{\mathrm{d}\mu}\bigg|_{\mu=0,t=t_0} \\ &= a(r_0,t_0)\partial_r\Theta(r_0,t_0). \end{aligned} \tag{3.2}$$

From the form of the above, it's clear that $a(r_0, t_0)\partial_r\Theta(r_0, t_0)$ is an eigenvalue of L with constant associated eigenfunction. To see that it is actually the principal eigenvalue, we know that the only constant spherical harmonic is when $\ell = 0$, hence the eigenvalue with associated constant eigenfunction is simple and therefore the principal eigenvalue. □

Remark 3.1.1. We saw how the normal variation with respect to a constant speed deformation corresponded to a partial derivative in the radial coordinate, however it remains to be seen what the variation along the timelike normal looks like. To do this, we borrow the notation of variations from [1], defining the variation of the geometric object ν on a surface T in the direction p^α by $\delta_p\nu = \frac{\partial \nu}{\partial \tau}$ for any one-parameter family of surfaces T_τ with $T_0 = T$ and where $p^\alpha \partial_{x^\alpha} = \frac{\partial}{\partial \tau}|_{\tau=0}$. In particular, $\delta_{\psi s}\Theta_{(+)} = L\psi$ for a function ψ on $\mathcal{S}$.

Recalling from (2.47), the Raychaudhuri equation tells us

$$\delta_{\ell_+}\Theta_{(+)} = -(\|\chi_+\|^2 + R_{\alpha\beta}\ell_+^\alpha\ell_+^\beta) = -W. \tag{3.3}$$

Since $\ell_+ = n + s$, we can use the additive property of variations to find

$$\begin{aligned} -W &= \delta_{\ell_+}\Theta_{(+)} \\ &= (\delta_n + \delta_s)\Theta_{(+)} \\ &= \delta_n\Theta_{(+)} + L(1) \\ &= \delta_n\Theta_{(+)} \end{aligned} \tag{3.4}$$

then using the ADM decomposition, assuming $(\frac{\partial}{\partial t})^\alpha = fn^\alpha$ is normal to the Σ_t where f is the lapse function, we have

$$\frac{\partial}{\partial t}\Theta_{(+)} = -fW. \tag{3.5}$$

As described in section 2.6, the authors of [1–3] proved results about the existence of MOTTs in a spacetime satisfying an appropriate energy condition dependent on the stability of a MOTS living on a leaf of the spacetime foliation, and in the general case, a genericity condition constructed from the variation of the null expansion along the inward null normal of the MOTS. While these results are useful, they don't show the full picture. Recalling the merger of distinct MOTS as depicted in fig. 1.1, at time $T_{\text{bifurcate}}$, a new MOTS suddenly appears enclosing the original surfaces, then it bifurcates into an inner and outer branch as the data is evolved. This situation is an example of theorem 2.6.2, however the original statement of the theorem doesn't specify a way to distinguish whether the MOTT will correspond to a creation, an annihilation, or even something degenerate. We're interested in extending this result by deriving a geometric condition local to the MOTS which will give us the tools to characterize the evolutionary behaviour of the MOTT. Let us start by stating results analogous to theorems 2.6.1 and 2.6.2 under the assumption of spherical symmetry.

Theorem 3.1.2. *Let $(\mathcal{M}, g)$ be a spherically symmetric spacetime foliated by spacelike hypersurfaces Σ_t. Assume that in some leaf Σ_0, a 2-sphere denoted (r_0, t_0) is a MOTS. If $\frac{\partial \Theta}{\partial r}(r_0, t_0) \neq 0$, then (r_0, t_0) is contained in a MOTT $(r(t), t)$ where $r(t)$ is a smooth function defined on a neighbourhood of t_0 with $r(t_0) = r_0$.*

This result tells us that the MOTS (r_0, t_0) with non-vanishing principal eigenvalue is contained in a MOTT of the form $(r(t), t)$ which (at least locally) evolves smoothly through time.

Proof. Since (r_0, t_0) is a MOTS, it necessarily satisfies $\Theta(r_0, t_0) = 0$ and by hypothesis it has non-vanishing normal variation $\partial_r \Theta(r_0, t_0) \neq 0$. By the implicit function theorem, there is an open neighbourhood V of t_0 and a smooth function $r(t)$ with $r(t_0) = r_0$ such that $\Theta(r(t), t) = 0$ for all $t \in V$. □

Remark 3.1.2. Theorem 3.1.2 is an analogue of the MOTT existence result considered in [1, 2]. This result is more restrictive than the original result as it requires spherical symmetry, however in a spherical setting, it turns out to be a much stronger result. We recall that the original result proved the existence of a MOTT which evolved smoothly provided the MOTS stability operator L was invertible. In the spherically symmetric case however, we only required that the principal eigenvalue was non-vanishing with no further constraints on higher eigenvalues.

Let us now state an analogue of theorem 2.6.2, again restricted to the setting of spherical symmetry.

Theorem 3.1.3. *Suppose* (r_0, t_0) *is a MOTS with* $\partial_r \Theta(r_0, t_0) = 0$ *and* $\partial_t \Theta(r_0, t_0) \neq 0$. *We then have the following results:*

1. *The MOTS* (r_0, t_0) *is contained in a MOTT* $(r, \tau(r))$ *where* $\tau(r)$ *is a smooth function defined on a neighbourhood of* r_0 *such that* $\tau(r_0) = t_0$ *and* $\tau'(r_0) = 0$.
2. *Letting* $\lambda_p(r)$ *denote the principal eigenvalue of the MOTS foliating the MOTT,* $\lambda_p'(r_0)$ *has the same sign as* $\partial_r^2 \Theta(r_0, t_0)$.
3. *Further assuming the spacetime satisfies the null energy condition,* $\tau''(r_0)$ *has the same sign as* $\partial_r^2 \Theta(r_0, t_0)$.

Proof. (1) Since $\partial_t \Theta(r_0, t_0) \neq 0$, the implicit function theorem tells us that there is an open neighbourhood U of r_0 and a smooth function $t = \tau(r)$ with $\tau(r_0) = t_0$ such that

$$\Theta(r, \tau(r)) = 0 \tag{3.6}$$

for all $r \in U$, hence (r_0, t_0) is contained in a MOTT of the form $(r, \tau(r))$. Taking a derivative of (3.6) and evaluating at (r_0, t_0), we find that

$$\begin{aligned} \partial_r \Theta(r_0, t_0) + \partial_t \Theta(r_0, t_0) \tau'(r_0) &= 0 \\ \tau'(r_0) &= -\frac{\partial_r \Theta}{\partial_t \Theta}(r_0, t_0) \\ &= 0. \end{aligned} \tag{3.7}$$

This result tells us that the MOTT is tangent to the time slice Σ_{t_0} at the MOTS, precisely the result from [3].

(2) Let $\lambda_p(r)$ denote the principal eigenvalue of the MOTS $(r, \tau(r))$ such that $\lambda_p(r_0) = 0$. We recall from theorem 3.1.1 that the principal eigenvalue is related to the first variation of the null expansion by

$$\lambda_p(r) = a(r, \tau(r)) \frac{\partial \Theta}{\partial r}(r, \tau(r)). \tag{3.8}$$

Taking a derivative, we obtain

$$\begin{aligned}\lambda_p'(r_0) &= \Big(\partial_r a(r_0,t_0) + \partial_t a(r_0,t_0)\tau'(r_0)\Big)\frac{\partial\Theta}{\partial r} + a(r_0,t_0)\Big(\frac{\partial^2\Theta}{\partial r^2}(r_0,t_0) + \frac{\partial^2\Theta}{\partial r\partial t}(r_0,t_0)\tau'(r_0)\Big)\\ &= a(r_0,t_0)\frac{\partial^2\Theta}{\partial r^2}(r_0,t_0). \end{aligned} \tag{3.9}$$

As a result, $\lambda_p'(r_0)$ has the same sign as $\partial_r^2\Theta(r_0,t_0)$.

(3) Since the MOTT has a critical point, the next step is to consider the second variation to determine the type of critical point. Taking a second derivative of (3.6), we obtain

$$\begin{aligned} 0 &= \Big(\partial_r^2\Theta + 2\partial_r\partial_t\Theta\tau'(r_0) + \partial_t^2\Theta(\tau')^2(r_0) + \partial_t\Theta\tau''(r_0)\Big)(r_0,t_0)\\ \tau''(r_0) &= -\frac{\partial_r^2\Theta + 2\partial_r\partial_t\Theta\tau' + \partial_t^2\Theta(\tau')^2}{\partial_t\Theta}(r_0,t_0)\\ &= -\frac{\partial_r^2\Theta}{\partial_t\Theta}(r_0,t_0). \end{aligned} \tag{3.10}$$

We recall that $\partial_t\Theta(r_0,t_0) = -fW$. Since we assume the spacetime satisfies the null energy condition, we know that $W \geq 0$. Taking the lapse $f > 0$, we have $\partial_t\Theta(r_0,t_0) < 0$ (since $\partial_t\Theta(r_0,t_0) \neq 0$ by hypothesis) and so the above tells us that $\tau''(r_0)$ has the same sign as $\partial_r^2\Theta(r_0,t_0)$, and hence the type of critical point becomes a sign condition on the second derivative of the function $\tau(r)$. □

Remark 3.1.3. The results of theorem 3.1.3 can be summarized as follows:

1. A MOTS satisfying $\partial_r\Theta = 0$ (marginally stable) and $\partial_t\Theta \neq 0$ (the genericity condition) is contained in a MOTT, expressed as the graph $(r,\tau(r))$, tangent to the time slice, precisely the case from theorem 2.6.2

2. Thinking of the principal eigenvalue as a function of r, it turns out that its first derivative is proportional to the second variation of $\Theta_{(+)}$ at the MOTS.

3. The second derivative of $\tau(r)$ being proportional to the second variation of $\Theta_{(+)}$ at the MOTS means that the evolutionary behaviour of the MOTT is dependent on the sign of a smooth function defined on a neighbourhood of the MOTS. In particular, whether the MOTT corresponds to a creation, an annihilation, or something degenerate depends on the sign of $\partial_r^2\Theta(r_0,t_0)$.

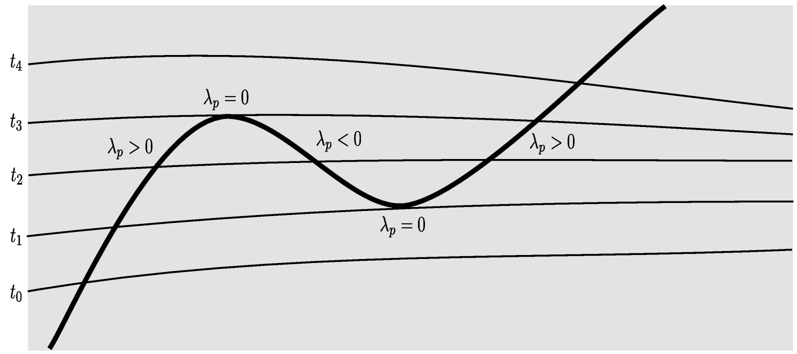


Figure 3.1: Schematic diagram of cubic MOTT weaving through time which intersects the t_1 and t_3 time slices at marginally stable MOTSs which bifurcate into strictly stable (blue) and unstable (red) branches.

Corollary 3.1.4. *In addition to the assumptions from theorem 3.1.3, further suppose that $\partial_r^2\Theta(r_0, t_0) \neq 0$, then $\lambda_p'(r_0) \neq 0$ and at the marginally stable MOTS (r_0, t_0), the MOTT will bifurcate into a strictly stable and an unstable branch. In the case $\partial_r^2\Theta > 0$, the MOTS corresponds to a creation event and will bifurcate into the future while if $\partial_r^2\Theta < 0$, the MOTS corresponds to an annihilation and will bifurcate into the past as seen in figs. 2.7 and 3.1. In the case of a future bifurcation, the outer branch will be strictly stable while the inner branch is unstable, and vice versa for a past bifurcation.*

Corollary 3.1.4 supposes that the second variation of $\Theta(r, t)$ doesn't vanish, however we can loosen this restriction a bit, only requiring that the m-th variation doesn't vanish for some natural number m. The result can then be stated as:

Corollary 3.1.5. *Let (r_0, t_0) be a MOTS as in theorem 3.1.3 contained in a MOTT $(r, \tau(r))$ and suppose there is a natural number m such that*

$$\frac{\partial\Theta}{\partial r}(r_0, t_0) = \frac{\partial^2\Theta}{\partial r^2}(r_0, t_0) = \cdots = \frac{\partial^{m-1}\Theta}{\partial r^{m-1}}(r_0, t_0) = 0, \ \frac{\partial^m\Theta}{\partial r^m}(r_0, t_0) \neq 0, \tag{3.11}$$

similar to the notion of "finitely stable" as defined in [32]. Then $\tau'(r_0) = \cdots = \tau^{(m-1)}(r_0) = 0$ and $\tau^{(m)}(r_0)$ has the same sign as $\partial_r^m\Theta(r_0, t_0)$.

Proof. By the implicit function theorem, there is a smooth function $\tau(r)$ defined on a neighbourhood of r_0 such that $\Theta(r, \tau(r)) = 0$ for all r locally. Computing the first variation, we again find that

$$\tau'(r_0) = -\frac{\partial_r \Theta}{\partial_t \Theta}(r_0, t_0) = 0. \tag{3.12}$$

The MOTT is once again tangent to the hypersurface Σ_{t_0}, however since the second normal variation of $\Theta(r,t)$ vanishes, we need to consider higher variations. Using the hypothesis that the first $(m-1)$ variations vanish, we obtain

$$\begin{aligned}
\tau''(r_0) &= -\frac{\partial_r^2 \Theta}{\partial_t \Theta}(r_0, t_0) = 0 \\
\tau^{(3)}(r_0) &= -\frac{\partial_r^3 \Theta}{\partial_t \Theta}(r_0, t_0) = 0 \\
&\vdots \\
\tau^{(m-1)}(r_0) &= -\frac{\partial_r^{m-1} \Theta}{\partial_t \Theta}(r_0, t_0) = 0 \\
\tau^{(m)}(r_0) &= -\frac{\partial_r^{m} \Theta}{\partial_t \Theta}(r_0, t_0),
\end{aligned} \tag{3.13}$$

hence $\tau^{(m)}(r_0)$ has the same sign as $\partial_r^m \Theta(r_0, t_0)$ since $\partial_t \Theta(r_0, t_0) < 0$ by the genericity condition. $\square$

Remark 3.1.4. We can interpret the above result in terms of the derivative test from calculus. There are two main cases to consider:

1. If m is even and $\tau^{(m)}(r_0) < 0$ (> 0), then (r_0, t_0) is a local maximum (minimum) of the MOTT.

2. If m is odd and $\tau^{(m)}(r_0) < 0$ (> 0), then (r_0, t_0) is a strictly decreasing (increasing) point of inflection.

In the same way as before, we let $\lambda_p(r)$ denote the principal eigenvalue of $(r, \tau(r))$

such that $\lambda_p(r_0) = 0$. Taking derivatives again, we obtain

$$\begin{aligned}
\lambda_p'(r_0)\psi &= \frac{\partial^2\Theta}{\partial r^2}(r_0, t_0) = 0 \\
\lambda_p''(r_0)\psi &= \frac{\partial^3\Theta}{\partial r^3}(r_0, t_0) = 0 \\
&\vdots \\
\lambda_p^{(m-2)}(r_0)\psi &= \frac{\partial^{m-1}\Theta}{\partial r^{m-1}}(r_0, t_0) = 0 \\
\lambda_p^{(m-1)}(r_0)\psi &= \frac{\partial^{m}\Theta}{\partial r^{m}}(r_0, t_0),
\end{aligned} \tag{3.14}$$

and once again, the $(m-1)$ derivative of $\lambda_p(r_0)$ has the same sign as $\partial_r^{m-1}\Theta(r_0, t_0)$ and hence $\tau^{(m)}(r_0)$.

3.2 Intersections of Marginally Outer Trapped Tubes

In the previous section, we introduced two characterizations of a spherical MOTT: either as the zero set of the null expansion, or as the disjoint union of smooth curves in the $(r,t)-$plane such that $\Theta(r(s), t(s)) = 0$ for all s. If the MOTT is smooth, these characterizations are equivalent, however a breakdown occurs when the zero set of $\Theta(r,t)$ contains intersections. In chapter 4 we will consider a specific example of a spherically symmetric spacetime in which this situation can be ruled out entirely, however the same can't be said in the general case. In this section, we will construct a spherically symmetric spacetime satisfying an appropriate energy condition that produces a null expansion $\Theta(r,t)$ whose zero set contains intersections.

It should be noted that the intersections of MOTTs currently being discussed should not be confused with the self-intersecting MOTS discussed in [25].

To begin, let us consider a general spherically symmetric spacetime with metric

$$g_{\alpha\beta}dx^\alpha dx^\beta = -A(r,t)dt^2 + 2B(r,t)dtdr + D(r,t)dr^2 + P^2(r)d\Omega^2 \tag{3.15}$$

where A, B, D, P are smooth functions. In this spacetime, we consider the spacelike hypersurface Σ with unit timelike normal

$$n_\alpha dx^\alpha = -\sqrt{\frac{B^2 + AD}{D}} dt = -\sqrt{\frac{F}{D}} dt \tag{3.16}$$

where we let $F = B^2 + AD$ to simplify notation. The induced metric on this slice is

$$h_{ab} dx^a dx^b = D dr^2 + P^2 d\Omega^2 \tag{3.17}$$

and the extrinsic curvature is

$$K_{ab} dx^a dx^b = \frac{(B\partial_r D + D\partial_t D - 2D\partial_r B) dr^2 - 2BPP' d\Omega^2}{2\sqrt{DF}}. \tag{3.18}$$

We now consider the spacelike 2-sphere $\mathcal{S}$ embedded in the spacetime with unit spacelike normal

$$s_\alpha dx^\alpha = \frac{B}{\sqrt{D}} dt + \sqrt{D} dr \tag{3.19}$$

and induced metric

$$q_{AB} dx^A dx^B = P^2 d\Omega^2. \tag{3.20}$$

The extrinsic curvature of this 2-surface is given by

$$J_{AB} dx^A dx^B = \frac{PP'}{\sqrt{D}} d\Omega^2, \tag{3.21}$$

and the mean curvature is hence

$$\begin{aligned} J &= \frac{2\partial_r P}{P\sqrt{D}} \\ &= \frac{2P'}{P\sqrt{D}}. \end{aligned} \tag{3.22}$$

We let the null normals to the 2-surface be

$$\ell^\pm_\alpha dx^\alpha = \frac{\pm B - \sqrt{F}}{\sqrt{D}} dt \pm \sqrt{D} dr \tag{3.23}$$

Finally, the outward null expansion scalar of the 2-surface is given by

$$\begin{aligned}\Theta_{(+)} &= q^{\alpha\beta}\nabla_\alpha \ell^+_\beta \\ &= \frac{2(F - B\sqrt{F})P'}{FP\sqrt{D}}.\end{aligned} \tag{3.24}$$

Let us verify the null energy condition using the null normals found above. We again simplify the notation by letting

$$\begin{aligned}G &= 2\partial_r B - \partial_t D \\ Q &= A\partial_r D + D\partial_r A.\end{aligned}$$

We then obtain

$$\begin{aligned}R_{\alpha\beta}\ell^\alpha_+\ell^\beta_+ = \frac{1}{PDF^2}\Bigg\{&-4P''B^4 + \Big[4P''\sqrt{F} + 2P'G\Big]B^3 \\ &+ 2\Big[P'\Big(Q - G\sqrt{F}\Big) - 3AP''D\Big]B^2 \\ &+ \Big[P'\Big(AD\big(G - 2\partial_t D\big) - D^2\partial_t A - 2Q\sqrt{F}\Big) \\ &\quad + 4P''AD\sqrt{F}\Big]B \\ &+ \Big[P'\Big(Q + 2D\partial_t B + 2\sqrt{F}\partial_t D\Big) - 2P''AD\Big]AD\Bigg\}.\end{aligned} \tag{3.25}$$

Looking at the form of $\Theta_{(+)}$, we know that the surface $\mathcal{S}$ is a MOTS if $\sqrt{F} = B$, that is, if $AD = 0$. We next want to see when the MOTS is marginally stable, so taking the r derivative and evaluating on $\mathcal{S}$, we obtain

$$\begin{aligned}
\partial_r \Theta_{(+)}|_{\mathcal{S}} &= \frac{2P'}{B^2 P\sqrt{D}}\left(\partial_r F - \sqrt{F}\partial_r B - \frac{B\partial_r F}{2\sqrt{F}}\right) \\
&= \frac{2P'}{B^2 P\sqrt{D}}\bigg(2B\partial_r B + A\partial_r D + D\partial_r A - B\partial_r B \\
&\qquad\qquad - \frac{2B\partial_r B + A\partial_r D + D\partial_r A}{2}\bigg) \\
&= \frac{P'(A\partial_r D + D\partial_r A)}{B^2 P\sqrt{D}} \\
&= \frac{QP'}{B^2 P\sqrt{D}},
\end{aligned} \tag{3.26}$$

so we see that $\mathcal{S}$ is a marginally stable MOTS if

$$AD = 0 \tag{3.27}$$

$$A\partial_r D + D\partial_r A = 0. \tag{3.28}$$

Applying these conditions to (3.25), we find that it reduces to

$$R_{\alpha\beta}\ell_+^\alpha \ell_+^\beta = -\frac{DP'\partial_t A}{PB^3}, \tag{3.29}$$

and thus if the null energy condition holds, then we must have

$$\frac{P'D\partial_t A}{PB^3} \leq 0 \tag{3.30}$$

on $\mathcal{S}$, so we gain the new condition $\partial_t A \leq 0$ on $\mathcal{S}$.

To this point, our conditions on the metric functions are

$$A(0,0) = \partial_r A(0,0) = \partial_t A(0,0) = 0 \tag{3.31}$$

$$\partial_t A(r,t) \leq 0. \tag{3.32}$$

To satisfy the condition that $g_{\alpha\beta}$ is a Lorentzian metric, we require the metric functions to satisfy

$$\det g = -(AD + B^2)P^4 \sin^2\theta < 0. \tag{3.33}$$

In a local neighbourhood of $\mathcal{S}$, we let $D = 1$, A be bounded, and we let B^2 be large enough to satisfy the Lorentzian signature of $g_{\alpha\beta}$.

In this spacetime, the zero set of $\Theta(r,t)$ is equivalent to the zero set of $A(r,t)$, so the MOTT has intersections if the zero set of $A(r,t)$ has intersections. To satisfy the null energy condition we also require that $\partial_t A(r,t) \leq 0$. A function that satisfies these conditions is

$$A(r,t) = -t(r-r_0)^2. \tag{3.34}$$

We have thus obtained a physically reasonable and spherically symmetric spacetime containing a marginally stable MOTS $\mathcal{S}$ that evolves into a intersecting MOTT of the form $\{r = r_0\} \cup \{t = 0\}$ as desired.

Chapter 4

Vaidya Spacetime

In chapter 3, we reviewed the basic construction of a spherically symmetric spacetime and remarked on the simplifications the symmetry provides to variations, and as a result, the MOTS stability condition. In this chapter, we will apply the results from the previous chapter using the *Vaidya spacetime*, the simplest example of a spherically symmetric, dynamical spacetime.

4.1 Finding MOTTs in Vaidya

4.1.1 Construction of Vaidya spacetime

The static spherically symmetric spacetime that solves the vacuum Einstein equations is the Schwarzschild spacetime with metric

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2 \tag{4.1}$$

where $f(r) = 1 - \frac{2m}{r}$ and $d\Omega^2$ is the metric on the 2-sphere. In fact, there is a result called Birkhoff's theorem which states that any spherically symmetric solution to the vacuum Einstein equations must be static and asymptotically flat, thus its exterior solution must be given by the Schwarzschild metric [24].

The Schwarzschild metric is static, meaning it has a timelike Killing vector field ξ^α and there is a spacelike hypersurface Σ that is hypersurface orthogonal to the orbits of the isometry, or by Frobenius's theorem, ξ^α satisfies

$$\xi_{[\alpha}\nabla_\beta\xi_{\gamma]} = 0, \tag{4.2}$$

or using the notation of forms [31]

$$\xi \wedge d\xi = 0. \tag{4.3}$$

As one may notice, the Schwarzschild metric has a singularity at $r = 2m$; this however is simply a coordinate singularity and it can be eliminated through a coordinate transformation adapted to radial null geodesics. We introduce the so-called tortoise coordinate

$$r^* = r + 2m\log\left|\frac{r}{2m} - 1\right|. \tag{4.4}$$

As seen from the above construction, the tortoise coordinate will go off to $-\infty$ as the radial coordinate r approaches the supposed singularity at $2m$. Although we can choose coordinates that make the metric well-behaved at $r = 2m$, the surface is still of interest. If we transform the metric into coordinates adapted to radial null geodesics such that the metric is regular at $r = 2m$, it turns out that for $r < 2m$ every radial null geodesic has decreasing r and reaches the curvature singularity $r = 0$ in finite affine parameter. Since the tangent vector to timelike curves lie inside null cones, then observers travelling along radial timelike curves will also reach $r = 0$. It turns out that the same result holds for any timelike or null curve in $r < 2m$, hence no signal can be sent from $r < 2m$ to $r > 2m$. The surface $r = 2m$ therefore acts as a barrier, preventing any observer in the region $r > 2m$ from seeing what happens in $r < 2m$, hence $r = 2m$ is the black hole *event horizon*.

In the standard Schwarzschild coordinates, as an observer approaches the event horizon at $r = 2m$, its coordinate velocity will continually slow down, and will in fact never reach the surface in finite Schwarzschild time. We can fix this behaviour by introducing the ingoing Eddington-Finkelstein coordinate $v = t + r^*$ and replacing the original Schwarzschild time coordinate t [26]. Under this transformation, signals travelling into the black hole along radial null geodesics will have constant coordinate velocity, and will be able to cross the event horizon.

We can then write the Schwarzschild metric in ingoing Eddington-Finklestein coordinates (v, r, θ, ϕ) giving us

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dv^2 + 2dvdr + r^2 d\Omega^2. \tag{4.5}$$

The *Vaidya* metric is the simplest non-static generalization of Schwarzschild originally introduced in [30]. It describes the non-empty exterior spacetime of a spherically symmetric mass which is either emitting or absorbing null dust. To obtain the ingoing Vaidya metric, we begin with the metric (4.5) and find that it is physically reasonable to extend the constant mass parameter to a function of the null coordinate v. The ingoing Vaidya metric is thus given by

$$g_{\alpha\beta}dx^\alpha dx^\beta = -\left(1 - \frac{2m(v)}{r}\right)dv^2 + 2dvdr + r^2 d\Omega^2 \tag{4.6}$$

[29]. The ingoing Vaidya metric is a solution to the Einstein equations with stress-energy tensor

$$T_{\alpha\beta} = \frac{m'(v)}{4\pi r^2}k_\alpha k_\beta \tag{4.7}$$

where $k_\alpha = -dv$ is tangent to ingoing null geodesics and a prime denotes a derivative with respect to v [24]. This spacetime satisfies the null energy condition provided the mass function is non-decreasing.

4.1.2 MOTS in Vaidya

We consider a spacelike hypersurface Σ of constant $T \equiv v - r$. Such a hypersurface will have unit timelike normal

$$n_\alpha dx^\alpha = -\sqrt{\frac{r}{r + 2m(v)}}(dv - dr) \tag{4.8}$$

and induced metric

$$h_{ab}dx^a dx^b = \frac{r + 2m(v)}{r}dr^2 + r^2 d\Omega^2. \tag{4.9}$$

As a tensor on $\mathcal{M}$, the induced metric takes the form

$$h_{\alpha\beta}dx^\alpha dx^\beta = \frac{4m^2(v)}{r(r + 2m(v))}dv^2 + \frac{4m(v)}{r + 2m(v)}dvdr + \frac{r}{r + 2m(v)}dr^2 + r^2 d\Omega^2. \tag{4.10}$$

The extrinsic curvature tensor for this hypersurface is

$$K_{ab}dx^a dx^b = \frac{2m^2(v) + 2rm(v) - r^2 m'(v)}{r^{5/2}\sqrt{r + 2m(v)}} dr^2 - \frac{2\sqrt{r}m(v)}{\sqrt{r + 2m(v)}} d\Omega^2. \tag{4.11}$$

Next we let the closed 2-surface $\mathcal{S}$ that bounds Σ be the 2-sphere of constant r. The unit normal to this surface (in $\mathcal{M}$) is

$$s_\alpha dx^\alpha = \sqrt{\frac{r}{r + 2m(v)}} \left(\frac{2m(v)}{r} dv + dr \right), \tag{4.12}$$

or in Σ

$$s_a dx^a = \sqrt{1 + \frac{2m(v)}{r}} dr. \tag{4.13}$$

The induced metric on $\mathcal{S}$ is

$$q_{AB} dx^A dx^B = r^2 d\Omega^2. \tag{4.14}$$

The extrinsic curvature tensor for $\mathcal{S}$ in Σ is

$$J_{AB} dx^A dx^B = \frac{r^{3/2}}{\sqrt{r + 2m(v)}} d\Omega^2. \tag{4.15}$$

The mean curvature (trace of the extrinsic curvature) of $\mathcal{S}$ is then

$$J = q^{AB} J_{AB} = \frac{2}{\sqrt{r(r + 2m(v))}} \tag{4.16}$$

and the trace of the extrinsic curvature on Σ with respect to the metric q is

$$\mathrm{tr}_q K = q^{ab} K_{ab} = -\frac{4m(v)}{r^{3/2}\sqrt{r + 2m(v)}} \tag{4.17}$$

where $q^{ab} = h^{ab} - s^a s^b$.

From the two normals we've so far computed, we construct future-directed outward and inward pointing unit null normals to $\mathcal{S}$:

$$\ell_\pm = n \pm s. \tag{4.18}$$

In our coordinate chart, they are given by

$$\ell^{\pm}_{\alpha} dx^{\alpha} = \sqrt{\frac{r}{r+2m(v)}} \left(\frac{\pm 2m(v) - r}{r} dv + (1 \pm 1) dr \right). \tag{4.19}$$

Using the null normals, we can decompose the extrinsic curvature tensor of $\mathcal{S}$ into two null second fundamental forms $\chi^{\pm}_{\alpha\beta} = \nabla_{\alpha} \ell^{\pm}_{\beta}$. The null expansion scalar is defined as $\Theta_{(\pm)} = q^{\alpha\beta} \nabla_{\alpha} \ell^{\pm}_{\beta}$, or equivalently by $\Theta_{(\pm)} = q^{ab} K_{ab} \pm q^{AB} J_{AB}$. The components of $\chi^{+}_{\alpha\beta}$ tangent to $\mathcal{S}$ (evaluated on $\mathcal{S}$) are

$$\begin{aligned}
\chi^{+}_{\theta\theta} &= \frac{\sqrt{r}(r - 2m(v))}{\sqrt{r + 2m(v)}} \\
\chi^{+}_{\phi\phi} &= \frac{\sqrt{r}(r - 2m(v))}{\sqrt{r + 2m(v)}} \sin^2 \theta.
\end{aligned}$$

The norm of this tensor is given by

$$\begin{aligned}
\|\chi_{+}\|^2_q &= q^{\alpha\mu} q^{\beta\nu} \chi^{+}_{\alpha\beta} \chi^{+}_{\mu\nu} \\
&= \frac{2(r - 2m(v))^2}{r^3 (r + 2m(v))}.
\end{aligned}$$

When evaluated on $\mathcal{S}$, we find that the outer null expansion is

$$\Theta_{(+)} = \frac{2(r - 2m(v))}{r^{3/2} \sqrt{r + 2m(v)}} \tag{4.20}$$

which vanishes when $r = 2m(v)$, implying the 2-sphere $r = 2m(v)$ is a MOTS.

4.1.3 Mass of Vaidya Black Hole

Before we start discussing the stability of the MOTS $\mathcal{S}$, let us first talk about the mass of the spacetime.

As in [24], we define by M the gravitational mass of an asymptotically-flat spacetime when $\mathcal{S}_t$ is a 2-sphere taken to infinity:

$$M = -\frac{1}{8\pi} \lim_{\mathcal{S}_t \to \infty} \oint_{\mathcal{S}_t} (k - k_0) \, \mathrm{d}\mathcal{S} \tag{4.21}$$

where $\mathrm{d}\mathcal{S} = \sqrt{\det q} d^2x$ is the surface element on $\mathcal{S}_t$, k is the trace of the extrinsic curvature of $\mathcal{S}_t$ embedded in the time slice Σ_t, and k_0 is the trace of the extrinsic curvature of $\mathcal{S}_t$ embedded in flat space.

In the case of Vaidya, we recall that the extrinsic curvature of the 2-sphere $\mathcal{S}(t, R)$ embedded in Σ_t is given by (4.16) and takes the form

$$\begin{aligned} k &= \frac{2}{R}\left(1 + \frac{2m(v)}{R}\right)^{-1/2} \\ &= \frac{2}{R}\left(1 - \frac{m(v)}{R} + \mathcal{O}(R^{-2})\right) \end{aligned} \tag{4.22}$$

where $R \gg 2m(v)$ and we used the binomial approximation in the second line. As a sphere embedded in flat space, $\mathcal{S}(t, R)$ has extrinsic curvature $k_0 = \frac{2}{R}$. Taking the difference of the extrinsic curvatures, we obtain

$$k - k_0 = -\frac{2m(v)}{R^2} + \mathcal{O}(R^{-3}). \tag{4.23}$$

Putting everything together, we find that the gravitational mass for Vaidya looks like:

$$\begin{aligned} M &= \lim_{\mathcal{S}_t \to \infty} \left(-\frac{1}{8\pi} \oint_{\mathcal{S}_t} \left(-\frac{2m(v)}{R^2} + \mathcal{O}(R^{-3})\right) R^2 \sin\theta \, \mathrm{d}^2x\right) \\ &= \lim_{\mathcal{S}_t \to \infty} [m(v) + \mathcal{O}(R^{-1})]. \end{aligned} \tag{4.24}$$

The mass was defined as the result of the integral taking the limit of $\mathcal{S}_t$ to infinity, however there are several ways of reaching infinity. The first we will discuss is *spatial infinity*, which we get when we keep t constant and take the limit $R \to \infty$. In this case, the mass given by (4.21) is called the *ADM* mass and it represents all of the mass contained in the spacetime. For Vaidya, the ADM mass will look like

$$\begin{aligned} M_{\mathrm{ADM}}(t) &= \lim_{R \to \infty} [m(R + t) + \mathcal{O}(R^{-1})] \\ &= m(\infty). \end{aligned} \tag{4.25}$$

Let us now take the limit of $\mathcal{S}_t$ to *null infinity*. The new limiting procedure corresponds to a distinct notion of mass called the *Bondi-Sachs* mass [7, 27]. To define the notion of null infinity, we must use the null coordinates $u = t - r$ (outgoing) and

$v = t + r$ (ingoing). In this case, the 2-surface becomes a surface of constant u and v which we denote $\mathcal{S}(u, v)$. Null infinity corresponds to taking the limit $v \to \infty$ while keeping u fixed. The Bondi-Sachs mass can be interpreted as the mass remaining in the spacetime at outgoing time u after emission of gravitational radiation. The physical significance of the Bondi-Sachs mass comes from the fact that when an isolated body emits radiation, the rate of change of $M_{\mathrm{BS}}(u)$ is directly related to the outward flux of radiated energy. Letting F denote the flux, the Bondi-Sachs mass satisfies

$$\frac{\mathrm{d}M_{\mathrm{BS}}}{\mathrm{d}u} = -\oint_{\mathcal{S}(u, v\to\infty)} F\sqrt{\det q}\,\mathrm{d}^2x. \tag{4.26}$$

As found in the original work by Bondi, van der Burg, and Metzner in [7], the flux is manifestly non-negative, that is $F \geq 0$. Hence the Bondi-Sachs mass decreases with time.

In the case of a Vaidya black hole, we find that the Bondi-Sachs mass is

$$M_{\mathrm{BS}} = m(\infty) \tag{4.27}$$

which is precisely the ADM mass. This is expected since the ingoing Vaidya metric describes the spacetime containing a black hole that is absorbing null dust and is not emitting any radiation, hence the outward flux of radiated energy will be zero and no energy leaves the spacetime.

4.1.4 Stability of MOTS in Vaidya

Now that we know $\mathcal{S}$ is a MOTS if $r = 2m(v)$, our next step is to verify if it's stable. Since Vaidya is spherically symmetric, the stability operator is once again self-adjoint, simplifying things considerably. Using the result from theorem 3.1.1, we know that in spherical symmetry, variations reduce to simple partial derivatives and in particular the principal eigenvalue of the stability operator takes the form

$$\lambda_p = a\frac{\partial\Theta}{\partial r}\bigg|_{\mathcal{S}} \tag{4.28}$$

for a positive function a on $\mathcal{S}$.

We find that the principal eigenvalue of L up to a constant scale factor is given by

$$\begin{aligned}
\frac{\partial \Theta}{\partial r}\bigg|_{r=2m(v)} &= \frac{1}{r^{3/2}\sqrt{r+2m(v)}}\bigg(2(1-2m'(v)) - \frac{3(r-2m(v))}{r} \\
&\qquad\qquad - \frac{(r-2m(v))(1+2m'(v))}{(r+2m(v))}\bigg)\bigg|_{r=2m(v)} \\
&= \frac{(1-2m'(v))\sqrt{2}}{4m^2(v)}
\end{aligned} \tag{4.29}$$

where a prime denotes a derivative with respect to v. From the above expression, we can say that the MOTS $r = 2m(v)$ is stable when $m'(v) \leq \frac{1}{2}$ where equality implies the MOTS is only marginally stable. Recall from (4.7) that the typical energy conditions are satisfied when $m'(v) \geq 0$ for all v. The results we're trying to derive are for a marginally stable MOTS in the initial data set, so we require $m'(v_0) = 1/2$ in the constant $v_0 = r_0 + t_0$ slice.

When $\mathcal{S}$ is marginally stable, we know that the principal eigenvalue, and hence the first derivative of the null expansion, vanishes. This tells us that $\mathcal{S}$ corresponds to a critical point of a curve in the (r,t)-plane (because of spherical symmetry) which is a MOTT. Since $\mathcal{S}$ is a critical point of the MOTT, our next step is to find out what the MOTT looks like nearby, so we take a second derivative. When evaluated at the marginally stable MOTS (r_0, t_0), i.e. $r = 2m(r+t)$ and $m'(r_0+t_0) = 1/2$, the second derivative of $\Theta(r,t)$ takes the form

$$\frac{\partial^2 \Theta}{\partial r^2}(r_0, t_0) = \frac{-4m''(r_0+t_0)}{r_0^{3/2}\sqrt{r_0+2m(r_0+t_0)}}. \tag{4.30}$$

We now have three possible cases:

1. If $m''(r_0+t_0) < 0$ then the MOTS (r_0, t_0) is a local maximum and corresponds to a past bifurcation, i.e. an annihilation;

2. If $m''(r_0+t_0) > 0$ then the MOTS (r_0, t_0) is a local minimum and corresponds to a future bifurcation, i.e. a creation;

3. If $m''(r_0+t_0) = 0$ then the MOTS (r_0, t_0) is a degenerate case and we require higher order derivatives to characterize the MOTT.

The initial data set the MOTS $\mathcal{S}$ lies in is a hypersurface of constant $t = v - r$. We can express the null expansion as

$$\Theta(r,t) = \frac{F(r,t)}{G(r,t)} \tag{4.31}$$

where $F(r,t) = 2(r - 2m(r+t))$ and $G(r,t) = r^{3/2}\sqrt{r + 2m(r+t)}$. Since (r_0, t_0) is a marginally stable MOTS, we know that $F(r_0, t_0) = 0$ and $m'(r_0 + t_0) = \frac{1}{2}$. The differential of $\Theta(r,t)$ at (r_0, t_0) is then

$$\begin{aligned} D\Theta(r_0,t_0) &= \frac{1}{G(r_0,t_0)} \begin{bmatrix} \partial_r F(r_0,t_0) & \partial_t F(r_0,t_0) \end{bmatrix} \\ &= \frac{1}{G(r_0,t_0)} \begin{bmatrix} 2(1 - 2m'(r_0+t_0)) & -4m'(r_0+t_0) \end{bmatrix} \\ &= \frac{1}{G(r_0,t_0)} \begin{bmatrix} 0 & -2 \end{bmatrix}. \end{aligned} \tag{4.32}$$

Since the $\partial_r\Theta(r_0,t_0) = 0$ and $\partial_t\Theta(r_0,t_0) \neq$, we see from theorem 3.1.3 there is a smooth function $\tau(r)$ such that the graph $(r, \tau(r))$ is a MOTT containing the MOTS (r_0, t_0), and since $\tau'(r_0) = 0$, we know the MOTT is tangent to (r_0, t_0).

4.1.5 Results about MOTTs

In chapter 3, we mentioned how the MOTT passing through $r = 2m(v)$ in the Vaidya spacetime doesn't contain any intersections. Let us now prove that result.

Theorem 4.1.1. *The zero level set of $\Theta(r,t)$ does not contain any intersections, that is the zero set does not contain any curves that intersect at a point.*

Proof. $(r(s), \tau(s))$ being a MOTT means that $\Theta(r(s), \tau(s)) = 0$ for all s in a neighbourhood of s_0 where $(r(s_0), \tau(s_0)) = (r_0, t_0)$ which is a MOTS. For Vaidya, the differential of Θ never vanishes, meaning Θ has no critical points. This means that the implicit function theorem always applies and so we can always find a unique smooth curve $r(t)$ or $\tau(r)$ passing through a MOTS, hence the neighbourhood of any point in the zero set is always a smooth curve. As a result, the zero set is a disjoint union of smooth curves, and thus there are no intersections. $\square$

Theorem 4.1.2. *The zero level set of $\Theta(r,t)$ does not contain any closed loops.*

Proof. Suppose the zero set of $\Theta(r,t)$ contains a closed loop and suppose $\Theta > 0$ inside the loop. Since the region bounded by the loop is compact and Θ is continuous, the extreme value theorem states that Θ must achieve a local max and min in the loop, that is it must have a critical point somewhere in the loop, however as in the previous proof, the differential of Θ never vanishes and thus it has no critical points, a contradiction. Therefore the zero set can't contain any closed loops. $\square$

Example 4.1.1. Finally, let us consider an example of a MOTT that lies entirely in the hypersurface, one of the possibilities allowed by theorem 2.6.2. We consider a Vaidya spacetime with linear mass function $m(v) = \frac{1}{2}v + c$ for some constant $c \in \mathbb{R}$. Considering the MOTS equation $r = 2m(r + t_0)$, we see that it reduces to

$$t_0 + 2c = 0. \tag{4.33}$$

Since r cancels, we see that if (4.33) is satisfied, there are infinite solutions for r, whereas if the condition is not satisfied, there are zero solutions. In the case where $c = -\frac{1}{2}t_0$, every 2-sphere of radius $r > 0$ is a (marginally stable) MOTS in the hypersurface Σ_{t_0}, so since the genericity condition always holds for a marginally stable MOTS in Vaidya, the implicit function theorem tells us that there exists a smooth function $\tau(r)$ such that $\Theta(r, \tau(r)) = 0$ for all r. Since every value of r corresponds to a MOTS in the time slice t_0, the function $\tau(r) \equiv t_0$, hence the MOTT denoted by $(r, \tau(r))$ lies entirely in Σ_{t_0}.

4.2 Physically reasonable Vaidya spacetime with prescribed MOTTs

In the previous section, it was observed that for any MOTS in Vaidya, since both variations of the null expansion never vanished identically, we could always use the implicit function theorem to find a MOTT of the form $(r(s), \tau(s))$ containing the MOTS. In particular, a marginally stable MOTS is contained in a MOTT of the form $(r, \tau(r))$, with the bifurcation condition of the MOTS reducing to a derivative test of $\tau(r)$. The goal then is to construct a MOTT exhibiting interesting behaviour, however we have to determine whether such a MOTT could actually exist in a physically reasonable Vaidya spacetime. It turns out that if $\tau(r)$ is a smooth function that satisfies a condition on its first derivative, we can find a mass function corresponding to a physically reasonable Vaidya spacetime for which $(r, \tau(r))$ is a MOTT.

Theorem 4.2.1. *If $\tau(r)$ is smooth with $\tau(0) = 0$ and $\tau'(r) > -1$, then there exists an $m(v)$ with $m'(v) > 0$ for all v and $m(v) > 0$ for all $v > 0$ such that $(r, \tau(r))$ represents a MOTT in the Vaidya spacetime with mass function $m(v)$.*

Proof. The graph $(r, \tau(r))$ represents a MOTT in the Vaidya spacetime with mass function $m(v)$ if and only if (4.34) is satisfied

$$r = 2m(r + \tau(r)). \tag{4.34}$$

Define $v(r) = r + \tau(r)$ and take a derivative to obtain

$$v'(r) = 1 + \tau'(r) > 0 \tag{4.35}$$

since $\tau'(r) > -1$ for all r by hypothesis. Since $\tau(r)$ is smooth, $v(r)$ must also be smooth, and since it has non-zero derivative, the inverse function theorem applies, telling us that $v(r)$ is invertible and has smooth inverse with derivative

$$\left(v^{-1}\right)'(r + \tau(r)) = \frac{1}{v'(r)} > 0 \tag{4.36}$$

for all r.

Now, since $\tau(0) = 0$, we must have

$$v(0) = 0, \tag{4.37}$$

and thus by a property of inverses, we also have

$$v^{-1}(0) = 0. \tag{4.38}$$

Since $v^{-1}(0) = 0$ and v^{-1} is increasing for all v, we must have that $v^{-1} > 0$ for all $v > 0$.

Returning to (4.34), we identify

$$v^{-1}(r + \tau(r)) = 2m(r + \tau(r)), \quad \text{i.e. } m = \frac{1}{2}v^{-1}, \tag{4.39}$$

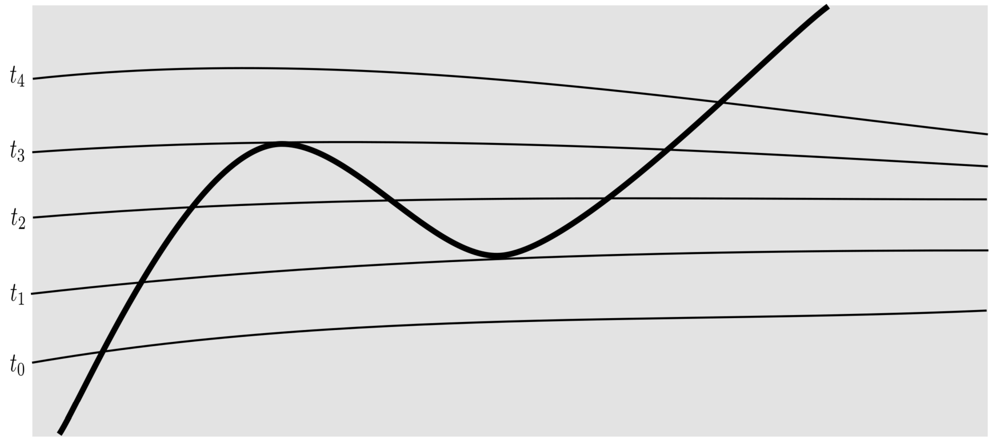


Figure 4.1: Schematic picture of a cubic MOTT weaving through time, adapted from figure in [32]

thus proving that there indeed exists a mass function $m(v)$ that satisfies the hypothesis. $\square$

In this section, we will examine some applications of theorem 4.2.1 and obtain mass functions when given a polynomial MOTT.

Example 4.2.1. Our first objective is to find a Vaidya spacetime that produces a cubic MOTT as in fig. 4.1. We begin with a general cubic of the form

$$\tau(r) = ar^3 + br^2 + cr \tag{4.40}$$

and we suppose that $\tau(r)$ has two turning points at R and ℓR respectively for some $\ell > 1$. We further suppose that R corresponds to a past bifurcation while ℓR corresponds to a future bifurcation, that is $\tau'(R) = 0 = \tau'(\ell R)$, $\tau''(R) < 0$ and $\tau''(\ell R) > 0$. Using the condition that the turning points are critical points of $\tau(r)$, we find the first coefficient is given by

$$a = \frac{-2b}{3(\ell+1)R}. \tag{4.41}$$

Using the condition on the second derivatives, we find that $b < 0$ so we let $b = -k^2$ for some real number k. To obtain an expression for c, we once again look at the first derivative at R, giving us

$$\begin{aligned}\tau'(r) &= \frac{-2b}{(\ell+1)R}r^2 + 2br + c \\ 0 &= \frac{2k^2R}{\ell+1} - 2k^2R + c \\ c &= \frac{2k^2\ell R}{\ell+1}.\end{aligned} \tag{4.42}$$

So far, our cubic MOTT takes the form

$$\tau(r) = \frac{2k^2}{3(\ell+1)R}r^3 - k^2r^2 + \frac{2k^2\ell R}{\ell+1}r. \tag{4.43}$$

By the hypothesis of theorem 4.2.1, there is a mass function $m(v)$ for a physically reasonable spacetime such that $(r, \tau(r))$ is a MOTT provided $\tau'(r) > -1$. Computing the derivative, we obtain the expression

$$\tau'(r) = \frac{2k^2}{(\ell+1)R}r^2 - 2k^2r + \frac{2k^2\ell R}{\ell+1}. \tag{4.44}$$

To get a condition on k^2 that makes (4.44) greater than -1, we look at the discriminant of the quadratic $\tau'(r)+1$ and see where it's negative.

When simplified, the discriminant of the quadratic reduces to

$$\Delta = 4k^2\frac{k^2R(\ell-1)^2 - 2(\ell+1)}{R(\ell+1)^2} \tag{4.45}$$

which is negative provided

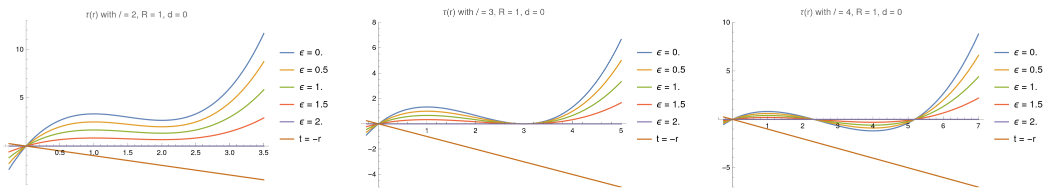
$$k^2 = \frac{(2-\varepsilon)(\ell+1)}{R(\ell-1)^2} \tag{4.46}$$

for some $\varepsilon \in (0,2)$.

Thus provided k^2 is of the form (4.46), the cubic MOTT (4.43) is contained in a physically reasonable Vaidya spacetime. Figures 4.2a to 4.2c depict the MOTT for different values of ℓ where we vary ε to observe how the MOTT changes for different values of k^2.

We can repeat the procedure for higher degree polynomials as well.

Example 4.2.2. Suppose we want a quartic MOTT with two future bifurcations at



(a) $\tau(r)$ with $\ell = 2$ (b) $\tau(r)$ with $\ell = 3$ (c) $\tau(r)$ with $\ell = 4$

Figure 4.2: A cubic MOTT in Vaidya

$r = R$ and $r = mR$ and a single past bifurcation at $r = \ell R$ where $m > \ell$. We start with a MOTT of the form

$$\tau(r) = ar^4 + br^3 + cr^2 + dr. \tag{4.47}$$

Using the fact that the derivative vanishes at the points of bifurcation, we can express the metric functions as

$$\begin{aligned} a &= -\frac{d}{4\ell m R^3} \\ b &= \frac{d(1+\ell+m)}{3\ell m R^2} \\ c &= -\frac{d(\ell+m+\ell m)}{2\ell m R}. \end{aligned} \tag{4.48}$$

Since $r = R$ corresponds to a future bifurcation, we recall that $\tau''(R) > 0$, which is satisfied provided $d < 0$ thus we let $d = -k^2$. We must now obtain a condition on k^2 that guarantees that $\tau'(r) > -1$ for all $r > 0$. In this case, $\tau'(r)$ is a cubic translated by $-k^2$, so to satisfy the condition we simply require $k^2 < 1$.

Putting everything together, we find that the MOTT takes the form

$$\tau(r) = (1-\varepsilon)\left(\frac{1}{4\ell m R^3}r^4 - \frac{(1+\ell+m)}{3\ell m R^2}r^3 + \frac{(\ell+m+\ell m)}{2\ell m R}r^2 - r\right). \tag{4.49}$$

Figure 4.3 depicts the MOTT (4.49) for varying ε.

4.3 Vaidya MOTT with glued Schwarzschild ends

From theorem 4.2.1, provided a smooth function $\tau(r)$ satisfied some conditions, we could always find a mass function such that the Vaidya spacetime with mass function

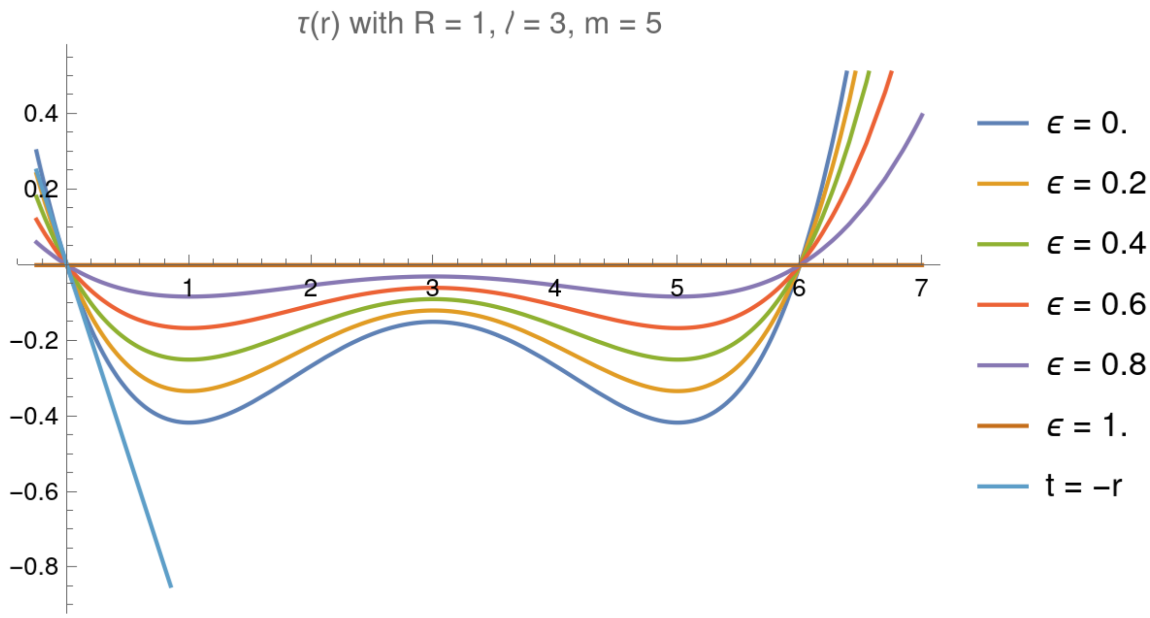


Figure 4.3: $\tau(r)$ with $\ell = 3$ and $m = 5$

$m(v)$ contains the MOTT characterized by $\tau(r)$. As seen in the cubic and quartic examples studied in section 4.2, this mass function becomes negative for $v < 0$ leading to the question of whether the spacetime is actually physically reasonable. However, since we're only interested in the local behaviour around $v > 0$, we can simply cut the portion we don't like and instead 'glue' a constant MOTT from Schwarzschild to the end points. Working in this direction, we can define the mass function as the piece-wise continuous function

$$m(v) = \begin{cases} M_-, & v \leq v_- \\ M(v), & v_- < v < v_+ \\ M_+, & v \geq v_+ \end{cases} \tag{4.50}$$

where $M(v)$ increases smoothly from M_- to M_+. This mass function describes a Schwarzschild spacetime with mass M_-, which between null times v_- and v_+ is accreting mass from in-falling shells of null dust, and then eventually settles down to a Schwarzschild black hole of mass M_+, which is the ADM mass of the black hole. As depicted in fig. 4.4, the MOTT weaving through spacetime is obtained from 'gluing' solutions from different regimes.

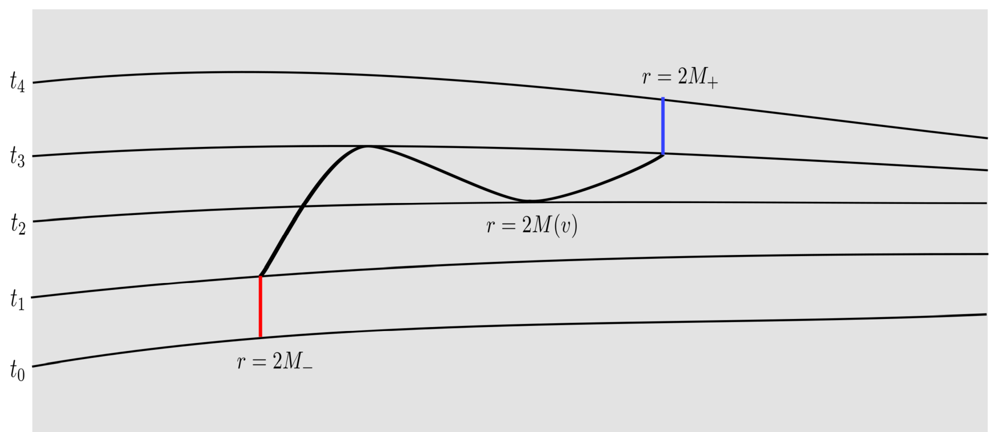


Figure 4.4: Schematic of a MOTT in spacetime. The red line represents the MOTT in Schwarzschild with mass M_-, the black curve represents the MOTT in Vaidya, and the blue line represents the MOTT in Schwarzschild with mass M_+.

Chapter 5

Conclusion

In this thesis, we have studied the time evolution of black hole horizons by way of quasi-local characterizations in the form of MOTTs. In particular, we looked at established results on the existence of MOTTs dependent on the stability of a MOTS in an initial data set, and motivated by recent numerical models of the collision and merger of binary black holes, we set out to derive a local geometric condition to distinguish the evolutionary behaviour of a MOTT tangent to a marginally stable MOTS. We started in chapter 3 with a discussion on general spherically symmetric spacetimes in which we proved that normal variations with constant speed reduced to partial derivatives, greatly simplifying the MOTS stability condition, and then we stated and proved existence results for MOTTs in spherical symmetry analogous to the established results. It turned out that for the strictly stable case, although restricted to a spherically symmetric setting, yielded a slightly stronger result than the original. In the marginally stable case, we used an implicit function theorem argument to prove the existence of a MOTT given by the graph of a smooth function. It was found that the second variation of the null expansion scalar had the same sign as the second derivative of the smooth function, giving us the desired local geometric condition to distinguish the evolutionary behaviour. Furthermore, it was found that if the MOTT bifurcated, then one branch was strictly stable while the other was unstable. The chapter concluded with a brief discussion on the possibilities of intersecting MOTTs in spherically symmetric spacetimes. Writing the metric in the standard form with a cross term, it was found that an intersecting MOTT existed if the zero set of the metric coefficient of the timelike coordinate had intersections.

In chapter 4 we considered the Vaidya spacetime as an example of a spherically symmetric spacetime containing a dynamical black hole. We observed that for a marginally stable MOTS, the evolutionary behaviour was controlled by the second derivative of the mass function for the spacetime. In particular, we considered an example of Vaidya with a linear mass function, finding that the MOTT evolving from a marginally stable MOTS must lie entirely in the initial hypersurface. The connection between the mass function and the MOTT led to the main result of the chapter which stated that for any smooth function whose first derivative had a lower bound of -1, we could find a corresponding mass function such that a Vaidya spacetime with the mass function would be physically reasonable and contain a MOTT given by the graph of the smooth function.

The work done in this thesis can be extended and further developed in a number of interesting ways. The obvious next step would be to generalize the existence result for the marginally stable case to an arbitrary spacetime and obtain a general form of the MOTS bifurcation parameter. Keeping with the spherically symmetric setting, an interesting next problem would be to consider other examples of spacetimes containing dynamical black holes such as the Tolman-Bondi solution and see if a result similar to theorem 4.2.1 is possible.

Appendix A

Derivation of MOTS Stability Operator

The goal is to determine how the null expansion $\Theta_{(+)}$ changes as the surface $\mathcal{S}$ is varied arbitrarily. The variation is defined by a smooth spacetime vector field X defined along $\mathcal{S}$. Specifically, there is a smooth $\Phi : (-\varepsilon, \varepsilon) \times \mathcal{S} \to \mathcal{M}$ such that for fixed μ, Φ_μ is an immersion and for fixed $p \in \mathcal{S}$, $\Phi(\mu, p)$ is a family of curves labelled by μ with tangent vector $\frac{\partial}{\partial\mu}|_p$. Define the one-parameter family of deformed surfaces $\mathcal{S}_\mu \equiv \Phi_\mu(\mathcal{S})$. Considering only deformations normal to the surface, we let $\left(\frac{\partial}{\partial\mu}\right)^a = \psi s^a \equiv X^a$ where ψ is a smooth function on $\mathcal{S}$ controlling the size of the deformation. By definition, $d\mu(X) = 1$, thus $(d\mu)_a = \frac{1}{\psi}s_a \equiv Y_a$.

By the torsion-free condition of the metric connection,

$$\begin{aligned} \mathcal{D}_b Y_a = \mathcal{D}_b (d\mu)_a = \mathcal{D}_b \mathcal{D}_a \mu = \mathcal{D}_a \mathcal{D}_b \mu \\ = \mathcal{D}_a Y_b. \end{aligned} \tag{A.1}$$

Then

$$\begin{aligned} s^a \mathcal{D}_a s_b &= s^a \mathcal{D}_a (\psi Y_b) \\ &= (s^a \mathcal{D}_a \psi) Y_b + \psi s^a \mathcal{D}_a Y_b \\ &= \frac{1}{\psi} s_b s^a \mathcal{D}_a \psi + \psi s^a \mathcal{D}_b \frac{1}{\psi} s_a \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\psi} s_b s^a \mathcal{D}_a \psi + \psi s^a s_a \mathcal{D}_b \frac{1}{\psi} + s^a \mathcal{D}_b s_a \\
&= \frac{1}{\psi} s_b s^a \mathcal{D}_a \psi - \frac{1}{\psi} \mathcal{D}_b \psi \\
&= \frac{1}{\psi} (s^a s_b - h^a{}_b) \mathcal{D}_a \psi \\
&= -\frac{1}{\psi} q^a{}_b \mathcal{D}_a \psi \\
&= -\frac{1}{\psi} \mathring{\eth}_b \psi. \qquad \text{(A.2)}
\end{aligned}
$$

Recall that the null expansion can be expressed in terms of quantities on $\mathcal{S}$ as given by (2.39). Then to compute the variation, we start with the mean curvature term, $J = q^{ab}\mathcal{D}_a s_b$:

$$
\begin{aligned}
\frac{\mathrm{d}J}{\mathrm{d}\mu} &= X^c \mathcal{D}_c (q^{ab} \mathcal{D}_a s_b) \qquad \text{(A.3)} \\
&= \psi s^c \mathcal{D}_c ([h^{ab} - s^a s^b] \mathcal{D}_a s_b) \\
&= \psi h^{ab} s^c \mathcal{D}_c \mathcal{D}_a s_b \qquad \text{(A.4)}
\end{aligned}
$$

where in (A.4), we used the fact that $s^a \mathcal{D}_b s_a = 0$ to eliminate the second term. Then by the Ricci identity, we find

$$
\begin{aligned}
\frac{\mathrm{d}J}{\mathrm{d}\mu} &= \psi h^{ab} s^c (R_{cabd} s^d + \mathcal{D}_a \mathcal{D}_c s_b) \\
&= -\psi R_{cd} s^c s^d + \psi h^{ab} s^c \mathcal{D}_a \mathcal{D}_c s_b. \qquad \text{(A.5)}
\end{aligned}
$$

Considering the second term of (A.5), we have

$$
\begin{aligned}
h^{ab} s^c \mathcal{D}_a \mathcal{D}_c s_b &= (q^{ab} + s^a s^b) s^c \mathcal{D}_a \mathcal{D}_c s_b \qquad \text{(A.6)} \\
&= q^{ab} s^c \mathcal{D}_a \mathcal{D}_c s_b + s^a s^b s^c \mathcal{D}_a \mathcal{D}_c s_b \\
&= q^{ab} \mathcal{D}_a (s^c \mathcal{D}_c s_b) - q^{ab} (\mathcal{D}_a s^c)(\mathcal{D}_c s_b) + s^a s^c \mathcal{D}_a (s^b \mathcal{D}_c s_b) - s^a s^c (\mathcal{D}_a s^b)(\mathcal{D}_c s_b) \\
&= q^{ab} \mathcal{D}_a (s^c \mathcal{D}_c s_b) - (h^{ab} - s^a s^b)(\mathcal{D}_a s^c)(\mathcal{D}_c s_b) - s^a s^c (\mathcal{D}_a s^b)(\mathcal{D}_c s_b) \\
&= q^{ab} \mathcal{D}_a \left(-\frac{1}{\psi} \mathring{\eth}_b \psi \right) - (\mathcal{D}^b s^c)(\mathcal{D}_c s_b) - (s^a \mathcal{D}_a s^b)(s^c \mathcal{D}_c s_b) \qquad \text{(A.7)} \\
&= q^{ab} \left(-\frac{1}{\psi} \mathcal{D}_a \mathring{\eth}_b \psi + \frac{1}{\psi^2} \mathcal{D}_a \psi \mathring{\eth}_b \psi \right) - (\mathcal{D}^b s^c)(\mathcal{D}_c s_b) - \left(-\frac{1}{\psi} \mathring{\eth}^b \psi \right) \left(-\frac{1}{\psi} \mathring{\eth}_b \psi \right)
\end{aligned}
$$

$$= -\frac{1}{\psi}\mathring{\partial}^b\mathring{\partial}_b\psi + \frac{1}{\psi^2}\mathring{\partial}^b\psi\mathring{\partial}_b\psi - \frac{1}{\psi^2}\mathring{\partial}^b\psi\mathring{\partial}_b\psi - (\mathcal{D}^b s^c)(\mathcal{D}_c s_b)$$

$$= -\frac{1}{\psi}\Delta_q\psi - (\mathcal{D}^b s^c)(\mathcal{D}_c s_b) \tag{A.8}$$

where Δ_q is the Laplacian on $\mathcal{S}$ and using (A.2) in (A.7). Recall that the extrinsic curvature of $\mathcal{S}$ is given by $J_{ab} = q_a{}^c\mathcal{D}_c s_b$, therefore

$$\begin{aligned} J_{ab}J^{ab} &= q_a{}^c\mathcal{D}_c s_b q^{ad}\mathcal{D}_d s^b \\ &= (h_a{}^c - s_a s^c)(h^{ad} - s^a s^d)\mathcal{D}_c s_b\mathcal{D}_d s^b \\ &= (h^{cd} - s^c s^d - s^d s^c + s^c s^d)\mathcal{D}_c s_b\mathcal{D}_d s^b \\ &= q^{cd}\mathcal{D}_c s_b\mathcal{D}_d s^b. \end{aligned} \tag{A.9}$$

Note that

$$\begin{aligned} J_{ab}J^{ab} &= J_{ab}e^a{}_A e^b{}_B J^{AB} \\ &= J_{AB}J^{AB}. \end{aligned}$$

Thus we have from (A.5)

$$\frac{\mathrm{d}J}{\mathrm{d}\mu} = -\Delta_q\psi - J_{AB}J^{AB}\psi - \psi R_{ab}s^a s^b. \tag{A.10}$$

Recalling the Gauss relation (2.27):

$$\begin{aligned} {}^{\Sigma}R &= h^{ac}h^{bd}R_{abcd} \\ &= (q^{AC}e^a{}_A e^c{}_C + s^a s^c)(q^{BD}e^b{}_B e^d{}_D + s^b s^d)R_{abcd} \\ &= q^{AC}q^{BD}e^a{}_A e^b{}_B e^c{}_C e^d{}_D R_{abcd} + 2s^b s^d R_{abcd}q^{AC}e^a{}_A e^c{}_C \\ &= q^{AC}q^{BD}(R_{ABCD} + J_{AD}J_{BC} - J_{AC}J_{BD}) + 2s^b s^d R_{bd} \\ &= {}^{\mathcal{S}}R - J^2 + J_{AB}J^{AB} + 2R_{bd}s^b s^d \\ R_{bd}s^b s^d &= \frac{1}{2}({}^{\Sigma}R - {}^{\mathcal{S}}R + J^2 - J_{AB}J^{AB}) \end{aligned} \tag{A.11}$$

where ${}^{\Sigma}R$ is the scalar curvature of Σ and ${}^{\mathcal{S}}R$ is the scalar curvature of $\mathcal{S}$. Then combining terms, we find that the first variation of the mean curvature of $\mathcal{S}$ is given by

$$\frac{\mathrm{d}J}{\mathrm{d}\mu} = -\Delta_q\psi - \psi J_{AB}J^{AB} - \frac{1}{2}({}^{\Sigma}R - {}^{\mathcal{S}}R + J^2 - J_{AB}J^{AB})\psi$$
$$= -\Delta_q\psi - \frac{1}{2}({}^{\Sigma}R - {}^{\mathcal{S}}R + J^2 + J_{AB}J^{AB})\psi. \tag{A.12}$$

The next term we must consider is the variation of $\mathrm{tr}_q\,K$, the trace of K_{ab} over $\mathcal{S}$.

$$\frac{\mathrm{d}(\mathrm{tr}_q\,K)}{\mathrm{d}\mu} = \psi s^c\mathcal{D}_c\Big(K - K_{ab}s^a s^b\Big) \tag{A.13}$$
$$= \psi\Big(s^c\mathcal{D}_cK - s^c\mathcal{D}_cK_{ab}s^a s^b\Big)$$
$$= \psi\Big(s^c\mathcal{D}_cK - s^a s^b s^c\mathcal{D}_cK_{ab} - K_{ab}s^a s^c\mathcal{D}_cs^b - K_{ab}s^b s^c\mathcal{D}_cs^a\Big)$$
$$= \psi\Big(s^c\mathcal{D}_cK - s^a(h^{ab} - q^{ab})\mathcal{D}_cK_{ab} - 2K_{ab}s^a s^c\mathcal{D}_cs^b\Big)$$
$$= \psi s^c\Big(\mathcal{D}_cK - \mathcal{D}_bK_c{}^b\Big) + \psi s^a q^{bc}\mathcal{D}_cK_{ab} - 2\psi K_{ab}s^a\Big(-\frac{1}{\psi}\mathring{\partial}^b\psi\Big)$$
$$= -\psi G_{\alpha\beta}s^\alpha n^\beta + \psi\Big(q^{bc}\mathcal{D}_cK_{ab}s^a - q^{bc}K_{ab}\mathcal{D}_cs^a\Big) + 2q^b{}_aK_{bc}s^c\mathring{\partial}^a\psi \tag{A.14}$$
$$= -\psi G_{\alpha\beta}s^\alpha n^\beta + \psi q^{bc}\Big[\mathcal{D}_c(q^d{}_b + s_bs^d)K_{ad}s^a - K_{ab}(q^a{}_d + s_ds^a)\mathcal{D}_cs^d\Big] + 2\omega^C\mathring{\partial}_C\psi$$
$$= -\psi G_{\alpha\beta}s^\alpha n^\beta + \psi\mathring{\partial}_C\omega^C - \psi K_{AB}J^{AB} + \psi(K_{ab}s^a s^b)J + 2\omega^C\mathring{\partial}_C\psi \tag{A.15}$$

where

$$\omega_A = e^\beta{}_A s^\alpha\nabla_\alpha n_\beta$$
$$= e^\beta{}_A K_{\beta\alpha}s^\alpha \tag{A.16}$$

is a one-form on Σ pulled back to $\mathcal{S}$. In (A.14), we used the momentum constraint (2.33).

Our interest is in quantities on $\mathcal{S}$, in particular we want to isolate the purely transverse component of the extrinsic curvature tensor:

$$\tilde{K}_{ab} = q^c{}_a q^d{}_b K_{cd}$$
$$= (h^c{}_a - s^c s_a)(h^d{}_b - s^d s_b)K_{cd} \tag{A.17}$$
$$= K_{ab} - s^d K_{ad}s_b - s^c K_{cb}s_a + s^c s^d K_{cd}s_a s_b$$

and so the extrinsic curvature tensor can be decomposed into its transverse and longitudinal components as

$$K_{ab} = \tilde{K}_{ab} + K_{ad}s^d s_b + K_{cb}s^c s_a - K(s,s)s_a s_b \tag{A.18}$$

where $K(s,s) = K_{ab}s^a s^b$. Furthermore, the transverse component can be decomposed into its irreducible components:

$$\tilde{K}_{ab} = \frac{\operatorname{tr}_q \tilde{K}}{2} q_{ab} + \sigma_{ab} \tag{A.19}$$

where σ_{ab} is the symmetric trace-free component. Taking the trace of (A.18), we obtain

$$\begin{aligned} h^{ab}K_{ab} &= h^{ab}\Big(\tilde{K}_{ab} + K_{ad}s^d s_b + K_{cb}s^c s_a - K(s,s)s_a s_b\Big) \\ &= h^{ab}\tilde{K}_{ab} + 2K(s,s) - K(s,s) \\ &= \operatorname{tr}_q \tilde{K} + K(s,s) \\ &= \operatorname{tr} K. \end{aligned} \tag{A.20}$$

Recalling the Hamiltonian constraint, we need $K_{ab}K^{ab}$ and K^2:

$$\begin{aligned} K_{ab}K^{ab} &= h^{ac}h^{bd}K_{ab}K_{cd} \\ &= (q^{ac} + s^a s^c)(q^{bd} + s^b s^d)K_{ab}K_{cd} \\ &= (q^{ac}q^{bd} + 2q^{ac}s^b s^d + s^a s^c s^b s^d)K_{ab}K_{cd} \\ &= \|\tilde{K}\|^2 + 2\omega^a K_{ab}s^b + K(s,s)^2 \\ &= \|\tilde{K}\|^2 + 2e^a{}_A\omega^A K_{ab}s^b + K(s,s)^2 \\ &= \|\tilde{K}\|^2 + 2\|\omega\|^2 + K(s,s)^2 \end{aligned} \tag{A.21}$$

$$K^2 = (\operatorname{tr}_q \tilde{K})^2 + 2(\operatorname{tr}_q \tilde{K})K(s,s) + K(s,s)^2. \tag{A.22}$$

The Hamiltonian constraint thus takes the form:

$$\begin{aligned} {}^{\Sigma}R &= K_{ab}K^{ab} - K^2 + 2G_{\alpha\beta}n^\alpha n^\beta \\ &= \|\tilde{K}\|^2 + 2\|\omega\|^2 - (\operatorname{tr}_q \tilde{K})^2 - 2(\operatorname{tr}_q \tilde{K})K(s,s) + 2G_{\alpha\beta}n^\alpha n^\beta. \end{aligned} \tag{A.23}$$

Returning to (A.12), we have

$$\begin{aligned} \frac{\mathrm{d}J}{\mathrm{d}\mu} &= -\Delta_q\psi + \frac{1}{2}({}^{\mathcal{S}}R - J^2 - \|J\|^2)\psi \\ &\qquad - \frac{1}{2}(\|\tilde{K}\|^2 + 2\|\omega\|^2 - (\operatorname{tr}_q \tilde{K})^2 - 2(\operatorname{tr}_q \tilde{K})K(s,s) + 2G_{\alpha\beta}n^\alpha n^\beta)\psi. \end{aligned} \tag{A.24}$$

We recall that $\Theta_{(+)} = \operatorname{tr}_q K + J$, so combining (A.12) and (A.15), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}\Theta_{(+)}}{\mathrm{d}\mu} &= \frac{\mathrm{d}J}{\mathrm{d}\mu} + \frac{\mathrm{d}(\operatorname{tr}_q K)}{\mathrm{d}\mu} && \text{(A.25)}\\
&= -\Delta_q^2\psi + 2\omega^A \mathring{\partial}_A\psi + \frac{1}{2}({}^{\mathcal{S}}R - J^2 - \|J\|^2)\psi \\
&\qquad - \frac{1}{2}\Big(\|\tilde{K}\|^2 + 2\|\omega\|^2 - (\operatorname{tr}_q \tilde{K})^2 - 2(\operatorname{tr}_q \tilde{K})K(s,s) + 2G_{\alpha\beta}n^\alpha n^\beta\Big)\psi \\
&\qquad - \Big(G_{\alpha\beta}r^\alpha n^\beta - \mathring{\partial}_A\omega^A + \tilde{K}_{AB}J^{AB} - JK(s,s)\Big)\psi && \text{(A.26)}\\
&= -\Delta_q\psi + 2\omega^A \mathring{\partial}_A\psi + \Big(\frac{{}^{\mathcal{S}}R}{2} - \|\omega\|^2 - G_{\alpha\beta}\ell^\alpha n^\beta + \mathring{\partial}_A\omega^A\Big)\psi \\
&\qquad + (\operatorname{tr}_q \tilde{K} + J)K(s,s)\psi \\
&\qquad - \frac{1}{2}\Big(2\tilde{K}_{AB}J^{AB} + J^2 + \|J\|^2 + \|\tilde{K}\|^2 - (\operatorname{tr}_q \tilde{K})^2\Big)\psi && \text{(A.27)}\\
&= -\Delta_q\psi + 2\omega^A \mathring{\partial}_A\psi + \Big(\frac{{}^{\mathcal{S}}R}{2} - \|\omega\|^2 - G_{\alpha\beta}\ell^\alpha n^\beta + \mathring{\partial}_A\omega^A\Big)\psi \\
&\qquad + \Theta_{(+)}K(s,s)\psi + \frac{1}{2}\Big((\operatorname{tr}_q \tilde{K})^2 - J^2\Big)\psi - \frac{1}{2}\|\tilde{K} + J\|^2\psi && \text{(A.28)}\\
&= -\Delta_q\psi + 2\omega^A \mathring{\partial}_A\psi + \Big(\frac{{}^{\mathcal{S}}R}{2} - \|\omega\|^2 - G_{\alpha\beta}\ell^\alpha n^\beta + \mathring{\partial}_A\omega^A - \frac{1}{2}\|K^{(\ell)}\|^2\Big)\psi \\
&\qquad + \Theta_{(+)}\Big(K(s,s) + \frac{\Theta_{(-)}}{2}\Big)\psi && \text{(A.29)}
\end{aligned}
$$

where in (A.28) we used the definition of the null expansion to re-write the $\operatorname{tr}_q \tilde{K} + J$ term, then in (A.29) we used

$$
(\operatorname{tr}_q \tilde{K})^2 - J^2 = (\operatorname{tr}_q \tilde{K} + J)(\operatorname{tr}_q \tilde{K} - J) = \Theta_{(+)}\Theta_{(-)}. \tag{A.30}
$$

Furthermore,

$$
\begin{aligned}
(\tilde{K}_{ab} + J_{ab})(\tilde{K}^{ab} + J^{ab}) &= K^{(\ell)}_{ab}K^{(\ell)ab} \\
&= K^{(\ell)}_{AB}K^{(\ell)AB}
\end{aligned} \tag{A.31}
$$

where $K^{(\ell)}$ is the null second fundamental form of $\mathcal{S}$ with respect to ℓ_+. Similar to $\tilde{K}$, $K^{(\ell)}$ can be decomposed into its irreducible components:

$$
\begin{aligned}
K^{(\ell)}_{AB} &= \frac{\operatorname{tr} K^{(\ell)}}{2}q_{AB} + \sigma^{(\ell)}_{AB} \\
&= \frac{\Theta_{(+)}}{2}q_{AB} + \sigma^{(\ell)}_{AB}
\end{aligned} \tag{A.32}
$$

where $\sigma^{(\ell)}_{AB}$ is the trace-free symmetric component. Then

$$\|K^{(\ell)}\|^2 = \frac{(\Theta_{(+)})^2}{2} + \|\sigma^{(\ell)}\|^2, \tag{A.33}$$

so we find that the first variation of the null expansion is

$$\begin{aligned}\frac{\mathrm{d}\Theta_{(+)}}{\mathrm{d}\mu} = -\Delta_q\psi + 2\omega^A\mathring{\partial}_A\psi + \Big(\frac{{}^{\mathcal{S}}R}{2} - \|\omega\|^2 - G_{\alpha\beta}\ell^\alpha n^\beta + \mathring{\partial}_A\omega^A - \frac{1}{2}\|\sigma^{(\ell)}\|^2\Big)\psi \\ + \Big(K(s,s)\Theta_{(+)} + \frac{\Theta_{(+)}\Theta_{(-)}}{2} - \frac{(\Theta_{(+)})^2}{4}\Big)\psi.\end{aligned} \tag{A.34}$$

If $\mathcal{S}$ is a MOTS then we necessarily have $\Theta_{(+)} = 0$ and hence

$$\begin{aligned}\frac{\mathrm{d}\Theta_{(+)}}{\mathrm{d}\mu}\Big|_{\Theta_{(+)}=0} = -\Delta_q\psi + 2\omega^A\mathring{\partial}_A\psi \\ + \Big(\frac{{}^{\mathcal{S}}R}{2} - \|\omega\|^2 - G_{\alpha\beta}\ell^\alpha n^\beta + \mathring{\partial}_A\omega^A - \frac{1}{2}\|\sigma^{(\ell)}\|^2\Big)\psi.\end{aligned} \tag{A.35}$$

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