Invasion Speed Determinacy for Wave Propagations in Partial Differential Equation Models Arising from Population Biology

by

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Abstract

This thesis aims at developing the study of invasion speed determinacy for wave propagation in partial differential equations arising from population biology. Along this direction, we first investigate a reaction-diffusion-advection equation in a cylindrical domain with a Fisher-KPP type nonlinearity. Using the upper and/or lower solutions method, we obtain sufficient conditions under which the linear or nonlinear selection is realized when the model is prescribed with Neumann boundary conditions and Dirichlet boundary conditions, respectively. To study the invasion speed determinacy of a system, we investigate a reaction-diffusion-advection population model arising in stream ecology. We concentrate on how the spreading speed (the minimal wave speed) is impacted by the Allee effect in the model. Linear and nonlinear selection mechanisms for the spreading speed are first defined, and the determinacy is further established by way of the upper and lower solution method. It is found that the nonlinear determinacy is realized if there exists a lower solution with a faster decay.

For a multiple species population system having diffusion, individual species possibly invade into the far end with different spreading speeds. Predicting or determining them (the fast and slow-spreading speeds) becomes challenging. Hence, we first analyze a cooperative Lotka-Volterra system, which admits a single or multiple spreading speeds (co-speed or fast-slow speeds). We successfully derive a necessary and sufficient condition for this particular model to determine whether the system has a single spreading speed or multiple spreading speeds. We define the linear and nonlinear speed selection mechanism for each case and derive new conditions to classify the speed selection. After studying the former three particular models arising from population biology, we further, in the last part, present the speed selection mechanism for an abstract time-periodic monotone semiflow. At the end of this thesis, we present our future work.

To my dearest parents
for always loving and supporting me.

Lay summary

The wave propagation, found in partial differential equation models, has wide applications in physics, chemistry, and biology. The invasion speed (spreading speed, mathematically) is an essential characteristic of the waves. This thesis focuses on studying the invasion speed determinacy for wave propagation by presenting three typical models from population biology and studying the spreading speeds of time-periodic semiflows.

We start with a reaction-diffusion-advection equation in an infinite cylindrical domain with a Fisher-KPP type nonlinearity. We prescribe this model with either Neumann boundary conditions or Dirichlet boundary conditions to have broader applications. Using the upper and lower solutions method, we obtain sufficient conditions under which the spreading speed is linearly or nonlinearly selected. Numerically, we provide two examples that match our theoretical result. We then proceed to investigate systems of partial differential equations by a population model from stream ecology. We concentrate on how the Allee effect impacts the spreading speed. We define the linear and nonlinear speed selection mechanisms, derive conditions to classify them and perform several numerical simulations that illustrate our discovery.

When a model consists of multiple species, individual species may invade at different speeds. To study such a phenomenon, we consider a cooperative Lotka-Volterra model. This model admits either a single spreading speed or multiple spreading speeds based on different parameter sets. We prove a sufficient and necessary condition to decide which case will happen. Then, we define the speed selection mechanism for each case and derive conditions to classify the selection mechanisms by the upper and lower solutions method. For each case, we also perform numerical simulations to confirm our results. Accumulating all the knowledge learned from the above three models, we finally present a study of the speed selection mechanism for time-periodic semiflows.

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Statement of contribution

Chapters 2–5 of this thesis consist of the following papers:

Chapter 2: Zhe Huang and Chunhua Ou, Speed selection for traveling waves of a reaction- diffusion-advection equation in a cylinder, Physica D: Nonlinear Phenomena, 402:132225, 2020.

Chapter 3: Zhe Huang and Chunhua Ou, Speed determinacy of traveling waves to a stream-population mode with Allee effect, SIAM J. Appl. Math., 80(4):1820–1840, 2020.

Chapter 4: Zhe Huang and Chunhua Ou, Determining Individual Invasion Speed for the Cooperative Lotka-Volterra System, in progress, 2021.

Chapter 5: Zhe Huang and Chunhua Ou, Determining spreading speeds for abstract time-periodic monotone semiflows, in progress, 2021.

The author performed the work of the above papers under the supervision of Professor Chunhua Ou.

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Chapter 1

Introduction

The wave propagation, described by partial differential equation models, has always been a hot topic for applied mathematical research due to its various applications in practice areas, especially population invasion in biology. The invasion speed for wave propagation is an essential character to describe the waves. Thus, in this thesis, we focus on the determinacy of the invasion speed.

In the present introduction, we attempt to give a clear picture of invasion speed determinacy by presenting a prototype model which admits traveling waves – the famous Fisher-KPP equation. This model is given as

$$\begin{cases} u_t = u_{xx} + f(u), \ x \in \mathbb{R}, \ t > 0, \\ u(0, x) = u_0(x). \end{cases} \tag{1.0.1}$$

For the history and development of this model, we refer to [3, 4, 24, 37, 68, 82] and references therein. In the model, the function $u(t, x) \in [0, 1]$ represents the population density of a species at location x and time t. The growth function $f(u)$ is assumed to satisfy

$$f(0) = f(1) = 0, \ f'(0) > 0, \ f'(1) < 0 \text{ and } f(u) > 0 \text{ for } u \in (0, 1).$$

To investigate the wave propagation of such a model, we focus on the traveling wave solutions. A traveling wave solution is a solution of special type satisfying

$$u(t, x) = U(z), \ z = x - ct,$$

where $U(z)$ is called the wave profile, z is the wave coordinate, and $c \geqslant 0$ is the wave speed. It is well-known that there exists a critical number c^* so that the equation (1.0.1) has traveling wave solutions for all $c \geqslant c^*$, see, e.g., [3, 4, 24, 37]. The critical number c^* is the spreading speed in the sense that

$$\lim_{t\to\infty,\ x>(c^*+\varepsilon)t} u(t,x) = 0, \quad \lim_{t\to\infty,\ x<(c^*-\varepsilon)t} u(t,x) = 1, \tag{1.0.2}$$

for any $\varepsilon > 0$, when the initial data $u_0(x)$ has compact support. The solution with this kind of initial data will converge to a traveling wave solution with speed c^*. To estimate the spreading speed c^*, we linearize (1.0.1) around 0 to obtain a linear speed denoted as $c_0 = 2\sqrt{f'(0)}$. It is known that, see, e.g., [43],

$$c^* \geqslant c_0.$$

The determinacy of spreading speed is to find conditions to decide which selection occurs: a linear selection $c^* = c_0$ or a nonlinear selection $c^* > c_0$. It is known that when f satisfies a subhomogeneous condition (sublinear condition, in some context)

$$f(u) \leqslant f'(0)u,$$

the system (1.0.1) has a spreading speed $c^* = c_0$, that is, the linear selection is realized. This subhomogeneous condition for the growth function has important applications. A representative example of the growth function is the Logistic growth function given as $f(u) = u(1-u)$, which clearly satisfies the subhomogeneous condition; thus, it follows that $c^* = c_0 = 2$. However, when an Allee effect appears, e.g., $f(u) = u(1-u)(1+\rho u)$, the inequality $f(u) \leqslant f'(0)u$ fails when $\rho > 1$ and $u \in (0, \frac{\rho-1}{\rho})$. Thus, the equality $c^* = c_0$ may not happen.

We here show a numerical simulation to give a visual idea. Figure 1.1 is drawn by choosing $f(u) = u(1-u)$. The left panel of Figure 1.1 is the initial data. As time increases, we see in the right panel of Figure 1.1, a wave-like function with a stationary form propagates to the right. To find its corresponding speed, we use the level set method. After running the simulation a few minutes so that a stable wave-like solution has appeared, we extract the solutions' data. Then, we obtain a number of solutions $u(t,x)$ at different time t, shown in Figure 1.2. Thus, we find

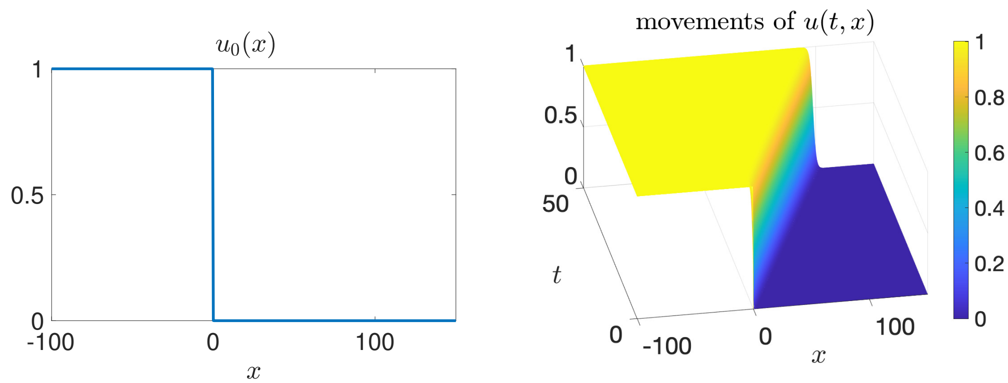


Figure 1.1: Traveling wave solutions of the Fisher-KPP equation with $f(u) = u(1-u)$.

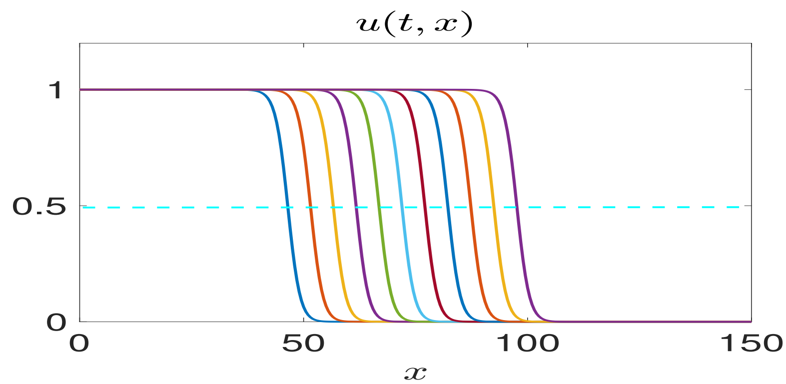


Figure 1.2: Traveling wave solutions at different time.

the positions of $x(t)$ for $u(t, x) = 0.5$ in each curve, that is, the intersections of those colorful curves and the cyan dashed line which denotes $u = 0.5$ in Figure 1.2. Thus, we can compute the speed as $c^* = 2.00047$, which indicates a linear selection $c^* = c_0$. Using the same initial condition and the level set method, we find the speed increases to $c^* = 2.1374 > 2$ when $f(u) = u(1-u)(1+4u)$. This indeed implies that a nonlinear selection is realized.

The investigation of spreading speed c^* when the subhomogeneous condition does not hold is a challenging problem. In this thesis, we study the invasion speeds of three partial differential equation models arising in population biology as well as a time-periodic abstract semiflow. A summary of our research work is given below.

1.1 A Reaction-Diffusion-Advection Equation in a Cylinder

In the first project, we consider a reaction-diffusion-advection equation in a cylindrical domain with a Fisher-KPP type nonlinearity. The equation is given as

$$\begin{cases} u_t = u_{xx} + \Delta_y u + \alpha(y) u_x + f(u), \ (x,y) \in \mathbb{R} \times \Omega, \ t > 0, \\ Bu = 0, \ (x,y) \in \mathbb{R} \times \partial\Omega, \\ u(x,y,0) = u_0(x,y), \ (x,y) \in \mathbb{R} \times \Omega. \end{cases}$$

We prescribe the equation with either the Neumann boundary condition (i.e., $\partial_\nu u(x,y,t) = 0$) or the Dirichlet boundary condition (i.e., $u(x,y,t) = 0$). When the initial condition is chosen as a step function in x and satisfying its own boundary condition in y, this model has a traveling wave, propagating along x-direction, with the form $u(x,y,t) = U(x - ct, y)$.

Since it is well-known that there exists a minimal wave speed $c_{\min}$ such that a traveling wave solution exists if and only if $c \geqslant c_{\min}$, we mainly here are concerned with the linear or nonlinear selection mechanism for the minimal speed. By using the upper and/or lower solutions method, we establish the speed selection mechanism. To have a more direct understanding of the speed selection mechanism, we present two examples in Chapter 2. One application has a cubic reaction term $f(u) = u(1-u)(1+\rho u)$ and $\Omega = (-L_y, L_y)$ and the other one is a subcritical quintic Ginzburg-Landau equation, that is, $f(u) = \mu u + u^3 - u^5$ and $\Omega = (-L_y, L_y)$. We obtain sufficient conditions under which the linear or nonlinear selection is realized for both boundary conditions. Numerical simulations are carried out and illustrate our theoretical results.

1.2 A Stream-Population Model with Allee Effect

In the second model of this thesis, we consider a reaction-advection-diffusion population model from stream ecology. In 2005, Pachepsky, Lutscher, Nisbet, and Lewis [67] proposed a model as

$$\begin{cases} \frac{\partial u}{\partial t} = -\sigma u + \mu v - \alpha \frac{\partial u}{\partial x} + d \frac{\partial^2 u}{\partial x^2}, \\ \frac{\partial v}{\partial t} = +\sigma u - \mu v + (1-v)v. \end{cases}$$

This model describes one species living in stream with a drift flow. By dividing the total population into two interacting compartments: individuals residing on the benthos (the bottom of the stream) and individuals drifting in the flow, their model explains the population persistence very well. However, as the model shown, the only nonlinear growth function is subhomogeneous due to $v(1-v) \leqslant v$ when $v \in [0,1]$; thus, the invasion speed of this species has to be its linear speed obtained by linearizing the model around zero.

In Chapter 3, we extend the above model to the following one with the reaction term possibly having the Allee effect and the residing individuals having a weak diffusive behavior:

$$\begin{cases} \frac{\partial u}{\partial t} = -\sigma u + \mu v - \alpha \frac{\partial u}{\partial x} + d \frac{\partial^2 u}{\partial x^2}, \\ \frac{\partial v}{\partial t} = +\sigma u - \mu v + f(v) + \epsilon \frac{\partial^2 v}{\partial x^2}. \end{cases}$$

With the appearance of the Allee effect, $f(v)$ may not satisfy the subhomogeneous condition ($f(v) \leqslant f'(0)v$). A typical example for $f(v)$ is $v(1-v)(1+\rho v)$ where ρ is an Allee factor (see [84]).

We first define the linear and nonlinear selection mechanisms for the minimal speed (or the spreading speed). Then, by way of the upper and lower solutions method, we establish the determinacy of the speed. It is found that the nonlinear determinacy is realized if there exists a lower solution with a faster decay. By constructing appropriate trial functions, novel results are obtained. At the end of this chapter, numerical simulations are carried out to illustrate our discovery.

1.3 A Cooperative Lotka-Volterra System

Cooperation in a population system can result in the existence of a co-existence (win-win) equilibrium. When diffusion is incorporated, individual species possibly invade into the far end with different spreading speeds. Predicting or determining them (the fast and slow spreading speeds) becomes challenging. The third model we choose is a cooperative Lotka-Volterra system

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u(1 - u + b_1 v), \\ v_t = d_2 v_{xx} + r_2 v(1 - v + b_2 u), \end{cases}$$

with $d_i,\ r_i,\ b_i > 0$ $(i = 1, 2)$, $b_1 b_2 < 1$ and $d_1 r_1 > d_2 r_2$. Under these parameter conditions, this model admits single or multiple spreading speeds (co-speed or fast-slow speed).

In the case of single spreading speed, the two species share a common invasion speed, and nonnegative traveling wave profiles exist, either connecting the co-existence state or the extinction equilibrium, if and only if the wave speed is not less than the common speed. Predicting or determining the invasion speed is more linked to the linearized system at the extinction state.

The existence of multiple spreading speeds indicates new connections of traveling wave profiles into some intermediate states. Due to this, the determinacy of each spreading speed focuses on not only the extinction states but also the corresponding intermediate states. Based on the constructions of upper-lower solutions, we derive new results determining the fast-slow invasive speeds.

When a model does admit multiple spreading speeds, the analysis of it is complicated. To have a clear map for the cooperative Lotka-Volterra model, we perform several numerical simulations that confirm our theoretical predictions. Our numerical simulations also show the existence of traveling wave with a terrace.

1.4 Speed Determinacy for Abstract Time-Periodic Monotone Semiflows

The last chapter is devoted to studying the speed selection mechanisms of traveling waves to an abstract time-periodic monotone semiflow, which is of monostable type with weak compactness and admits boundary equilibria in the phase space. We study various cases when a single spreading speed exists or there exist multiple spreading speeds (fast and slow spreading speeds) and provide a series of conditions to classify the linear and nonlinear selection. When the single spreading speed coincides with the so-called minimal wave speed, we can find a necessary and sufficient condition for the nonlinear selection. Furthermore, by way of comparison principle, we give a bound estimate for the minimal wave speed when it is nonlinearly selected. We apply our results to four time-periodic models: a delayed diffusive equation, a stream population model with the benthic zone, a nonlocal dispersal Lotka-Volterra model, and a reducible cooperative system.

Chapter 2

Speed Selection for Traveling Waves of a Reaction-Diffusion-Advection Equation in a Cylinder

2.1 Introduction

In this chapter, we investigate the speed selection mechanism for traveling wave solutions to a reaction-diffusion-advection equation in an infinite cylindrical domain. The equation we consider is in the following form

$$\begin{cases} u_t = u_{xx} + \Delta_y u + \alpha(y)u_x + f(u),\ (x,y) \in \mathbb{R}\times\Omega,\ t>0, \\ Bu = 0,\ (x,y)\in\mathbb{R}\times\partial\Omega, \\ u(x,y,0) = u_0(x,y),\ (x,y)\in\mathbb{R}\times\Omega. \end{cases} \tag{2.1.1}$$

Here $\Omega \subset \mathbb{R}^{n-1}(n \geqslant 2)$ is a bounded smooth domain. The boundary condition "$Bu = 0$" denotes either the Neumann boundary condition, i.e., $\partial_\nu u(x,y,t) = 0$ for $(x,y)\in\mathbb{R}\times\partial\Omega$, which implies there is no flux of u across the wall of the cylinder, or the Dirichlet boundary condition, i.e., $u(x,y,t) = 0$ for $(x,y)\in\mathbb{R}\times\partial\Omega$, which means the value of u is fixed at zero on the wall of the cylinder. The third term $\alpha(y)u_x$ on the right hand side is a predetermined transport term, or a driving flow, in the x-direction, and the function $\alpha(y)$ is always assumed to be bounded. The reaction term $f : \mathbb{R} \to \mathbb{R}$ is assumed to be a C^2 function with the properties: $f(0) = f(1) = 0$, $f'(0)$ and $f'(1)$ with $f'(1) < 0$.

There are three typical types of function f in applications:
(A1) $f > 0$ on $(0, 1)$;
(A2) for some $\theta \in (0, 1)$, $f = 0$ on $[0, \theta]$ and $f > 0$ on $(\theta, 1)$;
(A3) for some $\theta \in (0, 1)$, $f < 0$ on $(0, \theta)$, $f(\theta) = 0$, and $f > 0$ on $(\theta, 1)$.
Actually, when (A1) or (A3) occurs, these semilinear parabolic equations have many applications in biology, such as population dynamics, gene developments and so on. For more details and descriptions, please see [3, 4, 7, 22, 23, 64]. When (A1) or (A2) occurs, such equations also arise in the study of flame propagation in a tube. For a detailed derivation and physical discussion, we refer readers to [3,7,10,11,39,56,58,92].

Here, we focus on the so-called traveling wave solutions. The traveling wave solutions are defined as solutions of the form

$$u(x, y, t) = U(\xi, y), \ \xi = x - ct. \tag{2.1.2}$$

Here, $U(\xi, y)$ is called the wave profile, and ξ is the wave variable, and $c \in \mathbb{R}$ is the speed of the wave, which is to be determined. After substituting the solution form (2.1.2) into Equation (2.1.1), we find the equation for $U(\xi, y)$ as

$$U_{\xi\xi} + \Delta_y U + [\alpha(y) + c]U_\xi + f(U) = 0. \tag{2.1.3}$$

The traveling wave solutions are required to satisfy the limiting conditions

$$\lim_{\xi\to+\infty} U(\xi, y) = 0, \ \ \lim_{\xi\to-\infty} U(\xi, y) = \beta(y) \not\equiv 0, \tag{2.1.4}$$

uniformly for $y \in \bar{\Omega}$, where the non-negative limiting state $\beta(y)$ is the solution of

$$\begin{cases} \Delta_y U + f(U) = 0, \ y \in \Omega, \\ BU = 0, \ y \in \partial\Omega. \end{cases} \tag{2.1.5}$$

Clearly, if the Neumann boundary condition occurs, it is easy to have $\beta(y) \equiv 1$. On the other hand, in the case of the Dirichlet boundary condition, we can have only one non-negative solution $\beta(y)$ with $0 < \beta(y) < 1$ for $y \in \Omega$ under some mild condition (i.e., the zero solution is linearly unstable) and this can be shown later.

Before stating our main results, we review relevant references on the traveling wave solutions of (2.1.3)-(2.1.4). There is a vast list of literature on the theory related to the existence of the traveling wave solutions in such an equation. For example, in [20, 43, 44], the authors studied the theory of asymptotic speeds of spreading in terms of abstract monotonic systems. In particular, in [7,10,25,69,80,81], the authors considered the existence and uniqueness of the traveling wave solutions in a cylindrical domain.

The most related works to ours are [10] and [44]. In [10], Berestycki and Nirenberg considered Equation (2.1.3)-(2.1.4) prescribed by the Neumann boundary condition. When $f(U)$ satisfies (A2) or (A3) respectively, the authors proved the existence of a traveling wave solution (c, U) and then used the sliding method to further prove the uniqueness of such a solution. Here, the uniqueness is up to a translation, i.e., if there exist solutions (c, U) and (c', U'), then $c' = c$ and $U'(\xi, y) = U(\xi + \tau, y)$ for some real constant τ. For the case (A1), the authors proved that there exists a critical number (or the minimum number) $c^* \in \mathbb{R}$ such that the solution (c, U) exists for c being any value in $[c^*, +\infty)$ and also showed that, if $f(s) \leqslant f'(0)s$ for $0 < s < 1$, this critical number c^* is explicitly determined by Ω, $\alpha(y)$ and the value of $f'(0)$.

In section 6 of [44], Liang and Zhao focused on investigating the theory of spreading speeds and traveling waves for abstract monostable evolution systems. They proved that the spreading speed c^* coincides with the minimal wave speed with a result that traveling wave solutions, connecting β and 0, exist for all $c \geqslant c^*$. When f satisfies the subhomogeneous condition in the sense that $f(\varrho s) \geqslant \varrho f(s)$ for all $\varrho \in [0, 1]$ and $0 \leqslant s \leqslant 1$, they obtained a formula for the speed c^*.

Based on the results in [10, 43, 44], in the case (A1), we know that there always exists a minimal wave speed $c_{\min}$ such that (2.1.1) has a traveling wave solution if $c \geqslant c_{\min}$ and no traveling wave solution exists if $c < c_{\min}$. To proceed, we only consider the case (A1) in this chapter and denote the minimal wave speed $c_{\min}$ as

$$c_{\min} := \inf\{c : \text{the system (2.1.3)-(2.1.4) has a non-negative solution } U(\xi, y)\}.$$

With the understanding that the minimal wave speed is always the spreading speed of biological invasion, it is natural to ask how to determine the speed $c_{\min}$. To estimate it, first by the standard linearization analysis near the zero solution, we will obtain a linear system and the linear speed c_0 in the next section, where $c_{\min} \geqslant c_0$ will be shown. Furthermore, it was numerically observed that depending on the nonlinearity $f(u)$, the wave speed $c_{\min}$ is either equal to or greater than the linear speed c_0. Thus, to distinguish the two different cases, we give the following classification of the speed selection mechanism.

Definition 2.1.1. *The speed selection mechanism for (2.1.3)-(2.1.4) is called a linear selection if $c_{min} = c_0$; otherwise, it is called a nonlinear selection if $c_{min} > c_0$.*

When the space dimension is confined in one dimension, the speed selection can be found in [18, 63, 70, 75, 87, 90] and the references therein. But in higher dimensions with a non-constant convection term, there are not many references on such a topic. In this chapter, we shall focus on the speed selection of the monotone traveling wave solution connecting β to 0 under the condition when zero solution is linearly unstable and β is linearly stable. To see the linear stability, we linearize (2.1.5) near one of the steady states (using ψ to denote either 0 or β) and consider the corresponding eigenvalue problem as

$$\begin{cases} \Delta_y \phi + f'(\psi)\phi = \mu_1(\psi)\phi, \ y \in \Omega, \\ B\phi = 0, \ y \in \partial\Omega, \end{cases}$$

where $\mu_1(\psi)$ is the principal eigenvalue. We say that ψ is linearly stable if $\mu_1(\psi) < 0$ and linearly unstable if $\mu_1(\psi) > 0$. Thus, to have such a monotone traveling wave solution, we further require f to satisfy the following conditions:

(A4) If (2.1.3)-(2.1.4) is prescribed by the Neumann boundary condition, then we require $f'(0) > 0$ and $f'(1) < 0$;

(A5) If (2.1.3)-(2.1.4) is prescribed by the Dirichlet boundary condition, then we require $\mu_1(0) > 0$, and $\mu_1(\beta) < 0$.

Under these conditions, we can confirm that there exists a unique solution $\beta(y)$ to (2.1.5) satisfying $0 < \beta(y) \leqslant 1$.

With the application of the upper and lower solution method, we are able to establish the linear and/or nonlinear selection mechanism for our system. The detail is shown in Sections 3 and 4 which are valid for both Neumann and Dirichlet boundary conditions. We also find a sufficient condition for the nonlinear selection mechanism to our model under the Neumann boundary condition. We should emphasize that our investigations greatly extend the conclusions in [47, 70, 87].

The rest of this chapter is organized as follows. In Section 3.2, we perform the local analysis near zero to find the linear speed c_0. In Section 3.3, we study the speed selection mechanism and present the main result. Then, we give two applications in Section 3.4, one with a cubic nonlinearity and the other with a subcritical quintic Ginzburg-Landau equation in a cylindrical domain. Finally, in Section 3.5 we summarize the obtained results and discuss some open problems. Section 3.6 is an appendix to illustrate the upper and lower solutions method used in our model.

2.2 Local analysis near zero

Linearizing Equation (2.1.3) near zero gives

$$\begin{cases} U_{\xi\xi} + \Delta_y U + (\alpha(y) + c)U_\xi + f'(0)U = 0, \xi \in (-\infty, \infty),\ y \in \Omega, \\ BU = 0,\ y \in \partial\Omega. \end{cases} \tag{2.2.1}$$

Then, letting $U = \varphi(y)e^{-\lambda\xi}$ for some non-negative function $\varphi(y)$ and a real constant λ, we obtain an eigenvalue problem

$$\begin{cases} \Delta_y \varphi + [\lambda^2 - \lambda(\alpha(y) + c) + f'(0)]\varphi = 0,\ y \in \Omega, \\ B\varphi = 0,\ y \in \partial\Omega. \end{cases} \tag{2.2.2}$$

To further discuss the above problem, we denote

$$L_\lambda = \Delta_y + \left[\lambda^2 - \lambda(\alpha(y) + c) + f'(0)\right]. \tag{2.2.3}$$

Then solving the problem (2.2.2) can be regarded as seeking the non-negative solution(s) of $L_\lambda\varphi = 0$ with the boundary condition $B\varphi = 0$. Let $\mu(\lambda)$ be the principal eigenvalue of the operator L_λ, and we consider the following eigenvalue problem

$$L_\lambda \psi = \mu(\lambda)\psi, \quad B\psi|_{y\in\partial\Omega} = 0, \tag{2.2.4}$$

for some non-negative non-zero function $\psi(y)$, $y \in \Omega$. It is clear that to find the solution of (2.2.2) is equivalent to find (c, λ) such that $\mu(\lambda) = 0$, with the corresponding eigenfunction $\psi(y)$ as the solution. For the eigenvalue problem (2.2.4), we have the following results.

(1) When $\lambda \to 0$, $L_\lambda \varphi \to \Delta_y \psi + f'(0)\psi$. From (A4) or (A5), we have $\mu(0) > 0$.

(2) When $\lambda \to +\infty$, we have $\lambda^2 - \lambda(\alpha(y) + c) + f'(0) > M$ for any large positive number M. In this case, by comparison, we have $\mu(+\infty) > 0$ for both boundary conditions.

Furthermore, due to the convexity of the function "$\lambda^2 - \lambda(\alpha(y) + c) + f'(0)$" with respect to λ, it is easy to have the following proposition.

Proposition 2.2.1. *The principal eigenvalue $\mu(\lambda)$ defined in (2.2.2) is convex with respect to $\lambda > 0$.*

Proof. Due to the term λ^2, through a direct computation, it follows that $\mu\left(\frac{\lambda_1+\lambda_2}{2}\right) \leqslant \frac{1}{2}\left(\mu(\lambda_1) + \mu(\lambda_2)\right)$. This proves the result. □

From Equation (2.2.4), it is clear to see that μ is decreasing in c. Thus, we can define

$$c_0 := \min\{c|\ c \in \mathbb{R} \text{ such that } \mu(\lambda) = 0 \text{ has a solution } \lambda \in (0, +\infty)\}.$$

Now, in view of the above proposition, we can arrive at the following theorem.

Theorem 2.2.2. *For the eigenvalue problem (2.2.4), there exists a critical number $c_0 \in \mathbb{R}$ such that*

(1) when $c < c_0$, there is no positive λ such that $\mu(\lambda) = 0$, and (2.2.2) has no non-negative non-zero solution;

(2) when $c = c_0$, there is only one positive λ_0 such that $\mu(\lambda) = 0$, and (2.2.2) has one

solution $\varphi_0 = \psi_0$, where ψ_0 is the principal eigenfunction corresponding to $\lambda = \lambda_0$ in (2.2.4);

(3) when $c > c_0$, there exist $\lambda_1(c)$ and $\lambda_2(c)$ with $\lambda_2(c) > \lambda_1(c) > 0$ such that $\mu(\lambda_i(c)) = 0, i = 1, 2$, and (2.2.2) has two solutions $\varphi_j = \psi_j$ when $\lambda = \lambda_j(c)$, where ψ_j is the principal eigenfunction corresponding to $\lambda = \lambda_j(c)$ in (2.2.4) for $j = 1, 2$.

Remark 2.2.3. *Near $\xi = \infty$, equation (2.1.3) is approximated by the linear equation (2.2.1). From the above theorem, we can see that $c \geqslant c_0$ is a necessary condition for (2.1.3)-(2.1.4) to have a non-negative traveling wave solution. Therefore, $c_{min} \geqslant c_0$. Moreover, $\lambda_2(c) > \lambda_0(c_0) > \lambda_1(c) > 0$ if $c > c_0$.*

2.3 The speed selection

In this section, we study the speed selection mechanism for (2.1.3)-(2.1.4) through the upper and lower solutions method. The key point is to construct a pair of suitable upper and lower solutions. The definition of an upper (or a lower) solution and the details of this method are shown in the Appendix section. To begin with, we denote the left hand side of Equation (2.1.3) as

$$\mathcal{L}(U) := U_{\xi\xi} + \Delta_y U + (\alpha(y) + c)U_\xi + f(U). \tag{2.3.1}$$

For any $c = c_0 + \epsilon_1$ with $\epsilon_1 > 0$, we have two pairs of solutions $(\lambda_1(c), \varphi_1)$ and $(\lambda_2(c), \varphi_2)$ with $\lambda_2(c) > \lambda_1(c) > 0$ for (2.2.2) by Theorem 2.2.2. Then we define a continuous function $\overline{U}(\xi, y)$ as the solution of the following equation

$$\overline{U}_\xi = -\lambda_1(c)\overline{U}\left(1 - \frac{\overline{U}^\gamma}{\beta^\gamma}\right), \tag{2.3.2}$$

where $\gamma > 0$ is a parameter to be determined. Considering the boundary conditions as $\overline{U}(\xi, y) \sim \beta(y)$ when $\xi \to -\infty$, and $\overline{U}(\xi, y) \sim \varphi_1(y)e^{-\lambda_1(c)\xi} \to 0$ when $\xi \to +\infty$, we will obtain the formula for $\overline{U}$ as

$$\overline{U} = \frac{\beta\varphi_1}{[\beta^\gamma e^{\lambda_1(c)\gamma\xi} + \varphi_1^\gamma]^{\frac{1}{\gamma}}}. \tag{2.3.3}$$

It is easy to see that $0 \leqslant \overline{U} \leqslant \beta$ for all $(\xi, y) \in \mathbb{R} \times \Omega$ and

$$\overline{U}_{\xi\xi} = \lambda_1^2(c)\overline{U}\left(1 - \frac{\overline{U}^\gamma}{\beta^\gamma}\right)\left(1 - (\gamma+1)\frac{\overline{U}^\gamma}{\beta^\gamma}\right). \tag{2.3.4}$$

By substituting the formulas of $\overline{U}$, $\overline{U}_\xi$, $\overline{U}_{\xi\xi}$ and $\Delta_y \overline{U}$ into (2.3.1), and after a tedious computation, we finally obtain

$$\mathcal{L}(\overline{U}) = \frac{\overline{U}^{(\gamma+1)}}{\beta^\gamma}\left(1 - \frac{\overline{U}^\gamma}{\beta^\gamma}\right)\left\{ -(\gamma+1)\,\lambda_1^2(c) - (\gamma+1)\,\frac{\varphi_1^2}{\beta^2}\left[\nabla\left(\frac{\beta}{\varphi_1}\right)\right]^2 + G_1(\xi, y)\right\}, \tag{2.3.5}$$

where

$$G_1(\xi, y) = \frac{[f(\overline{U}) - f'(0)\overline{U}] + \left(\frac{\overline{U}^{\gamma+1}}{\beta^\gamma}\right)\left[f'(0) - \frac{f(\beta)}{\beta}\right]}{\frac{\overline{U}^{\gamma+1}}{\beta^\gamma}\left(1 - \frac{\overline{U}^\gamma}{\beta^\gamma}\right)}. \tag{2.3.6}$$

It is clear that if $\epsilon_1 \to 0$, then $c \to c_0$, $\lambda_1(c) \to \lambda_0(c_0)$ and $\varphi_1 \to \varphi_0$. Thus, for $\epsilon_1 \ll 1$, in the sense of Definition 2.6.1 and Lemma 2.6.2, the function $\overline{U}$ is an upper solution to (2.3.1) if

$$\max_{(\xi,y)\in\mathbb{R}\times\Omega} G_1(\xi, y) < (\gamma+1)\lambda_0^2(c_0) + (\gamma+1)\,\frac{\varphi_0^2}{\beta^2}\left[\nabla\left(\frac{\beta}{\varphi_0}\right)\right]^2. \tag{2.3.7}$$

Consequently, we have the following lemma for an upper solution.

Lemma 2.3.1. *Suppose $c = c_0 + \epsilon_1$ with ϵ_1 being a sufficiently small positive number. If the inequality (2.3.7) holds, then the function $\overline{U}$, defined in (2.3.3), is an upper solution to system (2.1.3)-(2.1.4) with $\overline{U}(-\infty, y) = \beta(y)$ and $\overline{U}(+\infty, y) = 0$.*

Remark 2.3.2. *To have the above lemma hold, we need the boundedness of G_1 (at least being bounded from above). Indeed, $G_1(\xi, y)$ is continuous on $(\xi, y) \in \mathbb{R} \times \Omega$. Thus it suffices to find $\lim\limits_{\xi\pm\infty} G_1(\xi, y)$ and determine whether they are bounded. As*

$\xi \to -\infty$, i.e., $\overline{U} \to \beta$, we have

$$\lim_{\xi\to-\infty} G_1(\xi,y) = \lim_{\overline{U}\to\beta}\left\{\frac{f(\overline{U}) - \frac{\overline{U}^{\gamma+1}}{\beta^{\gamma+1}}f(\beta)}{\frac{\overline{U}^{\gamma+1}}{\beta^{\gamma}}\left(1-\frac{\overline{U}^{\gamma}}{\beta^{\gamma}}\right)} - \frac{f'(0)}{\frac{\overline{U}^{\gamma}}{\beta^{\gamma}}}\right\} = -\frac{f'(\beta)}{\gamma} + \frac{\gamma+1}{\gamma\beta}f(\beta) - f'(0). \tag{2.3.8}$$

The last equality is obtained by L'Hospital's rule. For $\xi \to +\infty$, i.e., $\overline{U} \to 0$, we have

$$\begin{aligned}\lim_{\xi\to+\infty} G_1(\xi,y) &= \lim_{\overline{U}\to 0}\left\{\frac{f(\overline{U}) - f'(0)\overline{U}}{\frac{\overline{U}^{\gamma+1}}{\beta^{\gamma}}\left(1-\frac{\overline{U}^{\gamma}}{\beta^{\gamma}}\right)} + \frac{f'(0) - \frac{f(\beta)}{\beta}}{1-\frac{\overline{U}^{\gamma}}{\beta^{\gamma}}}\right\} \\ &= \lim_{\overline{U}\to 0}\frac{f''(U)}{\gamma(\gamma+1)\frac{\overline{U}^{\gamma-1}}{\beta^{\gamma}} - 2\gamma(2\gamma+1)\frac{\overline{U}^{2\gamma-1}}{\beta^{2\gamma}}} + f'(0) - \frac{f(\beta)}{\beta}\end{aligned} \tag{2.3.9}$$

The boundedness of the above term depends on the choice of γ and the formula of $f(u)$. Actually, we give the following results.

(1) If $f''(0)$ exists, then by choosing $\frac{1}{2} \leqslant \gamma \leqslant 1$, we find that G_1 is bounded for all $-\infty < \xi < +\infty$.

(2) If $U = 0$ is a solution for $f''(U) = 0$ with multiplicity k, $k = 1, 2, \cdots$, then by choosing $\gamma = k + 1$, we also find that G_1 is bounded for all $-\infty < \xi < +\infty$.

According to Theorem 2.6.4 in the Appendix, to obtain the existence of traveling wave solution $U(\xi, y)$, we also need to find a lower solution to Equation (2.1.3) when $c = c_0 + \epsilon_1$. For this purpose, define a continuous function $\underline{U}(\xi, y)$ as

$$\underline{U}(\xi,y) = \max\{0, \varphi_1(y)(1 - Me^{-\delta\xi})e^{-\lambda_1(c)\xi}\}. \tag{2.3.10}$$

Here, $(\lambda_1(c), \varphi_1)$ has the same meaning as in $\overline{U}$ from Lemma 2.3.1. We fix a small $\delta > 0$ such that $\lambda_1 + \delta < \lambda_2$ and the constant $M > 0$ is to be determined. Let $\xi_0 = \frac{\ln M}{\delta}$, it is easy to see that $\underline{U}$ satisfies the following:

(1) When $\xi \leqslant \xi_0$, $\underline{U} = 0$;

(2) When $\xi > \xi_0$, $\underline{U} = \varphi_1(1 - Me^{-\delta\xi})e^{-\lambda_1\xi}$.

Notice that $\max_{\xi\in\mathbb{R}} \underline{U}(\xi, y) = \frac{\delta\varphi_1(y)}{\lambda_1+\delta}\left[\frac{\lambda_1}{M(\lambda_1+\delta)}\right]^{\frac{\lambda_1}{\delta}} \ll 1$ when M is sufficiently large. Furthermore, we can obtain the following lemma.

Lemma 2.3.3. *When $c = c_0 + \epsilon_1$, the function defined in (2.3.10) is a lower solution to the system (2.1.3)-(2.1.4).*

Proof. If $\xi \leqslant \xi_0$ (i.e., $\underline{U} = 0$), a direct computation gives $\mathcal{L}(\underline{U}) = 0$. If $\xi > \xi_0$, by substituting the formula of $\underline{U}$, we obtain the following:

$$
\begin{aligned}
\mathcal{L}(\underline{U}) &= -e^{-(\delta+\lambda_1)\xi} M L_{(\lambda_1+\delta)}\varphi_1 + f(\underline{U}) - f'(0)(1 - Me^{-\delta\xi})\varphi_1 e^{-\lambda_1\xi} \\
&= -e^{-(\delta+\lambda_1)\xi} M L_{(\lambda_1+\delta)}\varphi_1 + f(\underline{U}) - f'(0)\underline{U} \\
&\geqslant 0
\end{aligned}
\tag{2.3.11}
$$

provided M is sufficiently large. Note that, in the last inequality, we have used the fact that $L_{\lambda_1}\varphi_1 = 0$ and $L_{(\lambda_1+\delta)}\varphi_1 < 0$ when $\lambda_1 + \delta < \lambda_2$, and $[f(\underline{U}) - f'(0)\underline{U}] \sim O(\varphi_1^2 e^{-2\lambda_1\xi})$ as $\underline{U}$ is close to 0. By (2.3.11), Definition 2.6.1 and Lemma 2.6.2, it then follows that there exist positive numbers δ and $M = M(\delta)$ such that $\underline{U}$ is a lower solution of (2.1.3)-(2.1.4) when $c = c_0 + \epsilon_1$. This completes the proof. $\square$

Now, with the construction of an upper and a lower solution above, it is easy to find a ξ_1 so that $\bar{U}(\xi-\xi_1)$ is still an upper solution with $0 \leqslant \underline{U} \leqslant \overline{U}(\xi-\xi_1)$. Therefore, we are ready to give our results for the linear speed selection.

Theorem 2.3.4. *When (2.3.7) is satisfied, the minimal wave speed c_{min} of the system (2.1.3)-(2.1.4) is linearly selected, i.e., $c_{min} = c_0$.*

Proof. When $c = c_0 + \epsilon_1$, by Lemma 2.3.1 and Lemma 2.3.3, we have a pair of an upper and a lower solution. Thus, the existence of a monotone traveling wave solution U of (2.1.3)-(2.1.4) with the speed $c = c_0 + \epsilon_1$ follows from Theorem 2.6.4 and the traveling wave solution satisfies $U(+\infty, y) = 0$ and $U(-\infty, y) = \beta(y)$.

In the case when $c = c_0$, a limiting argument can be applied to obtain the existence of traveling waves.

To be exact, we choose a sequence $\{c_n\}$ such that $c_n \in (c_0, c_0+1]$ and $\lim_{n\to+\infty} c_n = c_0$. For instance, we can choose $c_n = c_0 + \frac{1}{n}$ which clearly satisfies the requirement. Corresponding to each c_n, by the above arguments and Theorem 2.6.4, there exists a monotone decreasing traveling wave solution $U_n(\xi, y)$ of (2.1.3)-(2.1.4). Since $U_n(\xi + \bar{\xi}_0, y)$, $\bar{\xi}_0 \in \mathbb{R}$ is also such a solution, by translation we can always assume $U_n(0, y_0) = \frac{1}{2}\beta(y_0)$ for a given $y_0 \in \Omega$.

Notice that $U_n(\xi, y)$ is uniformly bounded, that is, $|U_n(\xi, y)| \leqslant \beta(y) \leqslant \max \beta(y)$, $\forall(\xi, y) \in \mathbb{R} \times \bar{\Omega}$, $n \geqslant 1$. According to Theorem 2.6.4, U_n is the fixed point of the solution map $T_{-ct}Q_t$, that is, $T_{-ct}Q_t[U_n](x, y) = U_n(x, y)$. Moreover, $\{T_{-ct}Q_t[U_n]\}_{n\geqslant 1}$ is precompact. It then follows that there exists a convergent subsequence of U_n, say $\{U_{n_k}\}_{k\geqslant 1}$, converging to a function $W \in \mathcal{C}_\beta$ as $k \to +\infty$. That is, there exists a function W satisfying $Q_t[W](x, y) = W(x - c_0 t, y) = W(\xi, y)$, or equivalently the equation

$$W_{\xi\xi} + \Delta_y W + (\alpha(y) + c_0)W_\xi + f(W) = 0,\ (\xi, y) \in \mathbb{R} \times \Omega.$$

Clearly, $W(\xi, y)$ is non-increasing in $\xi \in \mathbb{R}$ and $W(0, y_0) = \frac{1}{2}\beta(y_0)$. Moreover, $W(\xi, y)$ connects β to 0 with $W(-\infty, y) = \beta(y)$ and $W(+\infty, y) = 0$ for all $y \in \Omega$. Consequently, when (2.3.7) is satisfied, (2.1.3)-(2.1.4) has a monotone traveling wave solution connecting $\beta(y)$ to 0 with $c = c_0$. The proof is complete. $\square$

Next, we want to investigate the nonlinear speed selection. To proceed, we first prove the following lemma.

Lemma 2.3.5. *For $c_1 > c_0$, suppose that there exists a lower solution $\underline{U}(\xi, y)$ to system (2.1.3)-(2.1.4), which is non-increasing in ξ and satisfies $0 < \underline{U} < \beta(y)$ and*

$$\underline{U} \sim \varphi_2(y)e^{-\lambda_2(c_1)\xi}$$

as $\xi \to +\infty$, where $(\lambda_2(c_1), \varphi_2)$ is defined in Theorem 2.2.2 and $\xi = x - c_1 t$, i.e., $\underline{U}(\xi, y)$ has the faster decay rate near positive infinity. Then there is no traveling wave solution to system (2.1.3)-(2.1.4) connecting $\beta(y)$ to 0 with speed $c \in [c_0, c_1)$.

Proof. By this assumption, there exists a lower solution $\underline{U}(x - c_1 t, y)$ with $c_1 > c_0$ to

$$u_t = u_{xx} + \Delta_y u + \alpha(y) u_x + f(u), \tag{2.3.12}$$

with initial data

$$u(x, y, 0) = \underline{U}(x, y).$$

By way of contradiction, we assume that, for some $c \in [c_0, c_1)$, there exists a monotonic traveling wave solution $U(x - ct, y)$, which connects $\beta(y)$ to 0 and has initial data as

$$u(x, y, 0) = U(x, y).$$

We should note that if $c = c_0$, then we have traveling wave solutions for all $c > c_0$. Thus we can always assume that $c \in (c_0, c_1)$.

Following the calculations from the previous section (see, e.g., from (2.2.1) to (2.2.4)), it is easy to find the asymptotic behavior of $U(x - ct, y)$ with

$$U(\xi, y) \sim C_1 \varphi_1(y) e^{-\lambda_1(c)\xi} + C_2 \varphi_2(y) e^{-\lambda_2(c)\xi}, \ \xi \to \infty,$$

for $C_1 > 0$, or $C_1 = 0, C_2 > 0$. A rigorous proof of this can be obtained by the comparison principle and the linearization of the model. Moreover, we have $\lambda_2(c_1) > \lambda_2(c) > \lambda_0(c_0) > \lambda_1(c) > \lambda_1(c_1)$ when $c \in [c_0, c_1)$. Thus, we can always assume $\underline{U}(x, y) \leqslant U(x, y)$ for $(x, y) \in \mathbb{R} \times \Omega$ (by shifting of U if necessary). Since $\underline{U}(\xi, y), \xi = x - ct$, is assumed to be a lower solution to Equation (2.3.12) and $\underline{U}(x, y) \leqslant U(x, y)$, by comparison, we have

$$\underline{U}(x - c_1 t, y) \leqslant U(x - ct, y), \ (x, y, t) \in \mathbb{R} \times \Omega \times \mathbb{R}_+. \tag{2.3.13}$$

Now, if we fix $\xi_1 = x - c_1 t$, then $\underline{U}(\xi_1, y) > 0$ is fixed. On the other hand, from

$U(x-ct,y)$, it is clear to see

$$U(x-ct,y)=U(\xi_1+(c_1-c)t,y)\sim U(+\infty,y)=0, \text{ as } t\to+\infty.$$

By (2.3.13), we therefore get $\underline{U}(\xi_1,y)\leqslant 0$. This is a contradiction. Thus, there is no traveling wave solution when $c\in[c_0,c_1)$. This completes the proof. $\square$

Remark 2.3.6. *This lemma implies that if there is a lower solution $\underline{U}$ satisfying $0<\underline{U}<\beta(y)$ and $\underline{U}\sim\varphi_2(y)e^{-\lambda_2(c_1)\xi}$ as $\xi\to+\infty$, for $c_1>c_0$, then the nonlinear selection is realized.*

Now, let $c_1=c_0+\epsilon_2$ and define a continuous function as follows

$$\underline{U}_1=\frac{\beta\varphi_2}{[\beta^\gamma e^{\lambda_2(c_1)\gamma\xi}+\varphi_2^\gamma]^{\frac{1}{\gamma}}}. \tag{2.3.14}$$

Similarly to the previous computations, we get

$$\mathcal{L}(\underline{U}_1)=\frac{\underline{U}_1^{(\gamma+1)}}{\beta^\gamma}\left(1-\frac{\underline{U}_1^\gamma}{\beta^\gamma}\right)\left\{-(\gamma+1)\,\lambda_2^2(c_1)-(\gamma+1)\,\frac{\varphi_2^2}{\beta^2}\left[\nabla\left(\frac{\beta}{\varphi_2}\right)\right]^2+G_2(\xi,y)\right\}, \tag{2.3.15}$$

where

$$G_2(\xi,y)=\frac{[f(\underline{U}_1)-f'(0)\underline{U}_1]+\left(\frac{\underline{U}_1^{\gamma+1}}{\beta^\gamma}\right)\left[f'(0)-\frac{f(\beta)}{\beta}\right]}{\frac{\underline{U}_1^{\gamma+1}}{\beta^\gamma}\left(1-\frac{\underline{U}_1^\gamma}{\beta^\gamma}\right)}. \tag{2.3.16}$$

To obtain a condition for the nonlinear selection, we will take $\underline{U}_1$ as the lower solution which satisfies $\underline{U}_1\sim\varphi_2(y)e^{-\lambda_2(c_1)\xi}$ as $\xi\to+\infty$. Notice that when $\epsilon_2\to 0$, we have $\lambda_2(c_1)\to\lambda_0(c_0)$ and $\varphi_2\to\varphi_0$. Thus, if the following condition

$$\min_{(\xi,y)\in\mathbb{R}\times\Omega}G_2(\xi,y)>(\gamma+1)\lambda_0^2(c_0)+(\gamma+1)\,\frac{\varphi_0^2}{\beta^2}\left[\nabla\left(\frac{\beta}{\varphi_0}\right)\right]^2 \tag{2.3.17}$$

is true, then the nonlinear selection is realized.

In the case of Neumann boundary conditions, we have $\beta(y)\equiv 1$, and thus (2.3.14) can be simplified as

$$\underline{U}_1=\frac{\varphi_2}{(e^{\lambda_2(c)\gamma\xi}+\varphi_2^\gamma)^{\frac{1}{\gamma}}}. \tag{2.3.18}$$

We thus have

$$\mathcal{L}(\underline{U}_1) = \underline{U}_1^{(\gamma+1)}\,(1-\underline{U}_1^{\gamma})\left\{-(\gamma+1)\,\lambda_2^2(c)-(\gamma+1)\left[\varphi_2\nabla\left(\frac{1}{\varphi_2}\right)\right]^2+G_2(\xi,y)\right\}, \tag{2.3.19}$$

and

$$G_2(\xi,y)=\frac{f(\underline{U}_1)-f'(0)\underline{U}_1+\underline{U}_1^{\gamma+1}f'(0)}{\underline{U}_1^{\gamma+1}\,(1-\underline{U}_1^{\gamma})}.$$

Moreover, when $\epsilon_2\to 0$, under the condition

$$\min_{(\xi,y)\in\mathbb{R}\times\Omega} G_2(\xi,y) > (\gamma+1)\lambda_0^2(c_0)+(\gamma+1)\left[\varphi_0\nabla\left(\frac{1}{\varphi_0}\right)\right]^2, \tag{2.3.20}$$

we are ready to have the nonlinear selection as follows.

Theorem 2.3.7. *If the inequality (2.3.20) is satisfied, then the minimal speed of system (2.1.3)-(2.1.4) prescribed by the Neumann boundary condition is nonlinearly selected.*

In the case of Dirichlet boundary condition, through similar analysis to that in Remark 2.3.2, we obtain

$$\lim_{\xi\to-\infty} G_2(\xi,y) = -\frac{f'(\beta)}{\gamma}+\frac{\gamma+1}{\gamma\beta}f(\beta)-f'(0)$$

or

$$\lim_{\xi\to-\infty} G_2(\xi,y) = -\frac{\beta}{\gamma}g'(\beta)-g(0)+g(\beta)$$

where $g(u)=f(u)/u$. This gives $\lim\limits_{y\to\partial\Omega}\lim\limits_{\xi\to-\infty} G_2(\xi,y)=0$. Thus (2.3.17) cannot be true, i.e., this choice of the lower solution (i.e., $\underline{U}_1$ in (2.3.14)) is not valid when (2.1.3)-(2.1.4) is prescribed by the Dirichlet boundary condition. We suspect that other challenging types of lower solutions need to be constructed. This will be a subject of our future study.

2.4 Applications

In this section, we apply the results of Section 3 to the reaction-diffusion model with a cubic reaction term and a subcritical quintic Ginzburg-Landau equation respectively. By applying numerical simulations to each case, we will find the linear wave speed, i.e., c_0 defined in Theorem 2.2.2 as well as the numerical minimal wave speed. Comparison of them is carried out to illustrate our theoretical results.

2.4.1 A cubic reaction term

The first application is a cubic reaction term given as $f(u) = u(1-u)(1+2\epsilon u)$ with $\epsilon \geqslant 0$ and $\Omega = (-L_y, L_y)$, that is, we consider traveling wave solutions of the following equation

$$u_t = u_{xx} + u_{yy} + \alpha(y)u_x + u(1-u)(1+2\epsilon u),\ (x,y) \in \mathbb{R} \times (-L_y, L_y),\ t > 0. \quad (2.4.1)$$

The corresponding wave profile (i.e., letting $u(t,x,y) = U(\xi, y)$ and $\xi = x - ct$) becomes

$$U_{\xi\xi} + U_{yy} + (\alpha(y) + c)\, U_\xi + U(1-U)(1+2\epsilon U) = 0, \quad (2.4.2)$$

satisfying

$$\lim_{\xi \to +\infty} U(\xi, y) = 0, \text{ and } \lim_{\xi \to -\infty} U(\xi, y) = \beta(y). \quad (2.4.3)$$

The speed selection of such an equation in one dimensional case was first considered by Hadeler and Rothe [29] in 1975. They studied the equation

$$u_t = u_{xx} + u(1-u)(1+2\epsilon u),\ \epsilon > -\frac{1}{2},\ x \in \mathbb{R},\ t > 0, \quad (2.4.4)$$

and obtained that the minimal speed of the traveling waves is linearly selected when $\epsilon \leqslant 1$ and nonlinearly selected when $\epsilon > 1$. For more details of this result, please refer to [29].

In the sequel, for the model (2.4.1) we always assume that $\epsilon > 0$ and also show that there exists a critical number of ϵ to classify the linear and nonlinear selection mechanism.

The reaction term f is smooth on $[0,1]$ and

$$f(0)=f(1)=0,\ f'(0)=1>0>f'(1)=-1-2\epsilon,\ \text{and } f(u)>0 \text{ for } u\in(0,1).$$

Thus f satisfies (A1), (A4) and (A5) for all ϵ. Moreover, there are equilibria 0 and a nonzero function $\beta(y)$ with $0\leqslant\beta(y)\leqslant 1$ for all $y\in\overline{\Omega}$. Since $-2-8\epsilon\leqslant f''(u)=4\epsilon-2-12\epsilon u\leqslant 4\epsilon-2$, we can choose $\gamma=1$ in (2.3.5). Then, by substituting the formula of f into Equation (2.3.5) and simplifying it, we obtain

$$\mathcal{L}(\overline{U})=\frac{\overline{U}^2}{\beta}\left(1-\frac{\overline{U}}{\beta}\right)\left\{-2\lambda_1^2(c)-2\frac{(\beta'\varphi_1-\beta\varphi_1')^2}{\varphi_1^2\beta^2}+2\epsilon\beta^2\right\}. \tag{2.4.5}$$

Here, $G_1(\xi,y)=2\epsilon\beta^2$ is clearly monotonic in ϵ. Thus, the condition (2.3.7) for the linear selection becomes

$$\epsilon<\min_{y\in\Omega}\left[\frac{\lambda_0^2(c_0)}{\beta^2}+\frac{(\beta'\varphi_0-\beta\varphi_0')^2}{\varphi_0^2\beta^4}\right]. \tag{2.4.6}$$

Similarly, the condition (2.3.20) for the nonlinear selection becomes

$$\epsilon>\max_{y\in\Omega}\left[\frac{\lambda_0^2(c_0)}{\beta^2}+\frac{(\beta'\varphi_0-\beta\varphi_0')^2}{\varphi_0^2\beta^4}\right]. \tag{2.4.7}$$

Next, we will show the existence of a threshold value of ϵ so that, when ϵ increases to cross through this critical value, the speed selection changes from linear to nonlinear. To this end, we want to prove the following lemma first.

Lemma 2.4.1. *Let (2.4.2)-(2.4.3) be prescribed by Neumann boundary conditions (or Dirichlet boundary conditions). If the wave speed is linearly selected when $\epsilon=\epsilon_l$ for some $\epsilon_l>0$, then it is linearly selected for all $\epsilon<\epsilon_l$.*

Proof. By this assumption, when $\epsilon=\epsilon_l$, we have U_l as a solution, which is decreasing in $\xi\in\mathbb{R}$, with $c=c_0+\epsilon_1$ to (2.4.2) for any small $\epsilon_1>0$. Thus, it satisfies

$$(U_l)_{\xi\xi}+(U_l)_{yy}+(\alpha(y)+c)(U_l)_\xi+U_l(1-U_l)(1+2\epsilon_lU_l)=0. \tag{2.4.8}$$

Then, by substituting $U_l(\xi, y)$ into (2.4.2) with $\epsilon < \epsilon_l$, we obtain

$$\begin{aligned} & (U_l)_{\xi\xi} + (U_l)_{yy} + (\alpha(y) + c)(U_l)_\xi + U_l(1 - U_l)(1 + 2\epsilon U_l) \\ = \;& (U_l)_{\xi\xi} + (U_l)_{yy} + (\alpha(y) + c)(U_l)_\xi + U_l(1 - U_l)(1 + 2\epsilon_l U_l - 2\epsilon_l U_l + 2\epsilon U_l) \\ = \;& -2U_l^2(1 - U_l)(\epsilon_l - \epsilon) \\ \leqslant \;& 0. \end{aligned} \tag{2.4.9}$$

This implies that U_l can be viewed as an upper solution to (2.4.2) for $\epsilon < \epsilon_l$. Then taking the lower solution defined in Lemma 2.3.3, we conclude that the wave speed is linearly selected for $\epsilon < \epsilon_l$. This completes the proof. □

From the above lemma, we can define the threshold value of ϵ as

$$\epsilon_c := \sup\{\epsilon | \text{ the linear speed selection of (2.4.2)-(2.4.3) is realized}\}. \tag{2.4.10}$$

Remark 2.4.2. *By the above definition, we have* $0 \leqslant \epsilon_c \leqslant \infty$*. Furthermore, if* $\epsilon_c = 0$*, then the interval* $0 < \epsilon \leqslant \epsilon_c$ *is empty, thus the nonlinear speed selection is realized for all* $\epsilon > 0$*; if* $\epsilon_c = \infty$*, then the linear speed selection is realized for all* $\epsilon \geqslant 0$*.*

Depending on the choice of boundary conditions, the critical value ϵ_c may differ. We start with the case where (2.4.2)-(2.4.3) is prescribed by Neumann boundary conditions, i.e., $U_y(\xi, -L_y) = U_y(\xi, L_y) = 0$. In this case, $\beta(y) \equiv 1$ and we have the following theorem about the value of ϵ_c.

Theorem 2.4.3. *If the system (2.4.2)-(2.4.3) is prescribed by the Neumann boundary condition, then*

$$\lambda_0^2(c_0) \leqslant \epsilon_c \leqslant \lambda_0^2(c_0) + \max_{y\in[-L,L]} \left(\frac{\varphi_0'}{\varphi_0}\right)^2,$$

where λ_0 *and* φ_0 *are defined in Theorem 2.2.2.*

Proof. For the Neumann boundary case, we have $\beta \equiv 1$; thus, (2.4.6) reduces to $\epsilon < \lambda^2(c_0)$ due to the fact that $\min(\varphi_0')^2 = 0$ (at the boundary). It leaves us to prove the linear selection in the case when $\epsilon = \lambda_0^2(c_0)$. To this end, we choose a sequence

$\epsilon_n \to \lambda_0^2(c_0)$. By Theorem 2.3.4, it follows that (2.4.2)-(2.4.3) has a monotone traveling wave solution when $c = c_0$ for any $\epsilon = \epsilon_n$. Due to the compactness of the solution map, a limiting argument gives the existence of traveling waves when $\epsilon = \lambda_0^2(c_0)$ for all $c \geq c_0$. In other words, when $\epsilon = \lambda_0^2(c_0)$, the minimal speed of (2.4.2)-(2.4.3) is linearly selected.

To obtain an upper bound of the critical value ϵ_c, we will concentrate on the nonlinear selection. From (2.4.7) and Theorem 2.3.7, it follows that the nonlinear selection is realized when $\epsilon > \lambda_0^2 + \max_{y\in\Omega}\left(\frac{\varphi_0'}{\varphi_0}\right)^2$. Consequently, combining those results, this theorem holds. □

Remark 2.4.4. *From Theorem 2.4.3, for the Nuemann boundary case with $\alpha(y) = 0$, we obtain that $\epsilon_c = 1$ since $\varphi_0 = 1$ and $\lambda_0 = 1$ under such a condition. This recovers the result of [29],*

For the case where (2.4.2)-(2.4.3) is prescribed by Dirichlet boundary conditions, i.e.,

$$U(\xi, -L_y) = U(\xi, L_y) = 0,$$

we have $0 \leqslant \beta(y) \leqslant 1$ for $y \in [-L_y, L_y]$. From (2.4.6), it immediately follows that (2.4.2) is linearly selected when $\epsilon = \lambda_0^2(c_0) < \min\limits_{y\in\Omega} \frac{\lambda_0^2(c_0)}{\beta^2}$. Furthermore, it is easy to see that $\min\limits_{y\in\Omega} \frac{\lambda_0^2(c_0)}{\beta^2} = \frac{\lambda_0^2(c_0)}{\max\limits_{y\in\Omega}(\beta^2)}$. Similarly to Theorem 2.4.3, we arrive at the following result for the linear selection.

Theorem 2.4.5. *Let the system (2.4.2)-(2.4.3) be prescribed by Dirichlet boundary conditions. Then the linear selection is realized for all $\epsilon \leqslant \bar{\epsilon}$, where $\bar{\epsilon} = \frac{\lambda_0^2}{\max\limits_{y\in\Omega}(\beta^2)}$.*

Let us now perform some numerical simulations on (2.4.2)-(2.4.3) using the Matlab software. To make our numeric method look more convincing, we first compare the numerical results with the accurate solution obtained in [29]. The authors have found that the formula of the minimal wave speed is

$$c_{\min} = \begin{cases} 2, \ \epsilon \leqslant 1, \\ \sqrt{\epsilon} + \sqrt{\frac{1}{\epsilon}}, \ \epsilon \geqslant 1, \end{cases} \tag{2.4.11}$$

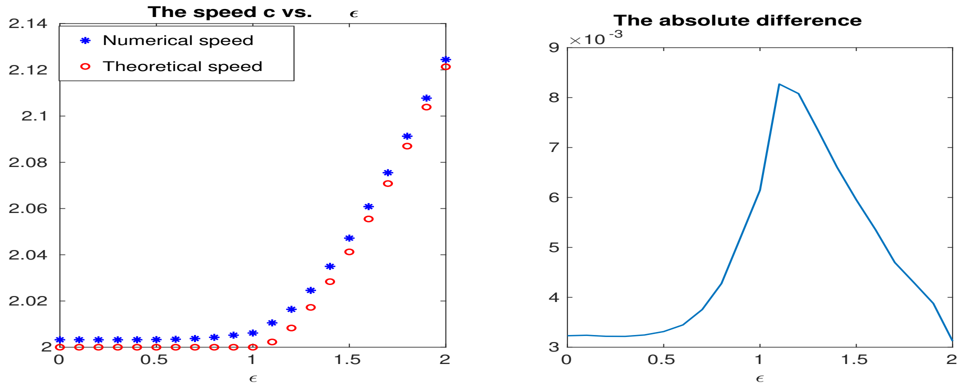


Figure 2.1: The speed comparison of numerical results and theoretical results. The figures show the speed for $\epsilon \in [0, 2]$.

and the traveling wave solution is a so-called Huxley's solution

$$u(x,t) = \frac{1}{1 + e^{\sqrt{\epsilon}(x-ct)}}, \text{ with } c = \sqrt{\epsilon} + \sqrt{\frac{1}{\epsilon}}.$$

The comparison results are summarized in Fig. 2.1. The figures show results related to the minimal wave speed. The left figure tells us that our numerically computed speeds matches the speeds predicted by the accurate formula (2.4.11); the right one shows the absolute difference between them, which are as small as $O(10^{-3})$. Thus, our numeric methods are reliable and will be explained in details in the following context.

Throughout simulations in the rest of this section, we fix $\alpha(y) = \sin(y)$ if not specified otherwise, and $L_y = 5\pi$. The simulations are also taken into two cases: one is the Neumann boundary condition case and the other one is the Dirichlet boundary condition case.

(1) When (2.4.2)-(2.4.3) is prescribed by the Neumann boundary condition, we do the following numerical computations. Through applying the central difference method to the eigenvalue problem (2.2.4), we determine that $c_0 = 2.58$ and $\lambda_0 = 0.93$. As we can see in Figure 2.2, the large one manifests the relation between μ and λ, which verifies the convexity of $\mu(\lambda)$ with respect to λ; the small one is an enlarged figure when $\lambda \in [0.6, 1.2]$, which implies $c_0 = 2.58$.

Furthermore, to obtain a traveling wave solution, we do numerical simulations on (2.4.1). By applying the central difference method on space variables, the 4th-order

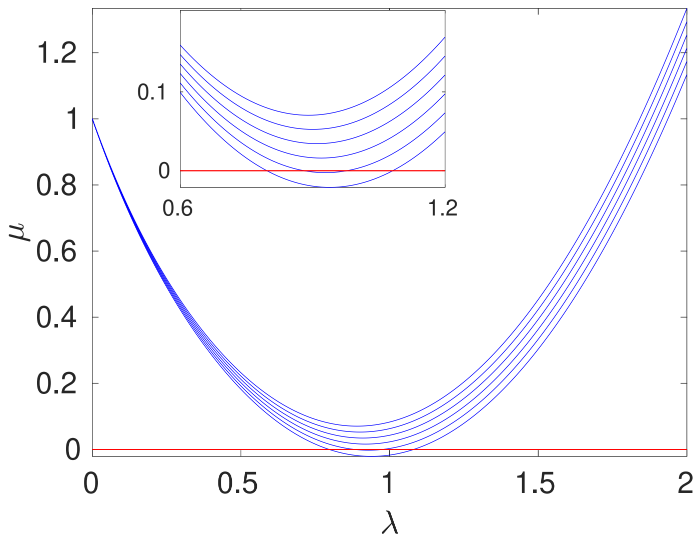


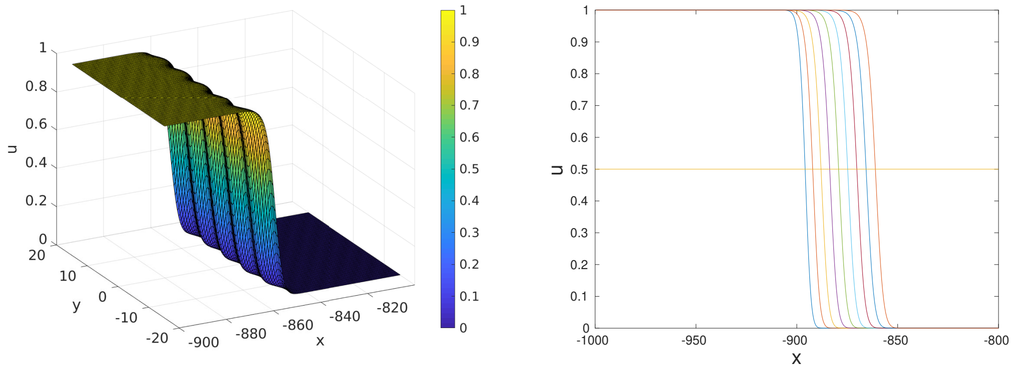
Figure 2.2: The relation between the principal eigenvalue $\mu(\lambda)$ and λ. From top to bottom, $c = 2.5,\ 2.52,\ 2.54,\ 2.56,\ 2.58$ and 2.6 respectively.

Runge-Kutta method on the time variable, and choosing an initial condition as

$$u_0(x,y) = \frac{1}{1+e^{10^5(x+x_0)}},\ (x,y) \in \mathbb{R} \times \overline{\Omega},\ x_0 = 900, \tag{2.4.12}$$

we will obtain a solution that stabilizes to a traveling wave solution. We conjecture without proof that the wave takes the minimal speed due to the fast decaying initial function. To have a stable wave profile, we start to store all the data after $t = 200$. As shown in Figure 2.3, the left panel is a 3-D figure that displays the shape of the solution; the right panel is obtained through fixing $y = 0$ and letting $t = 210, 211, \cdots, 220$. Actually, in Figure 2.3 (b), by letting $u(t,x,0) \equiv 0.5$, we can find the level set $x(t)$ for every t through linear interpolation, and use it to compute the spreading wave speed. Through this method, we calculate the minimal wave speed c whose result is shown in Figure 2.4. As we can see in this figure, the numerically computed speed $c_{\text{num}} \simeq c_0$ when $\epsilon \leqslant \lambda_0^2 = 0.865$. By substituting the value of c_0 and λ_0 into the eigenvalue problem (2.2.4), we can numerically solve φ_0 and by which we find $\max\limits_{y\in[-L,L]} \left(\frac{\varphi_0'}{\varphi_0}\right)^2 = 0.2205$. Therefore, by Theorem 2.4.3, the system is nonlinearly selected if $\epsilon > 1.091$, which has been verified by the figure. Actually, from the numerical simulation, we find that $\epsilon_c \simeq 1$.

(2) When (2.4.2)-(2.4.3) is prescribed by the Dirichlet boundary condition, we do



(a) (b)

Figure 2.3: Figure (a) depicts the solution of (2.4.1) with the Neumann boundary condition when $t = 220$. Figure (b) depicts the solution when $y = 0$, $t = 210, 211, \cdots, 220$. The parameter set corresponds to: $(x, y) \in [-1000, 1000] \times [-5\pi, 5\pi]$ and $x_0 = 900$.

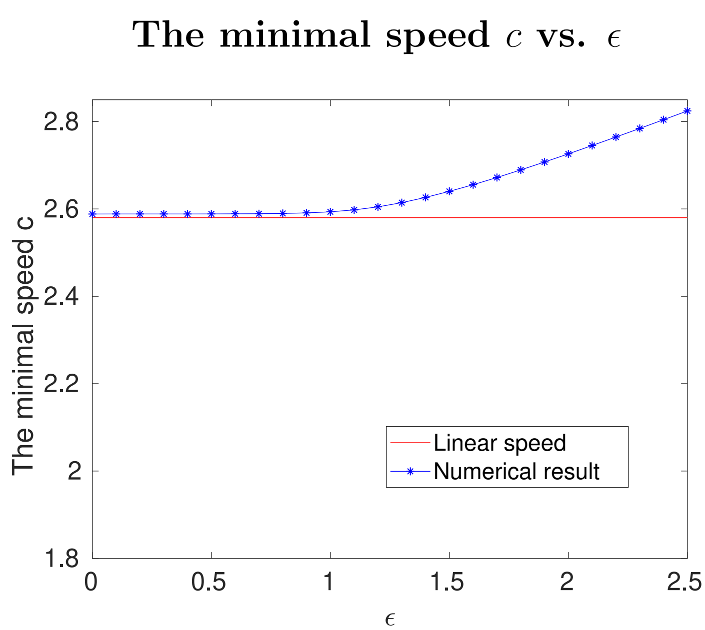


Figure 2.4: The relation between the asymptotic spreading speed c and ϵ. The blue line with stars denotes the numerically computed speed obtained by direct simulation, and the red line is $c_0 = 2.58$.

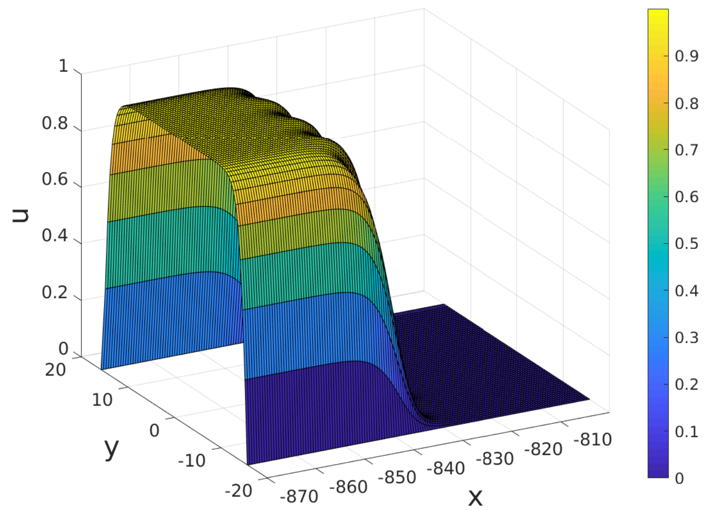


Figure 2.5: The solution of (2.4.1) with the Dirichlet boundary condition when $t = 220$.

similar simulations. The same method applied to the eigenvalue problem (2.2.4) with Dirichlet boundary conditions, we obtain $c_0 = 2.36$ and $\lambda_0 = 0.885$. Next, to obtain a traveling wave solution here, we choose the initial data as

$$u_0(x, y) = \frac{\cos(\pi y/2L_y)}{1 + e^{10^5(x+x_0)}}, \ (x, y) \in \mathbb{R} \times \overline{\Omega}, \ x_0 = 900. \tag{2.4.13}$$

Due to the zero boundary condition, the shape of a traveling wave solution in this case looks like an arch, which is quite different from the former one and is shown in Figure 2.5. Finally, using the same method as the one used in the previous case, we calculate the wave speed corresponding to different values of the parameter ϵ. The results are shown in Figure 2.6. As shown in the figure, there is a critical number ϵ_c such that the speed is linearly selected when $\epsilon \leqslant \epsilon_c$, and nonlinearly selected when $\epsilon > \epsilon_c$. Here, by the numerical simulation, we can see $\epsilon_c \simeq 0.8 > \lambda_0^2 = 0.783$.

To complete the numerical simulations for the cubic nonlinearity, we provide some discussions of the effect of $\alpha(y)$ on the critical number ϵ_c when the Neumann boundary condition occurs. When $\alpha(y) \equiv \alpha$ with α being a constant, through a direct computation, we find that the eigenfunction of (2.2.2) can be always normalized to

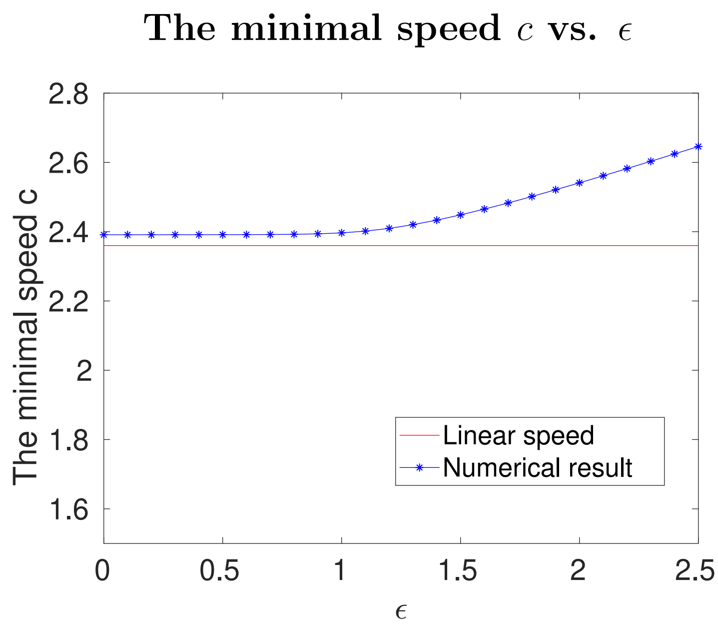


Figure 2.6: The relation between the minimal speed c and ϵ. The blue line with stars denotes the numerically computed speed obtained by direct numerical simulations and the red line is $c_0 = 2.36$.

be "$\varphi_0 = 1$" and the eigenvalue

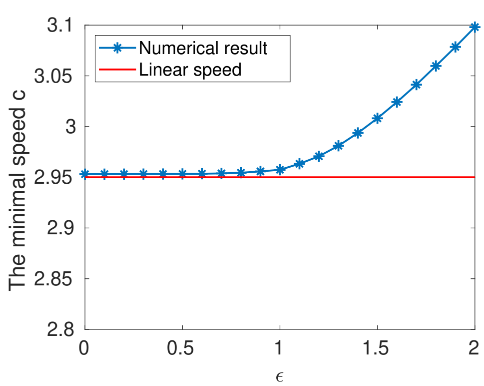
$$\lambda_0 = \frac{\alpha + c_0}{2} \equiv 1 \text{ where } c_0 = 2 - \alpha.$$

By Theorem 2.4.3, $\epsilon_c \equiv 1$ for all $\alpha \in \mathbb{R}$. In other words, α only affects the value of the linear speed c_0 but it does not affect the critical value ϵ_c.

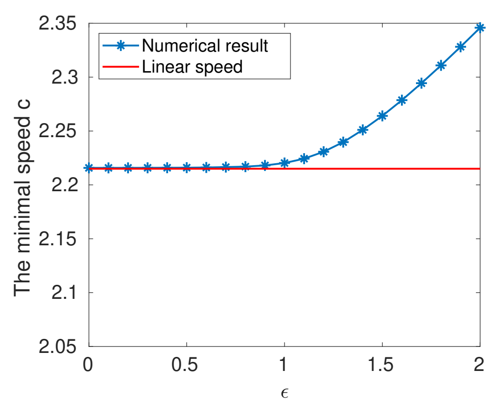
When $\alpha(y)$ is not a constant, with the help of numerical simulations, we also find that ϵ_c always equals to 1. We first give a table to manifest the influence of α on c_0, λ_0, and the range of ϵ_c by Theorem 2.4.3. As Table 2.1 shows, when $\alpha_2(y) \leqslant \alpha_1(y)$ for all $y \in [-L, L]$, $\lambda_{0,2} \geqslant \lambda_{0,1}$ while $\max\limits_{y\in[-L,L]} \left(\frac{\varphi'_{0,2}}{\varphi_{0,2}}\right)^2 \leqslant \max\limits_{y\in[-L,L]} \left(\frac{\varphi'_{0,1}}{\varphi_{0,1}}\right)^2$, where $\lambda_{0,i}$ $(i = 1, 2)$ denotes λ_0 corresponding to $\alpha_i(y)$ $(i = 1, 2)$ and the same notations are used for $\varphi_{0,i}$. The last column of Table 2.1 shows the range of ϵ_c. It is clear that all of them contain the value 1. Furthermore, we apply the same numerical method used for $\alpha(y) = \sin(y)$ to other two cases: (a) $\alpha(y) = 1.5\sin(y)$ and (b) $\alpha(y) = 0.5\sin(y)$. The details are shown in Figure 2.7. From those figures, we can see that $\epsilon_c = 1$ for both cases. It can be interesting to prove this result rigorously.

$\alpha(y)$	c_0	λ_0	$\max\limits_{y\in[-L,L]} \left(\frac{\varphi_0'}{\varphi_0}\right)^2$	the range of ϵ_c
$1.5\sin(y)$	2.95	0.914	0.3362	[0.8354,1.1713]
$1.25\sin(y)$	2.7	0.92	0.2747	[0.8464,1.1211]
$\sin(y)$	2.58	0.93	0.2195	[0.8649,1.0844]
$0.75\sin(y)$	2.4	0.938	0.1683	[0.8798,1.0481]
$0.5\sin(y)$	2.23	0.951	0.1191	[0.9044,1.0235]
$0.2\sin(y)$	2.08	0.974	0.0645	[0.9478,1.0132]
0	2	1	0	1

Table 2.1: The influence of $\alpha(y)$ on the range of ϵ_c.



(a)



(b)

Figure 2.7: The numerical speed c corresponding to different ϵ. Figure (a) is depicted when $\alpha(y) = 1.5\sin(y)$ while (b) is depicted when $\alpha(y) = 0.5\sin(y)$.

2.4.2 Subcritical quintic Ginzburg-Landau Equation

In our second application, we consider a subcritical quintic Ginzburg-Landau equation in a cylindrical domain. The equation is given by

$$u_t = u_{xx} + u_{yy} + \alpha(y)u_x + \mu u + u^3 - u^5,\ (x, y) \in \mathbb{R} \times \Omega,\ \mu > 0. \tag{2.4.14}$$

Here $f(u) = \mu u + u^3 - u^5$ and $\Omega = (-L_y, L_y)$. Thus, for traveling wave solutions, we mean $u(t, x, y) = U(\xi, y)$ where $\xi = x - ct$. Then, the equation for the wave profile is

$$U_{\xi\xi} + U_{yy} + (\alpha(y) + c)\, U_\xi + \mu U + U^3 - U^5 = 0, \tag{2.4.15}$$

satisfying

$$\lim_{\xi\to+\infty} U(\xi, y) = 0,\quad \lim_{\xi\to-\infty} U(\xi, y) = \beta(y) \leqslant \mu_+,\ y \in \Omega, \tag{2.4.16}$$

where

$$\mu_+ = \sqrt{\frac{1+\sqrt{1+4\mu}}{2}} > 1.$$

It is easy to have

$$f(0) = f(\mu_+) = 0,\ f'(0) = \mu > 0 > f'(\mu_+) = -2\mu_+^2\sqrt{1+4\mu_+}.$$

Clearly, f satisfies (A1) and (A4). Notice that $f'(0)$ depends on the parameter μ. Thus, we may require some extra conditions on μ for f to satisfy (A5) when (2.4.15) is prescribed by the Dirichlet boundary condition.

Since $f''(u) = 6u - 20u^3$ and $f'''(u) = 6 - 60u^2$, $u = 0$ is a solution of $f''(u) = 0$ with multiplicity $k = 1$. Following Remark 2.3.2, we will choose $\gamma = 2$ in (2.3.2). By substituting the formula of f into Equation (2.3.5) and simplifying it, we then obtain

$$\mathcal{L}(U_1) = \frac{U_1^3}{\beta^2}\left(1 - \frac{U_1^2}{\beta^2}\right)\left\{-3\lambda_1^2(c) - 3\frac{(\beta'\varphi_1 - \beta\varphi_1')^2}{\varphi_1^2\beta^2} + \beta^4\right\}, \tag{2.4.17}$$

and now $G_1 = \beta^4$. With the condition $0 \leqslant \beta(y) \leqslant \mu_+$ for $y \in \Omega$, we further have

$$\max_{(\xi,y)\in\mathbb{R}\times\Omega} G_1(\xi, y) \leqslant \mu_+^4 = \frac{1}{2} + \mu + \frac{1}{2}\sqrt{1+4\mu}. \tag{2.4.18}$$

Thus, the condition (2.3.7) for the linear selection becomes

$$\frac{1}{2}+\mu+\frac{1}{2}\sqrt{1+4\mu}<3\lambda_0^2(c_0). \tag{2.4.19}$$

We then have the following theorem.

Theorem 2.4.6. *When (2.4.15)-(2.4.16) is prescribed by Neumann (or Dirichlet) boundary conditions, the minimal wave speed is linearly selected if the inequality (2.4.19) holds.*

As for the nonlinear selection, we give a condition for the Neumann boundary condition case as follows. Substituting the formula of f into (2.3.20) gives $G_2=\beta^4=\mu_+^4=\frac{1}{2}+\mu+\frac{1}{2}\sqrt{1+4\mu}$. Then, we arrive at the following theorem.

Theorem 2.4.7. *When (2.4.15)-(2.4.16) is prescribed by Neumann boundary conditions, the minimal wave speed is nonlinearly selected if*

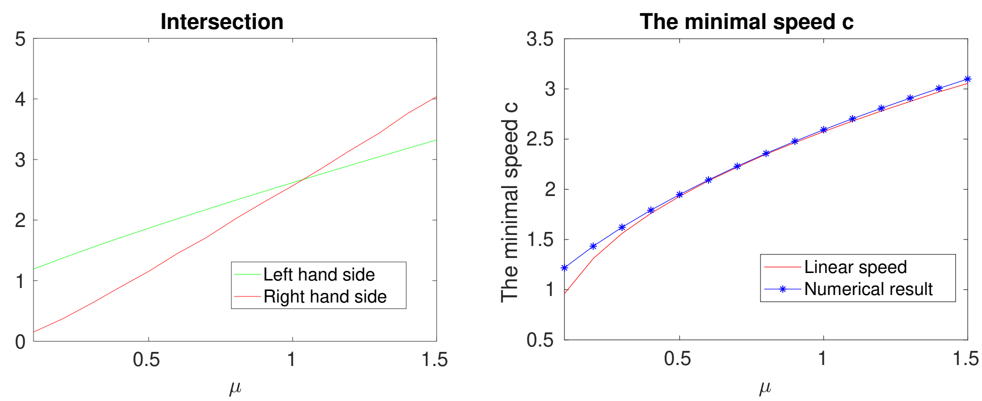
$$\frac{1}{2}+\mu+\frac{1}{2}\sqrt{1+4\mu}>3\lambda_0^2(c_0)+3\left(\frac{\varphi_0'}{\varphi_0}\right)^2, \tag{2.4.20}$$

where $\lambda_0(c_0)$ and $\varphi_0(c_0)$ are defined in Theorem 2.2.2.

Remark 2.4.8. *Actually, if the Neumann boundary condition case occurs with $\alpha(y)=0$, (2.4.19) and (2.4.20) imply that there is a critical value $\mu_c=0.75$ such that the minimal wave speed of (2.4.15) is linearly selected if $\mu\geqslant\mu_c$ and nonlinearly selected if $\mu<\mu_c$. This means, our results include the one in [47]. When $\alpha(y)\neq 0$, there is a gap between conditions (2.4.19) and (2.4.20), we conjecture that there exists a critical number μ_c and its exact value can be found by numerical simulations.*

Next, we perform numerical simulations on (2.4.15)-(2.4.16). Here, we also fix $\alpha(y)=\sin(y)$ and $L_y=5\pi$. Similarly, we apply the same method as that in the previous application and carry out simulations in two cases.

(1) We first do simulations for the Neumann boundary condition case. By direct calculations on (2.4.19), we obtain the left panel of Figure 2.8. In the figure, we use the green line to represent the left hand side of (2.4.19), i.e., $\frac{1}{2}+\mu+\frac{1}{2}\sqrt{1+4\mu}$, and



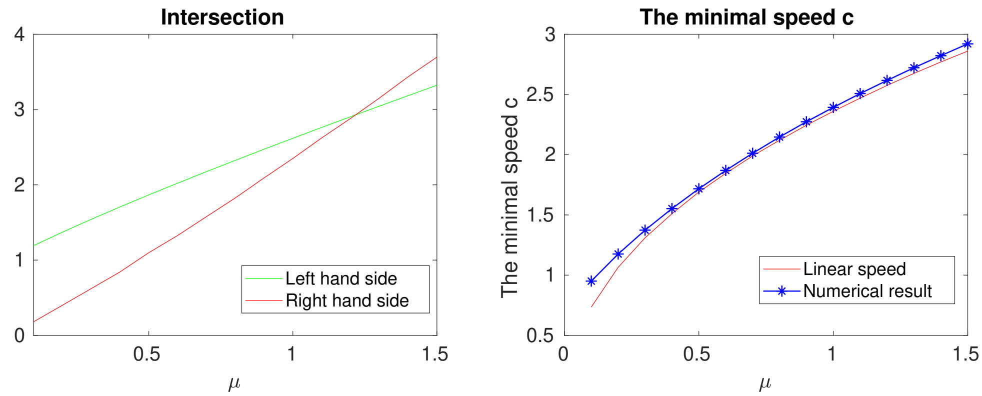
(a)

(b)

Figure 2.8: The numerical speed c corresponding to μ.

the red line to denote the right hand side, that is, $3\lambda_0^2$. Clearly, there is an intersection $\mu_c \simeq 1.1$ shown in Figure 2.8(b). Following Theorems 2.4.6 and 2.4.7, we expect that the system (2.4.14) is linearly selected when $\mu > \mu_c$ and nonlinearly selected when $\mu \leqslant \mu_c$. In the right panel of Figure 2.8, we illustrate the relation between c_{num} and c_0. By choosing the same initial condition given in Equation (2.4.12), we obtain the traveling wave solution for (2.4.14). The shape of this solution is similar to the one shown in Figure 2.3, so we will not repeat showing it here. To obtain a stable traveling wave solution, we record all the speed data after 200 seconds. On the other hand, c_0 is from (2.2.4) and its value differs as μ varies. Then, we use the blue line with stars to denote c_{num} and the red line to denote c_0. As we can see, the system is nonlinearly selected when $\mu \leqslant 1$ and linearly selected when $\mu > 1$. Thus, with the help of numerical simulations, we indeed have verified the theoretical results.

(2) In the Dirichlet boundary condition case, we carry out similar procedures. In the left panel of Figure 2.9, we find an intersection $\mu_d \simeq 1.25$ from (2.4.19). Following Theorem 2.4.6, we expect that the system (2.4.14) is nonlinearly selected when $\mu \leqslant \mu_d$. To verify this, we choose the same initial condition defined in (2.4.13) to obtain the traveling wave solution for (2.4.14). Again, we store all the data after 200 seconds and use the red line to denote c_0 while the blue line with stars to denote c_{num}. As we can see in the right panel of Figure 2.9, in our depicted region $\mu \in [0.1, 1.5]$, the blue line is always above the red one, which means the system is nonlinearly selected for all $\mu \in [0.1, 1.5]$. Thus, we have verified that the system is indeed nonlinearly selected when $\mu \leqslant \mu_d$.



(a)

(b)

Figure 2.9: In the left panel, the green line denotes the left hand side of (2.4.19), i.e., $\frac{1}{2} + \mu + \frac{1}{2}\sqrt{1+4\mu}$, and the red line denotes the right hand side, i.e., $3\lambda_0^2$. The right panel depicts the relation between the parameter μ and c_0 (red line) or c_{num} (blue line with stars).

2.5 Conclusion and discussion

In summary, by the upper and lower solutions method, we have obtained a speed selection mechanism (including linear and nonlinear) for traveling wave solutions of a reaction-diffusion-advection equation in a cylindrical domain. Precisely, we found conditions on the linear selection when the model is prescribed by Neumann (or Dirichlet) boundary conditions, see the inequality (2.3.7) and Theorem 2.3.4. We also give results on the nonlinear selection when the model is prescribed by Neumann boundary conditions, see the inequality (2.3.20) and Theorem 2.3.7. To see the speed selection mechanism more specifically, we gave two applications in Section 4. In each application, we obtained the corresponding simplified conditions for the speed selection mechanism and then verified them by direct numerical simulations.

We should emphasize that, because of our newly constructed upper and lower solutions, our results make a significant progress in the study of the speed selection in higher dimension models such as (2.1.1). These constructed solutions are more accurate for approaching the true traveling wave solutions. With this method, we extend the previous results in the Neumann boundary condition case, and even give a sufficient condition on the linear selection in the Dirichlet boundary condition case, which was thought to be very difficult to study.

There are many interesting but open problems related to the topic of speed selections. One open problem arising in this chapter is how to find a suitable lower solution to analyze the nonlinear selection in the Dirichlet boundary condition case. Furthermore, concerning the problem of wave speeds, it is interesting and challenging to find an estimation of $c_{\min}$ or even give the exact formula when the nonlinear selection is realized.

2.6 Appendix

The upper and lower solution method has proved to be a very powerful tool to investigate the existence of monotone traveling wave solutions (see e.g. [96]). This method was first introduced by [16] and Weinberger [85], independently, and has been extended by many academics, such as in [55, 96]. The main idea is as follows. By transforming the wave profile equation (2.1.3) or its original partial differential equation (2.1.1) into an integral one, we can define a monotone solution map. Then with the definition of the solution map, we can construct a pair of upper and lower solutions of (2.1.1) to set up an iteration scheme. Through the scheme, we then obtain the existence of traveling wave solutions of (2.1.1).

To proceed, we present the phase space used in our model. Let $\mathcal{C}$ $(\tilde{\mathcal{C}})$ be the set of all bounded continuous functions from $\mathbb{R} \times \Omega$ to $\mathbb{R}$ (or $\tilde{\mathcal{C}} = C(\mathbb{R}, X)$, where $X = C_0(\Omega)$), and $\mathcal{C}_\beta := \{\varphi \in \mathcal{C} : 0 \leqslant \varphi \leqslant \beta\}$ $(\tilde{\mathcal{C}}_\beta := \{\varphi \in \tilde{\mathcal{C}} : 0 \leqslant \varphi \leqslant \beta\})$. Here, $\mathcal{C}$ is used for the Neumann boundary condition case, while $\tilde{\mathcal{C}}$ is used for the Dirichlet boundary condition case. Since the process in each case is similar, we then only take the Neumann boundary condition case to show the scheme.

To obtain a monotone solution map, we let M_1 be a sufficiently large positive number such that $F_1(u) := f(u) + M_1 u$ is monotone in u. Thus, Equation (2.1.1) is equivalent to the following one:

$$u_t = u_{xx} + \Delta_y u + \alpha(y)u_x - M_1 u + F_1(u). \tag{2.6.1}$$

Next, we want to transform it into an integral form. To this end, we first investigate the corresponding homogeneous equation, that is,

$$u_t = u_{xx} + \Delta_y u + \alpha(y)u_x - M_1 u. \tag{2.6.2}$$

Let $\Gamma(t,x,y)$ (or $\tilde{\Gamma}(t,x,y)$) be the Green's function of (2.6.2) prescribed by the Neumann (or Dirichlet) boundary conditions (see, e.g., [27]). Then the solution of (2.6.2) with the initial value $u(0,\cdot)=\varphi(\cdot)$ can be expressed as

$$u(t,x,y)=\Gamma(t,x-x_0,y-y_0)*\varphi(x_0,y_0).$$

By the comparison principle (see, e.g., [49]), the above Green's function is monotone in u, that is, $\Gamma*u_1\geqslant\Gamma*u_2$ when $u_1\geqslant u_2$ for $(x,y)\in\mathbb{R}\times\overline{\Omega}$. Now, by variation of parameters, the equation (2.6.1) can be written in an integral form as

$$u(t,x,y)=\Gamma(t,x-x_0,y-y_0)*\varphi(x_0,y_0)+\int_0^t\Gamma(t-t_0,x-x_0,y-y_0)*F_1(u(t_0,x_0,y_0))dt_0, \tag{2.6.3}$$

where the initial data $\varphi\in\mathcal{C}_\beta$ and $*$ denotes the convolution as

$$\Gamma(t,x-x_0,y-y_0)*\varphi(x_0,y_0)=\int_{\mathbb{R}}\int_{\Omega}\Gamma(t,x-x_0,y-y_0)\cdot u_0(x_0,y_0)dy_0dx_0.$$

We define

$$Q_t[\varphi]=u(t,\cdot,\varphi).$$

It then follows that $\{Q_t\}_{t=0}^{\infty}$ is a semiflow on $\mathcal{C}_\beta$ with $Q_t(0)=0$ and $Q_t(\beta)=\beta$. Then, by a traveling wave solution of the map Q_t for each $t\geqslant 0$, we mean a special solution $U(x,y)$ satisfying

$$Q_t[U](x,y)=U(x-ct,y)$$

for some constant c, and $U(x,y)$ connecting β to 0 if $U(-\infty,y)=\beta(y)$ and $U(+\infty,y)=0$. Notice that, in the literature of Q_t, the minimal wave speed defined in the Introduction means that Q_t has a non-increasing traveling wave connecting β to 0 if and only if $c\geqslant c_{\min}$. Furthermore, for any $t\geqslant 0$, the solution map Q_t has the following properties:

(1) Q_t is monotone in the sense that $Q_t[U_1]\geqslant Q_t[U_2]$ whenever $U_1\geqslant U_2$ for $(x,y)\in\mathbb{R}\times\overline{\Omega}$;

(2) If $U\in\mathcal{C}_\beta$ is decreasing with respect to $\xi\in\mathbb{R}$, so is $Q_t[U]$;

(3) $Q_t[\mathcal{C}_\beta]$ is precompact in $\mathcal{C}_\beta$ (see, e.g., [43] for the Neumann boundary conditions and [44] for the Dirichlet boundary conditions).

Then, corresponding to the solution map Q_t, we introduce the definition of an upper (or a lower) solution. Given $x_0 \in \mathbb{R}$, we define the translation operator T_{x_0} by $T_{x_0}[U](x, y) = U(x - x_0, y)$.

Definition 2.6.1. *For any given c, a continuous function $u(x, y)$ is called an upper solution to the integral equation (2.6.3) if*

$$T_{-ct}\left[Q_t[u]\right](x, y) \leqslant u(x, y), \ \forall (x, y) \in \mathbb{R} \times \Omega.$$

A lower solution of (2.6.3) is defined by reversing the inequality.

In the following lemma, we give the inequality in Definition 2.6.1 in terms of the differential equation for the wave profile, since these differential form inequalities are straightforward in our analysis.

Lemma 2.6.2. *A continuous function $U(\xi, y) = T_{ct}[U](x, y)$, where $\xi = x - ct$, is twice continuously differentiable on $\mathbb{R} \times \Omega$ except finite many points ξ_i with*

$$U_\xi(\xi_i^+, y) \leqslant U_\xi(\xi_i^-, y), \ i = 1, 2, \cdots, m, \tag{2.6.4}$$

and

$$U_{\xi\xi} + \Delta_y U + [\alpha(y) + c]U_\xi + f(U) \leqslant 0, \ \forall (\xi, y) \in \mathbb{R} \setminus \{\xi_i\} \times \overline{\Omega}, \ i = 1, 2, \cdots, m. \tag{2.6.5}$$

Then, it is an upper solution of (2.6.3). A lower solution is obtained by reversing the afore-mentioned inequalities.

Proof. Suppose there is a solution $\overline{U}(\xi, y)$ satisfies (2.6.5). We denote

$$u(t, x, y) = \overline{U}(x - ct, y).$$

Substituting it into (2.6.1) gives $u_t = -c\overline{U}_\xi$, $u_{xx} = \overline{U}_{\xi\xi}$ and $\Delta_y u = \Delta_y \overline{U}$. Then, (2.6.5) implies

$$\begin{cases} u_t \geqslant u_{xx} + \Delta_y u + \alpha(y) u_x + f(u), \\ u(0, x, y) = \overline{U}(x, y). \end{cases} \tag{2.6.6}$$

Since $Q_t[\overline{U}](x,y)$ is the solution of (2.6.1) with an initial data as $\overline{U}(x,y)$. By the comparison principle (see, e.g., [49]), we then obtain $u(t,x,y) \geqslant Q_t[\overline{U}](x,y)$ for all $t \geqslant 0$. That is, $\overline{U}(x-ct,y) = T_{ct}[\overline{U}](x,y) \geqslant Q_t[\overline{U}](x,y)$ for all $t \geqslant 0$. Thus, $\overline{U}(x,y) \geqslant T_{-ct}[Q_t[\overline{U}]](x,y)$, which exactly meets the requirement for an upper solution in Definition 2.6.1. A similar proof can be applied to the lower solution of (2.6.3) if we reverse (2.6.4) and (2.6.5). This completes the proof. □

The existence of an upper and a lower solution to the system (2.6.3) will give the existence of an actual traveling wave solution. Indeed, for our problem, we assume the following hypothesis.

Hypothesis 2.6.3. *For $c > c_0$, there exists a monotone non-increasing upper solution $\overline{U}(x,y)$ with respect to x and a non-zero lower solution $\underline{U}(x,y)$ to the system (2.6.1) with the following properties:*
(1) $\underline{U}(x,y) \leqslant \overline{U}(x,y)$, for all $(x,y) \in \mathbb{R} \times \Omega$;
(2) $\overline{U}(-\infty,y) = \beta(y)$, $\overline{U}(+\infty,y) = 0$, for all $y \in \Omega$;
(3) $\underline{U}(-\infty,y) = \beta^*(y)$, $\underline{U}(+\infty,y) = 0$, where $0 \leqslant \beta^* \leqslant \beta$ for all $y \in \Omega$.

When the above hypothesis holds true, we can define an iteration scheme as

$$U_0(x,y) = \overline{U}(x,y),\ U_{n+1}(x,y) = T_{-ct}[Q_t[U_n]](x,y),\ n = 0,1,2,\cdots. \tag{2.6.7}$$

With the construction of upper and lower solutions and the iteration scheme, we then arrive at the following existence theorem for a traveling wave solution (see, e.g., [16, 43, 55] for the Neumann boundary condition case, and [44] for the Dirichlet boundary condition case).

Theorem 2.6.4. *If the hypothesis 2.6.3 holds true and Q_t is defined in (2.6.3), then the iteration (2.6.7) converges to a function $U(x,y)$. This function is a solution to (2.1.3)-(2.1.4) with $U(x-ct,y) = Q_t[U](x,y)$. Furthermore, $U(x-ct,y) = U(\xi,y)$ with $\xi = x-ct$ is non-increasing in $\xi \in \mathbb{R}$ with $U(-\infty,y) = \beta(y)$ and $U(+\infty,y) = 0$ uniformly for $y \in \Omega$.*

Chapter 3

Speed Determinacy of Traveling Waves to a Stream-Population Model with Allee Effect

3.1 Introduction

The study of the biological population of species in streams, rivers, and estuaries have been attracting considerable attention recently (see, e.g., [50, 61, 67, 73]). As in these investigations of stream ecology, the so-called "drift paradox" is an interesting phenomenon, according to which the species at any fixed location will not become extinct, even though there exists a downstream drift that washes away the species. Perhaps the first reasonable explanation was the theory of the colonization cycle proposed by Müller [61, 62]. Afterward, different from Müller's idea, Speirs and Gurney [73] further formulated a constant-coefficient scalar partial differential equation to describe the situation. Their model demonstrated a simplified one-dimensional representation of a species residing in a stream, a river or an estuary subject to advection (stream drift flow) and diffusion (random movement), with

$$\frac{\partial u}{\partial t} = g(u)u - \alpha \frac{\partial u}{\partial x} + d \frac{\partial^2 u}{\partial x^2}. \tag{3.1.1}$$

Here, $u(x,t)$ is the density of the species, $g(u)$ is the per capita growth rate of the population, α is the advection speed (i.e., the speed of the flow), and d is the diffusion coefficient. They concentrated on the role of diffusion, variable river flow direction, and the swimming of organisms to the persistence of the species.

Later, Pachepsky, Lutscher, Nisbet, and Lewis [67] extended (3.1.1) to a coupled system, investigating the persistence of benthic aquatic organisms. They assumed the total population to be divided into two interacting compartments: individuals residing on the benthos (the bottom of the stream) and individuals drifting in the flow. Their non-dimensional system is given by

$$\begin{cases} \frac{\partial u}{\partial t} = -\sigma u + \mu v - \alpha \frac{\partial u}{\partial x} + d \frac{\partial^2 u}{\partial x^2}, \\ \frac{\partial v}{\partial t} = +\sigma u - \mu v + (1-v)v, \end{cases} \tag{3.1.2}$$

where the newly introduced coefficients μ is the per capita rate at which individuals in the benthic population enter the drift; σ is the per capita rate at which the species returns to the benthic population from drifting, e.g., the number of the species that settle down to the benthos to give birth or find food. This separation has significant implications for the population persistence (for full details, please see [50,67]). Except for the persistence or the critical domain size, for such a model, academics were also interested in the propagation speed. Since the system includes advection, it can distinguish the propagation speed with two cases: downstream (same direction of advection) and upstream (opposite direction of advection). Clearly the downstream propagation speed increases with the advection, whereas the upstream speed decreases. However, from the mathematical point of view, the analysis for an upstream-facing traveling wave solution will be similar to that of the downstream's; thus we will only consider a downstream-facing traveling wave solution that demonstrates a situation where a species invades an uninhabited downstream terrain. The main model in this chapter is extended from (3.1.2) with the reaction term possibly having the Allee effect and the residing individuals having a weak diffusive behavior:

$$\begin{cases} \frac{\partial u}{\partial t} = -\sigma u + \mu v - \alpha \frac{\partial u}{\partial x} + d \frac{\partial^2 u}{\partial x^2}, \\ \frac{\partial v}{\partial t} = +\sigma u - \mu v + f(v) + \epsilon \frac{\partial^2 v}{\partial x^2}, \end{cases} \tag{3.1.3}$$

where ϵ is a small nonnegative number due to the fact that the population living in the benthos barely move horizontally; the reaction term $f(v)$ is a smooth function (say, with second-order derivative) satisfying $f(0) = f(1) = 0$, $f'(0) > 0 > f'(1)$ and $f(v) > 0$ for $v \in (0,1)$; d, σ, α, μ are positive constants with similar biological meanings to those in the model (3.1.2). The spatially homogeneous solutions to (3.1.3) are $e_0 = (0,0)$ and $e_1 = (\frac{\mu}{\sigma}, 1)$. Moreover, one can easily find that e_0 is unstable and e_1 is stable for the corresponding spatially homogeneous system.

To investigate the propagation phenomena, we change the model with the wave moving coordinates so as to introduce the following wave profile

$$u(x,t) = U(\xi),\ v(x,t) = V(\xi),\ \xi = x - ct, \tag{3.1.4}$$

where $c \geqslant 0$ is the unknown wave speed. Now, for a downstream-facing wave, the system for the wave profile is

$$\begin{cases} -cU' = -\sigma U + \mu V - \alpha U' + dU'', \\ -cV' = +\sigma U - \mu V + f(V) + \epsilon V'', \end{cases} \tag{3.1.5}$$

subject to

$$(U,V)(-\infty) = (\frac{\mu}{\sigma}, 1),\ (U,V)(+\infty) = (0,0). \tag{3.1.6}$$

A typical example for $f(V)$ is $V(1-V)(1+\rho V)$ which has an Allee factor ρ (see [84]), compared to the conventional Logistic growth.

By Theorems 4.1 and 4.2 in [43], it is known that there exists a critical number $c_{\min}$ defined as

$$c_{\min} := \inf\{c|\ c \in \mathbb{R} \text{ such that (3.1.5)-(3.1.6) has a nonnegative solution}\},$$

so that the system (3.1.5)-(3.1.6) has a nonnegative solution if and only if $c \geqslant c_{\min}$. Biologically and significantly, this speed is also equal to the asymptotic spreading speed that indicates the velocity of biological invasion. Usually, the exact value of this speed is difficult to determine, even for the simple Fisher-KPP scalar model with the Allee effect. What we are able to do is to find the speed for the linearized system around zero and use it to estimate the spreading speed. For instance, for our model, by linearizing the system (3.1.5) near zero, we can obtain the linear speed c_0 whose details will be shown in the next section, and by which, it can be seen that $c_{\min} \geqslant c_0$, a fact that is believed to be true for all cooperative systems. Whether they are equal becomes a challenging problem, and this results in the following definition of linear or nonlinear determinacy, classifying the speed selection.

Definition 3.1.1. *If $c_{min} = c_0$, we say the minimal speed of the system (3.1.5)-(3.1.6) is linearly selected; otherwise, if $c_{min} > c_0$, we say the minimal speed is nonlinearly selected.*

Currently, there are a few references working on the speed determinacy to scalar reaction-diffusion equations or the diffusive Lotka-Volterra competition model (see [1, 2, 47, 87]). As we notice that the variation principle in [47] does not work here, we will investigate the speed selection of by the upper-lower solution technique to the wave profile system coupled with the comparison principle to the partial differential equations (3.1.3). Our construction of the upper or lower solution is different from the classical upper (or lower) solution of [16] that is an exponential function (a solution to the linear system) capped by the positive constant solution, and it usually gives the mechanism of the linear speed determinacy, with a further requirement that the nonlinear model is bounded by its linearized system. Our new upper solution comes directly from the solution of a nonlinear system. It can effectively approximate the real wavefront and thus provides better or superior conditions for the linear speed selection. Furthermore, by analyzing the nature of the *pushed* wavefront (wavefront with $c_{\min} > c_0$), we will construct a lower solution with a fast decay rate to establish the nonlinear selection mechanism. The spreading speed is shown to be an increasing function of the Allee factor. Numerical simulations are carried out to obtain the linear speed, and to indicate the linear and nonlinear speed determinacy.

The remaining part of this chapter is organized as follows. In Section 4.2, we will study the wave profile behavior locally near the equilibrium e_0. In Section 4.3, we will present our main results for the speed selection mechanism. In Section 4.4, we will apply our results to a cubic reaction term to obtain further results, by choosing subtle forms of upper and lower solutions. In the last section of this chapter, we append the idea of the upper and lower solution method.

3.2 Linearization at $e_0 = (0, 0)$

In this section, we focus on the local analysis near e_0, i.e., $(U, V) = (0, 0)$. To begin with, we linearize system (3.1.5) near e_0 to derive the following system

$$\begin{cases} -cU' = dU'' - \alpha U' - \sigma U + \mu V, \\ -cV' = \epsilon V'' + \sigma U - \mu V + f'(0)V. \end{cases} \tag{3.2.1}$$

This can be regarded as a fourth-order linear differential system with constant coefficients. Let $(U, V) = (A_1, A_2)e^{-\lambda\xi}$ with $\lambda > 0$ and A_1, A_2 being constants. We then obtain the following eigenvalue problem:

$$\begin{cases} c\lambda A_1 = d\lambda^2 A_1 + \alpha\lambda A_1 - \sigma A_1 + \mu A_2, \\ c\lambda A_2 = \epsilon\lambda^2 A_2 + \sigma A_1 - \mu A_2 + f'(0)A_2. \end{cases} \tag{3.2.2}$$

For simplicity of notations, we denote it in a matrix form:

$$c\lambda A = \begin{pmatrix} d\lambda^2 + \alpha\lambda - \sigma & \mu \\ \sigma & \epsilon\lambda^2 - \mu + f'(0) \end{pmatrix} A, \tag{3.2.3}$$

where $A = (A_1 \ A_2)^T$. To solve the above eigenvalue problem, we first consider the eigenvalue problem of the right-side operator :

$$B(\lambda)A = k(\lambda)A, \tag{3.2.4}$$

where $k(\lambda)$ denotes the principal eigenvalue and

$$B(\lambda) = \begin{pmatrix} B_1(\lambda) & \mu \\ \sigma & B_2(\lambda) \end{pmatrix}, \ \ B_1(\lambda) = d\lambda^2 + \alpha\lambda - \sigma, \ B_2(\lambda) = \epsilon\lambda^2 - \mu + f'(0). \tag{3.2.5}$$

Clearly, to obtain a nonzero solution of (3.2.4), we require

$$k^2 - (B_1 + B_2)k + B_1B_2 - \sigma\mu = 0.$$

Thus, we obtain

$$k_\pm = \frac{(B_1 + B_2) \pm \sqrt{(B_1 + B_2)^2 - 4(B_1B_2 - \sigma\mu)}}{2}.$$

Notice that, the determinant $\Delta = (B_1+B_2)^2 - 4(B_1B_2 - \sigma\mu) = (B_1 - B_2)^2 + 4\sigma\mu > 0$. This means that $k_- < k_+$, and they both are real. Substituting B_1 and B_2 into it, the exact formula of k_+ is given by

$$k_+ = \frac{(d+\epsilon)\lambda^2 + \alpha\lambda - \sigma - \mu + f'(0) + \sqrt{[(d+\epsilon)\lambda^2 + \alpha\lambda - \sigma + \mu - f'(0)]^2 + 4\sigma\mu}}{2}. \tag{3.2.6}$$

Furthermore, since all the parameters are positive, from the above formula, we have the following result.

Proposition 3.2.1. $k_+ > 0$ *for all* $\lambda \in (0, +\infty)$.

The principal eigenvalue of the cooperative matrix $B(\lambda)$ is

$$k(\lambda) = k_+(\lambda), \tag{3.2.7}$$

where k_+ is defined in (3.2.6). Moreover, due to the term "$d\lambda^2$", it follows that k is convex with respect to λ (see, e.g., [13]).

From (3.2.3), we want to find c such that $c\lambda = k(\lambda)$ has a solution $\lambda \in (0, +\infty)$. It is not hard to find the following property of the function $k(\lambda)$.

Lemma 3.2.2. $k(\lambda)$ *defined in (3.2.7) is a real continuous and convex function with respect to* $\lambda \in \mathbb{R}$. *If we define*

$$c_0 = \inf_{\lambda \in (0,+\infty)} \frac{k(\lambda)}{\lambda} \in \mathbb{R}_+, \tag{3.2.8}$$

which is called the linear speed, then the equation $c\lambda = k(\lambda)$ *has*

(1) no solution if $c < c_0$*;*

(2) exactly one solution $\lambda_0(c_0)$ *if* $c = c_0$*;*

(3) two solutions $\lambda_1(c)$ *and* $\lambda_2(c)$ *with* $\lambda_1(c) < \lambda_2(c)$ *if* $c > c_0$.

Here, we manifest this lemma with a particular example, see Fig. 3.1. Letting $d = 3$, $\epsilon = 0.1$, $\alpha = 1$, $\mu = 1$, $\sigma = 3$, and $f'(0) = 1$, we obtain that $c_0 = 1.99456$ and $\lambda_0 = 0.6906$. In the figure, the black curve denotes $\frac{k(\lambda)}{\lambda}$. As we can see from the figure, there is no intersection when $c < c_0$ (see the yellow line); there is exactly one

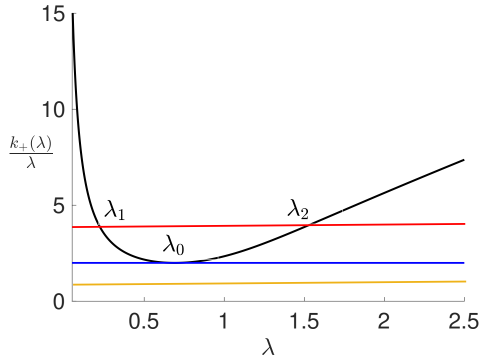


Figure 3.1: The function $\frac{k(\lambda)}{\lambda}$ vs. λ. This figure is obtained in the parameter set: $d = 3$, $\alpha = 1$, $\mu = 1$, $\sigma = 3$, $\epsilon = 0.1$, and $f'(0) = 1$. The black curve denotes the function $\frac{k(\lambda)}{\lambda}$, and the blue line is the value of $c_0 = 1.99456$.

intersection when $c = c_0$ (see the blue line); there are two intersections when $c > c_0$ (see the red line).

Moreover, based on the above lemma, we can give the exact exponential behavior of the waves $(U, V)(\xi)$ as $\xi \to +\infty$ in the following lemma.

Lemma 3.2.3. *Under the definition of c_0 in Lemma 3.2.2, for any $c > c_0$, the wave profile (if it exists) has the following asymptotic behavior*

$$\begin{pmatrix} U \\ V \end{pmatrix} \sim C_1 \begin{pmatrix} -\frac{\mu}{B_1(\lambda_1(c)) - c\lambda_1(c)} \\ 1 \end{pmatrix} e^{-\lambda_1(c)\xi} + C_2 \begin{pmatrix} -\frac{\mu}{B_1(\lambda_2(c)) - c\lambda_2(c)} \\ 1 \end{pmatrix} e^{-\lambda_2(c)\xi} \tag{3.2.9}$$

with $C_1 > 0$, or $C_1 = 0$, $C_2 > 0$. Here B_1 is defined in (3.2.5).

Proof. For any given $c > c_0$, the traveling wave satisfies $(U, V) \to (0, 0)$ as $\xi \to \infty$. Therefore, as $\xi \to \infty$, by way of asymptotic analysis, the leading term of (U, V) (still denote as (U, V)) satisfies (3.2.1). Therefore, via the characteristic equation of the linear system, the decaying solution of (3.2.1) can be obtained as in the right side of (3.2.9). In other words, the positive wave profile satisfies

$$\begin{pmatrix} U \\ V \end{pmatrix} = C_1 \begin{pmatrix} -\frac{\mu}{B_1(\lambda_1) - c\lambda_1} \\ 1 \end{pmatrix} e^{-\lambda_1\xi} + C_2 \begin{pmatrix} -\frac{\mu}{B_1(\lambda_2) - c\lambda_2} \\ 1 \end{pmatrix} e^{-\lambda_2\xi} + o(e^{-\lambda_1\xi}),$$

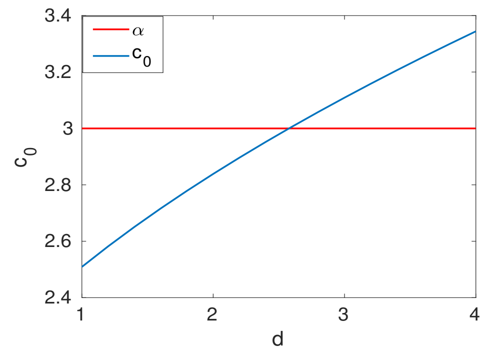


Figure 3.2: There relation between the linear speed c_0 and d. This figure is obtained when $\alpha = 3$, $\mu = 1$, $\sigma = 2$, $\epsilon = 0.1$, $f'(0) = 1$ and d varies from 1 to 4. The blue curve denotes the linear speed corresponding to each d, while the black line is the value of α.

with $C_1 > 0$, or $C_1 = 0,\ C_2 > 0$. This completes the proof. $\square$

Remark 3.2.4. *According to Lemma 3.2.3 and the eigenvalue problem (3.2.3), when $c > c_0$, the asymptotic behavior of the wave can also be given by*

$$\begin{pmatrix} U \\ V \end{pmatrix} \sim C_1 \begin{pmatrix} -\frac{B_2(\lambda_1(c)) - c\lambda_1(c)}{\sigma} \\ 1 \end{pmatrix} e^{-\lambda_1(c)\xi} + C_2 \begin{pmatrix} -\frac{B_2(\lambda_2(c)) - c\lambda_2(c)}{\sigma} \\ 1 \end{pmatrix} e^{-\lambda_2(c)\xi},$$

which is equivalent to (3.2.9).

Remark 3.2.5. *Lemma 3.2.2 implies $c_{min} \geqslant c_0$. It is impossible to expect a nonnegative wavefront for ξ near infinity when $c < c_0$ since λ has a non-trivial imaginary part, and $(0, 0)$ becomes a spiral point. When $c > c_0$, $\lambda_1(c)$ is decreasing in c and $\lambda_2(c)$ is increasing in c.*

From the formula of c_0 (see (3.2.8)), it is clear to see that c_0 is increasing in d. By numerical simulations, we show their relation in Fig. 3.2. It is interesting to observe that c_0 may even be less than α (the drift speed of the stream) when d is small enough, in which the species is fighting with the drift flow to stay via the choice of residing at the bottom.

Remark 3.2.6. *Moreover, if we normalize by setting $A_2 = 1$, then the eigenvalue* *problem (3.2.2) can be rewritten as*

$$\begin{cases} d\lambda^2 A_1 - (c-\alpha)\lambda A_1 = \sigma A_1 - \mu, \\ \epsilon\lambda^2 - c\lambda + f'(0) = -(\sigma A_1 - \mu). \end{cases}$$

When $c = c_0$, we have

$$A_1(c_0) = -\frac{\mu}{B_1(\lambda_0) - c_0\lambda_0}, \tag{3.2.10}$$

where λ_0 is given in Lemma 3.2.2 (2).

3.3 The speed selection mechanism

In this section, we want to study the speed selection mechanism of the system (3.1.5). The method used is the upper and lower solution technique (please see the Appendix section for details). Noticing that the first equation in (3.1.5) is always a linear equation in U, thus by the variation of parameters, we can solve U in terms of V as

$$U(\xi) = \frac{\mu}{d(\tau_2 - \tau_1)}\left\{\int_{-\infty}^{\xi} e^{\tau_1(\xi-s)}V(s)ds + \int_{\xi}^{\infty} e^{\tau_2(\xi-s)}V(s)ds\right\} := H(V), \tag{3.3.1}$$

where τ_1, τ_2 satisfy

$$d\tau^2 + (c-\alpha)\tau - \sigma = 0,$$

with

$$\tau_1 = \frac{-(c-\alpha) - \sqrt{(c-\alpha)^2 + 4\sigma d}}{2d} < 0 < \tau_2 = \frac{-(c-\alpha) + \sqrt{(c-\alpha)^2 + 4\sigma d}}{2d}. \tag{3.3.2}$$

For any $c > c_0$, by Lemmas 3.2.2 and 3.2.3, it is easy to verify that

$$H(e^{-\lambda_i(c)\xi}) = A_{1,i}(c)e^{-\lambda_i(c)\xi} \text{ and } H(1) = \frac{\mu}{\sigma},\ i = 1, 2,$$

where

$$A_{1,i}(c) = \frac{-\mu}{B_1(\lambda_i(c)) - c\lambda_i(c)}.$$

Clearly, for any given continuous function $V(\xi)$ satisfying $V(-\infty)=1$ and $V(+\infty)=0$, by (3.3.1), we have the existence of U subject to $U(-\infty)=\frac{\mu}{\sigma}$ and $U(+\infty)=0$.

For simplicity of notations, we denote

$$\begin{aligned} L_1(U,V) &:= dU''+(c-\alpha)U'-\sigma U+\mu V, \\ L_2(U,V) &:= \epsilon V''+cV'+\sigma U-\mu V+f(V). \end{aligned}$$

By the U's formula, (3.1.5) reduces to a non-local equation

$$\begin{cases} \epsilon V''+cV'+\sigma U-\mu V+f(V)=0, \\ V(-\infty)=1,\ V(+\infty)=0, \end{cases} \tag{3.3.3}$$

where $U=H(V)$ is the integral given in (3.3.1).

From now on, we will focus on constructing a pair of suitable upper and lower solutions to the above V-equation (see Theorem 6.4).

For any $c=c_0+\varepsilon_1$, by Lemma 3.2.2, there exist $0<\lambda_1(c)<\lambda_2(c)$. Inspired by Lemma 3.2.3, we proceed to construct upper or lower solutions with suitable decaying behaviors. Let

$$\bar V(\xi)=\frac{\bar k_v}{[1+(\bar k_v e^{\lambda_1(c)\xi})^m]^{\frac{1}{m}}},\quad m\geqslant 1,\ \bar k_v\geqslant 1. \tag{3.3.4}$$

It is easy to see that this $\bar V$ function has the asymptotic behaviors: $\bar V\sim e^{-\lambda_1(c)\xi}$ as $\xi\to+\infty$ and $\bar V\to\bar k_v$ as $\xi\to-\infty$. Then, through a simple computation, its first and second derivatives are found as follows:

$$\bar V'=-\lambda_1(c)\bar V\left(1-\bar V_1^m\right)\ \text{and}\ \bar V''=\lambda_1^2(c)\bar V\left(1-\bar V_1^m\right)\left(1-(m+1)\bar V_1^m\right),$$

where $\bar V_1=\frac{\bar V}{\bar k_v}$. Substituting all the above formulas into the left-hand side of (3.3.3), we obtain

$$\begin{aligned} L_2(\bar U,\bar V) &= \bar V^2(1-\bar V_1^m)\Bigg\{-(m+1)\epsilon\lambda_1^2(c)\frac{1}{\bar k_v}\bar V_1^{m-1}+\frac{\sigma[\frac{H(\bar V)}{\bar V}-A_1(1-\bar V_1^m)]-\mu\bar V_1^m}{\bar V(1-\bar V_1^m)} \\ &\quad +\frac{\frac{f(\bar V)}{\bar V}-f'(0)(1-\bar V_1^m)}{\bar V(1-\bar V_1^m)}\Bigg\} \\ &=: \bar V^2(1-\bar V_1^m)\cdot J_{\lambda_1}(m,\bar k_v). \end{aligned}$$

In view of the definition of an upper solution (see Def. 3.6.1 and Lemma 3.6.2 for details) and $\lambda_1 \to \lambda_0$ as $\varepsilon \to 0$, we can easily derive that the continuous function $\bar{V}$ given by (3.3.4) is an upper solution to (3.3.3) if

$$J_{\lambda_0}(m, \bar{k}_v) < 0, \tag{3.3.5}$$

with m and $\bar{k}_v$ suitably chosen. Now, we summarize the above discussion into the following lemma.

Lemma 3.3.1. *If the inequality (3.3.5) holds, then the continuous function $\bar{V}$ given by (3.3.4) is an upper solution to (3.3.3) (i.e., $L_2(\bar{U}, \bar{V}) \leqslant 0$).*

To apply Theorem 3.6.4 on (3.3.3), we need to construct a lower solution to (3.1.5) when $c = c_0 + \varepsilon_1$. To this end, we define a continuous function $\underline{V}$ as

$$\underline{V} = \begin{cases} e^{-\lambda_1(c)\xi}(1 - Me^{-\varepsilon_2 \xi}), & \xi > \xi_0, \\ 0, & \xi \leqslant \xi_0, \end{cases} \tag{3.3.6}$$

where $0 < \varepsilon_2 \ll 1$, M is a positive number to be determined, and $\xi_0 = \frac{\log M}{\varepsilon_2}$.

Lemma 3.3.2. *When $c = c_0 + \varepsilon_1$, there exists $0 < \varepsilon_2 \ll 1$ and $M \gg 1$ such that the pair of continuous functions $(\underline{U}, \underline{V})(z)$, where $\underline{V}$ is defined in (3.3.6) and $\underline{U} = H(\underline{V})$ is defined by (3.3.1), is a lower solution to the system (3.1.5)-(3.1.6).*

Proof. To prove the chosen function satisfying the definition of a lower solution, we need to show that for all $\xi \in \mathbb{R}$,

$$\begin{aligned} &d\underline{U}'' + (c - \alpha)\underline{U}' - \sigma\underline{U} + \mu\underline{V} \geqslant 0, \\ &\epsilon\underline{V}'' + c\underline{V}' + \sigma\underline{U} - \mu\underline{V} + f(\underline{V}) \geqslant 0. \end{aligned}$$

Notice that the first inequality is always true for all $\xi \in \mathbb{R}$, and the second one holds for $\xi \leqslant \xi_0$. As for $\xi > \xi_0$, by direct substitution, we have

$$\begin{aligned}
& \epsilon \underline{V}'' + c\underline{V}' + \sigma \underline{U} - \mu \underline{V} + f(\underline{V}) \\
= \;& e^{-\lambda_1(c)\xi}[\epsilon\lambda_1^2(c) - c\lambda_1(c) + \sigma A_1 - \mu + f'(0)] - Me^{-(\lambda_1(c)+\varepsilon_2)\xi}[\epsilon(\lambda_1(c)+\varepsilon_2)^2 \\
& -c(\lambda_1(c)+\varepsilon_2) - \mu + f'(0)] - \sigma M H(e^{-(\lambda_1(c)+\varepsilon_2)\xi}) + [f(\underline{V}) - f'(0)\underline{V}] \\
> \;& -Me^{-(\lambda_1(c)+\varepsilon_2)\xi}[\epsilon(\lambda_1(c)+\varepsilon_2)^2 - c(\lambda_1(c)+\varepsilon_2) + \sigma A_1 - \mu + f'(0)] + [f(\underline{V}) - f'(0)\underline{V}]
\end{aligned}$$

The last inequality is guaranteed by $H(e^{-(\lambda_1(c)+\varepsilon_2)\xi}) < A_1 e^{-(\lambda_1(c)+\varepsilon_2)\xi}$, which can be derived by direct computation. For the last line, it is easy to see that the first term is always positive when ε_2 is sufficiently small. By choosing M to be sufficiently large, we can have $\xi_0 > 0$ and $\underline{V} \ll 1$ so that $[f(\underline{V}) - f'(0)\underline{V}] \sim O(e^{-2\lambda_1(c)\xi})$; thus the first term dominates the second one. Hence, the proof is complete. □

The condition $\underline{V}'(\xi_0^-) \leqslant \underline{V}'(\xi_0^+)$ can be easily verified, and by translation if necessary, we can have $\underline{U}(\xi) \leqslant \bar{U}(\xi)$ and $\underline{V}(\xi) \leqslant \bar{V}(\xi)$ for $\xi \in (-\infty, +\infty)$. Then we conclude that $(\bar{U}, \bar{V})(\xi)$ and $(\underline{U}, \underline{V})(\xi)$ are a pair of upper and lower solutions respectively. By Theorem 3.6.4, we obtain the following linear selection result.

Theorem 3.3.3. *(Linear selection) When (3.3.5) is satisfied, the minimal speed of the system (3.1.5)-(3.1.6) is linearly selected (i.e., $c_{min} = c_0$).*

We then turn to study the nonlinear selection through the upper and lower solutions method. The key observation is that, when a lower solution has an asymptotic behavior $e^{-\lambda_2\xi}$ (i.e., the faster decay rate) as $\xi \to +\infty$, the nonlinear selection will be realized. We give the following theorem as a justification.

Theorem 3.3.4. *For a given $c_1 > c_0$, assume that there exist a pair of nonnegative functions $(\underline{U}, \underline{V})(\xi)$ with $\xi = x - c_1 t$, as a pair of lower solutions to the partial differential system*

$$\begin{cases} u_t = du_{xx} - \alpha u_x - \sigma u + \mu v, \\ v_t = \epsilon v_{xx} + \sigma u - \mu v + f(v). \end{cases} \tag{3.3.7}$$

We further suppose that $\underline{V}(\xi)$ is monotone, and satisfies

$$\limsup_{\xi \to -\infty} \underline{V}(\xi) < 1,$$

and has the asymptotic behavior $Ce^{-\lambda_2 \xi}$ as $\xi \to +\infty$, for some positive constant C. Then there exists no traveling solution to (3.1.5)-(3.1.6) for $c \in [c_0, c_1)$.

Proof. We prove here by contradiction. Assume that there exists a monotone traveling wave solution $(U, V)(\xi)$, $\xi = x - ct$ with $c \in [c_0, c_1)$, subject to the initial conditions

$$u(x, 0) = U(x) \text{ and } v(x, 0) = V(x).$$

We should note that if $c = c_0$, then we have traveling wave solutions for all $c > c_0$ by Theorems 4.1 and 4.2 in [43]. Thus we can always assume that $c \in (c_0, c_1)$.

Moreover, (U, V) satisfies (3.1.5), and their decaying behavior near $+\infty$ can be easily analyzed (see, e.g., Section 2). By the monotonicity of $\lambda_1(c)$ and $\lambda_2(c)$ in terms of c, we can always assume (by shifting if necessary) that $(\underline{U}, \underline{V})(x) \leqslant (U, V)(x)$, $\forall x \in \mathbb{R}$. Since $(\underline{U}, \underline{V})(x - c_1 t)$ is a lower solution to the system (3.3.7) with the initial data $(\underline{U}, \underline{V})(x)$, by comparison, we obtain

$$\underline{U}(x - c_1 t) \leqslant U(x - ct), \text{ and } \underline{V}(x - c_1 t) \leqslant V(x - ct), \tag{3.3.8}$$

for all $(x, t) \in \mathbb{R} \times \mathbb{R}_+$. Now, if we fix $\xi = x - c_1 t$, then $\underline{V}(\xi) > 0$ is fixed. On the other hand, from $V(x - ct)$, it is clear to see

$$V(x - ct) = V(\xi + (c_1 - c)t) \to V(+\infty) = 0, \text{ as } t \to +\infty.$$

By (3.3.8), we thus get $\underline{V}(\xi) \leqslant 0$. This is a contradiction. Therefore, there is no traveling wave solution for $c \in [c_0, c_1)$. This completes the proof. $\square$

Remark 3.3.5. *Due to the above theorem, for the nonlinear selection, we only need* *to find a lower solution that has an asymptotic behavior* $e^{-\lambda_2(c)\xi}$, *as* $\xi \to +\infty$ *for some* $c > c_0$.

Now, suppose $\underline{V}_2$ has the following form:

$$\underline{V}_2(\xi) = \frac{\underline{k}_v}{[1 + (\underline{k}_v e^{\lambda_2(c)\xi})^m]^{\frac{1}{m}}}, \quad m \geqslant 1,\ 0 < \underline{k}_v < 1. \tag{3.3.9}$$

Clearly, this function connects $\bar{k}_v$ to 0 and has the asymptotic behavior $e^{-\lambda_2\xi}$ as $\xi \to +\infty$. By substituting the above formula into the left-hand side of (3.3.3), we obtain

$$\begin{aligned} L_2(\underline{U}_2, \underline{V}_2) &= \underline{V}_2^2(1 - \underline{V}_1^m)\Big\{ -(m+1)\epsilon\lambda_2^2(c)\frac{1}{\underline{k}_v}\underline{V}_1^{m-1} \\ &\quad + \frac{\sigma[\frac{H(\underline{V}_2)}{\underline{V}_2} - A_1(1 - \underline{V}_1^m)] - \mu\underline{V}_1^m}{\underline{V}_2(1 - \underline{V}_1^m)} + \frac{\frac{f(\underline{V}_2)}{\underline{V}_2} - f'(0)(1 - \underline{V}_1^m)}{\underline{V}_2(1 - \underline{V}_1^m)}\Big\} \\ &=: \underline{V}_2^2(1 - \underline{V}_1^m) \cdot J_{\lambda_2}(m, \underline{k}_v), \end{aligned}$$

where $\underline{V}_1 = \frac{\underline{V}_2}{\underline{k}_v}$. For suitably chosen m and $\underline{k}_v$, it follows that $\underline{V}_2$ is a lower solution to (3.3.3) (i.e., $L_2(\underline{U}_2, \underline{V}_2) \geqslant 0$) if

$$J_{\lambda_2}(m, \underline{k}_v) > 0. \tag{3.3.10}$$

Then, by Lemma 3.3.4, the following result holds.

Theorem 3.3.6. *(Nonlinear selection) If the inequality (3.3.10) holds for some* m *and* $\underline{k}_v$, *then the minimal speed of traveling waves to the system (3.1.5)-(3.1.6) is nonlinearly selected.*

3.4 Applications

In this section, we will apply the linear and nonlinear selection theorems proved in the previous section to the model with a cubic nonlinear reaction term, i.e., $f(v) = v(1-v)(1+\rho v)$ with ρ being a nonnegative constant. This cubic reaction term can be viewed as the classical logistic growth with a weak Allee effect (see [84]) and can be applied to model a lot of biological phenomena. We want to investigate how the Allee effect impacts the spreading speed. In current references such as [20, 91], they require that $f(v)$ is sublinear in the sense that $f(v) \leqslant f'(0)v$, and thus a linear selection result is obtained. Following this, we immediately obtain that the minimal wave speed is linearly selected when $\rho \leqslant 1$. Now, with our methods, conclusions on the speed selection can be considerably extended. To proceed, we start with the system of the wave profile

$$\begin{cases} dU'' + (c-\alpha)U' - \sigma U + \mu V = 0, \\ \epsilon V'' + cV' + \sigma U - \mu V + (1-V)(1+\rho V)V = 0, \\ (U,V)(-\infty) = \left(\frac{\mu}{\sigma}, 1\right), \ (U,V)(+\infty) = (0,0). \end{cases} \tag{3.4.1}$$

With the values of d, ϵ, μ, σ, α and $f'(0)$ being fixed, we first show the existence of a threshold $\bar{\rho}$ so that, when ρ increases to cross over this critical value, the speed selection changes from linear to nonlinear. To see this, we will prove the following lemma.

Lemma 3.4.1. *If the minimal wave speed of (3.4.1) is linearly selected when $\rho = \rho_l$ for some ρ_l, then it is linearly selected for all $\rho < \rho_l$.*

Proof. From the assumption that $\rho = \rho_l$, we have (U_l, V_l) as a pair of solutions, which are decreasing with respect to $\xi \in \mathbb{R}$, with $c = c_0 + \varepsilon_1$ to (3.4.1) for any small $\varepsilon_1 > 0$. Thus, they satisfy

$$\begin{cases} dU_l'' + (c-\alpha)U_l' - \sigma U_l + \mu V_l = 0, \\ \epsilon V_l'' + cV_l' + \sigma U_l - \mu V_l + (1-V_l)(1+\rho_l V_l)V_l = 0. \end{cases}$$

Then, by substituting (U_l, V_l) into (3.4.1) with $\rho < \rho_l$, we see that the first equation is always zero and the second one becomes

$$\begin{aligned} & \epsilon V_l'' + cV_l' + \sigma U_l - \mu V_l + (1-V_l)(1+\rho V_l)V_l \\ = \ & (1-V_l)V_l^2(\rho - \rho_l) < 0. \end{aligned}$$

This means that (U_l, V_l) is an upper solution to (3.4.1) for $\rho < \rho_l$. Then, by taking the lower solution defined in Lemma 3.3.2, we conclude that the minimal wave speed is linearly selected for all $\rho < \rho_l$. This completes the proof. □

From the above lemma, we can define the threshold value of ρ as

$$\overline{\rho} := \sup\{\rho \,|\, \text{the linear speed selection is realized for (3.4.1)}\}.$$

Although we obtained the existence of the threshold $\bar{\rho}$, its exact value is hard to derive. In practice, we want to give an estimate of it. Moreover, the exact formula of U in terms of V (see (3.3.1)) is too complicated to determine the conditions in the speed selection, so we will establish some novel upper (lower) solutions to the U-equation simultaneously, i.e., $L_1(\bar{U}, \bar{V}) \leqslant 0$ ($L_1(\underline{U}, \underline{V}) \geqslant 0$), instead of using the formula $H(V)$.

To carry on, we numerically compute the value of $\frac{U}{V} = \frac{H(V)}{V}$, where V is defined in (3.3.4) with $m = 2$, $\bar{k}_v = 1$, and $c = c_0$. An example is shown in Fig. 3.3. These figures are depicted when $d = 3$, $\epsilon = 0.1$, $\alpha = 1$, $\mu = 1$, and $\sigma = 3$. With the parameter set, we find that $A_1 = 0.44325$, $c_0 = 1.9945625$, and $\lambda_0 = 0.6906$. The left panel shows the functions of V and $U = H(V)$. The right panel shows the value of $\frac{U}{V}$. As we can see, $\frac{U}{V} \to \frac{\mu}{\sigma}$ as $\xi \to +\infty$, $\frac{U}{V} \to A_1$ as $\xi \to -\infty$, and the curve looks like a vertical parabola. When $m = 1$, similar phenomena can happen. Inspired by this observation, we will construct innovative approximate formulas of U in terms of V, which are much simpler than the abstract one $U = H(V)$. The details are shown as follows.

Motivated by this observation, we first give results on the speed selection by using the trial function $U = V \cdot (A_1 + bV + aV^2)$ with $b = \frac{\mu}{\sigma} - A_1 - a$ and $a \in \mathbb{R}_+$ to be determined. We give the following notations to state our theorems more fluently. Denote

$$\begin{aligned} h_{c_0}(a) := {} & a^2 \left\{33d^2\lambda_0^4 + 6d\lambda_0^3(c_0 - \alpha) + 9\lambda_0^2(c_0 - \alpha)^2 + 48d\lambda_0^2\sigma\right\} + a\Bigg\{ 12d^2\lambda_0^4\frac{\mu}{\sigma} \\ & -108d^2\lambda_0^4 A_1(c_0) - 60d\lambda_0^3\left(\frac{\mu}{\sigma} - A_1(c_0)\right)(c_0 - \alpha)\Bigg\} + 36d^2\lambda_0^4\left(\frac{\mu}{\sigma} - A_1(c_0)\right)^2 \end{aligned} \tag{3.4.2}$$

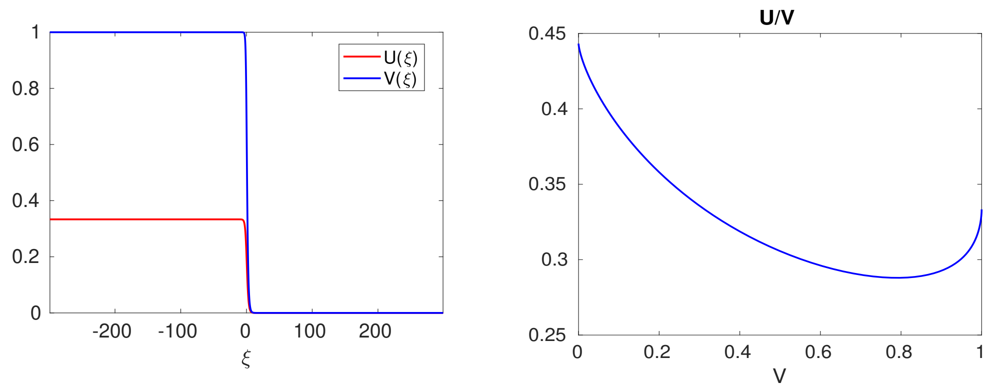


Figure 3.3: The functions U, V and $\frac{U}{V}$. These figures are obtained in the parameter set: $d = 3$, $\alpha = 1$, $\mu = 1$, $\sigma = 3$, $\epsilon = 0.1$, and $f'(0) = 1$.

and

$$\begin{cases} a_3(c_0,\lambda_0) = \frac{2d\lambda_0\left(\frac{\mu}{\sigma}-A_1(c_0)\right)}{5d\lambda_0-(c_0-\alpha)}, \ a_4(c_0,\lambda_0) = \frac{d\lambda_0^2\left(\frac{4\mu}{\sigma}-6A_1(c_0)\right)-2\lambda_0(c_0-\alpha)\left(\frac{\mu}{\sigma}-A_1(c_0)\right)}{4d\lambda_0^2-2\lambda_0(c_0-\alpha)-\sigma}, \\ a_5(c_0,\lambda_0) = \frac{2\left[d\lambda_0^2\frac{\mu}{\sigma}+\lambda_0(c_0-\alpha)\left(\frac{\mu}{\sigma}-A_1(c_0)\right)\right]}{\left[-d\lambda_0^2-\lambda_0(c_0-\alpha)+\sigma\right]}. \end{cases} \tag{3.4.3}$$

Notice that $h_{c_0}(a)$ is a quadratic polynomial and $h_{c_0}(0) = 36d^2\lambda_0^4\left(\frac{\mu}{\sigma} - A_1(c_0)\right)^2 > 0$; thus, if $h_{c_0}(a) = 0$ has solutions $a_1(c_0,\lambda_0)$ and $a_2(c_0,\lambda_0)$, then they must satisfy that $0 < a_1(c_0,\lambda_0) \leqslant a_2(c_0,\lambda_0)$ or $a_1(c_0,\lambda_0) \leqslant a_2(c_0,\lambda_0) < 0$. Due to the requirement $a > 0$, we only consider the former case. Furthermore, define the sets

$$\begin{cases} S_1(c_0,\lambda_0) := \{a : a \leqslant a_1(c_0,\lambda_0) \text{ or } a \geqslant a_2(c_0,\lambda_0)\}, \\ \text{and } S_1'(c_0,\lambda_0) := \{a : a_1(c_0,\lambda_0) < a < a_2(c_0,\lambda_0)\}, \\ S_2(c_0,\lambda_0) := \{a : a < a_3(c_0,\lambda_0)\} \text{ and } S_2'(c_0,\lambda_0) := \{a : a > a_3(c_0,\lambda_0)\}, \\ S_3(c_0,\lambda_0) := \{a : a \leqslant a_4(c_0,\lambda_0)\} \text{ and } S_3'(c_0,\lambda_0) := \{a : a \geqslant a_4(c_0,\lambda_0)\}, \\ S_4(c_0,\lambda_0) := \{a : a \leqslant a_5(c_0,\lambda_0)\} \text{ and } S_4'(c_0,\lambda_0) := \{a : a \geqslant a_5(c_0,\lambda_0)\}. \end{cases} \tag{3.4.4}$$

To proceed, we summarize the above notations into an assumption as follows.

(**H1**) Let c_0, λ_0, $A_1(c_0)$, and $h_{c_0}(a)$ be defined in (3.2.8), Lemma 3.2.2, (3.2.10), and (3.4.2), respectively. Assume that $h_{c_0}(a) = 0$ has two nonnegative solutions $0 \leqslant a_1(c_0,\lambda_0) < a_2(c_0,\lambda_0)$ and then define a_i $(i = 1,\cdots,5)$ and S_j, S_j' $(j = 1,\cdots,4)$ as shown in (3.4.3) and (3.4.4), respectively.

Theorem 3.4.2. *Let the assumption (**H1**) hold. Define*

$$\bar{M}(c_0, \lambda_0) := \bar{M}_1 \cup \bar{M}_2 \cup \bar{M}_3 \cup \bar{M}_4 \cap \{a : a > 0\},$$

where

$$\begin{cases} \bar{M}_1 := (S_2(c_0, \lambda_0) \cap S_3(c_0, \lambda_0)) \cup S_1'(c_0, \lambda_0), \ \ \bar{M}_2 := (S_2(c_0, \lambda_0) \cap S_3'(c_0, \lambda_0)) \cup S_1'(c_0, \lambda_0), \\ \bar{M}_3 := (S_2'(c_0, \lambda_0) \cap S_3(c_0, \lambda_0)) \cup S_1'(c_0, \lambda_0), \ \ \bar{M}_4 := (S_2'(c_0, \lambda_0) \cap S_3'(c_0, \lambda_0)) \cup S_1'(c_0, \lambda_0). \end{cases}$$

Then the linear selection is realized if there exists a positive constant $a \in \bar{M}$ and

$$\rho \leqslant \sigma \bar{a} + 2\epsilon \lambda_0^2, \ \ \text{where } \bar{a} = \sup \bar{M}. \tag{3.4.5}$$

Proof. When $c = c_0 + \varepsilon_1$, let $\bar{V}$ be defined in (3.3.4) with $m = 1$ and $\bar{k}_v = 1$ (which implies $\bar{V}_1 = \bar{V}$). Define

$$\bar{U} = \bar{V} \cdot [A_1(c) + b\bar{V} + a\bar{V}^2], \ \ a > 0, \tag{3.4.6}$$

where $b = \frac{\mu}{\sigma} - A_1(c) - a$ and a is to be determined. Here, we emphasize that such a $\bar{U}$ function satisfies $\frac{\bar{U}}{\bar{V}} \to \frac{\mu}{\sigma}$ as $\xi \to -\infty$ and $\frac{\bar{U}}{\bar{V}} \to A_1(c)$ as $\xi \to +\infty$. In the following context, we denote $\lambda_1 = \lambda_1(c)$ and $A_1 = A_1(c)$ for short unless otherwise specified. Then, through tedious computations, we obtain the first and second derivative of $\bar{U}$ as follows:

$$\bar{U}' = -\lambda_1 \bar{V}(1-\bar{V})(A_1+2b\bar{V}+3a\bar{V}^2)\bar{U}'' = \lambda_1^2 \bar{V}(1-\bar{V})[A_1+(4b-2A_1)\bar{V}+(9a-6b)\bar{V}^2-12a\bar{V}^3]$$

and

$$\bar{U}'' = \lambda_1^2 \bar{V}(1-\bar{V})[A_1 + (4b - 2A_1)\bar{V} + (9a - 6b)\bar{V}^2 - 12a\bar{V}^3].$$

By substituting $\bar{U}$, $\bar{U}'$ and $\bar{U}''$ into L_1, we obtain

$$L_1(\bar{U},\bar{V}) = \bar{V}^2(1-\bar{V})G_1(\bar{V}), \tag{3.4.7}$$

where

$$\begin{aligned} G_1(\bar{V}) &= -12d\lambda_1^2 a\bar{V}^2 + \bar{V}\cdot[d\lambda_1^2(9a-6b) - 3\lambda_1 a(c-\alpha)] + d\lambda_1^2(4b-2A_1) - 2\lambda_1(c-\alpha)b + \sigma a \\ &= -12d\lambda_1^2 a\bar{V}^2 + 3\lambda_1\bar{V}\left[(5d\lambda_1 - (c-\alpha))a - 2d\lambda_1(\frac{\mu}{\sigma} - A_1)\right] \\ &\quad + a\left[-4d\lambda_1^2 + 2\lambda_1(c-\alpha) + \sigma\right] + d\lambda_1^2\left(\frac{4\mu}{\sigma} - 6A_1\right) - 2\lambda_1(c-\alpha)\left(\frac{\mu}{\sigma} - A_1\right). \end{aligned}$$

It is clear that $\bar{G}_1(\bar{V})$ is a parabolic function, which opens down, in $\bar{V}$. Through a direct computation, its determinant can be found as

$$\begin{aligned} \Delta = a^2\left\{33d^2\lambda_1^4 + 6d\lambda_1^3(c-\alpha) + 9\lambda_1^2(c-\alpha)^2 + 48d\lambda_1^2\sigma\right\} + a\Bigg\{12d^2\lambda_1^4\frac{\mu}{\sigma} \\ -108d^2\lambda_1^4 A_1 - 60d\lambda_1^3\left(\frac{\mu}{\sigma} - A_1\right)(c-\alpha)\Bigg\} + 36d^2\lambda_1^4\left(\frac{\mu}{\sigma} - A_1\right)^2, \end{aligned}$$

which is $h_c(a)$ by replacing c_0 and λ_0 with c and λ_1 in $h_{c_0}(a)$. When ε_1 is small enough and by assumption, the equation $h_c(a) = 0$ has two roots $0 \leqslant a_1(c,\lambda_1) \leqslant a_2(c,\lambda_1)$. Then, there are two cases to discuss.

When $a_1(c,\lambda_1) < a < a_2(c,\lambda_1)$ (i.e., $a \in S_1'(c,\lambda_1)$), it follows that $h_c(a) \leqslant 0$. In other words, $\Delta < 0$, which implies that $G_1(\bar{V}) = 0$ has no solution. Therefore, $L_1(\bar{U},\bar{V}) \leqslant 0$ if $a \in S_1(c,\lambda_1)$.

When $0 \leqslant a \leqslant a_1(c,\lambda_1)$ or $a \geqslant a_2(c,\lambda_1)$ (i.e., $a \in S_1(c,\lambda_1)$), it immediately obtains that $\Delta \geqslant 0$. Thus, under this condition, $G_1(\bar{V}) = 0$ must have solutions. Furthermore, if the symmetric axis of $G_1(\bar{V})$ is less than zero and $G_1(0) \leqslant 0$, then $L_1(\bar{U},\bar{V}) \leqslant 0$. The first condition means

$$(5d\lambda_1 - (c-\alpha))a - 2d\lambda_1(\frac{\mu}{\sigma} - A_1) < 0.$$

The second condition ($G_1(0) \leqslant 0$) shows that

$$a\left[-4d\lambda_1^2 + 2\lambda_1(c-\alpha) + \sigma\right] + d\lambda_1^2\left(\frac{4\mu}{\sigma} - 6A_1\right) - 2\lambda_1(c-\alpha)\left(\frac{\mu}{\sigma} - A_1\right) \leqslant 0.$$

When $5d\lambda_1 - (c-\alpha) > 0$ and $4d\lambda_1^2 - 2\lambda_1(c-\alpha) - \sigma > 0$, then

$$a < a_3(c,\lambda_1) \text{ and } a \geqslant a_4(c,\lambda_1). \tag{3.4.8}$$

Thus, when $a \in S_1(c,\lambda_1) \cap S_2(c,\lambda_1) \cap S_3'(c,\lambda_1)$, we have $L_1 \leqslant 0$. Summarizing the above discuss, we obtain that if

$$\begin{aligned} & a \in (S_1(c,\lambda_1) \cap S_2(c,\lambda_1) \cap S_3'(c,\lambda_1)) \cup S_1'(c,\lambda_1) \\ = \; & (S_2(c,\lambda_1) \cap S_3'(c,\lambda_1)) \cup S_1'(c,\lambda_1) = \bar{M}_1(c,\lambda_1), \end{aligned}$$

then $L_1(\bar{U},\bar{V}) \leqslant 0$. It is clear to see that, depending on the signs of $5d\lambda_1 - (c-\alpha)$ and $4d\lambda_1^2 - 2\lambda_1(c-\alpha) - \sigma$, we will obtain sets $\bar{M}_2(c,\lambda_1)$, $\bar{M}_3(c,\lambda_1)$, and $\bar{M}_4(c,\lambda_1)$. In summary, if $a \in \bar{M}(c,\lambda_1)$, then $L_1(\bar{U},\bar{V}) \leqslant 0$.

By inserting $\bar{U}$-formula into L_2, we have

$$L_2(\bar{U},\bar{V}) = \bar{V}^2(1-\bar{V})(-2\epsilon\lambda_1^2 - \sigma a + \rho).$$

Now, it is clear to see that, if $\rho \leqslant \sigma\bar{a}_1 + 2\epsilon\lambda_1^2$ with $\bar{a}_1 = \sup \bar{M}(c,\lambda_1)$, then $L_2 \leqslant 0$. Thus, $(\bar{U},\bar{V})$ is a pair of upper solutions when $a \in \bar{M}(c,\lambda_1)$ and $\rho < \sigma\bar{a} + 2\epsilon\lambda_1^2$ hold. Combining a pair of lower solutions from Lemma 3.3.2 and using Theorem 3.6.4, we obtain the existence of $(U,V)(\xi)$ when $c = c_0 + \varepsilon_1$, which implies the linear selection of (3.4.1). Then, a limiting argument can show that the linear selection is realized when $a \in \bar{M}(c_0,\lambda_0)$ and $\rho \leqslant 2\epsilon\lambda_0^2 + \sigma\bar{a}$. This completes the proof. $\square$

Remark 3.4.3. *If $h_{c_0}(a) = 0$ has no solution when $a > 0$, then the above theorem still holds by replacing $S_1' = \phi$ where ϕ is the empty set.*

Since the minimal wave speed is always linearly selected when $\rho \leqslant 1$, it immediately implies the following corollary.

Corollary 3.4.4. *Let **(H1)** be true. The minimal wave speed is linearly selected if $a \in \bar{M}(c_0, \lambda_0)$ and*

$$\rho \leqslant \max\{\sigma\bar{a} + 2\epsilon\lambda_0^2, 1\}. \tag{3.4.9}$$

For the nonlinear selection, we first give the following theorem.

Theorem 3.4.5. *Let the assumption **(H1)** hold and*

$$\underline{M}(c_0, \lambda_0) := (\underline{M}_1 \cup \underline{M}_2 \cup \underline{M}_3 \cup \underline{M}_4) \cap \{a : a > 0\},$$

where

$$\begin{cases} \underline{M}_1 := S_1(c_0,\lambda_0) \cap S_3(c_0,\lambda_0) \cap S_4(c_0,\lambda_0), \ \underline{M}_2 := S_1(c_0,\lambda_0) \cap S_3'(c_0,\lambda_0) \cap S_4(c_0,\lambda_0), \\ \underline{M}_3 := S_1(c_0,\lambda_0) \cap S_3(c_0,\lambda_0) \cap S_4'(c_0,\lambda_0), \ \underline{M}_4 := S_1(c_0,\lambda_0) \cap S_3'(c_0,\lambda_0) \cap S_4'(c_0,\lambda_0). \end{cases}$$

Then the nonlinear selection is realized if there exists $a \in \underline{M}$ and

$$\rho > \sigma\underline{a} + 2\epsilon\lambda_0^2, \ \textit{where } \underline{a} = \inf \underline{M} > 0. \tag{3.4.10}$$

Proof. When $c = c_0 + \varepsilon_2$, let $\underline{V}$ be defined in (3.3.9) with $m = 1$ and $\underline{k}_v = 1$, and

$$\underline{U} = \underline{V}[A_1(c) + b\underline{V} + a\underline{V}^2], \ a > 0,$$

with $b = \frac{\mu}{\sigma} - A_1(c) - a$ and a is to be determined. For simplicity, we will denote $\lambda_2 = \lambda_2(c)$ and $A_1 = A_1(c)$ unless otherwise specified. With the help of calculations done in Theorem 3.4.2, we can relatively easily derive the following formulas for L_1:

$$L_1(\underline{U}, \underline{V}) = \underline{V}^2(1 - \underline{V})G_2(\underline{V}), \tag{3.4.11}$$

where

$$\begin{aligned} G_2(\underline{V}) &= -12d\lambda_2^2 a\underline{V}^2 + 3\lambda_2\underline{V}\left[(5d\lambda_2 - (c-\alpha))a - 2d\lambda_2(\frac{\mu}{\sigma} - A_1)\right] \\ &\quad + a\left[-4d\lambda_2^2 + 2\lambda_2(c-\alpha) + \sigma\right] + d\lambda_2^2\left(\frac{4\mu}{\sigma} - 6A_1\right) - 2\lambda_2(c-\alpha)\left(\frac{\mu}{\sigma} - A_1\right). \end{aligned}$$

Notice that $G_2(\underline{V})$ is a parabolic function in $\underline{V}$. Through a similar analysis on its determinant done in Theorem 3.4.2, we obtain that $G_2(\underline{V}) = 0$ has solutions when $a \in S_1(c, \lambda_2)$. Under this condition, the inequalities "$G_2(0) \geqslant 0$" and "$G_2(1) \geqslant 0$" assure $L_1 \geqslant 0$. That means,

$$G_2(0) = a\left[-4d\lambda_2^2 + 2\lambda_2(c-\alpha) + \sigma\right] + d\lambda_2^2\left(\frac{4\mu}{\sigma} - 6A_1\right) - 2\lambda_2(c-\alpha)\left(\frac{\mu}{\sigma} - A_1\right) \geqslant 0$$

and

$$G_2(1) = a\left[-d\lambda_2^2 - \lambda_2(c-\alpha) + \sigma\right] - 2\left[d\lambda_2^2\frac{\mu}{\sigma} + \lambda_2(c-\alpha)\left(\frac{\mu}{\sigma} - A_1\right)\right] \geqslant 0.$$

Depending on the sign of $-4d\lambda_2^2 + 2\lambda_2(c-\alpha) + \sigma$ and $-d\lambda_2^2 - \lambda_2(c-\alpha) + \sigma$, we have four cases. Since the analyses on those four cases are similar, we only present the case when $-4d\lambda_2^2 + 2\lambda_2(c-\alpha) + \sigma > 0$ and $-d\lambda_2^2 - \lambda_2(c-\alpha) + \sigma > 0$ in details. Under this condition,

$$a \geqslant \frac{d\lambda_2^2\left(\frac{4\mu}{\sigma} - 6A_1\right) - 2\lambda_2(c-\alpha)\left(\frac{\mu}{\sigma} - A_1\right)}{4d\lambda_2^2 - 2\lambda_2(c-\alpha) - \sigma}, \text{ and } a \geqslant \frac{2\left[d\lambda_2^2\frac{\mu}{\sigma} + \lambda_2(c-\alpha)\left(\frac{\mu}{\sigma} - A_1\right)\right]}{-d\lambda_2^2 - \lambda_2(c-\alpha) + \sigma}.$$

That means, if $a \in S_3'(c, \lambda_2) \cap S_4'(c, \lambda_2) \cap S_1(c, \lambda_2)$, then $L_1(\underline{U}, \underline{V}) \geqslant 0$. In other words, when $a \in \underline{M}_3(c, \lambda_2)$, we have $L_1(\underline{U}, \underline{V}) \geqslant 0$.

For V-equation, we obtain

$$L_2(\underline{U}, \underline{V}) = \underline{V}^2(1 - \underline{V})J_{\lambda_2}(\underline{V}) = \underline{V}^2(1 - \underline{V})(-2\epsilon\lambda_2^2 - \sigma a + \rho).$$

It is easy to see that if the strict inequality (3.4.10) holds, then $\rho > \sigma\underline{a} + 2\epsilon\lambda_2^2$ with

$\underline{a} = \inf \underline{M}$, that means, $L_2(\underline{U}, \underline{V}) > 0$. Therefore, we have found a pair of lower solution with the faster decay rate. If we take $\underline{k}_v = 1 - \eta$ for sufficiently small η, by continuity, the above derivation is still true. By Theorem 3.3.4, the nonlinear selection is realized. □

Since the ratio $\frac{U}{V}$ has a parabolic behavior as shown in the right panel in Fig. 3.3, we can give another approach to find conditions for the nonlinear selection.

Theorem 3.4.6. *Let* $\kappa = \frac{A_1(c_0)}{\frac{\mu}{\sigma} + A_1(c_0)}$. *Suppose that*

$$\begin{cases} 2\lambda_0(c_0 - \alpha)A_1(c_0) + \mu - 6d\lambda_0^2 A_1(c_0) > 0, \\ \sigma A_1(c_0) + 2\mu + 2\lambda_0(c_0 - \alpha)A_1(c_0) > 0, \\ -6d\lambda_0^2 A_1^2(c_0) + 2A_1(c_0)\left(\frac{\mu}{\sigma} + A_1(c_0)\right)\left(2d\lambda_0^2 - \lambda_0(c_0 - \alpha)\right) + \sigma\left(\frac{\mu}{\sigma} + A_1(c_0)\right)^2 > 0, \\ -2d\lambda_0^2 - 2\lambda_0(c_0 - \alpha) + \sigma > 0. \end{cases} \tag{3.4.12}$$

Then the minimal wave speed of system (3.4.1) is nonlinearly selected if

$$\rho > 2\epsilon\lambda_0^2 + \frac{\mu\kappa}{1 - \kappa}, \tag{3.4.13}$$

where $A_1(c_0)$ *and* λ_0 *defined in (3.2.10) and Lemma 3.2.2, respectively.*

Proof. When $c = c_0 + \varepsilon_3$ with $\varepsilon_3 > 0$ being small, let $\underline{V}$ be defined in (3.3.9) with $m = 1$ and $\underline{k}_v = 1$. Define

$$\underline{U} = \underline{V} \cdot \max_{\xi \in \mathbb{R}} \left\{ A_1(c)(1 - \underline{V}),\ \frac{\mu}{\sigma}\underline{V} \right\} = \begin{cases} A_1(c)(1 - \underline{V})\underline{V},\ \xi \geqslant \xi_2, \\ \frac{\mu}{\sigma \underline{k}_v}\underline{V}^2,\ \xi < \xi_2, \end{cases}$$

where $\xi_2 \in \mathbb{R}$ such that $\underline{V}(\xi_2) = \frac{A_1(c)}{A_1(c) + \frac{\mu}{\sigma}}$. Thus, by substituting them into L_1 and L_2, we obtain

$$L_1(\underline{U},\underline{V}) = \begin{cases} \underline{V}^2\Big\{ -6d\lambda_2^2A_1\underline{V}^2 + \underline{V}\left[12d\lambda_2^2A_1 - 2\lambda_2(c-\alpha)A_1\right] + 2\lambda_2(c-\alpha)A_1 + \mu - 6d\lambda_2^2A_1\Big\}, \\ \hfill \underline{V}\in[0,\underline{V}(\xi_2)], \\ \frac{\mu}{\sigma}\underline{V}(1-\underline{V})\Big\{ -6d\lambda_2^2\underline{V}^2 + \underline{V}\left[4d\lambda_2^2 - 2\lambda_2(c-\alpha)\right] + \sigma\Big\},\ \underline{V}\in(\underline{V}(\xi_2),1] \end{cases}$$

and

$$L_2(\underline{U},\underline{V}) = \begin{cases} \underline{V}^2(1-\underline{V})\Big\{ -2\epsilon\lambda_2^2 + \frac{-\mu}{1-\underline{V}} + \rho\Big\},\ \underline{V}\in[0,\underline{V}(\xi_2)], \\ \frac{\mu}{\sigma}\underline{V}^2(1-\underline{V})\Big\{ -2\epsilon\lambda_2^2 + \frac{-\sigma A_1}{\underline{V}} + \rho\Big\},\ \underline{V}\in(\underline{V}(\xi_2),1]. \end{cases}$$

For L_1 part, let $G_3(\underline{V}) := -6d\lambda_2^2A_1\underline{V}^2 + \underline{V}\left[12d\lambda_2^2A_1 - 2\lambda_2(c-\alpha)A_1\right] + 2\lambda_2(c-\alpha)A_1 + \mu - 6d\lambda_2^2A_1$, which is a quadratic function in $\underline{V}$. The first inequality in (3.4.12) implies $G_3(0) \geqslant 0$, and

$$G_3(\underline{V}(\xi_2)) = \frac{\mu^2\left(-6d\lambda_2^2A_1 + 2\lambda_2(c-\alpha)A_1 + \mu\right) + \mu\sigma A_1\left(2\lambda_2(c-\alpha)A_1 + \sigma A_1 + 2\mu\right)}{(\mu+\sigma A_1)^2} \geqslant 0$$

provided by the first and second inequalities. Therefore, $G_3(\underline{V}) \geqslant 0$ for $\underline{V}\in[0,\underline{V}(\xi_2)]$. Then, denote $G_4(\underline{V}) := -6d\lambda_2^2\underline{V}^2 + \underline{V}\left[4d\lambda_2^2 - 2\lambda_2(c-\alpha)\right] + \sigma$, which is convex down in $\underline{V}$. Thus, it suffices to find the values of $G_4(\underline{V}(\xi_2))$ and $G_4(1)$. Through a direct computation and the third and fourth inequalities in (3.4.12), we obtain that $G_4(1) = -2d\lambda_2^2 - 2\lambda_2(c-\alpha) + \sigma \geqslant 0$ and

$$G_4(\underline{V}(\xi_2)) = \frac{-6d\lambda_2^2A_1^2}{\left(\frac{\mu}{\sigma}+A_1\right)^2} + \frac{A_1\left[4d\lambda_2^2 - 2\lambda_2(c-\alpha)\right]}{\left(\frac{\mu}{\sigma}+A_1\right)} + \sigma \geqslant 0.$$

As for L_2 part, it is not difficult to verify that $L_2(\underline{U},\underline{V}) \geqslant 0$ if $\rho \geqslant 2\epsilon\lambda_2^2 + \frac{\mu}{1-\underline{V}(\xi_2)}$ when $\xi \geqslant \xi_2$, and $\rho \geqslant 2\epsilon\lambda_2^2 + \frac{\sigma A_1}{\underline{V}(\xi_2)}$ when $\xi < \xi_2$. Notice that $\frac{\sigma A_1}{\underline{V}(\xi_2)} = \frac{\mu}{1-\underline{V}(\xi_2)}$. When ε_3 is small enough, (3.4.13) implies that if $\rho \geqslant 2\epsilon\lambda_2^2 + \frac{\mu}{1-\underline{V}(\xi_2)}$, then $L_2(\underline{U},\underline{V}) \geqslant 0$ for $\xi\in\mathbb{R}$. Thus, the nonlinear selection result follows. $\square$

Remark 3.4.7. *In this application, we only present conditions for the speed selection* *when $m = 1$. In fact, if $m = 2$ (the derivation is much more complicated), we can obtain the following result.*

Theorem 3.4.8. *Let*

$$F_{c_0}(a) := a^2\left[469d^2\lambda_0^4 + 135d\lambda_0^3(c_0-\alpha)\right] - a\left[128d^2\lambda_0^4\frac{\mu}{\sigma} + 7d^2\lambda_0^4 A_1(c_0)\right] + 64d^2\lambda_0^4\left(\frac{\mu}{\sigma} - A_1(c_0)\right)^2 \tag{3.4.14}$$

and $F_{c_0}(a) = 0$ has two roots $0 < a_m(c_0) < a_M(c_0)$ with $A_1(c_0)$ and λ_0 being defined in (3.2.10) and Lemma 3.2.2, respectively. Assume that

$$4d\lambda_0^2 - 2\lambda_0(c_0-\alpha) - \sigma > 0. \tag{3.4.15}$$

Then the system (3.4.1) is linearly selected if

$$\rho \leqslant 1 + \sigma A_1(c_0) - \mu + \sigma a_M(c_0). \tag{3.4.16}$$

We omit the proof, since it is similar to the previous one. Later, we will demonstrate a numerical example (the first one) in which the result in the choice of $m = 2$ may be better than that in the choice of $m = 1$ when $4d\lambda_0^2 - 2\lambda_0(c_0-\alpha) - \sigma > 0$.

To complete this section, we provide two numerical examples to manifest our theoretical results. In the first example, we choose the parameter set as $d = 3$, $\epsilon = 0.1$, $\mu = 1$, $\sigma = 3$ and $\alpha = 1$. In this set, we find that $c_0 = 1.9946$, $\lambda_0 = 0.6906$ and $A_1 = 0.4433$. Then, by a simple computation, we obtain that $4d\lambda_0^2 - 2\lambda_0(c_0-\alpha) - \sigma = 1.3495 > 0$, $5d\lambda_0 - (c_0-\alpha) = 9.36444 > 0$, $a_3 = -0.0486$, and $a_4 = -1.2942$. Through Theorem 3.4.2 and its corollary, the linear selection result is only valid if $\rho \leqslant 1$, but Theorem 3.4.8 can show an improvement. We find that $F_{c_0}(a) = 0$ has two solutions $a_m = 0.0231$ and $a_M = 0.0626$. Thus, by Theorem 3.4.8, the system under this parameter set is linearly selected when $\rho \leqslant 1.5175$. To find numerical speeds c_{num} corresponding to different values of ρ, we use the Matlab software to compute the solution of (3.1.3), where the initial conditions are

$$u(x,0) = \frac{\frac{\mu}{\sigma}}{1+e^{10x}} \quad \text{and} \quad v(x,0) = \frac{1}{1+e^{10x}} \tag{3.4.17}$$

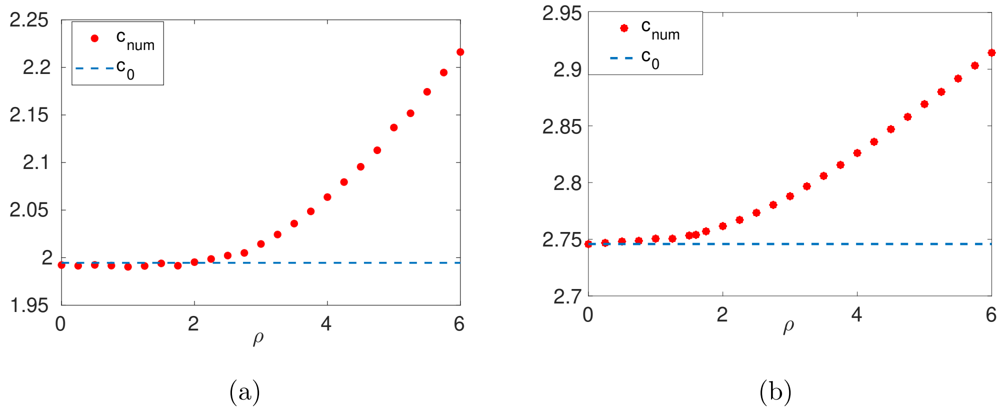


Figure 3.4: The relation between the spreading speed and ρ. (a) This figure is depicted when $d = 3$, $\epsilon = 0.1$, $\sigma = 3$, $\mu = 1$ and $\alpha = 1$. Here, $c_0 = 1.9945625$. (b) This figure is depicted when $d = 2$, $\epsilon = 0.2$, $\mu = 3$, $\sigma = 1$ and $\alpha = 2$. Here, $c_0 = 2.7458$.

such that they are steep enough to be close to the step functions. By [20, 91], the spreading speed of solutions with such initial data will evolve to $c_{\min}$, so our numerically computed c_{num}, obtained from the level set of the solution, would give an approximation to the minimal wave speed. The values of numerically computed speed are shown in Fig. 3.4 (a). As we can see, the critical value for ρ is $\bar{\rho} \simeq 2.2$. Furthermore, this result illustrates our theoretical results.

In the second example, we fix $d = 2$, $\epsilon = 0.2$, $\mu = 3$, $\sigma = 1$ and $\alpha = 2$. Under this choice of parameters, we can find that $c_0 = 2.7458$, $\lambda_0 = 0.3947$, and $A_1 = 3.0526$. Through a direct computation, it follows that $4d\lambda_0^2 - 2\lambda_0(c_0 - \alpha) - \sigma = -0.3424$, $5d\lambda_0 - (c_0 - \alpha) = 3.2012$, $-d\lambda_0^2 - \lambda_0(c_0 - \alpha) + \sigma = 0.3941$, $a_3 = -0.0259$, $a_4 = 5.6567$, and $a_5 = 4.6653$. Moreover, $h_{c_0} = 0$ has two solutions $a_1 = 0.0003$ and $a_2 = 1.4477$. Thus, (3.4.1) is linearly selected if $a \in [a_1, a_2]$ and $\rho \leqslant \sigma a_2 + 2\epsilon\lambda_0^2 = 1.51$ by Theorem 3.4.2. As for the nonlinear selection, by Theorem 3.4.5, we obtain that the nonlinear selection is realized if $a \geqslant a_4 = 5.6567$ and $\rho > 5.7190$. Using the same method as that in the first example, the numerical speeds (spreading speeds) are obtained and shown in Fig. 3.4 (b). As we can see in the figure, $\bar{\rho} \simeq 1.6$, which confirms our theoretical result.

Remark 3.4.9. *Finally, we would like to emphasize that the model here is completely different from the diffusive Lotka-Volterra competition system in [1, 2], where two species compete for the same resource. Here, we study a significant model describing* *a species in two different compartments or stages. The Allee effect appears in this model, while we cannot see this in [1, 2]. We focus here on how the spreading speed is impacted by the Allee strength. Furthermore, in [1, 2], the linearized system at* $(0, 0)$ *is decoupled so that the linear speed was given by a simple formula* $c_0 = 2\sqrt{1 - a_1}$*. For the construction of upper or lower solutions to the system, we can take* $V = kU$ *for different values of* k *or can assume that* V *admits different decay behavior than* U*. For the stream population model in this chapter, the linear system at* $(0, 0)$ *is irreducible and the linear speed is determined by an order-4 polynomial. No explicit formula* c_0 *can be obtained. To determine the spreading speed (the minimal speed), our numerical simulation indicates that the graph of* U/V *looks like a vertical parabola. This provides us insight to construct novel solution pairs with* $U/V = aV^2 + bV + c$*.*

3.5 Conclusion

In this chapter, we investigated the speed selection mechanism (linear and nonlinear) via the upper and lower solution method for traveling waves to a reaction-advection-diffusion model (3.1.3).

For this stream population model, we focus on how the spreading speed is impacted by the Allee effect. Here, the so-called asymptotic spreading speed (which represents a critical value of biological invasion) coincides with the minimal speed $c_{\min}$ of the traveling waves. However, its value is usually difficult to determine. We consider the case when the system is modeled with a weak Allee effect [84], i.e., with a growth function as $f(v) = v(1 - v)(1 + \rho v)$. For such a growth function, when $\rho > 1$ (i.e., $f(v) > f'(0)v$ when $v \in [0, 1]$), the per capita growth rate $(f(v)/v)$ of this species attains its maximum at an intermediate population size. The strength of the Allee effect increases in the parameter ρ. When $\rho = 0$, it reduces to the classical logistic growth. We are successful in establishing the relation between the spreading speed and the Allee effect. We also have proved that there exists a threshold value (a critical number) $\bar{\rho}$ to divide the speed selection. Specifically, our theoretical and numerical results show that the spreading speed is an increasing function of ρ. For given values of μ, σ, α, ϵ, d (through experiments), we can compute the linear speed c_0 and further estimate the threshold value $\bar{\rho}$ with analytic formulas.

In the novel construction of upper and lower solutions for the speed selection, we should emphasize that the parabolic formula for $\frac{U}{V}$ in terms of V is entirely new and totally different from the formula given in [1, 2] (where they only assumed a linear relation, i.e., $\frac{U}{V} = k$, and this idea doesn't work here). By this technique, we successfully establish explicit conditions for both the linear and the nonlinear selection: see Theorems 3.4.2 ($m = 1$) and 3.4.8 ($m = 2$) for the linear selection and Theorems 3.4.5-3.4.6 for the nonlinear selection.

We should also mention that all the coefficients of our main model are constant, but this is not essential in our method and idea. It can be interestingly extended to a more general case, such as where all the coefficients are time-periodic functions, and even with periodic habitats. Efforts on these aspects are currently in progress and will be presented in future publications.

3.6 Appendix

In this appendix, we will show the upper and lower solutions method in detail. This method is originated in [17, 85] and used to prove the existence of monotone traveling wave solutions to the partial differential equations. In the meantime, we can also apply it to derive the linear speed selection.

Let $\bar{M}_1$ be a sufficiently large positive number so that

$$F(U,V) = \sigma U - \mu V + f(V) + MV$$

is monotone in V. Then the wave equations in (3.1.5) are equivalent to

$$\begin{cases} dU'' + (c-\alpha)U' - \sigma U = -\mu V, \\ \epsilon V'' + cV' - MV = -F(U,V). \end{cases} \tag{3.6.1}$$

For the first equation, we have already solved it by (3.3.1). For the second equation, when $\epsilon > 0$, the integral form is given by

$$\begin{aligned} V(\xi) &= \frac{1}{\epsilon(\gamma_2-\gamma_1)}\left\{\int_{-\infty}^{\xi} e^{\gamma_1(\xi-s)}F(U(s),V(s))ds + \int_{\xi}^{+\infty} e^{\gamma_2(\xi-s)}F(U(s),V(s))ds\right\} \\ &=: T_2(U,V), \end{aligned} \tag{3.6.2}$$

where

$$\gamma_1 = \frac{c - \sqrt{c^2 + 4\epsilon M}}{2\epsilon} < 0 < \gamma_2 = \frac{c + \sqrt{c^2 + 4\epsilon M}}{2\epsilon}. \tag{3.6.3}$$

When $\epsilon = 0$,

$$V(\xi) = \frac{1}{c}\int_{\xi}^{+\infty} e^{\frac{M}{c}(\xi - s)} F(U(s), V(s))ds =: T_2(U, V). \tag{3.6.4}$$

Thus, the system (3.6.1) in an integral form reads

$$\begin{cases} U(\xi) = H(V) = T_1(U, V) \\ V(\xi) = T_2(U, V), \end{cases} \tag{3.6.5}$$

where $H(V)$ is defined by (3.3.1) and $T_2(U, V)$ is defined by (3.6.2) when $\epsilon > 0$ or (3.6.4) when $\epsilon = 0$. Then, with the integral form, we can define an upper (or a lower) solution.

Definition 3.6.1. *A pair of continuous functions $(U, V)(\xi)$ is an upper (a lower) solution to the integral system (3.6.5) if*

$$\begin{cases} U(\xi) \geqslant (\leqslant) T_1(U, V)(\xi), \\ V(\xi) \geqslant (\leqslant) T_2(U, V)(\xi). \end{cases}$$

Since the above integral forms are not practical in finding upper or lower solutions, we then give inequalities in terms of differential equations themselves that imply the Definition 3.6.1 in the following lemma.

Lemma 3.6.2. *A pair of continuous functions $(U, V)(\xi)$ which is differentiable on $\mathbb{R}$ except at finite numbers of points ξ_i, $i = 1, \cdots, n$, and satisfies*

$$\begin{cases} dU'' + (c - \alpha)U' - \sigma U + \mu V \leqslant 0, \\ \epsilon V'' + cV' + \sigma U - \mu V + f(V)V \leqslant 0, \end{cases}$$

for $\xi \neq \xi_i$, and $(U', V')(\xi_i^-) \geqslant (U', V')(\xi_i^+)$ for all ξ_i, is an upper solution to the integral *system (3.6.5). A lower solution can be defined by reversing all the inequalities.*

Proof. We give a proof for the upper solution, while a similar argument can be applied for the lower solution. From the above inequalities, we have

$$\begin{aligned} T_1(U,V)(\xi) &= \frac{\mu}{d(\tau_2-\tau_1)}\left\{\int_{-\infty}^{\xi} e^{\tau_1(\xi-s)}V(s)ds + \int_{\xi}^{\infty} e^{\tau_2(\xi-s)}V(s)ds\right\} \quad (3.6.6)\\ &\leqslant \frac{-\mu}{d(\tau_2-\tau_1)}\left\{\int_{-\infty}^{\xi} e^{\tau_1(\xi-s)}(dU''+(c-\alpha)U'-\sigma U)(s)ds\right.\\ &\quad \left.+\int_{\xi}^{\infty} e^{\tau_2(\xi-s)}(dU''+(c-\alpha)U'-\sigma U)(s)ds\right\}. \end{aligned}$$

By a similar calculation to that of [[55], proof of Lemma 2.5], we can show that

$$T_1(U,V)(\xi) \leqslant U(\xi).$$

The same result holds for $T_2(U,V)(\xi) \leqslant V(\xi)$. This implies that $(U,V)(\xi)$ is an upper solution to the system (3.6.5). The proof for the lower solution is the same and omitted. □

To move on to the upper and lower solutions method, we first assume the following hypothesis.

Hypothesis 3.6.3. *For a given $c > c_0$, assume there exists a monotone non-increasing upper solution $(\overline{U},\overline{V})(\xi)$ and a non-zero lower solution $(\underline{U},\underline{V})(\xi)$ to the system (3.6.5) with the following properties:*
(1) $(\underline{U},\underline{V})(\xi) \leqslant (\overline{U},\overline{V})(\xi)$, for all $\xi \in \mathbb{R}$;
(2) $(\overline{U},\overline{V})(+\infty) = (0,0)$ and $(\overline{U},\overline{V})(-\infty) = (\overline{k}_1,\overline{k}_2)$;
(3) $(\underline{U},\underline{V})(+\infty) = (0,0)$ and $(\underline{U},\underline{V})(-\infty) = (\underline{k}_1,\underline{k}_2)$, for $(0,0) \leqslant (\underline{k}_1,\underline{k}_2) \leqslant (\frac{\mu}{\sigma},1)$ and $(\overline{k}_1,\overline{k}_2) \geqslant (\frac{\mu}{\sigma},1)$ so that no other equilibrium solution to (3.1.5) exists in the set $\{(U,V)|(0,0) \leqslant (U,V) \leqslant (\overline{k}_1,\overline{k}_2)\}$.

Then, under the conditions of the above hypothesis, we can define an iteration scheme as

$$\begin{cases} (U_0, V_0) = (\overline{U}, \overline{V}), \\ U_{n+1} = T_1(U_n, V_n), \ n = 0, 1, 2, \cdots, \\ V_{n+1} = T_1(U_n, V_n), \ n = 0, 1, 2, \cdots. \end{cases} \tag{3.6.7}$$

At last, by the results in [17, 88], we can arrive at the following theorem, which shows the existence of an upper and a lower solution indicates the existence of the actual solution.

Theorem 3.6.4. *If Hypothesis 3.6.3 is true, then the iteration scheme (3.6.7) converges to a pair of non-increasing functions $(U, V)(\xi)$, which is a solution to the system (3.1.5) with $(U, V)(+\infty) = (0, 0)$ and $(U, V)(-\infty) = (\frac{\mu}{\sigma}, 1)$. Moreover, $(\underline{U}, \underline{V})(\xi) \leqslant (U, V)(\xi) \leqslant (\overline{U}, \overline{V})(\xi)$ for all $\xi \in \mathbb{R}$.*

Chapter 4

Spreading Speeds Determinacy for a Cooperative Lotka-Volterra System with Stacked Fronts

4.1 Introduction

The Lotka-Volterra cooperative model considered in this chapter is given by

$$\begin{cases} u_t = d_1 u_{xx} + r_1 u(1 - u + b_1 v), \\ v_t = d_2 v_{xx} + r_2 v(1 - v + b_2 u), \end{cases} \tag{4.1.1}$$

where all the parameters $(d_i,\ r_i,\ b_i)$ are positive with $b_1 b_2 < 1$. In the model, u and v stand for the population densities of two collaborated species at time $t > 0$ and location $x \in \mathbb{R}$; d_1 and d_2 are diffusion coefficients; r_1 and r_2 are the net birth rates; b_1 and b_2 represent the cooperation strengths. For applications of this model, see e.g., [34, 88]. It is easy to find four equilibria of (4.1.1) as

$$\mathbf{0} = (0, 0),\ \alpha_1 = (1, 0),\ \alpha_2 = (0, 1),\ \beta = \left(\frac{1 + b_1}{1 - b_1 b_2}, \frac{1 + b_2}{1 - b_1 b_2} \right) =: (e_1, e_2). \tag{4.1.2}$$

Among them, $\mathbf{0}$ is called the extinction state, α_1 and α_2 are the intermediate (or monoculture) states, and β is the co-existence state. Moreover, from its corresponding space-homogeneous ordinary differential system, i.e.,

$$\begin{cases} u' = r_1 u(1 - u + b_1 v), \\ v' = r_2 v(1 - v + b_2 u), \end{cases}$$

it is easy to see that $\mathbf{0}$ is unstable while β is stable; α_1 and α_2 are saddles.

Since this chapter focuses on the spreading phenomena of model (4.1.1), we follow the pioneering work [20, 41] and references therein to define the spreading speed(s) first. Let $\mathcal{C}$ denote the set of all bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}^2$, and $[\phi, \psi]_{\mathcal{C}} := \{w \in \mathcal{C} : \ \phi \leqslant w \leqslant \psi\}$. For more details including the ordering signs of $\leqslant$ and $\ll$, please see the aforementioned references.

Denote $\mathbf{w}_0(x) = (u_0, v_0)(x)$, and we know that system (4.1.1) generates a monotone semiflow $Q_t : [\mathbf{0}, \beta]_{\mathcal{C}} \to [\mathbf{0}, \beta]_{\mathcal{C}}$ defined by

$$Q_t[\mathbf{w}_0](x) = \mathbf{w}(t, x; \mathbf{w}_0) = (u, v)(t, x), \quad \forall (t, x) \in [0, +\infty) \times \mathbb{R},$$

where $\mathbf{w}(t, x; \mathbf{w}_0)$ is the unique solution of (4.1.1) satisfying $\mathbf{w}(0, \cdot; \mathbf{w}_0) = \mathbf{w}_0 \in [\mathbf{0}, \beta]_{\mathcal{C}}$. Let $\bar{\omega}$ be a vector with $0 \ll \bar{\omega} \ll \beta$ and choose the initial condition $\mathbf{w}_0$ satisfying: (a). $\mathbf{w}_0(x)$ is nonincreasing in x, (b). $\mathbf{w}_0(x) = 0$ for all $x \geqslant 0$, and (c). $\mathbf{w}_0(-\infty) = \bar{\omega}$. For a given real number c, a sequence $\{a_n(c; s)\}$ can be defined by the recursion

$$a_0(c; s) = \mathbf{w}_0(s), \ a_{n+1}(c; s) = \max\{\mathbf{w}_0(s), Q_1[a_n(c; \cdot)](s + c)\}.$$

It follows from [48] that this sequence converges to a continuous function $a(c; s)$ which is nonincreasing in both s and c with $a(c; -\infty) = \beta$, and $a(c; \infty)$ is an equilibrium of Q_1, i.e., $a(c; \infty) = \mathbf{0}, \ \alpha_1, \ \alpha_2$, or β.

Two critical numbers with biological implications can be defined as follows. The slowest spreading speed is

$$c^* = \sup\{c \in \mathbb{R} : \ a(c; \infty) = \beta\} \tag{4.1.3}$$

and it has the property such that for any $\epsilon > 0$,

$$\limsup_{t\to\infty,\ x\geqslant(c^*+\epsilon)t} \mathbf{w} < \beta \text{ and } \lim_{t\to\infty,\ x\leqslant(c^*-\epsilon)t} [\beta - \mathbf{w}] = 0. \tag{4.1.4}$$

The fastest spreading speed is

$$c_f^* = \sup\{c \in \mathbb{R} :\ a(c;\infty) > 0\} \tag{4.1.5}$$

and it has the property such that for any $\epsilon > 0$,

$$\lim_{t\to\infty,\ x\geqslant(c_f^*+\epsilon)t} \mathbf{w} = 0 \text{ and } \liminf_{t\to\infty,\ x\leqslant(c_f^*-\epsilon)t} \mathbf{w} > 0, \tag{4.1.6}$$

see e.g., [20,48,88]. It is clear that $c^* \leqslant c_f^*$. We call that a single spreading speed exists if $c^* = c_f^*$, which definitely will happen for a system without other equilibria between $\mathbf{0}$ and β, and multiple spreading speeds exist if $c^* < c_f^*$. Moreover, by [20,48], it follows that c^* and c_f^* are independent of the choice of the initial function. In population invasion, the existence of a single spreading speed means that all species invade the inhabited area at the same speed c^*, while the existence of multiple spreading speeds (i.e., $c^* < c_f^*$) can be interpreted as follows: no species spreads more slowly than c^* and at least one spreads at this speed, and no species spreads more quickly than c_f^* and at least one is equal to the speed c_f^*.

The above two speeds are related to an important biological phenomenon: traveling wave fronts. A traveling wave front of (4.1.1) is a special pattern-moving solution with the form

$$(u,v)(t,x) = (U,V)(z),\ z = x - ct. \tag{4.1.7}$$

Here, $U,\ V$ are moving profile that is nonincreasing in z, and $c \in \mathbb{R}$ is the wave speed to be determined. The system for the wave profile $(U,V)(z)$ $(z = x - ct)$ can be easily given by

$$\begin{cases} d_1U'' + cU' + r_1U(1 - U + b_1V) = 0, \\ d_2V'' + cV' + r_2V(1 - V + b_2U) = 0. \end{cases} \tag{4.1.8}$$

In terms of Q_t, the traveling wave $\mathbf{W}(x) = (U,V)(x)$ with speed c satisfies

$$Q_t[\mathbf{W}](x) = \mathbf{W}(x - ct) = (U,V)(x - ct), \quad \forall x \in \mathbb{R},\ t \geqslant 0. \tag{4.1.9}$$

Moreover, we say that this traveling wave connecting β if $(U,V)(-\infty) = \beta$. Since there are other equilibria between 0 and β, it is uncertain which equilibrium will be reached when $z \to \infty$. This means that $(U,V)(+\infty)$ could be α_1 or α_2 or $\mathbf{0}$ if no further conditions are restricted.

Following Theorem 4.2 in [20], the existence of traveling wave of (4.1.1) can be obtained as follows.

Theorem 4.1.1 (Theorem 4.2 [20])**.** *Let c^* and c^*_f be defined as in (4.1.4) and (4.1.6), respectively. Then the following statements are true:*

(1) For any $c \geqslant c^*$, (4.1.1) has a traveling wave solution $(U,V)(x-ct)$ connecting β to some fixed point β_1 except β, i.e., β_1 is one of $\mathbf{0}$, α_1, and α_2.

(2) For any $c \geqslant c^*_f$, either of the following holds:

(i) (4.1.1) has a traveling wave solution $(U,V)(x-ct)$ connecting β to $\mathbf{0}$;

(ii) (4.1.1) has a traveling wave solution $(U_1,V_1)(x-ct)$ connecting β to α and a traveling wave $(U_2,V_2)(x-ct)$ connecting α to $\mathbf{0}$, where $\alpha = \alpha_i$ ($i = 1$ or 2).

(3) For any $c < c^*$, (4.1.1) has no traveling wave connecting β, and for any $c < c^*_f$, there is no traveling wave connecting β to $\mathbf{0}$.

We would like to review important past applications on the model (4.1.1). Li, Weinberger, and Lewis ([41], Example 4.1) studied the spreading speeds c^* and c^*_f, and the existence of traveling wave solutions (as a typical example) under a strong condition where $d_1r_1 > d_2r_2e_2$ so that $c^* < c^*_f$. Later, Lin, Li, and Ma ([46], Theorem 5.11) proved the existence of traveling waves connecting β to $\mathbf{0}$ with $c > 2\sqrt{d_1r_1}$ plus some further conditions. More recently, Lin ([45], Theorem 3.1) showed $c^*_f = 2\sqrt{d_1r_1}$ and $c^* \geqslant 2\sqrt{d_2r_2(1+b_2)}$ if $d_1r_1 > d_2r_2e_2$, and also pointed out that there exist traveling waves, connecting β and $\mathbf{0}$, for $c > 2\sqrt{d_1r_1}$, if $d_1 \geqslant d_2$ with $r_1 \geqslant r_2$, or if $d_1 = d_2$ with $r_2(1+b_2) \geqslant r_1 > r_2$. For spreading speed determinacy to the Lotka-Volterra competitive model as well as cooperative systems with stacked fronts, we refer to [1, 2, 28, 32–34, 53, 88].

The purpose of this chapter is to investigate the speed selection mechanism for (4.1.1) either with a single spreading speed or with multiple spreading speeds. Throughout this chapter, we assume that

$$d_1r_1 \geqslant d_2r_2, \tag{4.1.10}$$

since the dynamics will be similar if the inequality is reversed. By relating the fastest and slowest spreading speeds to each species' individual spreading speed, under the condition (4.1.10), we find that the invasion speed of v-species is always not faster than that of u-species. A necessary and sufficient condition is established for the existence of a single common speed. The results show that the invasion speed of u is faster than that of v if and only if the spreading speed $c^*_{\alpha_1,\beta}$ of the system, confined to the phase space between α_1 and β, is less than $2\sqrt{d_1 r_1}$. We develop the theory of speed selection separately for each case and our results provide the determinacy of each spreading speed, no matter whether they are equal or not. The numerical simulations not only demonstrate our theoretical discovery, but also indicate new and interesting phenomena that show the existence of terrace-type wave patterns.

Our speed selection mechanism can help us greatly understand the movement of stacked fronts, an interesting phenomenon originally observed from combustion theory in [36] and mathematically investigated in [34]. In the case when $\hat{c}^*$ (the spreading speed for the system confined to the phase space $[\alpha_1, \beta]_{\mathcal{C}}$) is strictly less than $2\sqrt{d_1 r_1}$, stacked fronts (in the first species) consist of two parts that can move with different speeds so that the upper part is slower than the lower part (see also the numerical simulation Figure 4.1). In the case when $\hat{c}^* = 2\sqrt{d_1 r_1}$, it is found a stacked wavefront with upper and lower parts moving with the same speed that can be sometimes determined by the linear speed $2\sqrt{d_1 r_1}$. This results in the formation of a wavefront with a terrace, when the initial data are properly assigned, see Figure 4.7 in the simulation. On the other hand, when $\hat{c}^* > 2\sqrt{d_1 r_1}$, no existence of terrace can be found and the whole solution will finally evolve into a traveling wave connecting β and zero, with a common speed that can be linearly determined by $2\sqrt{d_1 r_1}$ (see e.g., Figure 4.8).

The rest of this chapter is organized as follows. Section 4.2 is devoted to the study of the individual spreading speed of each species and some preliminaries related to our model. Section 4.3 provides a necessary and sufficient condition to decide whether (4.1.1) has a single spreading speed or multiple spreading speeds. The determinacy of multiple spreading speeds is provided in Section 4.4, while the selection of a single spreading speed is presented in Section 4.5. Numerical results will be provided in Section 4.6. The last section contains further discussions.

4.2 Individual spreading speed

To better understand c^* and c_f^*, we introduce the definition of the spreading speed of each species. For model (4.1.1) with initial data satisfying (a), (b), and (c), the spreading speed c_u^* of the species $u(t,x)$ is a constant such that

$$\liminf_{t\to\infty,\ x\leqslant(c_u^*-\epsilon)t} u(t,x)>0,\quad \lim_{t\to\infty,\ x\geqslant(c_u^*+\epsilon)t} u(t,x)=0,\quad \forall\ \epsilon>0, \tag{4.2.1}$$

and similarly, the spreading speed c_v^* of $v(t,x)$ is a constant such that

$$\liminf_{t\to\infty,\ x\leqslant(c_v^*-\epsilon)t} v(t,x)>0,\quad \lim_{t\to\infty,\ x\geqslant(c_v^*+\epsilon)t} v(t,x)=0,\quad \forall\ \epsilon>0, \tag{4.2.2}$$

Remark 4.2.1. *In the above definitions, it implies that there exist two small positive constants $\eta_u,\ \eta_v>0$ such that*

$$\lim_{t\to\infty,\ x\leqslant(c_u^*-\epsilon)t} u(t,x)\geqslant\eta_u,\quad \lim_{t\to\infty,\ x\leqslant(c_v^*-\epsilon)t} v(t,x)\geqslant\eta_v,\quad \forall\ \epsilon>0.$$

According to their definitions, the fastest and slowest spreading speeds of (4.1.1) can be related to the two individual spreading speeds in the following proposition.

Proposition 4.2.2.

$$c_f^*=\max\{c_u^*,\ c_v^*\},\ c^*=\min\{c_u^*,\ c_v^*\}. \tag{4.2.3}$$

If $c_u^*=c_v^*$, then (4.1.1) has a single spreading speed, i.e., $c^*=c_f^*$. Otherwise, it has multiple spreading speeds, i.e., $c^*<c_f^*$.

The proof of this proposition is straightforward and we omit it here.

If we forget about the condition (4.1.10), we have the following lemma.

Lemma 4.2.3. *Without the restriction (4.1.10), the following statements hold.*

(1) If $c_u^*>c_v^*$, then for $c\in[c_v^*,c_u^*)$, (4.1.1) has a traveling wave $(U,V)(x-ct)$ connecting β to $\alpha_1=(1,0)$. Furthermore, $c_u^*=2\sqrt{d_1r_1}$.

(2) If $c_v^*>c_u^*$, then for $c\in[c_u^*,c_v^*)$, (4.1.1) has a traveling wave $(U,V)(x-ct)$

connecting β *to* $\alpha_2 = (0,1)$*. Moreover,* $c_v^* = 2\sqrt{d_2 r_2}$*.*

(3) There is an estimate for c_u^* *and* c_v^* *as:* $2\sqrt{d_1 r_1 e_1} \geqslant c_u^* \geqslant 2\sqrt{d_1 r_1}$ *and* $2\sqrt{d_2 r_2 e_2} \geqslant c_v^* \geqslant 2\sqrt{d_2 r_2}$*.*

Proof. We start with proving (1). Under the condition of (1), by Proposition 4.2.2 and Theorem 4.1.1, we know that for any $c \in [c_v^*, c_u^*)$, (4.1.1) has a traveling wave $\mathbf{W}(x-ct) = (U,V)(x-ct)$ with $\mathbf{W}(-\infty) = \beta$ and $\mathbf{W}(+\infty) = \beta_1 (< \beta)$ (being an equilibrium). Next, we want to show $\beta_1 = \alpha_1$.

Now, choose $\mathbf{w}_0(x) = (u_0, v_0)(x)$ with the properties (a), (b), and (c) such that

$$\mathbf{w}_0(x) \leqslant \mathbf{W}(x),\ x \in \mathbb{R}.$$

Then, by comparison, we obtain that

$$(u,v)(t,x) = Q_t[\mathbf{w}_0](x) \leqslant Q_t[\mathbf{W}](x) = \mathbf{W}(x-ct),$$

Let $\epsilon = c_u^* - c$. Thus, for $x = (c + \frac{\epsilon}{2})t = (c_u^* - \frac{\epsilon}{2})t$ and $t \to \infty$, we have

$$\beta_1 = \mathbf{W}(+\infty) \geqslant \lim_{t\to+\infty,\ x=(c_u^*-\frac{\epsilon}{2})t} Q_t[\mathbf{w}_0](x).$$

On the other hand, (4.2.1) with its remark shows that

$$\liminf_{t\to\infty,\ x=(c_u^*-\frac{\epsilon}{2})t} u(t,x) \geqslant \eta_u > 0. \tag{4.2.4}$$

This implies that the u-coordinate of β_1 is always positive when $c \in [c_v^*, c_u^*)$. Combining with $\beta_1 < \beta$, it immediately follows that $\beta_1 = \alpha_1 = (1,0)$. This proves the first statement of (1).

To prove the second statement of (1), we consider the case where $c > c_v^*$, i.e., $c = c_v^* + \epsilon$ for some $\epsilon > 0$. Formula (4.2.2) indicates that $\lim\limits_{t\to\infty,\ x\geqslant(c_v^*+\epsilon)t} v(t,x) = 0$. Thus, with $t \to \infty$ and $x \geqslant (c_v^* + \epsilon)t$, we have a limiting system associated with

(4.1.1) (in which $v \equiv 0$) as follows:

$$u_t = d_1 u_{xx} + r_1 u(1-u). \tag{4.2.5}$$

This is the famous Fisher-KPP equation. It is well-known that (4.2.5) has a spreading speed $c_l^* = 2\sqrt{d_1 r_1}$ and there exists also traveling wave $U_1(x-ct)$ connecting 1 to 0 if and only if $c \geqslant c_l^* = 2\sqrt{d_1 r_1}$. Therefore, $c_u^* = c_l^* = 2\sqrt{d_1 r_1}$, which verifies the second statement.

As to part (2), the proof is similar and omitted.

For part (3), as in the proof of (1), we already know that $c_u^* \geqslant 2\sqrt{d_1 r_1}$ via linearization. Since $v \leqslant e_2$ is always true, by the comparison principle, u cannot spread more quickly than the wave generated by letting $v \equiv e_2$ in the first equation of (4.1.1). The resulting equation is again a Fisher-KPP equation. Thus, $c_u^* \leqslant 2\sqrt{d_1 r_1(1+b_1 e_2)} = 2\sqrt{d_1 r_1 e_1}$. In summary, the spreading speed of u has the range $2\sqrt{d_1 r_1} \leqslant c_u^* \leqslant 2\sqrt{d_1 r_1 e_1}$.

Because $0 \leqslant u \leqslant e_1$, by letting $u = 0$ (or $u = e_1$) in the second equation of (4.1.1) and using the comparison principle, we can similarly show that $2\sqrt{d_2 r_2} \leqslant c_v^* \leqslant 2\sqrt{d_2 r_2 e_2}$. This completes the proof. □

In this chapter, since the parameter condition (4.1.10) is assumed to be true, we can further have the following lemma.

Lemma 4.2.4. *When (4.1.10) holds (i.e., $d_1 r_1 \geq d_2 r_2$), it always follows that*

$$c_u^* \geqslant c_v^*. \tag{4.2.6}$$

Proof. By contradiction, we assume $c_u^* < c_v^*$. Then in view of Lemma 4.2.3 (2), we find that $c_v^* = 2\sqrt{d_2 r_2}$. This leads to a contradiction since the inequality $2\sqrt{d_1 r_1} \leqslant c_u^* < c_v^* = 2\sqrt{d_2 r_2}$ disagrees with our condition (4.1.10). The proof is complete. □

Remark 4.2.5. *From the above lemma, the condition (4.1.10) implies that u-species spreads more quickly than (at least the same speed as) v-species. Biologically, it means* *that u-species has a stronger intrinsic spreading ability. Similarly, reversing (4.1.10) reserves (4.2.6).*

Next, we introduce the definition of speed determinacy (selection). Since (4.1.1) indeed admits either a single spreading speed, or multiple spreading speeds, we will divide our analysis of them separately.

For the single spreading speed case, i.e., $c_u^* = c_v^*$, there always exists a traveling wave connecting zero.

Through linearizing (4.1.8) around $\mathbf{0}$, we obtain a reducible system (whose definition can be found in [88])

$$\begin{cases} d_1 U'' + cU' + r_1 U = 0, \\ d_2 V'' + cV' + r_2 V = 0. \end{cases}$$

For the first equation, a direct analysis shows that

$$c_0^u = 2\sqrt{d_1 r_1}, \tag{4.2.7}$$

where c_0^u is the minimal speed so that the first equation has nonnegative traveling wave solutions. By comparison (see [43]), it always follows that

$$c_u^* \geqslant c_0^u. \tag{4.2.8}$$

Similarly, from the second equation, we have that

$$c_v^* \geqslant c_0^v = 2\sqrt{d_2 r_2}, \tag{4.2.9}$$

where c_0^v is the minimal speed so that the second equation has nonnegative traveling wave solutions.

Now, we are ready to give the definition of speed selection corresponding to the single spreading speed case, based on the linearized system at zero.

Definition 4.2.6. *When (4.1.1) has a single spreading speed, we say that c_u^* is linearly selected if $c_u^* = c_0^u$, and nonlinearly selected if $c_u^* > c_0^u$; c_v^* is linearly selected if $c_v^* = c_0^v$, and nonlinearly selected if $c_v^* > c_0^v$.*

Remark 4.2.7. *Under the condition $d_1r_1 > d_2r_2$, we see that $c_0^u > c_0^v$. Thus, if $c_v^* = c_u^*$, we can always have that c_v^* is nonlinearly selected due to $c_v^* = c_u^* \geqslant c_0^u > c_0^v$.*

For the case where $c_u^* > c_v^*$, by Lemma 4.2.3, we know that when $c \in [c_v^*, c_u^*)$, (4.1.1) has a traveling wave $(U,V)(x-ct)$ with $(U,V)(-\infty) = \beta$ and $(U,V)(+\infty) = \alpha_1$. To indicate such a wave, we introduce an auxiliary system by using the change of variables $w = u - 1$ and $v = v$. Hence the auxiliary system can be obtained as

$$\begin{cases} w_t = d_1 w_{xx} + r_1(w+1)(-w + b_1 v), \\ v_t = d_2 v_{xx} + r_2 v(1 + b_2 - v + b_2 w). \end{cases} \tag{4.2.10}$$

Clearly, the auxiliary system has only two nonnegative equilibria

$$\hat{\beta} = (e_1 - 1, e_2) \text{ and } \mathbf{0} = (0,0).$$

Then, following a similar analysis for the original system (4.1.1), (4.2.10) generates a monotone semiflow $\hat{Q}_t : [\mathbf{0}, \hat{\beta}]_{\mathcal{C}} \to [\mathbf{0}, \hat{\beta}]_{\mathcal{C}}$ defined by

$$\hat{Q}_t[\hat{\mathbf{w}}_0](x) = \hat{\mathbf{w}}(t,x) = (w,v)(t,x),$$

where $\hat{\mathbf{w}}_0(x) = \hat{\mathbf{w}}(0,x)$. By [43] (or [20, 41]), it follows that (4.2.10) has a single spreading speed $\hat{c}^*$ (or $c^*_{\alpha_1,\beta}$) defined as

$$\hat{c}^* = \sup\{c \in \mathbb{R} : \ \hat{a}(c;\infty) > 0\}, \tag{4.2.11}$$

where

$$\hat{a}_0(c;s) = \hat{\mathbf{w}}_0(s), \ \hat{a}_{n+1} = \max\{\hat{\mathbf{w}}_0(s), \hat{Q}_1[\hat{\mathbf{a}}_n(c;\cdot)](s+c)\},$$

$\hat{a}(c;s) = \lim\limits_{n\to\infty} \hat{a}_n(c;s)$, and $\hat{\mathbf{w}}_0(s)$ satisfies (a), (b), and (c) with β replaced by $\hat{\beta}$. Furthermore, by the proof of Lemma 2.8 of [43], it then follows that $\hat{c}^*$ does not depend on the choice of $\hat{\mathbf{w}}_0(s)$ as long as it satisfies conditions (a), (b), and (c). For any $\epsilon > 0$, this spreading speed $\hat{c}^*$ has properties

$$\lim_{t\to\infty,\ x\geqslant(\hat{c}^*+\epsilon)t} \hat{\mathbf{w}}(t,x) = 0, \text{ and } \lim_{t\to\infty,\ x\leqslant(\hat{c}^*-\epsilon)t} |\hat{\beta} - \hat{\mathbf{w}}(t,x)| = 0. \tag{4.2.12}$$

The wave profile corresponding to system (4.2.10) is

$$(w, v)(t, x) = (W, V)(z),\ z = x - ct,$$

and the corresponding wave profile system can be obtained as

$$\begin{cases} d_1 W'' + cW' + r_1(W+1)(-W + b_1 V) = 0, \\ d_2 V'' + cV' + r_2 V(1 + b_2 - V + b_2 W) = 0, \end{cases} \tag{4.2.13}$$

subject to

$$(W, V)(-\infty) = \hat{\beta},\ (W, V)(\infty) = \mathbf{0}. \tag{4.2.14}$$

Such a wave solution with speed c satisfies

$$\hat{Q}_t[(W, V)](x) = (W, V)(x - ct).$$

Similar to Example 4.1 in [41], by Proposition 4.2.2 and Lemma 4.2.3, we have the following lemma to manifest the exact relation between $\hat{c}^*$ and c_v^*.

Lemma 4.2.8. *Under conditions (4.1.10) and $c_u^* > c_v^*$, we have*

$$c_v^* = \hat{c}^*, \tag{4.2.15}$$

where $\hat{c}^*$ is the spreading speed of the auxiliary system (4.2.10).

This lemma implies that we can use the information on $\hat{c}^*$ to further study the speed selection for c_v^*. A standard linearization analysis of (4.2.13) around $\mathbf{0}$ shows that

$$\hat{c}^* \geqslant 2\sqrt{d_2 r_2(1 + b_2)} =: c_{\alpha_1}^v, \tag{4.2.16}$$

where $c_{\alpha_1}^v$ is the minimal speed such that the corresponding linear system has a non-negative traveling wave solution.

We continue to find c_u^*. Under the condition $c_u^* > c_v^*$, c_u^* is determined by the limiting system

$$u_t = d_1 u_{xx} + r_1 u(1 - u)$$

which has a linear speed

$$c_0^u = c_l^* = 2\sqrt{d_1 r_1}, \tag{4.2.17}$$

where c_l^* is the spreading speed of the system (4.2.5).

Now, with the understanding of c_v^*, c_u^* and their corresponding linear speeds $c_{\alpha_1}^v$ and c_0^u, we can give the definition of speed selection when (4.1.1) has multiple spreading speeds.

Definition 4.2.9. *If (4.1.1) has multiple spreading speeds (i.e., $c_u^* > c_v^*$) and let $c_{\alpha_1}^v$ and c_0^u be defined in (4.2.16) and (4.2.17), respectively, we say that c_u^* is linearly selected if $c_u^* = c_0^u$ and nonlinearly selected if $c_u^* > c_0^u$, c_v^* is linearly selected if $c_v^* = c_{\alpha_1}^v$ and nonlinearly selected if $c_v^* > c_{\alpha_1}^v$.*

From the proof of Lemma 4.2.3, we have the following lemma to determine c_u^*.

Lemma 4.2.10. *Under conditions (4.1.10) and $c_u^* > c_v^*$, we have that*

$$c_u^* = c_l^* = 2\sqrt{d_1 r_1}, \tag{4.2.18}$$

where c_l^* is the spreading speed of the limiting system (4.2.5).

Remark 4.2.11. *Based on the above definition and lemma, it immediately follows that, if $c_u^* > c_v^*$, then c_u^* is always linearly selected, since $c_u^* = 2\sqrt{d_1 r_1} = c_0^u$,*

For the reader's convenience, we provide the following definition of an upper (or a lower) solution to (4.1.1).

Definition 4.2.12. *Assume that a pair of continuous functions $(U, V)(z)$, $z = x - ct$, is twice continuously differentiable on $\mathbb{R}$ except for finite m points z_i with*

$$U'(z_i^+) \leqslant U'(z_i^-),\ V'(z_i^+) \leqslant V'(z_i^-),\ i = 1, 2, \cdots, m, \tag{4.2.19}$$

and satisfy

$$\begin{cases} L_u[U,V] := d_1U'' + cU' + r_1U(1-U+b_1V) \leqslant 0, \\ L_v[U,V] := d_2V'' + cV' + r_2V(1-V+b_2U) \leqslant 0, \ \forall z \in \mathbb{R} \setminus \{z_i\}, \ i = 1,2,\cdots,m. \end{cases} \tag{4.2.20}$$

Then, this pair of functions is called an upper solution to (4.1.1). A lower solution is defined by reversing all the aforementioned inequalities.

Similarly, for the auxiliary system (4.2.10), an upper (or a lower) solution is obtained by replacing $L_u[U,V]$ and $L_v[U,V]$ with $\hat{L}_w[W,V]$ and $\hat{L}_v[W,V]$, respectively, where

$$\begin{cases} \hat{L}_w[W,V] := d_1W'' + cW' + r_1(W+1)(-W+b_1V), \\ \hat{L}_v[W,V] := d_2V'' + cV' + r_2V(1+b_2-V+b_2W). \end{cases} \tag{4.2.21}$$

4.3 Single or multiple spreading speeds: a necessary and sufficient condition

As our main model (4.1.1) may admit single spreading speed or multiple spreading speeds, we need to decide when $c_u^* = c_v^*$ or $c_u^* > c_v^*$. We will make use of the spreading speed $\hat{c}^*$ from the auxiliary system (4.2.10) to attack the problem.

A necessary and sufficient condition is obtained via the following theorem.

Theorem 4.3.1. *System (4.1.1) has multiple spreading speeds (i.e., $c_u^* > c_v^*$), if and only if $\hat{c}^* < 2\sqrt{d_1r_1}$, where $\hat{c}^*$, defined in (4.2.11), is the spreading speed for the system confined to the phase space $[\alpha_1,\beta]_{\mathcal{C}}$.*

Proof. If $c_u^* > c_v^*$, from Lemmas 4.2.3 and 4.2.8, we immediately obtain that

$$c_v^* = \hat{c}^* \text{ and } c_u^* = 2\sqrt{d_1r_1}. \tag{4.3.1}$$

The necessity is clear.

To prove the sufficiency, we use a contradiction argument. Suppose that $c_u^* = c_v^*$. By (4.2.7)-(4.2.9), we have that $c_v^* = c_u^* \geqslant 2\sqrt{d_1 r_1}$. By the definition of $\hat{c}^*$, we know that the auxiliary system (4.2.10) has a traveling wave $(W(x-\hat{c}^*t), V(x-\hat{c}^*t))$. Back to (4.1.1), it has a traveling wave $(U(x-\hat{c}^*t), V(x-\hat{c}^*t))$ satisfying $V(+\infty) = 0$. By comparison, this implies that $\hat{c}^* \geqslant c_v^*$. This contradicts the condition $\hat{c}^* < 2\sqrt{d_1 r_1}$; the proof is complete. $\square$

Next, we will find some specific sufficient conditions to determine whether a single or multiple spreading speeds exist. The result is shown in the following theorem.

Theorem 4.3.2. *Let c_u^*, c_v^*, $\hat{c}^*$, and $c_{\alpha_1}^v$ be defined in (4.2.1), (4.2.2), (4.2.11), and (4.2.16), respectively. The following statements for (4.1.1) hold.*

(I) If $d_1 r_1 > d_2 r_2 e_2$, then $c_u^* > c_v^*$.

(II) Suppose $d_2 r_2 e_2 \geqslant d_1 r_1 > 2 d_2 r_2 (1 + b_2)$. It follows that

(i) if $\hat{c}^* = c_{\alpha_1}^v$, then $c_u^* > c_v^*$;

(ii) if $\hat{c}^* > c_{\alpha_1}^v$ but it has an upper bound $\hat{c}_2$ satisfying $2\sqrt{d_1 r_1} > \hat{c}_2 > c_{\alpha_1}^v$, then $c_u^* > c_v^*$;

(iii) if $\hat{c}^* > c_{\alpha_1}^v$ and it has a lower bound $\hat{c}_1$ satisfying $\hat{c}_1 \geqslant 2\sqrt{d_1 r_1}$, then $c_u^* = c_v^*$.

(III) If $d_2 r_2 (1 + b_2) \geqslant d_1 r_1$, then $c_u^* = c_v^*$.

Proof. For (I), the result directly follows from Lemma 4.2.3 (3), since from the lemma, we have that

$$c_u^* \geqslant 2\sqrt{d_1 r_1} > 2\sqrt{d_2 r_2 e_2} \geqslant c_v^*.$$

Due to the necessary and sufficient condition found in Theorem 4.3.1, the proof of (II) is relatively straightforward. For (II)(i), it means that $\hat{c}^* = 2\sqrt{d_2 r_2 (1 + b_2)} < 2\sqrt{d_1 r_1}$. Therefore, multiple spreading speeds occur. Similarly, (II)(ii) and (II)(iii) can be obtained through Theorem 4.3.1.

The proof to (III) is also straightforward. Because $\hat{c}^* \geqslant 2\sqrt{d_2 r_2 (1 + b_2)}$ (i.e., (4.2.16)), combining the condition of (III), we have that $\hat{c}^* \geqslant 2\sqrt{d_1 r_1}$. Then, by Theorem 4.3.1, it is clear that $c_v^* = c_u^*$. $\square$

Remark 4.3.3. *In the above lemma, we list (II)(i) separately from (II)(ii) since the equality* $\hat{c}^* = c^v_{\alpha_1}$ *means that* $\hat{c}^*$ *is linearly selected when the auxiliary system (4.2.10) is considered. It is worth to be discussed individually.*

Remark 4.3.4. *If* $d_1 r_1 = d_2 r_2$*, a single spreading speed always exists by (III) of the above theorem.*

4.4 Determinacy of multiple spreading speeds

In this section, we study the determinacy of individual spreading speed when multiple spreading speeds exist. We will use the upper and lower solutions method to do the speed selection analysis. Since the construction of an upper or a lower solution is based on the asymptotic behaviors near the unstable equilibrium, we start with investigating the corresponding linear system.

Linearizing (4.2.13) near $\mathbf{0}$ gives

$$\begin{cases} d_1 W'' + cW' - r_1 W + r_1 b_1 V = 0, \\ d_2 V'' + cV' + r_2(1+b_2)V = 0. \end{cases} \tag{4.4.1}$$

Let $(W,V)(z) = (C_w, C_v)e^{-\hat{\mu} z}$, where $C_w,\ C_v$ are nonnegative constants, and $\hat{\mu} > 0$. Substituting it into the above linear system produces an eigenproblem

$$\begin{pmatrix} d_1\hat{\mu}^2 - c\hat{\mu} - r_1 & r_1 b_1 \\ 0 & d_2\hat{\mu}^2 - c\hat{\mu} + r_2(1+b_2) \end{pmatrix} \begin{pmatrix} C_w \\ C_v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.4.2}$$

Setting the right-bottom diagonal element to be zero, we get

$$\hat{\mu}_1(c) = \frac{c - \sqrt{c^2 - 4d_2 r_2(1+b_2)}}{2d_2},\ \hat{\mu}_2(c) = \frac{c + \sqrt{c^2 - 4d_2 r_2(1+b_2)}}{2d_2}, \tag{4.4.3}$$

and we have that $C_w = C_v \cdot \frac{-r_1 b_1}{d_1\hat{\mu}_i^2 - c\hat{\mu}_i - r_1}$, where $i = 1, 2$.

Letting the top-left diagonal element equal to zero, we obtain a positive root

$$\hat{\mu}_3(c) = \frac{c + \sqrt{c^2 + 4d_1 r_1}}{2d_1}. \tag{4.4.4}$$

To have a positive solution (W, V), we need $c \geqslant 2\sqrt{d_2 r_2(1+b_2)}$, that is, $c \geqslant c^v_{\alpha_1}$. When $c \geqslant c^v_{\alpha_1}$, it is easy to see that $0 < \hat{\mu}_1(c) \leqslant \hat{\mu}_2(c)$ with $\hat{\mu}_1(c)$ being decreasing in c while $\hat{\mu}_2(c)$ being increasing; $\hat{\mu}_3(c) > 0$ is always increasing in c. Then, for simplicity, we can set $C_v = 1$ or $C_v = 0,\ C_w = 1$. Thus, for $c > c^v_{\alpha_1}$, the decaying positive solution (W, V) has following asymptotic behavior:

$$\begin{pmatrix} W \\ V \end{pmatrix} \sim \hat{C}_1 \begin{pmatrix} k_1(\hat{\mu}_1) \\ 1 \end{pmatrix} e^{-\hat{\mu}_1 z} + \hat{C}_2 \begin{pmatrix} k_1(\hat{\mu}_2) \\ 1 \end{pmatrix} e^{-\hat{\mu}_2 z} + \hat{C}_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\hat{\mu}_3 z}, \quad \text{as } z \to +\infty, \tag{4.4.5}$$

where

$$k_1(\hat{\mu}_i) = -\frac{r_1 b_1}{d_1 \hat{\mu}_i^2 - c\hat{\mu}_i - r_1}, \ i = 1, 2, \tag{4.4.6}$$

and $\hat{C}_1 > 0$ or $\hat{C}_1 = 0,\ \hat{C}_2,\ \hat{C}_3 > 0$. Here, in (4.4.5), we assume that $\hat{\mu}_1,\ \hat{\mu}_2$ and $\hat{\mu}_3$ are not equal. If two of them are equal, then a similar but modified formula can be derived. The result follows from the standard phase plane analysis (see, e.g., [28, 35, 60]). Alternatively, the method of successive approximation (see, e.g., [52]) can be used to prove (4.4.5), and we leave it to interested readers.

Next, we will use the well-known upper and lower solutions method to investigate the classification of the speed selection for the auxiliary system (4.2.10).

Lemma 4.4.1. *For the spreading speed $\hat{c}^*$ of system (4.2.10), the following statements are true.*

(1) $\hat{c}^*$ is linearly selected (i.e., $\hat{c}^* = c^v_{\alpha_1}$), if for $c = c^v_{\alpha_1}$, there exists a pair of continuous, positive and nonincreasing functions $(\overline{W}, \overline{V})(z)$ being an upper solution to (4.2.13)-(4.2.14) and satisfying

$$\lim_{z\to-\infty} (\overline{W}, \overline{V}) \geqslant \hat{\beta} \ \text{and} \ \lim_{z\to\infty} (\overline{W}, \overline{V}) = (0, 0). \tag{4.4.7}$$

(2) $\hat{c}^*$ is nonlinearly selected with $\hat{c}^* \geqslant \hat{c}_1 > c^v_{\alpha_1}$, if for $c = \hat{c}_1$, there exists a pair of continuous, nonnegative and nonincreasing functions $(\underline{W}, \underline{V})(z)$ being a lower solution and satisfying

$$\lim_{z\to-\infty} (\underline{W}, \underline{V}) \ll \hat{\beta}, \ \text{and} \ \underline{V} \sim e^{-\hat{\mu}_2 z} \ \text{as} \ z \to \infty, \tag{4.4.8}$$

where $\hat{\mu}_2 = \hat{\mu}_2(\hat{c}_1)$ is defined in (4.4.3).

(3) $\hat{c}^*$ has an upper bound by $\hat{c}^* \leqslant \hat{c}_2$ with $\hat{c}_2 > c^v_{\alpha_1}$, if there exists a pair of continuous

and positive functions $(\overline{W}_2, \overline{V}_2)(x - \hat{c}_2 t)$ *being an upper solution and satisfying*

$$\lim_{z\to-\infty}(\overline{W}_2, \overline{V}_2) \geqslant \hat{\beta} \text{ and } \overline{V}_2 \sim e^{-\hat{\mu}_2 z}, \text{ as } z \to \infty,$$

where $\hat{\mu}_2 = \hat{\mu}_2(\hat{c}_2)$ *is defined by (4.4.3).*

Proof. (1) To prove $\hat{c}^* = c^v_{\alpha_1}$, we only need to show $\hat{c}^* \leqslant c^v_{\alpha_1}$ due to $\hat{c}^* \geqslant c^v_{\alpha_1}$ (since (4.2.16) is well-known). Suppose that we have an upper solution $(\bar{W}, \bar{V})(x - c^v_{\alpha_1} t)$ satisfies (4.4.7). Then, recall the process to define $\hat{c}^* = \sup\{c : \ \hat{a}(c; +\infty) = \hat{\beta}\}$, see (4.2.11). Let $c = c^v_{\alpha_1}$, we can define the sequence $\{\hat{a}_n\}$ and its limit $\hat{a}$ by

$$\begin{cases} \hat{a}_0(c;x) = \hat{\mathbf{w}}_0(x), \\ \hat{a}_{n+1}(c,x) = \max\{\hat{a}_0(c;x), \ \hat{Q}_1[\hat{a}_n(c;\cdot)](x+c)\}, \\ \hat{a}(c;x) = \lim\limits_{n\to\infty} a_n(c;x), \end{cases}$$

where the initial data $\hat{\mathbf{w}}_0(x)$ satisfies (a), (b), (c) by replacing β with $\hat{\beta}$ and

$$\hat{\mathbf{w}}_0(x) < (\overline{W}, \overline{V})(x), \ \forall \ x \in \mathbb{R}.$$

An induction shows that $\hat{a}_n(c;x) \leqslant (\overline{W}, \overline{V})(x), \ n \geqslant 1$, which implies that $\hat{a}(c;x) \leqslant (\overline{W}, \overline{V})(x)$, and hence,

$$\hat{a}(c; +\infty) = \lim_{x\to\infty} \hat{a}(c;x) \leqslant \lim_{x\to\infty}(\overline{W}, \overline{V})(x) = (0, 0).$$

Since $\hat{c}^*$ is independent of the initial data $\hat{\mathbf{w}}_0$, the definition of $\hat{c}^*$, see (4.2.11), shows that $\hat{c}^* \leqslant c = c^v_{\alpha_1}$. This completes the proof of (1).

(2) To prove the second statement, we shall use the way of contradiction. Based on the definition of $\hat{c}^*$, it is well-known that (4.2.10) has traveling waves for any $c \geq \hat{c}^*$. To the contrary, assume $\hat{c}^* < \hat{c}_1$. Then there exists a $c \in (c^v_{\alpha_1}, \hat{c}_1)$, such that (4.2.10) has a monotone traveling wave solution $(W, V)(x - ct)$ connecting $\hat{\beta}$ to $\mathbf{0}$. Clearly,

as $x \to -\infty$, (4.4.8) shows that $(\underline{W}, \underline{V})(x) < (W, V)(x)$. Using the monotonicity of $\hat{\mu}_1$ and $\hat{\mu}_2$, see (4.4.3), it is always true that $\underline{V}(x) < V(x)$ as $x \to \infty$. Thus, we can always assume that $\underline{V}(x) < V(x)$ for $x \in \mathbb{R}$, by shifting if necessary.

Now, we claim that, for any given non-increasing and continuous function $V(z)$, where $z = x - ct$ with $c \geqslant c^v_{\alpha_1}$, and $V(-\infty) = a > 0$, $V(+\infty) = 0$, there exists a nonincreasing function $W(z)$ satisfying

$$\begin{cases} d_1 W'' + cW' + r_1(1+W)(b_1 V - W) = 0, \\ W(-\infty) = b_1 a, \ W(+\infty) = 0. \end{cases} \tag{4.4.9}$$

To see this, we will apply the upper and lower solutions method. It is easy to see that $\bar{W} = b_1 a$ is an upper solution due to $\hat{L}_w[\bar{W}, V] \leqslant 0$, while $\underline{W} = 0$ is a lower solution since $\hat{L}_w[\underline{W}, V] \geqslant 0$; thus, the result follows. Moreover, since the reaction term $r_1(1+W)(b_1V - W)$ is monotone in V, by comparison, we obtain that $W(V_1) \geqslant W(V_2)$ if $V_1 \geqslant V_2$ for $z \in \mathbb{R}$.

The above claim combing the condition $\underline{V} < V$ implies that

$$(\underline{W}, \underline{V})(x) \leqslant (W, V)(x), \ \forall x \in \mathbb{R}.$$

Hence, by comparison, we have

$$\underline{\hat{\mathbf{w}}}(x - \hat{c}_1 t) \leqslant \hat{Q}_t[\underline{\hat{\mathbf{w}}}] \leqslant \hat{Q}_t[\hat{\mathbf{w}}] = \hat{\mathbf{w}}(x - ct),$$

where $\hat{\mathbf{w}}(x) = (W, V)(x)$. Since $\underline{\hat{\mathbf{w}}}(x) = (\underline{W}, \underline{V})(x) \not\equiv 0$ nonincreasing in x, there is $x_1 \in \mathbb{R}$ such that $\underline{\hat{\mathbf{w}}}(x_1) > 0$. For x, t satisfying $x - \hat{c}_1 t = x_1$ and $t \to \infty$, we can derive $\underline{\hat{\mathbf{w}}}(x_1) \leqslant 0$. This is a contradiction.

(3) When $\hat{c}^* > c^v_{\alpha_1}$ holds, we proceed to find an upper bound of $\hat{c}^*$. Again, by the fact that $\hat{c}^*$ is independent of the choice of the initial condition $\hat{w}_0(x)$ as long as it satisfies (a), (b), and (c), through replacing $c^v_{\alpha_1}$ and $(\bar{W}, \bar{V})$ with $\hat{c}_2$ and $(\bar{W}_2, \bar{V}_2)$ in the proof of (1), we can show that $\hat{c}^* \leqslant \hat{c}_2$. The proof is completed. $\square$

Based on the above lemma, we proceed to find some specific conditions to classify the speed selection of system (4.2.10) by picking up some trial functions.

Theorem 4.4.2. *If $\frac{d_1}{d_2} \leqslant 2$, then the spreading speed of system (4.2.10) is linearly selected, i.e., $\hat{c}^* = c^v_{\alpha_1}$.*

Proof. For $c = c^v_{\alpha_1}$, we have $\hat{\mu}_1(c) = \hat{\mu}_2(c) = \hat{\mu}_0 = \sqrt{\frac{r_2(1+b_2)}{d_2}}$ and

$$0 < k_1 = k_1(\hat{\mu}_0) = \frac{b_1}{1 - \frac{r_2(1+b_2)}{r_1}\left(\frac{d_1}{d_2} - 2\right)} \leqslant b_1$$

provided that $\frac{d_1}{d_2} \leqslant 2$. Then, define

$$\bar{V}(z) = \frac{e_2}{1 + e^{\hat{\mu}_0 z}}, \text{ and } \bar{W}(z) = (e_1 - 1)\frac{\bar{V}}{e_2} = b_1 \bar{V}(z). \tag{4.4.10}$$

It is easy to see that $\bar{V} \to e_2$, $\bar{W} \to e_1 - 1$ as $z \to -\infty$, and $\bar{V} \to 0$, $\bar{W} \to 0$ as $z \to +\infty$. Through a direct computation, we find the first and second derivatives of $\bar{V}$ as

$$\bar{V}' = -\hat{\mu}_0 \bar{V}(1 - \bar{V}_1), \; \bar{V}'' = \hat{\mu}_0^2 \bar{V}(1 - \bar{V}_1)(1 - 2\bar{V}_1), \quad \text{where} \;\; \bar{V}_1 = \frac{\bar{V}}{e_2}. \tag{4.4.11}$$

Substituting them all into $\hat{L}_w[W, V]$ and $\hat{L}_v[W, V]$ (see, 4.2.21) gives

$$\begin{aligned} \hat{L}_w[\bar{W}, \bar{V}] &= d_1 \hat{\mu}_0^2 b_1 \bar{V}(1 - \bar{V}_1)(1 - 2\bar{V}_1) - c\hat{\mu}_0 b_1 \bar{V}(1 - \bar{V}_1) + 0 \\ &= \frac{\bar{V}^2}{e_2}(1 - \bar{V}_1)\left\{-2d_1\hat{\mu}_0^2 + r_1 \frac{1 - \frac{b_1}{k_1}}{\bar{V}_1}\right\} \end{aligned}$$

and

$$\begin{aligned} \hat{L}_v[\bar{W}, \bar{V}] &= d_2 \hat{\mu}_0^2 \bar{V}(1 - \bar{V}_1)(1 - 2\bar{V}_1) - c\hat{\mu}_0 \bar{V}(1 - \bar{V}_1) + r_2 \bar{V}(1 + b_2 - \bar{V} + b_2 \bar{W}) \\ &= \frac{\bar{V}^2}{e_2}(1 - \bar{V}_1)\left\{-2d_2\hat{\mu}_0^2\right\}. \end{aligned}$$

Since $1 - \frac{b_1}{k_1} \leqslant 0$, it is clearly that $\hat{L}_w[\bar{W}, \bar{V}] \leqslant 0$ and $\hat{L}_v[\bar{W}, \bar{V}] \leqslant 0$. Then, by Lemma

4.4.1 (1), we have found an upper solution satisfying (4.4.7) with $c = c^v_{\alpha_1}$. Thus, $\hat{c}^* = c^v_{\alpha_1}$, which completes the proof. $\square$

Let $\bar{V}$ be defined as (4.4.10) and

$$\bar{W} = \min\{e_1 - 1,\ k_1\bar{V}\} = \begin{cases} e_1 - 1,\ z \leqslant z_1, \\ k_1\bar{V},\ z > z_1, \end{cases}$$

where $z_1 \in \mathbb{R}$ such that $\bar{V}(z_1) = \frac{e_1-1}{k_1} = \frac{b_1}{k_1}e_2 < e_2$. We can have the following result.

Theorem 4.4.3. *If*

$$\begin{cases} \frac{d_1}{d_2} > 2, \\ \max\left\{\frac{d_1}{d_2} - \frac{r_1}{r_2(1+b_2)},\ \frac{b_2k_1}{1-b_1b_2}\right\} < 2, \\ \ \textit{where } k_1 = \frac{b_1}{1-\frac{r_2(1+b_2)}{r_1}\left(\frac{d_1}{d_2}-2\right)}, \end{cases} \tag{4.4.12}$$

then $\hat{c}^*$ is linearly selected, i.e., $\hat{c}^* = c^v_{\alpha_1}$.

Proof. For $c = c^v_{\alpha_1}$, we still have $\hat{\mu}_0 = \sqrt{\frac{r_2(1+b_2)}{d_2}}$ and

$$k_1(\hat{\mu}_0) = \frac{b_1}{1 - \frac{r_2(1+b_2)}{r_1}\left(\frac{d_1}{d_2} - 2\right)} > b_1$$

since $0 < \frac{d_1}{d_2} - 2 < \frac{r_1}{r_2(1+b_2)}$. Let $\bar{V}$ be defined as (4.4.10) and

$$\bar{W} = \min\{e_1 - 1,\ k_1\bar{V}\} = \begin{cases} e_1 - 1,\ z \leqslant z_1, \\ k_1\bar{V},\ z > z_1, \end{cases}$$

where $z_1 \in \mathbb{R}$ such that $\bar{V}(z_1) = \frac{e_1-1}{k_1} = \frac{b_1}{k_1}e_2 < e_2$. Thus, when $z \leqslant z_1$, we have

$$\hat{L}_w[\bar{W}, \bar{V}] = 0 + 0 + r_1e_1(-e_1 + 1 + b_1\bar{V}) \leqslant r_1e_1(-e_1 + 1 + b_1e_2) = 0,$$

and

$$\begin{aligned}
\hat{L}_v[\bar{W},\bar{V}] &= d_2\hat{\mu}_0^2\bar{V}(1-\bar{V}_1)(1-2\bar{V}_1) - c\hat{\mu}_0\bar{V}(1-\bar{V}_1) + r_2\bar{V}(1+b_2-\bar{V}+b_2(e_1-1)) \\
&= \frac{\bar{V}^2}{e_2}(1-\bar{V}_1)\left\{-2d_2\hat{\mu}_0^2 + \frac{r_2b_1b_2e_2}{\bar{V}_1}\right\} \\
&\leqslant \frac{\bar{V}^2}{e_2}(1-\bar{V}_1)\left\{-2d_2\hat{\mu}_0^2 + \frac{r_2b_1b_2e_2}{\frac{b_1}{k_1}}\right\} \text{ since } \bar{V}_1 \in [\frac{b_1}{k_1}, 1] \text{ as } z \leqslant z_1 \\
&= \frac{\bar{V}^2}{e_2}(1-\bar{V}_1)\left\{r_2(1+b_2)[\frac{b_2k_1}{1-b_1b_2} - 2]\right\} \leqslant 0,
\end{aligned}$$

The last inequality holds by the second inequality from (4.4.12). When $z > z_1$, through a direct computation, we obtain that

$$\begin{aligned}
\hat{L}_w[\bar{W},\bar{V}] &= d_1\hat{\mu}_0^2k_1\bar{V}(1-\bar{V}_1)(1-2\bar{V}_1) - c\hat{\mu}_0k_1\bar{V}(1-\bar{V}_1) + r_1(k_1\bar{V}+1)(-k_1\bar{V}+b_1\bar{V}) \\
&= \frac{k_1\bar{V}^2}{e_2}(1-\bar{V}_1)\left\{-2d_1\hat{\mu}_0^2 + r_1\frac{\frac{b_1}{k_1} - 1 + b_1e_2 - k_1e_2}{1-\bar{V}_1}\right\}
\end{aligned}$$

is less than zero since $b_1 < k_1$. Also,

$$\begin{aligned}
\hat{L}_v[\bar{W},\bar{V}] &= d_2\hat{\mu}_0^2\bar{V}(1-\bar{V}_1)(1-2\bar{V}_1) - c\hat{\mu}_0\bar{V}(1-\bar{V}_1) + r_2\bar{V}(1+b_2-\bar{V}+b_2k_1\bar{V}) \\
&= \frac{\bar{V}^2}{e_2}(1-\bar{V}_1)\left\{-2d_2\hat{\mu}_0^2 + r_2\frac{(1+b_2)-e_2(1-k_1b_2)}{1-\bar{V}_1}\right\} \\
&< \frac{\bar{V}^2}{e_2}(1-\bar{V}_1)\left\{-2d_2\hat{\mu}_0^2 + r_2(1+b_2)\frac{1-\frac{1-b_2k_1}{1-b_1b_2}}{1-\frac{b_1}{k_1}}\right\} \text{ since } \bar{V}_1 \in [\mathbf{0}, \frac{b_1}{k_1}) \text{ as } z > z_1 \\
&= \frac{\bar{V}^2}{e_2}(1-\bar{V}_1)\left\{r_2(1+b_2)\left[\frac{b_2k_1}{1-b_1b_2} - 2\right]\right\} \leqslant 0.
\end{aligned}$$

This means that we have found an upper solution satisfying (4.4.7) with $c = c^v_{\alpha_1}$, which implies $\hat{c}^* = c^v_{\alpha_1}$. □

With the help of above theorems, we immediately obtain the following speed selection theorem for the original system (4.1.1).

Theorem 4.4.4. *If $d_1r_1 > d_2r_2(1+b_2)$ and one of the following holds:*

$$\frac{d_1}{d_2} \leqslant 2 \tag{4.4.13}$$

or,

$$\begin{cases} \frac{d_1}{d_2} > 2, \ \ \max\left\{\frac{d_1}{d_2} - \frac{r_1}{r_2(1+b_2)}, \ \frac{b_2k_1}{1-b_1b_2}\right\} < 2, \\ \textit{where } k_1 = \frac{b_1}{1-\frac{r_2(1+b_2)}{r_1}\left(\frac{d_1}{d_2}-2\right)}, \end{cases} \tag{4.4.14}$$

then (4.1.1) has multiple spreading speeds and both of them are linearly selected, i.e., $c_v^* = c_{\alpha_1}^v$ and $c_u^* = c_0^u$.

Proof. Since under conditions (4.4.13) or (4.4.14), by Theorems 4.4.2 and 4.4.3, it immediately follows that $\hat{c}^* = 2\sqrt{d_2r_2(1+b_2)} < 2\sqrt{d_1r_1}$. Through Theorem 4.3.1, $c_u^* > c_v^*$. Then, by Lemma 4.2.8 and Remark 4.2.11, the linear selection result follows. The proof is completed. □

The conditions in the above theorem imply that $\hat{\mu}_3(c_{\alpha_1}^v) > \hat{\mu}_0(c_{\alpha_1}^v)$. From (4.4.5), it follows that W and V from the traveling wave $(W,V)(x - c_{\alpha_1}^v t)$ have the same exponential decaying rate $\hat{\mu}_0$. When $\hat{\mu}_3(c_{\alpha_1}^v) < \hat{\mu}_0(c_{\alpha_1}^v)$, W may have a different exponential decaying rate from V as $z \to \infty$. We will present an example in Section 6 (see, Example 6.2) to show that, under such a condition, both the linear and nonlinear selection may happen depending on the parameters.

4.5 Determinacy of the single spreading speed

When $\hat{c}^* \geqslant 2\sqrt{d_1r_1}$, the system has a single spreading speed $c_u^* = c_v^*$. Based on the linearized system at zero, it is easy to know that c_v^* is always nonlinearly selected. Now we want to determine whether c_u^* is linearly or nonlinearly selected. Indeed, let $(U,V)(z) = (C_u e^{-\mu z}, C_v e^{-\mu z})$ with $C_u,\ C_v,\ \mu$ being positive numbers. By inserting it into the linear system, we find the eigenproblem as

$$\begin{cases} [d_1\mu^2 - c\mu + r_1]C_u = 0, \\ [d_2\mu^2 - c\mu + r_2]C_v = 0. \end{cases}$$

The first equation with $C_u > 0$ gives

$$\mu_1(c) = \frac{c - \sqrt{c^2 - 4d_1 r_1}}{2d_1}, \ \mu_2(c) = \frac{c + \sqrt{c^2 - 4d_1 r_1}}{2d_1}, \tag{4.5.1}$$

and the second equation with $C_v > 0$ gives

$$\mu_3(c) = \frac{c - \sqrt{c^2 - 4d_2 r_2}}{2d_2}, \ \mu_4(c) = \frac{c + \sqrt{c^2 - 4d_2 r_2}}{2d_2}. \tag{4.5.2}$$

We also want to obtain a positive solution. Thus $c \geqslant \max\{2\sqrt{d_1 r_1},\ 2\sqrt{d_2 r_2}\} = 2\sqrt{d_1 r_1}$, that is, $c \geqslant c_0^u$. Clearly, when $c \geqslant c_0^u$, $0 < \mu_1(c) \leqslant \mu_2(c)$, $0 < \mu_3(c) \leqslant \mu_4(c)$, and $\mu_{1,3}(c)$ are decreasing while $\mu_{2,4}(c)$ is increasing with respect to c. Hence, for $c > c_0^u$, the decaying positive solution $(U, V)(z)$ behaves like

$$\begin{pmatrix} U \\ V \end{pmatrix} \sim \begin{pmatrix} C_1 e^{-\mu_1(c)z} + C_2 e^{-\mu_2(c)z} \\ C_3 e^{-\mu_3(c)z} + C_4 e^{-\mu_4(c)z} \end{pmatrix}, \text{ as } z \to \infty, \tag{4.5.3}$$

where $C_1 > 0$ or $C_1 = 0,\ C_2 > 0$ while $C_3 > 0$ or $C_3 = 0,\ C_4 > 0$.

Lemma 4.5.1. *Suppose $c_u^* = c_v^*$. The following statements are true.*

(1) c_u^* is linearly selected, i.e., $c_u^* = c_0^u$, if for $c = c_0^u$, there exists a pair of continuous and positive functions $(\overline{U}, \overline{V})$ being an upper solution to (4.1.9) and satisfying

$$\lim_{z\to-\infty} (\overline{U}, \overline{V}) \gg 0 \text{ and } \lim_{z\to\infty} \overline{U} = 0. \tag{4.5.4}$$

Furthermore, if $\lim_{z\to\infty} (\overline{U}, \overline{V}) = (0, 0)$, then there exists a traveling wave connecting β and zero.

(2) c_u^* is nonlinearly selected and $c_u^* \geqslant c_1 > c_0^u$, if for $c = c_1$, there exists a pair of continuous and non-increasing functions $(\underline{U}, \underline{V})$ being a lower solution to (4.1.9) and satisfying

$$\lim_{z\to-\infty} (\underline{U}, \underline{V}) \ll \beta \text{ and } (\underline{U}, \underline{V}) \sim (e^{-\mu_2(c_1)z}, e^{-\mu_4(c_1)z}) \text{ as } z \to \infty, \tag{4.5.5}$$

where $\mu_2(c_1),\ \mu_4(c_1)$ are defined in (4.5.1) and (4.5.2), respectively.

(3) c_u^* has an upper bound $c_u^* \leqslant c_2$ with $c_2 > c_0^u$, if there exists a pair of continuous and positive functions $(\overline{U}_2, \overline{V}_2)(x - c_2 t)$ being an upper solution to (4.1.9) and satisfying

$$\lim_{z \to -\infty} (\overline{U}_2, \overline{V}_2) \gg 0 \text{ and } \overline{U}_2 \sim e^{-\mu_2(c_2) z}, \text{ as } z \to \infty, \tag{4.5.6}$$

where $\mu_2(c_2)$ is defined in (4.5.1).

Proof. (1) Since $(\bar{U}, \bar{V})(-\infty) \gg 0$, we can choose the initial data $\mathbf{w}_0(x)$ satisfying (a), (b), (c), and

$$\mathbf{w}_0(x) \leqslant (\bar{U}, \bar{V})(x) \text{ for } x \in \mathbb{R}, \text{ and } \mathbf{w}_0(-\infty) \gg 0.$$

Then, because $(\bar{U}, \bar{V})(x - c_0^u t)$ is an upper solution to (4.1.1) with the initial data $(\bar{U}, \bar{V})(x)$, and by comparison, we obtain that

$$(u, v)(t, x; \mathbf{w}_0) \leqslant (\bar{U}, \bar{V})(x - c_0^u t) \text{ for all } (t, x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{4.5.7}$$

Via the linearization, $c_u^* \geqslant c_0^u$ is always true, see (4.2.7)-(4.2.8). To the contrary, suppose that $c_u^* > c_0^u$. Let $\epsilon = c_u^* - c_0^u > 0$, then $x = (c_u^* - \frac{\epsilon}{2})t = (c_0^u + \frac{\epsilon}{2})t$. From (4.5.7), we obtain that

$$\lim_{t \to \infty,\ x = (c_u^* - \frac{\epsilon}{2})t} u(t, x) \leqslant \lim_{t \to \infty,\ x = (c_0^u + \frac{\epsilon}{2})t} \bar{U}(x - c_0^u t) = 0. \tag{4.5.8}$$

This is a contradiction through the definition of c_u^*, see (4.2.1). Thus, we must have $c_u^* = c_0^u$ if an upper solution satisfies (4.5.4).

From Proposition 4.2.2 and the assumption $c_u^* = c_v^*$, it follows that $c^* = c_u^* = c_v^*$. By Theorem 4.1.1 and (4.5.7) with the condition $(\bar{U}, \bar{V})(+\infty) = (0, 0)$, we immediately obtain that (4.1.1) has a traveling wave connecting β to zero when $c \geqslant c_u^*$.

(2) We suppose for the sake of contradiction that $c_u^* < c_1$. Then, under the condition $c_u^* = c_v^*$ and Theorem 4.1.1, we can assume that, for $c \in [c_u^*, c_1)$, (4.1.1) has a traveling wave $(U, V)(x - ct)$ connecting β to some equilibrium β_1 where β_1 can be

$\mathbf{0}$, α_1 or α_2.

We first start with the case when $\beta_1 = \mathbf{0}$, i.e., (U, V) connecting β to zero. Near the equilibrium zero, we have the asymptotic behavior defined in (4.5.3). Since $\mu_2(c_1) > \mu_2(c)$ and $\mu_4(c_1) > \mu_4(c)$ (see, (4.5.1) and (4.5.2)), it immediately follows that $(\underline{U}, \underline{V})(x) < (U, V)(x)$ as $x \to \infty$. By a shifting if necessary, we get

$$(\underline{U}, \underline{V})(x) \leqslant (U, V)(x), \text{ for all } x \in \mathbb{R}.$$

Since $(\underline{U}, \underline{V})$ is a lower solution to (4.1.1) and by comparison, we obtain

$$(\underline{U}, \underline{V})(x - c_1 t) \leqslant (U, V)(x - ct) \text{ for all } (x, t) \in \mathbb{R} \times \mathbb{R}_+. \tag{4.5.9}$$

Fixing $z_1 = x - c_1 t$, we have that $\underline{V}(z_1) > 0$. On the other hand,

$$V(x - ct) = V(z_1 + (c_1 - c)t) \to 0 \text{ as } t \to \infty.$$

It follows that $\underline{V}(z_1) \leqslant 0$, which is a contradiction.

In the case when $\beta_1 = \alpha_1 = (1, 0)$, the assumed traveling wave (U, V) connects β to α_1. Thus, $U(x) \geqslant 1 > \underline{U}(x)$ as $x \to \infty$. For the other component V, through linearizing (4.1.1) around α_1, we find the asymptotic behavior of V, which is near zero, as

$$V \sim C_1 e^{-\mu_1^{\alpha_1}(c)z} + C_2 e^{-\mu_2^{\alpha_1}(c)z},$$

where $C_1 > 0$, or $C_1 = 0$, $C_2 > 0$, and

$$\mu_1^{\alpha_1}(c) = \frac{c^2 - \sqrt{c^2 - 4d_2 r_2(1 + b_2)}}{2d_2}, \ \mu_2^{\alpha_1}(c) = \frac{c^2 + \sqrt{c^2 - 4d_2 r_2(1 + b_2)}}{2d_2},$$

for $c > 2\sqrt{d_2 r_2(1 + b_2)}$. This is the same as the auxiliary system (4.2.10), see (4.4.5). Then, it is easy to see that $\mu_4(c_1) > \mu_4(c) = \frac{c^2 + \sqrt{c^2 - 2d_2 r_2}}{2d_2} > \mu_2^{\alpha_1}(c) > \mu_1^{\alpha_1}(c)$. Thus, when $x \to +\infty$, it follows that $\underline{V}(x) < V(x)$. Then, through a shifting, we shall have

$(\underline{U}, \underline{V})(x) \leqslant (U, V)(x),$ for all $x \in \mathbb{R}$. The relation shown in (4.5.9) is still valid, and a contradiction follows.

Lastly, the proof of the case when $\beta_1 = \alpha_2 = (0, 1)$ is similar to the above one. In this case it is clear that $V(x) > \underline{V}(x)$ for all $x \in \mathbb{R}$. Through a linear analysis of (4.1.1) near α_2, we can obtain that $\underline{U}(x) < U(x)$ for all $x \in \mathbb{R}$. Thus, (4.5.9) still holds. Then, by applying the arguments below (4.5.9) to U and $\underline{U}$, we also obtain a contradiction. Therefore, $c_u^* \geqslant c_1$ when such a lower solution exists at $c = c_1$.

(3) By replacing c_0^u, $\bar{U}$ and $\bar{V}$ with c_2, $\bar{U}_2$ and $\bar{V}_2$ respectively in the proof of part (1), we shall find $c_u^* \leqslant c_2$. The proof is complete. □

Based on the above lemma, we then proceed to construct some appropriate upper and lower solutions to find specific conditions to classify the speed selection. For simplicity, we denote

$$\mu_0 := \mu_1(c_0^u) = \sqrt{\tfrac{r_1}{d_1}},\ \bar{\mu}_3 := \mu_3(c_0^u) = \tfrac{\sqrt{d_1 r_1} - \sqrt{d_1 r_1 - d_2 r_2}}{d_2},\ \bar{\mu}_4 := \mu_4(c_0^u) = \tfrac{\sqrt{d_1 r_1} + \sqrt{d_1 r_1 - d_2 r_2}}{d_2}. \tag{4.5.10}$$

Theorem 4.5.2. *If $d_2 r_2 < d_1 r_1 \leqslant d_2 r_2 (1 + b_2)$ and*

$$\frac{1}{\sqrt{d_1 r_1} + \sqrt{d_1 r_1 - d_2 r_2}} < \sqrt{\frac{r_1}{d_1 r_2^2}} < \frac{1}{\sqrt{d_1 r_1} - \sqrt{d_1 r_1 - d_2 r_2}}, \tag{4.5.11}$$

then system (4.1.1) has a single spreading speed and $c_u^* = c_0^u$.

Proof. By Theorem 4.3.2 (III), we have that $c_u^* = c_v^*$; thus, we then focus on finding suitable upper solutions satisfying Definition 4.2.12.

Let $c = c_0^u$, and μ_0, $\bar{\mu}_3$, $\bar{\mu}_4$ be defined in (4.5.10). Then, define

$$\bar{U} = \frac{e_1}{1 + e^{\mu_0 z}}, \tag{4.5.12}$$

whose first and second derivatives can be found as

$$\bar{U}' = -\mu_0 \bar{U}(1 - \bar{U}_1),\ \bar{U}'' = \mu_0^2 \bar{U}(1 - \bar{U}_1)(1 - 2\bar{U}_1),\ \text{where } \bar{U}_1 = \frac{\bar{U}}{e_1}, \tag{4.5.13}$$

and set $\bar{V} = e_2\bar{U}_1 = \frac{e_2}{e_1}\bar{U}$. Substituting all the formulas into $L_u[U,V]$ and $L_v[U,V]$ gives

$$\begin{aligned} L_u[\bar{U},\bar{V}] &= d_1\mu_0^2\bar{U}(1-\bar{U}_1)(1-2\bar{U}_1) - c\mu_0\bar{U}(1-\bar{U}_1) + r_1\bar{U}(1-\bar{U}+b_1e_2\bar{U}_1) \\ &= \frac{\bar{U}^2}{e_1}(1-\bar{U}_1)\left\{-2d_1\mu_0^2\right\} \leqslant 0, \end{aligned}$$

and

$$\begin{aligned} L_v[\bar{U},\bar{V}] &= d_2\mu_0^2\frac{e_2}{e_1}\bar{U}(1-\bar{U}_1)(1-2\bar{U}_1) - c\mu_0\frac{e_2}{e_1}\bar{U}(1-\bar{U}_1) + r_2\frac{e_2}{e_1}\bar{U}(1-e_2\bar{U}_1+b_2\bar{U}) \\ &= \frac{e_2}{e_1}\bar{U}(1-\bar{U}_1)\left\{d_2\mu_0^2 - c\mu_0 + r_2 - 2d_2\mu_0^2\bar{U}_1\right\} < 0. \end{aligned}$$

Here we have made use of (4.5.11). Since (4.5.11) implies $\bar{\mu}_3 < \mu_0 < \bar{\mu}_4$, the quadratic function $d_2\mu_0^2 - c\mu_0 + r_2 < 0$. Then, $(\bar{U},\bar{V})(x - c_0^u t)$ forms a pair of upper solutions satisfying (4.2.19). By Lemma 4.5.1 (1), the proof is complete. □

Theorem 4.5.3. *If $d_2r_2 < d_1r_1 < d_2r_2(1+b_2)$ and*

$$\begin{cases} \frac{1}{\sqrt{d_1r_1}+\sqrt{d_1r_1-d_2r_2}} > \sqrt{\frac{r_1}{d_1r_2^2}}, \\ p = \frac{\bar{\mu}_3}{\mu_0},\ M_1 = \max\{p-1,\ 1\},\ e_2 \leqslant \frac{d_2\bar{\mu}_3^2(1+\frac{1}{p})}{r_2M_1}, \end{cases} \tag{4.5.14}$$

then system (4.1.1) has a single spreading speed and c_u^* is linearly selected, that is, $c^* = c_f^* = c_u^* = c_0^u$.

Proof. Let $c = c_0^u$ and $\bar{U}$ be the same as in Theorem 4.5.2, i.e., $\bar{U} = \frac{e_1}{1+e^{\mu_0 z}}$ and $\bar{U}_1 = \frac{\bar{U}}{e_1}$. For V-part, we define

$$\bar{V} = e_2\bar{U}_1^p,\ p = \frac{\bar{\mu}_3}{\mu_0} > 1, \tag{4.5.15}$$

and then the first and the second derivatives of $\bar{V}$ are

$$\begin{cases} \bar{V}' = -p\mu_0 \bar{V}(1-\bar{U}_1) = -\bar{\mu}_3 \bar{V}(1-\bar{U}_1), \\ \bar{V}'' = (p\mu_0)^2 \bar{V}(1-\bar{U}_1)(1-(1+\frac{1}{p})\bar{U}_1) = \bar{\mu}_3^2 \bar{V}(1-\bar{U}_1)(1-(1+\frac{1}{p})\bar{U}_1). \end{cases} \tag{4.5.16}$$

We point out that such a function has an asymptotic behavior:

$$\bar{V}(-\infty) \to e_2,\ \bar{V}(z) \sim e^{-\bar{\mu}_3 z},\ \text{as } z \to \infty.$$

By inserting all the above formulas into $L_u[U,V]$ and $L_v[U,V]$ defined in (4.2.20), we obtain that

$$\begin{aligned} L_u[\bar{U},\bar{V}] &= d_1\mu_0^2 \bar{U}(1-\bar{U}_1)(1-2\bar{U}_1) - c\mu_0 \bar{U}(1-\bar{U}_1) + r_1 \bar{U}(1-\bar{U}+b_1 e_2 \bar{U}_1^p) \\ &= \frac{\bar{U}^2}{e_1}(1-\bar{U}_1)\left\{-2d_1\mu_0^2 + r_1 b_1 e_2 \frac{\bar{U}_1^{p-1}-1}{1-\bar{U}_1}\right\}, \end{aligned}$$

and

$$\begin{aligned} L_v[\bar{U},\bar{V}] &= d_2\bar{\mu}_3^2 \bar{V}(1-\bar{U}_1)(1-(1+\frac{1}{p})\bar{U}_1) - c\bar{\mu}_3 \bar{V}(1-\bar{U}_1) + r_2 \bar{V}(1-e_2\bar{U}_1^p + b_2\bar{U}) \\ &= \bar{U}_1 \bar{V}(1-\bar{U}_1)\Big\{-d_2\bar{\mu}_3^2(1+\frac{1}{p}) + r_2 e_2 \frac{1-\bar{U}_1^{p-1}}{1-\bar{U}_1}\Big\}. \end{aligned}$$

In $L_u[\bar{U},\bar{V}]$, since $p>1$, the fraction $\frac{\bar{U}_1^{p-1}-1}{1-\bar{U}_1}$ is less than zero and we always have $L_u[\bar{U},\bar{V}] \leqslant 0$. To determine the sign of $L_v[\bar{U},\bar{V}]$, we consider the monotonicity of the function $h(u) = \frac{1-u^{p-1}}{1-u}$ for $u \in [0,1]$. Through a direct analysis (by finding h'), we find that h is decreasing when $1 < p \leqslant 2$ while increasing when $p > 2$. Moreover, $h(0) = 1$ and $h(1) = p-1$. Thus, we have the following two statements:

(i) when $1 < p \leqslant 2$,

$$L_v[\bar{U},\bar{V}] \leqslant \bar{U}_1\bar{V}(1-\bar{U}_1)\Big\{-d_2\bar{\mu}_3^2(1+\frac{1}{p}) + r_2 e_2\Big\} \leqslant 0,\ \text{if } e_2 \leqslant \frac{d_2\bar{\mu}_3^2(1+\frac{1}{p})}{r_2}.$$

(ii) when $2 < p$,

$$L_v[\bar{U},\bar{V}] \leqslant \bar{U}_1\bar{V}(1-\bar{U}_1)\Big\{ -d_2\bar{\mu}_3^2(1+\frac{1}{p}) + r_2e_2(p-1)\Big\} \leqslant 0, \text{ if } e_2 \leqslant \frac{d_2\bar{\mu}_3^2(1+\frac{1}{p})}{r_2(p-1)}.$$

The second inequality in (4.5.14) guarantees (i) and (ii).

As a result, we have found a pair of upper solutions satisfying Lemma 4.5.1 (1); thus, $c_u^* = c_0^u$. The proof is complete. □

4.6 Numerical Simulations

In this section, we shall use the MATLAB software to simulate our model and numerically find the spreading speeds to better understand and demonstrate our obtained results. For the first three examples, we choose the following step functions as the initial data:

$$u_0(x) = \begin{cases} 0.5, & x < -95, \\ 0, & x \geqslant -95, \end{cases} \quad \text{and } v_0(x) = \begin{cases} 0.5, & x < -95, \\ 0, & x \geqslant -95. \end{cases}$$

Numerically, the solutions with such initial data evolve into traveling waves profile with the spreading speed(s). The detailed simulations are shown as follows.

Example 6.1 For the first example, we take

$$d_1 = 1,\ d_2 = 1,\ r_1 = 4,\ r_2 = 1,\ b_1 = 1,\ b_2 = 0.5.$$

Under such a choice, it is easy to find that $e_1 = 4$, $e_2 = 3$, and $d_1r_1 > d_2r_2e_2$; thus, this example belongs to the case (I) in Theorem 4.3.2, i.e., the case of multiple spreading speeds. Furthermore, by a direct computation, we find that

$$c_0^u = 4,\ c_{\alpha_1}^v = 2\sqrt{1.5} = 2.44948974278. \tag{4.6.1}$$

Since $\frac{d_1}{d_2} - 2 < 0$, we expect that $c_v^* = c_{\alpha_1}^v$ and $c_u^* = c_0^u$ by Theorem 4.4.4. The simulation is shown in Figure 4.1. The 3-D movements of u and v are shown in the left column; the 2-D figures in the right column depict the same movements from the top view, which show the dynamics more clear.

To calculate the numerical speeds, we use the level sets shown in Figure 4.2. The level set chosen to compute c_u^* is $u \equiv 0.5$; see the line in the left figure. Since c_v^* is the spreading speed of the traveling wave connects $\beta = (4,3)$ to $\alpha_1 = (1,0)$, the speeds found by $u \equiv 3$ in the left figure and $v \equiv 1.5$ in the right figure are the same. By finding their x-positions and dividing by the corresponding time unit, we find the numerically-computed fastest speed $\tilde{c}_u^*$ as $\tilde{c}_u^* = 4.0634$ and the numerically-computed slowest speed $\tilde{c}_v^*$ as $\tilde{c}_v^* = 2.4467$. Therefore, this example shows the linear selection of the spreading speeds in multiple spreading speeds case and agrees to our result.

Example 6.2 This example aims at showing that a nonlinear selection indeed exists when $c_u^* > c_v^*$. Let

$$d_1 = 4,\ d_2 = 0.1,\ r_1 = 2,\ r_2 = 1,\ b_1 = 0.2.$$

The last parameter b_2 varies from 0.1 to 4.5. It is easy to see that e_1 and e_2 are increasing in b_2 and $d_1 r_1 > d_2 r_2 e_2$ for all chosen b_2; thus, by Theorem 4.3.2, $c_u^* > c_v^*$. Through a direct computation, we can verify that $\hat{\mu}_3(c_{\alpha_1}^v) < \hat{\mu}_0(c_{\alpha_1}^v)$ in such a parameter set. Here, we present a figure to see it more clear, see the left panel of Figure 4.3. As shown in the figure, the red solid line is always above the blue one, which implies the inequality. Then, using the same initial data and numerical methods as in the former example, we can find the numerical spreading speeds for different b_2. From Remark 4.2.11, $c_u^* = 2\sqrt{d_1 r_1}$ is always linearly selected, so we then only focus on c_v^*. The result is shown in the right panel of Figure 4.3. Since the linear speed $c_{\alpha_1}^v$ is increasing in b_2, we depict it in the same figure as well. From the picture, we observe that the numerical speed is very close to the linear speed when $b_2 < 1.5$ and has an obvious increment as b_2 becomes larger than 1.5. Thus, from the numerical simulation, we can see that c_v^* is linearly selected when $b_2 < 1.5$ and the nonlinear selection is indeed realized when $b_2 > 1.5$.

Example 6.3 In this example, we take

$$d_1 = 1,\ d_2 = 1,\ r_1 = 2,\ r_2 = 1,\ b_1 = 1,\ b_2 = 0.5,$$

so that $e_1 = 4$, $e_2 = 3$, and $d_2 r_2 e_2 > d_1 r_1 > d_2 r_2 (1 + b_2)$. Thus, this example belongs to case (II) in Theorem 4.3.2. Notice that $d_1/d_2 = 1 < 2$, $\hat{c}^* = c_{\alpha_1}^v < 2\sqrt{d_1 r_1}$; thus, by

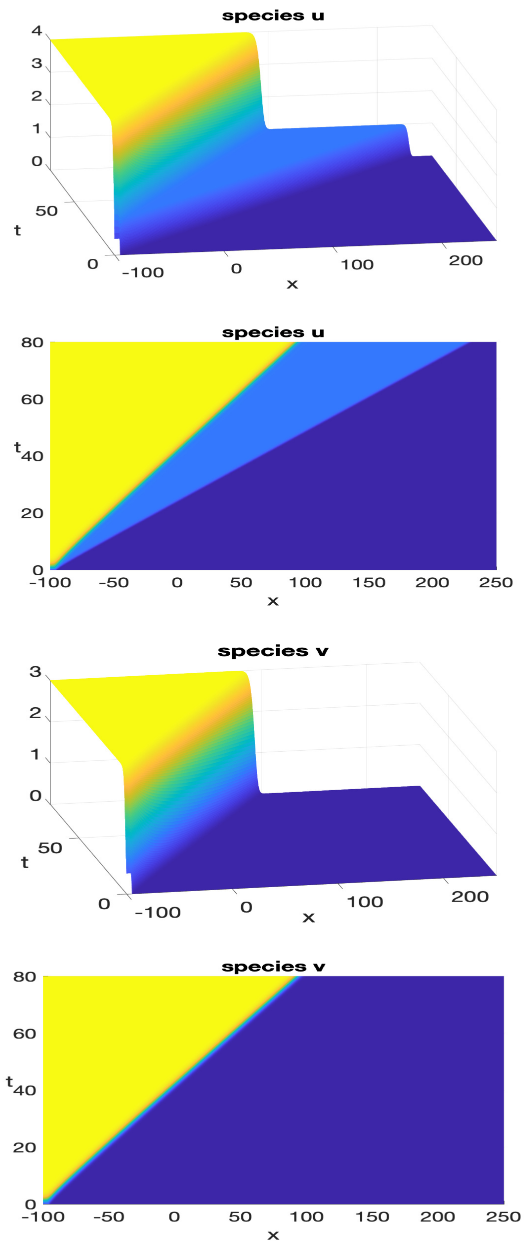


Figure 4.1: (Color online) Simulation of $(u, v)(t, x)$ when $d_1 r_1 > d_2 r_2 e_2$. Figures in the left column present the movements of u and v as time increases. Figures in the right column are the same movements but from the top view.

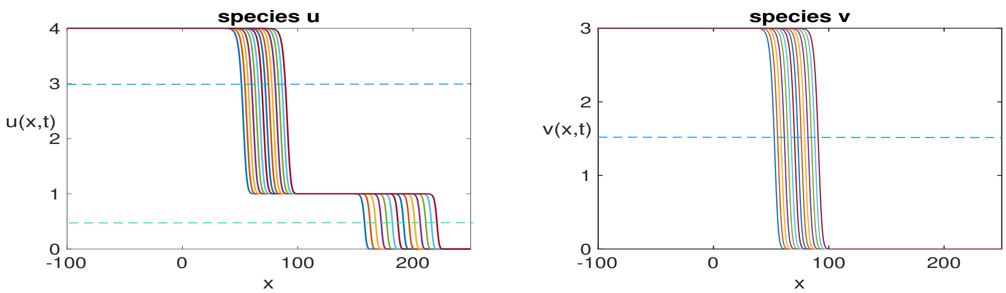


Figure 4.2: (Color online) Snapshots of u and v's movements. The left figure is for u while the right one is for v.

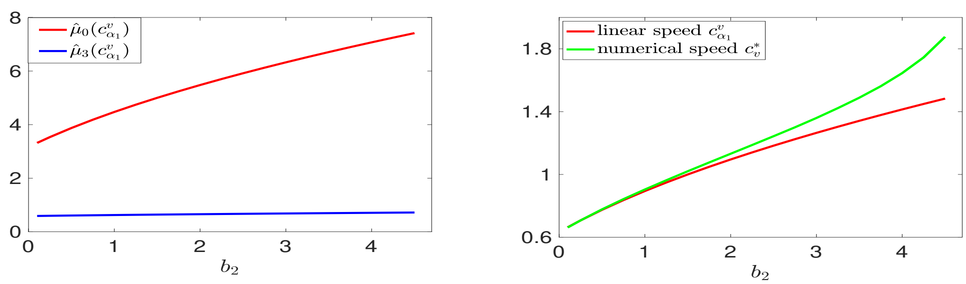


Figure 4.3: (Color online) A nonlinear selection example when $c_u^* > c_v^*$. The left panel depicts the relation between $\hat{\mu}_0$ and $\hat{\mu}_3$ as b_2 increases. The right panel draws the numerical speed c_v^* and the linear speed $c_{\alpha_1}^v$ for different b_2.

Theorem 4.4.4, we expect that $c_v^* = c_{\alpha_1}^v < c_u^* = c_0^u$. Through a simple computation, we obtain that

$$c_0^u = 2.8284, \; c_{\alpha_1}^v = 2.4495, \tag{4.6.2}$$

The simulation is shown in Figure 4.4. As we can see in the picture, we do observe different terraces in u. By the same method used in Example 6.1, we find the numerically computed speeds are $\tilde{c}_u^* = 2.8310$ and $\tilde{c}_v^* = 2.4392$. That means both spreading speeds are linearly selected.

Example 6.4 We then show an example whose parameters satisfy case (III). Let

$$d_1 = 1, \; d_2 = 1, \; r_1 = 1.5, \; r_2 = 1, \; b_1 = 0.5, \; b_2 = 1.$$

Then, we have $e_1 = 3$, $e_2 = 4$, and $d_2 r_2 (1 + b_2) > d_1 r_1$ which clearly indicates case (III). By Theorem 4.3.2, this is the single spreading speed case. Then, we only need to focus on c_0^u and decaying exponential rates near $(0, 0)$. By a simple computation,

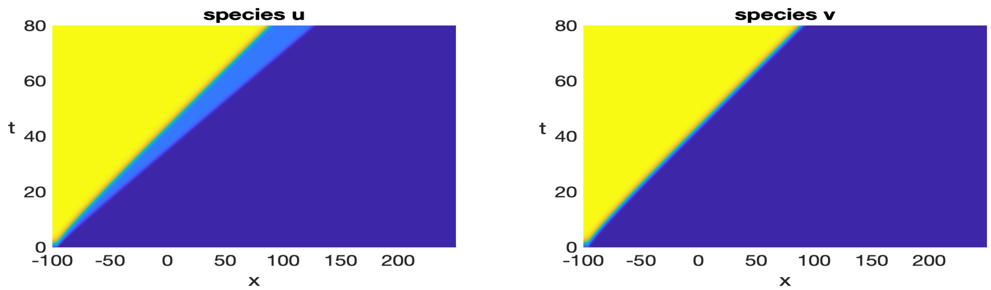


Figure 4.4: (Color online) Simulation of $(u, v)(t, x)$ when $d_2r_2e_2 > d_1r_1 > d_2r_2(1+b_2)$. The left figure is for u while the right one is for v.

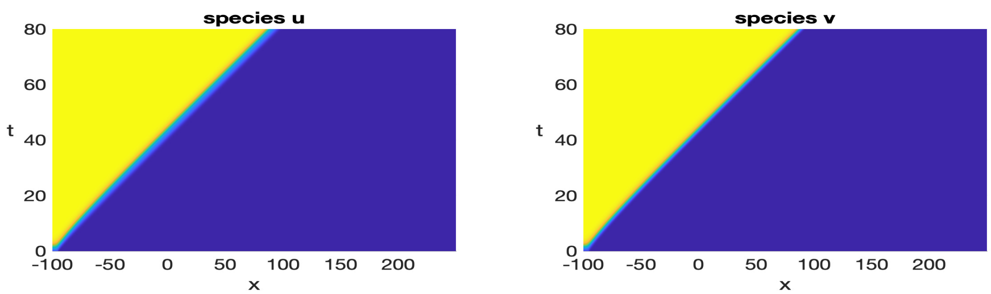


Figure 4.5: (Color online) Simulation of $(u, v)(t, x)$ when $d_2r_2(1 + b_2) > d_1r_1 > d_2r_2$. The left figure is for u while the right one is for v.

we find that

$$c_0^u = 2.4495, \;\; \mu_0 = 1.22475, \;\; \bar{\mu}_3 = 0.5176, \;\; \bar{\mu}_4 = 1.9318.$$

Clearly, $\bar{\mu}_3 < \mu_0 < \bar{\mu}_4$ which implies (4.5.11); thus, by Theorem 4.5.2, we wish to see the single spreading speed exists and $c_u^* = c_0^u$. The simulation outcome is depicted in Figure 4.5. As the picture has shown, there is only one connection from $\beta = (3, 4)$ to $\mathbf{0}$, which implies the single spreading speed case. Numerically, we find the speeds for both species are $\tilde{c}_u^* = 2.4487$ and $\tilde{c}_v^* = 2.4430$; thus, we have that $\tilde{c}_u^* = \tilde{c}_v^* \simeq c_0^u$. This example gives a numerical demonstration of our main result.

The remaining two examples are trying to show some interesting phenomena. The initial data chosen for the former examples decays simultaneously, by which, we mean that $u_0(x)$ and $v_0(x)$ reach 0 at the same position, i.e., $u_0(x) = 0 = v_0(x)$ when

$x \geqslant -95$. Now, we choose another step function as follows:

$$u_0(x) = \begin{cases} 1.2, & x < -400, \\ 1, & -400 \leqslant x < 0, \\ 0, & x \geqslant 0, \end{cases} \quad \text{and } v_0(x) = \begin{cases} 0.5, & x < -400, \\ 0, & x \geqslant -400. \end{cases}$$

The plotting is shown in Figure 4.6. As seen in the picture, $u_0(x)$ has a terrace $u \equiv 1$ when $-400 \leqslant x < 0$ and v_0 reaches zero first. The two examples below will use these initial data.

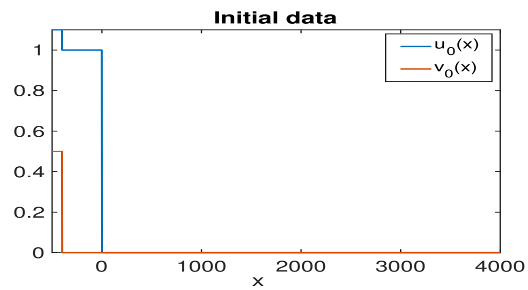


Figure 4.6: (Color online) The initial data for Examples 6.5 and 6.6.

Example 6.5 Choose

$$d_1 = 1,\ d_2 = 1,\ r_1 = 2,\ r_2 = 1,\ b_1 = 0.2,\ b_2 = 1,$$

then $e_1 = 1.5$, $e_2 = 2.5$, and $c_0^u = 2.8284$. It is clear that $d_1 r_1 = d_2 r_2 (1+b_2)$. Applying Theorems 4.3.2 (III) and 4.4.2, we obtain $2\sqrt{d_2 r_2(1+b_2)} = \hat{c}^* = c_u^* = 2\sqrt{d_1 r_1}$. The simulation is plotting in Figure 4.7. Numerically, we find that $\tilde{c}_u^* = 2.8299$ and $\tilde{c}_v^* = 2.83$; thus, $\tilde{c}_u^* = \tilde{c}_v^* = c_0^u$. In the picture, we can see that the terrace where $u = 1$ appeared in the initial data exists all the time. This phenomenon is reasonable. Since above the terrace $u = 1$, the traveling wave connects β to α_1. With the chosen initial condition, this upper wave (connecting β and α_1) propagates at its own spreading speed $\hat{c}^*$ at the beginning, but $\hat{c}^*$ also equals c_u^*. That means, the upper part cannot catch up with or pass the one ahead of it. Thus, the terrace keeps happening there. In fact, for (u_0, v_0) satisfying (a), (b), (c) (see the Introduction), as long as u_0 reaches zero ahead of v_0, one can always observe such a phenomenon when $\hat{c}^* = c_u^*$.

The last example shows that the upper part can indeed catch up with the one ahead of it, and finally the whole picture merges into a traveling wave with the slower

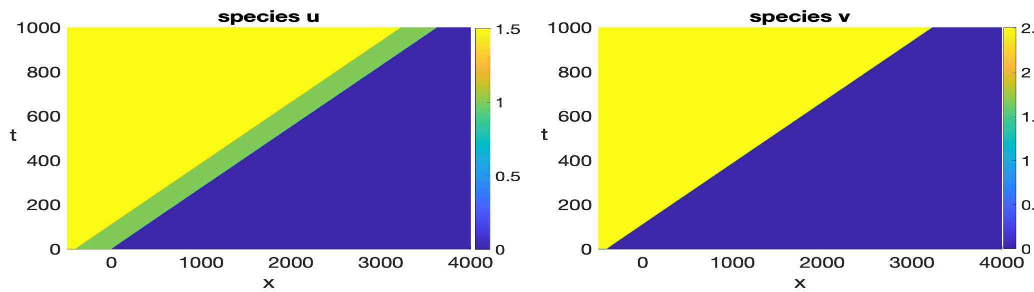


Figure 4.7: (Color online) Simulation of $(u, v)(t, x)$ for case $d_2r_2(1 + b_2) = d_1r_1$. The left figure is for u while the right one is for v.

speed, connecting β to $\mathbf{0}$ without the appearance of terrace.

Example 6.6 Choose

$$d_1 = 1,\ d_2 = 1,\ r_1 = 2,\ r_2 = 1,\ b_1 = 0.2,\ b_2 = 4,$$

then $e_1 = 6$, $e_2 = 25$, and $c_0^u = 2.8284$. Now, $d_2r_2(1 + b_2) > d_1r_1$ implies $c^* = c_f^*$ by Theorem 4.3.2. Through a simple computation, it is easy to find that

$$\bar{\mu}_3 = 0.4142 < \mu_0 = 1.4142 < \bar{\mu}_4 = 2.4142.$$

Then by Theorem 4.5.2, $c_u^* = c_0^u$. Based on the chosen initial data, when we consider the system restricted between α_1 and β (i.e., the auxiliary system), this traveling wave must propagate with the speed $\hat{c}^* \geqslant 2\sqrt{d_2r_2(1 + b_2)} = 4.472 > c_u^*$. It implies that the upper part moves faster than the lower one and they will merge somewhere. The simulation is shown in Figure 4.8. As we can see, around $t = 225$, they combine together as a traveling wave solution connecting β to $\mathbf{0}$, with the numerical spreading speed as the linear speed of u, i.e., $\tilde{c}_u^* = 2.824 \simeq c_0^u$.

4.7 Discussion

Propagation dynamics have extensive applications in practical areas such as population invasion in biology and combustion propagation in physics. Among the studies of the moving patterns, the investigation of the speed selection mechanism is challenging, especially for the case when multiple spreading speeds exist.

First of all, we would like to point out that our study in this chapter focuses

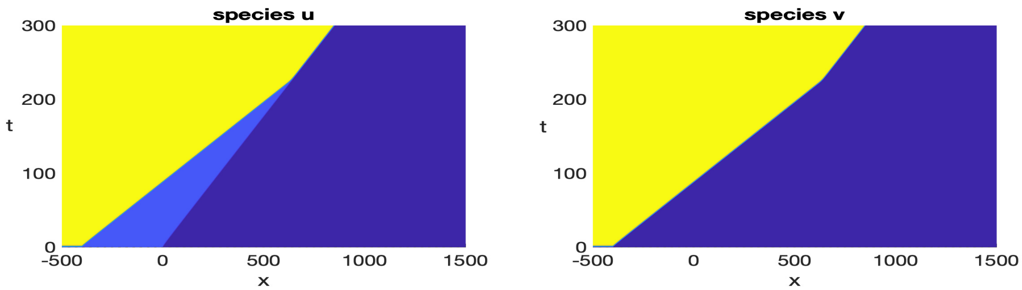


Figure 4.8: (Color online) Simulation of $(u, v)(t, x)$. The left figure is for u while the right one is for v.

on the speed selection mechanism for the individual spreading speed of each species. It provides a way to understand better the connection patterns of traveling wave profiles. We should also emphasize that our definition of the selection mechanism has been significantly developed than those in the classical case, see Definitions 4.2.6 and 4.2.9. Moreover, these definitions can be further extended to a system with n-species interaction, an interesting topic that will be studied in the future.

We also should mention that, for a system with only two equilibria β and zero (like the auxiliary wave profile system (4.2.13)-(4.2.14)), there exists a single spreading speed c^* and traveling wave profiles connecting β and zero always exist as long as $c \geq c^*$. However, for our main model (4.1.1), even when there exists a single spreading speed, the traveling wave may have different connections since the case (2)(ii) in Theorem 4.1.1 is not excluded. Thus, we need to find an upper solution connecting to zero so as to prove the existence of traveling waves connecting β to $\mathbf{0}$. The details can be found in Lemma 4.5.1 (1), see the conclusion under (4.5.4). Similar results can also be seen in [45, 46]. However, from our numerical simulation experience, we find that when $c_u^* = c_v^*$, with the initial data (satisfying (a) (b) and (c)) being properly assigned, the solution always stabilizes to a traveling wave connecting β to zero with the spreading speed. Thus, we propose a conjecture:

(**H**) if $c_u^* = c_v^*$, then for any $c \geqslant c_u^*$, (4.1.1) has a traveling wave connecting β to $\mathbf{0}$.

This is left for interested readers.

Finally, our numerical simulations show the existence of traveling waves with a terrace. This type of profile looks like a joint (gluing) or connection of two different traveling waves. For some parameter range, these terrace-like wave profiles initially appear and finally merge to form a classical traveling wave without any terrace. Our speed selection mechanism helps us better understand when and how this will happen.

Chapter 5

Determining spreading speeds for abstract time-periodic monotone semiflows

5.1 Introduction

Since the pioneering work of Fisher [24], and Kolmogorov, Petrovskii and, Piskunov (KPP) [37], traveling phenomena have been widely investigated in many practical fields such as biological invasions, combustion theory, and propagation of chemical materials modeled by reaction-diffusion equations, nonlocal dispersal systems and discrete lattice systems for evolution of a single population species in [3, 8–10, 24, 37], or for interactions of multiple species in [1, 2, 4, 20, 40, 43, 44, 48, 51, 83, 88]. In 1982, by using the language of dynamical system, Weinberger [86] generalized the above approaches by studying wave propagation for an abstract semiflow Q_t (or Q_1 with $t = 1$) in the so-called monostable case (when the zero equilibrium is unstable and the positive equilibrium, say β, is stable in the spatially-homogeneous environment). This abstract idea was further investigated in [20, 43, 44, 48, 53, 88]. By a traveling wave of the semiflow, we mean a special solution W, connecting zero and β, and satisfying

$$Q_t[W](x) = W(x - ct) \tag{5.1.1}$$

for some wave speed c. In the monostable case where only two equilibria exist, there is a crucial speed called the spreading speed c^*, which was firstly proposed by Aronson and Weinberger [3] in 1975 and was then thoroughly investigated by Lui [48]. This speed is an essential number for species invasion outwardly when the initial data have compact support, which can be recurringly defined by Lui's formula or alternatively can be understood as the limit of the solution level set over a time unit. In the case when only two equilibria exist, as seen in the Fisher-KPP model, there exists the so-called minimal speed $c_{\min}$ so that the dynamical equation (5.1.1) has a traveling wave solution, connecting zero and β, if and only if $c \geqslant c_{\min}$. Thanks to the paper of Liang and Zhao in [43], it was proved that these two speeds are amazingly equal (i.e., $c^* = c_{\min}$), although one may have difficulty in obtaining its explicit formulas. To estimate the spreading speed which is biologically significant, researchers resort to study the associated linear semiflow M_t obtained from the nonlinear map Q_t after linearizing at the equilibrium zero. A linear (spreading) speed c_0 can be readily derived from the characteristic equation of M_t and a comparison argument (see, e.g., [43]) always shows $c_{\min} \geqslant c_0$. Whether they are equal or not has becomes a challenging problem. We say that the minimal speed (or the spreading speed) is linearly selected if $c_{\min} = c_0$, and nonlinearly selected if $c_{\min} > c_0$. Physically or biologically, the traveling wave with the single spreading speed (the minimal speed) is called *pulled* wave if linear selection is realized, and *pushed* wave if nonlinear selection is realized. The dynamics behind a nonlinear selection can be understood in this way: a pushed wave is determined by its nonlinear "interior part," that is, it is pushed from behind by the whole system. Similarly, a pulled wave is determined only by the contribution of linearization about the unstable state, being pulled along by the dynamical force of the linear part at the far end.

As can be seen from [19, 20, 40, 88], most models concerning the interaction of multiple species may possess one or more equilibria between zero and the positive equilibrium β, possibly on the boundary of a box bounded by zero and β in the phase space, and this makes the speed selection problem even more challenging. Since the existence of more equilibria, a complicated model may admit more than one spreading speed: a slowest spreading speed c^* and a fastest one c^*_f (see, e.g. [48, 88], or (5.2.5) and (5.2.6) in Section 2 for details). When they are equal, we say a single spreading speed exists. As before, if the existence of traveling waves connecting zero and β is considered, due to the potential boundary equilibira, we are not sure whether there exists a minimal wave speed $c_{\min}$ so that a traveling wave solution of (5.1.1) exists if and only if $c \geqslant c_{\min}$. Furthermore, the linear semiflow M_t may be reducible and it has a fastest linear speed c_0^f. Depending on the situation whether a single spreading speed exists or not, we are interested in establishing the speed selection mechanism of c^*, and c_f^* as well as individual spreading speed of each species.

Contrast to traditional research topics as the existence, uniqueness and stability of traveling waves, the study of speed selection is challenging (see [18, 29, 29, 72, 74, 75, 77–79, 87] and references therein). W. Saarloos [77, 78] used the theory of linear and nonlinear marginal stability to investigate the speed selection, but all his arguments are formal (not rigorous). Lucia, Muratov, and Novaga [47] applied the variational principle and successfully established the speed selection mechanisms for a generalized KPP-Fisher model. However, this method is only valid for a scalar reaction-diffusion model or a model possessing a variational structure. Weinberger, Lewis, and Li [88] considered the recursion $u_{n+1} = Q[u_n]$ for a map Q and provided some sufficient conditions for linear selection under the condition when a single spreading speed exists. They didn't study nonlinear selection or linear selection when multiple spreading speeds appear. For further study on linear selection with a condition that the nonlinear system is bounded by its rival linear one, we refer to [5, 6, 20, 42, 54, 94].

Most recently, Ma and Ou [53] focused on the case when the map Q (or Q_t) is compact and has only two fixed points: zero and β, which implies that a single spreading speed always exists. They revealed an essential property about the nonlinear selection that a pushed traveling wave exists if and only if it decays to zero exponentially at the far end with a fast rate. Such a result is completely new in an abstract case. By constructing upper or lower solutions with particular decaying rates, they established a series of new easy-to-apply results on speed selection mechanism and extended their application to several practical models.

All aforementioned contributions are focused on traveling propagating in a homogeneous environment. However, in the natural environment, inhomogeneities are often present. For example, when formulating a model, seasonality and sunlight strength both introduce time-periodicity. Related references can be found in [6, 21, 42, 54, 91, 94] for time-periodic dynamical models.

In this chapter, we are concerned with the speed selection mechanism for traveling waves of an abstract time-periodic semiflow $\{Q_t\}_{t\geqslant 0}$, which is of monostable type with weak compactness and admits multiple equilibria. The difficulties and challenges in the study are mainly from below: (i) A time-periodic model is much more complicated than the traditional constant-coefficient one; (ii) The existence of extra equilibria between zero and β induces the possible non-uniqueness of the spreading speeds, and if a model indeed has two spreading speeds, to our knowledge, no references are focusing on classifying the speed selection currently; (iii) In the case when its linear system (around zero) is reducible so that the principal eigenvalue of linear map M_t doesn't possess a strongly positive eigenvector, problems arise in determining the linear speed, especially in the case where multiple linear spreading speeds exist; (iv) As to the fast decay of the pushed wave to the nonlinear semiflow Q_t, we need to determine whether the whole species components takes the same faster decay rate, or if not, which species will decay at a fast rate.

The remaining of this chapter aims at overcoming those difficulties and is organized as follows. We shall give the existence of the spreading speeds as well as traveling waves (see Lemma 5.2.4) in Section 5.2. Section 5.3 considers the linear semiflow (around zero) and establishes the linear speeds. Section 5.4 contains our main result on the speed selection, in which, to better solve (ii) and (iii), the study has been divided into three main cases: from subsections 5.4.1 to 5.4.3. We apply our results to four typical time-periodic models in Section 5.5: a delayed and diffusive equation, a stream population model with a benthic zone, a nonlocal dispersal Lotka-Volterra competitive model, and a reducible cooperative system. The last section presents a conclusion and some future study directions.

5.2 Spreading speeds and traveling waves of Q_t

Before investigating the speed selection mechanism for the abstract time-periodic semiflow Q_t, we need to establish the definition of spreading speeds as well as the existence of traveling waves. The idea was originated from [3,4,48] and further extended in [20, 41, 42, 88].

Let Ω be a compact metric space with metric d, $\mathbb{R}^l$ be the l-dimensional Euclidean space, and $\mathcal{X} = C(\Omega, \mathbb{R}^l)$. We endow $\mathcal{X}$ with the maximum norm $||\cdot||$ and define the positive cone as $\mathcal{X}_+ = C(\Omega, \mathbb{R}^l_+)$. Then $(\mathcal{X}, \mathcal{X}_+, ||\cdot||)$ is a Banach lattice. We use $\mathcal{M}$ to denote all the nonincreasing and bounded functions from $\mathcal{H}$ to $\mathcal{X}$, where $\mathcal{H} = \mathbb{R}$ or $\mathbb{Z}$. Any element in $\mathcal{X}$ can be viewed as a "constant" function in $\mathcal{M}$. Then we equip $\mathcal{M}$ with the compact open topology in the sense that $u_n \to u$ in $\mathcal{M}$ means that the sequence of $u_n(s)$ converges to $u(s)$ in $\mathcal{X}$ uniformly for s in any compact set of $\mathcal{H}$. For the definitions of orderings $>, \geqslant, \gg$, please see [20]. We say $\mathcal{S}$ of $\mathcal{M}$ is bounded if $\{|\phi(x)| : \phi \in \mathcal{S},\ x \in \mathcal{H}\}$ is bounded. For any subset $A \subset \mathcal{M}$ and $s \in \mathcal{H}$, we define $A(s) := \{u(s) : u \in A\}$. Moreover, we use the Kuratowski measure of noncompactness in $\mathcal{X}$ (see, e.g., [15]), that is,

$$\alpha(B) := \inf\{r : B \text{ has a finite cover of diameter} < r\},$$

for any bounded set $B \subset \mathcal{X}$. It is easy to derive that $\alpha(B) = 0$ if and only if B is precompact.

Since this chapter mainly concerns about the abstract time-periodic semiflow, we first introduce its definition. Let $\omega \in \mathcal{T}$ be the time period, where $\mathcal{T} = \mathbb{R}_+$ or $\mathbb{Z}_+$. Assume $\beta : \mathcal{T} \to \mathrm{Int}(\mathcal{X}_+)$ is continuous and ω-periodic in $t \in \mathcal{T}$, i.e., $\beta(t) = \beta(t+\omega)$. Then for any $t \in \mathcal{T}$, define

$$\mathcal{M}_{\beta(t)} := \{u \in \mathcal{M} : 0 \leqslant u(t) \leqslant \beta(t)\}, \text{ for } t \in \mathcal{T},$$

and assume that the map $Q_t : \mathcal{M}_{\beta(0)} \to \mathcal{M}_{\beta(t)}$ satisfies $Q_t[0] = 0$ and $Q_t[\beta(0)] = \beta(t)$.

Definition 5.2.1. *A family of mappings $\{Q_t\}_{t\in\mathcal{T}}$ is said to be an ω-periodic monotone semiflow from $\mathcal{M}_{\beta(0)} \to \mathcal{M}_{\beta(t)}$ if the following properties hold:*
(i) $Q_0[u] = u$, $\forall u \in \mathcal{M}_{\beta(0)}$;
(ii) $Q_{t+\omega}[u] = Q_t[Q_\omega[u]]$, $\forall\ t \in \mathcal{T}$ and $u \in \mathcal{M}_{\beta(0)}$;
(iii) $Q_t[u]$ is jointly continuous in (t, u) on $\mathcal{T} \times \mathcal{M}_{\beta(0)}$;
(iv) $Q_t[u] \geqslant Q_t[v]$ for all $t \in \mathcal{T}$ whenever $u \geqslant v$ in $\mathcal{M}_{\beta(0)}$.

The map

$$P = Q_\omega$$

is called the Poincaré map associated with this periodic semiflow. Clearly, $P[0] = 0$ and $P[\beta(0)] = \beta(\omega) = \beta(0)$, that means, 0 and $\beta(0)$ are two fixed points of P. Thus P maps $\mathcal{M}_\beta$ into itself, where we denote $\beta = \beta(0)$ for short.

Definition 5.2.2. *We say that $W(t, x-ct)$ is a periodic traveling wave solution of the ω-periodic semiflow $\{Q_t\}_{t\geqslant 0}$ with speed c if $W(t, z)$ is ω-periodic in t and nonincreasing* *in* $z \in \mathbb{R}$*, and there exists a countable subset* $\Sigma \subset \mathbb{R}$ *such that*

$$Q_t[W(0,\cdot)](x) = W(t, x-ct), \ \forall t \geqslant 0, \ x \in \mathbb{R}/\Sigma, \tag{5.2.1}$$

and

$$W(t,\pm\infty) \text{ exist such that } Q_t[W(0,\pm\infty)] = W(t,\pm\infty). \tag{5.2.2}$$

To proceed, we define a translation operator T_y on $\mathcal{M}_\beta$ as: for any $y \in \mathbb{R}$, $T_y[u](x) = u(x-y)$, $\forall\ x \in \mathcal{H}$. By [20, 42], we further assume that the ω-periodic semiflow Q_t satisfies the following assumptions:

(A1) (Translation invariance) $T_y \circ Q_t = Q_t \circ T_y$ for each $t \in \mathcal{T}$ and any $y \in \mathbb{R}$.

(A2) (Point-α-contraction) There exists a real number $k \in [0,1)$ such that for any $\mathcal{U} \subset \mathcal{M}_\beta$, $\alpha(P[\mathcal{U}](0)) \leqslant k\alpha(\mathcal{U}(0))$.

(A3) (Monostability) $P : \mathcal{X}_+ \to \mathcal{X}_+$ satisfies $P[0] = 0$ and $P[\beta] = \beta$ with $\lim_{n\to\infty} P^n[\varpi] = \beta$ for any $\varpi \in \mathcal{X}_+$ with $0 \ll \varpi \leq \beta$. The map P may also admit other boundary fixed points lying between β and 0. Biologically, it means that at least one of the species is extinct.

In view of (A1), it follows that (A2) is equivalent to $\exists\, k \in [0,1)$ such that $\alpha(P[\mathcal{U}](x)) \leqslant k\alpha(\mathcal{U}(x))$ for any $\mathcal{U} \subset \mathcal{M}_\beta$ and $x \in \mathcal{H}$. Note that the assumption (A2) is much weaker than the classical compact assumption; that means, if $P[\mathcal{M}_\beta]$ is precompact in $\mathcal{M}_\beta$, then it satisfies (A2) by choosing $k = 0$. For more interpretations of this assumption, please refer to [20]. The assumption (A3) implies that β is the minimal strictly positive equilibrium of P; biologically, there is no other all-species coexistence equilibrium below β.

To define the spreading speed of Q_t, we employ the idea in [42]. We first define the spreading speed of $P = Q_\omega$. Let $\varpi \in \mathcal{X}_\beta$ with $0 \ll \varpi \ll \beta$, and choose ϕ to be a continuous function from $\mathbb{R}$ to $\mathcal{X}$ with following properties:

(B1) ϕ is a nonincreasing function;

(B2) $\phi(x) = 0$ for $x \geq 0$;

(B3) $\phi(-\infty) = \varpi$.

Let c be a given real number, we define an operator R_c by

$$R_c[a](s) := \max\{\phi(s), T_{-c}[P[a]](s)\}, \tag{5.2.3}$$

and a sequence of functions $\{a_n(c;s)\}_{n\geqslant 0}$ by

$$a_0(c;s)=\phi(s),\ \ a_{n+1}(c;s)=R_c[a_n(c;\cdot)](s). \tag{5.2.4}$$

Then, following the idea of [20] which originates from Lui [48], the sequence has the following properties:
(a) $a_n\leqslant a_{n+1}\leq\beta$;
(b) $a_n(c;s)$ is nondecreasing in n and nonincreasing in both s and c, and continuous in (c,s);
(c) For each n, $a_n(c;-\infty)\geqslant P^n[\varpi]$ and $a_n(c;+\infty)=0$;
(d) $\lim\limits_{n\to\infty} a_n(c;s)=a(c;s)$ pointwise and $a(c;s)$ is nonincreasing in s and c;
(e) $a(c;-\infty)=\beta$ and $a(c;+\infty)$ exists in $\mathcal{X}_\beta$ and is a fixed point of P.

Define

$$c^*:=\sup\{c:a(c;+\infty)=\beta=\beta(0)\}, \tag{5.2.5}$$

and

$$c_f^*:=\sup\{c:a(c;+\infty)>0\}. \tag{5.2.6}$$

Clearly, $c^*\leqslant c_f^*$. We call c^* as the slowest spreading speed and c_f^* the fastest spreading speed. If $c^*=c_f^*$, then we say that this system has a single spreading speed.

Remark 5.2.3. *The above definition implies that the slowest spreading speed is a number c^* so that*

$$\lim_{x\leq(c^*-\varepsilon)n} P^n(\phi)=\beta,\ \ \limsup_{x\geq(c^*+\varepsilon)n} P^n(\phi)<\beta,\quad \text{as } n\to\infty \tag{5.2.7}$$

for any small positive ε, and the fastest speed is a number c_f^* so that

$$\liminf_{x\leq(c_f^*-\varepsilon)n} P^n(\phi)>0,\ \ \lim_{x\geq(c_f^*+\varepsilon)n} P^n(\phi)=0,\quad \text{as } n\to\infty \tag{5.2.8}$$

for any small positive ε. When $\mathcal{H}=\mathbb{Z}$, we prefer to define the two speeds in this manner.

Back to Q_t, it is easy to see that there exist two spreading speeds $\frac{c_f^*}{\omega}$ and $\frac{c^*}{\omega}$. Now, applying Theorem 3.8 in [20] and Theorems 2.1, 2.2 in [42], the existence of an

ω-periodic traveling waves can be summarized into the following lemma.

Lemma 5.2.4. *(see, [20,42]) Let Q_t be an ω-periodic semiflow satisfy (A1)-(A3), in which $P = Q_\omega$, and c^*, c_f^* be defined in (5.2.5) and (5.2.6) respectively. Then the following statements are valid:*

(1) For any $c \geqslant c^*/\omega$, Q_t has a left-continuous traveling wave $W(t, x-ct)$ connecting $\beta(t)$ to an ω-periodic function $\alpha(t) < \beta(t)$, where $\alpha(t)$ satisfies $Q_t[\alpha(0)] = \alpha(t)$.

(2) If, in addition, 0 is isolated from the other equilibrium, then for any $c \geqslant c_f^*/\omega$, either of the following holds true.

(2.1) There exists a traveling wave $W(t, x-ct)$ connecting $\beta(t)$ to 0;

(2.2) P has two ordered fixed points $\alpha_1(0)$ and $\alpha_2(0)$ in $\mathcal{X}_\beta/\{0, \beta\}$ such that Q_t has a left-continuous ω-periodic traveling wave $W_1(t, x-ct)$ connecting $\alpha_1(t)$ to 0 and a left-continuous traveling wave $W_2(t, x-ct)$ connecting $\beta(t)$ to $\alpha_2(t)$.

(3) For any $c < c^*/\omega$, Q_t has no traveling wave connecting $\beta(t)$, and for any $c < c_f^*/\omega$, Q_t has no ω-periodic traveling wave connecting $\beta(t)$ to 0.

Based on the above lemma and following Theorem 3.1 from [41], we can further have the following lemma, which gives sufficient conditions for $c^* = c_f^*$.

Lemma 5.2.5. *Under the conditions of Lemma 5.2.4, if (2.2) in Lemma 5.2.4 can be excluded, then Q_t has a single spreading speed. Moreover, if $P = Q_\omega$ has only two fixed points 0 and β in $\mathcal{X}_\beta$, then $c^* = c_f^*$.*

Remark 5.2.6. *Besides the sufficient condition for a single spreading speed mentioned in the above lemma, if P satisfies the conditions of Theorem 3.1 or 3.2 in [88], then $c^* = c_f^*$.*

5.3 Linear speed(s) near the extinction state

In this section, we introduce the definition of linear speed(s) near the extinction state 0 for the abstract semiflow Q_t, which gives an estimate of the spreading speeds.

Assume that for any $t \geqslant 0$,

(A4) Q_t is Fréchet-differentiable around any $\varphi(t) \in [0, \beta(t)]$.

Now, let M_t be the linearization operator of Q_t around 0 in the following sense:

$$M_t[\varphi] = \lim_{\rho \to 0} \frac{Q_t[\rho\varphi]}{\rho}. \tag{5.3.1}$$

Assume that M_t is a map from $\mathcal{M}$ to $\mathcal{M}$, and satisfies the properties of translation invariance, jointly continuous and point-α-contraction, i.e., (A1) and (A2) hold by replacing Q_t by M_t and $P = Q_\omega$ by M_ω respectively. Next, we introduce further hypotheses for M_t:

(C1) M_t is a positive operator, that is, $M_t[v] \geqslant 0$ whenever $v > 0$.

(C2) Define a linear map $B^t_\mu : \mathcal{X} \to \mathcal{X}$ as

$$B^t_\mu[v] = M_t[e^{-\mu x} v](x = 0),$$

where μ is a positive real number and v is a real vector. Let B^ω_μ be the Poincaré map associated with B^t_μ. We shall assume that B^ω_μ is a $l \times l$ matrix and is in Frobenius form, whose definition can be found in [88]. Without loss of generality, we further assume that it has N_0 elements of diagonal blocks. Let $\lambda_i(\mu)$ ($i = 1, \cdots, N_0$) be the principal eigenvalue of the i-th diagonal block of B^ω_μ. Among all the principal eigenvalues, we always assume that $\lambda_1(0) > 1$, $\lambda_i(0) \neq 1$ for all i. Then, for the whole system, we denote

$$I_0 := \{ i \,|\, \lambda_i(0) > 1, \text{ for } 1 \leqslant i \leqslant N_0 \},$$

and $I_1 := \{1, \cdots, N_0\} \backslash I_0$.

(C3) Let $M_{t,\varphi}$ be the linearization of Q_t around $\varphi(t)$, where $0 \leqslant \varphi(t) \leqslant \beta(t)$. In particular, define a linear map $B^t_{\gamma,\beta(t)} : \mathcal{X} \to \mathcal{X}$ by

$$B^t_{\gamma,\beta(t)}[\phi] = M_{t,\beta(t)}[e^{\gamma x}\phi](x = 0).$$

Then $B^\omega_{\gamma,\beta}$ is the Poincaré map associated with $B^t_{\gamma,\beta(t)}$. We also assume that $B^\omega_{\gamma,\beta}$ is in Frobenius form and has N_β diagonal blocks. Then, for each j-th block, there exists a simple principal eigenvalue denoted as $\bar{\lambda}_j(\gamma)$ $(j = 1, \cdots, N_\beta)$. Furthermore, since β is a stable fixed point for $P|_{\mathcal{X}_+}$ (see (A3)), $\bar{\lambda}_j(0)$ is further assumed to be less than 1 for all j.

By a similar argument to the spreading speed analyzed previously, when some boundary equilibria occur, the matrix B_μ^ω may be reducible and the definition of single linear speed c_0 in [53] needs to be developed. Thus, we introduce the idea from [88] to handle this difficulty. Notice that, by (C2), for the uppermost (first) block, we assume that there always exists

$$c_1^\omega = \inf_{\mu>0} \frac{\ln \lambda_1(\mu)}{\mu} = \frac{\ln \lambda_1(\bar{\mu}^1)}{\bar{\mu}^1}, \quad \text{and} \quad c_0^1 = \frac{c_1^\omega}{\omega}$$

for a finite number $\bar{\mu}^1$. Moreover, if $N_0 > 1$ in (C2), we can also define

$$c_i^\omega = \inf_{\mu>0} \frac{\ln \lambda_i(\mu)}{\mu} = \frac{\ln \lambda_i(\bar{\mu}^i)}{\bar{\mu}^i}, \ i \in I_0,$$

where $\bar{\mu}^i$ is assumed to be finite. Among all of them, there must exist the maximum number of the above speed, denoted as

$$c_\sigma^\omega := \max_{i \in I_0} \{c_i^\omega\}, \quad \text{for some } \sigma \in I_0.$$

Thus, Q_t has a faster linear spreading speed, defined as

$$c_0^f = \frac{c_\sigma^\omega}{\omega}. \tag{5.3.2}$$

Now, we turn to find properties of λ_i. In (C2), for each i, $\lambda_i(\mu)$ is log convex with respect to μ (see, [48]); thus, it is easy to arrive at the following lemma.

Lemma 5.3.1. *(1) For each $i \in I_0$, the following two statements are true.*

(a) For any $c > c_i^\omega/\omega$, there exist two positive numbers $\mu_1^i(c) < \bar{\mu}^i < \mu_2^i(c)$ such that

$$c = \frac{1}{\omega} \cdot \frac{\ln \lambda_i(\mu_1^i)}{\mu_1^i} = \frac{1}{\omega} \cdot \frac{\ln \lambda_i(\mu_2^i)}{\mu_2^i}, \tag{5.3.3}$$

and $\frac{1}{\omega} \cdot \frac{\ln \lambda_i(\mu)}{\mu} < c$ for any $\mu \in (\mu_1^i, \mu_2^i)$. Moreover, $\mu_1^i(c)$ is a decreasing *function while* $\mu_2^i(c)$ *is an increasing function in* c.

(b) *for* $c = c_i^\omega/\omega$, $\mu_1^i(c) = \mu_2^i(c) = \bar{\mu}^i$.

(2) *For each* $i \in I_1$ *and any* $c > c_0^1$*, similarly there exists an unique solution* $\mu^i = \mu_3^i$ *such that*

$$-\mu^i c\omega + \ln \lambda_i(\mu^i) = 0. \tag{5.3.4}$$

Moreover, $\mu_3^i(c)$ *is increasing in* c.

Remark 5.3.2. *It is clear that if* $c \geqslant c_0^f$*, (1)(a) and (b) in the above lemma are also true due to the definition of* c_0^f.

5.4 Speed selection mechanism

We have defined four essential numbers: c^*, c_f^*, c_0^1, and c_0^f. However, their explicit formulas are usually unknown. Therefore, we proceed to study the determinacy of the spreading speeds by the two linear speeds c_0^1 and c_0^f that are derived by the characteristic equations of the linear semiflow M_t.

Define a projection operator $\mathcal{P}_i$ as $\mathcal{P}_i[w]$, which takes the same coordinate value as w in the directions corresponding to the i-th diagonal block of B_0^ω, and zero values in the other direction components. Since $Q_t[\mathcal{P}_\sigma[w]] \leqslant Q_t[w]$ for $t \in \mathcal{T}$, following the idea from [88], it is easy to see that

$$c_0^f \leqslant c_f^*/\omega.$$

From the classical scaler Fisher-KPP model with $f(t,u) \equiv f(u)$, say in [26,47,53], we know that pushed and pulled traveling waves possess different exponential decaying behaviors around the unstable zero equilibrium. In [53], under time-homogeneous coefficient environment with only two fixed points, if the linear map M_t is irreducible or the principal eigenvalue possess a strongly positive eigenvector in the whole space, the authors rigorously proved that nonlinear selection is realized if and only if the pushed wave decays to zero at the far end in a faster rate. For a time-periodic semiflow with more fixed points such that a single spreading speed may not exist, or a case that the principal eigenvalue of linear map M_t doesn't possess a strongly positive eigenvector (e.g., in the reducible case), the study of speed determinacy becomes complex and challenging. Due to this, we want to divide our study into the following three cases.

(I). The linear system around zero is irreducible, that is, the matrix B_0^ω is irreducible.

(II). The linear system around zero is reducible, with $\lambda_i(0) < 1$ for all $i > 1$, that is, $N_0 = 1$ in (C2) or $I_0 = \{1\}$.

Biologically, this means that in the absence of other species, the first species is unstable near zero and it becomes the source of the invasion.

(III). The linear system around zero is reducible, and $\lambda_i(0) > 1$ for some $i > 1$, that is, $N_0 > 1$ in (C2) or $I_0 \backslash \{1\} \neq \emptyset$.

Biologically, this means that, in the absence of other species, there exists at least one other species (except the first one) that is unstable near zero. This results in the existence of non-unique spreading speeds and a competition between them may happen.

5.4.1 Case (I): an irreducible linear system

We begin with investigating the case when the linear system M_t is irreducible. In this case, we will have a single spreading speed.

Let $\{Q_t\}_{t\in\mathcal{T}}$ be an ω-periodic semiflow satisfying (A1)-(A4). When B_0^ω is irreducible, we find that, by comparison principle, P cannot have a fixed point α (with the first component to be zero) between zero and β. Indeed, to the contrary, if there is such α, then the first component of $B_0^\omega(\varepsilon\alpha)$, for any small ε, must be positive due to fact that B_0^ω is positive and irreducible (in fact, the monotone operator B_0^ω is strongly positive). Since B_0^ω is a linearization of P, we can derive that the first component $P(\varepsilon\alpha)$ must be positive for a sufficiently small ε, so is the first component of $P(\alpha)$ by comparison principle. This is a contradiction. As such, there exist only two fixed points zero and β for P and a single spreading speed exists.

With the understanding that (2.2) in Lemma 5.2.4 is excluded, we have the following result.

Lemma 5.4.1. *If B_0^ω is irreducible, then we have*

$$\frac{c^*}{\omega} = \frac{c_f^*}{\omega}, \;\; c_0^f = c_0^1 =: c_0.$$

Thus, the speed determinacy definition of this spreading speed is given as follows.

Definition 5.4.2. *When B_0^ω is irreducible, we say the single spreading speed of Q_t is linearly selected if $c_f^*/\omega = c_0$ and nonlinearly selected if $c_f^*/\omega > c_0$.*

Assume that a traveling wave W_c exists for $c > c_0$, connecting β and zero. Since $W_c(t, x) \to 0$ uniformly for $t \in \mathbb{R}$, as $x \to \infty$, the asymptotic behaviors of the wave satisfying (5.2.1) can be obtained by the linearized system at zero as well as its corresponding characteristic equation. Since the structure of the nonlinear map near $x = \infty$ is geometrically hyperbolic in the sense that no real part of the eigenvalue (decay rate) is zero, the behavior of the wavefront can be readily derived by the linear wave-profile equation via the theory of asymptotic analysis. From the linear semiflow, we can derive that, for any $c > c_0$, the asymptotic behavior of a positive traveling wave solution W_c (if it exists) is given by

$$W_c(t, x) = C_1 e^{-\mu_1(c)x} \zeta_{\mu_1(c)}(t) + C_2 e^{-\mu_2(c)x} \zeta_{\mu_2(c)}(t) \text{ as } x \to \infty, \tag{5.4.1}$$

where $C_1 > 0$ or $C_1 = 0,\ C_2 > 0$. Here, $\mu_i = \mu_i^1$ $(i = 1, 2)$ is defined in Lemma 5.3.1, ζ_{μ_i} is the strongly positive eigenfunction corresponding to μ_i. For a rigorous proof of such a behavior, please refer to [12, 30, 65, 66].

Now, we first give a necessary and sufficient condition for the nonlinear selection.

Theorem 5.4.3. *Let $\{Q_t\}_{t\in\mathcal{T}}$ satisfy (A1)-(A4) and B_0^ω is irreducible. The following results hold true:*

(i). There exists a critical number

$$c_{\min} = \frac{c^*}{\omega} = \frac{c_f^*}{\omega}$$

such that Q_t has an ω-periodic wave solution $W_c(t, x)$, connecting zero and β, if and only if for $c \geqslant c_{\min}$.

(ii). Assume that the wave W_c *has a continuous derivative* $W_c'(t,x)$*, and the prime "$'$" denotes the derivative with respect to the second variable. Moreover, we further assume that* $M_{\omega,W_{c_{\min}}}$ *has a simple principal eigenvalue with a strongly positive eigenfunction. Then, the following two statements are equivalent:*

*(**F1**)* $c_{\min} > c_0$*, that means, the nonlinear selection is realized;*

*(**F2**) there exists a speed* $\bar{c} > c_0$ *such that* Q_t *has an* ω*-periodic traveling wave* $W_{\bar{c}}(t,x)$ *(see, Definition 5.2.2), having the following property:*

$$W_{\bar{c}}(t,x) = Ce^{-\mu_2(\bar{c})x}\zeta_{\mu_2(\bar{c})}(t), \text{ as } x \to +\infty,$$

for some positive constant C *and* $\mu_2(\bar{c}) = \mu_2^1(\bar{c})$ *is defined in (5.3.3) and* $\zeta_{\mu_2(\bar{c})}(t)$ *is the corresponding eigenfunction.*

Proof. (i). The proof of this part follows from Lemmas 5.2.5 and 5.4.1.

(ii). The proof is similar to that in [53] except that we need keep in mind that the semiflow Q_t is now time-periodic. To make the chapter self-contained, we state it in two steps: one for the sufficiency and the other one for the necessity.

Step 1 (the sufficiency, from (***F2***) to (***F1***)): We shall show that Q_t has no traveling waves for any c in $[c_0, \bar{c})$ by way of contradiction.

For $c \in (c_0, \bar{c})$ (maybe close to $\bar{c}$), to the contrary, we suppose that Q_t has an ω-periodic traveling wave $W_c(t,x)$. By (5.4.1), we know that, when $c > c_0$, $W_c(0,x)$ has the following behavior:

$$W_c(0,x) \sim W_0 \sim C_1 e^{-\mu_1(c)x}\zeta_{\mu_1(c)}(0) + C_2 e^{-\mu_2(c)x}\zeta_{\mu_2(c)}(0), \text{ as } x \to \infty,$$

with $C_1 > 0$ or $C_1 = 0,\ C_2 > 0$. As discussed in Lemma 5.3.1, $\mu_1(c)$ and $\mu_2(c)$ are continuous and monotone in c, i.e., $\mu_1(\bar{c}) < \mu_1(c) < \mu_0 < \mu_2(c) < \mu_2(\bar{c})$. This leads to a conclusion $W_c(0,x) \gg W_{\bar{c}}(0,x)$ when x is near the positive infinity.

Near the negative infinity, i.e., $x \to -\infty$, we also have a claim that $W_{\bar{c}}(0,x) \ll W_c(t,x)$ as $x = -\infty$, if $c < \bar{c}$. To see this, we want to use the wave profile equation as well as its linearization at $\beta(t)$. As such, asymptotically, we let the asymptotic behavior of $W_c(t,x)$ near negative infinity to be

$$W_c(t,x) \sim \beta(t) - e^{\gamma x}\zeta_\gamma(t)$$

for some positive number γ and ω-periodic function $\zeta_\gamma(t)$. This γ is dependent on c and we want to derive the relationship. By (C3), $B^\omega_{\gamma,\beta}$ has N_β diagonal blocks, and there exists a principal eigenvalue $\bar{\lambda}_j(\gamma)$ ($j = 1, \cdots, N_\beta$) for each block. From the linearization of the equation $Q_t[W_c(t,\cdot)](x) = W_c(t, x-ct)$ at $\beta(t)$, we can derive that $\gamma c\omega + \ln \bar{\lambda}_j(\gamma) = 0$, $j = 1, \cdots, N_\beta$. Since $\bar{\lambda}_j(0) < 1$ and $\ln \bar{\lambda}_j(\gamma)$ is convex regarding to γ, one can find that, for each $j = 1, \cdots, N_\beta$, there exists a unique γ_j solving this equation, and each $\gamma_j = \gamma_j(c)$ is decreasing in c for $c \geq c_0$.

If all the components of W_c decay at a same exponential rate, i.e., if $B^\omega_{\gamma,\beta}$ has a strongly positive principal eigenvector $\zeta_{\bar{\gamma}}(0)$ with $\bar{\gamma} = \min\limits_{1\leqslant j\leqslant N_\beta}\{\gamma_j\}$, then

$$W_c(0,x) \sim \beta(0) - e^{\bar{\gamma}x}\zeta_{\bar{\gamma}}(0), \text{ as } x \to -\infty.$$

Moreover, $\bar{\gamma}(\bar{c}) < \bar{\gamma}(c)$ since $c < \bar{c}$. Thus, $W_c(0,x) \gg W_{\bar{c}}(0,x)$ as $x \to -\infty$.

If $B^\omega_{\gamma,\beta}$ does not have a strongly positive eigenvector for the whole system, then for each j-th block, the decay rate $\gamma_j = \gamma_j(c)$ is decreasing in c. We still can conclude that $W_c(0,x) \gg W_{\bar{c}}(0,x)$ as $x \to -\infty$.

Therefore, by a shift of distance ξ_0 for x, we can make $W_c(0, x+\xi_0)$ satisfy

$$\bar{W}_c(0,x) = W_c(0, x+\xi_0) \gg W_{\bar{c}}(0,x).$$

Then, by the monotonicity of P, it follows that

$$\bar{W}_c(0, x-cn\omega) = P^n[\bar{W}_c(0,\cdot)](x) \geq P^n[W_{\bar{c}}(0,\cdot)](x) = W_{\bar{c}}(0, x-\bar{c}n\omega) \qquad (5.4.2)$$

for $x \in \mathcal{H}$. Fixing some $z_0 = x - \bar{c}n\omega$, then $W_{\bar{c}}(0, x-\bar{c}n\omega) = W_{\bar{c}}(0, z_0) \gg 0$. On the other hand, we will have

$$\bar{W}_c(0, x - cn\omega) = \bar{W}_c(0, z_0 + (\bar{c} - c)n\omega) \to 0, \text{ as } n \to \infty,$$

which contradicts to the inequality (5.4.2). Therefore, Q_t has no traveling waves when $c \in (c_0, \bar{c})$.

Finally, if for $c = c_0$, Q_t has traveling waves, then Q_t has traveling waves for all $c \geqslant c_0$ by Lemmas 5.2.4 and 5.2.5. choosing a $c \in (c_0, \bar{c})$, we can repeat the above process to get a contradiction. The proof of this part is complete.

Step 2: (the necessity, from (***F1***) to (***F2***)). By assumption, we need to prove that, if $c_{\min} = c_f^*/\omega > c_0$, then the ω-periodic traveling wave $W_{c_{\min}}$ of Q_t has the following property:

$$W_{c_{\min}}(t, x) \sim C_2 e^{-\mu_2(c_{\min})x} \zeta_{\mu_2(c_{\min})}(t) \text{ as } x \to \infty,$$

for some constant $C_2 > 0$. Here again, we prove it by way of contradiction. Thus, to the contrary, at $t = 0$ we assume that

$$W_{c_{\min}}(0, x) \sim C_3 e^{-\mu_1(c_{\min})x} \zeta_{\mu_1(c_{\min})}(0) \text{ as } x \to \infty, \tag{5.4.3}$$

for some positive constant C_3 and eigenvector $\zeta_{\mu_1(c_{\min})}(0)$. Then we want to prove that, under this assumption, P does have a traveling wave $W_c(0, x)$ satisfying

$$P[W_c(0, \cdot)](x) = W_c(0, x - c\omega), \text{ or } T_{-c\omega} P[W_c(0, \cdot)](x) = W_c(0, x) \tag{5.4.4}$$

for some speed $c = c_{\min} - \delta$ with $\delta > 0$ being sufficiently small. Such a result implies that c_f^* is not the minimal speed of P, which induces a contradiction to the definition of $c_{\min}$. Under the assumption (5.4.3), we introduce a weighted function by

$$\bar{W}(x) = W_{c_{\min}}(0, x)\varrho(x), \text{ where } \varrho(x) = \frac{1}{1 + \delta e^{[\mu_1(c) - \mu_1(c_{\min})]x} \cdot \frac{\zeta_{\mu_1(c)}(0)}{\zeta_{\mu_1(c_{\min})}(0)}}.$$

The division and multiplication in the above formula are componentwise, so $\bar{W}(x)$ is well-defined componentwisely. We shall emphasize that the modified function $\bar{W}$ is close to $W_{c_{\min}}$ when δ is sufficiently small, but it has different decaying rates near positive infinity (indeed, $\bar{W}(x)$ has the same decay rate as $W_c(0,x)$ if it exists). Now, we want to apply a perturbation argument under the assumption (5.4.3) to prove the existence of W_c to (5.4.4) for sufficiently small δ. To proceed, we set

$$W_c = \bar{W}(x) + W_1 \tag{5.4.5}$$

and then back-substitute it into (5.4.4) to have

$$T_{-c\omega}P[\bar{W}(x) + W_1] = \bar{W}(x) + W_1,$$

where $W_1 = W_1(0,x)$ is a function to be determined. Through a direct computation and simplification, the equation of W_1 is obtained as

$$W_1 = T_{-c_{\min}\omega}M_{\omega,W_{c_{\min}}}[W_1] + F_0 + M_\omega^\delta[W_1] + F_h[W_1], \tag{5.4.6}$$

where

$$F_0 = T_{-c\omega}P[\bar{W}(x)] - \bar{W}(x),$$

$$M_\omega^\delta[W_1] = \left(T_{-c\omega}M_{\omega,\bar{W}(x)} - T_{-c_{\min}\omega}M_{\omega,W_{c_{\min}}}\right)[W_1],$$

and

$$F_h[W_1] = T_{-c\omega}P[\bar{W}(x) + W_1] - T_{-c\omega}P[\bar{W}(x)] - T_{-c\omega}M_{\omega,\bar{W}(x)}[W_1].$$

Here $M_{\omega,W_{c_{\min}}}$ and $M_{\omega,\bar{W}(x)}$ denote the Fréchet derivative of P around $W_{c_{\min}}$ and $\bar{W}(x)$ respectively, i.e., $M_{\omega,W_{c_{\min}}}$ is defined by

$$M_{\omega,W_{c_{\min}}}[\varphi] = \lim_{\eta\to 0}\frac{P[W_{c_{\min}} + \eta\varphi] - P[W_{c_{\min}}]}{\eta}.$$

Now, the existence of solution W_1 to (5.4.6) implies the existence of W_c to (5.4.4);

thus, we then focus on investigating solutions to (5.4.6). Through a simple estimate, we observe that $M_\omega^\delta[W_1] = O(\delta)W_1$, $F_0 = O(\delta)$ with $F_0 = o(e^{-\mu_1(c_{\min})x})$ as $x \to \infty$, $F_h = O(W_1^2)$ which is a higher order term.

To further study the existence of solutions to (5.4.6), we notice that W_1 is in the space $\mathcal{M}_0$, where

$$\mathcal{M}_0 = \{u(0,\cdot) \in \mathcal{M} : u(0,\pm\infty) = 0\}.$$

By assumption (ii) and a direct computation, we find that the operator $T_{-c_{\min}\omega}M_{\omega,W_{c_{\min}}}$ has a simple principal eigenvalue $\lambda = 1$ and its corresponding eigenvector is $\bar{u} = W'_{c_{\min}} \in \mathcal{M}_0$, which represents the first derivative of $W_{c_{\min}}(0,x)$ with respect to x. It is easy to find that $W'_{c_{\min}}$ and $W_{c_{\min}}$ have a common asymptotic behavior as $x \to \infty$, that is,

$$W'_{c_{\min}} \sim C_4 e^{-\mu_1(c_{\min})x} \quad \text{as} \quad x \to \infty$$

for some vector C_4.

To find a solution of (5.4.6), we need to omit this eigenvector $\bar{u}$, so we define a weighted space $\mathcal{W}$ by

$$\mathcal{W} = \{u \in \mathcal{M}_0 : ue^{\mu_1(c_{\min})x} = o(1) \text{ as } x \to \infty\}.$$

It is clear that the space $\mathcal{W}$ has excluded the eigenvector $\bar{u} = W'_{c_{\min}}$, and $T_{-c_{\min}\omega}M_{\omega,W_{c_{\min}}}$ has no eigenvalue $\lambda = 1$ in this space. By assumption, the operator $T_{-c_{\min}\omega}M_{\omega,W_{c_{\min}}}$ has $\lambda = 1$ as its simple principal eigenvalue with a strongly positive eigenfunction in $\mathcal{W}$, we know that $I - T_{-c_{\min}\omega}M_{\omega,W_{c_{\min}}}$ has a bounded inverse in $\mathcal{W}$, where I is the identity operator. Applying the well-known inverse function theorem in the abstract space $\mathcal{W}$, we obtain a conclusion: there exists a small number $\delta_0 > 0$ such that for any $\delta \in [0, \delta_0)$, the equation (5.4.6) has a solution W_1 in $\mathcal{W}$. Moreover, for sufficiently small δ, the positivity of the solution W_c is guaranteed. Thus, we have proved the existence of W_c to (5.4.4) when $c = c_{\min} - \delta$. This completes the proof. $\square$

Remark 5.4.4. *Theorem 5.4.3 reveals an essential property of the nonlinear selection,* *i.e., the pushed wave W_{c^*} has the fast decaying rate. The condition in case (I) can be satisfied by many biological and physical models, such as the stream population model(see, [91]).*

In practice, it is not easy to verify Theorem 5.4.3, since we are not easy to find an exact traveling wave with a particular decay rate, so we shall provide the following easy-to-apply condition for the nonlinear selection.

Theorem 5.4.5. *Assume that $\{Q_t\}_{t\in\mathcal{T}}$ satisfy (A1)-(A4), M_t satisfies (C1)-(C3), and B_0^ω is irreducible. If for $c_1 > c_0$, suppose that there exists a continuous ω-time-periodic function $\underline{W}(t,x)$ satisfying*

$$0 \ll \underline{W}(t,x) \ll \beta(t), \ \ \limsup_{x\to-\infty} \underline{W}(t,x) \ll \beta(t), \ \ \underline{W}(t,x) = e^{-\mu_2(c_1)x}\zeta_{\mu_2(c_1)}(t) \ \text{ as } x \to \infty,$$

and

$$Q_t[\underline{W}(0,\cdot)](x) \geqslant \underline{W}(t, x - c_1 t),$$

where $\mu_2(c_1) = \mu_2^1(c_1)$ is defined in (5.3.3). Then $c_{\min} \geqslant c_1$ and no traveling waves exist for $c \in [c_0, c_1)$. In other words, the nonlinear selection is realized.

Proof. We prove this theorem by way of contradiction, which is similar to Step 1 of Theorem 5.4.3; thus, it is omitted. □

Corollary 5.4.6. *Under the assumptions that $\{Q_t\}_{t\in\mathcal{T}}$ satisfy (A1)-(A4), M_t satisfies (C1)-(C3), and B_0^ω is irreducible, if there exists an ω-periodic function $\underline{W}(t,x) = \frac{\beta(t)}{1+e^{\bar{\mu}x}/\zeta_{\bar{\mu}}(t)} := \left(\frac{\beta^1(t)}{1+e^{\bar{\mu}x}/\zeta_{\bar{\mu}}^u(t)}, \cdots, \frac{\beta^l(t)}{1+e^{\bar{\mu}x}/\zeta_{\bar{\mu}}^l(t)}\right)$, where $\bar{\mu} = \bar{\mu}^1$ defined in Lemma 5.3.1, is a strongly-strict lower solution in the sense that*

$$Q_t[\underline{W}(0,\cdot)](x) \gg \underline{W}(t, x - c_0 t),$$

then the nonlinear selection is realized, i.e., $c_{\min} > c_0$.

Proof. By the continuity, there is a constant number c_1 which is slightly larger but sufficiently close to c_0, and an ω-time-periodic function $\beta_1(t)$, which is approaching

$\beta(t)$ from below, such that $\underline{W}_1(t,x) = \frac{\beta_1(t)}{1+e^{\mu_2(c_1)x}/\zeta_{\mu_2(c_1)}(t)}$ is a lower solution in the sense that $Q_t[\underline{W}_1(0,\cdot)](x) \geqslant \underline{W}_1(t, x-c_1t)$. Then, the result can be directly deduced by Theorem 5.4.5. Thus, the proof is complete. □

When an ω-periodic semiflow is known to be nonlinearly selected, i.e., $c_{\min} = c_f^*/\omega > c_0$, it is usually hard to obtain the exact formula for $c_{\min}$. Even for constant-coefficient cases, there are only limited results, e.g., [29, 47]. We want to give an estimate of $c_{\min}$ when the nonlinear selection is realized. It is easy to see that Theorem 5.4.5 has already given a lower bound for $c_{\min}$, i.e., $c_{\min} \geqslant c_1$. Now, we will give the following theorem for an upper bound of $c_{\min}$.

Theorem 5.4.7. *(upper bound for the minimal speed) When $\{Q_t\}_{t\in\mathcal{T}}$ satisfy (A1)-(A4), M_t satisfies (C1)-(C3), and B_0^ω is irreducible, for $c_2 > c_0$, if there exists a continuous positive ω-time-periodic function $\overline{W}(t,x) \in \mathcal{M}_{\beta(t)}$, satisfying*

$$\liminf_{x\to-\infty} \overline{W}(t,x) \gg 0,\ \overline{W}(t,x) = e^{-\mu_2(c_2)x}\zeta_{\mu_2(c_2)}(t) \text{ as } x\to\infty, \tag{5.4.7}$$

and

$$Q_t[\overline{W}(0,\cdot)](x) \leqslant \overline{W}(t, x-c_2t), \tag{5.4.8}$$

where $\mu_2(c_2) = \mu_2^1(c_2)$ is defined in (5.3.3), then $c_{\min} = c_f^*/\omega \leqslant c_2$.

Proof. Recall that $c_f^* := \sup\{c : a(c;+\infty) > 0\}$, i.e., (5.2.6), where $a(c;s) = \lim_{n\to\infty} a_n(c;s)$ with

$$a_0(c,s) = \phi,\ a_{n+1}(c,s) = R_c[a_n],\ \text{and } R_c[a](s) = \max\{\phi(s), T_{-c}[P[a]](s)\}.$$

By [20], c_f^* is independent of choice of $a_0(c;-\infty) = \varpi \in \mathcal{X}$ as long as $0 \ll \varpi \ll \beta$. Thus, we can let ϖ be small enough to have $\overline{W}$, or a shift of $\overline{W}$, to be an upper solution of P satisfying

$$a_0(c_2\omega; s) = \phi(s) \leqslant \overline{W}(0,s).$$

Following (5.2.3), (5.2.4), (5.4.7), and (5.4.8), and then by induction, we obtain that

$$a_{n+1}(c_2\omega; s) \leqslant \overline{W}(0, s), \text{ for all } n \geqslant 0.$$

Moreover, $a(c_2\omega; +\infty) = 0$. By (5.2.6), we have $c_f^* \leqslant c_2\omega$; thus, $c_{\min} \leqslant c_2$. $\square$

With the nonlinear selections constructed in above theorems, we now study the linear selection.

Theorem 5.4.8 (Linear Selection)**.** *Suppose that $\{Q_t\}_{t\in\mathcal{T}}$ satisfy (A1)-(A4), M_t satisfies (C1)-(C3), and B_0^ω is irreducible. Further assume that there exists an ω-periodic, continuous, and positive function $\overline{W}(t, x)$ satisfying*

$$\liminf_{x\to-\infty} \overline{W}(t, x) \gg 0, \quad \lim_{x\to\infty} \overline{W}(t, x) = 0, \text{ and } Q_t[\overline{W}(0, \cdot)](x) \leqslant \overline{W}(t, x - c_0 t).$$

Then the linear selection is realized.

Proof. The main part of the proof is similar to that of Theorem 5.4.7. By choosing an initial data $\phi(x) \leqslant \overline{W}(0, x)$ for all $x \in \mathbb{R}$, then $c_f^*/\omega \leqslant c_0$ directly follows from the comparison principle. The proof of $c_f^*/\omega \geqslant c_0$ is rather trivial by [42] or [88]. Therefore, we have $c_f^*/\omega = c_0$. $\square$

We then present two corollaries to give an idea for choosing some suitable upper solutions to achieve a linear selection result.

Corollary 5.4.9. *Suppose that $\{Q_t\}_{t\in\mathcal{T}}$ satisfy (A1)-(A4), M_t satisfies (C1)-(C3), and B_0^ω is irreducible. If $W = e^{-\bar{\mu}x}\zeta_{\bar{\mu}}(t)$ is an upper solution of the wave profile equation, that is,*

$$Q_t[e^{-\bar{\mu}x}\zeta_{\bar{\mu}}(0)] \leqslant e^{-\bar{\mu}(x-c_0 t)}\zeta_{\bar{\mu}}(t)$$

where $\bar{\mu} = \bar{\mu}^1$ is defined in Lemma 5.3.1, then the linear selection is realized.

Corollary 5.4.10. *Let $\{Q_t\}_{t\in\mathcal{T}}$ satisfy (A1)-(A4), M_t satisfies (C1)-(C3), and B_0^ω be irreducible. Suppose that $\bar{W}(t, x) = \frac{\beta(t)}{1+e^{\bar{\mu}x}/\zeta_{\bar{\mu}}(t)} := \left(\frac{\beta^1(t)}{1+e^{\bar{\mu}x}/\zeta_{\bar{\mu}}^u(t)}, \cdots, \frac{\beta^l(t)}{1+e^{\bar{\mu}x}/\zeta_{\bar{\mu}}^l(t)}\right)$, where*

$\bar{\mu} = \bar{\mu}^1$ *is defined in Lemma 5.3.1, is an upper solution of the wave profile equation, i.e.,* $Q_t[\bar{W}(0,\cdot)](x) \leqslant \bar{W}(t, x - c_0 t)$. *Then the linear selection is realized.*

5.4.2 Case (II): B_0^ω is not irreducible and $I_0 = \{1\}$

The case when B_0^ω is not irreducible may admit boundary fixed points of $P = Q_\omega$ in $\mathcal{M}_\beta$. To deal with it, we begin with a simpler one (II), i.e., $\lambda_i(0) < 1$ for all $i \neq 1$. Then, we need the following assumption on B_0^ω so that the extinction state zero can be invaded.

(C4) B_0^ω has at each diagonal block at least one nonzero entry underneath the first diagonal block. This means when the populations are very small, an increase in population of any species in the first block increases the populations of all the other species in a finite number of time steps.

Correspondingly, we have the following assumption to Q_t.

(A5) For every fixed point $\alpha(t)$ of Q_t in $\mathcal{M}_{\beta(t)}$ other than $\beta(t)$, $\mathcal{P}_1[\alpha(t)] = 0$.

Notice that assumption (A5) is consistent to (C4): to have a semi-trivial equilibrium α that is not strongly positive, we have to require all the components in the first block to be zero. Otherwise all the population of other blocks will follow the growth of the first block to stabilize into the unique positive equilibrium β.

The condition of (II) immediately shows that

$$c_0^f = c_0^1 = \frac{c_1^\omega}{\omega}.$$

Then, due to the importance of the uppermost block, we give the following notation.

Denote the components corresponding to the uppermost block of B_0^ω as U-*system*. That means, if $W(t,x)$ is a traveling wave solution of Q_t satisfying Definition 5.2.2, then it can be expressed as $W(t,x) = (U,V)(t,x)$, and V collects the rest components except U. Correspondingly, $\beta(t) = (\beta_u, \beta_v)(t)$ satisfies $W(t,-\infty) = (U,V)(t,-\infty) = (\beta_u, \beta_v)(t)$.

Next, we provide a condition so that a single spreading exists.

Lemma 5.4.11. *Assume that* $\{Q_t\}_{t\in\mathcal{T}}$ *satisfy (A1)-(A5),* M_t *satisfies (C1)-(C4), and Case (II) hold. Suppose that* $c_{\min}^\alpha$ *is the minimal speed so that the semiflow* Q_t *has* *no traveling waves conecting $\beta(t)$ to $\alpha(t)$ if $c < c^{\alpha}_{\min}$, where $\alpha(0)$ is any fixed point of $P = Q_\omega$ other than zero and β. Suppose that for any fixed point $\alpha_1(0) = (0, \hat{\alpha}(0))$ of P, the reductive system of Q_t restricted on $\mathcal{M}_{[0,\alpha_1(0)]} \to \mathcal{M}_{[0,\alpha_1(t)]}$ has no traveling wave, connecting $(0, \hat{\alpha}(t))$ and $(0, 0)$, for $c \geq c^{\alpha}_{\min}$. Then Q_t has a single spreading speed.*

Proof. The proof is trivial, since item (2.2) in Lemma 5.2.4 can be excluded. □

With the above lemma, we therefore have the following qualities for this case:

$$c_0 = c_1^{\omega}/\omega, \text{ and } c^*/\omega = c_f^*/\omega.$$

And the definition of the speed selection for this case is given as follows.

Definition 5.4.12. *When Case (II) and Lemma 5.4.11 hold, the single spreading speed of Q_t is said to be linearly selected if $c_f^*/\omega = c_0$ and nonlinearly selected if $c_f^*/\omega > c_0$.*

Then, we specify the asymptotic behaviors of traveling wave U near positive infinity. For $c > c_0$, if $W_c = (U_c, V_c)$ is a traveling wave solution, then U_c has the following behavior:

$$U_c(t, x) = C_1 e^{-\mu_1(c)x} \zeta_{\mu_1(c)}(t) + C_2 e^{-\mu_2(c)x} \zeta_{\mu_2(c)}(t) \text{ as } x \to \infty, \tag{5.4.9}$$

where $C_1 > 0$ or $C_1 = 0,\ C_2 > 0$. Here, $\mu_i = \mu_i^1$ $(i = 1, 2)$ is defined in Lemma 5.3.1, ζ_{μ_i} are the strongly positive eigenfunction corresponding to μ_i.

Then, we give a necessary and sufficient condition for the nonlinear selection.

Theorem 5.4.13. *Suppose that the conditions in Lemma 5.4.11 hold.*

(i). It follows that

$$c_0 = c_1^{\omega}/\omega, \text{ and } c^*/\omega = c_f^*/\omega =: c_{\min},$$

where $c_{\min}$ is the critical number so that Q_t has ω-periodic traveling wave solutions $W_c(t, x)$, connecting β and 0, if and only if $c \geqslant c_{\min}$.

(ii). Assume that the traveling wave $W_c(t,x)$ has a continuous derivative $W_c'(t,x)$ where $'$ denotes the derivative with respect to x. Moreover, the map $M_{\omega,W_{c_{\min}}}$ has a simple principal eigenvalue with a strogly positive eigenfunction. Then, the following two statements are equivalent:

(F1') *$c_{\min} > c_0$, that means, the nonlinear selection is realized;*

(F2') *There exists a speed $\bar{c} > c_0$ such that Q_t has an ω-periodic traveling wave $W_{\bar{c}}(t,x) = (U_{\bar{c}}, V_{\bar{c}})(t,x)$ connecting zero and β, with the following property*

$$U_{\bar{c}}(t,x) = Ce^{-\mu_2(\bar{c})x}\zeta^u_{\mu_2(\bar{c})}(t), \text{ as } x \to +\infty,$$

for some positive constant C.

Proof. (i). The proof of this part follows from Lemma 5.4.11.

(ii). We start with the sufficiency. That means, if (**F2'**) holds for some $\bar{c} > c_0$, then there is no traveling wave for $c \in [c_0, \bar{c})$ and $c_{\min} = \bar{c}$. As before, we only need to prove that under (**F2'**), there is no traveling wave solution for $c \in (c_0, \bar{c})$. For this purpose, to the contrary, we suppose that a positive W_c exists for some $c \in (c_0, \bar{c})$ (maybe close to $\bar{c}$). If there is only one stable block in B_0^ω, i.e., $I_1 = 2$, by the characteristic equation of the linear system M_t, similar to (5.4.9) we know the asymptotic behavior of W_c near $x = \infty$ is

$$W_c = \begin{pmatrix} U_c \\ V_c \end{pmatrix} \sim C_1 \begin{pmatrix} \zeta^u_{\mu_1(c)}(t) \\ \zeta^v_{\mu_1(c)}(t) \end{pmatrix} e^{-\mu_1(c)x} + C_2 \begin{pmatrix} \zeta^u_{\mu_2(c)}(t) \\ \zeta^v_{\mu_2(c)}(t) \end{pmatrix} e^{-\mu_2(c)x} + C_3 \begin{pmatrix} 0 \\ \zeta^v_{\mu_3(c)}(t) \end{pmatrix} e^{-\mu_3(c)x},$$

if μ_3 is not equal to μ_1 or μ_2, where $C_1 > 0, C_3 > 0$, or $C_1 = 0, C_2 > 0, C_3 > 0$, and $W_{\bar{c}}$ is

$$W_{\bar{c}} = \begin{pmatrix} U_{\bar{c}} \\ V_{\bar{c}} \end{pmatrix} \sim C_2 \begin{pmatrix} \zeta^u_{\mu_2(\bar{c})}(t) \\ \zeta^v_{\mu_2(\bar{c})}(t) \end{pmatrix} e^{-\mu_2(\bar{c})x} + C_3 \begin{pmatrix} 0 \\ \zeta^v_{\mu_3(\bar{c})}(t) \end{pmatrix} e^{-\mu_3(\bar{c})x}.$$

By the continuity and monotonicity of $\mu_i(c), i = 1, 2, 3$, for x near positive infinity, we always have $W_c \gg W_{\bar{c}}$ due to $\mu_1(c) < \mu_2(c) < \mu_2(\bar{c})$ and $\mu_3(c) < \mu_3(\bar{c})$ from Lemma 5.3.1. By the same arguments as that in Theorem 5.4.3, we also obtain $W_c \gg W_{\bar{c}}$

near $x = -\infty$. Moreover, this result can follow from the same arguments, even in the case that μ_3 is equal to μ_1 or μ_2, or I_1 contains more elements. Thus, by a translation ξ_0, we have

$$\bar{W}_c(t, x) = W_c(t, x + \xi_0) > W_{\bar{c}}(t, x), \text{ for all } x \in \mathbb{R}, \ t \in \mathbb{R}_+.$$

Fixing some $z_0 = x - \bar{c}n\omega$, we know that $W_{\bar{c}}(0, x - \bar{c}n\omega) = W_{\bar{c}}(0, z_0) \gg 0$. On the other hand, we will have

$$\bar{W}_c(0, cn\omega) = \bar{W}_c(0, z_0 + (\bar{c} - c)n\omega) \to 0, \text{ as } n \to \infty.$$

This is a contradiction. Therefore, Q_t has no traveling waves when $c \in (c_0, \bar{c})$. The proof of this part is complete.

Step 2: If $c_{\min} > c_0$, we need to prove the wave $U_{c_{\min}}$ in the ω-periodic traveling wave $W_{c_{\min}} = (U_{c_{\min}}, V_{c_{\min}})$ of Q_t has the following property

$$U_{c_{\min}}(t, x) \sim C_2 e^{-\mu_2(c_{\min})x} \zeta_{\mu_2(c_{\min})}(t) \text{ as } x \to \infty.$$

Assume to the contrary, if we have

$$U_{c_{\min}}(t, x) \sim C_1 e^{-\mu_1(c_{\min})x} \zeta_{\mu_1(c_{\min})}(t) \text{ as } x \to \infty,$$

for some positive constant C_1. Then, we want to prove that P does have a traveling wave $W_c(0, x)$ satisfying

$$T_{-c\omega} P \left[\begin{pmatrix} U \\ V \end{pmatrix} (0, \cdot) \right] (x) = \begin{pmatrix} U \\ V \end{pmatrix} (0, x). \tag{5.4.10}$$

for some speed $c = c_{\min} - \delta$, where $\delta > 0$ is sufficiently small. This results in a contradiction.

Similar to Step 2 in Theorem 5.4.3, we then use a perturbation argument to prove the existence of W_c to (5.4.10), where $c = c_{\min} - \delta$ for sufficiently small δ. We define

$$\bar{U}(x) = U_{c_{\min}}(0,x)\varrho(x),\ \bar{V}(x) = V_{c_{\min}}(0,x),\ \text{where } \varrho(x) = \frac{1}{1 + \delta e^{[\mu_1(c)-\mu_1(c_{\min})]x} \cdot \frac{\zeta^u_{\mu_1(c)}(0)}{\zeta^u_{\mu_1(c_{\min})}(0)}}.$$

$$W_c = \bar{W}(x) + W_1$$

Substituting it into (5.4.4), we have

$$T_{-c\omega}P[\bar{W}(x) + W_1] = \bar{W}(x) + W_1,$$

where $W_1 = W_1(0,x)$ is a function to be determined. The rest of proof is similar to that in Step 2 of Theorem 5.4.3 by defining a weighted space by

$$\begin{aligned}\mathcal{W} = \{v \in \mathcal{M}_0 : v = (v_1, v_2)^T &\text{ where } v_1 \text{ has the same dimension as } U\text{-system, and} \\ &v_1 \cdot e^{\mu_1(c_{\min})x} = o(1) \text{ as } x \to \infty\}.\end{aligned}$$

Therefore, we obtain that, there exists a small number $\delta_0 > 0$ such that for any $\delta \in [0, \delta_0)$, (5.4.10) has a solution W_c with $c = c_{\min} - \delta$, which is a contradiction to the definition of c^*. The proof is complete. $\square$

We then give sufficient conditions for linear or nonlinear selections.

For the whole system $W = (U, V)$, U is an invader and drives W to invade onto the zero solution. This inspires us to make the following reasonable assumption, which indicates the existence of V in terms of U.

(A6) We consider the equation $Q_t[W(0,\cdot)](x) = W(t, x - ct)$, and $W = (U, V)$. For $c \geqslant c_0^1$ and any given continuous function $U = U(t,z)$ ($z = x - ct$), which is ω-periodic in t and nonincreasing in z, satisfies $U(t, -\infty) = a(t) \leqslant \beta_u(t)$ for some ω-periodic function $0 \ll a(t)$, $U(t, +\infty) = 0$. We assume that the V-system always has a solution $V(t,z)$ satisfying $V(t,-\infty) \leqslant \beta_v(t)$ and $V(t,+\infty) = 0$, which is also ω-periodic in t and nonincreasing in z. Denote the solution as $V = V(U)$. Moreover, it is monotone in U; in other words, $V(U_1) \geqslant V(U_2)$ if $U_1 \geqslant U_2$ for $(t,z) \in \mathcal{T} \times \mathbb{R}$.

Remark 5.4.14. *Assumption (A6) is raised for a technique reason, and it can be proved by many biological models, e.g., a famous Lotka-Volterra model, see [51, 83].*

Theorem 5.4.15. *Assume that the conditions in Lemma 5.4.11 hold and Q_t further satisfies (A6). If for $c_1 > c_0$, suppose that there exists a continuous ω-time-periodic function $\underline{W}(t,x) = (\underline{U}, \underline{V})(t,x)$ satisfying*

$$0 \ll \underline{U}(t,x) \ll \beta_u(t), \ \limsup_{x\to-\infty} \underline{U}(t,x) \ll \beta_u(t), \ \underline{U}(t,x) = e^{-\mu_2(c_1)x}\zeta^u_{\mu_2(c_1)}(t) \ \text{as } x \to \infty,$$

and

$$Q_t[\underline{W}(0,\cdot)](x) \geqslant \underline{W}(t, x - c_1 t),$$

where $\mu_2(c_1) = \mu_2^1(c_1)$ is defined in (5.3.3). Then $c_{\min} \geqslant c_1$ and no traveling waves exist for $c \in [c_0, c_1)$. In other words, the nonlinear selection is realized.

Proof. Due to the fact that $\underline{V} \leqslant V(\underline{U})$, the proof is similar to Step 1 of Theorem 5.4.3; thus, it is omitted. $\square$

Theorem 5.4.16. *(upper bound for the minimal speed) Suppose that the conditions in Lemma 5.4.11 hold and Q_t further satisfies (A6). For $c_2 > c_0$, if there exists a continuous positive ω-time-periodic function $\overline{W}(t,x) = (\overline{U}, \overline{V})(t,x)) \in \mathcal{M}_{\beta(t)}$, satisfying*

$$\liminf_{x\to-\infty} \overline{U}(t,x) \gg 0, \ \overline{U}(t,x) = e^{-\mu_2(c_2)x}\zeta^u_{\mu_2(c_2)}(t) \ \text{as } x \to \infty, \tag{5.4.11}$$

and

$$Q_t[\overline{W}(0,\cdot)](x) \leqslant \overline{W}(t, x - c_2 t), \tag{5.4.12}$$

where $\mu_2(c_2) = \mu_2^1(c_2)$ is defined in (5.3.3), then $c_{\min} = c_f^*/\omega \leqslant c_2$.

Proof. Due to the fact that $\overline{V} \geq V(\overline{U})$, the proof here is similar to that of Theorem 5.4.7 $\square$

In this case, we can also give conditions for linear selection.

Theorem 5.4.17 (Linear Selection)**.** *Let the conditions in Lemma 5.4.11 be true and the assumption (A6) holds. Further assume that there exists an ω-periodic, continuous, and positive function $\overline{W}(t,x)$ satisfying*

$$\liminf_{x\to-\infty} \overline{W}(t,x) \gg 0, \quad \lim_{x\to\infty} \overline{W}(t,x) = 0, \ \textit{and}\ Q_t[\overline{W}(0,\cdot)](x) \leqslant \overline{W}(t, x - c_0 t).$$

Then the linear selection is realized.

Proof. The proof is similar to that in the proof of Theorem 5.4.8 □

Corollary 5.4.18. *Assume that $\{Q_t\}_{t\in\mathcal{T}}$ satisfy (A1)-(A6), M_t satisfies (C1)-(C5), and Case (II) holds. If an ω-time-periodic function $\overline{W} = (\overline{U}, \overline{V})^T$ (where the dimension of $\overline{U}$ equals to that of U-system) has the formula*

$$\overline{U}(t,x) = \frac{\beta_u(t)}{1 + \frac{e^{\bar{\mu}x}}{\zeta^u_{\bar{\mu}}(t)}}, \ \overline{V} = V(\overline{U}) \ , \ \textit{and satisfies} \ \ Q_t[\overline{W}(0,\cdot)](x) \leqslant \overline{W}(t, x - c^1_0 t),$$

then the linear selection is realized.

5.4.3 Case (III): B^ω_0 is not irreducible and $I_0 \setminus \{1\} \neq \emptyset$

The previous two cases indicate that Q_t has a single spreading speed. When B^ω_0 is not irreducible with $I_0 \setminus \{1\} \neq \emptyset$, we shall see in this section that Q_t may have a single spreading speed or multiple spreading speeds.

To better understand c^* and c^*_f, we introduce the individual spreading speed of each species. Let Q_t satisfy (A1)–(A4) and initial data ϕ satisfy (B1)–(B3), the individual spreading speed c^*_j ($1 \leqslant j \leqslant l$) of the j-th species is a constant such that

$$\lim_{n\to\infty, x\geqslant(c^*_j+\epsilon)n} [P^n(\phi)]_j(x) = 0, \quad \lim_{n\to\infty, x\leqslant(c^*_j-\epsilon)n} [P^n(\phi)]_j(x) \geqslant \eta_j > 0, \tag{5.4.13}$$

where $P = Q_\omega$, $\eta_j > 0$ is a constant and $\epsilon > 0$ is small.

According to their definitions, the slowest and fastest spreading speeds can be related to each individual spreading speed in the following proposition.

Proposition 5.4.19.

$$c^* = \min_{1\leqslant j\leqslant l}\{c_j^*\},\ \ c_f^* = \max_{1\leqslant j\leqslant l}\{c_j^*\}. \tag{5.4.14}$$

For a specific model, both cases $c^* < c_f^*$ and $c^* = c_f^*$ can happen. We will establish speed determinacy separately.

When Q_t has a single spreading speed with $c^* = c_f^*$, we want to indicate how they are determined by the speeds of the linearized system at zero. Assume $c_0^f > c_0^i$ for any $i \in I_0$. We can see that $c^*/\omega > c_0^i$ for any $i \in I_0$. From Section 3, we have the definition c_0^f and $c_f^*/\omega \geqslant c_0^f$. We want to see whether $c^* = c_f^*$ can be determined by the linear speed c_0^f. Therefore, the definition of speed selection for the fastest spreading speed is given as follows.

Definition 5.4.20. *Suppose that Q_t has a single spreading speed with $c^* = c_f^*$. We say c_f^* is linearly selected if $c_f^*/\omega = c_0^f$ and nonlinearly selected if $c_f^*/\omega > c_0^f$.*

Let $I_0 = \{1, i_1, \cdots, i_N\}$ for $1 < i_1, i_2, \cdots, i_N \leqslant N_0$, $U = (U_1, U_{i_1}, \cdots, U_{i_N})$ and $W = (U, V)$. We assume that (A6) is true. We have the following theorem.

Theorem 5.4.21. *Assume that Q_t satisfies (A1)-(A4) and (A6), and the linear map M_t satisfies (C1)-(C3). Then $c^*,\ c_f^*,\ c_0^f$ are defined. When B_0^ω is not irreducible with $I_0\backslash\{1\} \neq \emptyset$ and $c^* = c_f^*$, the following statements are valid.*

(1) If for $c = c_0^f$, there exists a continuous and positive functions $\bar{W}$ satisfying

$$\lim_{x\to\infty} \bar{W}(t,x) = 0,\ \ \textit{and}\ Q_t[\bar{W}(0,\cdot)](x) \leqslant \bar{W}(t, x-ct). \tag{5.4.15}$$

Then Q_t has a traveling wave connecting β to zero and c_f^* is linearly selected, i.e., $c_f^*/\omega = c_0^f$.

(2) If for $c = c_1 > c_0^f$, there exists a pair of continuous and non-increasing functions $(\underline{U}, \underline{V})$ being a lower solution to Q_t and satisfying

$$\lim_{z\to-\infty} (\underline{U}, \underline{V}) < \beta\ \textit{and}\ \underline{U}_i \sim e^{-\mu_2^i(c_1)z}\ \textit{as}\ z \to \infty, \tag{5.4.16}$$

$i \in I_0$, where $\mu_2^i(c_1)$ are defined. Then c_f^* is nonlinearly selected with $c_f^*/\omega > c_0^f$.

Next, we proceed to study the case when Q_t has multiple spreading speeds, i.e., $c^* < c_f^*$. We need to consider the linearization of the semiflow around each $\alpha(t)$. We give the following assumption.

(C5) Let $M_{t,\alpha(t)}$ be the linearization of Q_t around $\alpha(t)$, where $\alpha(t)$ satisfies $Q_t[\alpha(0)] = \alpha(t)$. Define a linear map $B^t_{\gamma,\alpha(t)} : \mathcal{X} \to \mathcal{X}$ by

$$B^t_{\mu,\alpha(t)}[\xi] = M_{t,\alpha(t)}[e^{-\mu x}\xi](x = 0),$$

where $\mu > 0$ is a constant, and ξ is a real vector. Assume that $B^\omega_{\mu,\alpha(0)}$ is in Frobenius form and has N_0^α elements of diagonal blocks. Let $\lambda_i^\alpha(\mu)$ $(i = 1, \cdots, N_0^\alpha)$ be the principal eigenvalue of the i-th diagonal block of $B^\omega_{\mu,\alpha(0)}$. Among all the principal eigenvalues, we always assume that $\lambda_1^\alpha(0) > 1$, $\lambda_i^\alpha(0) \neq 1$ for all i. Then, for the whole system, we denote

$$I_0^\alpha := \{i \,|\, \lambda_i(0) > 1, \text{ for } 1 \leqslant i \leqslant N_0\},$$

and $I_1^\alpha := \{1, \cdots, N_0^\alpha\} \backslash I_0^\alpha$.

Then for each $\alpha(t)$, there exists a corresponding fast linear speed $c_\alpha^f = \max\limits_{i \in I_0^\alpha} \{c_\alpha^i\}$, where $c_\alpha^i = \frac{1}{\omega} \inf\limits_{\mu > 0} \frac{\lambda_i^\alpha(\mu)}{\mu}$.

Remark 5.4.22. *Lemma 5.3.1 is still true by replacing c_0^f with c_α^f around each α.*

Denote

$$c_1^* = \min_{1 \leqslant j \leqslant l} \left\{ \{c_j^* : 1 \leqslant j \leqslant l\} \backslash \{c^*\} \right\}.$$

Then, $c = c^* + \epsilon < c_1^*$ for sufficiently small $\epsilon > 0$, we assume that Q_t has a traveling wave connecting $\beta(t)$ to $\alpha_1(t)$ with speed c. Define

$$Q_t^{(\alpha_1,\beta)} := Q_t|_{\mathcal{M}_{[\alpha_1,\beta]}} : \mathcal{M}_{[\alpha_1(0),\beta(0)]} \to \mathcal{M}_{[\alpha_1(t),\beta(t)]}$$

Then $Q_t^{(\alpha_1,\beta)}$ satisfies (A1)-(A3) and has a single spreading speed $c^*_{\alpha_1} = c^*$, since there exists no fixed point between β and α_1.

Now, from (C5), corresponding to c^*, we have $c^*/\omega \geqslant c^f_{\alpha_1}$. Then the speed selection for the spreading speeds is defined as follows.

Definition 5.4.23. *Assume that the semiflow admits multiple spreading speeds. The slowest speed c^* is linearly selected if $c^*/\omega = c^f_{\alpha_1}$ and nonlinearly selected if $c^*/\omega > c^f_{\alpha_1}$. The fastest speed c^*_f is linearly selected if $c^*_f/\omega = c^f_0$ and nonlinearly selected if $c^*_f/\omega > c^f_0$.*

To determine c^*, we can study the semiflow Q_t in the phase space (α_1, β) as in Case (I) and (II). Similar results can hold.

To determine the fastest speed, we can study a reduced system. Indeed, let $c^*_2 = \max_{1\leqslant i\leqslant l}\left\{\{c^*_i\}\backslash\{c^*_f\}\right\}$. Assume that there exists α_2 so that $\lim_{n\to\infty,(c^*_f-\epsilon)n\geqslant x\geqslant(c^*_2+\epsilon)n} |P^n(\phi)(x) - \alpha_2(0)| = 0$ for sufficiently small $\epsilon > 0$. Then by restricting Q_t in the phase space $[0, \alpha_2(0)]$, that is,

$$Q_t^{(0,\alpha_2)} := Q_t|_{\mathcal{M}_{[0,\alpha_2]}} : \mathcal{M}_{[0,\alpha_2(0)]} \to \mathcal{M}_{[0,\alpha_2(t)]},$$

we can study the fastest speed selection as in cases (I) and (II). The detail is omitted. According to the properties of $Q_t^{(\alpha_1,\beta)}$ and $Q_t^{(0,\alpha_2)}$, we can further give a necessary and sufficient condition to decide whether Q_t has a single spreading speed or not.

Theorem 5.4.24. *Assume that $Q_t^{(\alpha_1,\beta)}$ ($Q_t^{(0,\alpha_2)}$) has a single spreading speed $c^*_{\alpha_1}/\omega$ ($c^*_{\alpha_2}/\omega$) so that $Q_t^{(\alpha_1,\beta)}$ ($Q_t^{(0,\alpha_2)}$) has a traveling wave connecting β and α_1 (α_2 and 0) if and only if $c \geqslant c^*_{\alpha_1}/\omega$ ($c \geqslant c^*_{\alpha_2}/\omega$). Then Q_t has a single spreading speed, i.e., $c^* = c^*_f$, if and only if $c^*_{\alpha_1} \geqslant c^*_{\alpha_2}$.*

5.5 Applications

In this section, we will apply our results to four examples with time-periodic coefficients, and they are listed as follows: (1) a single delayed and diffusive equation corresponding to case (I); (2) a stream population model with the benthic zone corresponding to case (I), and it is a system with its linear system around zero being irreducible; (3) a non-local dispersal Lotka-Volterra model corresponding to case (II); (4) a reducible cooperative system corresponding to case (III).

5.5.1 A time-periodic diffusive equation with discrete delay

We begin with a periodic diffusive equation with a discrete delay. Let $\omega > 0$, $\tau > 0$ be two constants. The model is given as

$$\frac{\partial u}{\partial t} = d(t)\frac{\partial^2 u}{\partial x^2} + f(t, u, u_\tau),\ t > 0,\ x \in \mathbb{R}, \tag{5.5.1}$$

where $d(t) \geqslant \eta$ for some constant $\eta > 0$ and it is a bounded ω-periodic function; $f \in C^1(\mathbb{R}_3^+, \mathbb{R})$ is also ω-periodic in t; and $u_\tau = u(t-\tau, x)$. To investigate monostable periodic traveling wave solutions, we further require f to satisfy

(**D1**) $f(t,0,0) \equiv 0$, $\frac{\partial f(t,u,v)}{\partial v} > 0$, $\forall (t,u,v) \in \mathbb{R}_+^3$, and there is a real number $H > 0$ such that $f(t,H,H) \leqslant 0$.

Let $\mathcal{X} = C([-\tau, 0], \mathbb{R})$ and $\mathcal{M}$ be defined in Section 2 with $\mathcal{H} = \mathbb{R}$ and $l = 1$, that is, $\mathcal{M}$ is the space of all bounded and nonincreasing functions from $\mathbb{R}$ to $\mathcal{X}$. Using the periodic semiflow generated by the periodic heat equation $\frac{\partial u}{\partial t} = d(t)\frac{\partial^2 u}{\partial x^2}$ (see, e.g., Section 2 of [14]) and Theorem 2.2 of [71], it can be shown that (5.5.1) generates a monotone periodic semiflow $Q_t : \mathcal{M}_H \to \mathcal{M}_H$ defined by

$$Q_t[\phi](x)(\theta) = u(t+\theta, x; \phi),\ \theta \in [-\tau, 0],\ (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

where $u_t(\theta, x; \phi) = u(t+\theta, x; \phi)$ is the unique solution of (5.5.1) for any given $\phi \in \mathcal{M}_H$. It is easy to see that Q_t satisfies (A1). Let $\hat{Q}_t$ be the restriction of Q_t to $\mathcal{X}_H$. We can see that $\hat{Q}_t : \mathcal{X}_H \to \mathcal{X}_H$ is the periodic semiflow generated by

$$\frac{du}{dt} = f(t, u(t), u(t-\tau)),\ t \geqslant 0, \tag{5.5.2}$$

with the initial dada $u_0 = \phi \in \mathcal{X}_H$. To have a positive ω-periodic solution of (5.5.2), we need the following assumption:

(**D2**) $r_0 > 1$, where r_0 is the spectral radius of the Poincaré map $\hat{P}$ associated with

$$\frac{du}{dt} = f'_u(t,0,0)u(t) + f'_v(t,0,0)u(t-\tau),\ t \geqslant 0,$$

where f'_u (f'_v) denotes the partial derivative with repsect to the second (third) variable u (v) in $f(t,u,v)$.

By [[95], Theorem 2.1.2], (5.5.2) has a positive ω-periodic solution $\beta(t)$, which satisfies $\lim_{n\to\infty} \hat{P}^n[u] = \beta(0)$ when $u \in \mathcal{X}_H \backslash \{0\}$. Thus, assumption (A3) holds for $P = Q_\omega$ since $\hat{P} = P|_{\mathcal{X}_H}$.

Following Section 4 in [42], we obtain that Q_t satisfies (A2). Thus, Q_t satisfies all the assumptions (A1)-(A3) and the existence of traveling waves can be obtained directly.

Lemma 5.5.1. *Assume (**D1**) and (**D2**) hold. (5.5.1) has a single spreading speed c^* defined in (5.2.5) (i.e., $c^* = c_f^*$) so that (5.5.1) has a periodic traveling wave $U(t, x - ct)$, connecting β and 0, if and only if $c \geqslant c^*/\omega$.*

Now we focus on the speed selection through applying our theorems in Section 3. To proceed, let's work out the linear speed.

Let $\lambda(\mu)$ be the spectral radius of the Poincaré map associated with the following linear equation

$$\frac{du}{dt} = d(t)\mu^2 u(t) + f_u'(t, 0, t)u(t) + f_v'(t, 0, 0)u(t - \tau),\ t > 0.$$

(**D2**) implies that $\lambda(0) > 1$. Therefore, it is not hard to see that (C1)-(C3) hold. This is a scalar equation, and thus it belongs to case (I): the linear system is irreducible. Furthermore, following Theorem 5.4.3 (i), there exists a critical number $c_{\min} = c^*/\omega = c_f^*/\omega$.

To investigate this traveling wave, we let $u(t, x) = U(t, \xi)$ ($\xi = x - ct$). Then, the equation for this traveling wave profile is found as

$$\begin{cases} d(t)U_{\xi\xi} + cU_\xi + f(t, U, U(t, \xi + c\tau)) - U_t = 0, \\ U(t, -\infty) = \beta(t),\ U(t, +\infty) = 0. \end{cases} \tag{5.5.3}$$

Now we want to obtain the characteristic equation of the linearized system. Linearizing (5.5.3) around zero gives

$$d(t)U_{\xi\xi} + cU_\xi + f_u'(t, 0, 0)U + f_v'(t, 0, 0)U(t, \xi + c\tau) - U_t = 0.$$

Through letting $U = \zeta(t)e^{-\mu\xi}$ with $\zeta(t)$ being ω-periodic and integrating over a period, we obtain

$$h(\mu) := \bar{d}\mu^2 - c\mu + \bar{f}'_u + \bar{f}'_v e^{-\mu c\tau} = 0,$$

where $\bar{d} = \int_0^\omega d(t)dt/\omega$, $\bar{f}'_u = \int_0^\omega f'_u(t,0,0)dt/\omega$, and $\bar{f}'_v = \int_0^\omega f'_v(t,0,0)dt/\omega$.

Define

$$c_0 := \inf \{c \in \mathbb{R} \,|\, h(\mu) = 0 \text{ has a positive real solution}\}.$$

Theorem 5.5.2. *Let*

$$\bar{U}(t,\xi) = \frac{\beta(t)}{1 + \frac{e^{\bar{\mu}\xi}}{\zeta(t)}}.$$

If

$$-2\bar{\mu}^2 d(t) + \max_{\xi\in\mathbb{R}} \left\{ \frac{f\big(t,\bar{U},V\big) - f(t,\beta,\beta(t-\tau))\bar{U}_1 - F(t,\xi)}{\frac{\bar{U}^2}{\beta}\big(1-\bar{U}_1\big)} \right\} \leqslant 0, \tag{5.5.4}$$

where $F(t,\xi) = \bar{U}(1-\bar{U}_1)\cdot[f'_u(t,0,0) + f'_v(t,0,0)e^{-\bar{\mu}c_0\tau}]$, $\bar{U}_1 = \bar{U}/\beta$, $V(t,\xi) = \bar{U}(t,\xi + c_0\tau)$, $\bar{\mu} = \mu(c_0)$, *then* $c_{\min} = c_0$.

Proof. When $c = c_0$, we find formulas of $\bar{U}_\xi$, $\bar{U}_{\xi\xi}$, and $\bar{U}_t$, and substitute them all into the left-hand side of (5.5.3) to obtain

$$\begin{aligned}
& d(t)\bar{U}_{\xi\xi} + c\bar{U}_\xi + f(t,\bar{U},V) - \bar{U}_t \\
= \; & \frac{\bar{U}^2}{\beta}\left(1-\frac{\bar{U}}{\beta}\right)\left\{ -2\bar{\mu}^2 d(t) + \frac{f\big(t,\bar{U},V\big) - f(t,\beta,\beta(t-\tau))\bar{U}_1}{\frac{\bar{U}^2}{\beta}\big(1-\bar{U}_1\big)} - \frac{f'_u(t,0,0) + f'_v(t,0,0)e^{-\bar{\mu}c_0\tau}}{\bar{U}_1} \right\} \\
= \; & \frac{\bar{U}^2}{\beta}\left(1-\frac{\bar{U}}{\beta}\right)\left\{ -2\bar{\mu}^2 d(t) + \frac{f\big(t,\bar{U},V\big) - f(t,\beta,\beta(t-\tau))\bar{U}_1 - F(t,\xi)}{\frac{\bar{U}^2}{\beta}\big(1-\bar{U}_1\big)} \right\} \\
\leqslant \; & 0,
\end{aligned}$$

provided that (5.5.4) holds. This means, $\bar{U}$ is an upper solution of (5.5.3). Then, by choosing $\phi = \bar{U}(0, x - c_0\theta)$ and using the comparison principle, it immediately follows that $Q_t[\bar{U}(0,\cdot - c_0\theta)](x) \leqslant \bar{U}(t, x - c_0(t+\theta))$. Therefore, by Theorem 5.4.8, the linear selection is realized. $\square$

As for the nonlinear selection, we apply Corollary 5.4.5 and obtain the following result.

Theorem 5.5.3. *If $c_1 > c_0$, let $\underline{U}$ be defined as*

$$\underline{U}(t,\xi) = \frac{k\beta(t)}{1 + \frac{e^{\mu_2 \xi}}{\zeta_{\mu_2}(t)}},$$

where $0 < k < 1$. If

$$-2\mu_2^2 d(t) + \min_{\xi \in \mathbb{R}} \left\{ \frac{f\big(t, U, V\big) - f(t, \beta, \beta(t-\tau))U_1 - F_2(t,\xi)}{\frac{U^2}{\beta}(1-U_1)} \right\} > 0,$$

where $\mu_2 = \mu_2(c_1)$, $F_2(t,z) = U(1-U_1)\left[f_u'(t,0,0) + f_v'(t,0,0)e^{-\mu_2(c_1)c_1\tau}\right]$, $U_1 = U/\beta$, $V(t,\xi) = U(t,\xi + c_1\tau)$, then the nonlinear selection is realized, i.e., $c_{\min} \geqslant c_1 > c_0$.

The above theorem not only gives a condition such that the nonlinear selection is realized but also provides a lower bound of $c_{\min}$. Next, by applying Theorem 5.4.5, we are able to give an upper bound for $c_{\min}$ when $c_{\min} \geqslant c_1$.

Theorem 5.5.4. *For any $c_2 > c_0$, let U be defined as*

$$U(t,\xi) = \frac{\beta(t)}{1 + \frac{e^{\mu_2 \xi}}{\zeta_{\mu_2}(t)}},$$

if

$$-2\mu_2^2 d(t) + \max_{\xi \in \mathbb{R}} \left\{ \frac{f\big(t, U, V\big) - f_u'(t,0,0)U(1-U_1) - f_v'(t,0,0)e^{-\mu_2 c_2 \tau} U(1-U_1)}{\frac{U^2}{\beta}(1-U_1)} \right\} \leqslant 0,$$

where $\mu_2 = \mu_2(c_2)$, $U_1 = U/\beta$, $V(t,\xi) = U(t,\xi + c_2\tau)$, then $c_{\min} \leqslant c_2$.

5.5.2 A stream population system with a benthic zone

In this subsection, we study a periodic stream population model with the benthic zone, which can be used to handle the persistence of benthic aquatic organisms. The system is a monotonic coupled system given by

$$\begin{cases} u_t = -a(t)u + b(t)v - \sigma(t)u_x + d(t)u_{xx}, \\ v_t = a(t)u - b(t)v + v(1-v)(1+\rho v) + \varepsilon(t)v_{xx}. \end{cases} \tag{5.5.5}$$

Here, all the coefficients ($a(t)$, $b(t)$, $\sigma(t)$, $\varepsilon(t)$, and $d(t)$) are positive continuous ω-time-periodic functions with $\omega > 0$. In the model, $u(t)$ denotes the population density in the drift while $v(t)$ denotes the one resided on the benthic zone (benthos). The periodic coefficient $a(t)$ denotes the per capita rate at which individuals on the benthic zone entering the drift while $b(t)$ is the one describing the reverse direction; $\sigma(t)$ is the advection speed due to the drifting itself; and $d(t)$ is the diffusion coefficient from the drifting while $\varepsilon(t)$ is small (even can be zero) and represents the diffusion from the benthic zone. This model originates with constant coefficients from [67] and was extended by [50], in which, the authors considered temporal variability but with a linear birth function. Later, Yu and Zhao in [91] considered a nonlinear case, but with a subhomogeneous condition, i.e., replacing $v(1-v)(1+\rho v)$ by $f(t,v)v$ with $f(t,v) \leqslant f(t,0)$. However, the nonlinear function considered in (5.5.5) has a wider application in the study of Allee efect. Such a birth function can be seen in [29], and it corresponds to a so-called weak Allee effect [26, 76].

According to the phase settings in Section 2, we assume that $\mathcal{X} = \mathbb{R}^2$ and $\mathcal{M}$ be the space of all bounded and nonincreasing functions from $\mathbb{R}$ to $\mathcal{X}$. Using [91], it follows that (5.5.5) generates an ω-periodic monotone semiflow $Q_t : \mathcal{M}_+ \to \mathcal{M}_+$ defined by

$$Q_t[(\phi_1,\phi_2)](x) = (u(t,x;\phi_1), v(t,x;\phi_2)),\ \forall(\phi_1,\phi_2) \in \mathcal{M}_+,\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \quad (5.5.6)$$

and $P = Q_\omega$ is the corresponding Poincaré map. It is easy to see that (A1) (translation invariance) holds for Q_t, $t \geqslant 0$.

To see the existence of a traveling wave solution to (5.5.5), we first consider the spatially homogeneous system

$$\begin{cases} u_t = -a(t)u + b(t)v, \\ v_t = a(t)u - b(t)v + v(1-v)(1+\rho v). \end{cases} \tag{5.5.7}$$

Linearizing it around $(0,0)$ gives

$$\begin{cases} u_t = -a(t)u + b(t)v, \\ v_t = a(t)u - b(t)v + v. \end{cases} \tag{5.5.8}$$

To have a positive periodic solution, we need the following assumption.

(D3) Let r_1 be the principal Floquet multiplier of (5.5.8) and $r_1 > 1$.

By Theorem 2.1.2 in [95], (5.5.7) has two ω-periodic solution: $(0,0)$ and $(u^*(t), v^*(t)) \gg (0,0)$. Moreover, $\mathbf{0} = (0,0)$ is unstable while we assume that $\beta(t) = (u^*(t), v^*(t))$ is stable in the sense that $\lim_{n\to\infty} \hat{P}^n[\mathbf{w}] = \beta(0)$ when $\mathbf{0} \ll \mathbf{w} \leq \beta(0) = (u^*(0), v^*(0))$, where $\hat{P}$ is the Poincaré map associated with (5.5.7). Thus, (A3) holds for P since $P = \hat{P}$ when the phase space is restricted on $\mathcal{X}_+$.

By Lemma 2.2 in [91], it follows that Q_t satisfies (A2) (point-α-contraction). Furthermore, $Q_t : \mathcal{M}_{\beta(0)} \to \mathcal{M}_{\beta(t)}$ has only two spacially homogeneous periodic solutions; therefore, (2.2) in Lemma 5.2.4 is ruled out immediately. The existence of traveling waves is summarized into the following lemma.

Lemma 5.5.5. *Assume that **(D3)** holds. Then P has a single spreading speed $c^* = c_f^*$ and the minimal wave speed satisfies $c_{\min} = c^*/\omega$ such that (5.5.5) has a traveling wave, connecting β and zero, if and only if $c \geqslant c_{\min}$.*

To apply our theory on the speed selection mechanism, we first need to figure out the linear speed of system (5.5.5). Linearizing (5.5.5) around $\mathbf{0}$ gives

$$\begin{cases} u_t = -a(t)u + b(t)v - \sigma(t)u_x + d(t)u_{xx}, \\ v_t = a(t)u - b(t)v + v + \varepsilon(t)v_{xx}. \end{cases} \tag{5.5.9}$$

Let $(u,v)(t,x) = (w_1, w_2)(t)e^{-\mu x}$ with $\mu \in \mathbb{R}_+$, then we have

$$\begin{cases} w_1' = d(t)\mu^2 w_1 + \sigma(t)\mu w_1 - a(t)w_1 + b(t)w_2, \\ w_2' = \varepsilon(t)\mu^2 w_2 + a(t)w_1 - b(t)w_2 + w_2. \end{cases} \tag{5.5.10}$$

Let M_t be the solution map associated with (5.5.9), and B_μ^t be defined by M_t as in Section 2. It is clear to see that B_μ^t is the solution map of (5.5.10). Let $\lambda(\mu)$ be the principal Floquet multiplier of the linear map M_ω. Then it follows that $\lambda(\mu)$ is the principal eigenvalue of B_μ^ω. Thus, the linear speed of system (5.5.5) is

$$c_0 = \frac{1}{\omega} \inf_{\mu>0} \frac{\ln \lambda(\mu)}{\mu}. \tag{5.5.11}$$

Moreover, by the Floquet theory, there exists a positive ω-periodic functions $(\zeta_1, \zeta_2)(t)$ such that (5.5.10) has a solution $(w_1, w_2)(t) = (\zeta_1, \zeta_2)(t)e^{(\ln\lambda(\mu))t}$. Next, we consider the wave profile system, that is, let $(u, v)(t, x) = (U, V)(t, z)$ with $z = x - ct$; thus, the system for the wave profile (U, V) is given by

$$\begin{cases} d(t)U_{zz} + (c - \sigma(t))U_z - a(t)U + b(t)V - U_t = 0, \\ \varepsilon(t)V_{zz} + cV_z + a(t)U - b(t)V + V(1 - V)(1 + \rho V) - V_t = 0, \\ (U, V)(t, -\infty) = \beta(t), \ (U, V)(t, +\infty) = (0, 0). \end{cases} \tag{5.5.12}$$

Alternatively, we can find the linear speed from the wave profile equations. By way of asymptotic analysis, we can assume $(U, V) \sim (\zeta_1, \zeta_2)(t)e^{-\mu z}$ as $z \to \infty$. This gives

$$\begin{cases} d(t)\mu^2\zeta_1 - \mu(c - \sigma(t))\zeta_1 - a(t)\zeta_1 + b(t)\zeta_2 - \zeta_1' = 0, \\ \varepsilon(t)\mu^2\zeta_2 - c\mu\zeta_2 + a(t)\zeta_1 - b(t)\zeta_2 + \zeta_2 - \zeta_2' = 0, \end{cases} \tag{5.5.13}$$

It is easy to know that there exists a minimal linear speed c_0 so that the above equations have a strongly positive solution $(\zeta_1, \zeta_2)(t)$ if and only if $c \geq c_0$. For $c > c_0$, the asymptotic behavior of (U, V) as $z \to \infty$ is given by

$$\begin{pmatrix} U(t, z) \\ V(t, z) \end{pmatrix} \sim C_1 \begin{pmatrix} \zeta_{1,\mu_1}(t) \\ \zeta_{2,\mu_1}(t) \end{pmatrix} e^{-\mu_1(c)z} + C_2 \begin{pmatrix} \zeta_{1,\mu_2}(t) \\ \zeta_{2,\mu_2}(t) \end{pmatrix} e^{-\mu_2(c)z},$$

with $C_1 > 0$, or $C_1 = 0$, $C_2 > 0$. Here, ζ_{j,μ_i} $(i, j = 1, 2)$ is the positive ω-periodic functions satisfying

$$\begin{cases} d(t)\mu^2\zeta_1 - \mu(c - \sigma(t))\zeta_1 - a(t)\zeta_1 + b(t)\zeta_2 - \zeta_1' = 0, \\ \varepsilon(t)\mu^2\zeta_2 - c\mu\zeta_2 + a(t)\zeta_1 - b(t)\zeta_2 + \zeta_2 - \zeta_2' = 0, \end{cases}$$

with $\mu = \mu_i(c)$, respectively. Clearly, this model belongs to the case (I): an irreducible linear system.

For this particular case, we can prove that there exists a critical value of ρ for speed selection.

Lemma 5.5.6. *For given $a(t)$, $b(t)$, $\sigma(t)$, $\mu(t)$ and $d(t)$, (5.5.5) has a critical $\bar{\rho}$ such* *that the minimal wave speed is linearly selected if $\rho \leq \bar{\rho}$, and nonlinearly selected if $\rho > \bar{\rho}$.*

Proof. To prove the existence of $\bar{\rho}$, it suffices to prove the following claim.

Claim. If (5.5.5) is linearly selected for some $\rho = \rho_l$, then the linear selection is realized for all $\rho \leqslant \rho_l$.

By assumption, (5.5.5) has a traveling wave solution (U_l, V_l) with speed $c = c_0$ when $\rho = \rho_l$. Thus,

$$\begin{cases} d(t)U_{l,zz} + (c_0 - \sigma(t))U_{l,z} - a(t)U_l + b(t)V_l = 0, \\ \varepsilon(t)V_{l,zz} + c_0 V_{l,z} + a(t)U_l - b(t)V_l + (1 - V_l)(1 + \rho_l V_l)V_l = 0. \end{cases}$$

Then, we substitute (U_l, V_l) into (5.5.5) when $\rho < \rho_l$. It is easy to see that the first equation is always zero; the second one becomes

$$\begin{aligned} & \varepsilon(t)V_{l,zz} + c_0 V_{l,z} + a(t)U_l - b(t)V_l + (1 - V_l)(1 + \rho V_l)V_l \\ = \; & (1 - V_l)V_l^2(\rho - \rho_l) < 0. \end{aligned}$$

This means that (U_l, V_l) is an upper solution to (5.5.5) for $\rho < \rho_l$. By Theorem 5.4.8, we conclude that the minimal wave speed is linearly selected for all $\rho \leq \rho_l$. The proof is complete. □

Therefore, we can define

$$\bar{\rho} := \sup\{\rho \in \mathbb{R} \,|\, (5.5.5) \text{ is linearly selected}\}.$$

By the theory of [42] (see also in [91]), it is known that if $\rho \leqslant 1$, the system is linearly selected. The reason is that the Poincaré map is subhomogeneous under such a condition, i.e., $v(1-v)(1+\rho v) \leqslant v$. Although the existence of $\bar{\rho}$ has been proved in the above lemma, its explicit formula is unknown. Next, we will give an estimate of $\bar{\rho}$ for the speed selection by the theorems in our chapter. This also provide an estimate of the critical value.

Now, we choose a testing function as

$$V = \frac{v^*(t)}{1 + \frac{e^{\bar{\mu}z}}{\zeta_2(t)}}$$

and let

$$U = \frac{\zeta_1}{\zeta_2}V + m_2(t)V^2 + m_1(t)V^3,$$

where

$$m_2(t) = \left[\frac{u^*(t)}{v^*(t)} - \frac{\zeta_1(t)}{\zeta_2(t)}\right]\frac{1}{v^*(t)} - m_1(t)v^*(t).$$

Here, $\bar{\mu} = \bar{\mu}(c_0)$ and $\zeta_j = \zeta_{j,\bar{\mu}}$ $(j = 1, 2)$. Substituting them into the wave profile system (5.5.12), the first equation becomes

$$L_1(U, V) = \frac{V^2}{v^*(t)}\left(1 - \frac{V}{v^*(t)}\right)G_1(V) \tag{5.5.14}$$

where

$$\begin{aligned} G_1(t, z) \quad = \quad & -12d(t)\bar{\mu}^2 m_1(t)V^2 + V\Bigg[d(t)\bar{\mu}^2(9m_1(t)v^*(t) - 6m_2(t)) - 3\bar{\mu}(c_0 - \sigma(t))m_1(t)v^*(t) \\ & -3m_1(t)v^*(t)\frac{\zeta_2'}{\zeta_2}\Bigg] + d(t)\bar{\mu}^2(4m_2(t)v^*(t) - 2\frac{\zeta_2'}{\zeta_2}) - 2\bar{\mu}(c_0 - \sigma(t))m_2(t)v^*(t) \\ & +a(t)m_1(t)v^*(t) + \frac{v^{*'}}{v^*(t)}\left(\frac{\zeta_1}{\zeta_2} + 3m_1(t)v^{*2}\right) + m_1'(t)v^{*2} - 2\frac{\zeta_2'}{\zeta}m_2(t)v^*(t). \end{aligned}$$

For the second equation, we have

$$L_2(U, V) = \frac{V^2}{v^*(t)}\left(1 - \frac{V}{v^*(t)}\right)\left[-2\varepsilon(t)\bar{\mu}^2 - a(t)m_1v^*(t) - \frac{v^{*'}(t)}{V} + \rho v^{*2}(t)\right] \tag{5.5.15}$$

With the above computations and by Theorems 5.4.8, 5.4.5 and 5.4.7, we then have the following result for the speed selection.

Theorem 5.5.7. *Let $L_1(U, V)$ and $L_2(U, V)$ be defined in (5.5.14) and (5.5.15) respectively. System (5.5.5) has a following speed selection mechanism.*

(1) It is linearly selected, i.e., $c = c_0$ *where* c_0 *is defined in (5.5.11), if*

$$\max_{(t,z)\in[0,\omega]\times\mathbb{R}} G_1(V) \leqslant 0 \text{ and } \rho \leqslant \max_{(t,z)\in[0,\omega]\times\mathbb{R}} \left\{ 2\mu^2 \frac{\varepsilon(t)}{v^{*2}(t)} + \frac{a(t)m_1(t)}{v^*(t)} + \frac{{v^*}'(t)}{v^{*2}(t)V(t,z)} \right\}. \tag{5.5.16}$$

(2) It is nonlinearly selected, i.e., $c > c_0$*, if*

$$\min_{(t,z)\in[0,\omega]\times\mathbb{R}} G_1(V) > 0 \text{ and } \rho > \min_{(t,z)\in[0,\omega]\times\mathbb{R}} \left\{ 2\mu^2 \frac{\varepsilon(t)}{v^{*2}(t)} + \frac{a(t)m_1(t)}{v^*(t)} + \frac{{v^*}'(t)}{v^{*2}(t)V(t,z)} \right\}. \tag{5.5.17}$$

Remark 5.5.8. *The idea dealing with this example can be easily extended by replacing* $v(1-v)(1+\rho v)$ *by a general function* $f(t,v)$ *with one positive zero (e.g.,* $f(t,0) = f(t,1) = 0$*). It is left to interested readers.*

5.5.3 A nonlocal dispersal Lotka-Volterra model

In this subsection, we investigate a nonlocal dispersal Lotka-Volterra model, which can reflect case (II). The model is given by

$$\begin{cases} \frac{\partial u}{\partial t} = \big[J_1 * u - u\big](t,x) + u(r_1(t) - a_1(t)u - b_1(t)v), \\ \frac{\partial v}{\partial t} = \big[J_2 * v - v\big](t,x) + v(r_2(t) - a_2(t)u - b_2(t)v). \end{cases} \tag{5.5.18}$$

Here, all the coefficients $r_i(t)$, $a_i(t)$, $b_i(t)$ are nonnegative continuous ω-periodic functions with $\omega > 0$ being a constant; $J_i * w = \int_{-\infty}^{+\infty} J_i(x-y)\cdot w(t,y)dy,\ i = 1,2$. Here, we require the kernel J_i satisfying

(D4) For $i = 1,2$, $\int_{\mathbb{R}} J_i(y)dy = 1$, $J_i(y) = J_i(-y)$, and $\int_{\mathbb{R}} J_i(y)e^{\mu y}dy < \infty$ for every $\mu > 0$.

When we consider the space-homogeneous system, that is,

$$\begin{cases} \frac{du}{dt} = u(r_1(t) - a_1(t)u - b_1(t)v), \\ \frac{dv}{dt} = v(r_2(t) - a_2(t)u - b_2(t)v), \end{cases}$$

it is readily seen that it has three nonnegative ω-periodic solutions: $e_0 = (0,0)$ (locally unstable), $e_1 = (p(t), 0)$ (locally stable), and $e_2 = (0, q(t))$ (locally unstable) under conditions:

$$\int_0^\omega r_1(t)dt > \int_0^\omega b_1(t)q(t)dt, \quad \int_0^\omega r_2(t)dt < \int_0^\omega a_2(t)p(t)dt, \tag{5.5.19}$$

where

$$\begin{cases} p(t) = \dfrac{p_0 e^{\int_0^t r_1(s)ds}}{1 + p_0 \int_0^t e^{\int_0^s r_1(\tau)d\tau} a_1(s)ds}, \quad p_0 = \dfrac{e^{\int_0^\omega r_1(s)ds} - 1}{\int_0^\omega e^{\int_0^s r_1(\tau)d\tau} a_1(s)ds}, \\ q(t) = \dfrac{q_0 e^{\int_0^t r_2(s)ds}}{1 + q_0 \int_0^t e^{\int_0^s r_2(\tau)d\tau} b_2(s)ds}, \quad q_0 = \dfrac{e^{\int_0^\omega r_2(s)ds} - 1}{\int_0^\omega e^{\int_0^s r_2(\tau)d\tau} b_2(s)ds}. \end{cases}$$

We further assume that there exists no other positive ω-periodic solutions. We are interested in the existence of monotone traveling waves connecting e_1 to e_2. To simplify (5.5.18) and obtain a cooperative system (under which, our results in Section 3 can be applied directly), we let $\tilde{u}(t,x) = \frac{u(t,x)}{p(t)}$, $\tilde{v}(t,x) = 1 - \frac{v(t,x)}{q(t)}$ and drop the tilde to obtain

$$\begin{cases} u_t = \big[J_1 * u - u\big] + u\big[a_1(t)p(t)(1-u) - b_1(t)q(t)(1-v)\big], \\ v_t = \big[J_2 * v - v\big] + (1-v)\big[a_2(t)p(t)u - b_2(t)q(t)v\big]. \end{cases} \tag{5.5.20}$$

Corresponding to the settings in Section 2, we have periodic solutions $\beta = (1,1)$, $\alpha = (0,1)$, and $0 = (0,0)$, originating from $(p(t),0)$, $(0,0)$, and $(0,q(t))$, respectively.

Now, let $\mathcal{X} = \mathbb{R}^2$ and $\mathcal{X}_\beta$, $\mathcal{M}$ and $\mathcal{M}_\beta$ be defined as in Section 2. Denote $w = (u,v)^T$, and $F(t,w)$ denotes the right-hand side vector field of (5.5.20) without the nonlocal dispersal term. Then, denote $\tilde{Q}_t$ as the solution semigroup of the linear nonlocal dispersal equation $u_t = J * u - u$. By [20, 89],

$$\tilde{Q}_t[\phi](x) = e^{-t}\Sigma_{m=0}^\infty \frac{t^m}{m!} a_m(\phi)(x),$$

where $a_0(\phi) = \phi$ and $a_m(\phi) = J * a_{m-1}(\phi)$ $\forall m \geqslant 1$. Hence, (5.5.20) can be written into an integral form:

$$w(t,x) = \tilde{Q}_t[w(0,\cdot)](x) + \int_0^t \tilde{Q}_t\big[F[s, w(s,\cdot)]\big](x)ds.$$

By [20], it follows that $\forall \phi \in \mathcal{M}_\beta$, (5.5.20) has a unique mild ω-periodic solution $u(t,x;\phi)$ with $u(0,x;\phi) = \phi$.

Let Q_t be the solution map of (5.5.20), i.e., $Q_t[\phi] = u(t, \cdot; \phi)$, satisfying Definition 5.2.1, and $P = Q_\omega$ be the associated Poincaré map. It is easy to see that (A1) and (A3) hold for Q_t. Following the arguments in the proof of Theorem 5.1 in [20], it follows that (A2) holds. Applying [Theorem 5.3, [20]] to P, (2.2) in Lemma 5.2.4 is excluded. Therefore, we obtain that $c_{\min} = c^*/\omega = c_f^*/\omega$, and the existence of traveling waves is given below.

Lemma 5.5.9. *Assume (5.5.19) and (**D4**) hold. Let $c_{\min} = c_f^*/\omega$ be defined in (5.2.6) associated with Q_t. Then (5.5.20) has an ω-periodic traveling wave $(U, V)(t, x-ct)$ connecting β to 0 if and only if $c \geqslant c_{\min}$.*

Next, we go further for its speed selection mechanism. First, we can find the system for the wave profile (by setting $u(t, x) = U(t, \xi)$ and $v(t, x) = V(t, \xi)$ $(\xi = x - ct)$) as

$$\begin{cases} [J_1 * U - U] + cU_\xi + U[a_1(t)p(t)(1-U) - b_1(t)q(t)(1-V)] - U_t = 0, \\ [J_2 * V - V] + cV_\xi + (1-V)[a_2(t)p(t)U - b_2(t)q(t)V] - V_t = 0, \\ (U, V)(t, -\infty) = (1, 1), \ (U, V)(t, +\infty) = (0, 0). \end{cases} \tag{5.5.21}$$

Linearizing (5.5.21) near zero, and letting $(U, V) = (\zeta^u(t), \zeta^v(t))e^{-\mu\xi}$ with $\mu > 0$ and ζ^u, ζ^v being ω-periodic give

$$\begin{pmatrix} f_\mu(t) - c\mu & 0 \\ a_2(t)p(t) & g_\mu(t) - c\mu \end{pmatrix} \begin{pmatrix} \zeta^u \\ \zeta^v \end{pmatrix} = 0, \tag{5.5.22}$$

where

$$\begin{cases} f_\mu(t) = [\int_{-\infty}^{\infty} J_1(y)e^{\mu y}dy - 1] + \big[a_1(t)p(t) - b_1(t)q(t)\big] - \frac{\zeta^{u'}}{\zeta^u}, \\ g_\mu(t) = [\int_{-\infty}^{\infty} J_2(y)e^{\mu y}dy - 1] - b_2(t)q(t) - \frac{\zeta^{v'}}{\zeta^v}. \end{cases}$$

Integrating the diagonal elements in (5.5.22) from 0 to ω gives two characteristic equations

$$f_\mu^\omega := \int_{-\infty}^{\infty} J_1(y)e^{\mu y}dy - 1 + \overline{a_1p - b_1q} = c\mu,$$

and

$$g_\mu^\omega := \int_{-\infty}^{\infty} J_2(y)e^{\mu y}dy - 1 - \overline{b_2q} = c\mu,$$

where $\overline{u}$ is defined as $\overline{u} = \frac{\int_0^\omega u(t)dt}{\omega}$ for any ω-periodic function$u$. Clearly, $f_0^\omega = \overline{a_1p - b_1q} > 0$

by (5.5.19) and $g_0^\omega = -\overline{b_2 q} < 0$, it immediately follows that

$$c_0 = \inf_{\mu>0} \frac{\int_{-\infty}^{\infty} J_1(y)e^{\mu y}dy - 1 + \overline{a_1 p - b_1 q}}{\mu}. \tag{5.5.23}$$

The right-hand side attains its infinitum at some finite value $\bar{\mu}$. Moreover, it is clear to see that for $c > c_0$, there exists $\mu_1(c)$ and $\mu_2(c)$ such that $c = f_{\mu_1}^\omega/\mu_1 = f_{\mu_2}^\omega/\mu_2$, and μ_1 is decreasing while μ_2 is increasing in c. The asymptotic behavior of (U, V) is

$$\begin{pmatrix} U \\ V \end{pmatrix} \sim C_1 \begin{pmatrix} \zeta_{\mu_1}^u(t) \\ \zeta_{\mu_1}^v(t) \end{pmatrix} e^{-\mu_1 \xi} + C_2 \begin{pmatrix} \zeta_{\mu_2}^u(t) \\ \zeta_{\mu_2}^v(t) \end{pmatrix} e^{-\mu_2 \xi} + C_3 \begin{pmatrix} 0 \\ \zeta_{\mu_3}^v(t) \end{pmatrix} e^{-\mu_3 \xi},$$

when μ_3 is not equal to μ_1 and μ_2, where μ_3 is the unique solution of $g_\mu^\omega = c\mu$ for any given $c \geqslant c_0$ and increasing in c.

In summary, this model can be classified into case (II). To apply our theorems under this case, we want to prove (A6), the following lemma provides a justification.

Lemma 5.5.10. *Let (**D4**) and (5.5.19) hold. For $c \geqslant c_0$ (see, (5.5.23)), and a given continuous ω-periodic function $U(t,\xi)$ ($\xi = x - ct$) satisfying $U(t,\infty) = 0$, $U(t,-\infty) \geqslant \frac{b_2(t)q(t)}{a_2(t)p(t)}$, ω-periodic in t, and nonincreasing in ξ, there exists an ω-periodic function $V(t,\xi)$, which is also nonincreasing in z, solving*

$$\begin{cases} V_t = [J_2 * V - V] + cV_\xi + (1-V)\big(a_2(t)p(t)U(t,\xi) - b_2(t)q(t)V\big), \\ V(t,-\infty) = 1, \ V(t,\infty) = 0, \\ V(t,\xi) = V(t+\omega,\xi). \end{cases}$$

Moreover, V is monotone in U.

Proof. By letting $W(t,z) = 1 - V(t,\xi)$, $z = -\xi$, we obtain that

$$\begin{cases} W_t = [J_2 * W - W] - cW_z + b_2(t)q(t)W[R(t,z) - W], \\ W(t,-\infty) = 1, \ W(t,+\infty) = 0, \\ W(t,z) = W(t+\omega,z). \end{cases} \tag{5.5.24}$$

Here, $R(t,z) = 1 - \frac{a_2(t)p(t)}{b_2(t)q(t)}U(t,\xi)$. Thus, $R(t,-\infty) = 1 > 0 > R(t,+\infty) > -\infty$. Following [Theorem 1.1, [93]], we immediately get the existence of W to (5.5.24); therefore, the existence of V is obtained. Moreover, the monotonicity of V in U is guaranteed by the positivity of $(1-V)$ and $a_2(t)p(t)$. The proof is complete. $\square$

Remark 5.5.11. *In [93], the authors proved such an existence result through the upper and lower solutions method. They constructed a trivial upper solution $\bar{W} = 1$ and a nontrivial lower solution from a lower system with a combustion-type nonlinearity. We should mention that a lower solution can also be proved by following the ideas in [51, 83], in which, a lower nontrivial solution is constructed from a lower system with a bistable nonliearity.*

Next, we will provide some specific conditions on the speed selection and give an estimate on $c_{\min}$ when it is nonlinearly selected. Since the formula $V = V(U)$ is too complicated (cannot be found explicitly), we will give testing functions of U and V simultaneously. The results are presented as follows.

Theorem 5.5.12. *Let*

$$\bar{U}(t,\xi) = \frac{1}{1 + \frac{e^{\bar{\mu}\xi}}{\zeta^u_{\bar{\mu}}(t)}},\ \bar{V} = \min\{1, k\bar{U}\},\ \textit{with } k \geqslant 1,$$

where $\bar{\mu} = \mu(c_0)$ and $\xi_1(t)$ be an ω-periodic function satisfying $k\bar{U}(t,\xi_1) = 1$. If

$$\Gamma_{1,\bar{\mu}} := \max\left\{\left[\int_{-\infty}^{\infty} J_1(\xi-y)\frac{\bar{U}(t,y) - e^{\bar{\mu}y}\bar{U}(1-\bar{U})}{\bar{U}^2(1-\bar{U})}dy - \frac{1}{1-\bar{U}}\right] + G_1(t,z)\right\} \leqslant 0, \tag{5.5.25}$$

and

$$\Gamma_{2,\bar{\mu}} := \max\left\{\int_{\mathbb{R}}\left[J_2(\xi-y)\frac{\bar{U}(t,y)}{\bar{U}(1-\bar{U})} - J_1(y)e^{\bar{\mu}y}\right]dy - \frac{\bar{U}}{1-\bar{U}} + G_2(t,z)\right\} \leqslant 0, \tag{5.5.26}$$

where

$$\begin{cases} G_1(t,z) = -\frac{(b_1(t)q(t)+1)\bar{U} - b_1(t)q(t)}{\bar{U}(1-\bar{U})}, \\ G_2(t,z) = -a_1(t)p(t) + b_1(t)q(t) + \frac{(1-k\bar{U})(\frac{a_2(t)p(t)}{k} - b_2(t)q(t))}{1-\bar{U}}, \end{cases}$$

then the linear selection is realized.

Proof. Substituting formulas of $\bar{U}$ and $\bar{V}$ into the left-hand side of (5.5.18), and through a tedious computation, we have

$$\begin{aligned}
&\left[\int_{-\infty}^{\infty} J_1(y)\bar{U}(t,\xi-y)dy-\bar{U}\right]+c_0\bar{U}_\xi+\bar{U}\left[a_1(t)p(t)(1-\bar{U})-b_1(t)q(t)(1-\bar{V})\right]-\bar{U}_t\\
=\ &\left[\int_{-\infty}^{\infty} J_1(y)\bar{U}(t,\xi-y)dy-\bar{U}\right]-c_0\bar{\mu}\bar{U}(1-\bar{U})\\
&+\bar{U}\left[a_1(t)p(t)(1-\bar{U})-b_1(t)q(t)(1-\bar{V})\right]-\frac{\zeta_{\bar{\mu}}^{u\prime}}{\zeta_{\bar{\mu}}^{u}}\bar{U}(1-\bar{U})\\
=\ &\bar{U}^2(1-\bar{U})\left\{\left[\int_{-\infty}^{\infty} J_1(\xi-y)\frac{\bar{U}(t,y)-e^{\bar{\mu}y}\bar{U}(1-\bar{U})}{\bar{U}^2(1-\bar{U})}dy-\frac{1}{1-\bar{U}}\right]+b_1(t)q(t)\frac{\bar{V}-\bar{U}}{\bar{U}(1-\bar{U})}\right\},
\end{aligned}$$

and

$$\begin{aligned}
&\left[J_2*\bar{V}-\bar{V}\right]+c\bar{V}_\xi+(1-\bar{V})(a_2(t)p(t)\bar{U}-b_2(t)q(t)\bar{V})-\bar{V}_t\\
=\ &k\bar{U}(1-\bar{U})\left\{\int_{\mathbb{R}}\left[J_2(\xi-y)\frac{\bar{U}(t,y)}{\bar{U}(1-\bar{U})}-J_1(y)e^{\bar{\mu}y}\right]dy-\frac{\bar{U}}{1-\bar{U}}+G_2(t,z)\right\}
\end{aligned}$$

Here, $\bar{U}=\bar{U}(t,\xi)$. Notice that, for $t\in[0,\omega]$,

$$\frac{\bar{V}-\bar{U}}{\bar{U}(1-\bar{U})}=\begin{cases}\frac{1}{\bar{U}},\ \xi\le\xi_1(t),\\[4pt] \frac{k-1}{1-\bar{U}},\ \xi>\xi_1(t).\end{cases}$$

Thus, it is less than $\frac{1}{\bar{U}}$ for $\xi\in\mathbb{R}$. Then, by (5.5.25) and (5.5.26), the above two equations are less than 0. This implies that such a choice of $\bar{U}$ and $\bar{V}$ forms upper solutions to (5.5.21). Therefore, the linear selection result directly follows from Theorem 5.4.8. This completes proof. □

Theorem 5.5.13. *For some $c_1>c_0$, let $\mu_2=\mu_2(c_1)$, and*

$$\underline{U}(t,\xi)=\frac{\underline{k}}{1+\frac{e^{\mu_2 z}}{\zeta_{\mu_2}^{u}(t)}},\ \underline{V}=\frac{\underline{U}}{\underline{k}},\ \textit{with } 0<\underline{k}<1.$$

The nonlinear selection is realized if

$$\min\left[\int_{-\infty}^{\infty} J_1(\xi-y)\frac{\underline{U}-e^{\mu_2 y}\underline{V}(1-\underline{V})}{\underline{k}\underline{V}^2(1-\underline{V})}dy-\frac{1}{1-\underline{V}}\right]+\frac{a_1(t)p(t)\,(1-k)}{1-\underline{V}}\geqslant 0,$$

and

$$\min\left\{\int_{\mathbb{R}}\left[J_2(\xi-y)\frac{\underline{U}}{\underline{U}(1-\underline{V})}-J_1(y)e^{\mu_2 y}\right]dy-\frac{\underline{V}}{1-\underline{V}}+G_3(t,z)\right\}\geqslant 0,$$

where

$$G_3(t,z)=-a_1(t)p(t)+b_1(t)q(t)+\underline{k}a_2(t)p(t)-b_2(t)q(t).$$

Furthermore, for some $c_2 \geqslant c_1 > c_0$, let a pair of ω-time-periodic functions be defined as

$$U(t,z)=\frac{1}{1+\frac{e^{\mu_2 z}}{\zeta^u_{\mu_2}(t)}},\ V=\min\{1,\,k_2U\},\ \textit{with } k_2\geqslant 1,$$

where $\mu_2=\mu_2(c_2)$, and there exist $\xi_2(t)$ such that $k_2U(t,\xi_2)=1$. Replace $\bar{\mu}$, $\bar{U}$, $\bar{V}$ with μ_2, U, V in Γ_{1,μ_2} and Γ_{2,μ_2} (whose definitions are seen in (5.5.25) and (5.5.26)), respectively. If the inequalities $\Gamma_{1,\mu_2}\leqslant 0$ and $\Gamma_{2,\mu_2}\leqslant 0$ hold, then $c_1\leqslant c_{\min}\leqslant c_2$.

Remark 5.5.14. *Here, we only choose $V = kU$ to show how to apply our theorems in Section 3. To gain some sharper conditions, we can use the idea in the stream population example and choose*

$$V=AU+BU^2+CU^3$$

with suitably chosen parameters: A, B, and C. For U, we can use

$$U_\xi=-\mu U(1-U^m),\ m>0,\ \textit{or } U_\xi=-\mu U(1-\frac{1}{1-\ln U}).$$

We leave this for interesting readers.

5.5.4 A reducible cooperative system

Finally we will work on a reducible periodic cooperative system, which may admit multiple spreading speeds so that it provides an example for Case (III). The system is given by

$$\begin{cases}\frac{\partial u}{\partial t}=d_1(t)\frac{\partial^2 u}{\partial x^2}+f(t,u),\\ \frac{\partial v}{\partial t}=d_2(t)\frac{\partial^2 v}{\partial x^2}+g(t,v)+k(t)u.\end{cases}\tag{5.5.27}$$

Here, $d_1(t)$, $d_2(t)$, $k(t)$, $f(t,u)$, and $g(t,v)$ are all ω-periodic Hölder-continuous functions, and $f(t,u)$ and $g(t,v)$ are Lipschitz continuous in the second variable. Moreover, $d_i(t) \geqslant \eta > 0$ $(i = 1,2)$, and $k(t) > 0$, f and g are assumed to satisfy

(D5) $f(t,0) = g(t,0) = f(t,1) = g(t,1) = 0$, $f(t,u) > 0$, $g(t,v) > 0$ for all $0 < u, v < 1$, and $f'_u(t,0) > 0$, $g'_v(t,0) > 0$ where f'_u, g'_v denote the derivative with respect to the second variable.

Following (5.4.13) in Case (III), we can define the individual spreading speeds for species u (c_u^*) and v (c_v^*) as

$$\begin{cases} \lim\limits_{t\to\infty,\ x\geqslant(c_u^*/\omega+\epsilon)t} u(t,x) = 0, & \lim\limits_{t\to\infty,\ x\leqslant(c_u^*/\omega-\epsilon)t} u(t,x) \geqslant \eta_u > 0, \\ \lim\limits_{t\to\infty,\ x\geqslant(c_v^*/\omega+\epsilon)t} v(t,x) = 0, & \lim\limits_{t\to\infty,\ x\leqslant(c_v^*/\omega-\epsilon)t} v(t,x) \geqslant \eta_v > 0, \end{cases} \tag{5.5.28}$$

for any small $\epsilon > 0$ and η_u, η_v being positive constants.

When $f(t,u) = u(1-u^2)$, $g(t,v) = v(1-v^2)$, $d_1(t) \equiv 1$, $d_2(t) \equiv d$, and $k(t) \equiv K$, this model was studied in [74]. Notice that, under these conditions, the nonlinearities are subhomogeneous (i.e., $f(u) \leqslant f'(0)u$ and $g(v) \leqslant g'(0)v$); thus, c_u^* is always linearly selected by [42]. Moreover, in [74], the authors claimed that (i) if $d < 1$, then species u and v share a common spreading speed spread $c_u^* = c_v^* = 2$; (ii) if $d > 1$, then species u and v have different spreading speeds: $c_u^* = 2$ and $c_v^* = 2\sqrt{d}$.

Next, we will remove the subhomogeneous condition and study a time-periodic system. To apply our theorems, we need to identify $\beta(t)$ and $\alpha(t)$ (if any); thus, let

$$\frac{dv}{dt} = g(t,v) + k(t)$$

have a unique positive ω-periodic solution $v^*(t) \geqslant 1$ for $t \in [0,\omega]$. Define $\beta(t) = (1, v^*(t))$. Then, between $\beta(t)$ and 0, (5.5.27) has three spacial homogeneous but ω-periodic solutions:

$$\beta(t) = (1, v^*(t)),\ \alpha(t) = (0,1),\ 0 = (0,0), \tag{5.5.29}$$

with $\beta(t)$ being stable and 0 being unstable.

Following [[31], Chapter II], the ω-periodic heat equation $\frac{\partial \mathbf{u}}{\partial t} = D(t)\Delta\mathbf{u}$ generates an

ω-periodic semiflow denoted as $\hat{Q}(t,s)$ $(0 \leqslant s \leqslant t \leqslant T)$ such that the solution of (5.5.27) can be represented in an integral form as

$$\mathbf{u}(t,\cdot;\phi) = \hat{Q}(t,0)\phi + \int_0^t \hat{Q}(t,s)\mathbf{f}(s,\mathbf{u}(s,\cdot;\phi))ds, \tag{5.5.30}$$

where $\mathbf{u} = (u,v)^T$, $D(t) = \mathrm{diag}(d_1(t), d_2(t))$, and $\mathbf{f} = (f,g)^T$. Define $Q_t[\phi] := \mathbf{u}(t,\cdot;\phi)$ and it satisfies Definition 5.2.1. Moreover, notice that (5.5.27) is a cooperative system and $\mathbf{f}$ is Lipschitz continuous in $u \in \mathcal{C}_1$ and Hölder continuous in $t \in \mathbb{R}_+$. It is easy to verify (A1), (A3) and (A4) for Q_t and $P = Q_\omega$. Moreover, P is a compact map, thus (A2) is satisfied automatically. Therefore, by Lemma 5.2.4, we have the existences of c^*, c_f^*, and traveling waves for Q_t. It has been shown that, in the constant case, the inequality $c^* < c_f^*$ can be true under some further requirements (e.g., $d > 1$). Following the procedure in Case (III), we then relate the slowest and fastest spreading speeds to the individual spreading speeds and investigate the determinacy of them.

A traveling wave solution takes the form $u(t,x) = U(t,\xi),\ v(t,x) = V(t,\xi),\ \xi = x - ct$. Thus, by substituting the wave profiles into (5.5.27), we obtain

$$\begin{cases} d_1(t)U_{\xi\xi} + cU_\xi + f(t,U) - U_t = 0, \\ d_2(t)V_{\xi\xi} + cV_\xi + g(t,V) + k(t)U - V_t = 0. \end{cases} \tag{5.5.31}$$

Linearizing (5.5.31) around 0 and then substituting $(U,V)^T = (\zeta^u, \zeta^v)e^{-\mu z}$ into it, we obtain an eigen problem

$$\begin{pmatrix} \tilde{f}_\mu(t) - c\mu & 0 \\ k(t) & \tilde{g}_\mu(t) - c\mu \end{pmatrix} \begin{pmatrix} \zeta^u \\ \zeta^v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{5.5.32}$$

where $\tilde{f}_\mu(t) = d_1(t)\mu^2 + f'_u(t,0) - \frac{\zeta^{u\prime}}{\zeta^u}$ and $\tilde{g}_\mu(t) = d_2(t)\mu^2 + g'_v(t,0) - \frac{\zeta^{v\prime}}{\zeta^v}$. Here, from the linear system (5.5.32), it is clear to see that (5.5.27) belongs to Case (III): B_0^ω is reducible with $I_0 \setminus \{1\} \neq \emptyset$. Integrating the diagonal elements of (5.5.32) leads to

$$\tilde{f}_\mu^\omega := \mu^2 \int_0^\omega d_1(t)dt + \int_0^\omega f'_u(t,0)dt = c\mu\omega$$

and

$$\tilde{g}_\mu^\omega := \mu^2 \int_0^\omega d_2(t)dt + \int_0^\omega g_u'(t,0)dt = c\mu\omega.$$

Due to **(D5)**, $\tilde{f}_0^\omega > 0$ and $\tilde{g}_0^\omega > 0$; therefore, we have

$$c_0^u := \frac{1}{\omega} \inf_{\mu>0} \frac{\tilde{f}_\mu^\omega}{\mu} = 2\sqrt{d_1 f_u'(t,0)}, \text{ and } c_0^v := \frac{1}{\omega} \inf_{\mu>0} \frac{\tilde{g}_\mu^\omega}{\mu} = 2\sqrt{d_2 g_u'(t,0)}.$$

For $c > \max\{c_0^u, c_0^v\}$, the asymptotic behaviors near 0 can be found as

$$\begin{pmatrix} U \\ V \end{pmatrix} \sim C_1 \begin{pmatrix} \zeta_{\mu_1}^u(t) \\ \zeta_{\mu_1}^v(t) \end{pmatrix} e^{-\mu_1\xi} + C_2 \begin{pmatrix} \zeta_{\mu_2}^u(t) \\ \zeta_{\mu_2}^v(t) \end{pmatrix} e^{-\mu_2\xi} + C_3 \begin{pmatrix} 0 \\ \zeta_{\mu_3}^v(t) \end{pmatrix} e^{-\mu_3\xi} + C_4 \begin{pmatrix} 0 \\ \zeta_{\mu_4}^v(t) \end{pmatrix} e^{-\mu_4\xi},$$

if $\mu_i = \mu_i(c)$, $i = 1,2,3,4$ are not equal.

Similarly, through linearizing (5.5.31) near $\alpha = (0,1)$, there exists a single linear speed satisfying

$$c_\alpha^f = c_0^u. \tag{5.5.33}$$

In this special case, we notice that the U-equation has been decoupled from the system and has only two equilibrium. Thus, by applying Lemma 5.2.4, we obtain the following lemma.

Lemma 5.5.15. *Let **(D5)** be true. Then the following two statements hold.*

(1) Let c_u^* be the spreading speed u-species, then it is the critical number in the sense that the U-equation in (5.5.27) has a traveling wave $U(t,\xi)$ satisfying $U(t,-\infty) = 1$, $U(t,+\infty) = 0$, and nonincreasing in ξ if and only if $c \geqslant c_u^*/\omega \geqslant c_0^u$.

(2) There exists a critical number $\hat{c}_v$ such that

$$\begin{cases} d_2(t)V_{\xi\xi} + cV_\xi + g(t,V) - V_t = 0, \\ V(t,-\infty) = 1, \ V(t,+\infty) = 0, \end{cases} \tag{5.5.34}$$

has solutions if and only if $c \geqslant \hat{c}_v/\omega \geqslant c_0^v$.

Then, according to the definitions of c_u^*, c_v^* and $\hat{c}_v$, we have the following lemma to determine their relation.

Lemma 5.5.16. *Let **(D5)** hold and c_u^*, c_v^* are the spreading speeds of u and v species. Then the following results are true.*

(i). $c_v^* \geq \hat{c}_v$.

(ii). $c_v^* \geq c_u^*$.

(iii). $c_u^* = c_v^*$ *if and only if* $c_u^* \geq \hat{c}_v$.

Proof. (i). Since the term $k(t)u$ is always nonnegative, by comparison, the result follows.

(ii). Assume to the contrary that $c_v^* < c_u^*$. Let $x = (\frac{c_v^*+c_u^*}{2})t$ and $t \to \infty$. From the second equation of the system, we obtain a contradiction $0 = k(t)u(t,x) > 0$. Thus, it is always that $c_v^* \geqslant c_u^*$.

(iii). If $c_u^* \geq \hat{c}_v$, we will derive $c_u^* = c_v^*$. Otherwise, we will get $c_u^* < c_v^*$ by (ii). As such, from the limiting equation of the second equation, we get $c_v^* = \hat{c}_v$. This is a contradiction. The remainder part can be similarly proved. □

Following Definitions 5.4.20 (single) and 5.4.23 (multiple) in Case (III), and Lemma 5.5.16, we proceed to give a speed selection theorem, which shows sufficient conditions to decide the speed selection mechanism for (5.5.27) with either a single spreading speed or multiple spreading speeds.

Theorem 5.5.17. *Let **(D5)** hold. The following statements are true.*

(1) $c_u^*/\omega = c_0^u$, *if*

$$\max\left\{-2d_1(t)\bar{\mu}_u^2 + \frac{f(t,\bar{U}) - f_u'(t,0)\bar{U} - \bar{U}}{\bar{U}^2(1-\bar{U})}\right\} \leqslant 0, \tag{5.5.35}$$

where $\bar{\mu}_u = \mu_0^u(c_0^u)$ *and* $\bar{U}(t,z) = 1/[1+\exp\{\bar{\mu}_u z\}/\zeta_{\bar{\mu}_u}^u(t)]$.

(2) $\hat{c}_v/\omega = c_0^v$ *if*

$$\max\left\{-2d_2(t)\bar{\mu}_v^2 + \frac{g(t,\bar{V}) - g_v'(t,0)\bar{V} - \bar{V}}{\bar{V}^2(1-\bar{V})}\right\} \leqslant 0, \tag{5.5.36}$$

where $\bar{\mu}_v = \mu_0^v(c_0^v)$ *and* $\bar{V}(t,z) = 1/[1+\exp\{\bar{\mu}_v z\}/\zeta_{\bar{\mu}_v}^v(t)]$.

(3) Let (5.5.35) and (5.5.36) hold. If, in addition,

$$\overline{d_1 f_u'(t,0)} < \overline{d_2 g_v'(t,0)}, \tag{5.5.37}$$

then (5.5.27) has multiple spreading speeds and both of them are linearly selected, that is, $c_u^*/\omega = c_0^u < c_0^v = c_v^*/\omega$.

(4) When (5.5.35) and (5.5.36) hold, and besides,

$$\overline{d_1 f_u'(t,0)} > \overline{d_2 g_v'(t,0)}, \tag{5.5.38}$$

then (5.5.27) has a spreading speed with c_u^* *being linearly selected while* c_v^* *being nonlinearly selected in the sense that* $c_v^*/\omega = c_u^*/\omega = c_0^u > c_0^v$.

(5) $c_u^*/\omega > c_0^u$ *if*

$$\min\left\{-2d_1(t)\bar{\mu}_u^2 + \frac{f(t,U_1) - f_u'(t,0)}{U_1^2(1-U_1)}\right\} > 0, \tag{5.5.39}$$

where $U_1 = 1/[1+\exp\{\bar{\mu}_u z\}/\zeta_{\bar{\mu}_u}^u(t)]$ *and* $\bar{\mu}_u = \bar{\mu}_u(c_0^u)$.

(6) $\hat{c}_v/\omega > c_0^v$ *if*

$$\min\left\{-2d_2(t)\bar{\mu}_v^2 + \frac{g(t,V_1) - g_v'(t,0)}{V_1^2(1-V_1)}\right\} > 0, \tag{5.5.40}$$

where $V_1 = 1/[1+\exp\{\bar{\mu}_v z\}/\zeta_{\bar{\mu}_v}^v(t)]$ *and* $\bar{\mu}_v = \bar{\mu}_v(c_0^v)$.

(7) Let (5.5.39) and (5.5.40) be true. If (5.5.27) has a single spreading speed, then $c_u^* = c_v^*$ *and they are both nonlinearly selected in the sense that* $c_u^*/\omega > c_0^u$ *and* $c_v^*/\omega > c_0^v$. *Otherwise, (5.5.27) has multiple spreading speeds, then* $c_u^* < c_v^*$ *and both of them are nonlinearly selected in the sense that* $c_u^*/\omega > c_0^u$ *and* $c_v^*/\omega > \max\{c_0^u,\ c_0^v\}$.

Proof. (1) Since the condition (5.5.35) implies that $\bar{U}(t, x - c_0^u t)$ is an upper solution to U-equation. By Theorem 5.4.8, the result follows.

(2) Similar to (1), when the limiting system (5.5.34) is considered, the condition (5.5.36) indicates the desired result.

(3) Under conditions (5.5.35), (5.5.36) and (5.5.37), we have that $c_u^*/\omega = c_0^u < c_0^v = \hat{c}_v/\omega$. Thus, following Lemma 5.5.16 (iii), we immediately obtain that (5.5.27) has multiple spreading speeds, i.e., $c_u^* < c_v^*$. Moreover, we have that $c_v^*/\omega = \hat{c}_v/\omega = c_0^v$.

(4) The inequalities (5.5.35), (5.5.36) and (5.5.38) implies that $c_u^*/\omega = c_0^u > c_0^v = \hat{c}_v/\omega$. Thus, we get that $c_u^* = c_v^*$ by Lemma 5.5.16 (iii). Moreover, it follows that $c_v^*/\omega = c_u^*/\omega = c_0^u > c_0^v$.

(5) From Corollary 5.4.6, it immediately follows that $c_u^*/\omega > c_0^u$ when (5.5.39) holds.

(6) Similar to (5), when the limiting system (5.5.34) is considered, (5.5.40) implies the desired result.

(7) Combining Lemma 5.5.16 (i) and (ii), it is always that

$$c_v^* \geqslant \max\{c_u^*,\ \hat{c}_v\}.$$

When conditions (5.5.39) and (5.5.40) hold, the rest of this proof is straightforward. If the reducible system (5.5.27) has a single spreading speed, then $c_u^*/\omega > c_0^u$ and $c_v^*/\omega \geqslant \hat{c}_v/\omega > c_0^v$. If (5.5.27) has multiple spreading speeds, then $c_u^*/\omega > c_0^u$ and $c_v^*/\omega \geqslant \max\{c_u^*/\omega,\ \hat{c}_v/\omega\} > \max\{c_0^u,\ c_0^v\}$. □

To finish this example, we give a small discussion here. Since this is an example to present the usage of our theory, we only give a quite general discussion here. More specific discussions of such a typical model to learn the speed selection are left for interesting readers.

5.6 Conclusion

In this chapter, we have investigated the speed selection mechanism for traveling waves to monotone periodic semiflows in the monostable case and successfully improved the current results on the linear selection and made a breakthrough on the nonlinear selection. The improvements of this chapter mainly focus in three aspects: the results are applicable to time-periodic reaction-diffusion models even having boundary equilibria; the conditions for the linear selection have been improved from the classical one, which requires the monotone semiflow can be governed by its corresponding linear map; the results for the nonlinear selection are novel.

In Section 5.2, we introduced some preliminaries to establish the existence of traveling waves. Then, we examined the linear speed in Section 5.3. Our main theoretical result was presented in Section 5.4. We gave our definition of the speed selection, and considered three cases to investigate the mechanism further (see, subsections 5.4.1–5.4.3). Here is a brief review of new results in this chapter: we found a sufficient and necessary condition of the nonlinear selection and provided an estimate of the speed when the nonlinear selection occurs for Case (I) and (II); a fairly detailed investigation (containing both linear and nonlinear selection) has been provided for Case (III). Four applications are carried out in Section 5.5, and they covered all the cases discussed in Section 5.4.

Chapter 6

Future Work

Research of the invasion speed determinacy for wave propagation as well as traveling waves is explosively expanding. The models considered in this thesis have either constant coefficients or time-periodic coefficients. Thus, a direct extension is to investigate the speed selection of time-space periodic monotone models. The existence of a traveling wave of time-space periodic monotone semiflows can be found in [19]. It is worthy of extending our speed determinacy discussion in Chapter 5 to the time-space periodic case since the time-space periodic environment is typical in biology.

In Chapters 4 and 5, we discussed the case where a model admits multiple spreading speeds. However, we only considered the speed determinacy for the slowest and fastest spreading speeds. A possible extension would be to investigate the speed determinacy for all spreading speeds when a model has more than two speeds, which is indeed observed in combustion phenomena, see, e.g., [34, 57].

In this thesis, we always assume that $f'(0) > 0$ so that $c_0 > 0$, while it is possible to have $f'(0) = 0$. A typical example is a degenerate Fisher equation, i.e., (1.0.1) with $f(u) = u^m(1-u)$, $m > 1$. From [38], it has a critical number $c^*(m) > 0$ so that the degenerate Fisher equation has a traveling wave connecting 1 to 0 if and only if $c \geqslant c^*(m)$. Moreover, it is easy to see that this equation has a linear speed $c_0 = 0$. Thus, the invasion speed of this equation is always nonlinearly selected. It would be challenging to determine the spreading speed of a degenerate system.

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