MARGINALLY OUTER TRAPPED SURFACES IN REISSNER–NORDSTRÖM SPACETIME

by

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Abstract

Looping marginally outer trapped surfaces, MOTS, have been found and seem to be a key phenomenon in black hole interiors during binary black hole mergers. These looping surfaces have also been found in the much simpler (and static) Schwarzschild black holes. The results presented in this thesis include observations of looping MOTS in charged, Reissner-Nordström black holes, where we find looping, cusping, and wiggling MOTS behavior depending on the charge of the black hole and stability of its inner horizon.

Declaration

The work presented in this thesis was conducted by me along with the guidance of my supervisor Dr. Ivan Booth, as well as Kam To Billy Chan. All figures within this thesis have been produced by me, using Mathematica code produced with the help of Kam To Billy Chan. Results found will also be my own work or from collaborative work in which I contributed significantly, unless I state otherwise. A paper in which this work presents itself in is the reference below.

- Robie A. Hennigar, Kam To Billy Chan, Liam Newhook, Ivan Booth, "The Interior MOTSs of Spherically Symmetric Black Holes", *Phys.Rev.D* 105 (2022), arXiv:2111.09373v1

Acknowledgements

I give my thanks to Dr. Booth for the opportunity to research such an interesting and wonderous area, and for the assistance and guidance provided. I'd also like to say thanks to Kam To Billy Chan for the guidance he has given me throughout the course of this research.

Table of Contents

List of Figures

Chapter 1

Introduction

One of the most exciting predictions of general relativity is the existence of black holes. Much is known about the region outside these objects, but less is known of the mechanisms within. Understanding how such objects merge is of importance and there have already been some observations of such events using gravitational waves [1].

Two key structures that are tracked to help understand mergers are event horizons and apparent horizons. The behavior of the event horizons in these mergers has long been understood [2]. Apparent horizons, the focus of this thesis, have been much less understood. These are surfaces from which light nether expands nor contracts, and these are known to form and annihilate throughout the course of the merger [3].

Single black holes, like the charged one that is the core of this work, have proven to be a way to examine a simpler case than the mergers but they still give some insight into how horizons evolve. This is because small, singular black holes, have quite a strong gravitational field near the horizons, while larger black holes have a weaker field at the horizons. So, when we consider a merger between a very large black hole and a very small black hole, the most important thing is the small one, meaning we can examine singular cases of small black holes to gain insight into how the mergers may behave [4].

The charged black hole is also important for another reason. The most common black holes in our universe are rotating black holes, meaning rotating black holes are important to study, but they are quite complicated. Charged black holes, however, while not nearly as complicated, share some characteristics with the rotating case. Most notably they also contain an inner horizon that is characterized by a region with gravitational repulsion. Thus, the charged black hole is perfect for a simpler intensive examination.

1.1 Marginally Outer Trapped Surfaces Explained

A large stride was made in the study of mergers when it was found that they contain horizon-like structures that self-intersect [3][5]. These self-intersecting surfaces are given the name Looping Marginally Outer Trapped Surfaces or Looping MOTS for short.

To understand a regular (not necessarily looping) MOTS, imagine a surface in any spacetime you like, perhaps just a sphere, resting in an ordinary Minkowski spacetime. Now imagine placing light bulbs all over this surface and then flicking them on. Light rays will head both radially inwards and outwards, just flying off as they please. However if we introduce a strong enough gravitational field somewhere inside our constructed surface, then when we flicked on the bulbs, even the “outward” moving light rays will be pulled back in if the escape velocity exceeds the speed of light. Turning to black holes, it is known that light that has gone past the event horizon of a black hole, does not leave. In our mental concoction this corresponds to the outward oriented light falling back down toward the source of the gravitational force. Finally, imagine that the force is only just strong enough to hold the light rays constant on our surface, meaning that when we flick our bulbs on, the light rays remain stationary. This is precisely a MOTS. A looping MOTS is a MOTS that intersects itself.

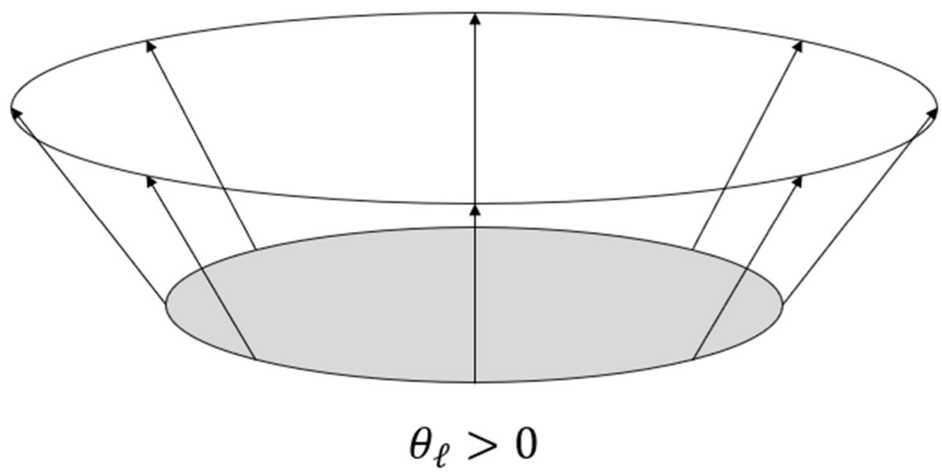

FIG. 1.1. Depiction of a “regular” surface, from which the light rays would expand freely.

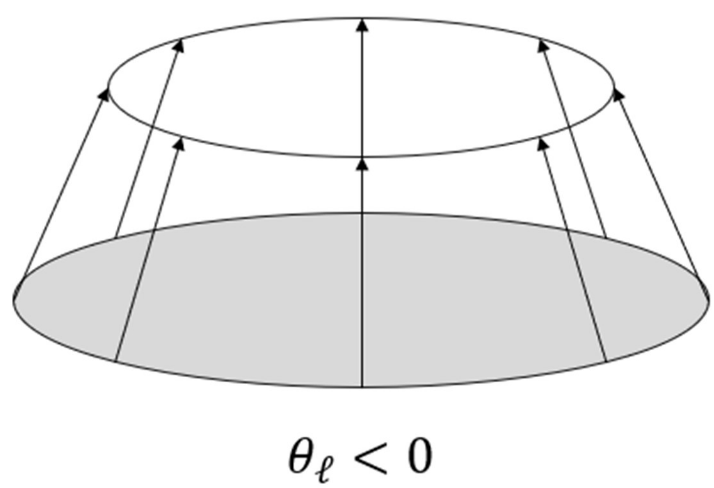

FIG. 1.2. Depiction of an outer trapped surface, from which the light rays are drawn inward.

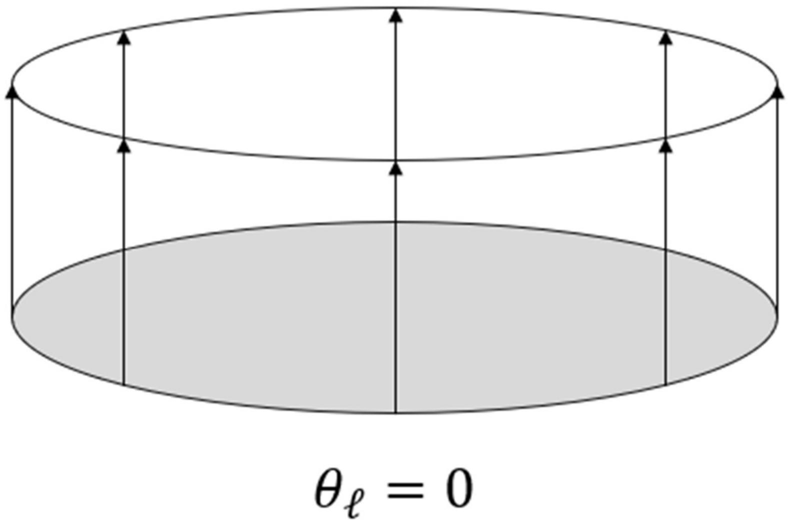

FIG. 1.3. The Marginally Outer Trapped Surface (MOTS). Surface in which light rays positioned on top of the surface remain stationary on the surface.

1.2 Apparent Horizons and MOTSs

For all these surfaces, something akin to a second derivative test for maxima/minima tells us about the stability of the surface. This is referred to as the stability operator, and the number of negative eigenvalues to this operator will be the indicator of the stability. Stability in this sense does not mean what it does when talking about, say, orbits in a Newtonian sense. In this context stability indicates two major things about surfaces. First if a surface is stable (the stability operator has no negative eigenvalues), then it forms a boundary between outer and inner trapped regions. Second, a stable surface cannot be smooth(ly) deformed [6]. The derivation of this operator is beyond the scope of this thesis, so the interested reader is directed to work by my collaborators [6].

1.3 Roadmap

Shortly after the discovery of self-intersecting MOTSs within black hole mergers, they were also found within singular black holes [3]. The simplest case is the black hole in the Schwarzschild spacetime, a black hole with no charge or spin. Large numbers of self-intersecting MOTSs were found within the Schwarzschild black hole. In this work, I will be examining the Reissner-Nordström, RN for short, black hole, which has a charge, and will be examined in a new coordinate system which spans the inside and outside of the black hole with spacelike "instants" of time.

I will begin by re-examining the Schwarzschild black hole to demonstrate the basic ideas and inner workings of the simulations used to produce images of MOTS. This will be done through examining the generalized Painlevé-Gullstrand (PG for short) metric and setting the charge to zero, effectively giving the Schwarzschild black hole we desire. This practice also proves useful in ensuring that the code is working as intended. Proceeding on, I remove restrictions from the code and sift through slices in time for set charge values. This will be the main work, and I will present a few important observations made throughout the course of this research.

Chapter 2

Methods

2.1 Metrics

First, an understanding of metrics is required, specifically the metric in question, the generalized PG metric. From this understanding, I can construct a framework of how we will find the MOTS of interest. Initially though, I will begin by giving a general explanation of metrics and related topics by using the Schwarzschild metric.

The Schwarzschild metric is,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \qquad (1)$$

Intuitively this so-called line element describes how to calculate arclengths in the given coordinate system. The Schwarzschild metric describes a spherical, uncharged black hole and is an exact solution to Einstein's equations. There are a few things to notice here with the major one being that the dr^2 component of the metric diverges at $r = 2M$ and is degenerate at $r = 0$. Thus, this coordinate system is not good for understanding the physics near or inside the horizon. This can be remedied through altering our coordinate system. A common set of coordinates to examine the Schwarzschild black hole without the limitations are the PG coordinates, which have the following line element [7],

$$ds^2 = -f dt^2 + 2\sqrt{1-f}\,dt dr + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2. \tag{2}$$

While standard Schwarzschild coordinates are tied to stationary observers (and hence fail at the horizon where no such observers can exist), these are tied to infalling observers and so are good both inside and outside the black hole. In this coordinate system,

$$f = 1 - \frac{2M}{r}. \tag{3}$$

If you examine the components of the PG metric with this function, you will see that there is no longer a divergence at $r = 2M$, yet there still is one for the point $r = 0$.

2.2 Penrose Diagrams

These coordinate systems can be quite well described by what is known as Penrose diagrams, or portions of them. In these diagrams we see light rays follow along 45-degree angles either from the positive or negative horizontal axis. The axes are spacelike and timelike for the x and y respectfully.

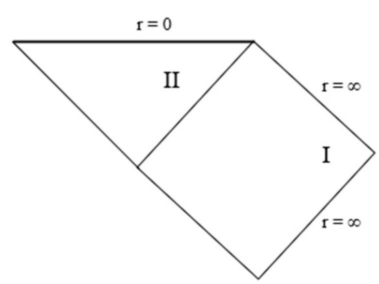

FIG. 2.1. Penrose diagram illustrating our universe (I), and the black hole interior (II).

In Figure 2.1, I aim to show the general idea of a Schwarzschild black hole in a basic Schwarzschild coordinate system. The observer here is stationary along the $r = \infty$ line and cannot see things inside the horizon (no light rays can cross out of the horizon). Note too that the $r = constant$ obsrevers must approach the speed of light as $r \rightarrow 2M$. There are no such observers inside the black hole.

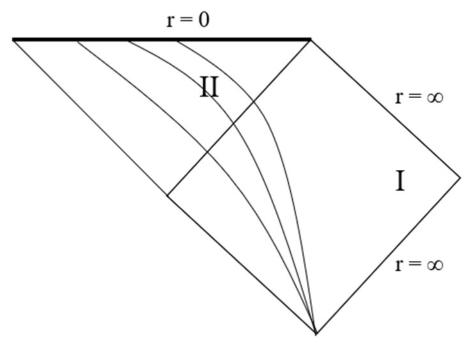

FIG. 2.2. Second Penrose diagram illustrating a PG observers' penetration of the outer horizon.

By contrast, in Figure 2.2 we see that PG observers fall through the outer horizon and so give us the ability to map the interior of the Schwarzschild type black hole.

2.3 Reissner-Nordström Described

We now move onto the main black hole type considered in this work, which is the RN black hole. This black hole has a charge as mentioned previously, and this does complicate things slightly. Due to the charge, an inner horizon is formed, and the interior of this region has repulsive gravity. This means that observers entering this region will not be able to reach all the way in before being repulsed outward. It is worth noting that this inner horizon is not a stable surface like the outer horizon is. Mathematically this is because the surface has negative eigenvalues associated with it, but more simply that is has trapped surfaces outside and untrapped ones inside. This inner horizon is then a problem because if we cannot enter the region deeply enough, we cannot map it in its entirety. Thus, we require an acceleration of our coordinate system, strong enough to push through the charged region of the black hole. This is achieved by the generalized PG coordinates.

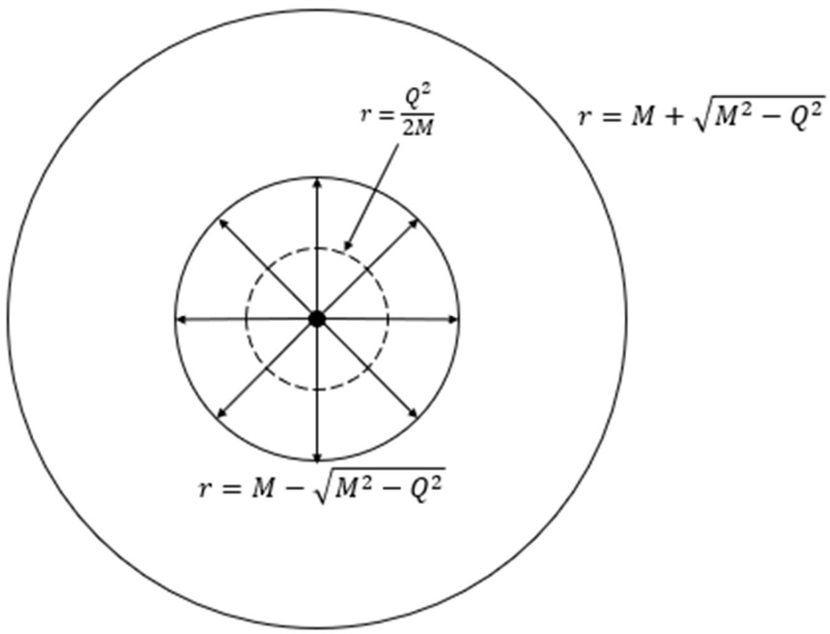

FIG. 2.3. An illustration of the RN black hole, where we see the inner horizon with lines indicating the repulsive gravity at $r = M - \sqrt{M^2 - Q^2}$. The $r = \frac{Q^2}{2M}$ indicates the region in which basic PG coordinates cannot penetrate. Finally, $r = M + \sqrt{M^2 - Q^2}$ is the outer horizon of the black hole.

As you can see from Figure 2.3, we have a region that cannot be penetrated by the original PG coordinates. We can see this by examining the PG metric, which is equation (2), if we consider the metric function for the RN black hole,

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{4}$$

When you place this in equation (2), you will see that under the square root you will have an undefined zone, due to the argument needing to be positive or zero, at,

$$r = \frac{Q^2}{2M}. \tag{5}$$

Standard PG coordinates are constructed by setting them to free-fall into the black hole. However, the repulsive gravity prevents them from reaching the singularity. To solve this problem, we instead consider coordinates that accelerate into the black hole [6]. These take the form,

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + 2\sqrt{\frac{2Mr}{r^2 + Q^2}}dtdr + \frac{r^2}{r^2 + Q^2}dr^2 + r^2 d\theta^2$$

$$+r^2 \sin^2\theta \, d\phi^2 \tag{5}$$

which is our generalized PG coordinate system. As you can see, this has already been fitted with the RN metric function, and there are no longer any singularities to worry about other than $r = 0$. Previously, I said it was beneficial to examine the Schwarzschild case to test if the results I obtain are what I'd expect, as well as to see if the code was running as intended. If you examine equation (5), and set $Q = 0$, you will notice that the metric reduces back to the PG metric with the Schwarzschild metric function. This way of testing the validity of my code proved quite useful since I could compare the reduced case to previous work [3].

2.4 Finding MOTSs Through Numerical Solving

As mentioned previously, MOTSs are surfaces in which the outward expansion of light is zero. Thus, we can find equations that describe this expansion, and set them to zero to determine where MOTSs are in a system. I shall go through this process in a simplified manner to give some slight insight into how the code for finding MOTSs works. First, we recognize that we are finding two dimensional spacelike surfaces which are axisymmetric about the z axis within our preferred spacetime. This means that surfaces must cross that axis at a 90-degree angle. We consider two null vectors, one that points outward from the surface and another that points inward, both of which are future directed [6]. Using this and a few other considerations, including that we hold time constant, looking at slices of time, we arrive at a pair of two non-linear 2nd order ordinary differential equations. I solve the equations for r and θ using Mathematica's NDSolve and then plot $\rho = r\mathrm{Cos}(\theta)$ vs $z = r\mathrm{Sin}(\theta)$.

$$\ddot{r} = -\frac{p'\dot{r}^2 - 2r\dot{\theta}^2}{2p} + \frac{r\dot{\theta}\kappa}{\sqrt{p}}, \tag{6}$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} - \frac{\sqrt{p}\dot{r}\kappa}{r}. \tag{7}$$

Where κ and p are given by,

$$\kappa = -\frac{1}{r\sqrt{p}}\left[p\dot{r}\,\mathrm{Cot}(\theta) - r\dot{\theta}\right]$$

$$+\frac{1}{2r\sqrt{p(1-pf)}}\left[rp^2\dot{r}^2f' + r\dot{r}^2p' - 2\left(r^2\dot{\theta}^2+1\right)(1-pf)\right], \quad (8)$$

and,

$$p(r) = \frac{r^2}{(r+Q)^2}. \quad (9)$$

For details of the derivation of these equations, see [6].

2.5 Shooting Method

I have described how to find equations for MOTSs, so now I will touch on the process of using the code to produce a MOTS. First, recall that I said these are surfaces that are axisymmetric about the z axis. Thus, any surface must leave and touch the axis at a 90-degree angle. What we do is shoot from certain values of r, the (coordinate) distance from the singularity of the black hole, which will actually be a value laying on the z axis, and we require for something to truly be a MOTS, it must close, that is, it should return to the z-axis at a right angle. Generally, we begin by looking for the outer horizon of the black hole and work our way inward. This is because the value for the outer horizon is known for the black hole, so it is a good way to test and see if everything is working. Then, as we move toward the center of the black hole, we will see the surface shoot off and curl inward, until eventually wrapping around again until it once again curls. This process continues on, but we are looking for the point right between the curling inward, and curling outward moments, this is where we will find a closed MOTS. As an example, I show some plots of finding a MOTS in the Schwarzschild case while testing the code.

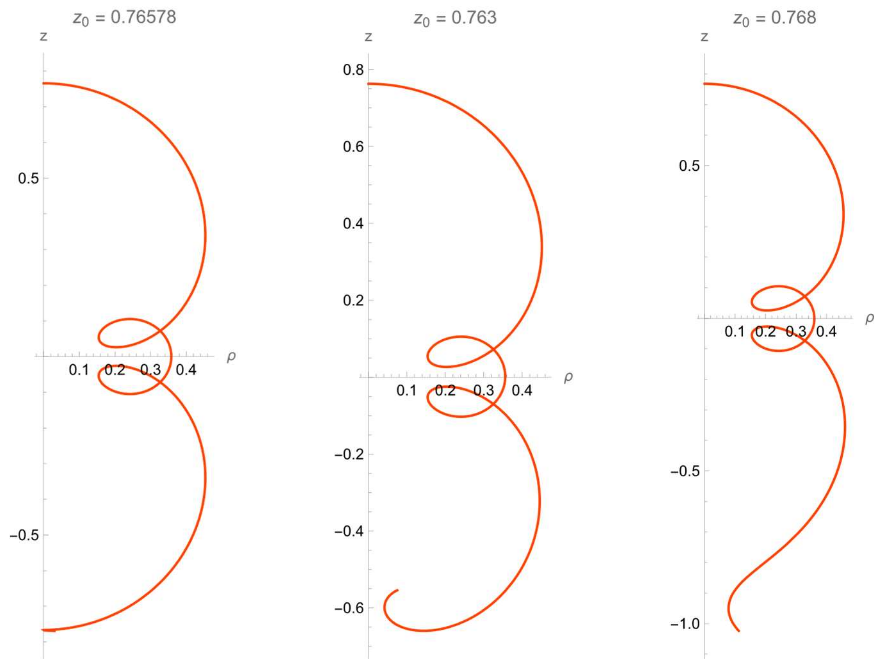

FIG. 2.4. Depiction of the shooting method used to find MOTSs by hand. In this case, finding a two looping MOTS for the Schwarzschild black hole.

As you can see from Figure 2.4, we shoot off at $z = 0.768$, we see that the curve curls out and will continue to diverge. We then shoot off from a little bit of a smaller value, $z = 0.762$, we see that the surface is now curling inward. Thus, the point in which the surface touches back down and becomes closed in between those two values. We eventually find the closed MOTS at $z = 0.764$. This shooting method is all performed by hand, as it does not take long to find these surfaces. There is a possibility for automation but because the act of finding these looping surfaces takes such little time, I decided to continue finding the surfaces by hand.

Chapter 3

Results

3.1 Schwarzschild Case

Here I will present the MOTSs found when reducing the charge value to zero. These are the MOTSs for the Schwarzschild case in the Generalized PG metric. I have plotted the first three MOTSs that I found along with the outer horizon.

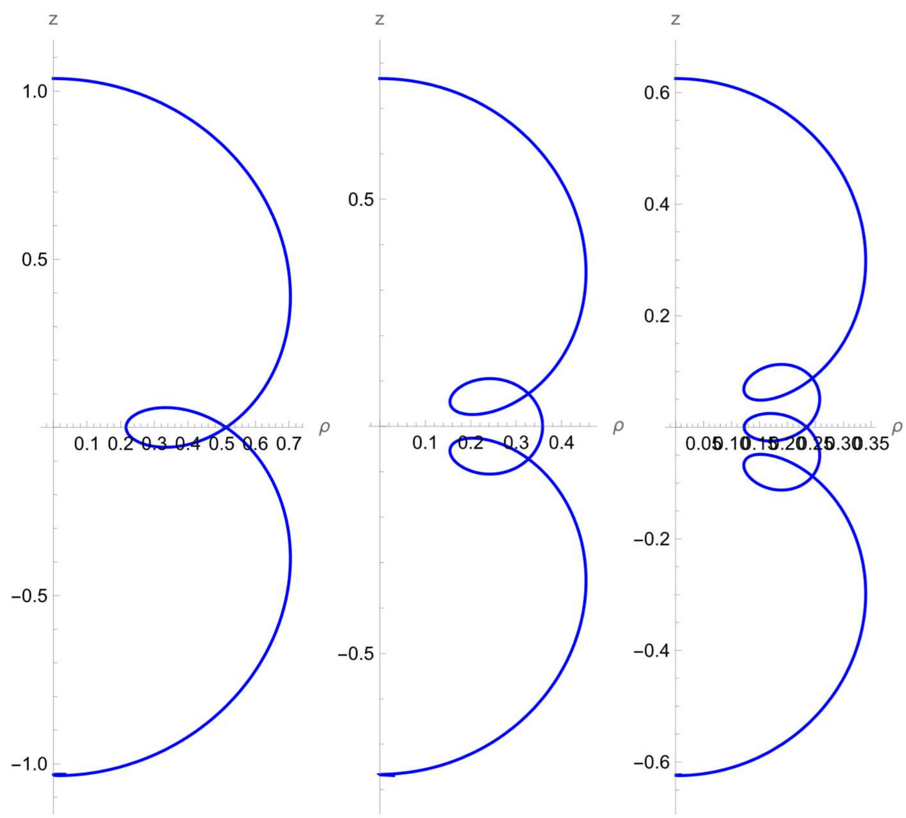

FIG. 3.1. One, two, and three looping MOTSs for the Schwarzschild case.

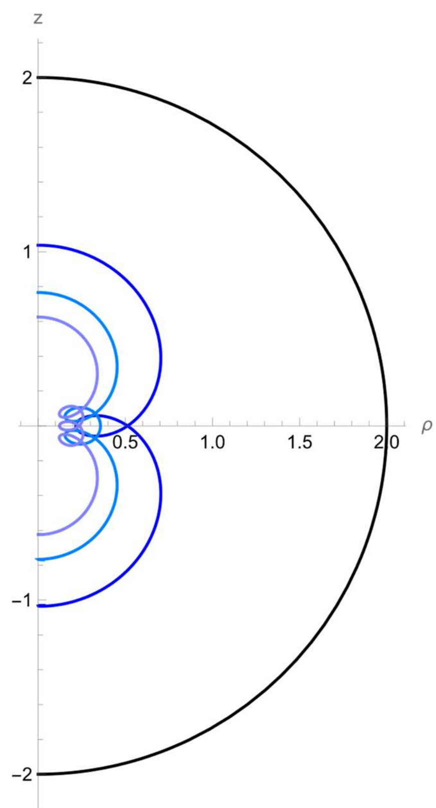

FIG. 3.2. One, two, and three looping MOTSs for Schwarzschild plotted together, along with outer horizon, which is depicted by the black line, to show scale.

Figure 3.2 depicts the results obtained through the methods stated above. To ensure that these are what I would expect, I examine results from a previous paper [3]. The results match, so all is working as hoped. It is known from previous work [3], that there are likely an infinite number of these surfaces within the Schwarzschild black hole; one for each integer number of intersections.

3.2 Reissner-Nordström Results

This is the main work of the research conducted. Ultimately, I make three important observations from the simulations. The first was found when examining the inner horizon of the black hole, which is where the gravitational force flips. When shooting off from the z axis near the inner horizon, we see a sort of "wiggling". The line moves in and out wiggling about the inner horizon. Doing this for some small deviations from the z value of the inner horizon will give us quite a pretty thing to look at, which is akin to geodesic deviation [8].

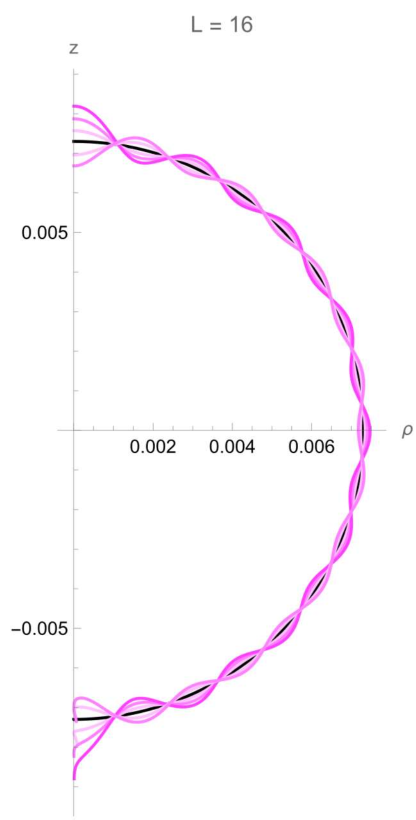

FIG. 3.3. MOTSodesic deviation about the inner horizon for L = 16, shown to have 16 intersections.

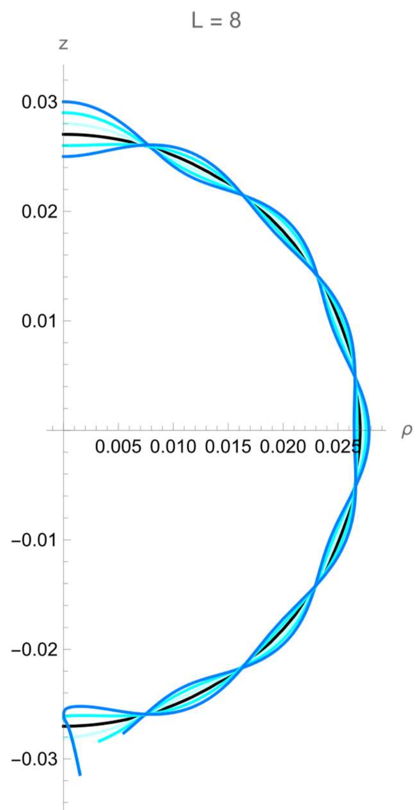

FIG. 3.4. MOTSodesic deviation about inner horizon at $L = 8$. Seen here again that we have number of intersections equal to L value.

We see in Figure 3.3 and Figure 3.4 this sort of geodesic deviation [8], which I'll refer to as "MOTSodesic deviation", with sixteen and eight intersections respectively. These intersections can be related to a value denoted as L. This value has a connection to the stability operator mentioned in section 1.2 and is proportional to the charge of the black hole.

$$Q = \frac{2\sqrt{L^2 + L + 1}}{L^2 + L + 2} \tag{6}$$

Knowing that there is a relation between the charge of the black hole and this L value, I thought it proper to categorize my work by it. Doing so, along with examining these MOTSodesic deviations about the inner horizon, lead to the understanding that at positive integer values of L, we have intersections equal to L. One thing to note is that when L is not an integer, say $L = 3.5$ we will have 4 intersections, meaning the 4th intersection always begins just after $L = 3$.

This is not only a phenomenon found around the inner horizon. While examining looping MOTSs for certain values of L, I also found that as you find increasing numbers of loops, the number of intersections increases, until the maximum number which is equal to L.

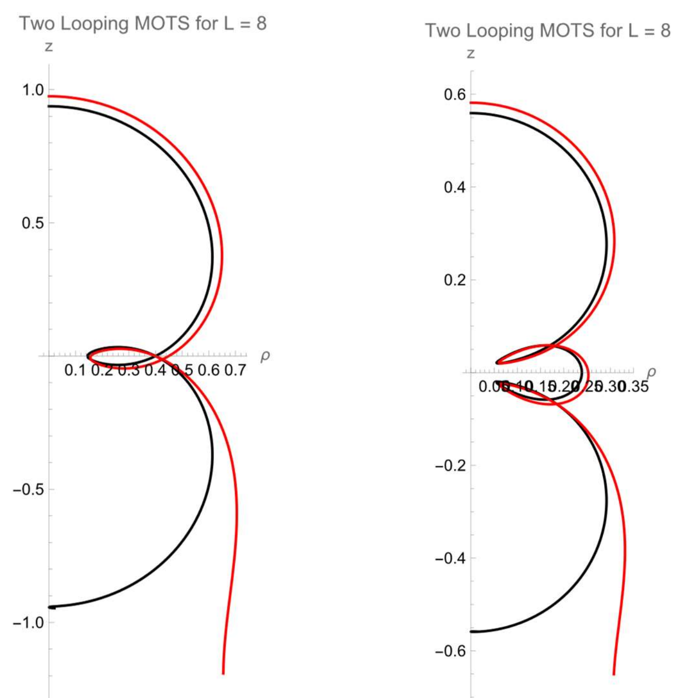

FIG. 3.5. MOTSodesic deviation about the one-looping and two-looping MOTSs for the value $L = 8$.

As you can see from Figure 3.5, the one-looping has 2 intersections, while the two-looping has 4 intersections. This example is for the $L = 8$ case. If you continue to do this for the rest of the self-intersecting MOTS, until they no longer appear, and you hit the inner horizon, you will see the intersections increase until their maximum value. Ultimately, this is just a nice connection to make but it does bring us to our next point, which is the annihilation/emission of MOTSs.

3.3 Annihilation and Emission of MOTSs

As stated previously, there are annihilation/emission events within the charged black hole. When examining the charged black hole for a certain value of charge, we will find a finite number of MOTSs. If we then lower the charge, which will decrease the size of the inner horizon, we will be able to identify more MOTSs. In turn, if we increase the charge from its original value, increasing the size of the inner horizon, we will find fewer MOTSs. This is due to annihilation/emission events from the inner horizon. I say annihilation/emission because how this is perceived depends on if you examine the events from the view of decreasing charge or increasing charge.

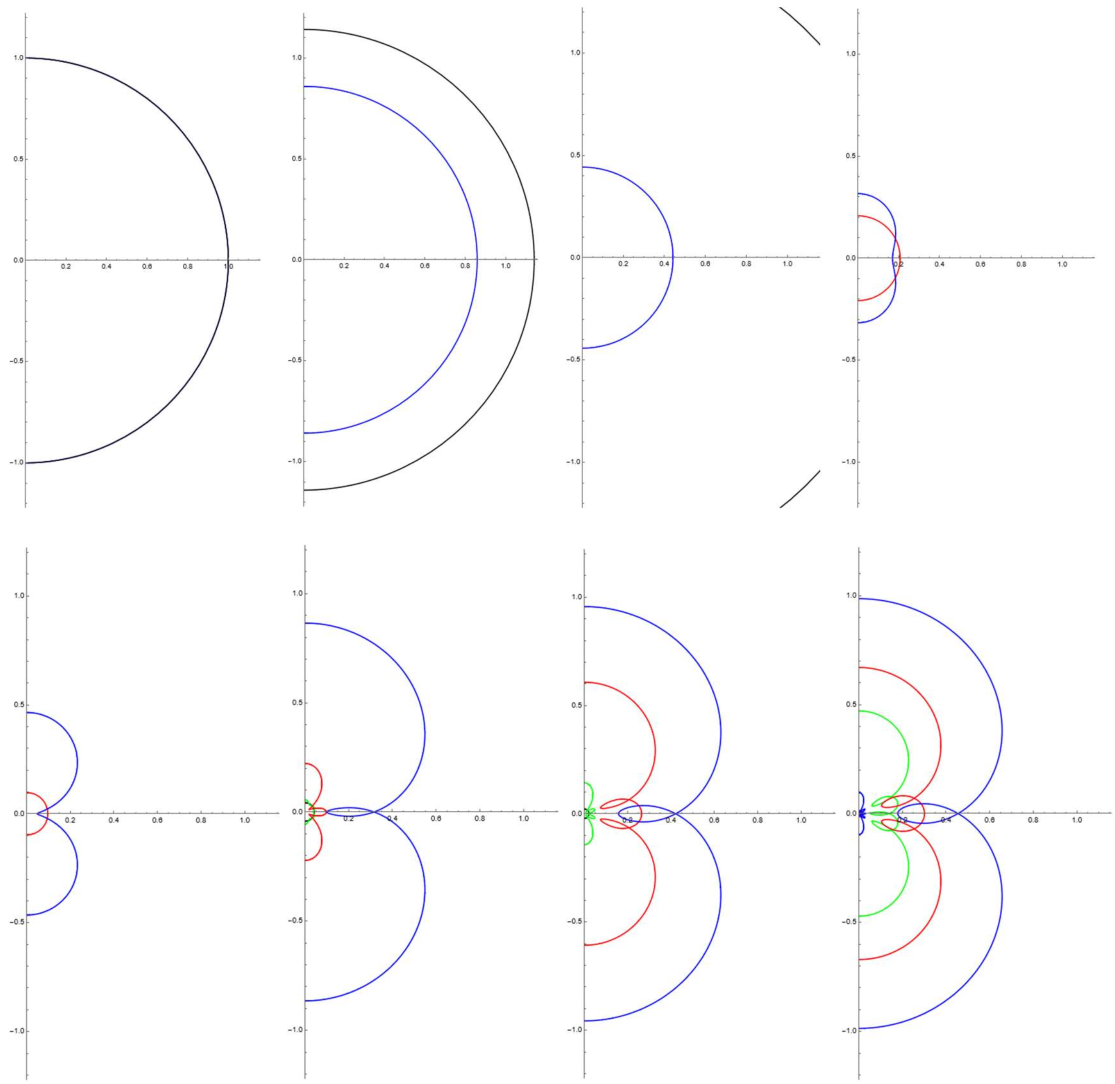

FIG. 3.6. Progression of charged black hole as charge goes from high values to low. Beginning has black outer horizon. As progression continues, semi-circle indicated where inner horizon lies, MOTSs are seen to be emitted as charge decreases.

As you can see from Figure 3.6, we see the blue, red, and green traces being emitted as the charts progress. This is an example of the charge going from quite high values to the point where the inner horizon and the outer horizon nearly touch, down to very low values of charge. The colored traces are then the MOTSs that are emitted as the inner horizon shrinks. Looking through these values of charge is akin to a quasistatic process within thermodynamics in which we are looking at equilibrium states of the system. That is, we change the charge in a manner such that when we stop, we are looking at an equilibrium state.

3.4 Transition from Looping to Wiggling

When closely examining the annihilation/emission events shown in section 3.3, I find that the MOTSs proceed interestingly. As the MOTSs are approached by the inner horizon whilst charge is increasing, we will see the MOTSs lose their looping characteristics, in favor for something more like wiggles, as shown in Figure 3.7. Even more so, as the charge progresses upward, these wiggles will become ever more apparent, until the MOTS eventually merges with the inner horizon, as shown in Figure 3.8. When examining these MOTSs further, the surfaces are being emitted from within the inner horizon as charge decreases in the black hole [6].

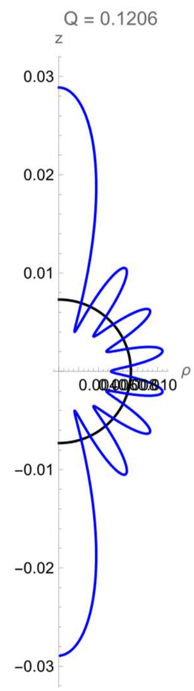

FIG. 3.7. MOTS that has lost the looping characteristic. This is referred to a wiggling MOTS that has intersected with the inner horizon.

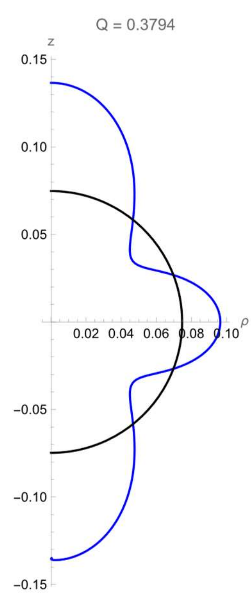

FIG. 3.8. Example of MOTS at Q = 0.3794. This MOTS is near the point of merging with the inner horizon.

Chapter 4

Conclusion

Through extensive examination of the Reissner-Nordström black hole I found a few key results. The first is that there is a connection between the stability operator and the number of intersections that MOTSodesics deviations make with the inner horizon. The second is that for finite values of charge there will be a finite number of MOTSs present, where in the limit of charge going to zero there will be an infinite number of MOTSs. Third, we see events of annihilation/emission during the increasing or decreasing of charge in the Reissner-Nordström black hole. Lastly, there is a transition of looping MOTSs to wiggling MOTSs as the charge of the black hole increases and the inner horizon approaches the MOTSs. These observations can provide grounds for further research, especially for the surfaces' behaviour within and near the inner horizon, as well as bolster the idea that studying these systems will allow us to not only understand the merging of black holes more, but the general internal geometries of black holes as well.

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