On compactness properties of subgroups

by

© **Shivam Arora**

A thesis submitted to the School of Graduate Studies
in partial fulfillment of the requirements for the degree
of Doctor of Philosophy.

Department of Mathematics and Statistics
Memorial University

June 2022

St. John's, Newfoundland and Labrador, Canada

Abstract

The class of locally compact groups has been widely studied in group theory, representation theory, and harmonic analysis. There is a current program of extending geometric techniques used in the study of discrete groups to this larger class [Wil94, KM08, CCMT15, CdlH16]. This thesis is part of that program. We use geometric methods to study the compactness properties of subgroups in the class of topological groups containing a compact open subgroup. This class includes discrete groups, profinite groups, and totally disconnected locally compact groups as subclasses.

In the first project, we study discrete hyperbolic groups. Finitely presented subgroups of hyperbolic groups are not necessarily hyperbolic; the first examples of this phenomenon were constructed by Brady [Bra99]. In contrast, for hyperbolic groups of integral cohomological dimension at most two, finitely presented subgroups are hyperbolic; this is a result of Gersten [Ger96b]. We extend this result to hyperbolic groups with rational cohomological dimension bounded by two. This applies to examples of groups constructed by Bestvina and Mess, fully describing the nature of their finitely presented subgroups, which was previously unknown.

In the second project, we extend Gersten's result for totally disconnected locally compact (TDLC) groups. In particular, we prove that closed compactly presented subgroups of hyperbolic TDLC groups of discrete rational cohomological dimension bounded by two are hyperbolic. We also characterize hyperbolic TDLC groups in terms of isoperimetric inequalities and study small cancellation quotients of amalgamated free products of profinite groups over open subgroups.

In the last project, we study *coherence* of topological groups. A group is *coherent* if every compactly generated subgroup is compactly presented. We prove that amalgamated free products of coherent groups over compact open subgroups are coherent. We also show that certain small cancellation quotients of these groups are also coherent,

generalizing a result of McCammond and Wise [MW05]. In order to prove the main results, we study relative hyperbolicity for topological groups containing compact open subgroups with respect to finite collection of open subgroups, and extend some results of Osin [Osi06].

To my parents who started all this and to everyone I met ever since...

Lay summary

A *group* in mathematics is an abstraction of the concept of symmetry. For any object, the collection of all its transformations forms a group. For instance, the collection of all the moves of a Rubik's cube is a group. The subject of this thesis falls under the domain of Geometric group theory, which aims to study infinite groups from the perspective of geometry. For instance, many properties of groups can be understood by studying graphs, called *Cayley graphs*; and these graphs are unique up to *quasi-isometry*. Intuitively, this means that all Cayley graphs of a finitely generated group look the same from far away. A finite set of points and a single point; or a discrete line and a continuous line, are some simple examples of objects that look the same from far enough distance. They are in fact, *quasi-isometric* under the formal framework. This opens up a whole paradigm where geometric properties that are preserved under quasi-isometry can be attributed to groups.

An interesting property of groups that we study in this thesis is δ-hyperbolicity, where δ is any real number. Intuitively, a finitely generated group is hyperbolic if any triangle in its Cayley graph is thin. To illustrate hyperbolicity, imagine a planet where every person on it can see up to δ unit distance. If this planet is δ-hyperbolic, then among any three neighboring house properties on this planet, none of them can have privacy. In particular, any part of each house can be seen from at least one of the other neighbour's property. Groups with such underlying geometry have very interesting properties and have been studied widely by mathematicians over the past three decades. In this thesis, we study the behavior of *subgroups* of certain hyperbolic groups. A *subgroup* is a subcollection of the group that itself forms a group. In general, a subgroup need not have the same geometric properties as the group. We study hyperbolic groups among other groups and provide results where subgroups preserve the geometric structure of the ambient group.

Acknowledgements

I would first like to thank my supervisor, Dr. Eduardo Martínez-Pedroza, from whom I learned a better way of doing math, which includes research, teaching, communicating, and writing math. I will forever be thankful to him for accepting me as a student. His enthusiasm for maths and life has made this program some of the best years of my education. He is the kind of person I will always strive to be.

I am grateful to all the following people:

To Dr. Ilaria Castellano for teaching me about the discrete rational cohomology, to Dr. G. C. Cook for showing us a better proof to a result, to Dr. Sam Hughes for very useful comments on the work, and to Dr. Luis Jorge for multiple fruitful discussions.

To the professors in the department who taught me courses during my Ph.D. program: Dr. Tom Baird, Dr. Alexander Bihlo, Dr. Graham Cox, and Dr. Mikhail Kotchetov. To my supervisory committee, Dr. Hari Kunduri and Dr. Marco Merkli, for their support and suggestions during my program.

To MSRI, University of North Carolina, Casa Matemática Oaxaca, Universite de Rennes for generous travel grants to attend the schools and workshops during my program. To AARMS for the graduate scholarship. To Dr. Yorck Sommerhäuser for inviting me to participate in the mini-courses organized through the Atlantic Algebra Centre.

To Dr. Ivan Booth, Dr. Danny Dyer, and Dr. J.C. Loredo-Osti for sharing their love for math outreach and giving me opportunities to volunteer with them in various such activities during my program. To the graduate officer Dr. Asokan M. Variyath, and the staff of the Department of Mathematics and Statistics for the supportive environment.

To late Dr. I.B.S. Passi for introducing geometric group theory to me, Dr. Krishnendu Gongopadhyay, Dr. Pranab Sardar, and Dr. Jeetendra Aggarwal for being my mentors.

To Irene Pasquinelli, Suraj Krishna, Meenakshy Jyothis, Chris Chia, and Mica McCurdy for giving suggestions on the draft of the thesis.

Lastly, I thank all the people in my life for sharing their priceless love and time with me.

Statement of contribution

The work in this thesis is done under the supervision of Dr. Eduardo Martínez-Pedroza. The thesis is written in the *manuscript style*, and Chapters 3-5 in this thesis consist of the following manuscripts. The research on these papers was shared jointly, with all authors contributing equally, except where specified otherwise below:

- Chapter 3: This work is in collaboration with Dr. Eduardo Martínez-Pedroza and is published in [AMP20] which appeared as: S. Arora and E. Martínez-Pedroza. *Subgroups of word hyperbolic groups in rational dimension 2.* Groups, Geometry, and Dynamics, 2020 EMS Press 15 (2021), 83-100. doi: 10.4171/GGD/592

- Chapter 4: This project was done in collaboration with Dr. I. Castellano and Dr. E. Martínez-Pedroza. There was a further contribution by Dr. G. C. Cook to give a different proof to one of the propositions that helped us strengthen the main result. The work is now published in [ACCCMP21] which appeared as: S. Arora, I. Castellano, G. Corob Cook, and E. Martínez-Pedroza. *Subgroups, hyperbolicity and cohomological dimension for totally disconnected locally compact groups.* Journal of Topology and Analysis (2021 Mar), 1–27.

- Chapter 5: This work is in collaboration with Dr. Eduardo Martínez-Pedroza and is ready for submission. S. Arora and E. Martínez-Pedroza. *Topological groups with a compact open subgroup, Coherence and Relative hyperbolicity.*

Other mathematical work during the Ph.D. program that is not included in the thesis:

- S. Arora and E. Martínez-Pedroza. *Fixed Point Sets in Diagrammatically Reducible Complexes.* arXiv:2107.01254 (2021).

Table of contents

List of symbols

$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational numbers
$\mathbb{R}$	Real numbers
$\mathbb{Q}_p$	The field of p-adic numbers
$\mathbb{Z}_p$	The ring of p-adic integers

Chapter 1

Introduction

Geometric group theory aims to study groups using topological and geometric techniques. The underlying principle is that many properties of groups can be understood by studying associated spaces. There are numerous examples of spaces associated to groups that have been paramount in the study of groups including *Cayley graphs, classifying spaces*, *presentation complexes*, *homogeneous spaces*, *buildings* etc. [Geo08]. Some of the pioneering ideas of using geometry and topology to study groups were introduced in the early 20$^{\text{th}}$ century in the works of J.H.C. Whitehead, E. Van Kampen, J. R. Stallings, and many others. For instance, Max Dehn [Deh11] in 1911, used the geometry of the hyperbolic space to solve the *word problem* for the fundamental groups of surfaces. Small Cancellation theory [Gre60] and Bass-Serre theory [Ser80] are some of the other key examples of early theories using such methods.

By the late 20$^{\text{th}}$ century, many interesting techniques and results in this direction emerged from different areas of mathematics; for instance, the Mostow rigidity theorem from the study of lattices in Lie groups [Mos68], the study of Kleinian groups and 3-manifolds [Thu82], and the study of groups as formal languages in the work of Cannon, Thurston et al. [ECH$^+$92]. Gromov's influential monograph on hyperbolic groups [Gro87] and his theorem characterizing virtually nilpotent groups in terms of geometric attributes [Gro81], are considered instrumental to the development of geometric group theory as an independent subject of study.

Geometric group theory currently has large overlaps with homological algebra, topology, geometry, formal language theory, probability, dynamics, and more. The techniques in geometric group theory have been highly successful in solving some major long-standing problems in mathematics, for instance, the Virtual Haken Conjecture [Ago13] and Tarski's problem on elementary theory of free groups [KM98, Sel01].

Although most tools in geometric group theory were developed to study discrete groups, recent works have shown that these techniques can also be successfully applied to study locally compact topological groups [KM08, CCMT15, CdlH16]. The study of locally compact groups has been fundamental in the context of harmonic analysis, ergodic theory, and representation theory. For a locally compact group G, the connected component of the identity, denoted by G_0, is a connected locally compact normal subgroup, and we have a short exact sequence $1 \to G_0 \to G \to G/G_0 \to 1$, where the quotient group G/G_0 is a totally disconnected and locally compact group, often abbreviated as a TDLC group. So, in principle, the study of locally compact groups can be divided into studying connected locally compact groups and TDLC groups. The connected case has been studied exhaustively and is understood better, especially after the solution to Hilbert's fifth problem; it is known that connected locally compact groups are inverse limits of Lie groups, see [Tao14] for an exposition.

However, no similar understanding of the structure of TDLC groups exists. The class of TDLC groups comprises a large collection of seemingly unrelated subclasses of groups like discrete groups, profinite groups, algebraic groups over p-adic numbers, and the automorphism groups of locally finite graphs, among others. The geometric approach to understanding locally compact groups has gathered a new interest in their study. In this thesis, we continue this approach.

The general theme of the thesis is to understand the behavior of subgroups of locally compact groups. Specifically, the results of the thesis concerns *compactness properties* of subgroups. *Compactness properties* are the properties of groups that are trivially satisfied by compact groups, like being compactly generated, admitting compact presentation, etc. This is a natural generalization of *finiteness properties* for discrete groups, like being finitely generated, admitting finite presentation, etc. These properties have well-defined geometric interpretations and a natural hierarchical generalization, called F_n-properties, studied in [Bro94] and [CC20], for discrete groups and TDLC groups, respectively.

In general, compactness properties of subgroups can be quite different from that of the ambient group. For certain classes of groups, an additional hypothesis on the dimension of groups can ensure that the compactly presented subgroups inherit the geometric structure of the ambient groups; or that the compactly generated subgroups are in fact compactly presented. This thesis comprises three projects covered in Chapters 3, 4, and 5; that provide positive results in this direction. In the first project, we study discrete groups, and in the last two projects, we study these properties for topological groups.

In the first project, presented in Chapter 3, we study discrete hyperbolic groups. These groups were introduced by Gromov in his monograph [Gro87] and have been the centre of interest in geometric group theory, see Chapter 2 for the definition. Examples of hyperbolic groups include free groups, fundamental groups of surfaces with genus at least two, one-relator groups with torsion, fundamental groups of Riemannian manifolds of negative sectional curvature, special linear group $SL_2(\mathbb{Z})$, among others, see [DK18].

Hyperbolic groups, though defined geometrically, have an interesting algebraic structure. For instance, they are finitely presented, (in fact are of type F_n for all $n \geq 1$ [Rip82]), have solvable word problem [Deh12, BH99], have finitely many finite subgroups up to conjugacy [BsG95, Bra00], etc. Interestingly, in a certain sense *almost all finitely presented groups are hyperbolic*, see [Os92] for a formal statement. It is thus natural to investigate whether a subgroup of a hyperbolic group is hyperbolic. In general, the answer is negative; for instance, the commutator subgroup of the free group over two generators is not even finitely generated. Rips [Rip82] constructed the first examples of hyperbolic groups with finitely generated subgroups that are not hyperbolic. However, in 1996, Gersten proved the following result in the positive direction,

Theorem 1.0.1 (Gersten [Ger96b])**.** *Let G be a hyperbolic group such that $cd_{\mathbb{Z}}(G) = 2$. If H is a finitely presented subgroup, then H is hyperbolic.*

Note that $cd_{\mathbb{Z}}(G)$ represents the cohomological dimension of the group with respect to the ring of integers. For the definition, refer Chapter 2. Gersten's result turned out to be sharp for integral cohomological dimension 2, since Noel Brady in [Bra99] proved that this phenomenon fails for higher dimensions. In particular, he constructed hyperbolic groups with integral cohomological dimension 3, that contain finitely presented subgroups that are not hyperbolic.

In this project, we extend Gersten's result for rational cohomological dimension. In fact, there exist hyperbolic groups with integral cohomological dimension 3 and rational cohomological dimension 2. The first examples of such groups were discovered by Bestvina and Mess [BM91] based on methods given by Davis and Januszkiewicz [DJ91]. This class also contains finite index subgroups of hyperbolic Coxeter groups, examples that were discovered by Dranishnikov [Dra99]. The nature of finitely presented subgroups of groups in this class was not known before. In particular we prove that

Theorem 1.0.2. [AMP20] *Let G be a hyperbolic group such that $\mathsf{cd}_{\mathbb{Q}}(G) \leq 2$. If H is a finitely presented subgroup, then H is hyperbolic.*

For the proof of Theorem 1.0.1, Gersten's argument uses *homological isoperimetric inequalities* for groups with respect to the ring of integers and relies upon the existence of finite-dimensional Eilenberg–MacLane complexes corresponding to the groups. Homological isoperimetric inequalities with respect to the integers have been studied by many authors, see [You11, Fle98, HMP16] for example. For this project, we study homological isoperimetric inequalities in arbitrary dimensions with respect to the rational numbers. Since direct topological constructs are not available in the generality we work, we use tools from the field of homological algebra.

Another motivation of this project is to generalize Gersten's result to groups admitting torsion, specifically, to the class of hyperbolic groups G admitting a 2-dimensional classifying space for proper actions, denoted as $\underline{E}G$. A model for $\underline{E}G$ is a G-CW-complex X with the property that, for each subgroup H the subcomplex of fixed points is contractible if H is finite, and empty if H is infinite. The minimal dimension of a model for $\underline{E}G$ is denoted by $\underline{\mathsf{gd}}(G)$. Considering the cellular chain complex with rational coefficients of a model for $\underline{E}G$ with minimal dimension shows that

$$\mathsf{cd}_{\mathbb{Q}}(G) \leq \underline{\mathsf{gd}}(G).$$

This inequality implies the following corollary.

Corollary 1.0.3. [AMP20] *If G is a hyperbolic group such that $\underline{\mathsf{gd}}(G) \leq 2$, then any finitely presented subgroup is hyperbolic.*

The results of this project are published in [AMP20], and this article is presented in Chapter 3.

Hyperbolic groups, though widely studied for the discrete case, can be studied for the more general class of locally compact groups, see [CCMT15]. In the second project, which is presented in Chapter 4, we study hyperbolicity for locally compact groups with totally disconnected topology, i.e. TDLC groups. By van Dantzig's Theorem [VD36], all TDLC groups admit a basis of compact open subgroups. This allows extension of some of the techniques from geometric group theory to this setting. For instance, every compactly generated topological group with a compact open subgroup admits a cocompact action on a locally finite graph with compact open vertex stabilizers. These graphs can be chosen uniquely up to quasi-isometry. This is analogous to the Cayley graph for finitely generated groups, and these graphs are called Cayley-Abels graph [KM08]. As a consequence, hyperbolicity for these groups can be defined as follows: A compactly generated TDLC group is said to be *hyperbolic* if any Cayley-Abels graph of G is δ-hyperbolic for some real number δ. For an example of a hyperbolic topological group, consider the following:

Example 1.0.4. For any prime p, the Cayley-Abels graph of the special linear group $SL_2(\mathbb{Q}_p)$ over the p-adic numbers is quasi-isometric to a tree [KM08, Example 2], and hence it is a hyperbolic TDLC group.

A group G is *compactly presented* if it admits a group presentation with a compact generating set and relators of bounded length. A discrete hyperbolic group can be completely characterized by isoperimetric inequalities, see [Ger96b, Theorem 3.1]. In this project, we characterize TDLC hyperbolic groups in terms of weak isoperimetric inequalities. In particular,

Theorem 1.0.5. [ACCCMP21] *A compactly generated TDLC-group G is hyperbolic if and only if G is compactly presented and satisfies the weak linear isoperimetric inequality.*

The main goal of this project is to extend Gersten's result to hyperbolic TDLC groups. In [CW16a], Castellano and Weigel introduced *rational discrete cohomology* for TDLC groups, a cohomology theory that captures the geometry of the actions of TDLC groups on Cayley-Abels graphs. This allows the notion of rational cohomological dimensions to be defined for TDLC groups. We extend some of the techniques developed in the first project to this framework, and prove the following generalization of Theorem 1.0.2:

Theorem 1.0.6. [ACCCMP21] *Let G be a hyperbolic TDLC-group with $\mathrm{cd}_{\mathbb{Q}}(G) \leq 2$. Every compactly presented closed subgroup H of G is hyperbolic.*

The above theorem can be applied, for instance, to the following geometric scenario.

Corollary 1.0.7. [ACCCMP21] *Let X be a locally finite 2-dimensional simplicial $CAT(-1)$-complex. If $\mathrm{Aut}(X)$ acts with finitely many orbits on X, then every compactly presented closed subgroup of $\mathrm{Aut}(X)$ is a hyperbolic TDLC-group.*

We further use small-cancellation theory to illustrate hyperbolic TDLC groups satisfying the hypothesis of Theorem 1.0.6.

Theorem 1.0.8. [ACCCMP21] *Let $A *_C B$ be the amalgamated free product of the profinite groups A, B over a common open subgroup C. Let R be a finite symmetrized subset of $A *_C B$ that satisfies the $C'(1/12)$ small cancellation condition. Then the quotient $G = (A *_C B)/\langle\langle R \rangle\rangle$ is a hyperbolic TDLC-group with $\mathrm{cd}_{\mathbb{Q}}(G) \leq 2$.*

The results of this project are published in [ACCCMP21].

In the last project, presented in Chapter 5, we study coherence among topological groups. A topological group is said to be *coherent* if every open compactly generated subgroup is compactly presented. Compact groups are the easiest examples of coherent topological groups. In the case of discrete groups, the study of coherent groups has been a topic of wide interest, motivated by long outstanding conjectures like Baumslag's conjecture [Bau74]. Finite groups, polycyclic groups, free groups, surface groups are some of the examples of discrete coherent groups. A remarkable result proven by Scott [Sco73] and Shalen (unpublished) independently shows that fundamental group of manifolds of dimension at most three are coherent. A recent survey by Wise [Wis20] reviews the progress in the field of coherent groups and poses current open questions.

We focus on coherence in the class of locally compact groups with a compact open subgroup. Recall that all TDLC groups are particular examples of these groups. In the discrete case, amalgamated free products of coherent groups over finite groups are coherent [MPW11b]. In this project, we generalize that as follows:

Theorem 1.0.9 (Combination of Coherence Groups)**.** [AMP22] *If $G = A *_C B$ is a topological group that splits as an amalgamated free product of two coherent open subgroups A and B with compact intersection C, then G is coherent.*

In particular, groups splitting as the amalgamated free products over compact open subgroups are coherent groups. This implies that $SL_2(\mathbb{Q}_p)$ is coherent since it splits as an amalgamated free product of compact groups over a common open subgroup. One can also iterate the construction, for instance: The amalgamated free product $SL_2(\mathbb{Q}_p) *_{SL_2(\mathbb{Z}_p)} SL_2(\mathbb{Q}_p)$ of two copies of $SL_2(\mathbb{Q}_p)$ along the compact open subgroup of $SL_2(\mathbb{Z}_p)$ is a coherent group. We are able to import McCammond and Wise's perimeter method in the class of locally compact groups with a compact open subgroup:

Theorem 1.0.10. [AMP22] *Let $A *_C B$ be a topological group that splits as an amalgamated free product of two coherent open subgroups A and B with compact intersection C. Suppose $r \in A *_C B$ is not conjugate into A or B. If m is sufficiently large and r^m satisfies the $C'(1/6)$ small cancellation condition, then the quotient group $G = (A *_C B)/\langle\langle r^m \rangle\rangle$ is coherent.*

This result in the case that A and B are free groups is a result of McCammond and Wise [MW02, Theorem 8.3], and the generalization where A and B are coherent discrete groups is a result in [MPW11b, Theorem 1.8].

In order to prove the above results, we extend some of the classical theory of discrete groups studied by Osin in [Osi06] to topological groups with a compact open subgroup. In particular, we introduce the notion of relative compact generation, relative compact presentation, and relative hyperbolicity. The relative notions concern groups along with a collection of their subgroups; in particular, we focus on what we refer to as proper pairs:

Definition 1.0.11 (Proper pair)**.** A pair $(G, \mathcal{H})$ is called a *proper pair* if

1. G is a topological group with a compact open subgroup;
2. $\mathcal{H}$ is a finite collection of open subgroups of G;
3. No pair of distinct subgroups in $\mathcal{H}$ are conjugate in G.

We develop the notion of compact generation of a topological group relative to a finite collection of open subgroups. A topological group G is *compactly generated relative to a collection of subgroups* $\mathcal{H}$ if there is a compact subset $K \subset G$ such that G is algebraically generated by $K \cup \bigcup \mathcal{H}$. This is a natural generalization of the work of Osin in [Osi06].

We complement the algebraic notion of relative compact generating set with an equivalent notion of a relative compact generating graph, see Definition 5.4.4. It is particularly suited for the study of topological groups since it encodes the topology of the group, see Proposition 5.4.3, and it allows for a natural application of geometric techniques.

We prove that topological groups compactly generated relative to a finite collection of open subgroups act cocompactly and discretely on connected graphs. In particular, we prove the following topological characterization.

Theorem 1.0.12 (Topological Characterization)**.** [AMP22] *Let $(G, \mathcal{H})$ be a proper pair. The following statements are equivalent:*

1. *G is compactly generated relative to $\mathcal{H}$.*

2. *G admits a compact generating graph relative to $\mathcal{H}$.*

3. *There is a discrete, connected cocompact G-graph Γ with compact edge stabilizers, vertex stabilizers are either compact or conjugates of subgroups in $\mathcal{H}$, every $H \in \mathcal{H}$ is the G-stabilizer of a vertex, and any pair of vertices with the same G-stabilizer $H \in \mathcal{H}$ are in the same G-orbit if H is non-compact.*

We define the connected G-graph satisfying the condition (3) in Theorem 1.0.12, as a *relative Cayley-Abels graph* of G with respect to $\mathcal{H}$, see also Definition 5.4.10. These graphs are not necessarily locally finite; however, we show that they are pairwise quasi-isometric. There is a generalization of locally finite graphs introduced by Bowditch known as *fine* graphs [Bow12]; a graph is *fine* if for any pair of vertices u, v and any integer n, there are finitely many embedded n-paths from u to v.

Theorem 1.0.13 (Quasi-isometry Invariance)**.** [AMP22] *Let $(G, \mathcal{H})$ be a proper pair.*

1. *Any two relative Cayley-Abels graphs of G with respect to $\mathcal{H}$ are quasi-isometric.*

2. *If one relative Cayley-Abels graph of G with respect to $\mathcal{H}$ is fine, then all are fine.*

The above theorem allows for large-scale geometric techniques to apply. For example, we can define relative hyperbolicity for $(G, \mathcal{H})$ pairs: A topological group G

is said to be *relatively hyperbolic* with respect to $\mathcal{H}$ if there exists a relative Cayley-Abels graph of G with respect to $\mathcal{H}$ that is fine and hyperbolic. This is a generalization of relative hyperbolicity for finitely generated groups given by Bowditch [Bow12]. This definition of relative hyperbolicity for discrete groups has been studied in [MP16], [MPW11a]. There are multiple definitions of relatively hyperbolic groups in the discrete case. For a proof of equivalence, see [Hru10]. A generalization of a different definition of relative hyperbolicity for locally compact groups has been studied in [CCMT15].

Example 1.0.14. For examples of relative hyperbolic topological groups, consider the following. See Proposition 5.8.5 for details and Theorem 1.0.8 for a comparison.

1. Let $(\mathcal{G}, \Lambda)$ be a finite graph of topological groups with compact open edge groups. Then the fundamental group of $(\mathcal{G}, \Lambda)$ is relatively hyperbolic with respect to the vertex groups $\mathcal{G}_v$.

2. Let $A *_C B$ be the amalgamated free product of the topological groups A, B over a common open subgroup C. Let R be a finite symmetrized subset of $A *_C B$ that satisfies the $C'(1/12)$ small cancellation condition. Then the quotient $G = (A *_C B)/\langle\langle R \rangle\rangle$ is relatively hyperbolic with respect to $\{A, B\}$.

We generalize the concept of compact presentation for pairs $(G, \mathcal{H})$ in Definition 5.6.1, and we prove that:

Theorem 1.0.15 (Topological Characterization)**.** [AMP22] *Let $(G, \mathcal{H})$ be a proper pair. The following statements are equivalent:*

1. *G is compactly presented with respect to $\mathcal{H}$.*

2. *There exists a relative Cayley-Abels graph Γ of G with respect to $\mathcal{H}$ which is the 1-skeleton of a simply-connected cocompact discrete G-complex.*

We define the G-complex satisfying the condition (2) in Theorem 1.0.15, as *relative Cayley-Abels complex* of G with respect to $\mathcal{H}$. A stronger version of the above theorem is proven as Corollary 5.6.7, which generalizes a classical result by [Mac64] that a discrete group is finitely presented if and only if it acts cellularly, cocompactly, and with finite vertex stabilizers on a simply connected space.

We further prove that if G is relatively hyperbolic with respect to $\mathcal{H}$, then G is compactly presented relative to $\mathcal{H}$. We also generalize results of Osin [Osi06, Theorem 1.1 and Theorem 2.40] for topological pairs $(G, \mathcal{H})$.

Theorem 1.0.16. [AMP22] *Let $(G, \mathcal{H})$ be a proper pair. Suppose that G is compactly presented relative to $\mathcal{H}$.*

1. *If each $H \in \mathcal{H}$ is compactly presented then G is compactly presented.*

2. *If G is compactly generated, then each $H \in \mathcal{H}$ is compactly generated.*

The details of this project are presented in Chapter 5.

We conclude this introduction with an application obtained by combining results from Chapter 4 and Chapter 5. We will cover the future directions of research in Chapter 6.

Theorem 1.0.17. *Let $W = A *_C B$ be a topological group that splits as an amalgamated free product of profinite groups over a common open subgroup. Suppose $r \in A *_C B$ is not conjugate into A or B. If m is sufficiently large and r^m satisfies the $C'(1/6)$ small cancellation condition, then compactly generated subgroup of the group $G = W/\langle\langle r^m \rangle\rangle$ is hyperbolic.*

Proof. Let H be any compactly generated subgroup of G. By Theorem 1.0.10, G is coherent, hence H is compactly presented. By Theorem 1.0.8, G is a hyperbolic group with $\mathsf{cd}_{\mathbb{Q}}(G) \leq 2$. Therefore, Theorem 1.0.6 implies H is hyperbolic. $\square$

Observe that, by applying the coherence result Theorem 1.0.10, we get a stronger result, describing the nature of all compactly generated subgroups, as compared to just applying Theorem 1.0.8, which deals with compactly presented subgroups.

Chapter 2

Background

In this chapter, we recall some of the definitions used in this thesis and illustrate them with examples. Some of the technical definitions are included in the individual chapters. For detailed background, we refer to these texts [Geo08, BH99, Bro94].

Quasi-isometry

Quasi-isometry is an equivalence relation on metric spaces. It is a coarse version of Lipschitz equivalence and intuitively means that two spaces look the same from far enough distance. Formally, two metric spaces are said to be **quasi-isometric** if there exists a map $f : X \to Y$ and constants $\lambda \geq 1$, $k \geq 0$, and $c \geq 0$ such that

1. For any $p, q \in X$, we have
$$\frac{1}{\lambda}d(p,q) - k \leq d(f(p), f(q)) \leq \lambda d(p,q) + k$$

2. For every $y \in Y$ there exists $x \in X$ such that $d(y, f(x)) \leq c$.

Such a map f is said to be a *quasi-isometric embedding.* Some easy examples to observe here are the following:

- A singleton set and a finite set, as metric subspaces of $\mathbb{R}^n$, for any natural number n, are quasi-isometric.

- The real line with the usual metric and the set of integers as its subspace are quasi-isometric.

The metric spaces we will be working within this thesis are connected graphs. A connected graph can be considered as a metric space by assigning a unit length to each edge.

A *Cayley graph* for a finitely generated group is a directed graph with transitive and free action of the group by graph automorphisms. It is a classical result that all Cayley graphs of a finitely generated group are quasi-isometric to each other. A generalization of Cayley graphs for compactly generated topological groups with compact open subgroups is known as *Cayley-Abels graph*, see Section 4.4 for details. Any two Cayley-Abels graphs for a compactly generated topological group with a compact open subgroup are quasi-isometric, see [KM08] for a proof.

In this thesis, we will introduce *relative Cayley-Abels graphs* for pairs $(G, \mathcal{H})$, where G is a topological group with a compact open subgroup, and $\mathcal{H}$ is a finite collection of open subgroups. We will generalize the quasi-isometry result for these graphs in Section 5.5.

Cellular complexes

A map $Y \to X$ between CW complexes is *cellular* if its restriction to each open cell of Y is a homeomorphism onto an open cell of X. A CW-complex X is *cellular* provided that the attaching map of each open cell of X is cellular for a suitable subdivision.

An example of a cellular complex in which we are interested in this thesis is the *presentation complex*. It is a 2-dimensional cell complex associated to any presentation of a group G. The complex has a single vertex, and one loop at the vertex for each generator of G. There is one 2-cell for each relation in the presentation, with the boundary of the 2-cell attached along the appropriate word. The fundamental group of a presentation complex of a group is the group itself; higher dimensional cells can be attached to the presentation complex to get an Eilenberg–MacLane $K(G, 1)$ space for G, which has trivial homotopy groups for $n > 1$. The universal cover of the presentation complex is called *Cayley complex*, whose 1-skeleton is a Cayley graph of the group.

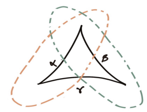

Figure 2.1: Note that $\gamma \subseteq N_\delta(\alpha) \cup N_\delta(\beta)$

In Section 5.6, we will introduce relative Cayley-Abels complexes for proper pairs $(G, \mathcal{H})$, which are 2-dimensional cellular complexes with relative Cayley-Abels graphs as 1-skeleton.

Hyperbolic groups

A finitely generated group is *hyperbolic* if any of its Cayley graph is a δ-*hyperbolic space*, for some real number δ. The δ-hyperbolicity, often called Gromov hyperbolicity, is a metric generalization of Riemannian manifolds with constant negative sectional curvature. For a geodesic metric space, being a δ-*hyperbolic space* means that all its geodesic triangles are δ-slim. Formally, a geodesic space X is Gromov-hyperbolic if there exists a real number δ such that for any geodesic triangle, each side is contained in the δ-neighbourhood of the other two sides, see Figure 2.1. It is a classical result that Gromov-hyperbolicity is a quasi-isometric invariant and hence is a well-defined property of a finitely generated group. Some of the examples of hyperbolic groups are finite groups, free groups, fundamental group of closed surfaces of genus grater than one, etc.

Projective resolution

A module P is *projective* if and only if for every surjective module morphism $f\colon N \to M$ and every module morphism $g\colon P \to M$, there exists a module morphism $h\colon P \to N$ such that $f \circ h = g$. Given a module M, a *projective resolution* of M is an infinite exact sequence of modules.

$$\cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0,$$

where all P_i are projective modules. It is said to have *length* m if

$$0 \to P_m \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0,$$

is exact.

We will use projective resolutions in Chapter 3 and Chapter 4, to define the dimensions of groups. In particular, the *cohomological dimension* $\mathsf{cd}_R(G)$ of a group G with respect to a commutative ring R is less than or equal to n if the trivial RG-module R has a projective resolution of length n. Here RG is the *group ring*, which is defined as follows: Suppose G is a group and R is a commutative ring, then the *group ring* RG is $\bigoplus_{g \in G} R(g)$, where $R(g)$ is the free R-module generated by the set $\{g\}$. The multiplication is defined as follows

$$(\sum_{g \in G} r_g g)(\sum_{g' \in G} s_{g'} g') = \sum_{g, g' \in G} r_g s_{g'} (gg')$$

These rings are usually very large; for example, the group ring for a commutative ring R and the group of integers $\mathbb{Z} = \langle t \rangle$ is isomorphic to the Laurent polynomials $R[t, t^{-1}]$ over the ring R. Investigating the properties of these rings is a very popular area of research, for instance see recent resolution to the long-standing Kaplansky's conjecture [Gar21].

For a group G and a commutative ring R, we can construct a projective resolution of the trivial RG-module R by considering the *cellular R-chain complex* corresponding to the universal cover $\tilde{X}$, where X is a $K(G, 1)$-space for the group G. Since G acts on $\widetilde{X}$ freely, each module in the chain complex is a free RG-module; and since $\widetilde{X}$ is contractible, the cellular chain complex is a resolution. In summary, for a group G and a commutative ring R, the cellular R-chain complex of the universal cover of a $K(G, 1)$-space of a group G, is a free resolution of the trivial RG-module R.

Totally disconnected locally compact groups

A *topological group* is a group with a structure of a topological space such that the group multiplication and inversion maps are continuous. All topological groups in this thesis are assumed to be Hausdorff. A topological space X is said to be *locally* *compact* if each point $x \in X$ has a compact neighbourhood, i.e. there exists a compact subset K and open subset U of X such that $x \in U \subseteq K$. A *locally compact group* is a topological group with locally compact and Hausdorff topology. The main objects of study in Chapter 5, are topological groups containing a compact open subgroup. Observe that these groups are implicitly locally compact.

A *totally disconnected and locally compact* group is a locally compact group such that the connected component of the identity is the identity. We will often refer to them as TDLC groups in this thesis. By van Dantzig's Theorem [VD36], all TDLC groups have a basis of compact open subgroups, and hence are particular examples of topological groups containing a compact open subgroup.

All discrete groups are TDLC groups. Non-discrete TDLC groups include *profinite groups.* A profinite group is a topological group that is isomorphic to the inverse limit of an inverse system of discrete finite groups. They are precisely the compact TDLC groups [Ser02, Proposition 0]. For example, the p-adic integers $\mathbb{Z}_p$, for any prime p. An important class of non-discrete TDLC groups is the class of non-Archimedean local fields. For instance, p-adic numbers $\mathbb{Q}_p$, and consequentially the general linear groups over $\mathbb{Q}_p$ are TDLC groups. We will use the fact that the ring of p-adic integers $\mathbb{Z}_p$ is a compact open subgroup of $\mathbb{Q}_p$ in Chapter 4 and Chapter 5.

Chapter 3

Subgroups of hyperbolic groups in low cohomological dimension

3.1 Abstract

A result of Gersten states that if G is a hyperbolic group with integral cohomological dimension $\mathsf{cd}_{\mathbb{Z}}(G) = 2$ then every finitely presented subgroup is hyperbolic. We generalize this result for the rational case $\mathsf{cd}_{\mathbb{Q}}(G) = 2$. In particular, our result applies to the class of torsion-free hyperbolic groups G with $\mathsf{cd}_{\mathbb{Z}}(G) = 3$ and $\mathsf{cd}_{\mathbb{Q}}(G) = 2$ discovered by Bestvina and Mess.

3.2 Introduction

The cohomological dimension $\mathsf{cd}_R(G)$ of a group G with respect to a ring R is less than or equal to n if the trivial RG-module R has a projective resolution of length n. Let $\mathbb{Q}$ denote the field of rational numbers. The main result of this chapter:

Theorem 3.2.1. *Let G be a hyperbolic group such that $\mathsf{cd}_{\mathbb{Q}}(G) \leq 2$. If H is a finitely presented subgroup, then H is hyperbolic.*

The analogous statement for $\mathsf{cd}_{\mathbb{Z}}(G)$ is a result of Steve Gersten that we recover as a consequence of the inequality

$$\mathsf{cd}_{\mathbb{Q}}(G) \leq \mathsf{cd}_{\mathbb{Z}}(G).$$

Corollary 3.2.2 (Gersten)**.** [Ger96b, Theorem 5.4] *Let G be a hyperbolic group such that $\mathsf{cd}_{\mathbb{Z}}(G) = 2$. If H is a finitely presented subgroup, then H is hyperbolic.*

The first motivation to generalize Gersten's result to the rational case is the existence of hyperbolic groups of integral cohomological dimension three and rational cohomological dimension two. The nature of finitely presented subgroups of groups in this class was not known. The first examples of such groups were discovered by Bestvina and Mess [BM91] based on methods by Davis and Januszkiewicz [DJ91]. The class also contains finite index subgroups of hyperbolic Coxeter groups, examples that were discovered by Dranishnikov [Dra99, Corollary 2.3]. We recall the nature of Bestvina-Mess examples in the following corollary.

Corollary 3.2.3. [BM91] *Let X be a finite polyhedral 3-complex such that*

- *X admits piecewise constant negative curvature cellular structure satisfying Gromov's link condition, and*
- *X is a 3-manifold (without boundary) in the complement of a single vertex whose link is a non-orientable closed surface.*

If $G = \pi_1 X$ then $\mathsf{cd}_{\mathbb{Q}}(G) = 2$, $\mathsf{cd}_{\mathbb{Z}}(G) = 3$ and any finitely presented subgroup of G is hyperbolic.

The statement of Corollary 3.2.2 is sharp in the sense that there exist hyperbolic groups of integral cohomological dimension three containing finitely presented subgroups that are not hyperbolic, the first example was found by Noel Brady [Bra99]. More recently, infinite families of hyperbolic groups of integral cohomological dimension three containing non-hyperbolic finitely presented subgroups have been constructed, see for example [Kro21].

Corollary 3.2.4. *If G is a hyperbolic group such that $\mathsf{cd}_{\mathbb{Z}}(G) = 3$ and it contains a non-hyperbolic finitely presented subgroup, then $\mathsf{cd}_{\mathbb{Q}}(G) = \mathsf{cd}_{\mathbb{Z}}(G)$.*

A second motivation of this project was to generalize Gersten's result to groups admitting torsion, specifically, to the class of hyperbolic groups G admitting a 2-dimensional classifying space for proper actions $\underline{E}G$. Recall that a model for $\underline{E}G$ is a G-CW-complex X with the property that for each subgroup H the subcomplex of fixed points is contractible if H is finite, and empty if H is infinite. The minimal dimension of a model for $\underline{E}G$ is denoted by $\underline{\mathsf{gd}}(G)$. Considering the cellular chain complex with rational coefficients of a model for $\underline{E}G$ with minimal dimension shows that

$$\mathsf{cd}_{\mathbb{Q}}(G) \leq \underline{\mathsf{gd}}(G).$$

This inequality implies the following corollary.

Corollary 3.2.5. *If G is a hyperbolic group such that $\underline{\mathsf{gd}}(G) \leq 2$, then any finitely presented subgroup is hyperbolic.*

The statement of Corollary 3.2.5 was known in the following cases:

- If G admits a $\mathsf{CAT}(-1)$ 2-dimensional model for $\underline{E}G$, see [HMP16, Corollary 1.5].
- If G admits a 2-dimensional model for $\underline{E}G$, and H is finitely presented with finitely many conjugacy classes of finite groups, a consequence of [MP17, Theorem 1.3].
- If G is a hyperbolic small cancellation group of type $C(7)$, $C(5){-}T(4)$, $C(4){-}T(5)$, $C(3){-}T(7)$ or $C'(1/6)$, see [Ger96b, Theorem 7.6].

We remark that for a group G satisfying the hypothesis of Corollary 3.2.5, the conclusion follows from Gersten's result 3.2.2 if, in addition, G is assumed to be virtually torsion free. It is an outstanding question whether hyperbolic groups are virtually torsion free [KW00].

Homological filling functions and the Proof of Theorem 3.2.1

Let R be a subring of $\mathbb{Q}$. The $(n+1)$-dimensional homological Filling Volume function over R of a cellular complex X is a function $\mathsf{FV}^{n+1}_{X,R}\colon \mathbb{N} \to \mathbb{R}$ describing the minimal volume required to fill integral cellular n-cycles with cellular $(n+1)$-chains with coefficients in R. For a formal definition, see Definition 3.4.2.

For a group G with a $K(G,1)$ model X with finite $(n+1)$-skeleton, the $(n+1)$-dimensional homological Filling Volume function over R of G, denoted by $\mathsf{FV}^{n+1}_{G,R}$, is defined as $\mathsf{FV}^{n+1}_{\tilde{X},R}$ where $\tilde{X}$ is the universal cover of X. This function depends only of the group G up to the equivalence relation on the set of non-decreasing functions $\mathbb{N} \to \mathbb{R}$ defined as $f \sim g$ if and only if $f \preceq g$ and $g \preceq f$, where $f \preceq g$ means there is $C > 0$ such that for all $k \in \mathbb{N}$,

$$f(k) \leq Cg(Ck + C) + Ck + C.$$

Recall that a group G is of type R-FP_n if the trivial RG-module R admits a partial projective resolution

$$P_n \to \cdots \to P_2 \to P_1 \to P_0 \to R \to 0$$

where each P_i is a finitely generated RG-module. In [HMP16], it is shown that to define $\mathsf{FV}^{n+1}_{G,\mathbb{Z}}$, it is enough to assume that the group G is of type $\mathbb{Z}$-FP_{n+1}. We prove that the same statement holds for $\mathsf{FV}^{n+1}_{G,R}$ in Section 3.4. The main technical result of this note is the following.

Theorem 3.2.6. *Let R be a subring of $\mathbb{Q}$. Let G be a group of type R-FP_{n+1} and suppose $\mathsf{cd}_R(G) = n+1$. Let $H \leq G$ be a subgroup of type R-FP_{n+1}. Then*

$$FV^{n+1}_{H,R} \preceq FV^{n+1}_{G,R}.$$

This theorem generalizes the main result of [HMP16], by considering an arbitrary subring of the rational numbers instead of only the ring of integers, and by replacing the topological assumptions F_{n+1} on G and H with the weaker hypothesis R-FP_{n+1}.

The main result of this note, Theorem 3.2.1, is a consequence of Theorem 3.2.6 and the characterization of hyperbolic groups stated below, which is credited to Gersten [Ger96a]. This characterization was revised by Mineyev [Min02, Theorem 7, statements (0) and (2)], and it was also revisited by Groves and Manning in [GM08, Theorem 2.30].

Theorem 3.2.7. [Min02, Theorem 7] [GM08, Theorem 2.30] *A group G is hyperbolic* *if and only if G is finitely presented and the rational filling function $\mathsf{FV}^2_{G,\mathbb{Q}}$ is bounded by a linear function, i.e., $\mathsf{FV}^2_{G,\mathbb{Q}}(k) \preceq k$.*

Proof of Theorem 3.2.1. Let G be a hyperbolic group such that $\mathsf{cd}_{\mathbb{Q}}(G) = 2$, and let H be a finitely presented subgroup. Theorem 3.2.7 implies that $\mathsf{FV}^2_{G,\mathbb{Q}}$ is bounded by a linear function. By Theorem 3.2.6, $\mathsf{FV}^2_{H,\mathbb{Q}} \preceq \mathsf{FV}^2_{G,\mathbb{Q}}$. It follows $\mathsf{FV}^2_{H,\mathbb{Q}}$ is bounded by a linear function. Then Theorem 3.2.7 implies that H is a hyperbolic group. $\square$

In view of Theorem 3.2.7, we raised the following question.

Question 3.2.8. *Let G be a $\mathbb{Q}$-FP_2 group and suppose $FV^2_{G,\mathbb{Q}}$ is bounded by a linear function. Is G a hyperbolic group?*

The analogous question obtained by replacing $\mathbb{Q}$ with $\mathbb{Z}$ is known to have a positive answer [Ger96b, Theorem 5.2]. One motivation behind this question is that a positive answer would imply that in our main result Theorem 3.2.1 H can be assumed to be $\mathbb{Q}$-FP_2 instead of being finitely presented. Recall that $\mathbb{Q}$-FP_2 condition is weaker than being finitely presented, see the examples in [BB97].

The rest of the note is devoted to the definition of homological filling function and the proof of Theorem 3.2.6. The argument is relatively self-contained, and uses and simplifies ideas from [HMP16]. The main contributions of the article beside the results stated above are:

1. The definition of filling functions for arbitrary subdomains of the rationals, since the definition in [HMP16] does not generalize directly, and
2. the replacement of topological arguments in [HMP16] by algebraic ones that allow us to prove certain statements under the weaker homological finiteness condition R-FP_{n+1} instead of the topological assumption F_{n+1}; see Proposition 3.5.1 which is a construction based on the homological mapping cylinders, and Remark 3.5.2.

Organization

Preliminary definitions are included in Section 3.3, specifically the notions of filling norms and bounded morphisms on modules over arbitrary normed rings. Section 3.4 discusses the generalization of homological filling functions defined over arbitrary subdomains of the rational numbers. The last section contains the proof of Theorem 3.2.6.

3.3 Filling Norms, Bounded morphisms

Let R be a ring and let $\mathbb{R}$ denote the ordered field of real numbers. A norm on R is a function $|\cdot|\colon R \to \mathbb{R}$ such that for any $r, r' \in R$

- $|r| \geq 0$ with equality if and only if $r = 0$,
- $|r + r'| \leq |r| + |r'|$, and
- $|r_1 r_2| \leq |r_1||r_2|$ for $r_1, r_2 \in R$.

A *normed ring* is a ring equipped with a norm.

From here on, assume that R is a normed ring. A norm on an R-module M is a function $\|\cdot\|\colon M \to \mathbb{R}$ such that for any $m, m' \in M$ and $r \in R$

- $\|m\| \geq 0$ with equality if and only if $m = 0$,
- $\|m + m'\| \leq \|m\| + \|m'\|$, and
- $\|rm\| \leq |r|\|m\|$.

A function $M \to \mathbb{R}$ that satisfies the last two conditions and has only non-negative values is called a pseudo-norm.

The ℓ_1-*norm* on a free R-module F with fixed basis Λ is defined as

$$\left\| \sum_{x \in \Lambda} r_x x \right\|_1 = \sum_{x \in \Lambda} |r_x|.$$

A free R-module with fixed basis is called a *based free module.*

Definition 3.3.1 (Filling norm)**.** A *filling norm* on a finitely generated R-module M is defined as follows. Let $\rho\colon F \to M$ be a surjective morphism of R-modules where F is a finitely generated free R-module with fixed basis Λ and induced ℓ_1-norm $\|\cdot\|_1$. The *filling norm on M* induced by ρ and Λ is defined as

$$\|m\|_M = \inf\{\|x\|_1 \colon x \in F, \rho(x) = m\}.$$

Remark 3.3.2. The following statements can be easily verified.

1. An ℓ_1-norm $\|\cdot\|_1$ on a finitely generated free R-module F is a filling norm.

2. A filling norm $\|\cdot\|$ on a finitely generated R-module M is a pseudo-norm, and is regular in the sense that
$$\|rm\| = |r|\|m\|$$
for any any $m \in M$ and $r \in R$ such that r is a unit and $|r||r^{-1}| = 1$.

Definition 3.3.3 (Bounded Morphism)**.** A morphism $f\colon M \to N$ between R-modules with norms $\|\cdot\|_M$ and $\|\cdot\|_N$ respectively is called *bounded (with respect to these norms)* if there exists a fixed constant $C > 0$ such that $\|f(a)\|_N \leq C\|a\|_M$ for all $a \in M$.

The following lemma appears in [MP17] for the case that R is a group ring. The proof for an arbitrary ring is analogous, we have included the argument for the convenience of the reader.

Lemma 3.3.4. [MP17, Lemma 4.6] *Morphisms between finitely generated R-modules are bounded with respect to filling norms.*

Proof. First observe that if $\tilde{\varphi}\colon A \to B$ is a morphism between finitely generated based free R-modules, then for $a \in A$,

$$\|\tilde{\varphi}(a)\|_B \leq C\|a\|_A,$$

where $\|\cdot\|_A$ and $\|\cdot\|_B$ are the corresponding ℓ_1-norms, the constant C is defined as $\max\{\|\tilde{\varphi}(a)\|_B \colon a \in \Lambda\}$ where Λ is the fixed basis of A.

Now we prove the statement of the lemma. Let $\varphi\colon P \to Q$ be a morphism between finitely generated R-modules, and let $\|\cdot\|_P$ and $\|\cdot\|_Q$ denote filling norms on P and Q respectively. Suppose A is a finitely generated based free R-module and that $\rho\colon A \to P$ induces the filling norm $\|\cdot\|_P$, and analogously assume that $\rho'\colon B \to Q$ induces the filling norm $\|\cdot\|_Q$. Then, since A is free, there is a morphism $\tilde{\varphi}\colon A \to B$ such that $\varphi \circ \rho = \rho' \circ \tilde{\varphi}$. Let C be the constant for $\tilde{\varphi}$ defined above. Let $p \in P$ and note that for any $a \in A$ such that $\rho(a) = p$,

$$\|\varphi(p)\|_Q \leq \|\tilde{\varphi}(a)\|_B \leq C\|a\|_A.$$

Hence $\|\varphi(p)\|_Q \leq C\|p\|_P$. $\square$

Two norms $\|\cdot\|$ and $\|\cdot\|'$ on an R-module M are said to be *equivalent* if there exists a constant $C > 0$ such that for all $m \in M$

$$C^{-1}\|m\| \leq \|m\|' \leq C\|m\|.$$

By considering the identity function on a finitely generated module M, the previous lemma implies:

Corollary 3.3.5. *Any two filling norms on a finitely generated R-module M are equivalent.*

Remark 3.3.6. Let M be a free R-module with basis Λ, and let N be a free R-submodule generated by a finite subset $\Lambda' \subseteq \Lambda$. Consider the induced ℓ_1-norms $\|\cdot\|_\Lambda$ and $\|\cdot\|_{\Lambda'}$ on M and N respectively.

1. The projection map $\pi\colon M \to N$ is bounded with respect to the induced ℓ_1-norms.
2. The inclusion map $\imath\colon N \to M$ preserves the induced ℓ_1-norms, in particular, it is bounded.

Lemma 3.3.7. *Let N be a finitely generated module with filling norm $\|\cdot\|_N$. Suppose that N is an internal direct summand of a free module F with an ℓ_1-norm $\|\cdot\|_1$. Then $\|\cdot\|_N \sim \|\cdot\|_1$ on N.*

Proof. Since N is a finitely generated module contained in F, there exist a finitely generated free submodule I of F which is an internal summand, $F = I \oplus J$, such that $N \subseteq I$, and the restriction of $\|\cdot\|_1$ to I is an ℓ_1-norm on I. Let $\iota\colon N \to I$ denote the inclusion and $\phi\colon F \to N$ denote the projection. By Lemma 3.3.4, both $\phi|_I\colon I \to N$ and $\iota\colon N \to I$ are bounded morphisms with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_N$; let C_1 and C_2 be the corresponding constants. Then

$$\|n\|_N = \|\phi(\iota(n))\|_N \leq C_1\|\iota(n)\|_1 \leq C_2C_1\|n\|_N$$

for all $n \in N$, and hence $\|\cdot\|_N \sim \|\cdot\|_1$ on N. □

For the rest of this section, let G be a group, let H be a subgroup, and as above let R be a ring with norm $|\cdot|$.

Remark 3.3.8. Let M be a free RG-module with ℓ_1-norm $\|\cdot\|_\Lambda$ induced by a free basis set Λ. Then M is a free RH-module and there exist a free RH-basis Λ_H of M such that the induced ℓ_1-norms $\|\cdot\|_\Lambda$ and $\|\cdot\|_{\Lambda_H}$ are equal.

Indeed, if S is a right transversal of the subgroup H in G, then $\Lambda_H = \{gx \colon x \in \Lambda, g \in S\}$ is a free RH-basis of M as an H-module, and the statement about the ℓ_1-norms holds.

Lemma 3.3.9. *Let M be a finitely generated and projective RG-module with filling norm $\|\cdot\|_M$ and let N be a finitely generated RH-module with filling norm $\|\cdot\|_N$. Suppose that N is an internal direct summand of M as an RH-module. Then $\|\cdot\|_N \sim \|\cdot\|_M$ on N.*

Proof. Let F be a finitely generated free based module with ℓ_1-norm $\|\cdot\|_1$, and let $\phi : F \to M$ be a surjective RG-morphism inducing the filling norm $\|\cdot\|_M$. Since M is projective, there exist an RG-morphism $j \colon M \to F$ such that $j \circ \phi = \mathrm{id}_M$. Lemma 3.3.4 implies that j and ϕ are bounded RG-morphisms. Therefore $\|\cdot\|_M \sim \|\cdot\|_1$ on M. Now consider F as an RH-module with the same ℓ_1-norm $\|\cdot\|_1$, see Remark 3.3.8. Since N is a direct summand of M as an RH-module, it is a direct summand of F as an RH-module. Then Lemma 3.3.7 implies $\|\cdot\|_N \sim \|\cdot\|_1$ on N. $\square$

3.4 Definition of Homological Filling Functions

In this section R denotes a subring of the rational numbers with the absolute value as a norm. Let G be a group. The group ring RG is a free module over R, observe that RG is a normed ring with ℓ_1-norm induced by the free R-basis G. From now on, we consider RG as a normed ring with this norm.

Definition 3.4.1 (Integral part)**.** Let P be a finitely generated RG-module. An *integral part* of P is a $\mathbb{Z}G$-submodule A that is finitely generated as a $\mathbb{Z}G$-module and generates P as an RG-module.

From here on, $[0, \infty]$ denotes the set of non-negative real numbers and infinity. The order relation as well as the addition operations are extended in the natural way.

Definition 3.4.2. The n^{th}*–filling function* of a group G of type R-FP_{n+1},

$$FV_{G,R}^{n+1}\colon \mathbb{N} \to [0,\infty],$$

is defined as follows. Let

$$P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \longrightarrow R \to 0, \tag{3.4.1}$$

be a partial projective resolution of finitely generated RG-modules of the trivial RG-module R. Let K_n be an integral part for $\ker(\partial_n)$, let $\|\cdot\|_{P_n}$ and $\|\cdot\|_{P_{n+1}}$ be filling norms for P_n and P_{n+1} respectively. Then

$$FV_{G,R}^{n+1}(k) = \max\left\{\ \|\gamma\|_{\partial_{n+1}}\colon \gamma \in K_n,\ \|\gamma\|_{P_n} \leq k\ \right\},$$

where

$$\|\gamma\|_{\partial_{n+1}} = \inf\left\{\|\mu\|_{P_{n+1}}\colon \mu \in P_{n+1},\ \partial_{n+1}(\mu) = \gamma\right\}.$$

By convention, define the maximum of the empty set as zero.

See Remark 3.4.8 on finiteness of $FV_{G,R}^{n+1}$. The rest of this section discusses the proof of the following theorem, which generalizes [HMP16, Theorem 3.5]. Consider the equivalence relation on the set of non-decreasing functions $\mathbb{N} \to [0,\infty]$ defined as $f \sim g$ if and only if $f \preceq g$ and $g \preceq f$, where $f \preceq g$ means there is $C > 0$ such that for all $k \in \mathbb{N}$,

$$f(k) \leq Cg(Ck + C) + Ck + C.$$

Theorem 3.4.3. *Let G be a group of type R-FP_{n+1}. Then the n^{th}–filling function $FV_{G,R}^{n+1}$ of G is well defined up to the equivalence relation $\sim$.*

The proof of Theorem 3.4.3 relies on the following basic structure theorem for subrings of $\mathbb{Q}$.

Proposition 3.4.4. *Let R be a subring of $\mathbb{Q}$. Then there is a set S of prime numbers in $\mathbb{Z}_+$ such that R consists of all fractions $\frac{a}{b}$ where $a \in \mathbb{Z}$ and b is a product of powers of elements of S.*

In the following proposition, which is a consequence of Proposition 3.4.4, we use the convention that for an element a of an RG-module A, and any $r \in R$, ra denotes the element $(re)a \in A$ where e is the identity element of G; moreover, the ring of integers $\mathbb{Z}$ is naturally identified with the subring of RG via $m \mapsto me$.

Proposition 3.4.5. *Let P and Q be finitely generated RG-modules. Then*

1. *If A is an integral part of a finitely generated module, then for all units $r \in R$, $rA = \{ra \colon a \in A\}$ is an integral part of a finitely generated module.*
2. *If $f \colon P \to Q$ is a morphism of RG-modules, and A and B are integral parts of P and Q respectively, then there exists a positive integer m which is a unit in RG and such that $f(mA) \subseteq B$.*

Proof. The first statement is immediate from the definition. For the second statement, let S be a finite generating set of A as a $\mathbb{Z}G$-module, and observe that S generates P as an RG-module. Let $F(S)$ be the free RG-module on S, let $\phi \colon F(S) \to P$, and let C be the $\mathbb{Z}G$-submodule of $F(S)$ generated by S, and observe that $\phi(C) = A$. Analogolusly, let T be a finite generating set of B as a $\mathbb{Z}G$-module, let $\psi \colon F(T) \to Q$, and let C' be the $\mathbb{Z}G$-submodule of $F(T)$ generated by T, and note that $\psi(C') = B$.

Since $F(S)$ is free, there is an RG-morphism $\eta \colon F(S) \to F(T)$ such that the following diagram commutes.

$$\begin{tikzcd}
F(S) \arrow[r, "\eta"] \arrow[d, two heads, "\phi"'] & F(T) \arrow[d, two heads, "\psi"] \\
P \arrow[r, "f"'] & Q
\end{tikzcd} \tag{3.4.2}$$

Note that $\eta \colon F(S) \to F(T)$ is described by a finite matrix with entries in RG. By Proposition 3.4.4, there is an integer m, which is a unit in R, such that the morphism $m\eta \colon F(S) \to F(T)$ given by $\alpha \mapsto m\eta(\alpha)$ has the property that $\eta(C) \subseteq C'$. By commutativity of the diagram $f \circ (m\phi) = \psi \circ (m\eta)$ and therefore $f(mA) \subseteq B$. $\square$

The following lemma is a strengthening of Proposition 3.4.5 that will be used in the last section.

Lemma 3.4.6. *Let $H \leqslant G$ be a subgroup and let P and Q be finitely generated RH and RG modules respectively. If $f \colon P \to Q$ is an RH-morphism, and A and B are* *integral parts of P and Q respectively, then there exists a positive integer m, which is a unit in R, such that $f(mA) \subseteq B$.*

Proof. Considering Q as an RH-module, the proof proceeds similar to 3.4.5 except that here $F(T)$ is infinitely generated and so the matrix is infinite. But observe that only finitely many entries are non-zero, so the same argument holds. $\square$

Proof of Theorem 3.4.3. The proof is divided into two steps. The second step is a small variation of the argument in [HMP16, Proof of Theorem 3.5] for which we only remark the changes.

Step 1. FV_G^{n+1} *(up to equivalence) does not depend on the choice of the integral part* K_n.

Let A and B be two integral parts of $\ker(\partial_n)$, and let FV_A and FV_B denote the corresponding n^{th}-filling functions of G. By Proposition 3.4.5, there exists a positive integer m, that is a unit in RG, such that $mA \subseteq B$. Let $\gamma \in A$ such that $\|\gamma\|_{P_n} \leq k$. Then, since m is a unit and $|m||m^{-1}| = 1$, $\|\gamma\|_{\delta_{n+1}} = \frac{1}{m}\|m\gamma\|_{\delta_{n+1}}$ and $\|m\gamma\|_{P_n} = m\|\gamma\|_{P_n} \leq mk$; see Remark 3.3.2. Observe that $m\gamma \in B$ therefore $\|\gamma\|_{\delta_{n+1}} \leq \frac{1}{m}FV_B(mk)$. Since γ was arbitrary, $FV_A(k) \leq \frac{1}{m}FV_B(mk)$. By symmetry we get the other inequality.

Step 2. FV_G^{n+1} *(up to equivalence) does not depend on the choice of the resolution* (3.4.1).

Let (P_*, ∂_*) and (Q_*, δ_*) be a pair of resolutions as in (3.4.1). Since any two projective resolutions of R are chain homotopy equivalent, there exist chain maps $f_i\colon P_i \to Q_i$, $g_i\colon Q_i \to P_i$, and a map $h_i\colon P_i \to P_{i+1}$ such that

$$\partial_{i+1} \circ h_i + h_{i-1} \circ \partial_i = g_i \circ f_i - Id.$$

By Proposition 3.4.5, there exist integral parts K_n and K_n' of $ker(\partial_n)$ and $ker(\delta_n)$ respectively, such that $f_n(K_n) \subseteq K_n'$. This ensures that the same argument in [HMP16, Proof of Theorem 3.5] works except for a minor change in the choice of the constant named β in the cited proof. Replace it by the following: "For $\epsilon < C$, choose $\beta \in Q_{n+1}$ such that $\delta_{n+1}(\beta) = f_n(\alpha)$ and $\|\beta\|_{Q_{n+1}} < \|f_n(\alpha)\|_{\delta_{n+1}} + \epsilon$". The rest of the proof proceeds in the same manner. $\square$

Remark 3.4.7 (Topological interpretation of filling functions)**.** Assume G admits a $K(G,1)$ model X with finite $(n+1)$-skeleton. The augmented cellular chain complex $C_*(X,R)$ of the universal cover $\widetilde{X}$ of X is a projective resolution of the trivial RG-module R by free modules. By considering the ℓ_1-norm of $C_i(X,R)$ induced by the basis consisting of i-dimensional cells of $\widetilde{X}$, the definition of $\mathsf{FV}^{n+1}_{G,R}$ using this resolution provides the interpretation $\mathsf{FV}^{n+1}_{G,R}$ as the minimal volume required to fill integral n-cycles with $(n+1)$-cellular chains with coefficients in R. Observe that

$$FV^{n+1}_{G,R} \leq FV^{n+1}_{G,\mathbb{Z}} \tag{3.4.3}$$

Remark 3.4.8. [Finiteness of $FV^{n+1}_{G,R}$] Assume that G admits a $K(G,1)$ model X with finite $(n+1)$-skeleton. By the main result of [FMP18], for every positive integer k, $FV^{n+1}_{G,\mathbb{Z}}(k) < \infty$. Then equation (3.4.3) implies that $FV^{n+1}_{G,R}(k) < \infty$ for any $k \geq 0$.

A positive answer to the following question in the case that $R = \mathbb{Z}$ is given in [FMP18].

Question 3.4.9. *Suppose that G is of type R-FP_{n+1}. Is $FV^{n+1}_{G,R}(k) < \infty$ for all $k \in \mathbb{N}$?*

Remark 3.4.10 (On the use integral part in Definition 3.4.2)**.** We note that the filling function $\mathsf{FV}^{n+1}_{G,\mathbb{Z}}$ was defined in [HMP16] by considering $ker(\partial_n)$ in lieu of its integral part. This approach does not work to define $\mathsf{FV}^{n+1}_{G,\mathbb{Q}}$ as the following example illustrates. Consider the group presentation $G = \langle x, y | [x,y] \rangle$ and let X be the universal cover of the presentation complex, i.e., the Cayley complex. In X consider the following cycles with rational coefficients $a_n = \frac{1}{4n}[x^n, y^n]$ for $n \in \mathbb{N}$. Then $\|a_n\|_1 = 1$ and by regularity $\|a_n\|_\partial = \frac{1}{4}n$, in particular

$$\max\{\|\gamma\|_{\partial_2} \colon \gamma \in Z_n(\widetilde{X}, \mathbb{Q}), \|\gamma\|_1 \leq 1\} = \infty,$$

and hence the approach in [HMP16] does not yield a well defined $FV^2_{G,\mathbb{Q}}(k)$. In contrast, using Definition 3.4.2, $FV^2_{G,\mathbb{Q}} \preceq FV^2_{G,\mathbb{Z}} \sim k^2$.

3.5 Proof of Theorem 3.2.6

The proof of Theorem 3.2.6 is discussed after the proof of the following proposition.

Proposition 3.5.1. *Suppose that* $cd_R(G) = n+1$*,* G *is of type* R*-*FP_{n+1}*, and* H *is a subgroup of* G *of type* R*-*FP_{n+1}*. Then for any partial projective resolution of the trivial* RH*-module* R *of finite type*

$$Q_{n+1} \to Q_n \to \cdots \to Q_0 \to R \to 0, \tag{3.5.1}$$

there is a projective resolution of the trivial RG*-module* R *of finite type*

$$0 \to M_{n+1} \to M_n \to \cdots \to M_0 \to R \to 0, \tag{3.5.2}$$

and injective morphisms $\imath_i \colon Q_i \to M_i$ *of* RH*-modules,* $0 \le i \le n$*, such that*

$$\begin{tikzcd} Q_n \arrow[r] \arrow[d, "\imath_n"] & \cdots \arrow[r] & Q_1 \arrow[r] \arrow[d, "\imath_1"] & Q_0 \arrow[r] \arrow[d, "\imath_0"] & R \arrow[d, "Id"] \\ M_n \arrow[r] & \cdots \arrow[r] & M_1 \arrow[r] & M_0 \arrow[r] & R. \end{tikzcd} \tag{3.5.3}$$

is a commutative diagram of RH*-modules, and the short exact sequences of* RH*-modules*

$$0 \longrightarrow Q_i \xrightarrow{\imath_i} M_i \longrightarrow S_i \longrightarrow 0 \tag{3.5.4}$$

split. In particular each S_i *is a projective* RH*-module.*

Remark 3.5.2. Proposition 3.5.1 replaces topological arguments in [HMP16], based on work of Gersten [Ger96b], that use topological mapping cylinders. The arguments there are relatively less involved. In the generality that we are working, it is not possible to rely on this type of constructions. We would need free cocompact actions on $(n+1)$-acyclic complexes for G and H, they are not known to exist under our hypothesis. Specifically, recall that a group G is of type FH_n, if G admits a cocompact action on an n-acyclic space X; in this case the action of G on the cellular chain complex of X induces a resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module. Hence FH_n implies FP_n. It is an open question whether groups of type FP_n are of type FH_n for $n \ge 3$, see [BB97].

The proof of the Proposition 3.5.1 is an application of the *mapping cylinder* of chain complexes from basic homological algebra that we recall below.

Let $B_* = \{B_i, d_i\}$ and $C_* = \{C_i, d_i'\}$ be two chain complexes of modules over some fixed ring, and let $f \colon B_* \to C_*$ be a chain map. Then the mapping cylinder

$M_* = \{M_i, d''_i\}$ is a chain complex where $M_i = C_i \oplus B_i \oplus B_{i-1}$ with

$$d''_i = \begin{pmatrix} d'_i & 0 & -f_{i-1} \\ 0 & d_i & id_{B_{i-1}} \\ 0 & 0 & -d_{i-1} \end{pmatrix}$$

In particular, $d''_i \colon M_i \to M_{i-1}$ is given by $(c, b, b') \mapsto (d'_i(c) - f_{i-1}(b'), d_i(b) + b', -d_{i-1}(b'))$. The natural inclusion $C_* \hookrightarrow M_*$ is a chain homotopy equivalence with homotopy inverse map $M_* \to C_*$ given by $(c, b, b') \mapsto c + f(b)$. Also note that, if both B_* and C_* consists of only finitely generated projective modules then the the same holds for M_*. For background on mapping cylinders see [Wei94].

Proof of Proposition 3.5.1. We split the proof into four steps.

Step 1. *Definition of the resolution* (3.5.2) *as a mapping cylinder*

Since $cd_R(G) = n+1$ and G is of type R-FP_{n+1}, there is a projective resolution of RG-modules of finite type

$$0 \to P_{n+1} \to P_n \to \ldots.. \to P_0 \to R \to 0, \tag{3.5.5}$$

see [Bro94, pg.199, Prop. 6.1].

The group ring RG is a free right RH-module. It follows that the extension of scalars functor from left RH-modules to left RG-modules $M \mapsto RG \otimes_{RH} M$ is exact. This functor also preserves finite generation and projectiveness. From the given resolution (3.5.1), we obtain a partial projective resolution of the RG-module $RG \otimes_{RH} R$ of finite type

$$RG \otimes_{RH} Q_n \to \cdots \to RG \otimes_{RH} Q_0 \to RG \otimes_{RH} R \to 0. \tag{3.5.6}$$

Consider the RG-morphism $\phi \colon RG \otimes_{RH} R \to R$ induced by

$$\phi \colon RG \times R \to R, \qquad (s, r) \mapsto \epsilon(s) r, \tag{3.5.7}$$

where $\epsilon \colon RG \to R$ is the augmentation map, $\epsilon(\sum r_i g_i) = \sum r_i$. Since each of the RG-modules $RG \otimes_{RH} Q_i$ is projective, there are RG-morphisms $f_i \colon RG \otimes_{RH} Q_i \to P_i$

such that

$$\begin{array}{ccccccc} RG\otimes_{RH} Q_n & \longrightarrow & \cdots & \longrightarrow & RG\otimes_{RH} Q_0 & \longrightarrow & RG\otimes_{RH} R \\ \big\downarrow{\scriptstyle f_n} & & & & \big\downarrow{\scriptstyle f_0} & & \big\downarrow{\scriptstyle \phi} \\ P_n & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & R. \end{array} \tag{3.5.8}$$

is a commutative diagram, see [Bro94, pg.22, Lemma 7.4].

Let $M_* = (M_i)$ be the mapping cylinder of the chain map $f = (f_i)$ where f_i is the RG-morphism defined above for $0 \le i \le n$, f_{n+1} is the morphism $0 \to P_{n+1}$, and f_i is the morphism $0 \to 0$ for any other value of i.

Observe that

$$M_i = P_i \oplus (RG \otimes_{RH} Q_i) \oplus (RG \otimes_{RH} Q_{i-1})$$

for $1 \le i \le n$, $M_0 = P_0 \oplus (RG \otimes_{RH} Q_0) \oplus 0$, $M_{n+1} = P_{n+1} \oplus 0 \oplus (RG \otimes_{RH} Q_n)$, and $M_i = 0$ for any other value of i. Hence all M_i are finitely generated and projective.

Let $P_* = (P_i)$ be the chain complex induced by (3.5.5), where $P_i = 0$ for $i > n+1$ and $i < 0$. Observe that P_* is the target of the chain map f. Since P_* and M_* are chain homotopic,

$$0 \to M_{n+1} \to M_n \to \cdots \to M_0 \to R \to 0,$$

is a projective resolution of finite type of the trivial RG-module R.

Step 2. *Definition of the injective RH-morphisms $\imath_i\colon Q_i \to M_i$.*

We have the following commutative diagram of RH-modules

$$\begin{array}{ccccccc} Q_n & \longrightarrow & \cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 \\ \big\downarrow{\scriptstyle \tau_n} & & & & \big\downarrow{\scriptstyle \tau_1} & & \big\downarrow{\scriptstyle \tau_0} \\ RG\otimes_{RH} Q_n & \longrightarrow & \cdots & \longrightarrow & RG\otimes_{RH} Q_1 & \longrightarrow & RG\otimes_{RH} Q_0 \\ \big\downarrow{\scriptstyle \jmath_n} & & & & \big\downarrow{\scriptstyle \jmath_1} & & \big\downarrow{\scriptstyle \jmath_0} \\ M_n & \longrightarrow & \cdots & \longrightarrow & M_1 & \longrightarrow & M_0. \end{array} \tag{3.5.9}$$

where $\tau_k\colon Q_k \to RG \otimes_{RH} Q_k$ is the natural inclusion given by $q \mapsto e \otimes q$ (here e

denotes the identity element of G), and the vertical arrows $\jmath_i\colon RG \otimes_{RH} Q_i \to M_i$ are the natural inclusions. Then define

$$\imath_i = \jmath_i \circ \tau_i$$

for $0 \leq i \leq n$, and observe that they are injective RH-morphisms.

Step 3. *Verifying commutative diagram* (3.5.3).

In view of the commutative diagram (3.5.9), we only need to verify that if $H_0(Q)$ and $H_0(M)$ denote the cokernels of $Q_1 \to Q_0$ and $M_1 \to M_0$ respectively then the RH-morphism $H(\imath_0)\colon H_0(Q) \to H_0(M)$ induced by $\imath_0$ is an isomorphism.

Before the argument, we remark that this is not immediate, it depends on the choice of the RG-morphism f_0; the available choices for f_0 depend on the choice of the RG-morphism $\phi\colon RG \otimes_{RH} R \to R$; our choice is defined by (3.5.7).

Let $H_0(P)$ denote the cokernel of $P_1 \to P_0$. Let $\tau_{-1}\colon R \to RG \otimes_{RH} R$ be defined by $r \mapsto e \otimes r$ where e denotes the identity element of G. Then $\phi \circ \tau_{-1}$ is the identity map on R. It follows that the induced RH-morphism $H_0(f_0 \circ \tau_0)\colon H_0(Q) \to H_0(P)$ is an isomorphism. Since $\kappa\colon M_* \to P_*$ given by $(p, q, q') \mapsto p + f(q)$ is a chain homotopy equivalence, $H(\kappa_0)\colon H_0(M) \to H_0(P)$ is an isomorphism. Observe that $H(f_0 \circ \tau_0)$ equals $H(\kappa_0) \circ H(\imath_0)$ and hence $H(\imath_0)$ is an isomorphism.

Step 4. *The exact sequence* (3.5.4) *splits, and each S_i is a projective RH-module.*

This is immediate since $\imath_i\colon Q_i \to M_i$ is the inclusion of a direct summand of M_i as an RH-module. Since restriction of scalars preserves projectiveness, M_i is projective as an RH-module and hence S_i is projective as well. □

Proof of Theorem 3.2.6. Consider projective resolutions as (3.5.1) and (3.5.2) as well as RH-morphisms $\imath_i\colon Q_i \to M_i$ as described in Proposition 3.5.1.

Let $M_* = (M_i, \delta_i^M)$ denote the chain complex induced by (3.5.2), with the assumption that $M_i = 0$ for $i > n$ and $i < 0$. Analogously, let $Q_* = (Q_i, \delta_i^Q)$ be the chain complex induced by (3.5.1), with the assumption that $Q_i = 0$ for $i > n$ and $i < 0$. Observe that we are not using the modules Q_{n+1} and M_{n+1} in the definition of Q_* and M_*. Let S_* be the quotient chain complex M_*/Q_*. Consider the induced chain map $\imath = (\imath_i)\colon Q_* \to M_*$.

We use the following notation. The kernel of δ_n^Q is denoted by $Z_n(Q)$. The n-homology group of the complex Q_* is denoted by $H_n(Q)$. Analogous notation is used for the other chain complexes.

Step 1. *The induced sequence*

$$0 \longrightarrow Z_n(Q) \xrightarrow{\imath_n} Z_n(M) \longrightarrow Z_n(S) \longrightarrow 0 \tag{3.5.10}$$

is exact and satisfies

- $Z_n(Q)$ *is a finitely generated* RH*-module .*
- $Z_n(M)$ *is a finitely generated and projective* RG*-module.*
- $Z_n(Q)$ *is a direct summand of* $Z_n(M)$ *as an* RH*-module.*

Observe that $H_{n+1}(Q)$ and $H_{n-1}(Q)$ are both trivial. The short exact sequence of chain complexes of RH-modules

$$0 \longrightarrow Q_* \xrightarrow{\imath} M_* \longrightarrow S_* \longrightarrow 0 \tag{3.5.11}$$

induces a long exact sequence

$$0 \longrightarrow H_n(Q) \xrightarrow{\imath_n} H_n(M) \longrightarrow H_n(S) \longrightarrow 0 \tag{3.5.12}$$

which is precisely (3.5.10).

The RH-module $Z_n(Q)$ is finitely generated since Q_{n+1} is a finitely generated RH-module and δ_{n+1}^Q maps Q_{n+1} onto $Z_n(Q)$.

That $Z_n(M)$ is a finitely generated and projective RG-module follows from a direct application of Schanuel's lemma [Bro94, pg.193, Lemma 4.4] to the exact sequences (3.5.2) and

$$0 \to Z_n(M) \to M_n \to \cdots \to M_0 \to R \to 0. \tag{3.5.13}$$

Finally, to show that $Z_n(Q)$ is a direct summand of $Z_n(M)$ as an RH-module, we argue that that $Z_n(S)$ is projective RH-module. Consider the sequence of RH-modules induced by S_*

$$0 \to Z_n(S) \to S_n \to \cdots \to S_0 \to 0. \tag{3.5.14}$$

Note that this sequence is exact by observing the long exact sequence of homologies induced by (3.5.11). Indeed, $H_i(Q)$ and $H_i(M)$ are trivial for $0 < i < n$, and $H(\imath)\colon H_0(Q) \to H_0(M)$ is an isomorphism by (3.5.3). Since each S_i is projective, exactness of (3.5.14) implies that $Z_n(S)$ is projective.

Step 2. $FV^{n+1}_{H,R} \preceq FV^{n+1}_{G,R}$.

Let $\|\cdot\|_{M_n}$ and $\|\cdot\|_{Z_n(M)}$ denote filling norms on the RG-modules M_n and $Z_n(M)$ respectively. Similarly, let $\|\cdot\|_{Q_n}$ and $\|\cdot\|_{Z_n(Q)}$ denote filling norms on RH-modules Q_n and $Z_n(Q)$. For the map $Z_n(Q) \xrightarrow{\imath_n} Z_n(M)$, by Lemma 3.4.6 there exist integral parts K and K' of $Z_n(Q)$ and $Z_n(M)$ respectively, such that K maps into K' by the morphism $\imath$.

Since $\imath\colon Q_n \to M_n$ is the inclusion of a direct summand of M_n as an RH-module, and M_n is a projective RH-module, Lemma 3.3.9 implies that $\|\cdot\|_{M_n} \sim \|\cdot\|_{Q_n}$ on Q_n. In particular, there is a constant C_0 such that

$$\|\imath_n(\gamma)\|_{M_n} \leq C_0 \|\gamma\|_{Q_n}$$

for every $\gamma \in Q_n$.

By Step 1, $\imath_n\colon Z_n(Q) \to Z_n(M)$ is the inclusion of a direct summand of $Z_n(M)$ as an RH-module, and $Z_n(M)$ is a projective RH-module. Lemma 3.3.9 implies $\|\cdot\|_{Z_n(M)} \sim \|\cdot\|_{Z_n(Q)}$ on $Z_n(Q)$. Hence there is $C_1 > 0$ such that

$$\|\gamma\|_{Z_n(Q)} \leq C_1 \|\imath_n(\gamma)\|_{Z_n(M)}$$

for every $\gamma \in Z_n(M)$, and $\rho \circ \imath$ is identity on $Z_n(Q)$.

Let $k \in \mathbb{N}$ and $\gamma \in K \subseteq Z(Q_n)$ such that $\|\gamma\|_{Q_n} \leq k$. Then

$$\|\gamma\|_{Z_n(Q)} \leq C_1 \|\imath_n(\gamma)\|_{Z_n(M)} \leq C_1 \, \mathsf{FV}^{n+1}_{G,R}(\|\imath_n(\gamma)\|_{M_n}) \leq C_1 \, \mathsf{FV}^{n+1}_{G,R}(C_0 \|\gamma\|_{Q_n})$$

Therefore $\mathsf{FV}^{n+1}_{H,R}(k) \leq C_1 \, \mathsf{FV}^{n+1}_{G,R}(C_0 k)$ for every k. $\square$

Chapter 4

Subgroups of totally disconnected locally compact topological groups

4.1 Abstract

This article is part of the program of studying large-scale geometric properties of totally disconnected locally compact groups, TDLC-groups, by analogy with the theory for discrete groups. We provide a characterization of hyperbolic TDLC-groups, in terms of homological isoperimetric inequalities. This characterization is used to prove that, for hyperbolic TDLC-groups with rational discrete cohomological dimension ≤ 2, hyperbolicity is inherited by compactly presented closed subgroups. As a consequence, every compactly presented closed subgroup of the automorphism group $\mathrm{Aut}(X)$ of a negatively curved locally finite 2-dimensional building X is a hyperbolic TDLC-group, whenever $\mathrm{Aut}(X)$ acts with finitely many orbits on X. Examples where this result applies include hyperbolic Bourdon's buildings.

4.2 Introduction

A locally compact group G is said to be *totally disconnected* if the identity is its own connected component. For an arbitrary locally compact group, the connected component of the identity is always a closed normal subgroup with a totally disconnected quotient. Therefore, in principle, the study of locally compact groups can be reduced to the study of the two subclasses formed by connected locally compact groups and totally disconnected locally compact groups. By the celebrated solution to Hilbert's fifth problem, connected locally compact groups are inverse limits of Lie groups. However, such a thorough understanding has not been achieved for the totally disconnected counterpart.

Hereafter, we use TDLC-group as a shorthand for totally disconnected locally compact group. The class of TDLC-groups has been a topic of interest in the last three decades since the work of G. Willis [Wil94], and of M. Burger and S. Mozes [BM97].

Large-scale properties of a TDLC-group G can be addressed by investigating a family of quasi-isometric locally finite connected graphs which are known as Cayley-Abels graphs of G; see § 4.4.1 for the definition and further details. Therefore, the theory of TDLC-groups becomes amenable to many tools from geometric group theory (see [Bau07, BSW08, Möl02] for example) and the notion of hyperbolic group carries over to the realm of TDLC-groups.

The motivation for this work is to gain a better understanding of the interaction between the geometric properties of the TDLC-group G and its cohomological properties by analogy with the discrete case. An investigation of this type was initiated by Castellano and Weigel in [CW16b, Cas20] where the *rational discrete cohomology* for TDLC-groups has been introduced and the authors have shown that many well-known properties that hold for discrete groups can be transferred to the context of TDLC-groups (in some cases after substantial work).

For a TDLC-group G, the representation theory used in [CW16b] leans on the notion of *discrete* $\mathbb{Q}G$*-module*, that is a $\mathbb{Q}G$-module M such that the action $G \times M \to M$ is continuous when M carries the discrete topology. In the case that G is discrete, any $\mathbb{Q}G$-module is discrete. Because of the divisibility of $\mathbb{Q}$, the abelian category ${}_{\mathbb{Q}G}\mathbf{dis}$ of discrete $\mathbb{Q}G$-modules has enough projectives. As a consequence, the notions of *rational discrete cohomological dimension*, denoted by $\mathrm{cd}_{\mathbb{Q}}(G)$, and *type* FP_n can be introduced for every TDLC-group G in the category ${}_{\mathbb{Q}G}\mathbf{dis}$ (see §4.3.3 for the necessary background). This opens up the possibility of investigating TDLC-groups by imposing some cohomological finiteness conditions.

The main result of this article is a subgroup theorem for hyperbolic TDLC-groups of rational discrete cohomological dimension at most 2.

Theorem 4.2.1. Let G be a hyperbolic TDLC-group with $\mathrm{cd}_{\mathbb{Q}}(G) \leq 2$. Every compactly presented closed subgroup H of G is hyperbolic.

This theorem generalizes the following two results for discrete groups:

- Finitely presented subgroups of hyperbolic groups of integral cohomological dimension less than or equal to two are hyperbolic. This is a result of Gersten [Ger96b, Theorem 5.4] which can be recovered as a consequence of the inequality $\mathrm{cd}_{\mathbb{Q}}(\_) \leq \mathrm{cd}_{\mathbb{Z}}(\_)$.
- Finitely presented subgroups of hyperbolic groups of rational cohomological dimension less than or equal to two are hyperbolic. This is a recent result in [AMP20] which is the analogue of Theorem 4.2.1 in the discrete case.

We remark that Brady constructed an example of a discrete hyperbolic group of integral cohomological dimension three that contains a finitely presented subgroup that is not hyperbolic [Bra99]. Hence the the dimensional bound on the results stated above is sharp.

In the discrete case, a class of hyperbolic groups of rational cohomological dimension two is given by groups admitting finite presentations with certain small cancellation conditions. This is also the case for TDLC-groups for small cancellation quotients of amalgamated free products of profinite groups. We refer the reader to Section 4.8 for details on the following result.

Theorem 4.2.2. Let $A *_C B$ be the amalgamated free product of the profinite groups A, B over a common open subgroup C. Let R be a finite symmetrized subset of $A *_C B$ that satisfies the $C'(1/12)$ small cancellation condition. Then the quotient $G = (A *_C B)/\langle\langle R \rangle\rangle$ is a hyperbolic TDLC-group with $\mathrm{cd}_{\mathbb{Q}}(G) \leq 2$.

In the framework of discrete groups, it is a result of Gersten that type FP_2 (over $\mathbb{Z}$) subgroups of hyperbolic groups of integral cohomological dimension at most two are hyperbolic [Ger96b, Theorem 5.4]. We raise the following question:

Question 1. Does Theorem 4.2.1 remain true if H is of type FP_2 but not compactly presented?

It is well known that if X is a locally finite simplicial complex then the group of simplicial automorphisms $\mathrm{Aut}(X)$ endowed with the compact open topology is a

TDLC-group [Cam96, Theorem 2.1]. If, in addition, X admits a $CAT(-1)$ metric and $\mathrm{Aut}(X)$ acts with finitely many orbits on X, then $\mathrm{Aut}(X)$ is a hyperbolic TDLC-group with $\mathrm{cd}_{\mathbb{Q}}(\mathrm{Aut}(X)) \leq \dim(X)$.

Corollary 4.2.3. Let X be a locally finite 2-dimensional simplicial $CAT(-1)$-complex. If $\mathrm{Aut}(X)$ acts with finitely many orbits on X, then every compactly presented closed subgroup of $\mathrm{Aut}(X)$ is a hyperbolic TDLC-group.

A discrete version of Corollary 4.2.3 was proved in [HMP14, Corollary 1.5] using combinatorial techniques. There are different sources of complexes X satisfying the hypothesis of Corollary 4.2.3 and such that $\mathrm{Aut}(X)$ is a non-discrete TDLC-group. For example:

- Bourdon's building $I_{p,q}$, $p \geq 5$ and $q \geq 3$, is the unique simply connected polyhedral 2-complex such that all 2-cells are right-angled hyperbolic p-gons and the link of each vertex is the complete bipartite graph $K_{q,q}$. These complexes were introduced by Bourdon [Bou97]. The natural metric on $I_{p,q}$ is $CAT(-1)$ and $\mathrm{Aut}(I_{p,q})$ is non-discrete.

- For an integer k and a finite graph L, a (k, L)-complex is a simply connected 2-dimensional polyhedral complex such that all 2-dimensional faces are k-gons and the link of every vertex is isomorphic to the graph L. A result of Świątkowski [Świ98, Main Theorem (1)] provides sufficient conditions on the graph L guaranteeing that if $k \geq 4$ then $\mathrm{Aut}(X)$ is a non-discrete group for any (k, L)-complex X. It is a consequence of Gromov's link condition, that a (k, L)-complex admits a $CAT(-1)$-structure for any k sufficiently large.

In order to prove Theorem 4.2.1, we follow ideas from Gersten [Ger96b]. We introduce the concept of *weak n-dimensional linear isoperimetric inequality* for TDLC-groups, which is a homological analogue in higher dimensions of linear isoperimetric inequalities. Profinite groups are characterized as TDLC-groups satisfying the weak 0-dimensional linear isoperimetric inequality: see Section 4.5. The weak 1-dimensional linear isoperimetric inequality is called from here on the *weak linear isoperimetric inequality.* The following result generalizes for TDLC-groups a well-known characterization of hyperbolicity in the discrete case [Ger96b, Theorem 3.1].

Theorem 4.2.4. A compactly generated TDLC-group G is hyperbolic if and only if G is compactly presented and satisfies the weak linear isoperimetric inequality.

The property of satisfying the weak n-dimensional linear isoperimetric inequality is inherited by closed subgroups under some cohomological finiteness conditions.

Theorem 4.2.5. Let G be a TDLC-group of type FP_∞ with $\mathrm{cd}_{\mathbb{Q}}(G) = n+1$ that satisfies the weak n-dimensional linear isoperimetric inequality. Then every closed subgroup H of G of type FP_{n+1} satisfies the weak n-dimensional linear isoperimetric inequality.

The major part of the paper is devoted to the proof of Theorem 4.2.5. Our strategy borrows ideas from [AMP20, Ger96b, HMP16]. Some remarks:

- In the case that G is discrete, Theorem 4.2.5 is a consequence of [AMP20, Theorem 1.7].
- The arguments in [AMP20], where the authors replace some topological techniques from [Ger96b, HMP16] with algebraic counterparts, carry over to the TDLC class under the stronger assumption that the subgroup H is open, see Remark 4.6.2.
- Currently, for TDLC-groups, there is no well studied notion of n-dimensional homological Dehn function as the definitions available in the discrete case, see for example [ABDY13, HMP16]. In contrast to the arguments in [AMP20], we avoid the use of these objects and provide a straight forward argument.

It is a simple verification that Theorem 4.2.1 follows by Theorems 4.2.5 and 4.2.4.

Proof of the Theorem 4.2.1. Since G is hyperbolic, Theorem 4.2.4 implies that G satisfies the weak linear isoperimetric inequality. By Theorem 4.2.5, H also satisfies the weak linear isoperimetric inequality. We can then apply Theorem 4.2.4 again to conclude the proof. $\square$

Locally compact hyperbolic groups

The monograph [CdlH16] by de Cornulier and de la Harpe laid out the foundations of the study of locally compact groups from the perspective of geometric group theory. In this context, a locally compact group is hyperbolic if it has a continuous proper cocompact isometric action on some proper geodesic hyperbolic metric space [CCMT15]; this generalizes the classical definition in the discrete case as well as the definition in the class of totally disconnected locally compact groups used in the present article. By analogy with the discrete case, the asymptotic dimension provides a quasi-isometry invariant of locally compact compactly generated groups. The question below suggests a possible generalization of Theorem 4.2.1 for the larger class of locally compact hyperbolic groups.

Question 2. Let G be a locally compact hyperbolic group such that $\operatorname{asdim} G \leq 2$. Are compactly presented subgroups of G hyperbolic?

We conclude the introduction of the article by verifying that the the above question has a positive answer for discrete hyperbolic groups. The argument provides a blueprint to answer the question in the positive in class of hyperbolic TDLC-groups using Theorem 4.2.1.

Theorem 4.2.6. Let G be a discrete hyperbolic group such that $\operatorname{asdim} G \leq 2$. Then every finitely presented subgroup of G is hyperbolic.

Proof. The main result of [AMP20] states that if $\mathrm{cd}_{\mathbb{Q}} G \leq 2$, then any finitely presented subgroup of G is hyperbolic. Therefore it is enough to verify the inequality

$$\mathrm{cd}_{\mathbb{Q}} G \leq \operatorname{asdim} G.$$

This inequality relies on important work of Buyalo and Lebedeva [BL07] and Bestvina and Mess [BM91] as explained below.

It is a result of Buyalo and Lebedeva, [BL07, Theorem 6.4], that the asymptotic dimension of every cobounded, hyperbolic, proper, geodesic metric space X equals the topological dimension of its boundary at infinity plus 1,

$$\operatorname{asdim} X = \dim \partial_\infty X + 1.$$

On the other hand, for a compact metrizable space Y, there is a notion of cohomological dimension $\dim_R Y$ over a ring R. It is known that if $\dim Y < \infty$ then

$$\dim_{\mathbb{Q}} Y \leq \dim_{\mathbb{Z}} Y = \dim Y,$$

see [BM91] for definitions and references.

Let $\partial_\infty G$ denote the Gromov boundary of G. Recall that $\partial_\infty G$ is a compact metrizable space with finite topological dimension, see for example [KB02]. It follows that

$$\dim_{\mathbb{Q}} \partial_\infty G \leq \dim \partial_\infty G.$$

The work of Bestvina and Mess [BM91, Corollary 1.4] implies that if $\mathrm{cd}_{\mathbb{Q}} G < \infty$ then

$$\mathrm{cd}_{\mathbb{Q}} G = \dim_{\mathbb{Q}} \partial_\infty G + 1. \tag{4.2.1}$$

Since discrete hyperbolic groups admit finite dimensional models for the universal space for proper actions (Rips complexes with large parameter, see [MS02] or [HOP14]), it follows that $\mathrm{cd}_{\mathbb{Q}} G < \infty$. Therefore $\mathrm{cd}_{\mathbb{Q}} G \leq \operatorname{asdim} G$. □

Remark 4.2.7. We expect a positive answer to Question 2 for hyperbolic TDLC-groups. Indeed, to obtain a positive answer it is enough to verify the following statement generalizing work of Bestvina and Mess [BM91, Corollary 1.4]:

Let G be a hyperbolic TDLC-group. If $\mathrm{cd}_{\mathbb{Q}} G < \infty$ then $\mathrm{cd}_{\mathbb{Q}} G = \dim_{\mathbb{Q}} \partial_\infty G + 1$, where $\mathrm{cd}_{\mathbb{Q}} G$ is the rational discrete cohomological dimension.

Then the proof of Theorem 4.2.6 works in the TDLC case by using Theorem 4.2.1 and that hyperbolic TDLC-groups have finite rational discrete cohomological dimension, see Proposition 4.4.7. An attempt to generalize the work of Bestvina and Mess in [BM91] for TDLC-groups is currently work in progress by the second author, F.W. Pasini and T. Weigel. In the generality of locally compact groups, the authors are not aware of a cohomology theory for locally compact groups that allow to pursue the techniques of this article.

Organization

Preliminary definitions regarding TDLC-groups and rational discrete modules are given in Section 4.3. Then Section 4.4 consists of definitions and some preliminary results on Cayley-Abels graphs, compact presentability and hyperbolicity for TDLC-groups. Section 4.5 introduces the weak n-dimensional linear isoperimetric inequality. Section 4.6 is devoted to the proof of Theorem 4.2.5. Finally, Section 4.7 relates hyperbolicity and the weak linear isoperimetric inequality and contains the proof of

Theorem 4.2.4.

4.3 TDLC-groups and rational discrete G-modules

Throughout this section G always denotes a TDLC-group. Note that a TDLC-group is Hausdorff. Discrete groups are TDLC-groups. Profinite groups are precisely compact TDLC-groups [Ser02, Proposition 0]. A fundamental result about the structure of TDLC-groups is known as van Dantzig's Theorem:

Theorem 4.3.1 (van Dantzig's Theorem, [VD36])**.** The family of all compact open subgroups of a TDLC-group G forms a neighbourhood system of the identity element.

Note that every Hausdorff topological group admitting such a local basis is necessarily TDLC. Hence the conclusion of van Dantzig's Theorem characterizes TDLC-groups in the class of Hausdorff topological groups.

For example, the non-Archimedean local fields $\mathbb{Q}_p$ and $\mathbb{F}_q((t))$ admit, respectively, the following local basis at the identity element:

1. $\{p^n\mathbb{Z}_p \mid n \in \mathbb{N}\}$, where $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x| \leq 1\} = \{x \in \mathbb{Q}_p \mid |x| < p\}$ is compact and open;
2. $\{t^n\mathbb{F}_q[[t]] \mid n \in \mathbb{N}\}$, where the norm is defined by $q^{-ord(f)}$.

4.3.1 Rational discrete G-modules

Let $\mathbb{Q}$ denote the field of rational numbers, and let ${}_{\mathbb{Q}G}\mathbf{mod}$ be the category of abstract left $\mathbb{Q}G$-modules and their homomorphisms. A left $\mathbb{Q}G$-module M is said to be *discrete* if the stabilizer

$$G_m = \{g \in G \mid g \cdot m = m\},$$

of each element $m \in M$ is an open subgroup of G. Equivalently, the action $G \times M \to M$ is continuous when M carries the discrete topology. The full subcategory of ${}_{\mathbb{Q}G}\mathbf{mod}$ whose objects are discrete $\mathbb{Q}G$-modules is denoted by ${}_{\mathbb{Q}G}\mathbf{dis}$. It was shown in [CW16b] that ${}_{\mathbb{Q}G}\mathbf{dis}$ is an abelian category with enough injectives and projectives.

4.3.2 Permutation $\mathbb{Q}G$-modules in ${}_{\mathbb{Q}G}\mathbf{dis}$

Let Ω be a non-empty left G-set. For $\omega \in \Omega$ let G_ω denote the pointwise stabilizer. The G-set Ω is called *discrete* if all pointwise stabilizers are open subgroups of G, and Ω is called *proper* if all pointwise stabilizers are open and compact.

The $\mathbb{Q}$-vector space $\mathbb{Q}[\Omega]$ - freely spanned by a discrete G-set Ω - carries a canonical structure of discrete left $\mathbb{Q}G$-module called the *discrete permutation $\mathbb{Q}G$-module induced by* Ω.

Note that a discrete permutation $\mathbb{Q}G$-module in ${}_{\mathbb{Q}G}\mathbf{dis}$ is a coproduct

$$\mathbb{Q}[\Omega] \cong \coprod_{\omega \in \mathcal{R}} \mathbb{Q}[G/G_\omega],$$

in ${}_{\mathbb{Q}G}\mathbf{dis}$, where $\mathcal{R}$ is a set of representatives of the G-orbits in Ω, and Ω is a discrete G-set.

A *proper permutation $\mathbb{Q}G$-module* is a discrete $\mathbb{Q}G$-module of the form $\mathbb{Q}[\Omega]$ where Ω is a proper G-set.

A proper permutation $\mathbb{Q}G$-module is a projective object in ${}_{\mathbb{Q}G}\mathbf{dis}$; see [CW16b]. The arguments of this article rely on the following characterization of projective objects in ${}_{\mathbb{Q}G}\mathbf{dis}$, a non-trivial result that in particular relies on Maschke's theorem on irreducible representations of finite groups, and Serre's results on Galois cohomology.

Proposition 4.3.2 ([CW16b, Corollary 3.3])**.** Let G be a TDLC-group. A discrete $\mathbb{Q}G$-module M is projective in ${}_{\mathbb{Q}G}\mathbf{dis}$ if, and only if, M is a direct summand of a proper permutation $\mathbb{Q}G$-module in ${}_{\mathbb{Q}G}\mathbf{dis}$.

Throughout the article, we only consider resolutions consisting of discrete permutation $\mathbb{Q}G$-modules, and we refer to this type of resolutions as *permutation resolutions in* ${}_{\mathbb{Q}G}\mathbf{dis}$. Analogously, a resolution that consists only of proper permutation modules is called a *proper permutation resolution in* ${}_{\mathbb{Q}G}\mathbf{dis}$. When the category is clear from the context, we will omit the term "in ${}_{\mathbb{Q}G}\mathbf{dis}$".

4.3.3 Rational discrete homological finiteness

Following [CW16b], we say that a TDLC-group G is of *type* FP_n ($n \in \mathbb{N}$) if there exists a partial proper permutation resolution in ${}_{\mathbb{Q}G}\mathbf{dis}$

$$\mathbb{Q}[\Omega_n] \longrightarrow \mathbb{Q}[\Omega_{n-1}] \longrightarrow \cdots \longrightarrow \mathbb{Q}[\Omega_0] \longrightarrow \mathbb{Q} \longrightarrow 0 \tag{4.3.1}$$

of the trivial discrete $\mathbb{Q}G$-module $\mathbb{Q}$ of *finite type*, i.e., every discrete left G-set Ω_i is finite modulo G or equivalently $\mathbb{Q}[\Omega_i]$ is finitely generated. Type FP_n in this paper will always mean over $\mathbb{Q}$, though the definition generalizes to finite type proper permutation resolutions over discrete rings other than $\mathbb{Q}$, where the proper permutation modules are no longer projective in general – see for example [CC20]. The group G is of *type* FP_∞ if it is FP_n for every $n \in \mathbb{N}$. Notice that having type FP_0 is an empty condition for a TDLC-group G. On the other hand, having type FP_1 is equivalent to be compactly generated (see [CW16b, Proposition 5.3]) and compact presentation implies type FP_2.

The *rational discrete cohomological dimension* of G, $\mathrm{cd}_{\mathbb{Q}}(G) \in \mathbb{N} \cup \{\infty\}$, is defined to be the minimum n such that the trivial discrete $\mathbb{Q}G$-module $\mathbb{Q}$ admits a projective resolution

$$0 \longrightarrow P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \mathbb{Q} \longrightarrow 0 \tag{4.3.2}$$

in ${}_{\mathbb{Q}G}\mathbf{dis}$ of length n. The rational discrete cohomological dimension reflects structural information on a TDLC-group G. For example, G is profinite if and only if $\mathrm{cd}_{\mathbb{Q}}(G) = 0$.

By composing the notions above, one says that G is of *type* FP if

(i) G is of type FP_∞, and

(ii) $\mathrm{cd}_{\mathbb{Q}}(G) = d < \infty$.

For a TDLC-group G of type FP, the trivial left $\mathbb{Q}G$-module $\mathbb{Q}$ possesses a projective resolution $(P_\bullet, \partial_\bullet)$ which is finitely generated and concentrated in degrees 0 to d. It is not known whether $(P_\bullet, \partial_\bullet)$ can be assumed to be a proper permutation resolution of finite length.

4.3.4 Restriction of scalars

Let H be a closed subgroup of the TDLC-group G. It follows that H is a TDLC-group and in particular the category ${}_{\mathbb{Q}H}\mathbf{dis}$ is well defined. The restriction of scalars from $\mathbb{Q}G$-modules to $\mathbb{Q}H$-modules preserves discretness. In other words there is a well defined *restriction functor*

$$\mathrm{res}_H^G(\_)\colon {}_{\mathbb{Q}G}\mathbf{dis} \to {}_{\mathbb{Q}H}\mathbf{dis}, \tag{4.3.3}$$

obtained by restriction of scalars via the natural map $\mathbb{Q}H \hookrightarrow \mathbb{Q}G$. The restriction is an exact functor which maps projectives to projectives. Indeed, for every proper permutation $\mathbb{Q}G$-module $\mathbb{Q}[\Omega]$, the discrete $\mathbb{Q}H$-module $\mathrm{res}_H^G(\mathbb{Q}[\Omega])$ is still a proper permutation module in ${}_{\mathbb{Q}H}\mathbf{dis}$. To simplify notation, for a discrete $\mathbb{Q}G$-module M, we may write M for $\mathrm{res}_H^G(M)$ when the meaning is clear.

4.4 Cayley-Abels graphs, Compact presentability and Hyperbolicity

4.4.1 Compactly generated TDLC-groups and Cayley-Abels graphs

In this article a graph is a 1-dimensional simplicial complex, hence graphs are undirected, without loops, and without multiple edges between the same pair of vertices.

A locally compact group is said to be *compactly generated* if there exists a compact subset that algebraically generates the whole group.

Proposition 4.4.1. [KM08, Theorem 2.2] A TDLC-group G is compactly generated if and only if it acts vertex transitively with compact open vertex stabilizers on a locally finite connected graph Γ.

A graph with a G-action as in the proposition above is called a *Cayley-Abels graph* for G. In [KM08] these graphs are referred to as *rough Cayley graphs* but the notion of Cayley-Abels graph traces back to Abels [Abe72].

As soon as the compactly generated TDLC-group G is non-discrete, the G-action on a Cayley-Abels graph is never free. That is to say, the action always has non-trivial vertex stabilizers. Nevertheless, these large but compact stabilizers play an important role in the study of the cohomology of G: they give rise to proper permutation $\mathbb{Q}G$-modules.

A consequence of van Dantzig's Theorem is the following.

Proposition 4.4.2. For a TDLC-group G the following statements are equivalent:

1. G is compactly generated.
2. There exists a compact open subgroup K of G and a finite subset S of G such that $K \cup S$ generates G algebraically.
3. There exists a finite graph of profinite groups (A, Λ) with a single vertex, together with a continuous open surjective homomorphism $\phi\colon \pi_1(A, \Lambda, \Xi) \to G$ such that $\phi|_{\mathcal{A}_v}$ is injective for all $v \in \mathcal{V}(\Lambda)$.

Proof. Note that if C is a compact set generating G and K is a compact open subgroup of G then there a finite subset $S \subset G$ such that the collection of left cosets $\{sK | s \in S\}$ covers C. Hence, by van Dantzig's Theorem, (1) implies (2). To show that (2) implies (3), consider the graph of groups with a single vertex and an edge for each element of S. The vertex group is K, and each edge group is $K \cap K^s$ with morphisms the inclusion and conjugation by s: see [CW16b, Proposition 5.10, proof of (a)]. That (3) implies (1) is immediate since G is a quotient of the compactly generated TDLC-group $\pi_1(A, \Lambda, \Xi)$. □

Note that in the terminology of the third statement of the above proposition, a Cayley-Abels graph for G can be obtained by considering the quotient of the (topological realisation as a 1-dimensional simplicial complex of the) universal tree of (A, Λ) by the kernel of ϕ.

4.4.2 Quasi-isometry for TDLC-groups and Hyperbolicity.

The edge-path metric on a Cayley-Abels graph Γ of a TDLC-group G induces a left-invariant pseudo-metric on G, by pulling back the metric of the G-orbit of a vertex of Γ. In the following proposition, we denote this pseudo-metric by dist_Γ.

Following [CdlH16], an action of a topological group G on a (pseudo-) metric space X is *geometric* if it satisfies:

- (Isometric) The action is by isometries;
- (Cobounded) There is $F \subset X$ of finite diameter such that $\bigcup_{g \in G} gF = X$;
- (Locally bounded) For every $g \in G$ and bounded subset $B \subset X$ there is a neighborhood V of g in G such that VB is bounded in X; and
- (Metrically proper) The subset $\{g \in G\colon \operatorname{dist}_X(x, gx) \leq R\}$ is relatively compact in G for all $x \in X$ and $R > 0$.

The following version of the Švarc-Milnor Lemma is a consequence of work by Cornulier and de la Harpe on locally compact groups; see [CdlH16, Corollary 4.B.11 and Theorem 4.C.5].

Proposition 4.4.3. Let G be a TDLC-group, let X be a geodesic (pseudo-) metric space, and let $x \in X$. Suppose there exists a geometric action of G on X. Then there is a Cayley-Abels graph Γ for G such that the map between the pseudo-metric spaces

$$(G, \operatorname{dist}_\Gamma) \to (X, \operatorname{dist}_X), \qquad x \mapsto gx$$

is a quasi-isometry.

This proposition implies the following result from [KM08, Theorem 2.7].

Corollary 4.4.4. The Cayley-Abels graphs associated to a compactly generated TDLC-group are all quasi-isometric to each other.

This quasi-isometric invariance of Cayley-Abels graphs allows us to define geometric notions for compactly generated TDLC-groups such as ends, number of ends or growth, by considering quasi-isometric invariants of a Cayley-Abels graph associated to G.

Definition 4.4.1. A TDLC-group G is defined to be *hyperbolic* if G is compactly generated and some (hence any) Cayley-Abels graph of G is hyperbolic.

For an equivalent definition of hyperbolic TDLC-group using (standard) Cayley graphs over compact generating sets see [BMW12] for details.

4.4.3 Compactly presented TDLC-groups

A locally compact group is said to be *compactly presented* if it admits a presentation $\langle K \mid R\rangle$ where K is a compact subset of G and there is a uniform bound on the length of the relations in R. Observe that being compactly presented implies being compactly generated. There are also an equivalent definition of compact presentation [CW16b, § 5.8] based on van Dantzig's Theorem in the context of Proposition 4.4.2.

Corollary 4.4.5. [CdlH16] A TDLC-group G is compactly presented if and only if

1. there exists a finite graph of profinite groups (A, Λ) with a single vertex together with a continuous open surjective homomorphism $\phi\colon \pi_1(A, \Lambda, \Xi) \to G$ such that $\phi|_{\mathcal{A}_v}$ is injective for all $v \in \mathcal{V}(\Lambda)$, and

2. the kernel of ϕ is finitely generated as a normal subgroup.

Proof. Note that the if direction is immediate since $\pi_1(A, \Lambda, \Xi)$ is compactly presented. Indeed, a group presentation of $\pi_1(A, \Lambda, \Xi)$ has as generators the formal union of the vertex group and a finite number of elements corresponding to the edges of the graph. The set of relations consists of the multiplication table of the vertex group and the HNN-relations; note that all these relations have length at most four. Since the kernel of ϕ is finitely generated as a normal subgroup, it follows that G is compactly presented.

For the only if direction, since G is compactly presented, in particular it is compactly generated and hence there is a finite graph of profinite groups (A, Λ) with the required properties for (1). It remains to show that the kernel of ϕ is finitely generated as a normal subgroup. By [CW16b, Proposition 5.10(b)], $\ker(\phi)$ is a discrete subgroup of $\pi_1(A, \Lambda, \Xi)$. Since $\pi_1(A, \Lambda, \Xi)$ is compactly generated and G is compactly presented, [CdlH16, Proposition 8.A.10(2)] implies that $\ker(\phi)$ is compactly generated as a normal subgroup; by discreteness it follows that $\ker(\phi)$ is finitely generated as a normal subgroup. □

Proposition 4.4.6. A TDLC-group G is compactly presented if and only if there exists a simply connected cellular G-complex X with compact open cell stabilizers , finitely many G-orbits of cells of dimension at most 2, and such that any element of G fixing a cell setwise fixes it pointwise (no inversions).

A G-complex with the properties stated in the above proposition is called a *topological model of G of type $\mathbb{F}_2$*.

Proof of Proposition 4.4.6. The equivalence of compact presentability and the existence of a topological model for $\mathbb{F}_2$ follows from standard arguments. That compact presentability is a consequence of the existence of the topological model follows directly from [BH99, I.8, Theorem 8.10]; for compact presentability implying the existence of such a complex see for example [CC20, Proposition 3.4]. □

The following result is well known for discrete hyperbolic groups. The proof in [BH99, III.Γ Theorem 3.21] carries over for hyperbolic TDLC-groups by considering the Rips complex on a Cayley-Abels graph instead of the standard Cayley graph.

Proposition 4.4.7. Let G be a hyperbolic TDLC-group. Then G acts on a simplicial complex X such that:

1. X is finite dimensional, contractible and locally finite;
2. G acts simplicially, cell stabilizers are compact open subgroups, and there are finitely many G-orbits of cells.
3. G acts transitively on the vertex set of X.

In particular, the topological realization of the barycentric subdivision of X is a topological model for $\mathbb{F}_2$, and hence G is compactly presented.

For a topological model X of G of type $\mathbb{F}_2$, by standard techniques we may add cells to kill higher homotopy, and get a contractible G-complex X' on which G acts simplicially with compact open stabilizers . Then the assumption on cell stabilizers implies that the collection of i-cells of X' is a proper G-set and hence $C_i(X', \mathbb{Q})$ is a proper permutation $\mathbb{Q}G$-module. Since X' is contractible, the augmented chain complex $(C_\bullet(X', \mathbb{Q}), \partial_\bullet)$ is a projective resolution of $\mathbb{Q}$ in ${}_{\mathbb{Q}G}\mathbf{dis}$ and, since $X'^{(2)} = X^{(2)}$ has finitely many orbits of cells, the chain complex is finitely generated in degrees 0, 1 and 2. In particular compactly presented TDLC-groups have type FP_2 in ${}_{\mathbb{Q}G}\mathbf{dis}$.

4.5 Weak n-dimensional isoperimetric inequality

4.5.1 (Pseudo-)Norms on vector spaces

Given a vector space V over a subfield $\mathbb{F}$ of the complex numbers, a *pseudo-norm* on V is a nonnegative-valued scalar function $\|\_\| : V \to \mathbb{R}_+$ with the following properties:

(N1) (Subadditivity) $\|u+v\| \leq \|u\| + \|v\|$ for all $u, v \in V$;

(N2) (Absolute Homogeneity) $\|\lambda \cdot v\| = |\lambda| \, \|v\|$, for all $\lambda \in \mathbb{F}$ and $v \in V$.

A pseudo-norm $\|\_\|$ on a vector space V is said to be a *norm* if it satisfies the following additional property:

(N3) (Point-separation) $\|v\| = 0, v \in V \Rightarrow v = 0$.

Let $f\colon (V, \|\_\|_V) \to (W, \|\_\|_W)$ be a linear function between pseudo-normed vector spaces. We say that f is *bounded* if there exists a constant $C > 0$ such that $\|f(v)\|_W \leq C\,\|v\|_V$ for all $v \in V$. In such a case, we write $\|\_\|_W \preceq^f \|\_\|_V$ when the constant C is irrelevant. Two different norms $\|\_\|$ and $\|\_\|'$ on V are said to be *equivalent*, $\|\_\| \sim \|\_\|'$, if $\|\_\| \preceq^{\mathrm{id}} \|\_\|' \preceq^{\mathrm{id}} \|\_\|$. From here on the relation $\preceq^{\mathrm{id}}$ will be denoted as $\preceq$.

4.5.2 ℓ_1-norm on permutation $\mathbb{Q}G$-modules

Let $\mathbb{Q}[\Omega]$ be a permutation $\mathbb{Q}G$-module. In particular, $\mathbb{Q}[\Omega]$ is a $\mathbb{Q}$-vector space with linear basis Ω. Therefore, the nonnegative-valued function

$$\|\_\|_1^{\Omega} : \mathbb{Q}[\Omega] \to \mathbb{Q}_+, \qquad \sum_{\omega\in\Omega} \alpha_\omega \omega \mapsto \sum_{\omega\in\Omega} |\alpha_\omega|, \tag{4.5.1}$$

defines a norm on $\mathbb{Q}[\Omega]$. As usual, we shall refer to $\|\_\|_1^{\Omega}$ as the ℓ_1-*norm* on $\mathbb{Q}[\Omega]$. Notice that $\|\_\|_1^{\Omega}$ is G-equivariant.

Proposition 4.5.1. Let $\phi\colon \mathbb{Q}[\Omega] \to \mathbb{Q}[\Omega']$ be a morphism of finitely generated permutation $\mathbb{Q}G$-modules. Then $\|\_\|_1^{\Omega'} \preceq^{\phi} \|\_\|_1^{\Omega}$.

Proof. This is a consequence of the G-invariance of the ℓ_1-norm and the fact that the modules are finitely generated. Indeed, the morphism ϕ is described by a finite matrix $A = (a_{ij})$ with entries in $\mathbb{Q}G$. Consider the ℓ_1-norm $\|\_\|_1$ on $\mathbb{Q}G$ and let $C = \max \|a_{ij}\|$. Then $\|\phi(x)\|_1^{\Omega'} \leq C \|x\|_1^{\Omega}$ for every $x \in \mathbb{Q}[\Omega]$. □

The above proposition will be used for discrete permutation modules over $\mathbb{Q}G$.

4.5.3 Filling pseudo-norms on discrete $\mathbb{Q}G$-modules

Let M be a finitely generated discrete $\mathbb{Q}G$-module. Since ${}_{\mathbb{Q}G}\mathbf{dis}$ has enough projectives, there exists a finitely generated proper permutation $\mathbb{Q}G$-module $\mathbb{Q}[\Omega]$ mapping onto M, that is, $\mathbb{Q}[\Omega] \overset{\partial}{\twoheadrightarrow} M$ and G acts on Ω with compact open stabilizers and finitely many orbits. The *filling pseudo-norm* $\|\_\|_\partial$ *on* M *induced by* ∂ is defined as

$$\|m\|_\partial = \inf\{\|x\|_1^{\Omega} \mid x \in \mathbb{Q}[\Omega], \partial(x) = m\}. \tag{4.5.2}$$

One easily verifies that $\|\_\|_\partial$ is subadditive and absolutely homogeneous. Note that

$$\|\_\|_\partial \preceq^{\partial} \|\_\|_1^{\Omega}. \tag{4.5.3}$$

It is an observation that an ℓ_1-norm on a finitely generated discrete permutation G-module $\mathbb{Q}[\Omega]$ is equivalent to a filling norm.

Proposition 4.5.2. Morphisms between finitely generated discrete $\mathbb{Q}G$-modules are bounded with respect to filling pseudo-norms.

Proof. Let $f : M \to N$ be a morphism of finitely generated discrete $\mathbb{Q}G$-modules. Since M and N are both finitely generated in ${}_{\mathbb{Q}G}\mathbf{dis}$, there exist morphisms $\mathbb{Q}[\Omega_1] \overset{\partial_1}{\twoheadrightarrow} M$ and $\mathbb{Q}[\Omega_2] \overset{\partial_2}{\twoheadrightarrow} N$ such that each $\mathbb{Q}[\Omega_i]$ is a finitely generated proper permutation module. By the universal property of $\mathbb{Q}[\Omega_1]$ as a projective object, there is $\phi\colon \mathbb{Q}[\Omega_1] \to \mathbb{Q}[\Omega_2]$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{Q}[\Omega_1] \arrow[r, dashed, "\phi"] \arrow[d, two heads, "\partial_1"'] & \mathbb{Q}[\Omega_2] \arrow[d, two heads, "\partial_2"] \\
M \arrow[r, "f"'] & N
\end{tikzcd}$$

For any $m \in M$ and any $\varepsilon > 0$, let $x_m \in \mathbb{Q}[\Omega_1]$ such that $\partial_1(x_m) = m$ and $\|x_m\|_1^{\Omega_1} \preceq^{\partial_1} \|m\|_{\partial_1} + \varepsilon$. Since $f(m) = \partial_2(\phi(x_m))$, one has

$$\begin{aligned}
\|f(m)\|_{\partial_2} &\preceq^{\partial_2} \|\phi(x_m)\|_1^{\Omega_2} && \text{by (4.5.3),} \\
&\preceq^{\phi} \|x_m\|_1^{\Omega_1} && \text{by Proposition 4.5.1,} \\
&\preceq^{\partial_1} \|m\|_{\partial_1} + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, we deduce $\|\_\|_{\partial_2} \preceq^f \|\_\|_{\partial_1}$. □

By considering the identity function on a finitely generated discrete $\mathbb{Q}G$-module M, the previous proposition implies:

Corollary 4.5.3. Let G be a TDLC-group. Any two filling pseudo-norms on a finitely generated discrete $\mathbb{Q}G$-module M are equivalent.

In particular, all the filling pseudo-norms on a finitely generated proper permutation $\mathbb{Q}G$-module $\mathbb{Q}[\Omega]$ are equivalent to $\|\_\|_1^{\Omega}$, and therefore they are all norms.

The former implies that each finitely generated discrete $\mathbb{Q}G$-module M admits a unique filling pseudo-norm up to equivalence. Therefore, by abuse of notation, we denote by $\|\_\|_M$ any filling pseudo-norm of M and we refer to $\|\_\|_M$ as the *filling pseudo-norm of* M.

4.5.4 Undistorted submodules

Let M be a discrete $\mathbb{Q}G$-module with a norm $\|\_\|$ and let N be a finitely generated discrete $\mathbb{Q}G$-submodule of M. Then N is said to be *undistorted with respect to* $\|\_\|$ if the restriction of $\|\_\|$ to N is equivalent to a filling norm on N. In the case that M is finitely generated and N is undistorted with respect to the filling norm $\|{-}\|_M$ we shall simply say that N is *undistorted* in M.

We note that in general it is not the case that finitely generated submodules of M are undistorted; we refer the reader to Section 4.7 for counter-examples.

Proposition 4.5.4. Let G be a TDLC-group. The filling pseudo-norm $\|{-}\|_P$ of a finitely generated projective discrete $\mathbb{Q}G$-module P is a norm. Moreover, if P is a direct summand of a finitely generated proper permutation module $\mathbb{Q}[\Omega]$, then P is undistorted in $\mathbb{Q}[\Omega]$.

Proof. Let $\mathbb{Q}[\Omega]$ be a finitely generated proper permutation module such that P is a direct summand of $\mathbb{Q}[\Omega]$; see Proposition 4.3.2. Let $\iota\colon P \to \mathbb{Q}[\Omega]$ be the inclusion and let $\pi\colon \mathbb{Q}[\Omega] \twoheadrightarrow P$ be the projection such that $\pi \circ \iota = \mathrm{id}_P$. Proposition 4.5.2 implies $\|{-}\|_1^\Omega \preceq^\iota \|{-}\|_P$ and $\|{-}\|_P \preceq^\pi \|{-}\|_1^\Omega$ on P. The former inequality implies that $\|{-}\|_P$ is a norm, and both of them imply that $\|{-}\|_P \sim \|{-}\|_1^\Omega$ on P. $\square$

More generally, this argument shows that a direct summand of any finitely generated discrete $\mathbb{Q}G$-module, with the filling norm, is undistorted.

We conclude the section with a technical result about bounded morphisms that will be used later and relies on the proof of the previous proposition.

Proposition 4.5.5. Let G be a TDLC-group and H a closed subgroup of G. Let M be a finitely generated and projective $\mathbb{Q}G$-module in ${}_{\mathbb{Q}G}\mathbf{dis}$ with filling norm $\|{-}\|_M$. Regard M as a $\mathbb{Q}H$-module via restriction, and suppose that N is a finitely generated direct summand of M in ${}_{\mathbb{Q}H}\mathbf{dis}$. Then N is an undistorted $\mathbb{Q}H$-module of M with respect to the norm $\|{-}\|_M$.

Proof. The $\mathbb{Q}H$-module N is projective since in the context we are working, restriction of M is projective and hence N is a direct summand of a projective $\mathbb{Q}H$-module.

By Proposition 4.5.4, M can be assumed to be a finitely generated proper permutation $\mathbb{Q}G$-module $\mathbb{Q}[\Omega]$. Note that the restriction of $\mathbb{Q}[\Omega]$ is a proper permutation $\mathbb{Q}[H]$-module.

Since N is finitely generated, there exists an H-subset Σ of Ω such that Σ/H is finite and N is a $\mathbb{Q}H$-submodule of $\mathbb{Q}[\Sigma]$. Since N and $\mathbb{Q}[\Sigma]$ are direct summands of

$\mathbb{Q}[\Omega]$ as $\mathbb{Q}H$-modules, it follows that N is a direct summand of the finitely generated proper permutation $\mathbb{Q}H$-module $\mathbb{Q}[\Sigma]$.

Proposition 4.5.4 implies that the pseudo-norm $\|{-}\|_N$ is a norm and $\|{-}\|_N \sim \|{-}\|_1^{\Sigma}$ on N. Since $\|{-}\|_1^{\Sigma} = \|{-}\|_1^{\Omega}$ on $\mathbb{Q}[\Sigma]$, it follows that $\|{-}\|_N \sim \|{-}\|_1^{\Omega}$ on the elements of N. □

4.5.5 Weak n-dimensional linear isoperimetric inequality

Let G be a TDLC-group of type FP_{n+1}. Then there exists a partial proper permutation resolution

$$\mathbb{Q}[\Omega_{n+1}] \xrightarrow{\delta_{n+1}} \mathbb{Q}[\Omega_n] \xrightarrow{\delta_n} \cdots \longrightarrow \mathbb{Q}[\Omega_1] \xrightarrow{\delta_1} \mathbb{Q}[\Omega_0] \longrightarrow \mathbb{Q} \longrightarrow 0 \tag{4.5.4}$$

of finite type, i.e. it consists of finitely generated discrete $\mathbb{Q}G$-modules. We say that G satisfies the *weak n-dimensional linear isoperimetric inequality* if $\ker(\delta_n)$ is an undistorted submodule of $\mathbb{Q}[\Omega_n]$. The special case for $n = 1$ is referred as the *weak linear isoperimetric inequality*.

Note that, by Proposition 4.5.2, $\|{-}\|_1^{\Omega_n} \preceq^{\imath} \|{-}\|_{\ker(\partial_n)}$ where $\imath\colon \ker(\partial_n) \to \mathbb{Q}[\Omega_n]$ is the inclusion. Hence, the weak n-dimensional linear isoperimetric inequality is equivalent to the existence of a constant $C > 0$ such that $\|{-}\|_{\ker(\partial_n)} \leq C\,\|{-}\|_1^{\Omega_n}$ on $\ker(\partial_n)$.

The proof of the following proposition is an adaption of the proof of [HMP16, Theorem 3.5] that we have included for the reader's convenience.

Proposition 4.5.6. For a TDLC-group G of type FP_{n+1}, the property of satisfying the weak linear n-dimensional isoperimetric inequality is independent of the choice of the proper permutation resolution of finite type in ${}_{\mathbb{Q}G}\mathbf{dis}$.

Proof. Let $(\mathbb{Q}[\Omega_i], \partial_i), (\mathbb{Q}[\Lambda_i], \delta_i)$ be a pair of proper permutation resolutions of $\mathbb{Q}$ which contain finitely generated modules for degrees $i = 0, \ldots, n+1$. Suppose G satisfies the weak n-dimensional linear isoperimetric inequality with respect to the resolution $(\mathbb{Q}[\Lambda_i], \delta_i)$. Hence there is $C > 0$ such that

$$\|x\|_{\ker(\delta_n)} \leq C\,\|x\|_1^{\Lambda_n}. \tag{4.5.5}$$

for all $x \in \ker(\delta_n)$.

Since any two projective resolutions of $\mathbb{Q}$ are chain homotopy equivalent, there exist chain maps $f\colon (\mathbb{Q}[\Omega_i], \partial_i) \to (\mathbb{Q}[\Lambda_i], \delta_i))$ and $g\colon (\mathbb{Q}[\Lambda_i], \delta_i) \to (\mathbb{Q}[\Omega_i], \partial_i)$, and a 1-differential $h\colon (\mathbb{Q}[\Omega_i], \partial_i) \to (\mathbb{Q}[\Omega_i], \partial_i)$ such that

$$\partial_{i+1} \circ h_i + h_{i-1} \circ \partial_i = g_i \circ f_i - \mathsf{Id}. \tag{4.5.6}$$

Diagrammatically, one has

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{Q}[\Omega_{n+1}] \arrow[r, "\partial_{n+1}"'] \arrow[d, shift right, "f_{n+1}"'] & \mathbb{Q}[\Omega_n] \arrow[r, "\partial_n"'] \arrow[l, bend right, "h_n"'] \arrow[d, shift right, "f_n"'] & \mathbb{Q}[\Omega_{n-1}] \arrow[r] \arrow[l, bend right, "h_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] & \cdots \\ \cdots \arrow[r] & \mathbb{Q}[\Lambda_{n+1}] \arrow[r, "\delta_{n+1}"'] \arrow[u, shift right, "g_{n+1}"'] & \mathbb{Q}[\Lambda_n] \arrow[r, "\delta_n"'] \arrow[u, shift right, "g_n"'] & \mathbb{Q}[\Lambda_{n-1}] \arrow[r] \arrow[u, shift right, "g_{n-1}"'] & \cdots \end{tikzcd} \tag{4.5.7}$$

Since g_{n+1}, f_n and h_n are morphisms between finitely generated discrete $\mathbb{Q}G$-modules, Proposition 4.5.2 applies and, therefore, the constant C defined above can be assumed to satisfy:

(D1) $\|g_{n+1}(\lambda)\|_1^{\Omega_{n+1}} \leq C\,\|\lambda\|_1^{\Lambda_{n+1}}$, for all $\lambda \in \mathbb{Q}[\Lambda_{n+1}]$;

(D2) $\|f_n(\omega)\|_1^{\Lambda_n} \leq C\,\|\omega\|_1^{\Omega_n}$, for all $\omega \in \mathbb{Q}[\Omega_n]$; and

(D3) $\|h_n(\omega)\|_1^{\Omega_{n+1}} \leq C\,\|\omega\|_1^{\Omega_n}$, for all $\omega \in \mathbb{Q}[\Omega_n]$.

We prove below that that there is a constant $D > 0$ such that for any $\alpha \in \ker(\partial_n)$ and $\epsilon > 0$

$$\|\alpha\|_{\ker(\partial_n)} \leq D\|\alpha\|_1^{\Omega_n} + D\epsilon.$$

Then it follows that G satisfies the weak n-dimensional linear isoperimetric inequality with respect to the resolution $(\mathbb{Q}[\Omega_i], \partial_i)$ by letting $\epsilon \to 0$.

Let $\alpha \in \ker(\partial_n)$ and $\epsilon > 0$. By the diagram (4.5.7), it follows that $f_n(\alpha) \in \ker(\delta_n) = \delta_{n+1}(\mathbb{Q}[\Lambda_{n+1}])$. Since $\mathbb{Q}[\Lambda_{n+1}]$ is finitely generated, we can consider the filling-norm $\|\_\|_{\ker(\delta_n)}$ to be induced by δ_{n+1}. Therefore, by the definition of the filling norm $\|\_\|_{\ker(\delta_n)}$ there is $\beta \in \mathbb{Q}[\Lambda_{n+1}]$ such that $\delta_{n+1}(\beta) = f_n(\alpha)$ and

$$\|\beta\|_1^{\Lambda_{n+1}} \leq \|f_n(\alpha)\|_{\ker(\delta_n)} + \epsilon. \tag{4.5.8}$$

By evaluating α in Equation 4.5.6, we can write

$$\begin{aligned} \alpha &= g_n(f_n(\alpha)) - \partial_{n+1}(h_n(\alpha)) \\ &= g_n(\delta_{n+1}(\beta)) - \partial_{n+1}(h_n(\alpha)) \\ &= \partial_{n+1}\left(g_{n+1}(\beta) - h_n(\alpha)\right). \end{aligned} \tag{4.5.9}$$

Hence

$$\begin{aligned} \|\alpha\|_{\ker(\partial_n)} &\leq \|g_{n+1}(\beta) - h_n(\alpha)\|_1^{\Omega_{n+1}} && \text{by (4.5.9) and definition of filling norm} \\ &\leq \|g_{n+1}(\beta)\|_1^{\Omega_{n+1}} + \|h_n(\alpha)\|_1^{\Omega_{n+1}} \\ &\leq C\|\beta\|_1^{\Lambda_{n+1}} + C\|\alpha\|_1^{\Omega_n} && \text{by inequalities (D1) and (D3)} \\ &\leq C\|f_n(\alpha)\|_{\ker(\delta_n)} + C\epsilon + C\|\alpha\|_1^{\Omega_n} && \text{by inequality (4.5.8)} \\ &\leq C^2\|f_n(\alpha)\|_1^{\Lambda_n} + C\|\alpha\|_1^{\Omega_n} + C\epsilon && \text{by inequality (4.5.5)} \\ &\leq C^3\|\alpha\|_1^{\Omega_n} + C\|\alpha\|_1^{\Omega_n} + C\epsilon && \text{by inequality (D2).} \quad \square \end{aligned}$$

4.5.6 Weak 0-Dimensional Linear Isoperimetric Inequality and Profinite Groups

As previously mentioned, a group is profinite if and only if it is a compact TDLC-group [Ser02, Proposition 0]. The following statement is a simple application of the definitions of this section.

Proposition 4.5.7. Let G be a TDLC-group. Then G is compact if and only if it is compactly generated and satisfies a weak 0-dimensional linear isoperimetric inequality.

The only if direction of the proposition is immediate. Indeed, if G is a compact TDLC-group, then the trivial G-module $\mathbb{Q}$ is projective in ${}_{\mathbb{Q}G}\mathbf{dis}$. In this case, one can read the weak 0-dimensional isoperimetric inequality from the resolution $0 \to \mathbb{Q} \to \mathbb{Q} \to 0$.

For the rest of the section, suppose that G is a TDLC-group satisfying a weak 0-dimensional linear isoperimetric inequality. Let Γ be a Cayley-Abels graph of G, let dist be the combinatorial path metric on the set of vertices V of Γ, and let E denote the set of edges of Γ. In order to prove that G is profinite, it is enough to show that V is finite.

Choose an orientation for each edge of Γ and consider the augmented rational cellular chain complex of Γ,

$$\mathbb{Q}[E] \xrightarrow{\delta} \mathbb{Q}[V] \xrightarrow{\varepsilon} \mathbb{Q} \to 0.$$

Since Γ is Cayley-Abels graph, this is a partial proper permutation resolution.

Following ideas in [FMP18], define a partial order $\preceq$ on $\mathbb{Q}[E]$ as follows. For $\nu, \mu \in \mathbb{Q}[E]$, $\nu = \sum_{e\in E} t_e e$ and $\mu = \sum_{e\in E} s_e e$, then $\nu \preceq \mu$ if and only if $t_e^2 \leq t_e s_e$ for every $e \in E$. Observe that if $\nu \preceq \mu$ then $\|\mu\|_1^E = \|\mu - \nu\|_1^E + \|\nu\|_1^E$; in particular $\|\nu\|_1^E \leq \|\mu\|_1^E$. An element $\nu \in \mathbb{Q}[E]$ is called integral if $t_e \in \mathbb{Z}$ for each e. Define analogously $\preceq$ on $\mathbb{Q}[V]$.

Lemma 4.5.8. Suppose that $\mu \in \mathbb{Q}[E]$ is integral and $\delta(\mu) = m(v-u)$ where $u, v \in V$ and m is a positive integer. Then there is an integral element $\nu \in \mathbb{Q}[E]$ such that $\delta(\nu) = v - u$ and $\nu \preceq \mu$ and $\|\nu\|_1^E \geq \operatorname{dist}(u, v)$.

Sketch of the proof. Suppose $\mu = \sum_{e\in E} s_e e$. Consider a directed multigraph Ξ (multiple edges between distinct vertices are allowed) with vertex set V and such that for each $e \in E$ if $s_e \geq 0$ then there are $|s_e|$ edges from a to b where $\delta(e) = b - a$; and if $s_e < 0$ then there are $|s_e|$ from b to a. The degree sum formula for directed graphs implies that u and v are in the same connected component of Ξ. It is an exercise to show that there is a directed path γ from u to v in Ξ that can be assumed to be injective on vertices. The path γ induces an element $\nu \in \mathbb{Q}[E]$ such that if $\nu = \sum_{e\in E} t_e e$ then $t_e = \pm 1$ and $\nu \preceq \mu$. Moreover γ induces a path in Γ from u to v and hence $\|\nu\|_1^E \geq \operatorname{dist}(u, v)$. $\square$

Suppose, for a contradiction, that V is an infinite set. Fix $v_0 \in V$. For every $n \in \mathbb{N}$, let $v_n \in V$ such that $\operatorname{dist}(v_0, v_n) \geq n$. Note that such a vertex v_n always exists since Γ is locally finite and connected. Let $\alpha_n = v_n - v_0$ and observe that $\alpha_n \in \ker(\varepsilon)$ and $\|\alpha_n\|_1^V = 2$. We will show that $\|\alpha_n\|_{\ker(\varepsilon)} \geq n$ for every n, and hence G cannot satisfy a weak 0-dimensional linear isoperimetric inequality. Fix $n \in \mathbb{N}$, and let $\mu = \sum_{e\in E} s_e e \in \mathbb{Q}[E]$ such that $\delta(\mu) = \alpha_n = v_n - v_0$.Then there is $m \in \mathbb{N}$ such that $m\mu$ is integral. Since $\delta(m\mu) = m(v_n - v_0)$, Lemma 4.5.8 implies that there is $\nu_1 \in \mathbb{Q}[E]$ such that $\delta(\nu_1) = v_n - v_0$ and $\nu_1 \preceq m\mu$ and $\|\nu\|_1 \geq \operatorname{dist}(v_0, v_n)$. Let

$\mu_1 = m\mu - \nu_1$ and note that μ_1 is integral, $\delta(\mu_1) = (m-1)(v_n - v_0)$, and

$$\|m\mu\|_1^E = \|\mu_1\|_1^E + \|\nu_1\|_1^E \geq \|\mu_1\|_1^E + \mathrm{dist}(v_0, v_n).$$

An induction argument on m then proves that $\|m\mu\|_1^E \geq m\,\mathrm{dist}(v_0, v_n)$ and hence $\|\mu\|_1^E \geq \mathrm{dist}(v_0, v_n)$. Since μ was an arbitrary element such that $\delta(\mu) = \alpha_n$, it follows tha that $\|\alpha_n\|_{\ker\delta} \geq \mathrm{dist}(v_0, v_n) \geq n$.

4.6 Proof of Subgroup Theorem

The proof of the theorem relies on the following lemma. Let G be a TDLC-group of type FP_n and H a closed subgroup of G of type FP_n.

Lemma 4.6.1. There are partial proper permutation resolutions

$$\mathbb{Q}[\Omega_n] \xrightarrow{\delta_n} \mathbb{Q}[\Omega_{n-1}] \to \cdots \to \mathbb{Q}[\Omega_0] \to \mathbb{Q} \to 0,$$

$$\mathbb{Q}[\Sigma_n] \xrightarrow{\partial_n} \mathbb{Q}[\Sigma_{n-1}] \to \cdots \to \mathbb{Q}[\Sigma_0] \to \mathbb{Q} \to 0$$

of $\mathbb{Q}$ in ${}_{\mathbb{Q}H}\mathbf{dis}$ and ${}_{\mathbb{Q}G}\mathbf{dis}$ respectively, satisfying the following properties.

1. $\Omega_0, \ldots, \Omega_n$ are finitely generated H-sets;

2. $\Sigma_0, \ldots, \Sigma_n$ are finitely generated G-sets;

3. restricting the G-action on each Σ_i to H, Ω_i is an H-subset of Σ_i via $\iota_i : \Omega_i \to \Sigma_i$;

4. the diagram

$$\begin{tikzcd}
\ker(\delta_n) \arrow[r] \arrow[d] & \mathbb{Q}[\Omega_n] \arrow[r, "\delta_n"] \arrow[d, "\mathbb{Q}[\iota_n]"] & \cdots \arrow[r] & \mathbb{Q}[\Omega_0] \arrow[r, "\delta_0"] \arrow[d, "\mathbb{Q}[\iota_0]"] & \mathbb{Q} \arrow[r] \arrow[d, equal] & 0 \\
\ker(\partial_n) \arrow[r] & \mathbb{Q}[\Sigma_n] \arrow[r, "\partial_n"] & \cdots \arrow[r] & \mathbb{Q}[\Sigma_0] \arrow[r, "\partial_0"] & \mathbb{Q} \arrow[r] & 0
\end{tikzcd}$$

   of $\mathbb{Q}[H]$-modules commutes;

5. $\mathrm{coker}(\ker(\delta_n) \to \ker(\partial_n))$ is a projective $\mathbb{Q}H$-module.

Proof. Take a partial proper permutation resolution

$$\mathbb{Q}[\Sigma_n] \xrightarrow{\partial_n} \mathbb{Q}[\Sigma_{n-1}] \to \cdots \to \mathbb{Q}[\Sigma_0] \to \mathbb{Q} \to 0$$

of $\mathbb{Q}$ in ${}_{\mathbb{Q}G}\mathbf{dis}$. We construct the required resolution

$$\mathbb{Q}[\Omega_n] \xrightarrow{\delta_n} \mathbb{Q}[\Omega_{n-1}] \to \cdots \to \mathbb{Q}[\Omega_0] \to \mathbb{Q} \to 0$$

in ${}_{\mathbb{Q}H}\mathbf{dis}$ by induction on n. So suppose we have already constructed a diagram

$$\begin{tikzcd}
\ker(\delta_{n-1}) \arrow[r] \arrow[d] & \mathbb{Q}[\Omega_{n-1}] \arrow[r, "\delta_{n-1}"] \arrow[d, "\mathbb{Q}[\iota_{n-1}]"] & \cdots \arrow[r] & \mathbb{Q}[\Omega_0] \arrow[r, "\delta_0"] \arrow[d, "\mathbb{Q}[\iota_0]"] & \mathbb{Q} \arrow[r] \arrow[d, equal] & 0 \\
\ker(\partial_{n-1}) \arrow[r] & \mathbb{Q}[\Sigma_{n-1}] \arrow[r, "\partial_{n-1}"] & \cdots \arrow[r] & \mathbb{Q}[\Sigma_0] \arrow[r, "\partial_0"] & \mathbb{Q} \arrow[r] & 0
\end{tikzcd}$$

satisfying the conditions for $n-1$ (this is trivial for the base case $n = 0$).

Write ι for the induced map $\ker(\delta_{n-1}) \to \ker(\partial_{n-1})$; by hypothesis, there is a map $\pi : \ker(\partial_{n-1}) \to \ker(\delta_{n-1})$ such that $\pi\iota$ is the identity on $\ker(\delta_{n-1})$. Since H has type FP_n, $\ker(\delta_{n-1})$ is finitely generated; pick a finite generating set $x_1, \ldots, x_k$ and pick a preimage y_i of each element x_i in $\mathbb{Q}[\Sigma_n]$, via the map $\mathbb{Q}[\Sigma_n] \xrightarrow{\partial_n} \ker(\partial_{n-1}) \xrightarrow{\pi} \ker(\delta_{n-1})$. Each y_i is a finite sum $\sum_{j=1}^{j_i} a_{ij}\alpha_{ij}$ with $\alpha_{ij} \in \Sigma_n$ and $a_{ij} \in \mathbb{Q}$. Now let Ω_n be the (finitely generated) H-subset of Σ_n generated by the α_{ij}. We get an induced map $\pi\partial_n\mathbb{Q}[\iota_n] : \mathbb{Q}[\Omega_n] \to \ker(\delta_{n-1})$ extending the commutative diagram as required; it only remains to check condition 5.

To see this, consider the following commutative diagram in ${}_{\mathbb{Q}H}\mathbf{dis}$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker(\delta_n) \arrow[r] \arrow[d, "\iota'"] & \mathbb{Q}[\Omega_n] \arrow[r, "\delta_n"] \arrow[d, "\mathbb{Q}[\iota_n]"] & \ker(\delta_{n-1}) \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & \ker(\partial_n) \arrow[r] \arrow[d] & \mathbb{Q}[\Sigma_n] \arrow[r, "\partial_n"] \arrow[d] & \ker(\partial_{n-1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{coker}(\iota') \arrow[r] \arrow[d] & \operatorname{coker}(\mathbb{Q}[\iota_n]) \arrow[r] \arrow[d] & \operatorname{coker}(\iota) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

Note that the diagram consists of exact rows and exact columns. Since $\mathbb{Q}[\Omega_n^H]$ is a direct summand of $\mathbb{Q}[\Sigma_n^G]$ in ${}_{\mathbb{Q}H}\mathbf{dis}$, it follows that each $\operatorname{coker}(\mathbb{Q}[\iota_n])$ is projective; $\operatorname{coker}(\iota)$ is projective by hypothesis. Then exactness of the bottom row implies that $\operatorname{coker}(\iota')$ is projective. □

Remark 4.6.2. In ${}_{\mathbb{Q}G}\mathbf{dis}$, it is possible to develop a homological mapping cylinder argument analogous to [AMP20, Proposition 4.1] that yields a similar conclusion to Lemma 4.6.1 but only for open subgroups of G. This argument was developed in a preliminary version of this article.

Proof of Theorem 4.2.5. Since G and H have type FP_n, we may use the partial proper permutation resolutions described in Lemma 4.6.1; we keep the notation from there. Because G has type FP_{n+1} and $\mathrm{cd}_{\mathbb{Q}}(G) = n+1$, $\ker(\partial_n)$ is finitely generated (in ${}_{\mathbb{Q}G}\mathbf{dis}$) and projective; because H has type FP_{n+1} and $\operatorname{coker}(\iota')$ is projective, $\ker(\delta_n)$ is a finitely generated (in ${}_{\mathbb{Q}H}\mathbf{dis}$) summand of $\ker(\partial_n)$. So:

1. $\|{-}\|_{\ker(\delta_n^H)} \sim \|{-}\|_{\ker(\partial_n^G)}$ on the elements of $\ker(\delta_n^H)$, by Proposition 4.5.5;
2. $\|{-}\|_1^{\Omega_n^H} \sim \|{-}\|_1^{\Sigma_n^G}$ on the elements of $\mathbb{Q}[\Omega_n^H]$, because Ω_n is a subset of Σ_n;
3. $\|{-}\|_{\ker(\partial_n^G)} \sim \|{-}\|_1^{\Sigma_n^G}$ on the elements of $\ker(\partial_n^G)$, because G satisfies the weak n-dimensional linear isoperimetric inequality.

Therefore $\|{-}\|_{\ker(\delta_n^H)} \sim \|{-}\|_1^{\Omega_n^H}$ on the elements of $\ker(\delta_n^H)$, i.e. H satisfies the weak n-dimensional isoperimetric inequality. □

4.7 Weak linear isoperimetric inequality and hyperbolicity

The notion of linear isoperimetric inequality was used to characterise discrete hyperbolic groups by Gersten [Ger96a]. Different generalizations of Gersten's result have been presented by various authors; see for example [GM08], [Min02] and [HMP16]. In particular, Manning and Groves [GM08] reformulated Gersten's argument to provide a homological characterization of simply connected hyperbolic 2-complexes by means of a *homological isoperimetric inequality.* Here we use results from [GM08] to provide an analogue characterization of hyperbolic TDLC-groups.

Let X be a complex with i-skeleton denoted by $X^{(i)}$. Consider the cellular chain complex $(C_\bullet(X,\mathbb{Q}),\partial_\bullet)$ of X with rational coefficients. Each vector space $C_i(X,\mathbb{Q})$ is $\mathbb{Q}$-spanned by the collection of i-cells σ of X. An i-chain α is a formal linear combination $\sum_{\sigma\in X^{(i)}} r_\sigma\sigma$ where $r_\sigma\in\mathbb{Q}$. The ℓ_1-norm on $C_i(X,\mathbb{Q})$ is defined as

$$\|\alpha\|_1^{X,i}=\sum|r_\sigma|,$$

where $|\_|$ denotes the absolute value function on $\mathbb{Q}$.

Definition 4.7.1 ([GM08, Definition 2.18] Combinatorial path)**.** Let X be a complex. Suppose I is an interval with a cellular structure. A *combinatorial path* $I\to X^{(1)}$ is a cellular path sending 1-cells to either 1-cells or 0-cells. A *combinatorial loop* is a combinatorial path with equal endpoints.

From here on, to simplify notation, the 1-chain induced by a combinatorial loop c in X is denoted by c as well.

Definition 4.7.2 ([GM08, Definition 2.28] Linear Homological isoperimetric inequality)**.** Let X be a simply connected complex. We say that X satisfies the *linear homological isoperimetric inequality* if there is a constant $K\geq 0$ such that for any combinatorial loop c in X there is some $\sigma\in C_2(X,\mathbb{Q})$ with $\partial(\sigma)=c$ satisfying

$$\|\sigma\|_1^{X,2}\leq K\|c\|_1^{X,1}. \tag{4.7.1}$$

Definition 4.7.3. Let G be a compactly presented TDLC-group. There exists a simply connected G-complex X with compact open cell stabilizers, the 2-skeleton $X^{(2)}$ is compact modulo G, the G-action is cellular and an element in G fixing a cell setwise fixes it already pointwise. The group G satisfies the *linear homological isoperimetric inequality* if X does.

The above definition is independent of the choice of X as a consequence of Proposition 4.5.6, the fact that a compactly presented TDLC-group has type FP_2, and the following statement.

Proposition 4.7.1. Suppose G is a compactly presented TDLC-group and X is a topological model of G of type $\mathbb{F}_2$. Then G satisfies the weak linear isoperimetric inequality if and only if X satisfies the linear homological isoperimetric inequality.

Proof. The augmented cellular chain complex $(C_\bullet(X,\mathbb{Q}),\partial_\bullet)$ of X is a proper partial permutation resolution of $\mathbb{Q}$ of type FP_2. The module $C_i(X,\mathbb{Q})$ is a proper permutation module and we can take as its filling norm $\|\text{–}\|_{C_i}$ the ℓ_1-norm induced by G-set of i-cells.

The weak linear isoperimetric inequality means that the filling norm $\|\text{–}\|_{Z_1}$ of $Z_1(X,\mathbb{Q})$ is equivalent to the restriction of $\|\text{–}\|_{C_1}$ to $Z_1(X,\mathbb{Q})$. Hence there is a constant $C>0$ such that $\|\text{–}\|_{Z_1}\leq C\,\|\text{–}\|_{C_1}$ on $Z_1(X,\mathbb{Q})$. To prove the linear homological isoperimetric inequality is enough consider non-trivial combinatorial loops, the inequality is trivial otherwise. Let c be a non-trivial combinatorial loop and let $\mu\in C_2(X)$ such that $\partial\mu=c$ and $\|\mu\|_{C_2}\leq\|c\|_{Z_1}+1$. In particular, $\|\mu\|_{C_2}\leq\|c\|_{Z_1}+\|c\|_{C_1}$, since $\|c\|_{C_1}$ is a positive integer. It follows that $\|\mu\|_{C_2}\leq(C+1)\,\|c\|_{C_1}$ for any non-trivial combinatorial loop.

Conversely, suppose that X satisfies the linear homological isoperimetric inequality for a constant C. Let $\gamma\in Z_1(X,\mathbb{Q})$. Then the filling norm on $\gamma\in Z_1(X,\mathbb{Q})$ is given by $\|\gamma\|_{Z_1}=\inf\{\|\mu\|_{C_2}:\mu\in C_2(X,\mathbb{Q}),\partial\mu=\gamma\}$. There is an integer m such that $m\gamma$ is an integer cycle. Then $m\gamma=c_1+c_2+\cdots+c_k$ where each c_i is a cycle induced by a combinatorial loop, and $\|m\gamma\|_{C_1}=\sum\|c_i\|_{C_1}$, see [Ger98, Lemma A.2]. Then there are 2-cycles $\sigma_i\in C_2(X,\mathbb{Q})$ such that $\partial\sigma_i=c_i$ and $\|\sigma_i\|_{C_2}\leq C\,\|c_i\|_{C_1}$. It follows that

$$\|m\gamma\|_{Z_1}\leq\left\|\sum_i\sigma_i\right\|_{C_2}\leq\sum_i\|\sigma_i\|_{C_2}\leq C\sum_i\|c_i\|_{C_1}=C\,\|m\gamma\|_{C_1}.$$

Since both $\|\text{–}\|_{Z_1}$ and $\|\text{–}\|_{C_1}$ are homogeneous (see (N2) in Section 4.5), the previous inequality implies that $\|\text{–}\|_{Z_1}\leq C\,\|\text{–}\|_{C_1}$ on $Z_1(X,\mathbb{Q})$. On the other hand, since the inclusion $Z_1(X,\mathbb{Q})\hookrightarrow C_1(X,\mathbb{Q})$ is bounded, there is another constant C' such that $\|\text{–}\|_{C_1}\leq C'\,\|\text{–}\|_{Z_1}$ on $Z_1(X,\mathbb{Q})$. Therefore the norms $\|\text{–}\|_{Z_1}$ and $\|\text{–}\|_{C_1}$ are equivalent on $Z_1(X,\mathbb{Q})$. □

Below we recall a characterization of hyperbolic simply connected 2-complexes from [GM08].

Proposition 4.7.2. [GM08, Proposition 2.23, Lemma 2.29, Theorem 2.30] Let X be a simply connected 2-complex.

1. If $X^{(1)}$ is hyperbolic, then X satisfies the linear homological isoperimetric inequality.

2. If there is a constant M such that the attaching map for each 2-cell in X has length at most M, and X satisfies a linear homological isoperimetric inequality; then $X^{(1)}$ is hyperbolic.

Proof of Theorem 4.2.4. Let G be a compactly generated TDLC-group. Suppose that G is hyperbolic. By Proposition 4.4.7, G is compactly presented and there is a topological model X of G of type F_2. By Proposition 4.4.3, the 1-dimensional complex $X^{(1)}$ is quasi-isometric to a Cayley-Abels graph of G. It follows that $X^{(1)}$ is hyperbolic. Hence, Propositions 4.7.1 and 4.7.2 imply that G satisfies the weak linear isoperimetric inequality.

Conversely, suppose that G is compactly presented and satisfies the weak linear isoperimetric inequality. Proposition 4.4.6 implies that there is a topological model X of G of type F_2. By Proposition 4.7.1, X satisfies the linear homological isoperimetric inequality. Since the G-action on the 2-skeleton $X^{(2)}$ has finitely many G-orbits of 2-cells, there is a constant M such that the attaching map for each 2-cell in X has length at most M. Then Proposition 4.7.2 implies that $X^{(1)}$ is hyperbolic. By Proposition 4.4.3, the Cayley-Abels graphs of G are hyperbolic. □

4.8 Small cancellation quotients of amalgamated free products of profinite groups

This section relies on small cancellation theory over free products with amalgamation as developed in Lyndon-Schupp textbook [LS01, Chapter V, Section 11]. Before we state the main result of the section, we recall some of the terminology.

Let A and B be groups, and let C be a common subgroup. A *reduced word*[1] is a sequence $x_1 \dots x_n, n \geq 0$, of elements of $A *_C B$ such that

1. Each x_i belongs to one of the factors A or B.
2. Successive x_i, x_{i+1} belong to different factors.
3. If $n > 1$, no x_i belongs to C.

[1] In [LS01, Section V Chapter 11] this is also called normal form.

4. If $n = 1$, then $x_1 \neq 1$.

A sequence $x_1 \ldots x_n$ is *semi-reduced* if it satisfies all the above items with (2) replaced by

(2') The product of successive x_i, x_{i+1} does not belong to C.

Every element of $A *_C B$ can be represented as the product of the elements in a reduced word. Moreover, if $x_1 \ldots x_n, n \geq 1$ is a reduced word, then the product $x_1 \cdots x_n$ is not trivial in $A *_C B$ (see [LS01, Theorem 2.6]). A reduced word $w = x_1 \ldots x_n$ is *cyclically reduced* if $n = 1$ or if x_n and x_1 are in different factors of $A *_C B$. The word w is *weakly cyclically reduced* if $n = 1$ or if $x_n x_1 \notin C$.

A subset R of words in $A *_C B$ is *symmetrized* if $r \in R$ implies r is weakly cyclically reduced and every weakly cyclically reduced conjugate of $r^{\pm 1}$ is also in R. A symmetrized subset R is *finite* if all elements represented by words in R belong to a finite number of conjugacy classes in $A *_C B$. Let R be a symmetrized subset of $A *_C B$. A word b is said to be a *piece* (relative to R) if there exist distinct elements r_1 and r_2 of R such that $r_1 = bc_1$ and $r_2 = bc_2$ in semi-reduced form.

$C'(\lambda)$: If $r \in R$ has semi-reduced form $r = bc$ where b is a piece, then $|b| < \lambda |r|$. Further, $|r| > 1/\lambda$ for all $r \in R$.

Theorem 4.8.1. Let $A *_C B$ be the amalgamated free product of the profinite groups A, B over a common open subgroup C. Let R be a finite symmetrized subset of $A *_C B$ that satisfies the $C'(1/12)$ small cancellation condition. Then the quotient $G = (A *_C B)/\langle\langle R \rangle\rangle$ is a hyperbolic TDLC-group with $\mathrm{cd}_{\mathbb{Q}}(G) \leq 2$.

Since the rational discrete cohomological dimension of a TDLC-group G is less or equal to the geometric dimension of a contractible G-CW-complex acted on by G with compact open stabilizers (see [CC20, Fact 2.7] for example), in order to prove the theorem, we construct a contractible cellular 2-dimensional G-complex X with compact open cell stabilizers, and such that its 1-skeleton is hyperbolic. The 1-skeleton is obtained as a quotient of the Bass-Serre tree of $A *_C B$, and then X is obtained by pasting G-orbits of 2-cells in one to one correspondence with conjugacy classes defined by R.

Recall that there exists a unique group topology on $A *_C B$ with the following properties (see [CdlH16, Proposition 8.B.9] for example): the natural homomorphisms $A \to A *_C B$, $B \to A *_C B$, and $C \to A *_C B$ are topological isomorphisms onto open subgroups of $A *_C B$. Moreover, $A *_C B$ is a TDLC-group, and in particular, all open subgroups of C form a local basis at the identity of compact open subgroups of $A *_C B$.

Proof. Let T be the Bass-Serre tree of $A *_C B$. Observe that T is locally finite, the action of $A *_C B$ on T is cobounded and it has compact open stabilizers. In particular, the action of $A *_C B$ on T is geometric and hence $A *_C B$ is a hyperbolic TDLC-group.

Let N denote the normal closure of R in $A *_C B$. By Greendlinger's lemma [LS01, Ch. V. Theorem 11.2], the natural morphisms $A \to G$ and $B \to G$ are monomorphisms. It follows that the subgroup N does not intersect A, B and C. Thus the action of N on the tree T is free, and as a consequence N is discrete. Hence N is closed in $A *_C B$, and G is a TDLC-group.

Let $X^{(1)}$ denote the quotient graph T/N. Since N was acting freely and cellularly on T, the quotient map $\rho\colon T \to X^{(1)}$ is a covering map. Since T is locally finite, $X^{(1)}$ is locally finite. Since the quotient map $A *_C B \to G$ is open, the action of $A *_C B$ on T induces an action of G on $X^{(1)}$ which is cobounded and has compact open stabilizers.

Let x_0 be a fixed vertex of T that we consider as the base point from now on. Since T is simply connected, there is a natural isomorphism from N to the fundamental group $\pi_1(X^{(1)}, \rho(x_0))$. Specifically, for each $g \in N$, let α_g be the unique path in T from x_0 to $g.x_0$. Let $\gamma_g = \rho \circ \alpha_g$ be the closed path in $X^{(1)}$ induced by α_g based at $\rho(x_0)$. Thus, the isomorphism from N to $\pi_1(X^{(1)}, \rho(x_0))$ is defined by $g \mapsto \gamma_g$.

We are ready to define X. For $g \in G$ and $h \in N$, let $g.\gamma_h$ be the translated closed path without an initial point, i.e., these are cellular maps from $S^1 \to X$. Consider the G-set $\Omega = \{g.\gamma_r \mid r \in R, g \in G\}$ of closed paths in $X^{(1)}$. Let X be the G-complex obtained by attaching a 2-cell to $X^{(1)}$ for every closed path in Ω. In particular, the pointwise G-stabilizer of a 2-cell of X coincides with the pointwise G-stabilizer of its boundary path and, therefore, compact and open. Then X is a discrete G-complex of dimension 2. Observe that the natural isomorphism from N to $\pi_1(X^{(1)}, \rho(x_0))$ implies that X is simply connected. Moreover, since R is finite, X is a cobounded 2-dimensional discrete G-complex.

We observe that X is a $C'(1/6)$ complex and in particular the one-skeleton of X

is a Gromov hyperbolic graph with respect to the path metric, this is a well known consequence, see [GS90]. Let R_1 and R_2 be a pair of distinct 2-cells in X such that the intersection of their boundaries contains an embedded path γ. We can assume, by translating by an element of G, that the base point of X is either the initial vertex of γ or the second vertex of γ. Let γ' be the sub-path of γ with initial point the base point of X; observe that $|\gamma| \leq |\gamma'| + 1$. Consider the boundary paths of R_1 and R_2 starting at the base point and oriented such that γ' is an initial sub-path of both of them. Consider the lifts of the chosen boundary paths of R_1 and R_2 in the tree T starting from the base point. They intersect along the lifting of the path γ'. Let us call this path $\hat{\gamma}$. Then the reduced word in $A *_C B$ corresponding to $\hat{\gamma}$ is a piece, and hence its length is bounded by $\frac{1}{12}|\partial R_i|$, for $i = 1, 2$. We have the following inequality

$$|\gamma| \leq |\gamma'| + 1 = |\hat{\gamma}| + 1 \leq \frac{1}{12}|\partial R_i| + \frac{1}{12}|\partial R_i|, \text{ for } i = 1, 2.$$

Hence X is a $C'(1/6)$ complex.

We conclude that the complex X is contractible by using a well known argument of Ol′shanskiĭ [Ol'91]. By a remark of Gersten [Ger87, Remark 3.2], if every spherical diagram in X is reducible, then X has trivial second homotopy group, and therefore X is contractible because it is simply connected. Let $S \to X$ be a spherical diagram, and suppose that it is not reducible. Consider the dual graph Φ to the cellular structure of S, specifically Φ is the graph whose vertices are the two cells of X and there is an edge between two vertices for each connected component of the intersection of the boundaries of the corresponding 2-cells. Observe that Φ is planar. Since X is a $C'(1/6)$ complex, the boundary paths of 2-cells are embedded paths and the intersection of the boundaries of any pair of 2-cells is connected, and hence Φ is simplicial. Also since X is $C'(1/6)$, every vertex of Φ has degree at least 6. Since a finite planar simplicial graph has at least one vertex of degree at most 5, we have reached a contradiction and therefore the diagram $S \to X$ has to be reducible. The above sketched argument can be found in [MP17, Proof of Theorem 6.3] in a different framework. $\square$

Chapter 5

Topological groups with a compact open subgroup, coherence and relative hyperbolicity

5.1 Abstract

The main objects of study in this article are pairs $(G, \mathcal{H})$ where G is a topological group with a compact open subgroup, and $\mathcal{H}$ is a finite collection of open subgroups. We develop geometric techniques to study the notions of G being compactly generated and compactly presented relative to $\mathcal{H}$. This includes topological characterizations in terms of discrete actions of G on complexes, quasi-isometry invariance of certain graphs associated to the pairs $(G, \mathcal{H})$ when G is compactly generated relative to $\mathcal{H}$, and extensions of known results for the discrete case. For example, generalizing results of Osin for discrete groups, we show that in the case that G is compactly presented relative to $\mathcal{H}$:

- if G is compactly generated, then each subgroup $H \in \mathcal{H}$ is compactly generated;
- if each subgroup $H \in \mathcal{H}$ is compactly presented, then G is compactly presented.

The article also introduces an approach to relative hyperbolicity for pairs $(G, \mathcal{H})$ based on Bowditch's work using discrete actions on hyperbolic fine graphs. For example, we

prove that if G is hyperbolic relative to $\mathcal{H}$ then G is compactly presented relative to $\mathcal{H}$.

As an application of the results of the article we prove results on coherence of groups. A topological group is coherent if every open compactly generated subgroup is compactly presented.

5.2 Introduction

This article is part of the program of generalizing geometric techniques in the study of discrete groups to the larger class of locally compact groups in the spirit of the book by Cornulier and de la Harpe on Metric geometry on locally compact groups [CdlH16].

As a convention, all topological groups considered in the article are Hausdorff, and all group actions on CW-complexes are assumed to be cellular. Throughout the article, we work in the class of topological groups with a compact open subgroup. Note that such groups are locally compact, and by van Dantzig's Theorem [VD36], all totally disconnected locally compact groups (TDLC groups) belong to this class. The class of TDLC-groups has been a topic of interest in the last three decades since the work of G. Willis [Wil94], of M. Burger and S. Mozes [BM00], and P.E Caprace and N. Monod [CM11]. TDLC groups include profinite groups, discrete groups, algebraic groups over non-archimedean local fields, and automorphism groups of locally finite graphs.

Let G be a topological group. The group G is *compactly generated* if it admits a compact generating set; it is *compactly presented* if it admits a standard presentation $\langle S \mid R\rangle$ with S a compact subset of G and R a set of words in S of uniformly bounded length. A topological group is said to be *coherent* if every open compactly generated subgroup is compactly presented.

The definition of coherence only considers open subgroups instead of the larger class of closed subgroups. One of the reasons is the following remark, used in some of our arguments, which is a consequence of the fact that quotient maps of topological groups are open but not necessarily closed (see Proposition 5.6.17).

Remark A. *Let G be a topological group and $N \trianglelefteq G$ be a compact normal subgroup. Then G is coherent if and only if G/N is coherent.*

The simplest coherent groups are compact groups. In the discrete case, any virtually free group is also coherent, a statement that can be generalized as follows:

Theorem B (Combination of Coherence Groups)**.** *If $G = A *_C B$ is a topological group that splits as an amalgamated free product of two coherent open subgroups A and B with compact intersection C, then G is coherent.*

There are classical examples that illustrate the above proposition, for instance the splitting of the discrete group $SL_2(\mathbb{Z})$ as $C_4 *_{C_2} C_6$ where C_n denotes a cyclic group of order n. In the non-discrete case, if $\mathbb{Q}_p$ denotes the field of p-adic numbers and $\mathbb{Z}_p$ the p-adic integers, the group $SL_2(\mathbb{Q}_p)$ splits as an amalgamated free product of two open subgroups isomorphic to the compact group $SL_2(\mathbb{Z}_p)$ along their common intersection, see for example [Ser80]. One can also iterate the construction, for instance the amalgamated free product $SL_2(\mathbb{Q}_p) *_{SL_2(\mathbb{Z}_p)} SL_2(\mathbb{Q}_p)$ of two copies of $SL_2(\mathbb{Q}_p)$ along the compact open subgroup of $SL_2(\mathbb{Z}_p)$ is a coherent group. Let us remark that coherence of $SL_2(\mathbb{Q}_p)$ follows directly from a result attributed to J.Tits, see [Pra82, Thm. T].

We are able to use McCammond and Wise's perimeter method in the class of locally compact groups with a compact open subgroup to obtain the following result.

Theorem C. *Let $A *_C B$ be a topological group that splits as an amalgamated free product of two coherent open subgroups A and B with compact intersection C. Suppose $r \in A *_C B$ is not conjugate into A or B. If m is sufficiently large and r^m satisfies the $C'(1/6)$ small cancellation condition, then the quotient group $G = (A *_C B)/\langle\langle r^m \rangle\rangle$ is coherent.*

This result in the case that A and B are free groups is a result of McCammond and Wise [MW02, Theorem 8.3], and the generalization where A and B are coherent discrete groups is a result of Wise and the second author [MPW11b, Theorem 1.8]. The proofs of these results rely on a technique known as the perimeter method developed in work McCammond and Wise [MW02]. Their work was motivated by the well known question of Gilbert Baumslag of whether all one relator groups are coherent [Bau74], see the survey on coherence by Wise [Wis20] and the recent work by Louder and Wilton [LW20].

The quest to prove Theorem C took us to develop versions in the framework of locally compact groups of other techniques in the study of discrete groups, specifically, tools that deal with properties of groups relative to collections of subgroups. We summarized this work below, where the main objects of study are pairs $(G, \mathcal{H})$ where G is a topological group with a compact open subgroup and $\mathcal{H}$ is a finite collection of open subgroups. Then we conclude the introduction with the proof of Theorem B. In order to avoid repetition, we introduced the following terminology.

Definition D (Proper pair). *A pair* $(G, \mathcal{H})$ *is called a* proper pair *if*

1. *G is a topological group with a compact open subgroup;*
2. *$\mathcal{H}$ is a finite collection of open subgroups of G;*
3. *No pair of distinct subgroups in $\mathcal{H}$ are conjugate in G.*

Note that for a proper pair $(G, \mathcal{H})$, we allow $\mathcal{H}$ to be the empty collection.

5.2.1 Compact relative generating sets.

A topological group G is *compactly generated relative to a collection of subgroups* $\mathcal{H}$ if there is a compact subset $K \subset G$ such that G is algebraically generated by $K \cup \bigcup \mathcal{H}$. An action of a topological group on a CW-complex by cellular automorphisms is called *discrete* if pointwise stabilizers of cells are open subgroups. A graph is a 1-dimensional CW complex, and the graph is simplicial if there are no loops or multiple edges between the same vertices. Relative compact generation is topologically characterized as follows.

Definition E (Cayley-Abels graph). *Let* $(G, \mathcal{H})$ *be a proper pair. A* Cayley-Abels graph *of G with respect to $\mathcal{H}$ is a connected cocompact simplicial G-graph Γ such that:*

1. *edge stabilizers are compact,*
2. *vertex stabilizers are either compact or conjugates of subgroups in $\mathcal{H}$,*
3. *every $H \in \mathcal{H}$ is the G-stabilizer of a vertex, and*
4. *any pair of vertices with the same G-stabilizer $H \in \mathcal{H}$ are in the same G-orbit if H is non-compact.*

Theorem F (Topological Characterization of Relative compact generation). *Let $(G, \mathcal{H})$ be a proper pair. The following statements are equivalent:*

1. *G is compactly generated relative to $\mathcal{H}$.*

2. *There exists a Cayley-Abels graph of G with respect to $\mathcal{H}$.*

In the case that G is a discrete group and $\mathcal{H}$ is empty, the above result is the well-know fact that a discrete group is finitely generated if and only if it acts properly and cocompactly on a connected graph. In the case that G is discrete and $\mathcal{H}$ is a finite collection of subgroups, the result appears implicit in the work of Hruska on relatively hyperbolic groups [Hru10] where there resulting graphs are called coned-off Cayley graphs. In the case that G is a TDLC group and $\mathcal{H}$ is empty, this is a result of Krön and Möller's [KM08] who show that the resulting graphs can be assumed to be vertex transitive and call them *rough Cayley graphs*; this graphs are also know as *Cayley-Abels graphs* after related work of Herbert Abels [Abe73]. In the case that G has a compact open subgroup and $\mathcal{H}$ is empty, the theorem is a result of Cornulier and de la Harpe [CdlH16, Proposition 2.E.9].

Relative Cayley-Abels graphs of proper pair $(G, \mathcal{H})$ are not necessarily locally finite graphs, however we show that they are pairwise quasi-isometric. There is a generalization of locally finite graphs introduced by Bowditch known as fine graphs [Bow12]; a graph is *fine* if for any pair of vertices u, v and any integer n, there are finitely many embedded n-paths from u to v.

Theorem G (Quasi-isometry invariance). *Let $(G, \mathcal{H})$ be a proper pair. If Γ and Δ are relative Cayley-Abels graphs of G with respect to $\mathcal{H}$, then*

1. *Γ and Δ are quasi-isometric, and*

2. *Γ is fine if and only if Δ is fine.*

This theorem in the case that G is a TDLC group and $\mathcal{H}$ is empty is a result of Krön and Möller's [KM08]. The quasi-isometry invariance of relative Cayley-Abels graphs allow us to define geometric invariants for pairs $(G, \mathcal{H})$ where G is a topological group with a compact subgroup and $\mathcal{H}$ is a finite collection of open subgroups. For example, hyperbolicity in the class of topological groups with a compact open subgroup can be defined as the groups that admit a hyperbolic Cayley-Abels graph (with respect to the empty collection) an approach considered in [ACCCMP21]. We use Theorem G to start the developing of a theory of relatively hyperbolic groups, see subsection 5.2.3.

The approach to hyperbolic groups for locally compact groups developed by Caprace, Cornulier, Monod and Tessera [CCMT15] when restricted to locally compact groups with a compact open subgroup provides an equivalent definition.

5.2.2 Compact relative presentations

Let G be a topological group and let $\mathcal{H}$ be a finite collection of open subgroups. We say that G is *compactly presented relative to* $\mathcal{H}$ if there is a short exact sequence

$$1 \to \langle\langle R \rangle\rangle \to \pi_1(\mathcal{G}, \Lambda) \xrightarrow{\phi} G \to 1$$

where $\pi_1(\mathcal{G}, \Lambda)$ is the fundamental group of a finite graph of groups $(\mathcal{G}, \Lambda)$ endowed with the topology induced by the vertex groups (see see Proposition 5.4.3), such that

- There are vertices $\{v_1, v_2, \cdots v_n\}$ of Λ and isomorphisms of topological groups $\phi_i \colon \mathcal{G}_{v_i} \to H_i$ such that

$$\begin{tikzcd} \mathcal{G}_{v_i} \arrow[r, hook, "i_{v_i}"] \arrow[d, "\phi_i"] & \pi_1(\mathcal{G}, \Lambda) \arrow[d, "\phi"] \\ H_i \arrow[r, hook] & G \end{tikzcd}$$

  is a commutative diagram.

- For every $v \in V(\Lambda)$, the map $\phi \circ i_v$ is injective.

- For each edge $e \in E(\Lambda)$ and each vertex $v \neq v_i$ in $V(\Lambda)$, the edge group $\mathcal{G}_e$ and the vertex group $\mathcal{G}_v$ are compact topological groups.

- ϕ is a continuous open surjective epimorphism whose restriction to each vertex group of $(\mathcal{G}, \Lambda)$ is injective.

- $\langle\langle R \rangle\rangle$ is a discrete normal subgroup generated by a finite subset R of $\pi_1(\mathcal{G}, \Lambda)$.

In this case, $\langle (\mathcal{G}, \Lambda, \phi) \mid R \rangle$ is called a *compact generalized presentation* of G with respect to $\mathcal{H}$.

In the case that G is compactly presented with respect to the empty collection, we say that G is *compactly presented*. This is equivalent to the definition stated at the beginning of the introduction, see Corollary 5.6.9 in the main body of the article.

In the case that G is discrete and $\mathcal{H}$ is empty, the definition is equivalent to G being the quotient of a virtually free group of finite rank by a normal subgroup generated by a finite number of elements. In the case that G is a discrete group and $\mathcal{H}$ is not empty, it is an observation that our definition of G being finitely presented relative to $\mathcal{H}$ coincides with the approach by Osin [Osi06]. We previously mentioned that in the case that G is a TDLC group and $\mathcal{H}$ is empty, this approach was used by Castellano and Weigel [CW16a, Cas20].

Theorem H (Topological Characterization of Relative compact presentation)**.** *Let G be a topological group with a finite collection $\mathcal{H}$ of open subgroups. The following statements are equivalent:*

1. *G is compactly presented with respect to $\mathcal{H}$.*
2. *There exists a relative Cayley-Abels graph Γ of G with respect to $\mathcal{H}$ which is the 1-skeleton of a simply-connected cocompact discrete G-complex.*

Note that for a discrete group G with an empty collection a stronger version of Theorem H holds in the sense that the second item can be expressed with an universal quantifier. In order to obtain this type of equivalence in our context we need to impose an additional hypothesis:

Corollary I. *Let $(G, \mathcal{H})$ be a proper pair. Suppose there is a fine relative Cayley-Abels graphs of G with respect to $\mathcal{H}$. The following statements are equivalent:*

1. *G is compactly presented with respect to $\mathcal{H}$.*
2. *Any relative Cayley-Abels graph of G with respect to $\mathcal{H}$ is the 1-skeleton of a simply-connected cocompact discrete G-complex.*

We are not aware whether the assumption on fineness of all Cayley-Abels graphs is necessary. The fineness property of a graph was placed in the context of isoperimetric inequalities in [MP16] (see also [HMPS21]). The complexes given by Corollary I, which are not necessarily locally finite satisfy the hypothesis of the following result.

Proposition J (Proposition 5.8.8)**.** [MP16, Propositions 2.1 and 2.6] *Let X be a cocompact simply-connected G-complex. Suppose that each edge of X is attached to finitely many 2-cells. Then the 1-skeleton of X is a fine graph if and only if the combinatorial Dehn function of X takes only finite values.*

Corollary I in the case that G is a TDLC group and $\mathcal{H}$ is the empty collection is a result of Castellano and Cook [CC20, Proposition 3.4], and for generally locally compact groups with empty $\mathcal{H}$, there is a version by Cornulier and de la Harpe, see [CdlH16, Corollary 8.A.9].

Theorem K. *Let $(G, \mathcal{H})$ be a proper pair. Suppose that G is compactly presented relative to $\mathcal{H}$.*

1. *If each $H \in \mathcal{H}$ is compactly presented then G is compactly presented.*
2. *If G is compactly generated, then each $H \in \mathcal{H}$ is compactly generated.*

In the case of the discrete groups, Theorem K is a result of Osin [Osi06, Theorem 1.1 and Theorem 2.40].

5.2.3 Relative hyperbolicity

Let (G, H) be a proper pair. The topological group G is *relatively hyperbolic* with respect to $\mathcal{H}$ if there exists a relative Cayley-Abels graph of G with respect to $\mathcal{H}$ that is fine and hyperbolic.

Remark L. *Let (G, H) be a proper pair. If G is hyperbolic relative to $\mathcal{H}$ then:*

1. *G is compactly generated relative to $\mathcal{H}$ by Theorem F; and*
2. *relative Cayley-Abels graphs of G with respect to $\mathcal{H}$ are fine and hyperbolic by Theorem G.*

Our definition of relative hyperbolicity when restricted to discrete groups coincides with the approach by Bowditch [Bow12] which is also equivalent to the approach by Osin [Osi06].

Theorem M. *Let $(G,\mathcal{H})$ be a proper pair. Suppose G is hyperbolic relative to $\mathcal{H}$. Then G is compactly presented relative to $\mathcal{H}$.*

Putting together Theorems K and M we obtain:

Corollary N. *Let $(G,\mathcal{H})$ be a proper pair. Suppose G is hyperbolic relative to $\mathcal{H}$.*

1. *If each $H \in \mathcal{H}$ is compactly presented then G is compactly presented.*
2. *If G is compactly generated, then each $H \in \mathcal{H}$ is compactly generated.*

Let us conclude the introduction with the proof of Theorem B that illustrates the results that have been stated.

5.2.4 Proof of Theorem B

Let T be the Bass-Serre tree of the splitting $G = A *_C B$. Observe that the G-action on T is discrete since A, B and C are open subgroups of G, and edge stabilizers are compact since C is compact.

Let Q be a compactly generated closed subgroup of G. If Q fixes a vertex of T, then coherence of A and B imply that Q is compactly presented.

Suppose that Q does not fix a vertex of T and let $U = Q \cap C$. Observe that U is a compact open subgroup of Q. Since Q is compactly generated and U is open, there is a finite subset $S \subset Q$ such that $S \cup U$ generates Q. Let e be an edge of T which is stabilized by U, let D the minimal connected subgraph of T containing e and $g.e$ for all $g \in S$. Since S is finite, D is a finite subtree. Since $S \cup U$ generates Q, it follows that $\Delta := \bigcup_{g\in Q} gD$ is a discrete cocompact Q-invariant subtree of T such that all edge stabilizers are compact. Let $\mathcal{H}$ be a collection of representatives of conjugacy classes of vertex Q-stabilizers of Δ, and note that $\mathcal{H}$ is finite and $Q \notin \mathcal{H}$. In particular, $(Q,\mathcal{H})$ is a proper pair, and Δ is a relative Cayley-Abels graph of Q with respect to $\mathcal{H}$. Since trees are hyperbolic and fine, it follows that Q is hyperbolic relative to $\mathcal{H}$. Since Q is compactly generated, Corollary N implies that each $H \in \mathcal{H}$ is a compactly generated group. On the other hand, observe that each $H \in \mathcal{H}$ is either compact or it is a closed subgroup of A or B up to conjugation in G. Since A and B are coherent, it follows that each $H \in \mathcal{H}$ is compactly presented. By the first part of Corollary N, it follows that Q is compactly presented.

Organization

The rest of the article is organized into seven sections. Each section contains the proof of one the theorems stated in the introduction, except Section 5.3 that contains a technical result about fineness. The mapping of results in the introduction with the main body of the article:

Definition E	Section 5.4	Definition 5.4.10 and Theorem 5.4.11
Theorem F		Theorem 5.4.9
Theorem G	Section 5.5	Corollary 5.5.3
Theorem H	Section 5.6	Theorem 5.6.6
Corollary I		Corollary 5.6.7
Theorem K	Section 5.7	Theorem 5.7.1
Theorem M	Section 5.8	Theorem 5.8.4
Proposition J		Proposition 5.8.8
Theorem C	Section 5.9	

5.3 Equivariant Edge Attachments and Fineness

This section revisits an argument from [MPR21] in order to prove that certain natural extensions of G-graphs preserve fineness. The main result of this section is Theorem 5.3.4.

Throughout the section, G denotes a topological group. All graphs in this section are 1-dimensional simplicial complexes. Let Γ be a simplicial graph, let v be a vertex of Γ, and let

$$T_v\Gamma = \{w \in V(\Gamma) \mid \{v, w\} \in E(\Gamma)\}.$$

denote the set of the vertices adjacent to v. For $x, y \in T_v\Gamma$, the *angle metric* $\angle_v(x, y)$ is the combinatorial length of the shortest path in the graph $\Gamma - \{v\}$ between x and y, with $\angle_v(x, y) = \infty$ if there is no such path.

Definition 5.3.1 (Bowditch fineness)**.** [Bow12] A simplicial graph Γ is *fine at* v if $(T_v\Gamma, \angle_v)$ is a locally finite metric space. A graph Γ is *fine* if it is fine at every vertex.

Definition 5.3.2 (Equivariant attachment of edges)**.** Let G be a group and let Γ and Δ be G-graphs.

1. Let $u \in V(\Gamma)$ and let $H \leq G$ be a subgroup. The G-graph Δ is *obtained from* Γ *by attaching an edge G-orbit with representative* $\{u, H\}$ if

$$V(\Delta) = V(\Gamma) \sqcup G/H, \qquad E(\Delta) = E(\Gamma) \sqcup \{\{gu, gH\} | g \in G\}$$

   where G/H denotes the G-set of left cosets of H in G.

2. Let $u, v \in V(\Gamma)$ distinct vertices. The G-graph Δ is *obtained from* Γ *by attaching a G-orbit of edges with representative* $\{u, v\}$ if

$$V(\Delta) = V(\Gamma), \qquad E(\Delta) = E(\Gamma) \cup \{\{g.u, g.v\} \mid g \in G\}$$

Definition 5.3.3 (Discrete G-graph)**.** Let G be a topological group and let Γ be a G-graph. If vertex and edge stabilizers are open subgroups, we say that Γ is a *discrete G-graph.*

Theorem 5.3.4. *Let Γ be a connected discrete G-graph with compact edge stabilizers. Let $u, v, a \in V(\Gamma)$, and let $H \leq G$ be a compact open subgroup. Let Δ be a G-graph obtained from Γ*

1. *by attaching a G-orbit of edges with representative $\{u, v\}$; or*

2. *by attaching a G-orbit of edges with representative $\{u, H\}$.*

If Γ is fine at a, then Δ is fine at a.

Corollary 5.3.5. *Let Γ_1 and Γ_2 be cocompact connected discrete G-graphs with compact edge stabilizers. Let $\mathcal{V}_\infty(\Gamma_i)$ be the set of vertices of Γ_i with non-compact stabilizer. If there is a G-equivariant bijection $\mathcal{V}_\infty(\Gamma_1) \xrightarrow{\eta} \mathcal{V}_\infty(\Gamma_2)$, then:*

1. *Γ_1 and Γ_2 are quasi-isometric, and*

2. *Γ_1 is fine if and only if Γ_2 is fine.*

Proof. Let Γ be the simplicial G-graph obtained by taking disjoint union of Γ_1 and Γ_2 identified along $\mathcal{V}_\infty(\Gamma_1)$ and $\mathcal{V}_\infty(\Gamma_2)$ through η. By cocompactness of Γ_2, the G-graph

Γ can be constructed from Γ_1 by finitely many G-edge attachments. Since $\mathcal{V}_\infty(\Gamma_2)$ contains all vertices of Γ_2 with non-compact stabilizer, we only need to equivariant edge attachment of edges satisfying the hypothesis of Theorem 5.3.4. By induction, Γ is fine if and only if Γ_1 is fine. It is not difficult to verify that the inclusion $\Gamma_1 \hookrightarrow \Gamma$ is a quasi-isometry, see subsection 5.3.5 for an argument. Analogously, Γ_2 is fine if and only if Γ is fine, and Γ_2 is quasi-isometric to Γ. $\square$

Theorem 5.3.4 addresses two constructions that preserve fineness of vertices, we will refer to them as the first and second case according to the enumeration in the statement. There are versions of Theorem 5.3.4 in the case of the **first construction** and under the assumption that G is discrete:

1. Bowditch shows that if Γ has finitely many G-orbits of vertices and edges and is fine, then Δ fine; see [Bow12, Lemma 4.5]. An alternative argument for this statement can be found in [MPW11a, Proof of Lemma 2.9].
2. The statement of Theorem 5.3.4 for the first construction in the case that G is discrete can be found in [MPR21, Proposition 4.2].

The proof of Theorem 5.3.4 for both constructions follows the same strategy as the argument in [MPR21, Proof of Proposition 4.2]. We only prove Theorem 5.3.4 for the second construction, i.e., the case that Δ is obtained by attaching a G-orbit of edges with representative $\{u, H\}$. This case has not been addressed even in the case that the group is discrete. While there is a significant overlap with the argument in [MPR21, Proof of Proposition 4.2], we decided to include a complete proof since there is number of additional lemmas that are required besides addressing topological matters arising from replacing finiteness of edge stabilizers with the assumption that edge stabilizers are compact and open. For the convenience of the reader we included some arguments from [MPR21] in some cases almost verbatim.

5.3.1 Preliminaries

Let us fix some notation for paths in a simplicial graph Γ. A *path* or an *edge-path* from a vertex v_0 to a vertex v_n of Γ is a sequence of vertices $[v_0, v_1 \ldots, v_n]$, where v_i and v_{i+1} are adjacent (in particular distinct) vertices for all $i \in \{0, \ldots, n-1\}$. The path is *embedded* if all vertices of the path are distinct. The *length* of a path is the total number of vertices in the sequence minus one. A path of length k is called a k-path. If $\alpha = [u_1, \dots, u_k]$ and $\beta = [v_1, \dots v_\ell]$ are paths with $u_k = v_1$, then $[\alpha, \beta]$ denotes the concatenated path $[u_1, \dots, u_k, v_2, \dots v_\ell]$.

5.3.2 Alternative approach to fineness

A path $[u, u_1 \dots, u_k]$ in a graph Γ is an *escaping path from u to v* if $v = u_k$ and $u_i \neq u$ for every $i \in \{1, \dots, k\}$. For vertices u and v of Γ and $k \in \mathbb{Z}_+$, define:

$$\vec{uv}(k)_\Gamma = \{w \in T_u\Gamma \mid w \text{ belongs to an escaping} \\ \text{path from } u \text{ to } v \text{ of length} \leq k\}.$$

Proposition 5.3.6. [MPR21, Lemma 4.4] *A graph Γ is fine at $u \in V(\Gamma)$ if and only if $\vec{uv}(k)_\Gamma$ is a finite set for every integer $k > 0$ and every vertex $v \in V(\Gamma)$.*

Remark 5.3.7. For vertices u and v of Γ and $k \in \mathbb{Z}_+$, if the set of vertices adjacent to v is $\{w_j \colon j \in J\}$ then

$$\vec{uv}(k+1)_\Gamma = \bigcup \{\vec{uw}(k)_\Gamma \colon w \in T_v\Gamma\}.$$

5.3.3 A Compactness argument

Proposition 5.3.8. *Let Γ be a connected discrete G-graph with compact edge stabilizers. Let $c = [x, y, z]$ be a 2-path and let $e = [u, v]$ be a 1-path. The set*

$$B = \{g.z \in V(\Gamma) \mid g \in G,\ g.x = u,\ g.y = v\}$$

is finite.

Proof. Suppose B is non empty. Then

$$A = \{g \in G \colon g.x = u,\ g.y = v\} = \bigsqcup_{g \in A} g(G_x \cap G_y \cap G_z)$$

is a nonempty subspace of G. Since Γ is a discrete G-graph, $G_x \cap G_y \cap G_y$ is an open subgroup; hence A is both open and closed in G. Fix an element $g_0 \in A$ and observe that

$$A \subset g_0(G_x \cap G_y).$$

Since Γ has compact edge stabilizers, $G_x \cap G_y$ is a compact subgroup. Therefore A is a closed subset of a compact set, hence A is compact.

Consider B as a discrete set, and consider the map

$$\varphi\colon A \to B, \qquad g \mapsto g.z.$$

Since $G_x \cap G_y$ is open, one easily verifies that φ is continuous. Note that φ is surjective. Since A is compact, B is finite. □

5.3.4 Proof of Theorem 5.3.4

Let Γ be a connected discrete G-graph with compact edge stabilizers. Let $u \in V(\Gamma)$, let $H \leq G$ be a compact open subgroup, and let Δ be the G-graph obtained from Γ by attaching a new G-orbit of an edge with representative $\{u, H\}$.

Lemma 5.3.9. *Δ is connected discrete G-graph with compact edge stabilizers.*

Proof. It is an observation that Δ is connected and the vertex H of Δ has G-stabilizer the subgroup H which is open by assumption. The edge $\{u, H\}$ of Δ has G-stabilizer $G_u \cap H$ which is open since both G_u and H are open, and it is compact since H is compact by assumption. Since any vertex or edge of Δ which is not in the G-orbits of the vertex H or the edge $\{u, H\}$ is in Γ, we have that Δ is a discrete G-graph. □

Lemma 5.3.10 (The vertex H has finite degree)**.** *The set $T_H\Delta$ of vertices of Δ adjacent to the vertex H is a finite subset of $V(\Gamma)$.*

Proof. By definition of Δ, every vertex adjacent to the vertex H is a vertex of Γ. Observe that the vertex H of Δ has stabilizer the subgroup H, and the edge $\{u, H\}$ has stabilizer $G_u \cap H$. Since all vertices of Δ adjacent to the vertex H are in the G-orbit of u, it follows that the vertex H has degree equal to the index of the subgroup $H \cap G_u$ in the group H. Since H is compact and $H \cap G_u$ is an open subgroup, the vertex H of Δ has finite degree. □

Lemma 5.3.11 (Fineness criterion for Δ)**.** *If $\vec{ab}(k)_\Delta$ is finite for every $k \geq 1$ and every $b \in V(\Gamma)$, then Δ is fine at a.*

Proof. Let v be a vertex of Δ and let $k \geq 1$. If v is not in the G-orbit of the vertex H, then by assumption $\vec{ab}(k)_\Delta$ is finite for every $k \geq 1$. Suppose that v is in the G-orbit of the vertex H. Then Lemma 5.3.10 implies that v has finite degree and $T_v\Delta = \{b_1, \dots, b_m\}$ is a subset of $V(\Gamma)$. Then Remark 5.3.7 implies that

$$\vec{av}(k)_\Delta = \bigcup_{i=1}^{k} \vec{ab_i}(k-1).$$

Since each $b_i \in V(\Gamma)$, the hypothesis implies that $\vec{ab_i}(k-1)$ is a finite set for each b_i. Therefore $\vec{av}(k)_\Delta$ is a finite set for every $v \in V(\Delta)$ and $k \geq 1$. By Proposition 5.3.6, Δ is fine at a. □

Suppose that Γ is fine at the vertex a. Let b be a vertex of Γ and let $k \geq 1$. We prove below that $\vec{ab}(k)_\Delta$ is finite. Observe that this implies that Δ is fine at a in view of Lemma 5.3.11. The argument follows the skeleton of the proof of [MPR21, Proposition 4.2].

The paths α_{ij} and the constants ℓ and n.

By Lemma 5.3.10,

$$T_H\Delta = \{v_1, v_2, \dots, v_m\} \subset V(\Gamma).$$

For each $v_i, v_j \in T_H\Delta$, let α_{ij} be a minimal length (embedded) path from v_i to v_j in Γ, note that such a path exists since Γ is connected. Let ℓ be an upper bound for the length of the paths α_{ij}, that is

$$|\alpha_{ij}| \leq \ell$$

for any $v_i, v_j \in T_H\Delta$. Let

$$n = k\ell. \tag{5.3.1}$$

The finite sets W_i and Z_i.

A subpath of length two of a path P is called a *corner of P*. Let

$$W_n = \vec{ab}(n)_\Gamma.$$

Let $j \leq n$ and suppose W_j has been defined. Let

$$Z_{j-1} = W_j \cup \{z \in T_a\Gamma \mid \exists w \in W_j \ \exists g \in G \ \exists v_i, v_j \in T_H\Delta \\ \exists c \text{ corner of } \alpha_{ij} \text{ such that } g.c = [z, a, w]\}.$$

$$W_{j-1} = \{w \in T_a\Gamma \mid \exists z \in Z_{j-1} \text{ such that } \angle_{T_a\Gamma}(z, w) \leq n\}.$$

Observe that

$$W_j \subseteq Z_{j-1} \subseteq W_{j-1}, \qquad \text{for all } 1 \leq j \leq n \tag{5.3.2}$$

Lemma 5.3.12. *If W_j is finite, then Z_{j-1} is finite.*

Proof. This is a consequence of the assumption that G acts discretely on Γ and edges have compact G-stabilizers. By contradiction, assume that Z_{j-1} is infinite and W_j is finite. Since there are finitely choices for α_{ij} and each of these paths has finitely many corners, the pigeon-hole argument shows that there is $w \in W_j$, there is an α_{ij}, and there is corner $c = [v, x, y]$ of α_{ij} such that the set

$$B = \{g.v \in V(\Gamma) \mid g \in G, \quad g.x = a, \quad g.y = w\}$$

is infinite. Since Γ is a discrete G-graph with compact edge stabilizers, Proposition 5.3.8 implies that B is finite, a contradiction. □

Lemma 5.3.13. *For $1 \leq j < n$, W_j and Z_j are finite subsets of $T_a\Gamma$. In particular, W_1 is finite.*

Proof. [MPR21, Proof of Lemma 4.6]. The conclusion follows by an inductive argument using Lemma 5.3.12 and the following pair of claims.

Claim 1: If Z_j is finite, then W_j is finite. Since Γ is fine at a, for each $z \in Z_{j-1}$, there are finitely $w \in T_a\Gamma$ such that $\angle_{T_a\Gamma}(w,z) \leq n$. Hence if Z_j is finite, then W_j is finite.

Claim 2. W_n is finite. By hypothesis, Γ is fine at a. Then Lemma 5.3.6 implies that $\vec{ab}_\Gamma(n) = W_n$ is finite. $\square$

Projecting paths from Δ to Γ.

Let δ be a k-path in Δ with initial vertex in Γ. An *α-replacement of δ* is a path γ in Γ obtained as follows: Replace each corner of δ of the form $[g.v_i, gH, g.v_j]$ for some $v_i, v_j \in T_H\Delta$ and $g \in G$ by the path $g.\alpha_{ij}$ (make a choice of g if necessary); if the terminal vertex of the resulting path is not in Γ then remove that vertex. Observe that γ is a path of length at most n, recall that n is defined in (5.3.1).

Lemma 5.3.14. *Let δ be an escaping k-path in Δ from a to b and let γ be an α-replacement. Then*

$$\delta \cap T_a\Gamma \subseteq \gamma \cap T_a\Gamma \subseteq W_1,$$

where $\delta \cap T_a\Gamma$ is the set of vertices of δ that belong to $T_a\Gamma$, and $\gamma \cap T_a\Gamma$ is defined analogously.

Proof. The following argument is taken almost verbatim from [MPR21, Proof of Lemma 4.7]. It only requires minor modifications due to our definition of α-replacement.

By construction, $\delta \cap T_a\Gamma \subseteq \gamma \cap T_a\Gamma$. Observe that γ is a path of the form

$$\gamma = [a, \gamma_1, a, \gamma_2, a, \ldots, a, \gamma_m],$$

where each γ_i is a path that does not contain the vertex a. Note that $m \leq n$ and that γ is not escaping when $m > 1$. In order to prove $\gamma \cap T_a\Gamma \subseteq W_1$ is enough to show that $\gamma_i \cap T_a\Gamma \subset W_i$ for $1 \leq i \leq m$ in view of (5.3.2) and that $\gamma \cap T_a\Gamma = \bigcup_{i=1}^m \gamma_i \cap T_a\Gamma$.

Let w_i and z_i denote the initial and terminal vertices of γ_i, respectively. The main observation: since δ is an escaping path from a, it follows that the corner $[z_i, a, w_{i+1}]$ of γ is a translation of a corner of a path α_{ij} for some $v_i, v_j \in T_H\Delta$.

Claim 1. $w_m \in W_m$. Note that $[a, \gamma_m]$ is an escaping path of length at most n from a to b in Γ. Therefore $w_m \in \vec{ab}_\Gamma(n) = W_n$. Since $m \leq n$, it follows that $w_m \in W_n \subseteq W_m$.

Claim 2. $z_{m-1} \in Z_{m-1}$. Note that $[z_{m-1}, a, w_m]$ is the translation of a corner of α_{ij}; since $w_m \in W_m$, we have that $z_{m-1} \in Z_{m-1}$.

Claim 3. If $z_i \in Z_i$ then $w_i \in W_i$. Indeed, since

$$\angle_{T_a\Gamma}(z_i, w_i) \leq |\gamma_i| \leq n$$

and $z_i \in Z_i$, it follows that $w_i \in W_i$.

Claim 4. If $w_{i+1} \in W_{i+1}$ then $z_i \in Z_i$. As $[z_i, a, w_{i+1}]$ is the translation of a corner of an α_{ij}, if $w_{i+1} \in W_{i+1}$, then by definition we have that $z_i \in Z_i$.

Claim 5. If $z_i \in Z_i$ then $\gamma_i \cap T_a\Gamma$ is a subset of W_i. Let $x \in \gamma_i \cap T_a\Gamma$. Observe that $\angle_{T_a\Gamma}(z_i, x) \leq n$. Since $z_i \in Z_i$, it follows that $x \in W_i$.

To conclude, observe that the first four claims imply that $z_i \in Z_i$ for $1 \leq i \leq m$. Then the last claim imply that $\gamma_i \cap T_a\Gamma$ is a subset of $W_i \subset W_1$ for $1 \leq i \leq m$. □

The finite set X_0

Let

$$X_0 = \{x \in \vec{ab}(k)_\Delta \mid x \notin T_a\Gamma\}.$$

Lemma 5.3.15. *X_0 is a finite set.*

Proof. If $x \in X_0$ then x is a translate of the vertex H of Δ. Hence if X_0 is nonempty, then a is adjacent to a translate of the vertex H. Suppose that X_0 is nonempty and without loss of generality, assume that a is adjacent to H, specifically

$$T_H\Delta = \{v_1, v_2 \ldots, v_m\}, \qquad \text{and} \qquad a \in T_H\Delta.$$

Suppose that X_0 is infinite. For each $x \in X_0$ choose an escaping path δ_x from a to b in Δ of length at most k. Then there is $g_x \in G$ and $v_{i_x}, v_{j_x} \in T_H\Delta$ such that $[g_x.v_{i_x}, g_xH, g_x.v_{j_x}]$ is the initial 2-subpath of δ_x.

Since $T_H\Delta$ is finite, by the Pigeon hole argument there is a fixed pair, after renumbering if necessary, $v_1, v_2 \in T_H\Delta$ such that

$$X_1 = \{x \in X_0 \mid v_{i_x} = v_1 \text{ and } v_{j_x} = v_2\}$$

is an infinite set. Observe that

$$g_x.v_1 = a \text{ and } g_x.H = x \qquad \text{for all } x \in X_1.$$

For $x \in X_1$, let γ_x be an α-replacement of δ_x that has $g_x.\alpha_{12}$ as an initial subpath. By Lemma 5.3.14,

$$\gamma_x \cap T_a\Gamma \subset W_1.$$

We showed that W_1 is finite, see Lemma 5.3.13. By the Pigeon-hole argument, there is a $z \in T_a\Gamma$ such that

$$X_2 = \{x \in X_1 \colon \gamma_x \text{ has } z \text{ as its second vertex.}\}$$

is an infinite set. Let w be the second vertex of α_{12}. Then for every $x \in X_2$, $g_x.v_1 = a$ and $g_x.w = z$. Since $g_x.H = x$ for all $x \in X_2$, it follows that

$$C = \{g.H \in V(\Delta) \mid g \in G, \quad g.w = z, \quad g.v_1 = a\}$$

is an infinite set. Since Δ is a discrete G-graph with compact edge stabilizers, $[w, v_1, H]$ is a 2-path in Δ, and $[a, z]$ is a 1-path of Δ; Proposition 5.3.8 implies that C is finite, a contradiction. □

Conclusion.

It is left to verify that $\vec{ab}(k)_\Delta$ is finite. Let δ be an escaping path from a to b in Δ of length $\leq k$. Then Lemma 5.3.14 implies that $\delta \cap T_a\Gamma \subseteq W_1$. Since any vertex in $\delta \cap T_a\Delta$ is either in $T_a\Gamma$ or X_0, it follows that $\delta \cap T_a\Delta \subseteq W_1 \cup X_0$. Since δ was arbitrary,

$$\vec{ab}(k)_\Delta \subseteq W_1 \cup X_0.$$

By Lemmas 5.3.13 and 5.3.15, $W_1 \cup X_0$ is a finite set, and hence $\vec{ab}(k)_\Delta$ is finite.

5.3.5 Remark on Quasi-isometry

Under the assumptions of Theorem 5.3.4, the inclusion $\Gamma \hookrightarrow \Delta$ is a G-equivariant quasi-isometry. Indeed, for any path δ in Δ with endpoints in Γ, its α-replacement γ

is a path in Γ with the same endpoints and $|\gamma| \leq \ell|\delta|$. Hence for any pair of vertices u, v of Γ, $\operatorname{dist}_\Gamma(u,v) \leq \ell \operatorname{dist}_\Delta(u,v) \leq \ell \operatorname{dist}_\Gamma(u,v)$. On the other hand, every vertex of Δ is adjacent to a vertex of Γ.

5.4 Relative generation of groups

Definition 5.4.1 (Relative Compact Generation)**.** A topological group G is *compactly generated relative to a collection of subgroups* $\mathcal{H}$ if there is a compact subset $A \subset G$ such that G is algebraically generated by $A \cup \bigcup \mathcal{H}$; in this case we say that A is a *compact generating set of* G *relative to* $\mathcal{H}$.

The Milnor-Svarc Lemma implies that a discrete group is finitely generated if and only if it acts cellularly, cocompactly and with finite vertex stabilizers on a connected graph [BH99, Proposition 8.19]. The main result of this section generalizes this fact for proper pairs $(G, \mathcal{H})$, see Definition D.

For a definition of graph of groups we refer the reader Serre's book on trees [Ser80].

Definition 5.4.2 (Graph of topological groups)**.** Let $\Lambda = (V, E, r)$ be a connected graph. A *graph of topological groups* $(\mathcal{G}, \Lambda)$ based on the graph Λ consists of

1. a topological group $\mathcal{G}_v$ for every vertex $v \in V$.
2. a topological group $\mathcal{G}_e$ for every edge $e \in E$.
3. an open continuous monomorphism $\eta_e : \mathcal{G}_e \to \mathcal{G}_v$ for every edge $e \in E$ and $v \in V$ such that $v \in r(e)$.

Denote by $\pi_1(\mathcal{G}, \Lambda, a)$ the fundamental group of the graph of groups $(\mathcal{G}, \Lambda)$, where a is a vertex of Λ, see [Bas93]. There are canonical monomorphisms

$$i_v : \mathcal{G}_v \hookrightarrow \pi_1(\mathcal{G}, \Lambda, a)$$

up to conjugation. Since the group is independent of the choice of a, we will simply denote it by $\pi_1(\mathcal{G}, \Lambda)$.

An *embedding* of topological spaces is a continuous map $f \colon X \to Y$ that is homeomorphism onto its image. An embedding is *open* if f is an open map. Specifically open embedding is an injective, open, and continuous map. The fundamental group of a topological graph of groups $(\mathcal{G}, \Lambda)$ admits a canonical topology:

Proposition 5.4.3. [CdlH16, Propositions 8.B.9 and 8.B.10] *Let $(\mathcal{G}, \Lambda)$ be a finite graph of topological groups. There exists a unique topology on $\pi_1(\mathcal{G}, \Lambda)$ such that $\mathcal{G}_v \hookrightarrow \pi_1(\mathcal{G}, \Lambda)$ is an open topological embedding for each vertex v. Moreover if vertex and edge groups are locally compact then $\pi_1(\mathcal{G}, \Lambda)$ is locally compact.*

From here on, all graphs of topological groups $(\mathcal{G}, \Lambda)$ are assumed to satisfy that Λ is a finite connected graph and we always consider $\pi_1(\mathcal{G}, \Lambda)$ as a topological group with the topology provided by Proposition 5.4.3.

Definition 5.4.4 (Compact generating graph)**.** Let G be a topological group with a finite collection $\mathcal{H} = \{H_1, H_2, \cdots H_n\}$ of open subgroups. A *compact generating graph of G relative to $\mathcal{H}$* is a triple $(\mathcal{G}, \Lambda, \phi)$ where $(\mathcal{G}, \Lambda)$ is a finite graph of topological groups, $\phi : \pi_1(\mathcal{G}, \Lambda) \to G$ is a continuous open surjective homomorphism, and the following properties hold:

1. There are vertices $\{v_1, v_2, \cdots v_n\}$ of Λ and isomorphims of topological groups $\phi_i \colon \mathcal{G}_{v_i} \to H_i$ such that

$$\begin{tikzcd} \mathcal{G}_{v_i} \arrow[r, hook, "i_{v_i}"] \arrow[d, "\phi_i"] & \pi_1(\mathcal{G}, \Lambda) \arrow[d, "\phi"] \\ H_i \arrow[r, hook] & G \end{tikzcd}$$

   is a commutative diagram.

2. For every $v \in V(\Lambda)$, the map $\phi \circ i_v$ is injective.

3. For each edge $e \in E(\Lambda)$ and each vertex $v \neq v_i$ in $V(\Lambda)$, the edge group $\mathcal{G}_e$ and the vertex group $\mathcal{G}_v$ are compact topological groups.

Example 5.4.5. Let G be a finitely generated group with a finite generating set $X = \{x_1, x_2, \cdots x_t\}$. Consider a compact generating graph $(\mathcal{G}, \Lambda, \phi)$ of G relative to the empty collection with Λ given by the Figure 5.1a; with vertex group and edge groups trivial. Note that $\pi_1(\mathcal{G}, \Lambda)$ is isomorphic to the free group $F(X)$ generated over X and $\phi\colon F(X) \to G$ is the natural quotient map.

Example 5.4.6. Let G be a discrete group finitely generated with respect to a finite collection of groups $\mathcal{H} = \{H_1, H_2 \cdots H_n\}$ in the sense of Osin [Osi06, Definition 2.1].

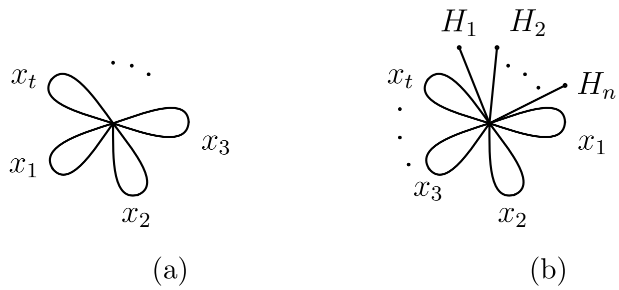

Figure 5.1: Compact generating graphs

In particular, there exists a finite set $X \subseteq G$ such that $(\bigcup_{i=1}^{n} H_i) \cup X$ generates G. Then $(\mathcal{G}, \Lambda, \phi)$ is a compact generating graph of G relative to $\mathcal{H}$ with Λ given by the Figure 5.1b; with edge groups and central vertex group trivial, and other vertex groups given by $H \in \mathcal{H}$. The homomorphism is the natural quotient map $\phi \colon \pi_1(\mathcal{G}, \Lambda) \to G$.

Example 5.4.7. Let G be a topological group with a compact open subgroup U and a finite set X such that $X \cup U$ generates G. Then $(\mathcal{G}, \Lambda, \phi)$ is a compact generating graph of G relative to the empty collection with Λ given by the Figure 5.1a; with vertex group U and edge groups given by $U \cap U^x$ for $x \in X$.

Remark 5.4.8 (Compact generating graph $\Longrightarrow$ Compact open subgroup)**.** Let G be a topological group with a compact generating graph $(\mathcal{G}, \Lambda, \phi)$ relative to a finite collection $\mathcal{H}$ of open subgroups. If Λ is a nontrivial graph (at least one edge) or $\mathcal{H}$ is empty, then G contains a compact open subgroup and hence G is locally compact. In particular, if G has no compact open subgroup, then $\mathcal{H} = \{G\}$.

A G-action on a cell complex X is called *discrete* if it is a cellular action such that the pointwise stabilizer of each cell is an open subgroup of G. A graph is a 1-dimensional cell complex.

Theorem 5.4.9 (Topological Characterization)**.** *Let $(G, \mathcal{H})$ be a proper pair. The following statements are equivalent:*

1. *There is a compact generating set of G relative to $\mathcal{H}$.*
2. *There exists a compact generating graph of G relative to $\mathcal{H}$.*
3. *There is a discrete, connected cocompact simplicial G-graph Γ with compact edge stabilizers, vertex stabilizers are either compact or conjugates of subgroups in $\mathcal{H}$,*

every $H \in \mathcal{H}$ is the G-stabilizer of a vertex, and any pair of vertices with the same G-stabilizer $H \in \mathcal{H}$ are in the same G-orbit if H is non-compact.

Theorem 5.4.9 relies on the notion of relative Cayley-Abels graph. Its definition uses Bass-Serre trees. For the rest of the article when considering a Bass-Serre tree, we mean the realization as a CW-complex of dimension at most one as described in the book by Serre [Ser80, Pg. 14].

Definition 5.4.10 (Relative Cayley-Abels graph)**.** Let $(\mathcal{G}, \Lambda, \phi)$ be a compact generating graph of a topological group G relative to a finite collection $\mathcal{H}$ of open subgroups, and let $\mathcal{T}$ be the corresponding Bass-Serre tree. The *relative Cayley-Abels graph* $\Gamma(\mathcal{G}, \Lambda, \phi)$ of G with respect to $\mathcal{H}$ corresponding to $(\mathcal{G}, \Lambda, \phi)$ is defined as the G-graph $\mathcal{T}/\ker(\phi)$. A G-graph isomorphic to $\Gamma(\mathcal{G}, \Lambda, \phi)$ for some compact generating graph $(\mathcal{G}, \Lambda, \phi)$ of G with respect to $\mathcal{H}$ is called a *relative Cayley-Abels graph of G with respect to $\mathcal{H}$.*

Theorem 5.4.11 (Characterization of Relative Cayley-Abels graphs)**.** *Let $(G, \mathcal{H})$ be a proper pair. For a G-graph Γ, the following statements are equivalent:*

1. *There is a compact generating graph (G, Λ, ϕ) of G relative to $\mathcal{H}$ such that Γ is the corresponding relative Cayley-Abels graph (up to isomorphism of G-graphs).*

2. *Γ is a connected discrete cocompact simplicial G-graph with compact edge stabilizers, vertex stabilizers are either compact or conjugates of subgroups in $\mathcal{H}$, every $H \in \mathcal{H}$ is the G-stabilizer of a vertex, and any pair of vertices with the same G-stabilizer $H \in \mathcal{H}$ are in the same G-orbit if H is non-compact.*

Moreover, if Γ satisfies the above conditions, then for any vertex $v \in \Gamma$, the stabilizer G_v is compact if and only if v has finite degree.

Example 5.4.12. The Cayley graph of a discrete group G with respect to a finite generating set X is defined as a graph Γ with vertex set $V(\Gamma) = G$ and edge set $E(\Gamma) = \{\{g, gx\} | g \in G, x \in X\}$. Consider a generating graph of G given by the graph in the Figure 5.1a. Then the corresponding relative Cayley-Abels graph is G-isomorphic to the Cayley graph with respect to the generating set X.

Example 5.4.13 (Farb's Coned-off Cayley Graph [Far98])**.** Let G be a group, let $\mathcal{H} = \{H_1, H_2 \cdots, H_n\}$ be a finite collection of subgroups, and let $X \subset G$ be a relative finite generating set of G with respect to $\mathcal{H}$. The *Coned-off Cayley graph* $\hat{\Gamma}(G, \mathcal{H}, X)$ of G with respect $\mathcal{H}$ is the graph $\hat{\Gamma}$ with the vertex set $V(\hat{\Gamma}) = (\bigcup_{i=1}^{n} G/H_i) \cup G$ and edge set $E(\hat{\Gamma}) = \{\{g, gx\} \mid g \in G, x \in X\} \cup \{\{k, gH\} \mid g \in G, H \in \mathcal{H}, k \in gH\}$. Consider a compact generating graph of G relative to $\mathcal{H}$ given by the graph of groups described in Figure 5.1b, and observe that the corresponding relative Cayley-Abels graph is G-isomorphic to a Coned-off Cayley graph $\hat{\Gamma}$.

Example 5.4.14 (Krön and Möller's Cayley-Abels graph [KM08])**.** Let G be a compactly generated totally disconnected locally compact group. Then G contains a compact open subgroup U and a finite subset $X \subset G$ such that any element of G is in a left coset wU where w is a word in X, see [KM08, Lemma 2]. The *Cayley-Abels graph* $\Gamma(G, U, X)$ is defined as a graph Γ with the vertex set $V(\Gamma) = G/U$ and the edge set $E(\Gamma) = \{\{gU, gxU\} \mid g \in G, x \in X\}$; the resulting graph is a connected vertex-transitive locally finite G-graph [KM08, Construction 1]. The quotient of Γ by G induces a compact generating graph of G with respect to the empty collection of the form Figure 5.1a with compact vertex group.

Let us record two immediate consequences of Theorem 5.4.9.

Corollary 5.4.15. *Let G be a topological group with a compact open subgroup. The following statements are equivalent:*

1. *G admits a compact generating set.*

2. *G admits a compact generating graph relative to the empty collection.*

Corollary 5.4.16 (Krön and Möller)**.** [KM08, Corollary 1] *Let G be a totally disconnected locally compact group. Suppose G acts on a connected locally finite graph Γ such that the stabilizers of vertices are compact open subgroups and G has only finitely many orbits on $V(\Gamma)$. Then G is compactly generated.*

The rest of the section is divided into four subsections. The proof of Theorem 5.4.11 is divided into two propositions which are the contents of subsections 5.4.1 and 5.4.2 respectively. Subsection 5.4.3 contains the proof of Theorem 5.4.11. Then we prove Theorem 5.4.9 in Subsection 5.4.4.

5.4.1 From Compact generating graph to discrete cococompact action on a graph.

Versions of the following proposition can be found in the literature, for example see [ACCCMP21, Proof of Theorem 7.1] and [HMPS21, Proposition 4.9].

Proposition 5.4.17. *Let $(\mathcal{G}, \Lambda, \phi)$ be a compact generating graph of G relative to a finite collection $\mathcal{H}$ of open subgroups, and let $\mathcal{T}$ be the corresponding Bass-Serre tree. Consider the relative Cayley-Abels graph $\Gamma = \Gamma(\mathcal{G}, \Lambda, \phi)$, the quotient map $\rho\colon \mathcal{T} \to \Gamma$, and the sequence*

$$1 \to \ker(\phi) \to \pi_1(\mathcal{G}, \Lambda) \xrightarrow{\phi} G \to 1.$$

Then the following statements hold:

1. *The group $\ker(\phi)$ is a discrete and closed subgroup of $\pi_1(\mathcal{G}, \Lambda)$ which acts freely on $\mathcal{T}$, and ρ is a covering map.*
2. *If x is a vertex of Γ, then there is a group isomorphism*
$$\ker(\phi) \to \pi_1(\Gamma, x) \qquad g \mapsto [\gamma_g],$$
where $\gamma_g = \rho \circ \alpha_g$ and α_g is the shortest path from x to $g.x$ in $\mathcal{T}$.
3. *The group G acts on Γ discretely, cocompactly, edge stabilizers are compact, and vertex stabilizers are either compact or conjugates of subgroups in $\mathcal{H}$.*
4. *For any $H \in \mathcal{H}$, there is a vertex $v \in \Gamma$ such that $G_v = H$.*
5. *Suppose $(G, \mathcal{H})$ is a proper pair. Then any pair of vertices with the same G-stabilizer $H \in \mathcal{H}$ are in the same G-orbit if H is non-compact.*
6. *A vertex of Γ has compact G-stabilizer if and only if it has finite degree.*

Recall that a group action on a complex has no inversions if for every cell its setwise stabilizer coincides with the pointwise stabilizer.

Lemma 5.4.18. *Let G be a group acting on a connected graph K discretely, cocompactly, and with compact edge stabilizers. Then for any vertex $v \in K$ incident to at least one edge, v has infinite degree if and only if its stabilizer G_v is non-compact.*

Proof. If the action has inversions, replace K with its barycentric subdivision. Let v be vertex in K and e be an edge incident to v, then there is bijection between the set of left cosets G_v/G_e and the G_v-orbit of e given by $gG_e \mapsto ge$. Let us suppose v has infinite degree. Since the action is cocompact, there exists an edge e adjacent to v with an infinite orbit. Thus $[G_v : G_e]$ is infinite. Since G_e is open, the cosets of G_e form an infinite cover of G_v with no finite subcover. Conversely suppose G_v is non-compact and let e be an edge adjacent to v. Since G_e is compact, $[G_v : G_e]$ is infinite. Thus v has infinite degree. $\square$

Proof of Proposition 5.4.17. Let $\widetilde{G} = \pi_1(\mathcal{G}, \Lambda)$ and let $\tilde{x} \in \mathcal{T}$ be any vertex. Since ϕ is injective on the vertex stabilizer $\widetilde{G}_{\tilde{x}}$, we have $\ker(\phi) \cap \widetilde{G}_{\tilde{x}} = 1$. Hence $\ker(\phi)$ acts freely and cellularly on $\mathcal{T}$. Therefore $\rho\colon \mathcal{T} \to \Gamma$ is a covering map, and $\ker(\phi)$ is a discrete and closed subgroup of $\widetilde{G}$. The isomorphism $\ker(\phi) \to \pi_1(\Gamma, x)$ is a well known consequence of covering space theory.

For item (3), we first observe the following properties about the $\widetilde{G}$-action on $\mathcal{T}$. The action is cocompact since $\mathcal{T}/\widetilde{G} \simeq \Lambda$ is a finite graph; and since vertex and edge stabilizers of $\mathcal{T}$ in $\widetilde{G}$ are isomorphic to vertex and edge groups, by Proposition 5.4.3, they are open subgroups of $\widetilde{G}$, in particular the $\widetilde{G}$-action on $\mathcal{T}$ is discrete. Moreover, by Definition 5.4.4, for the $\widetilde{G}$-action on $\mathcal{T}$, edge stabilizers are compact and vertex stabilizers are either compact or conjugates of subgroups in $\mathcal{H}$. Moreover, for every $H \in \mathcal{H}$, the tree T contains a vertex with $\widetilde{G}$-stabilizer equal to H.

Let N denote the normal subgroup $\ker(\phi)$ and let $\rho\colon \mathcal{T} \to \Gamma$ be the quotient map. The $\widetilde{G}$-action on $\mathcal{T}$ induces an action of $G = \widetilde{G}/N$ on $\Gamma = \mathcal{T}/N$. The covering map $\rho\colon \mathcal{T} \to \Gamma$ is equivariant with respect to ϕ and the restriction $\phi\colon \widetilde{G}_v \to G_{\phi(v)}$ is an isomorphism for any vertex v of T.

Let y be any vertex or edge in Γ, and let $\tilde{y}$ in $\mathcal{T}$ such that $\rho(\tilde{y}) = y$. Then $\widetilde{G}_{\tilde{y}}N$ is an open subgroup and $G_y = \phi(\widetilde{G}_{\tilde{y}}N) = \phi(\widetilde{G}_{\tilde{y}})$. Therefore G_y is open as ϕ is an open map, and if $\widetilde{G}_y$ is compact, then G_y is compact as well. To summarize, G-action on Γ is discrete; is cocompact since Λ is finite; has compact edge stabilizers, and each vertex stabilizer is either compact or a conjugate of some $H \in \mathcal{H}$, and each $H \in \mathcal{H}$ is the G-stabilizer of a vertex of Γ.

Now we prove item (5). Each subgroup $H \in \mathcal{H}$ is naturally identified with an isomorphic subgroup of $\widetilde{G}$, see Definition 1; to simplify notation, we assume H is a subgroup of $\widetilde{G}$ and ϕ restricted to H is the identity map.

Let u_1 and u_2 be vertices of Γ with the same G-stabilizer $H \in \mathcal{H}$ and suppose H is non-compact. Let v_1 and v_2 vertices of $\mathcal{T}$ mapping to u_1 and u_2 respectively. Suppose that v_1 has $\widetilde{G}$-stabilizer the vertex group $H_1^{g_1}$, and v_2 has $\widetilde{G}$-stabilizer $H_2^{g_2}$ where $g_1, g_2 \in \widetilde{G}$ and $H_1, H_2 \in \mathcal{H}$. Then $\phi(H_1^{g_1}) = H$ and $\phi(H_2^{g_2}) = H$; it follows that H_1, H_2 and H are subgroups in $\mathcal{H}$ that are pairwise conjugate in G. Since $(G, \mathcal{H})$ is a proper pair, $H_1 = H_2 = H$. Then $g_1^{-1}v_1$ and $g_2^{-1}v_2$ have both $\widetilde{G}$-stabilizer H. Since H is non-compact and open and all edge $\widetilde{G}$-stabilizer of the the tree $\mathcal{T}$ are compact open, it follows that $g_1^{-1}v_1$ and $g_2^{-1}v_2$ are the same vertex. Hence v_1 and v_2 are in the same $\widetilde{G}$-orbit, and therefore u_1 and u_2 are in the same G-orbit.

For the last item, Lemma 5.4.18 implies that each vertex of Γ with compact stabilizer has finite degree. □

5.4.2 From cocompact discrete action on a graph to compact generation

Proposition 5.4.19. *Let G be a topological group that acts on a connected graph Γ cocompactly, discretely, without inversions, and with compact edge stabilizers.*

Let S be a subset of vertices of Γ such that distinct elements are in distinct G-orbits, and all vertices of Γ with non-compact G-stabilizer are represented. Let $\mathcal{H} = \{G_v \mid v \in S\}$.

Then there exists a compact generating graph of G relative to $\mathcal{H}$ such that the corresponding relative Cayley-Abels graph and Γ are isomorphic as G-graphs.

Lemma 5.4.20. *Suppose G, H, and K are topological groups such that the diagram*

$$\begin{tikzcd} K \arrow[r, hook, "i"] \arrow[dr, "j"'] & H \arrow[d, "\phi"] \\ & G \end{tikzcd}$$

of homomorphism of groups commutes. If i, j are open, continuous. Then ϕ is continuous and open.

Proof. To prove that ϕ is continuous, we show that ϕ is continuous at 1. Let $U \subseteq G$

be an open set containing 1. Since j is continuous, $j^{-1}(U)$ is open. Let $W = i(j^{-1}(U))$. Then W is open since i is an open map; and by the commutative diagram, $1 \in W \subseteq \phi^{-1}(U)$. Since U was an arbitrary neighborhood of 1 in G, this shows that ϕ is continuous at 1. To prove that ϕ is open, first observe that if V is open in H and $V \subset i(K)$, then i being continuous and j being open imply that $\phi(V) = j(i^{-1}(V))$ is open in G. Moreover, for any $h \in H$, $\phi(h.V) = \phi(h).\phi(V)$ is open in G. For an arbitrary open subset U of H, let $U_h = U \cap h.i(K)$. Observe that $h^{-1}.U_h$ is open in H and $h^{-1}.U_h \subset V$. Therefore $\phi(U_h)$ is open in G, and hence $\phi(U) = \bigcup_{h \in H} \phi(U_h)$ is open in G. $\square$

Proof of Proposition 5.4.19. For discrete groups, the result follows from a well known construction of Bass [Bas93, Section 3] that we recall below. The case for topological groups follows from the same construction after addressing topological matters.

Let Λ be the quotient graph given by Γ/G, and let $r \colon \Gamma \to \Lambda$ be the quotient map. Choose a tree T and a connected graph S such that $T \subseteq S \subseteq \Gamma$, $r \colon T \to \Lambda$ is bijective on vertices, and $r \colon S \to \Lambda$ bijective on edges. For any $v \in \Lambda$ and $e \in \Lambda$, the vertex group $\mathcal{G}_v$ and edge group $\mathcal{G}_e$ are defined to be the vertex stabilizer $G_{v'}$ and the edge stabilizer $G_{e'}$, where v' and e' are preimages of v and e under r in T and S respectively. For every vertex $s \in S$, choose $g_s \in G$ such that $g_s s \in T$ and assume $g_s = 1$ if $s \in T$. Let e be an edge of Λ with endpoint vertices u and v. Let e' be the preimage of e in S with end points u' and w'. Since e' has at least one endpoint in T, without loss of generality assume that $u' \in T$. Define the morphism $\mathcal{G}_e \to \mathcal{G}_u$ to be the inclusion. The morphism from $\mathcal{G}_e \to \mathcal{G}_v$ is defined as $h \mapsto g_{w'} h g_{w'}^{-1}$. Thus $(\mathcal{G}, \Lambda)$ defines a graph of groups. Observe that if G is a topological group with discrete action on Γ then $G_{e'}$ and $G_{v'}$ are topological groups with the subspace topology and the morphisms $G_{e'} \to G_{v'}$ are open topological embeddings. In particular, $(\Lambda, \mathcal{G})$ is a graph of topological groups.

Fix a vertex $a \in \Lambda$ and let $a' \in \Gamma$ such that $r(a') = a$. Let $\widetilde{G} = \pi_1(\mathcal{G}, \Lambda, a)$ be the fundamental group of graph of groups, and let $\mathcal{T}$ be the corresponding Bass-Serre tree. Then by [Bas93, Theorem 3.6] there is a short exact sequence

$$1 \to \pi_1(\Gamma, a') \to \widetilde{G} \xrightarrow{\phi} G \to 1,$$

and a ϕ-equivariant morphism of graphs $\rho \colon \mathcal{T} \to \Gamma$, such that for any vertex $y \in \mathcal{T}$, the restriction of ϕ to stabilizers $\widetilde{G}_y \xrightarrow{\phi} G_{\rho(y)}$ is an isomorphism.

Now we construct a compact generating graph of G relative to $\mathcal{H}$. Consider the graph of topological groups $(\Lambda, \mathcal{G})$ constructed above. Since G acts on Γ cocompactly, $(\Lambda, \mathcal{G})$ is a finite topological graph of groups. The hypothesis on the G-stabilizers of vertices and edges of Γ imply that the edge groups of $(\Lambda, \mathcal{G})$ are compact and the vertex groups are either compact or in $\mathcal{H}$. Let $w \in \Gamma$ be any vertex and $\tilde{w} \in \mathcal{T}$ be one of its pre-images under ρ. Then $\widetilde{G}_{\tilde{w}} \xrightarrow{\phi} G_w$ is an isomorphism and we have the following commutative diagram.

$$\begin{array}{ccc} \widetilde{G}_{\tilde{w}} & \hookrightarrow & \tilde{G} \\ \big\downarrow & & \big\downarrow{\scriptstyle\phi} \\ G_w & \hookrightarrow & G \end{array}$$

Since $\widetilde{G}$-action on $\mathcal{T}$ is discrete, $\widetilde{G}_{\tilde{w}}$ is open in $\widetilde{G}$, and thus by Lemma 5.4.20, ϕ is continuous and open. Therefore $(\Lambda, \mathcal{G}, \phi)$ is a compact generating graph of G relative to $\mathcal{H}$. Since $\rho\colon \mathcal{T} \to \Gamma$ is a ϕ-equivariant morphism of graphs, the corresponding relative Cayley-Abels graph is isomorphic to Γ as a G-graph. $\square$

5.4.3 Proof of Theorem 5.4.11

That (1) implies (2) follows directly from Proposition 5.4.17. Conversely, suppose (2) holds. For each $H \in \mathcal{H}$, let v_H be a vertex in Γ with G-stabilizer H, and let $S = \{v_H \mid H \in \mathcal{H}\}$. Since no two distinct subgroups in $\mathcal{H}$ are conjugate, no two distinct elements of S are in same G-orbit. For any vertex v of Γ with non-compact G-stabilizer, G_v is conjugate to a subgroup in $\mathcal{H}$. Since any pair of vertices with the same G-stabilizer $H \in \mathcal{H}$ are in the same G-orbit if H is non-compact; it follows that every vertex of Γ with non-compact stabilizer is in the G-orbit of an element in S. By Proposition 5.4.19, there exists a compact generating graph $(\mathcal{G}, \Lambda, \phi)$ of G relative to $\mathcal{H}$ such that the corresponding relative Cayley-Abels graph is isomorphic to Γ. The statement for vertices of Γ on the equivalence on having compact G-stabilizer and having finite degree in Γ is part of the conclusion of Proposition 5.4.17.

5.4.4 Proof of Theorem 5.4.9

The equivalence of (2) and (3) is a direct consequence of Theorem 5.4.11. The equivalence of (1) and (3) follows from Proposition 5.4.22 stated below.

Proposition 5.4.21. [BH99, Theorem 8.10] *Let X be a topological space, let G be a group acting on X by homeomorphisms, and let U be an open subset such that $X = GU$. If X is connected, then the set $S = \{g \in G \mid g.U \cap U \neq \emptyset\}$ generates G.*

Proposition 5.4.22. *Let G be a topological group with a compact open subgroup U. Let $\mathcal{H}$ be a finite collection of open subgroups. The following statements are equivalent.*

1. *There is a compact subset $A \subset G$ such that $G = \langle A \cup \bigcup \mathcal{H} \rangle$.*

2. *There is a Cayley-Abels graph Γ of G relative to $\mathcal{H}$.*

Proof. Suppose there is a relative compact generating set A of G with respect to $\mathcal{H}$. Compactness of A implies that there is a finite subset $S \subset G$ such that $A \subset SU$. Define Γ as the G-graph with vertex set $V(\Gamma) = G/U \cup G/\mathcal{H}$ and edge set $E(\Gamma) = \{\{gU, gxU\} \mid g \in G,\ x \in S\} \cup \{\{gU, gH\} \mid g \in G,\ H \in \mathcal{H}\}$. Note that G acts discretely, there are only finitely many orbits of vertices and edges on Γ, edge stabilizers are compact open, vertex stabilizers are conjugates of U or $H \in \mathcal{H}$; moreover, for every $H \in \mathcal{H}$ there is a vertex in Γ with stabilizer equal to H. It is left to prove that Γ is connected. Indeed, if there is a path α in Γ from U to gU and $h \in X \cup \bigcup \mathcal{H}$, then there is a path from u to ghu, namely the concatenation of the path α and $g\beta$ where β is a path from u to hu given by

$$\beta = [u, H, h.u], \qquad \text{or} \qquad \beta = [U, hU]$$

if $h \in \bigcup \mathcal{H}$ or $h \in X$ respectively. By Theorem 5.4.11, Γ is a Cayley-Abels graph of G relative to $\mathcal{H}$.

Suppose Γ is a Cayley-Abels graph of G relative to H. Since Γ is connected and cocompact, Theorem 5.4.21 implies that G is generated by finite number of vertex stabilizers $G_{v_1}, \ldots, G_{v_k}$ and a finite set S. Then S together with the union of the G_{v_i} that are compact is a compact generating set of G relative to $\mathcal{H}$. □

5.5 Invariance of relative Cayley-Abels graphs

The main results of this section are Theorem 5.5.2 and Corollary 5.5.3. These results generalize the fact that any two Cayley graphs with respect to finite generating sets of a group are quasi-isometric [Bra99, Chapter I, Example 8.17(3)]. Note that these graphs are locally finite.

Definition 5.5.1. [Bow12] A graph Γ is said to be *fine* if each edge of Γ is contained in only finitely many circuits of length n for any n.

There is relationship between fineness and isoperimetric functions which was made explicit by Groves and Manning [GM08, Proposition 2.50, Question 2.51]. This relation has also been studied in [MP16, HMPS21].

Theorem 5.5.2. *Let $(G, \mathcal{H}_1)$ and $(G, \mathcal{H}_2)$ be proper pairs. Suppose the symmetric difference of $\mathcal{H}_1$ and $\mathcal{H}_2$ consists only of compact subgroups.*

If Γ_1 and Γ_2 are relative Cayley-Abels graphs of G with respect to $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then

1. *Γ_1 and Γ_2 are quasi-isometric; and*
2. *Γ_1 is fine if and only if Γ_2 is fine.*

The following corollary follows directly from Theorem 5.5.2.

Corollary 5.5.3 (Quasi-isometry Invariance of Relative Cayley-Abels Graphs)**.** *Let $(G, \mathcal{H})$ be a proper pair.*

1. *Any two relative Cayley-Abels graphs of G with respect to $\mathcal{H}$ are quasi-isometric.*
2. *If one relative Cayley-Abels graph of G with respect to $\mathcal{H}$ is fine, then all are fine.*

We also recover the following results as direct corollaries.

Corollary 5.5.4. [MP16, Theorem 1.4] *Let G be a discrete group finitely generated with respect to a finite collection $\mathcal{H}$ of subgroups. If the collection $\mathcal{H}$ is almost malnormal, then any two coned-off Cayley graphs are quasi-isometric.*

Note that the malnormality condition in the above corollary can be relaxed to require that $(G, \mathcal{H})$ is a proper pair.

Corollary 5.5.5. [KM08, Theorem 2] *Let G be a compactly generated topological group with a compact open subgroup. Then any two Cayley-Abels graphs are quasi-isometric.*

5.5.1 Proof of Theorem 5.5.2

Let us fix some notation. For a graph Γ denote by $\mathcal{V}_\infty(\Gamma)$ its set of vertices with infinite degree. For a collection of subgroups $\mathcal{H}$ of G, let $\mathcal{H}_\infty = \{H \in \mathcal{H} \mid \ H \text{ is non-compact}\}$ and $G/\mathcal{H}_\infty = \{gH \mid g \in G, H \in \mathcal{H}_\infty\}$.

Lemma 5.5.6. *Let $(G, \mathcal{H})$ be a proper pair. If Γ is a relative Cayley-Abels graph. Then there exists a G-equivariant bijection between $G/\mathcal{H}_\infty$ and $\mathcal{V}_\infty(\Gamma)$.*

Proof. By Theorem 5.4.11, for any $H_i \in \mathcal{H}_\infty$ there exists a vertex $v_i \in \Gamma$ such that $G_{v_i} = H_i$. Since $(G, \mathcal{H})$ is proper, no two distinct subgroups of $\mathcal{H}_\infty$ are conjugate. It follows that if H_i and H_j are distinct subgroups in $\mathcal{H}$, then v_i and v_j are in distinct G-orbits. By Theorem 5.4.11, each v_i has infinite degree in Γ; if w is a vertex of Γ with G-stabilizer a conjugate of H_i then w and v_i are in the same G-orbit; and any vertex in $V_\infty(\Gamma)$ has non-compact G-stabilizer. Consider the G-map $\pi \colon G/\mathcal{H}_\infty \to \mathcal{V}_\infty$ given by $gH_i \mapsto gv_i$. Then

1. π is well defined since $gH_i = fH_i$ iff $f^{-1}g \in H_i = G_{v_i}$ iff $gv_i = fv_i$.

2. π is injective. Suppose $gH_i, fH_j \in G/\mathcal{H}_\infty$ and $gv_i = fv_j$. Then v_i and v_j are in the same G-orbit and hence $H_i = H_j$. Thus $gv_i = fv_i$ implies $gH_i = fH_i$.

3. π is surjective. Let $w \in \mathcal{V}_\infty$. Then the G_w is non-compact and therefore G_w is conjugate to some $H_i \in \mathcal{H}_\infty$. Hence v_i and w are in the same G-orbit, say $w = gv_i$. Then $gH_i \in G/\mathcal{H}_\infty$ maps to w. □

Proof of Theorem 5.5.2. Let $(\mathcal{G}_1, \Lambda_1, \phi_1)$ and $(\mathcal{G}_2, \Lambda_2, \phi_2)$ be two compact generating graphs of G relative to $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and let $\Gamma_1(\mathcal{G}_1, \Lambda_1, \phi_1)$ and $\Gamma_2(\mathcal{G}_2, \Lambda_2, \phi_2)$ be their corresponding relative Cayley-Abels graphs. By Lemma 5.5.6, there exists a G-equivariant bijection η between $V_\infty(\Gamma_1)$ and $V_\infty(\Gamma_2)$. The result follows from Corollary 5.3.5. □

5.6 Compact relative presentations

A discrete group is finitely presented if and only if it acts cellularly, cocompactly and with finite vertex stabilizers on a simply connected space. This result is credited to [Mac64], see [BH99, Corollary 8.11]. The main results of this section, Theorem 5.6.6 and Corollary 5.6.7, generalize this fact for proper pairs $(G, \mathcal{H})$. We introduce the notions of compact generalized presentation and relative Cayley-Abels complex for pairs $(G, \mathcal{H})$ in order to state our main result.

Definition 5.6.1 (Compact generalized presentation)**.** Let G be a topological group and let $\mathcal{H}$ be a finite collection of open subgroups. A *compact generalized presentation of G relative to $\mathcal{H}$* is a pair

$$\langle\ (\mathcal{G}, \Lambda, \phi) \mid R\ \rangle \tag{5.6.1}$$

where $(\mathcal{G}, \Lambda, \phi)$ is a compact generating graph of G relative to $\mathcal{H}$, and $R \subseteq \pi_1(\mathcal{G}, \Lambda)$ is a finite subset such that $\langle\langle R \rangle\rangle = ker(\phi)$. In particular,

$$1 \to \langle\langle R \rangle\rangle \to \pi_1(\mathcal{G}, \Lambda) \xrightarrow{\phi} G \to 1$$

is a short exact sequence and ϕ is a continuous open surjective homomorphism.

The generalized presentations have been previously studied for TDLC groups [CW16a].

Definition 5.6.2. A topological group G is said to be *compactly presented relative to a finite collection of open subgroups $\mathcal{H}$* if there exists a compact generalized presentation relative to $\mathcal{H}$.

Remark 5.6.3 (Compact generalized presentation $\Longrightarrow$ Compact open subgroup)**.** If is G a topological group is compactly presented relative to a finite collection of open subgroups $\mathcal{H}$, then G has a compact open subgroup or $\mathcal{H} = \{G\}$. This is a consequence of Remark 5.4.8.

Example 5.6.4. We note examples of compact generalized presentation $\langle\ (\mathcal{G}, \Lambda, \phi) \mid R\ \rangle$ for some groups.

1. Let G be a finitely presented group with presentation $\mathcal{P} = \langle X | R \rangle$. Then a compact generalized presentation of G relative to the empty collection is given by $\langle\ (\mathcal{G}, \Lambda, \phi) \mid R\ \rangle$, where $(\mathcal{G}, \Lambda, \phi)$ is the graph of groups given in the Figure 5.1a.

$$\left\langle \begin{tikzcd} A \arrow[r, dash, "C"] & B \end{tikzcd} \;\middle|\; \mathbf{R} \right\rangle$$

Figure 5.2

2. Let G be a discrete group finitely generated with respect to a finite collection of groups $\mathcal{H} = \{H_1, H_2 \cdots H_n\}$. Let $X \subseteq G$ such that $(\bigcup_{i=1}^{n} H_i) \cup X$ generates G. Let $\langle X, \mathcal{H} | R \rangle$ be the relative presentation of G with respect to H as defined by Osin [Osi06]. Then a compact generalized presentation of G relative to H is given by $\langle\ (\mathcal{G}, \Lambda, \phi) \mid R\ \rangle$, where $(\mathcal{G}, \Lambda, \phi)$ is the graph of groups given in the Figure 5.1b.

3. Let $A *_C B$ be an amalgamated free product of groups A and B over a common compact open subgroup. Let R satisfy the $C'(1/12)$-small cancellation condition and let $N = \langle\langle R \rangle\rangle$ be the normal subgroup of G generated by R. Then N is a discrete group of $A *_C B$ by Proposition 5.4.17 (see also [ACCCMP21, Proof of Theorem 7.1]). Then a compact generalized presentation of $G = (A *_C B)/N$ relative to the collection $\{A, B\}$ is given by $\langle\ (\mathcal{G}, \Lambda, \phi) \mid R\ \rangle$, where $(\mathcal{G}, \Lambda, \phi)$ is the graph of groups given in the

Definition 5.6.5 (Relative Cayley-Abels complex)**.** Let G be a topological group compactly generated relative to a finite collection $\mathcal{H}$ of open subgroups. A *relative Cayley-Abels complex of G with respect to $\mathcal{H}$* is a discrete simply connected cocompact 2-dimensional G-complex with 1-skeleton a relative Cayley-Abels graph of G with respect to H

Theorem 5.6.6 (Topological Characterization)**.** *Let G be a topological group and let $\mathcal{H}$ be a finite collection of open subgroups. The following statements are equivalent:*

1. *G is compactly presented with respect to $\mathcal{H}$.*

2. *There exists a relative Cayley-Abels complex of G with respect to $\mathcal{H}$.*

The following corollary strengthen the previous theorem by imposing an additional hypothesis; it is proven in Section 5.6.2.

Corollary 5.6.7. *Let $(G, \mathcal{H})$ be a proper pair. Suppose there is a fine relative Cayley-Abels graphs of G with respect to $\mathcal{H}$. Then the following statements are equivalent:*

1. *G is compactly presented with respect to $\mathcal{H}$.*

2. *Any relative Cayley-Abels graph of G with respect to $\mathcal{H}$ is the 1-skeleton of a relative Cayley-Abels complex of G with respect to $\mathcal{H}$.*

Definition 5.6.8 (Compactly presented)**.** [CdlH16, Definition 8.A.1] A topological group G is *compactly presented* if G has a group presentation $\langle S|R\rangle$ with S compact and R a set of words in S of a bounded length.

Corollary 5.6.9 below states that compactly presented is equivalent to being compactly presented relative to the empty collection in the case that the group has a compact open subgroup. The proof of this statement is implicit in the results of [CdlH16] and [CW16a, Section 5.8]. A proof in the class of TDLC groups is discussed in [ACCCMP21, Corollary 3.5]. We discuss the argument in Section 5.6.3.

Corollary 5.6.9. *The following statements are equivalent for any topological group G.*

1. *G is compactly presented relative relative to the empty collection.*

2. *G has a compact open subgroup and is compactly presented.*

As a consequence, we also recover the following corollary from [ACCCMP21, Proposition 3.6], the proof of which relied on a result from [CC20, Proposition 3.4].

Corollary 5.6.10. [CC20] *A TDLC-group G is compactly presented if and only if there exists a simply connected cellular G-complex X with compact open cell stabilizers, finitely many G-orbits of cells of dimension at most 2, and such that elements of G fixing a cell setwise fixes it pointwise (no inversions)*

The rest of the section is subdivided into three subsections covering the proofs of the Theorem 5.6.6, Corollary 5.6.7, and Corollary 5.6.9 respectively.

5.6.1 Proof of the Theorem 5.6.6

The proof follows directly from Proposition 5.6.11 and Proposition 5.6.13 below.

Proposition 5.6.11 is a standard construction for discrete groups, see [BH99, Lemma 8.9], and it appears as [ACCCMP21, Theorem 7.1] for TDLC groups that split as an amalgamated free product over a compact open subgroup.

Proposition 5.6.11. *Let G be a topological group compactly presented relative to a finite collection $\mathcal{H}$ of open subgroups. Then there exists a relative Cayley-Abels complex X of G relative to $\mathcal{H}$ such that no distinct 2-cells of X have the same boundary.*

Proof. Let $\mathcal{P} = \langle\ (\mathcal{G}, \Lambda, \phi) \mid R\ \rangle$ be a compact generalized presentation of G relative to $\mathcal{H}$, and let $\mathcal{T}$ be the corresponding Bass-Serre tree. Consider a G-complex X constructed as follows. Let 1-skeleton $X^{(1)}$ be the relative Cayley-Abels graph $\Gamma = \mathcal{T}/\ker(\phi)$, and let $\rho\colon \mathcal{T} \to \Gamma$ be the natural quotient map. Suppose $x \in \Gamma$ be any vertex, then by Proposition 5.4.17, there is a group isomorphism from $\ker(\phi) \to \pi_1(\Gamma, x)$ given by $g \mapsto [\gamma_g]$, where γ_g is an combinatorial closed path in Γ based at x. For $g \in G$ and $h \in \ker(\phi)$, let $g\gamma_h$ be the translated closed path without an initial point, i.e., these are cellular maps from $S^1 \to X$. Consider the G-set $\Omega = \{g\gamma_r \mid r \in R, g \in G\}$ of closed paths in $X^{(1)}$.

Let X be the G-complex obtained by attaching a 2-cell to $X^{(1)}$ for every closed path in Ω. In particular, the pointwise G-stabilizer of a 2-cell of X coincides with the pointwise G-stabilizer of its boundary path. By Proposition 5.4.17, the G-action is discrete on X. Observe that the natural isomorphism from $\ker(\phi)$ to $\pi_1(X^{(1)}, \rho(x_0))$ implies that X is simply connected and since G acts cocompactly on $X^{(1)}$ and R is finite, X is a cocompact G-complex. Note that the definition of Ω, implies that no two distinct 2-cell of X have the same boundary. □

Example 5.6.12. Let $G = A *_C B/\langle\langle r^m \rangle\rangle$, where A and B are locally compact groups, C is a common compact open subgroup, and $r \in G$. Consider a compact generalized presentation $\langle\ (A *_C B, \phi) \mid R\ \rangle$ of G relative to $\{A, B\}$ as in Example 5.6.4. Let $\mathcal{T}$ be the Bass-Serre tree corresponding to $A *_C B$, and let $\phi\colon \pi_1(\mathcal{G}, \Lambda) \to G$ be the natural quotient map. Then the corresponding relative Cayley-Abels complex X is a simply connected 2-complex with 1-skeleton given by $\mathcal{T}/\ker(\phi)$. The G-complex X has a single orbit of 2-cells. Let D be a 2-cell such that $\phi(r) \subseteq G^D$, where G^D is the setwise stabilizer of D. Observe that G^D is the subgroup $\langle r \rangle$, and the G^D-translated of the path in $X^{(1)}$ corresponding to r cover the ∂D.

Proposition 5.6.13. *Let G be a topological group with compact generating graph $(\mathcal{G}, \Lambda, \phi)$ relative to a finite collection of open subgroups $\mathcal{H}$, and let Γ be the corresponding relative Cayley-Abels graph. If Γ is the 1-skeleton of a discrete simply connected cocompact G-complex X, then G has a compact generalized relative presentation relative to $\mathcal{H}$ with generating graph $(\mathcal{G}, \Lambda, \phi)$.*

Proof. Since X is simply connected, if there are no 2-cells in X, $\ker(\phi)$ is trival and there is compact generalized presentation $\langle\ (\mathcal{G},\Lambda,\phi)\ \rangle$. Suppose that $\ker(\phi)$ is not trivial. Since G acts cocompactly on X, there exists a finite collection $\{\Delta_1,\Delta_2,\cdots\Delta_k\}$ of G-orbit representatives of 2-cells in X. Let $\{\gamma_1,\gamma_2,\cdots,\gamma_k\}$ be their corresponding boundary paths. Let x_0 be a fixed vertex in $\mathcal{T}$ and for any γ_i, let α_i be a path in 1-skeleton of X from $\rho(x_0)$ to some fixed vertex of γ_i. Lift the concatenated paths $\alpha_i*\gamma_i$ to $\widetilde{\alpha_i*\gamma_i}$ in $\mathcal{T}$ starting at x_0. Then there exist $r_i\in\ker(\phi)$ such that $r_i.x_0$ is the endpoint of $\widetilde{\alpha_i*\gamma_i}$. As a consequence of Proposition 5.4.17, $R=\{r_1,r_2,\cdots,r_k\}$ is a finite set normally generating $\ker(\phi)$. Hence we have a compact relative presentation $\langle\ (\mathcal{G},\Lambda,\phi)\mid R\ \rangle$ of G. $\square$

5.6.2 Proof of the Corollary 5.6.7

We will use the following definition of large-scale simply connectedness introduced in [dlST19, Definition 1.3].

Definition 5.6.14. Let Γ be a connected graph. For $k\in\mathbb{N}$, define a 2-complex $\Omega_k(\Gamma)$ with 1-skeleton Γ and 2-cells as m-gons for any simple loop of length m up to cyclic permutation, where $0\leq m\leq k$.

Definition 5.6.15. [dlST19, Definition 1.3] A graph Γ is said to be k-simply connected if $\Omega_k(\Gamma)$ is simply connected. If there exists such a k, then we shall say that Γ is large-scale simply connected.

Remark 5.6.16. We note the following observations:

1. If Γ is a G-graph, then $\Omega_k(\Gamma)$ is a G-complex.

2. The large-scale simply connectedness for a graph is preserved by quasi-isometry. See [dlST19, Theorem 2.2].

3. If a graph Γ is a 1-skeleton of a simply connected cocompact G-complex, then Γ is large-scale simply connected.

Proof of the Corollary 5.6.7. (2) implies (1) is direct from Theorem 5.6.6. Conversely, let Γ be any relative Cayley-Abels graph of G with respect to $\mathcal{H}$. Since G is compactly presented, Theorem 5.6.6 implies there exists a relative Cayley-Abels graph Γ_1 that is large-scale simply connected, see Remark 5.6.16. Since there exists a fine relative Cayley-Abels graph of G with respect to $\mathcal{H}$. By Corollary 5.5.3, Γ is large-scale simply connected and fine. Observe that by Proposition 5.4.17, G acts cocompactly on Γ. Let $\mathcal{E}$ be a finite set of G-orbit representatives of edges in Γ. Since Γ is fine, for any $e \in \mathcal{E}$, there exists finitely many circuits of length at most k containing e. By construction, $\Omega_k(\Gamma)$ has finitely many 2-cells up to the G-action. Hence $\Omega_k(\Gamma)$ is a discrete simply connected cocompact G-complex with 1-skeleton Γ. □

5.6.3 Proof of Corollary 5.6.9

We will use the following proposition for the proof.

Proposition 5.6.17. [CdlH16, Prop. 8.A.10] *Let $1 \to N \to G \to Q \to 1$ be a short exact sequence of locally compact groups groups and continuous homomorphisms.*

1. *Assume that G is compactly presented and that N is compactly generated as a normal subgroup of G. Then Q is compactly presented.*
2. *Assume that G is compactly generated and that Q is compactly presented. Then N is compactly generated as a normal subgroup of G.*
3. *If N and Q are compactly presented, then so is G.*

Proof of Corollary 5.6.9. Suppose $\langle\ (\mathcal{G}, \Lambda, \phi) \mid R\ \rangle$ is a a compact generalized presentation of G relative to the empty collection. By Remark 5.6.3, G has a compact open subgroup. On the other hand, all vertex and edge groups of $(\mathcal{G}, \Lambda, \phi)$ are compact, and hence $\pi_1(\mathcal{G}, \Lambda)$ has an compact generating set S consisting of the union of all vertex groups and a finite set of elements (stable letters for HNN-extensions). Then a group presentation of $\pi_1(\mathcal{G}, \Lambda)$ with generating set S is obtained by considering all relations given by the multiplication tables of the vertex groups and the HNN relations; note that all relations have length at most four. Since R is a finite set, we have a bounded presentation for G. Conversely, assume G has a bounded presentation $\langle S|R\rangle$ and a compact open subgroup. Theorem 5.4.9 implies that G admits a compact generating graph $(\mathcal{G}, \Lambda, \phi)$ relative to the empty collection. Since $\pi_1(\mathcal{G}, \Lambda)$ is compactly generated and G is compactly presented, Proposition 5.6.17 implies that $\ker(\phi)$ is compactly generated as a normal subgroup. By Proposition 5.4.17, $\ker(\phi)$ is discrete and therefore finitely generated as a normal subgroup. □

Remark 5.6.18. Corollary 5.6.9 can also be obtained as a consequence of Theorem 5.6.6 and [BH99, Theorem 8.10]

5.7 Relatively compactly presented groups

In this section we prove two results whose restriction to discrete groups were proven by Osin in [Osi06, Theorem 2.40 and Theorem 1.1].

Theorem 5.7.1. *Let G be a topological group compactly presented relative to a finite collection $\mathcal{H}$ of open subgroups.*

1. *If each $H \in \mathcal{H}$ is compactly presented then G is compactly presented.*
2. *If G is compactly generated, then each $H \in \mathcal{H}$ is compactly generated.*

Remark 5.7.2. Note that the statements of Theorem 5.7.1 are trivial if G has no compact open subgroup since in this case Remark 5.6.3 implies that $\mathcal{H} = \{G\}$.

The section consists of three parts. The first subsection discusses the proof of Theorem 5.7.1(1), then the other two sections contain the proof of Theorem 5.7.1(2).

5.7.1 Normal forms

Let us recall the notion normal form for amalgamated free products and HNN extensions. For details we refer the reader to [LS01].

Normal form for amalgamated free product

Let $G = A *_C B$ be an amalgamated free product. Choose a system of representatives T_A of right cosets of C in A and a system of representatives T_B of right cosets of C in B. Assume that 1 represents the coset of C in A and B. A *normal* form in the amalgamated free product $A *_C B$ is a sequence $x_0 x_1 \cdots x_n$ such that

1. $x_0 \in C$.
2. $x_i \in T_A \setminus \{1\}$ or $x_i \in T_B \setminus \{1\}$. for $i \geq 1$, and consecutive terms x_i and x_{i+1} lie in distinct system of representatives.

Normal form for HNN extension

Let A be a group, let B and C be subgroups of A, and let $\alpha\colon C \to B$ be an isomorphism. Suppose G is the group defined as the HNN extension

$$G = A*_\alpha = \langle A, t \mid t^{-1}ct = \alpha(c), c \in C\rangle.$$

Choose a system of representatives T_C of right cosets of C in A and a system of representatives T_B of right cosets of B in A. Assume that 1 represents the coset of C and B. A *normal form* in an HNN extension is a $g_0 t_1^{\epsilon} g_1 \cdots t^{\epsilon_n} g_n$ such that

1. g_0 is an arbitrary element of G,
2. if $\epsilon_i = -1$, then $g_i \in T_B$,
3. if $\epsilon_i = 1$, then $g_i \in T_C$,
4. there is no consecutive subsequence $t^{-1}1t$.

Theorem 5.7.3. [LS01] *Suppose $G = A *_C B$ or $G = A*_\alpha$. Every element $g \in G$ can be uniquely written as the product of sequence in normal form.*

5.7.2 Proof of Theorem 5.7.1(1)

Lemma 5.7.4. *If $(\mathcal{G}, \Lambda)$ is a finite topological graph of groups with compactly presented vertex groups and compact edge groups, then $\pi_1(\Lambda, \mathcal{G})$ is compactly presented.*

Proof. It is enough to prove the result for the graph of groups corresponding to the free product with amalgamation and HNN extension. The general case follows by induction. Let Λ be a single edge with compactly presented vertex groups A and B and compact edge group C. Let $i_A\colon C \to A$ and $i_B\colon C \to B$ be open topological embeddings. A presentation of $\pi_1(\Lambda, \mathcal{G}) \simeq A*_C B$ is given by $\langle A, B \mid i_A(c)i_B(c)^{-1}$ for $c \in C\rangle$. Let $\langle S_A \mid R_A\rangle$ and $\langle S_B \mid R_B\rangle$ be bounded presentations of A and B over compact sets S_A and S_B respectively such that $C \subseteq S_A$ and $C \subseteq S_B$. Then $\langle S_A \cup S_B \mid i_A(c)i_B(c)^{-1}$ for $c \in C\rangle$ is a bounded presentation of G over a compact set, and hence G is compactly presented. For HNN extension, let Λ be an edge loop with compactly presented vertex group A and compact edge group C. Let $i_1\colon C \to A$ and $i_2\colon C \to A$ be open topological embeddings. Let $\langle S \mid R\rangle$ be a bounded presentation of A over compact set S such that $C \subseteq S$. Let y be a symbol not in S. Then a presentation of $\pi_1(\Lambda, \mathcal{G})$ is given by $\langle S, y \mid R, i_1(c)^{-1}yi_2(c)y^{-1}$ for $c \in C\rangle$. This is a bounded presentation over a compact set. Hence $\pi_1(\Lambda, \mathcal{G})$ is compactly presented. $\square$

Proof of Theorem 5.7.1(1). Suppose each $H \in \mathcal{H}$ is compactly presented. Let $\langle\, (\mathcal{G}, \Lambda, \phi) \mid R\,\rangle$ be a compact generalized presentation of G relative to $\mathcal{H}$, and let $\widetilde{G} = \pi_1(\Lambda, \mathcal{G})$ be the fundamental group of the graph of groups $(\Lambda, \mathcal{G})$. Since $(\mathcal{G}, \Lambda,)$ is a finite graph of topological groups with compactly presented vertex groups and compact edge groups, by Lemma 5.7.4 , $\widetilde{G}$ is compactly presented. Consider the short exact sequence $1 \to \ker(\phi) \to \widetilde{G} \to G \to 1$. Since R is finite, $\ker(\phi)$ is compactly generated as a normal subgroup. By Proposition 5.6.17, G is compactly presented. $\square$

5.7.3 Compactly generated topological graphs of groups.

In this part, we prove the following proposition which is a particular case of Theorem 5.7.1(2).

Proposition 5.7.5. *Let G be the fundamental group of a finite graph of topological groups $(\mathcal{G}, \Lambda)$ such that edge groups are compactly generated. If G is compactly generated, then vertex groups of $(\mathcal{G}, \Lambda)$ are compactly generated.*

This proposition can also be stated as follows:

Corollary 5.7.6. *Let G be a topological group acting discretely, cocompactly and without inversions on a tree such that the edge stabilizers are compactly generated. If G is compactly generated, then the vertex stabilizers are compactly generated.*

Corollary 5.7.6 in the case that G is a TDLC group and the edge stabilizers are compact is a result of Castellano [Cas20, Proposition 4.1]. The proof of Proposition 5.7.5 follows by induction on the number of edges of Λ, so it reduces to prove the result for amalgamated free products and HNN extensions. For definitions and results on normal forms, refer [LS01, Page 181].

Lemma 5.7.7. *Let $G = A *_C B$ be a topological group such that G and C are compactly generated, and C is an open subgroup containing a compact open subgroup. Then A is compactly generated.*

Proof. Let U be a compact open subgroup of C. Let C_A and C_B denote the copies of C in A and B respectively. Since G and C are compactly generated, there are finite subsets $X \subset G$ and $Y \subset C$ such that $G = \langle X \cup U \rangle$ and $C = \langle Y \cup U \rangle$. For each element of X choose a normal form; and let $Z \subset A$ consists of all $a \in A$ such that a appears in a chosen normal form of an element of X. Observe that Z is a finite set. We claim that $A = \langle Y \cup Z \cup U \rangle$ and hence A is compactly generated. Let A' be the subgroup of A generated by $Y \cup Z \cup U$. Since $C_A = \langle Y \cup U \rangle$, it follows that $C_A \leq A'$. Let $\psi \colon A' *_C B \to G$ the morphism induced by the inclusions $A' \hookrightarrow A \hookrightarrow G$ and $B \hookrightarrow G$. Then ψ is surjective since its image contains $\psi(X \cup U) = X \cup U$ which generates $A *_C B$. Note that ψ preserves the length of normal forms and $\psi(B) = B$; therefore surjectivity of ψ implies that $\psi(A') = A$. $\square$

Lemma 5.7.8. *Let A be a topological group, let B and C be open subgroups of A containing compact open subgroups and let $\alpha \colon C \to B$ be an isomorphism of topological groups. Let G be the TDLC group defined as the HNN extension*

$$G = A*_\alpha = \langle A, t \mid t^{-1}ct = \alpha(c), c \in C \rangle.$$

If G and C are compactly generated, then A is compactly generated.

Proof. Let U be a compact open subgroup in $C \cap B$. Since G, C, and B are compactly generated, there are finite subsets $X \subset G$, $Y \subset C$, and $W \subset B$ such that $G = \langle X \cup U \rangle$, $C = \langle Y \cup U \rangle$, and $B = \langle W \cup U \rangle$. For each element of X choose a normal form; and let $Z \subset A$ consists of all $a \in A$ such that a appears in a chosen normal form of an element of X. Observe that Z is a finite set. We claim that $A = \langle W \cup Y \cup Z \cup U \rangle$ and hence A is compactly generated. Let A' be the subgroup of A generated by $W \cup Y \cup Z \cup U$. Since $B = \langle W \cup U \rangle$ and $C = \langle Y \cup U \rangle$, it follows that $B \leq A'$ and $C \leq A'$. Consider the group $G' = A'*_\alpha$ and the morphism $\psi \colon G' \to G$ induced by the inclusions $A' \hookrightarrow A \hookrightarrow G$. Then ψ is surjective since its image contains $\psi(X \cup U) = X \cup U$ which generates G. Note that ψ preserves the length of normal forms; therefore surjectivity of ψ implies that $\psi(A') = A$. $\square$

5.7.4 Proof of the Theorem 5.7.1(2)

Definition 5.7.9 (Link)**.** Let X be a simplicial 2-complex and let σ be a cell in X. The *star* $\mathsf{st}(\sigma)$ of σ is defined as the set of all closed cell of X incident to σ. The *link* of σ in X is a subcomplex of X defined as the collection of closed cells of $\mathsf{st}(\sigma)$ that do not intersect σ. We will denote it as $\mathsf{link}_X(\sigma)$. If σ is a 0-cell, $\mathsf{link}_X(\sigma)$ can also be interpreted as a graph $\Gamma = \mathsf{link}_X(\sigma)$ defined with the set of vertices

$$V(\Gamma) = \{e \mid e \text{ is a 1-cell in } \mathsf{st}(\sigma) \text{ adjacent to } \sigma\}$$

and the set of edges $E(\Gamma) = \{(e_1, e_2)\}$, where e_1, e_2 are pair of edges in $\mathsf{st}(\sigma)$ adjacent to σ contained in the boundary of same 2-cell in $\mathsf{st}(\sigma)$.

Remark 5.7.10. We note the following facts without proof.

1. Let X be a 2-dimensional G-complex with cellular and cocompact action of G. Then G acts cocompactly on the barycentric subdivision of X.

2. Let X be a 2-dimensional G-complex. After sufficient subdivisions, X can be considered a simplicial G-complex. In particular, for any cell σ in X, $\mathsf{link}_X(\sigma)$ is well-defined and we can consider the G_σ-action on the $\mathsf{link}_X(\sigma)$.

3. Let X be a 2-dimensional simplicial G-complex and let X' be the barycentric subdivision of X. For any 0-cell $\sigma \in X$, there is an G_σ-equivariant isomorphism $\mathsf{link}_X(\sigma) \to \mathsf{link}_{X'}(\sigma)$.

Lemma 5.7.11. *Let X be a 2-dimensional cocompact simplicial G-complex without inversions. Let $\sigma \in X$ be a 0-cell. Then G_σ acts cocompactly on $\mathsf{link}_X(\sigma)$.*

Proof. Let X' be the barycentric subdivision of X with the induced cellular G-action. Then G acts on X' without inversions and, by Remark 5.7.10, the action is cocompact. Let $st(\sigma)$ be the set of closed cells of X incident to σ. Since the G-action on X is cocompact, there exists finitely many cells $\{\tau_i\}$ in $st(\sigma)$ such that for any $\tau \in st(\sigma)$ there exists $g_\tau \in G$ such that $g_\tau \tau = \tau_i$ for some i. By the definition of induced action on X', $g_\tau \sigma = \sigma$. Therefore, $g_\tau \in G_\sigma$. Hence G_σ acts cocompactly on $st(\sigma)$ and thus on $\mathsf{link}_{X'}(\sigma)$. By Remark 5.7.10, item (3), we conclude that G acts cocompactly on $\mathsf{link}_X(\sigma)$. □

Remark 5.7.12. [MW02, Definition 2.6] Let D be a disc diagram not homeomorphic to a disc, then D is either trivial, consists of a single 1-cell joining two 0-cells, or contains a cut 0-cell i.e. a 0-cell whose removal disconnects the diagram.

Remark 5.7.13. Let G be a group acting on a graph Γ without inversions. Let e be an edge with vertex representatives v, w. Let $H = G_v$. Then H acts on set of vertices adjacent to v, and $G_e = H_w$.

Lemma 5.7.14. *Let X be a simply connected 2-dimensional simplicial G-complex and let v be a 0-cell. Let $\mathcal{K}$ be the set of connected components of $\mathsf{link}_X(v)$ and Ω be the set of connected components of $X \setminus \{gv \mid g \in G\}$ whose closure contain v. Then there is a G_v-equivariant bijection between $\mathcal{K}$ and Ω.*

Proof. Since $\mathsf{link}_X(v)$ is subspace of $X \setminus \{gv \mid g \in G\}$, there is a natural G_v-equivariant inclusion $\eta\colon \mathcal{K} \to \Omega$. We claim that η is a bijection. Let $k_1, k_2 \in \mathcal{K}$ and x, y be vertices of X in k_1 and k_2 respectively. Suppose $\eta(k_1) = \eta(k_2)$. Then there exists a path p in $X \setminus \{gv \mid g \in G\}$ between x, y. Since $k_1, k_2 \in \mathcal{K}$, there exist paths p_1, p_2 in X joining v, x and y, v respectively. Let γ be the loop in X obtained by concatenating p_1, p, p_2. Choose γ to be of minimal length such that it is an embedding in X. Since X is simply connected, by [MW02, Lemma 2.17] there exists a disc diagram D and a reduced map $\phi : D \to X$ such that $\gamma = \phi \circ \gamma'$, where $\gamma' : S^1 \to D$ is the boundary cycle of D. Since γ is an embedding, by Remark 5.7.12, D is homeomorphic to a disc. Let w be a vertex in D mapping to v. Observe that ϕ induces a map between $link_D(w)$ and $link_X(v)$. Since $link_D(w)$ is path connected, there is path between x and y in $\mathsf{link}_X(v)$. Thus $k_1 = k_2$ and η is injective.

Let $\omega \in \Omega$. Then ω contains the interior of a cell σ incident to v. Let k be the connected component of $\mathsf{link}_X(v)$ containing $\mathsf{link}_X(v) \cap \sigma$. Then k maps to ω, and hence η is surjective. □

Proposition 5.7.15. *Let X be a simply connected 2-dimensional simplicial G-complex with discrete cocompact G-action such that every cell of dimension greater than 0 has compact stabilizer. Then for any vertex v, the setwise G_v-stabilizer of any connected component of $\mathsf{link}_X(v)$ is compactly generated.*

Proof. Let Δ be a connected connected component of $\mathsf{link}_X(v)$ and let K be its setwise G_v-stabilizer. That X is a discrete G-complex and stabilizers of 1-cells and 2-cells are compact, Δ is a connected discrete K-graph with compact cell stabilizers. By Lemma 5.7.11, we also have that Δ is a cocompact K-graph. Hence by Theorem 5.4.9, K is compactly generated. □

Proof of the Theorem 5.7.1(2). Let $\mathcal{P} = \langle\, (\mathcal{G}, \Lambda, \phi) \mid R \,\rangle$ be a compact relative generalized presentation of G with respect to the collection $\mathcal{H}$ of open subgroups, and let X be the barycentric subdivision of the corresponding relative Cayley-Abels complex. In particular, the G-action on X has no inversions, links of cells are well defined, and every cell of dimension greater than 0 has compact stabilizer.

Let $H \in \mathcal{H}$ and let v_H denote a 0-cell of X such that $G_{v_H} = H$, note that such vertex exists by Proposition 5.6.11. The proof is divided into two cases:

Case 1: v_H is not a cut-point in X.

By Lemma 5.7.14, $\mathsf{link}_X(v_H)$ is connected. In this case H acts on $\mathsf{link}_X(v_H)$ discretely, cocompactly and with compact stabilizers. By Proposition 5.7.15, H is compactly generated.

Case 2: v_H is a cut-point in X.

Observe that for any $g \in G$, the 0-cell gv_H is a cut-point in X. Let Ω be the set of connected components of the set $X \setminus \{gv_H \mid g \in G\}$. For any $\Delta \in \Omega$, denote $\overline{\Delta}$ be the closure of Δ in X. Construct a tree T corresponding to $X^{(1)}$ with $V(T) = \{gv_H \mid g \in G\} \cup \{v_\Delta \mid \Delta \in \Omega\}$ and $E(T) = \{\{gv_H, v_\Delta\} \mid gv_H \in \overline{\Delta}, g \in G, \Delta \in \Omega\}$. Observe that T is a tree with a natural G-action and $G_{v_H} = H$. We claim that the edge G-stabilizers in T are compactly generated. Without loss of generality, consider an edge $e = \{v_H, v_\Delta\}$ incident to vertex v_H in T. Then G_e is the setwise H-stabilizer of Δ. By Lemma 5.7.14, G_e is the setwise H-stabilizer of a connected component of $\mathsf{link}_X(v_H)$. Thus Proposition 5.7.15 implies H_Δ is compactly generated. Therefore G-stabilizers of edges of T are compactly generated; and since G is compactly generated, Proposition 5.7.5 implies vertex stabilizers are compactly generated. In particular, H is compactly generated. $\square$

5.8 Relatively hyperbolic groups

In this section, we generalize the notion of relatively hyperbolic group for proper pairs $(G, \mathcal{H})$. Our definition extends Bowditch's approach to relative hyperbolicity for discrete groups [Bow12]. The main result of the section is that relative hyperbolic groups admit compact relative presentations, see Theorem 5.8.4.

Definition 5.8.1. Let $(G, \mathcal{H})$ be a proper pair. The group G is *relatively hyperbolic* with respect to $\mathcal{H}$ if there exists a relative Cayley-Abels graph Γ of G with respect to $\mathcal{H}$ which is fine and hyperbolic.

Remark 5.8.2. Let $(G, \mathcal{H})$ be a proper pair. If G is a topological group with compact open subgroup, and G is hyperbolic relative to a finite collection $\mathcal{H}$ of open subgroups, then:

1. the group G is compactly generated relative to $\mathcal{H}$ by Theorem 5.4.9; and
2. every relative Cayley-Abels graph of G with respect to $\mathcal{H}$ is fine and hyperbolic by Theorem 5.5.3.

Example 5.8.3. Let G be the fundamental group of a finite graph of topological groups $(\mathcal{G}, \Lambda)$ with compact open edge groups. Then G is hyperbolic relative to the collection $\mathcal{H}$ of vertex groups $\mathcal{G}_v$. Indeed, the Bass-Serre tree of $(\mathcal{G}, \Lambda)$ is a relatively Cayley-Abels graph of G with respect to $\mathcal{H}$ which is hyperbolic and fine.

Theorem 5.8.4. *Let $(G, \mathcal{H})$ be a proper pair. If G is hyperbolic relative $\mathcal{H}$, then G is compactly presented relative to $\mathcal{H}$*

Small cancellation quotients of free products are a source of relatively hyperbolic groups in the discrete case [Osi06, Example(II) Page 4]. Proposition 5.8.5 generalizes this construction. For background on small cancellation quotients of amalgamated free products we refer the reader to the book by Lyndon and Schupp [LS01].

Proposition 5.8.5. *Let $A *_C B$ be a topological group that splits as an amalgamated free product over a common compact open subgroup C. If $R \subseteq A *_C B$ is symmetrized set satisfying $C'(\lambda)$ condition and $G = (A *_C B)/\langle\langle R \rangle\rangle$ then:*

1. *For the compact generalized presentation $\langle (A *_C B, \phi) \mid R \rangle$ of G relative to $\{A, B\}$, the presentation complex is $C'(2\lambda)$ small cancellation complex.*
2. *For $\lambda \geq 12$, any compact generating graph of G relative to $\{A, B\}$, the relative Cayley-Abels graph is fine and hyperbolic. In particular, G is relatively hyperbolic with respect to $\{A, B\}$.*

Corollary 5.8.6. *Let G be as in Proposition 5.8.5 and $R = \{r^m\}$, where r is a reduced and weakly reduced element in G. If $m \geq 12$ and R satisfies the $C'(1/12)$ small cancellation condition, then G is relatively hyperbolic with respect to $\{A, B\}$.*

The rest of the section is divided into two subsections containing the proofs of Theorem 5.8.4 and Proposition 5.8.5 respectively.

5.8.1 Proof of Theorem 5.8.4

We will use the following result by Bowditch.

Lemma 5.8.7. [Bow12, Proposition 3.1] *Let Γ be a hyperbolic graph with hyperbolicity constant k. Then there is a constant $n = n(k)$ with the following property. If $\Omega_n(\Gamma)$ is the 2-complex with 1-skeleton the graph Γ and such that each circuit of length at most n is the boundary of a unique 2-cell, then $\Omega_n(\Gamma)$ is simply-connected.*

Proof of Theorem 5.8.4. Let Γ be a Cayley-Abels graph of G relative to $\mathcal{H}$. By Remark 5.8.2, Γ is hyperbolic and fine. Let n be the constant given by Lemma 5.8.7. Since Γ is fine and G-cocompact, there are finitely many G-orbits of circuits of length at most n. Then $X = \Omega_n(\Gamma)$ is a discrete simply-connected cocompact 2-dimensional G-complex with 1-skeleton Γ. By Theorem 5.6.6, G is compactly presented with respect to $\mathcal{H}$. $\square$

5.8.2 Proof of Proposition 5.8.5

The Dehn filling function for a finitely presented group is defined in [Gro87]. It is a generalization of isoperimetric inequality. Moreover, hyperbolic groups are characterized as finitely presented groups with linear Dehn function, see [ABC+91, Section 2]. We will use the following result to prove fineness of certain complexes.

Proposition 5.8.8. [MP16] [HMPS21] *Let X be a cocompact simply-connected G-complex. Suppose that each edge of X is attached to finitely many 2-cells. Then the 1-skeleton of X is a fine graph if and only if the combinatorial Dehn function of X takes only finite values.*

The if direction is proved in [MP16, Proposition 2.1] for homological Dehn functions $FV_X(k)$. In [Ger96b, Section 3], Gersten observe that $FV_X(k) \preceq \delta^1_X(k)$ and hence the if direction of the proposition above follows. The proof of the only if direction is the same argument as [HMPS21, Proof of Lemma 2.3].

Proof of Proposition 5.8.5. G admits a a compact presentation $\langle\, (\mathcal{G}, \Lambda, \phi) \mid R \,\rangle$ relative to $\{A, B\}$, where $(\mathcal{G}, \Lambda)$ is a graph of groups with a single edge with A, B as vertex groups, C as edge group C. Let X be the corresponding presentation complex of G, see Example 5.6.12. If R is a symmetrized set satisfying $C'(\lambda)$ small cancellation condition, then X is $C'(2\lambda)$ complex, see [ACCCMP21, Theorem 7.1] for a proof. A classical result in [LS01, Theorem 11.2] implies that $\delta_X(k)$ is linear and hence $X^{(1)}$ is hyperbolic by [Gro87, Theorem 2.3.D]. Further $X^{(1)}$ is fine by Proposition 5.8.8. Therefore, G is relatively hyperbolic with respect to $\{A, B\}$. $\square$

5.9 Proof of Theorem C

5.9.1 Consequences of the McCammond and Wise's Perimeter method

A 2-complex is *M-thin*,

$$\#\{D \mid D \text{ is a 2-cell in } X \text{ and } e \text{ belongs to } \partial D\}\} \leq M$$

for any 1-cell e in X. A 2-complex is *uniformly circumscribed* if there is an integer L such that for each 2-cell D of X, the boundary cycle ∂D has length at most L.

Proposition 5.9.1. *Let X be a $C'(\lambda)$ small cancellation complex that is simply connected, uniformly circumscribed, M-thin, and $6\lambda M < 1$.*

1. [MPW11b, Theorem 3.3] *If Q acts faithfully on X, and is finitely generated relative to a finite collection of 0-cell stabilizers. Then there is a connected and quasi-isometrically embedded H-cocompact subcomplex of X.*

2. *If Q is a compactly generated topological group with a compact open subgroup which acts faithfully and discretely on X, 1-cell Q-stabilizers are compact, then Q is relatively hyperbolic with respect to a finite collection $\mathcal{K}$ of 0-cell Q-stabilizers. In particular, Q is compactly presented with respect to $\mathcal{K}$.*

3. *If H is a topological group with a compact open subgroup which acts faithfully and discretely on X, 1-cell H-stabilizers are compact, and 0-cell H-stabilizers are coherent, then H is coherent.*

Proof. To prove the second statement, let U be the Q-stabilizer of a 1-cell of X. Observe that U is a compact open subgroup of Q. Since Q is compactly generated, Q is finitely generated with respect to U. Hence Q is finitely generated with respect to the Q-stabilizer of a 0-cell of X. By the first statement of the proposition, there exists a connected Q-cocompact 1-dimensional subcomplex Y quasi-isometrically embedded in X. The Dehn function of the $C'(1/6)$ simply connected small cancellation complex X is linear [LS01, Ch.5 Thm 4.4] and hence the 1-skeleton $X^{(1)}$ is a hyperbolic graph by [Gro87, Theorem 2.3.D]. Further $X^{(1)}$ is fine graph by Theorem 5.8.8. Therefore, the Q-complex Y is also hyperbolic and fine. Since Q acts cocompactly on Y, by Proposition 5.4.19, there exists a finite collection $\mathcal{K}$ of representatives of conjugacy classes of 0-cells Q-stabilizers such that there exists a compact generating graph of Q with respect to $\mathcal{K}$ such that the corresponding relative Cayley-Abels graph is Q-isomorphic to Y. Observe that $(Q, \mathcal{K})$ is a proper pair. Thus Q is relatively hyperbolic with respect to $\mathcal{K}$ and by Theorem 5.8.4, Q is compactly presented with respect to $\mathcal{K}$.

To prove the third statement, let Q be a compactly generated open subgroup of H. The second statement shows Q is compactly presented with respect to a finite collection $\mathcal{K}$ of Q-stabilizers of 0-cells. Since Q is also compactly generated, Theorem 5.7.1 implies that every subgroup in $\mathcal{K}$ is compactly generated. Since Q is open in H, and the subgroups in $\mathcal{K}$ are open subgroups of H-stabilizers of 0-cells of X which are coherent; it follows that every subgroup in $\mathcal{K}$ is compactly presented. By Theorem 5.7.1, Q is compactly presented. □

5.9.2 Proof of Theorem C

Without loss of generality, assume that the normal form $x_0 x_1 x_2 \cdots x_\ell$ of r as an element of $A *_C B$ is a cyclically reduced word, that is, $x_1 \in A$ if and only if $x_k \in B$; see Section 5.7.2 for a definition of normal form. Since r is not conjugate into A or B, the assumption can be achieved by conjugating r if necessary. Let $|r|$ denote the length of the normal form, that is, $|r| = \ell$.

Consider a compact presentation

$$\langle\ (A *_C B, \phi) \mid R\ \rangle \tag{5.9.1}$$

of G relative to $\{A, B\}$, where $\phi\colon A *_C B \to G$ is a continuous epimorphism induced by the inclusions $A \hookrightarrow G$ and $B \hookrightarrow G$, and R is a finite symmetric set generated by r^m. Suppose that R satisfies the $C'(1/12)$ small cancellation condition. Let $\widetilde{G}$ denote $A *_C B$ and let $\mathcal{T}$ be the corresponding Bass-Serre tree.

Fix a vertex $y \in \mathcal{T}$, let γ be the unique fixed path from y to $r^2 y$, and let

$$M = k|r|, \quad \text{where} \quad k = \max\left\{[\widetilde{G}_t \colon \widetilde{G}_\gamma] \mid t \text{ is a 1-cell in the image of } \gamma\right\}. \tag{5.9.2}$$

Let us observe that k is finite. Observe that the pointwise $\widetilde{G}$-stabilizer $\widetilde{G}_\gamma$ of the path γ equals the intersection of the $\widetilde{G}$-stabilizers of edges of γ. Since γ is a finite path, $\widetilde{G}_\gamma$ is an open subgroup of $\widetilde{G}_t$. Since $\widetilde{G}_t$ is a compact group, the index $[\widetilde{G}_t \colon \widetilde{G}_\gamma]$ is finite.

Note that M is independent of m. Let X be the relative Cayley-Abels complex associated to (5.9.1), see Example 5.6.12.

Proposition 5.9.2. *The G-complex X is M-thin.*

The proof of Proposition 5.9.2 is postponed to the Subsection 5.9.3. Below we complete the proof of the Theorem C using this proposition.

Since r is reduced and weakly reduced, the symmetrized set R of elements of $A *_C B$ induced by r^m satisfies $C'(1/m)$ small cancellation condition. By Proposition 5.8.5, X is a $C'(2/m)$ small cancellation complex. By Proposition 5.9.2, X is M-thin. Since M is independent of m we can assume that

$$m \geq 12M.$$

In particular $m \geq 12$ and therefore Proposition 5.8.5 implies that the 1-skeleton of X is fine and hyperbolic. Since G acts cocompactly, X is uniformly circumscribed.

Consider the short exact sequence $1 \to N \to G \to Aut(X) \to 1$ induced by the G-action on X. Observe that N is a compact subgroup of G since G-stabilizers of cells of x are open subgroups, and 1-cell G-stabilizers are compact. Let $H = G/N$ and observe that H acts faithfully, discretely, cocompactly and with compact 1-cell stabilizers on X. Since H-stabilizers of 0-cells of X are conjugates in H of the subgroups A/N and B/N; by Remark A, X has coherent H-stabilizers of 0-cells. By Proposition 5.9.1, H is coherent and hence Remark A implies that G is coherent.

5.9.3 Proof of Proposition 5.9.2

Recall that $\mathcal{T}$ denotes the Bass-Serre tree of $A *_C B$, y is a fixed vertex of $\mathcal{T}$, and $\rho\colon \mathcal{T} \to X^{(1)}$ is the quotient map.

We use the following observation in the argument below: For a reduced and cyclically reduced word w of length $k > 0$, the edge-length of the unique embedded path in the tree $\mathcal{T}$ from y to $w.y$ is k.

By definition, X has a 2-cell D whose boundary path is the image by ρ of the path in $\mathcal{T}$ from y to $r^m y$. Let G^D denote the setwise stabilizer of D and G_D the pointwise stabilizer. Consider the finite set

$$\mathcal{U} = \{(t, D) \mid \text{ t is a 1-cell in } \partial D\}$$

and observe that $\mathcal{U}$ is a G^D-set. Let U be the set of G^D-orbits of elements of $\mathcal{U}$.

Claim. *The cardinality of U is bounded by $|r|$.*

Proof. Since $\phi(r)$ setwise stabilizes ∂D and no pair of distinct 2-cells of X have the same boundary, $\phi(r)$ is an element of G^D. Note that $\phi(r)$ has order m in G. Let q be the unique path in $\mathcal{T}$ from y to ry. Then $\rho(q)$ is a subpath of ∂D of length $|r|$. Since translates of the path $\rho(q)$ by $\langle \phi(r) \rangle$ cover ∂D, the set of elements (t, D) where t is a 1-cell in $\rho(q)$ contains representatives of all G^D-orbits of elements of $\mathcal{U}$. Hence U has at most $|r|$ elements. $\square$

Let e be a 1-cell in X. The thinness of e is given by the cardinality of the following G_e-set,

$$\mathcal{S}_e = \{(e, K) \mid K \text{ is a 2-cell in } X \text{ and } e \in \partial K\}.$$

Let S be the set of G_e-orbits in $\mathcal{S}_e$. We break the calculation of $|\mathcal{S}_e|$ into the following claims.

Claim. *There is an injective map from S to U*

Proof. Since there is a single G-orbit of 2-cells in X, for any $(e, K) \in \mathcal{S}_e$, there exists a $g_k \in G$ such that $g_k.K = D$. This defines a map $\psi\colon S \to U$ given by $(e, K) \mapsto (g_k.e, g_k.K)$. To prove the claim, we show that for any pair of elements of $\mathcal{S}_e$, they are in the same G_e-orbit in $\mathcal{S}_e/G_e$ if and only if their images under ψ are in the same G^D-orbit in $\mathcal{U}/G^D$.

Let (e, K) and (e, K') be elements of S_e. First suppose there is $f \in G_e$ such that $(e, K) = f.(e, K')$. Then for $h = g_k f g_{k'}^{-1} \in G^D$, we have $(g_k.e, D) = h.(g_{k'}.e, D)$. Conversely, suppose that there exists $h \in G^D$ such that $\psi(e, K) = h.\psi(e, K')$. That is $(g_k.e, g_k.K) = h.(g_{k'}.e, g_{k'}.K')$. Hence for $f = g_k^{-1} h g_{k'} \in G_e$, we have that $(e, K) = f.(e, K')$. To summarize, we have the following equation.

$$\psi(e, K) = h.\psi(e, K') \Leftrightarrow (g_k e, g_k.K) = h(g_{k'}.e, g_{k'}K) \Leftrightarrow (e, K) = g_k^{-1} h g_{k'}(e, K)$$

□

Claim. *If* $(e, K) \in \mathcal{S}_e$ *then* $[G_e \colon G_K] \le k$.

Proof. Recall that γ is the unique path between y and $r^2 y$ in $\mathcal{T}$. Consider the path $p = \rho(\gamma)$ in $X^{(1)}$. Note that p is a subpath of ∂D. We claim that $G_p = G_D$, where G_D is the pointwise stabilizer of D. Observe that $G_D \subseteq G_p$. Conversely, if $g \in G_p$, then $g.D$ is a 2-cell of X such that $|\partial D \cap \partial(g.D)| = 2|r| = \frac{2}{m}|\partial D|$ and hence by the small cancellation condition on X, $D = g.D$ pointwise and thus $g \in G_D$.

Let $(e, K) \in \mathcal{S}_e$ be any pair. Since there is a single orbit of 2-cells in X, there exists $g \in G$ such that $g.K = D$; and since G^D translates of p cover ∂D, g can be chosen such that $g.e$ is in the image of p. Let $\widetilde{g.e}$ be a 1-cell in γ such that $\rho(\widetilde{g.e}) = g.e$. Since ϕ restricted to the stabilizer of edges is an isomorphism, we have the following commutative diagram

$$\begin{tikzcd} \widetilde{G}_\gamma \arrow[r, hook] \arrow[d, "\phi"] & \widetilde{G}_{\widetilde{g.e}} \arrow[d, "\phi"] \\ G_p \arrow[r, hook] & G_{g.e}, \end{tikzcd}$$

and therefore,

$$[G_e \colon G_K] = [G_{g.e} \colon G_D] = [G_{g.e} \colon G_p] = [\widetilde{G}_{\widetilde{g.e}} \colon \widetilde{G}_\gamma] \le k.$$ □

The number of G_e-orbits in $\mathcal{S}_e$ is bounded by $|r|$ as a consequence of first two claims. On the other hand, each G_e-orbit in $\mathcal{S}_e$ is bounded by k by the last claim. The

Therefore, the cardinality of $\mathcal{S}_e$ is bounded by $|r|k = M$ and this concludes the proof of the Proposition 5.9.2.

Chapter 6

Conclusions and future work

We present the conclusion of the thesis in the form of future projects that relate to the results obtained in this thesis.

6.1 Asymptotic Dimension

In Chapter 4, we proved the following.

Theorem 6.1.1. [ACCCMP21] *Let G be a hyperbolic TDLC-group with* $\mathrm{cd}_{\mathbb{Q}}(G) \leq 2$. *Every compactly presented closed subgroup H of G is hyperbolic.*

Hyperbolicity can be studied for more general locally compact groups. In particular, a locally compact group is hyperbolic if it has a continuous proper cocompact isometric action on some proper geodesic hyperbolic metric space [CCMT15]; this generalizes the classical definition in the discrete case as well as the definition in the class of totally disconnected locally compact groups used in this thesis. By analogy with the discrete case, the asymptotic dimension provides a quasi-isometry invariant of locally compact compactly generated groups. The question below suggests a possible generalization of Theorem 6.1.1 for the larger class of locally compact hyperbolic groups.

Question 6.1.2. *Let G be a locally compact hyperbolic group such that* $\operatorname{asdim} G \leq 2$. *Are compactly presented subgroups of G hyperbolic?*

In the generality of locally compact groups, we are not aware of a cohomology theory for locally compact groups that allows extending the techniques used to prove

Theorem 6.1.1. However, using the results of Buyalo and Lebedeva [BL07] and Bestvina and Mess [BM91], we proved that this is true for the discrete case in Chapter 4.

Theorem 6.1.3. [ACCCMP21] *Let G be a discrete hyperbolic group such that* $\operatorname{asdim} G \leq 2$. *Then every finitely presented subgroup of G is hyperbolic.*

We believe that same can also be proven for the locally compact groups using the same strategy as the discrete case. It will require generalizing the work of Bestvina and Mess [BM91] for the locally compact groups.

6.2 Baumslag's Conjecture

The following question is one of the outstanding open problems in the group theory.

Question 6.2.1 (Baumslag)**.** *Is every one relator group coherent?*

The problem is proven to be true for partial cases, see [MW05] for example. Remarkable progress in this question was made by independent works of Louder, Wilton [LW20]; and Wise [Wis22], proving the following:

Theorem 6.2.2. *Every one relator group with torsion is coherent.*

In Chapter 5, we proved the following:

Theorem 6.2.3. *Let $A *_C B$ be a topological group that splits as an amalgamated free product of two coherent open subgroups A and B with compact intersection C. If $r \in A *_C B$ is not conjugate into A or B, then for any sufficiently large integer m, the quotient group $G = (A *_C B)/\langle\langle r^m \rangle\rangle$ is coherent.*

The above theorem relied upon the *perimeter method*, a tool developed to prove partial cases of Baumslag's conjecture [MW05]. We believe that the new tools developed in the recent works [LW20, Wis22] can be used to prove the following in the affirmative.

Question 6.2.4. *Let $A *_C B$ be a topological group that splits as an amalgamated free product of two coherent open subgroups A and B with compact intersection C. If $r \in A *_C B$ is not conjugate into A or B, is the quotient group $G = (A *_C B)/\langle\langle r^m \rangle\rangle$ coherent for any $m > 1$?*

6.3 Relatively hyperbolic groups

In Section 5.8, the definition of relatively hyperbolic groups for the pairs $(G, \mathcal{H})$ is a generalization of the definition by Bowditch [Bow12]. Relative hyperbolicity has been a popular topic of research in geometric group theory with different definitions, not known to be equivalent at the time, see [Osi06], [Gro87], [Far98], [DS05]. Note that not all of these definitions require groups to be finitely generated, and some are defined for any countable groups. All of these definitions are now known to be equivalent in complete generality, see [Hru10]. One of the generalizations of another definition of relative hyperbolicity for locally compact groups has been studied in [CCMT15]. A future direction of research is to extend other versions of the definition and verify if they are equivalent.

For instance, in [Osi06], Osin defined relative Dehn functions for pairs $(G, \mathcal{H})$, where G is a group and $\mathcal{H}$ is a finite collection of subgroups; and proved the following:

Theorem 6.3.1. *Let G be a finitely generated group and $\mathcal{H}$ be a finite collection of subgroups of G. Then the following conditions are equivalent.*

1. *G is finitely presented with respect to $\mathcal{H}$ and the corresponding relative Dehn function is linear.*
2. *G is hyperbolic relative to $\mathcal{H}$ in the sense of Bowditch.*

We studied homological Dehn functions for TDLC groups in the terms of weak isoperimetric inequalities in Chapter 4 and proved the following:

Theorem 6.3.2. [ACCCMP21] *A compactly generated TDLC-group G is hyperbolic if and only if G is compactly presented and satisfies the weak linear isoperimetric inequality.*

The theory of weak isoperimetric inequalities can be extended to define relative homological functions for proper pairs $(G, \mathcal{H})$ using the tools from Chapter 5 to prove the following.

Conjecture 6.3.3. *Let $(G, \mathcal{H})$ be a proper pair. The following are equivalent:*

1. *G is compactly presented relative to $\mathcal{H}$ and the corresonding relative homological function is linear.*

2. G is relatively hyperbolic to $\mathcal{H}$.

A subgroup H of a group G is *quasi-convex* if $H \to G$ is quasi-isometric embedding. Quasi-convex subgroups are an excellent source of examples of 'well-behaved' subgroups and are widely studied in the discrete case. The theory of relative quasi-convexity has been extended to groups with respect to the collection of subgroups, in the discrete case. We are currently extending that theory for locally compact groups with a compact open subgroup. We would also like to note that the subgroups in the Theorem 6.2.3 are in fact quasi-convex.

Bibliography

[ABC$^+$91] J. M. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro, and H. Short, *Notes on word hyperbolic groups*, Group theory from a geometrical viewpoint (Trieste, 1990), 1991, pp. 3–63. Edited by Short. MR1170363

[ABDY13] Aaron Abrams, Noel Brady, Pallavi Dani, and Robert Young, *Homological and homotopical Dehn functions are different*, Proc. Natl. Acad. Sci. USA **110** (2013), no. 48, 19206–19212. MR3153947

[Abe72] Herbert Abels, *Kompakt definierbare topologische Gruppen*, Math. Ann. **197** (1972), 221–233. MR0315039

[Abe73] ———, *Specker-Kompaktifizierungen von lokal kompakten topologischen Gruppen*, Math. Z. **135** (1973/74), 325–361. MR344375

[ACCCMP21] S. Arora, I. Castellano, G. Corob Cook, and E. Martínez-Pedroza, *Subgroups, hyperbolicity and cohomological dimension for totally disconnected locally compact groups*, Journal of Topology and Analysis (2021Mar), 1–27.

[Ago13] Ian Agol, *The virtual Haken conjecture*, Doc. Math. **18** (2013), 1045–1087. With an appendix by Agol, Daniel Groves, and Jason Manning. MR3104553

[AMP20] Shivam Arora and Eduardo Mart'inez-Pedroza, *Subgroups of word hyperbolic groups in rational dimension 2*, Groups, Geometry, and Dynamics (2020).

[AMP22] ———, *Topological groups with a compact open subgroup, coherence and relative hyperbolicity*, Preprint (2022).

[Bas93] H Bass, *Covering theory for graphs of groups*, Journal of pure and applied algebra (1993) (English).

[Bau07] Udo Baumgartner, *Totally disconnected, locally compact groups as geometric objects*, Geometric group theory, 2007, pp. 1–20.

[Bau74] Gilbert Baumslag, *Some problems on one-relator groups*, Proceedings of the Second International Conference on the Theory of Groups (Australian Nat. Univ., Canberra, 1973), 1974, pp. 75–81. Lecture Notes in Math., Vol. 372. MR0364463

[BB97] Mladen Bestvina and Noel Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** (1997), no. 3, 445–470. MR1465330

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR1744486

[BL07] S. V. Buyalo and N. D. Lebedeva, *Dimensions of locally and asymptotically self-similar spaces*, Algebra i Analiz **19** (2007), no. 1, 60–92. MR2319510

[BM00] Marc Burger and Shahar Mozes, *Lattices in product of trees*, Inst. Hautes Études Sci. Publ. Math. **92** (2000), 151–194 (2001). MR1839489

[BM91] Mladen Bestvina and Geoffrey Mess, *The boundary of negatively curved groups*, J. Amer. Math. Soc. **4** (1991), no. 3, 469–481. MR1096169

[BM97] Marc Burger and Shahar Mozes, *Finitely presented simple groups and products of trees*, C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), no. 7, 747–752. MR1446574

[BMW12] Udo Baumgartner, Rögnvaldur G Möller, and George A Willis, *Hyperbolic groups have flat-rank at most 1*, Israel Journal of Mathematics **190** (2012), no. 1, 365–388.

[Bou97] M. Bourdon, *Immeubles hyperboliques, dimension conforme et rigidité de Mostow*, Geom. Funct. Anal. **7** (1997), no. 2, 245–268. MR1445387

[Bow12] B. H. Bowditch, *Relatively hyperbolic groups*, Internat. J. Algebra Comput. **22** (2012), no. 3, 1250016, 66. MR2922380

[Bra00] Noel Brady, *Finite subgroups of hyperbolic groups*, Internat. J. Algebra Comput. **10** (2000), no. 4, 399–405. MR1776048

[Bra99] ———, *Branched coverings of cubical complexes and subgroups of hyperbolic groups*, J. London Math. Soc. (2) **60** (1999), no. 2, 461–480. MR1724853

[Bro94] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original. MR1324339

[BsG95] O. V. Bogopolʹskiĭ and V. N. Gerasimov, *Finite subgroups of hyperbolic groups*, Algebra i Logika **34** (1995), no. 6, 619–622, 728. MR1400705

[BSW08] Udo Baumgartner, Günter Schlichting, and George A Willis, *Geometric characterization of flat groups of automorphisms*, arXiv preprint arXiv:0807.5060 (2008).

[Cam96] Peter J Cameron, *Metric and topological aspects of the symmetric group of countable degree*, European Journal of Combinatorics **17** (1996), no. 2-3, 135–142.

[Cas20] Ilaria Castellano, *Rational discrete first degree cohomology for totally disconnected locally compact groups*, Mathematical Proceedings of the Cambridge Philosophical Society **168** (2020), no. 2, 361–377.

[CC20] I. Castellano and G. Corob Cook, *Finiteness properties of totally disconnected locally compact groups*, Journal of Algebra **543** (2020), 54–97.

[CCMT15] Pierre-Emmanuel Caprace, Yves Cornulier, Nicolas Monod, and Romain Tessera, *Amenable hyperbolic groups*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 11, 2903–2947. MR3420526

[CdlH16] Yves Cornulier and Pierre de la Harpe, *Metric geometry of locally compact groups*, EMS Tracts in Mathematics, vol. 25, European Mathematical Society (EMS), Zürich, 2016. Winner of the 2016 EMS Monograph Award. MR3561300

[CM11] Pierre-Emmanuel Caprace and Nicolas Monod, *Decomposing locally compact groups into simple pieces*, Math. Proc. Cambridge Philos. Soc. **150** (2011), no. 1, 97–128. MR2739075

[CW16a] I. Castellano and Th. Weigel, *Rational discrete cohomology for totally disconnected locally compact groups*, J. Algebra **453** (2016), 101–159. MR3465350

[CW16b] Ilaria Castellano and Thomas S. Weigel, *Rational discrete cohomology for totally disconnected locally compact groups*, J. Algebra **453** (2016), 101–159. MR3465350

[Deh11] M. Dehn, *Über unendliche diskontinuierliche gruppen*, Mathematische Annalen **71** (1911), no. 1, 116–144.

[Deh12] ———, *Transformation der Kurven auf zweiseitigen Flächen*, Math. Ann. **72** (1912), no. 3, 413–421. MR1511705

[DJ91] Michael W. Davis and Tadeusz Januszkiewicz, *Hyperbolization of polyhedra*, J. Differential Geom. **34** (1991), no. 2, 347–388. MR1131435

[DK18] Cornelia Druţu and Michael Kapovich, *Geometric group theory*, American Mathematical Society Colloquium Publications, vol. 63, American Mathematical Society, Providence, RI, 2018. With an appendix by Bogdan Nica. MR3753580

[dlST19] Mikael de la Salle and Romain Tessera, *Characterizing a vertex-transitive graph by a large ball*, J. Topol. **12** (2019), no. 3, 705–743. MR4072155

[Dra99] A. N. Dranishnikov, *Boundaries of Coxeter groups and simplicial complexes with given links*, J. Pure Appl. Algebra **137** (1999), no. 2, 139–151. MR1684267

[DS05] Cornelia Druţu and Mark Sapir, *Tree-graded spaces and asymptotic cones of groups*, Topology **44** (2005), no. 5, 959–1058. With an appendix by Denis Osin and Mark Sapir. MR2153979

[ECH+92] David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston, *Word processing in groups*, Jones and Bartlett Publishers, Boston, MA, 1992. MR1161694

[Far98] B. Farb, *Relatively hyperbolic groups*, Geom. Funct. Anal. **8** (1998), no. 5, 810–840. MR1650094

[Fle98] Jeffrey L. Fletcher, *Homological group invariants*, ProQuest LLC, Ann Arbor, MI, 1998. Thesis (Ph.D.)–The University of Utah. MR2697592

[FMP18] Joshua W. Fleming and Eduardo Martínez-Pedroza, *Finiteness of homological filling functions*, Involve **11** (2018), no. 4, 569–583. MR3778913

[Gar21] Giles Gardam, *A counterexample to the unit conjecture for group rings*, Ann. of Math. (2) **194** (2021), no. 3, 967–979. MR4334981

[Geo08] Ross Geoghegan, *Topological methods in group theory*, Graduate Texts in Mathematics, vol. 243, Springer, New York, 2008. MR2365352

[Ger87] S. M. Gersten, *Reducible diagrams and equations over groups*, Essays in group theory, 1987, pp. 15–73. MR919828

[Ger96a] ———, *A cohomological characterization of hyperbolic groups*, 1996.

[Ger96b] ———, *Subgroups of word hyperbolic groups in dimension* 2, J. London Math. Soc. (2) **54** (1996), no. 2, 261–283. MR1405055

[Ger98] ———, *Cohomological lower bounds for isoperimetric functions on groups*, Topology **37** (1998), no. 5, 1031–1072. MR1650363 (2000c:20063)

[GM08] Daniel Groves and Jason Fox Manning, *Dehn filling in relatively hyperbolic groups*, Israel J. Math. **168** (2008), 317–429. MR2448064

[Gre60] Martin Greendlinger, *Dehn's algorithm for the word problem*, Comm. Pure Appl. Math. **13** (1960), 67–83. MR124381

[Gro81] Mikhael Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. **53** (1981), 53–73. MR623534

[Gro87] M. Gromov, *Hyperbolic groups*, Essays in group theory, 1987, pp. 75–263. MR919829

[GS90] S. M. Gersten and H. B. Short, *Small cancellation theory and automatic groups*, Invent. Math. **102** (1990), no. 2, 305–334. MR1074477

[HMP14] Richard Gaelan Hanlon and Eduardo Martínez-Pedroza, *Lifting group actions, equivariant towers and subgroups of non-positively curved groups*, Algebr. Geom. Topol. **14** (2014), no. 5, 2783–2808. MR3276848

[HMP16] Richard Gaelan Hanlon and Eduardo Martínez Pedroza, *A subgroup theorem for homological filling functions*, Groups Geom. Dyn. **10** (2016), no. 3, 867–883. MR3551182

[HMPS21] Sam Hughes, Eduardo Martínez-Pedroza, and Luis Jorge Sánchez Saldaña, *Quasi-isometry invariance of relative filling functions*, 2021.

[HOP14] Sebastian Hensel, Damian Osajda, and Piotr Przytycki, *Realisation and dismantlability*, Geom. Topol. **18** (2014), no. 4, 2079–2126. MR3268774

[Hru10] G. Christopher Hruska, *Relative hyperbolicity and relative quasiconvexity for countable groups*, Algebr. Geom. Topol. **10** (2010), no. 3, 1807–1856. MR2684983

[KB02] Ilya Kapovich and Nadia Benakli, *Boundaries of hyperbolic groups*, Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001), 2002, pp. 39–93. MR1921706

[KM08] Bernhard Krön and Rögnvaldur G. Möller, *Analogues of Cayley graphs for topological groups*, Math. Z. **258** (2008), no. 3, 637–675. MR2369049

[KM98] O. Kharlampovich and A. Myasnikov, *Irreducible affine varieties over a free group. I. Irreducibility of quadratic equations and Nullstellensatz*, J. Algebra **200** (1998), no. 2, 472–516. MR1610660

[Kro21] Robert Kropholler, *Hyperbolic groups with finitely presented subgroups not of type F_3*, Geom. Dedicata **213** (2021), 589–619. MR4278346

[KW00] Ilya Kapovich and Daniel T. Wise, *The equivalence of some residual properties of word-hyperbolic groups*, J. Algebra **223** (2000), no. 2, 562–583. MR1735163

[LS01] Roger C. Lyndon and Paul E. Schupp, *Combinatorial group theory*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition. MR1812024 (2001i:20064)

[LW20] Larsen Louder and Henry Wilton, *One-relator groups with torsion are coherent*, Math. Res. Lett. **27** (2020), no. 5, 1499–1511. MR4216595

[Mac64] A. M. Macbeath, *Groups of homeomorphisms of a simply connected space*, Ann. of Math. (2) **79** (1964), 473–488. MR160848

[Min02] Igor Mineyev, *Bounded cohomology characterizes hyperbolic groups*, Q. J. Math. **53** (2002), no. 1, 59–73. MR1887670

[Möl02] Rögnvaldur G Möller, *Structure theory of totally disconnected locally compact groups via graphs and permutations*, Canadian Journal of Mathematics **54** (2002), no. 4, 795–827.

[Mos68] G. D. Mostow, *Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms*, Inst. Hautes Études Sci. Publ. Math. **34** (1968), 53–104. MR236383

[MP16] Eduardo Martínez-Pedroza, *A note on fine graphs and homological isoperimetric inequalities*, Canad. Math. Bull. **59** (2016), no. 1, 170–181. MR3451908

[MP17] Eduardo Martínez-Pedroza, *Subgroups of relatively hyperbolic groups of Bredon cohomological dimension 2*, J. Group Theory **20** (2017), no. 6, 1031–1060. MR3719315

[MPR21] Eduardo Martínez-Pedroza and Farhan Rashid, *A note on hyperbolically embedded subgroups*, Communications in Algebra **0** (2021), no. 0, 1–10, available at `https://doi.org/10.1080/00927872.2021.1983581`.

[MPW11a] Eduardo Martínez-Pedroza and Daniel T. Wise, *Relative quasiconvexity using fine hyperbolic graphs*, Algebr. Geom. Topol. **11** (2011), no. 1, 477–501. MR2783235 (2012c:20117)

[MPW11b] Eduardo Martínez-Pedroza and Daniel T. Wise, *Local quasiconvexity of groups acting on small cancellation complexes*, Journal of Pure and Applied Algebra **215** (2011), no. 10, 2396–2405.

[MS02] David Meintrup and Thomas Schick, *A model for the universal space for proper actions of a hyperbolic group*, New York J. Math. **8** (2002), 1–7. MR1887695

[MW02] Jonathan P. McCammond and Daniel T. Wise, *Fans and ladders in small cancellation theory*, Proceedings of the London Mathematical Society **84** (2002), no. 3, 599–644.

[MW05] J. P. McCammond and D. T. Wise, *Coherence, local quasiconvexity, and the perimeter of 2-complexes*, Geom. Funct. Anal. **15** (2005), no. 4, 859–927. MR2221153

[Ol'91] A. Yu. Ol'shanskiĭ, *Geometry of defining relations in groups*, Mathematics and its Applications (Soviet Series), vol. 70, Kluwer Academic Publishers Group, Dordrecht, 1991. Translated from the 1989 Russian original by Yu. A. Bakhturin. MR1191619

[Os92] A. Yu. Ol' shanskiĭ, *Almost every group is hyperbolic*, Internat. J. Algebra Comput. **2** (1992), no. 1, 1–17. MR1167524

[Osi06] Denis V. Osin, *Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems*, Mem. Amer. Math. Soc. **179** (2006), no. 843, vi+100. MR2182268 (2006i:20047)

[Pra82] Gopal Prasad, *Elementary proof of a theorem of Bruhat-Tits-Rousseau and of a theorem of Tits*, Bull. Soc. Math. France **110** (1982), no. 2, 197–202. MR667750

[Rip82] E. Rips, *Subgroups of small cancellation groups*, Bull. London Math. Soc. **14** (1982), no. 1, 45–47. MR642423

[Świ98] Jacek Świątkowski, *Trivalent polygonal corplexes of nonpositive curvature and Platonic symmetry*, Geom. Dedicata **70** (1998), no. 1, 87–110. MR1612350

[Sco73] G. P. Scott, *Finitely generated* 3*-manifold groups are finitely presented*, J. London Math. Soc. (2) **6** (1973), 437–440. MR380763

[Sel01] Zlil Sela, *Diophantine geometry over groups. I. Makanin-Razborov diagrams*, Publ. Math. Inst. Hautes Études Sci. **93** (2001), 31–105. MR1863735

[Ser02] Jean-Pierre Serre, *Galois cohomology, corrected reprint of the 1997 english edition*, Springer Monographs in Mathematics, Springer-Verlag, Berlin (2002), 94720–3840.

[Ser80] ———, *Trees*, Springer-Verlag, Berlin-New York, 1980. Translated from the French by John Stillwell. MR607504

[Tao14] Terence Tao, *Hilbert's fifth problem and related topics*, Graduate Studies in Mathematics, vol. 153, American Mathematical Society, Providence, RI, 2014. MR3237440

[Thu82] William P. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), no. 3, 357–381. MR648524

[VD36] D. Van Dantzig, *Zur topologischen Algebra. III. Brouwersche und Cantorsche Gruppen*, Compositio Math. **3** (1936), 408–426. MR1556954

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324

[Wil94] G. Willis, *The structure of totally disconnected, locally compact groups*, Math. Ann. **300** (1994), no. 2, 341–363. MR1299067

[Wis20] Daniel T. Wise, *An invitation to coherent groups* (Dylan Thurston, ed.), Princeton University Press, 2020.

[Wis22] ______, *Coherence, local indicability and nonpositive immersions*, J. Inst. Math. Jussieu **21** (2022), no. 2, 659–674. MR4386825

[You11] Robert Young, *Homological and homotopical higher-order filling functions*, Groups Geom. Dyn. **5** (2011), no. 3, 683–690. MR2813532